Theoretical Computer Science 244 (2000) 219–241
Theoretical Computer Science
www.elsevier.com/locate/tcs

Investigating topological chaos by elementary cellular automata dynamics ☆

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Received February 1998; revised October 1998
Communicated by M. Nivat

Abstract

We apply the two different definitions of chaos given by Devaney and by Knudsen for general discrete time dynamical systems (DTDS) to the case of elementary cellular automata, i.e., 1-dimensional binary cellular automata with radius 1. A DTDS is chaotic according to the Devaney's definition of chaos iff it is topologically transitive, has dense periodic orbits, and it is sensitive to initial conditions. A DTDS is chaotic according to the Knudsen's definition of chaos iff it has a dense orbit and it is sensitive to initial conditions. We enucleate an easy-to-check property (left or rightmost permutivity) of the local rule associated with a cellular automaton which is a sufficient condition for D-chaotic behavior. It turns out that this property is also necessary for the class of elementary cellular automata. Finally, we prove that the above mentioned property does not remain a necessary condition for chaoticity in the case of non elementary cellular automata.

1. Introduction

The notion of chaos is very appealing, and it has intrigued many scientists (see [1, 2, 10, 14, 22] for some works on the properties that characterize a chaotic process). There are simple deterministic dynamical systems that exhibit unpredictable behavior. Though counterintuitive, this fact has a very clear explanation. The lack of *infinite precision* in the description of the state of the system causes a loss of *information* which is dramatic for some processes which quickly loose their deterministic nature to assume a non deterministic (unpredictable) one. A chaotic phenomenon can indeed be viewed

☆ Partially supported by MURST 40% and 60% funds. A preliminary version of this paper has been presented to 3rd Italian Conf. on Algorithms and Complexity (CIAC'97), Lecture Notes in Computer Science, vol. 1203.

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0304-3975/00/$ - see front matter
PII: S0304-3975(98)00345-4

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                          Shift direction
                         <----------------

x =    ...????????  | 010101110 | ??????????...
          σ¹(x) =   | 10101110? |
                    | 0101110?? |
                    | 101110??? |
                    | 01110???? |
             ⋮      | 1110????? |
                    | 110?????? |
                    | 10??????? |
                    | 0???????? |
          σ⁹(x) =   | ????????? |
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Fig. 1. Finite precision combined with sensitivity to initial conditions causes unpredictability after a few iterations (x represents the state of the CA at time step 0, and $\sigma^i(x)$ the state at time step i).

as a deterministic one, in the presence of infinite precision, and as a nondeterministic one, in the presence of finite precision constraints. Thus one should look at chaotic processes as at processes merged into time, space, and precision bounds, which are the key resources in the science of computing.

A nice way in which one can analyze this finite/infinite dichotomy is by using cellular automata (CA) models. CA are dynamical systems consisting of a regular lattice of variables which can take a finite number of discrete values. The global state of the CA, specified by the values of all the variables at a given time, evolves in synchronous discrete time steps according to a given *local rule* which acts on the value of each single variable.

Consider the 1-dimensional CA $\langle X, \sigma\rangle$, where $X = \{0,1\}^{\mathbb{Z}}$ and σ is the left-shift map on X associating to any configuration $c \in \{0,1\}^{\mathbb{Z}}$ the next time step configuration $\sigma(c) \in \{0,1\}^{\mathbb{Z}}$ defined by

$$\forall i \in \mathbb{Z} \quad [\sigma(c)](i) = c(i+1).$$

In order to completely describe the elements of X, we need to operate on two-sided sequences of binary digits of infinite length. Assume for a moment that this is possible. Then the shift map is completely predictable, i.e., one can completely describe $\sigma^n(x)$, for any $x \in X$ and for any integer n. In practice, only finite objects can be computationally manipulated. Let $x \in X$. Assume we know a portion of x of length n (the portion between the two vertical lines in Fig. 1). One can easily verify that $\sigma^n(x)$ completely depends on the unknown portion of x. In other words, if we have finite precision, the shift map becomes unpredictable, as a consequence of the combination of the finite precision representation of x and the *sensitivity* of σ.

1.1. Chaos for discrete time dynamical systems

In the case of discrete time dynamical systems, $\langle X,F\rangle$, many definitions of chaos are based on the notion of sensitivity to initial conditions (see, e.g. [10, 13]). Here, we assume that the *phase space* X is equipped with a distance d and that the *next state map* $F:X\mapsto X$ is continuous on X according to the topology induced by the metric d. In particular, in the sequel, we assume that the metric space (X,d) is *perfect*, that is without isolated points.

Definition 1.1 (*Sensitivity*). A DTDS $\langle X,F\rangle$ is sensitive to initial conditions iff there exists $\delta>0$ such that

$$\forall x\in X\ \forall\varepsilon>0\ \exists y\in X\ \exists n\in\mathbb{N}:\quad (d(x,y)<\varepsilon \quad\text{and}\quad d(F^n(x),F^n(y))\geqslant\delta). \tag{1.1}$$

Constant δ is called the sensitivity constant.

Intuitively, a map is sensitive to initial conditions, or simply sensitive, if there exist points arbitrarily close to x which eventually separate from x by at least δ under iteration of F. We emphasize that not all points near x need eventually separate from x, but there must be at least one such point in every neighborhood of x. If a map possesses sensitive dependence on initial conditions, then for all practical purposes, the dynamics of the map defies numerical approximation. Small errors in computation which are introduced by round-off may become magnified upon iteration. The results of numerical computation of an orbit, no matter how accurate, may be completely different from the real orbit. As stressed above, the sensitive dependence on initial conditions require that for any state and any of its neighborhood there must exist at least a state whose dynamical evolution is affected of a δ unpredictability. In the case of **perfect** DTDS a stronger notion of sensitive dependence is the following one which, on the contrary, involves any pair of states.

Definition 1.2 (*Positive expansivity*). A DTDS $\langle X,F\rangle$ is *positively expansive* [1] iff there exists $\delta>0$ such that

$$\forall x,y\in X,\ x\neq y,\ \exists n\in\mathbb{N}:\quad d(F^n(x),F^n(y))\geqslant\delta. \tag{1.2}$$

Constant δ is called the expansivity constant.

In the case of *reversible* DTDS, i.e., DTDS $\langle X,F\rangle$ in which the next state map $F:X\mapsto X$ is one-to-one and onto, Definition 1.2 is enriched as follows (see [9] for application to ergodic theory in DTDS theory):

Definition 1.3 (*Expansivity*). A reversible DTDS $\langle X,F\rangle$ is *expansive* iff there exists $\delta>0$ such that

$$\forall x,y\in X,\ x\neq y,\quad \exists n\in\mathbb{Z}:\quad d(F^n(x),F^n(y))\geqslant\delta. \tag{1.3}$$

Constant δ is called the expansivity constant.

[1] When no confusion arises simply expansive.

In the case of continuous dynamical systems based on a metric space, there are many possible definitions of chaos, ranging from pure topological notions to notions of randomness in ergodic theory (which in any case involves a measure theoretic structure based on a metric space). In this paper we adopt the more general topological approach (for some links between the topological approach and the measure theoretical one, see for instance [4] and the references therein). We now recall some other properties which are central to topological chaos theory namely, having a *dense orbit*, *topological transitivity*, and *denseness of periodic points*.

Definition 1.4 (*Dense orbit*). A dynamical system $\langle X, F\rangle$ has a *dense orbit* iff

$$\exists x \in X\colon \forall y \in X \ \forall \varepsilon > 0 \ \exists n \in \mathbb{N}\colon \quad d(F^n(x), y) < \varepsilon. \tag{1.4}$$

For perfect DTDS the existence of a dense orbit implies topological transitivity.

Definition 1.5 (*Transitivity*). A dynamical system $\langle X, F\rangle$ is topologically *transitive* iff for all nonempty open subsets U and V of X, $\exists n \in \mathbb{N}$:

$F^n(U) \cap V \neq \emptyset$.

Intuitively, a topologically transitive map has points which eventually move under iteration from one arbitrarily small neighborhood to any other. As a consequence, the dynamical system cannot be decomposed into two disjoint closed sets which are invariant under the map (*undecomposability* condition).

Definition 1.6 (*Denseness of periodic points*). A dynamical system $\langle X, F\rangle$ has *dense periodic points* iff the set of all the periodic points of F defined by

$$Per(F) = \{x \in X \mid \exists k \in \mathbb{N}\colon F^k(x) = x\},$$

is a dense subset of X, i.e.,

$$\forall x \in X \ \forall \varepsilon > 0 \ \exists p \in Per(F)\colon \quad d(x, p) < \varepsilon. \tag{1.5}$$

Denseness of periodic points is often referred to as the *element of regularity* a chaotic dynamical system should exhibit.

The popular book by Devaney [10] isolates three components as being the essential features of topological chaos. They are formulated for a continuous map $F : X \mapsto X$, on some metric space (X, d).

Definition 1.7 (*D-chaos*). Let $F : X \mapsto X$, be a continuous map on a metric space (X, d). Then the dynamical system $\langle X, F\rangle$ is chaotic according to the Devaney's definition of chaos (D-chaotic) iff

(D_1): F is topologically transitive,
(D_2): F has dense periodic points (topological regularity), and
(D_3): F is sensitive to initial conditions.

It has been proved in [2] that for DTDS of infinite cardinality, transitivity and denseness of periodic points imply sensitivity to initial condition. As a consequence of this result, in order to prove that an *infinite* DTDS $\langle X, F\rangle$ is chaotic in the sense of Devaney, one has only to prove properties D_1 and D_2. A stronger result has been proved in [6] by one of the authors in the case of CA dynamical systems: topological transitivity implies sensitivity to initial conditions.

Knudsen in [14] proved that in the case of a dynamical system which is chaotic according to Devaney's definition, the restriction of the dynamics to the set of periodic points (which is clearly invariant) is Devaney's chaotic too. Due to the lack of non-periodicity this is not the kind of system most people would consider labeling chaotic. In view of these considerations, Knudsen proposed the following definition of chaos which excludes chaos without non-periodicity [14].

Definition 1.8 (*K-chaos*). Let $F : X \mapsto X$, be a continuous map on a metric space (X, d). Then the dynamical system $\langle X, F\rangle$ is chaotic according to the Knudsen's definition of chaos (K-chaotic) iff
(K_1): F has a dense orbit, and
(K_2): F is sensitive to initial conditions.

The two-sided shift dynamical system $\langle \mathscr{A}^{\mathbb{Z}}, \sigma\rangle$ on a finite alphabet $\mathscr{A}$ is a paradigmatic example of both Devaney's and Knudsen's chaotic system. In the case of compact and perfect DTDS, i.e., DTDS whose phase space is a compact and perfect metric space, we have the following result.

Proposition 1.1. *A compact and perfect DTDS $\langle X, F\rangle$ is topologically transitive iff it has a dense orbit. In addition, in this case the next state map $F : X \mapsto X$ is surjective.*

$$\boxed{\begin{array}{c}\textit{Transitive}\\ \equiv \\ \textit{One Dense Orbit}\end{array}} \overset{\textit{Compact}}{\Longrightarrow} \boxed{\textit{Surjective}} \tag{1.6}$$

As we will see later, the properties of being compact and perfect are the main features of the phase space of DTDS induced by CA local rules. As a consequence, the following immediately follows.
1. If a compact and perfect DTDS $\langle X, F\rangle$ is D-chaotic then it is K-chaotic.
2. In the case of a DTDS $\langle \mathscr{A}^{\mathbb{Z}}, F_f\rangle$ induced by a CA local rule f, the dynamical system is K-chaotic iff it is topologically transitive.

In the case of 1-dimensional CA dynamics, stronger definitions of chaos can be considered in order to distinguish *shift-like* chaotic behavior from more complex chaotic ones. We formulate it for perfect DTDS as follows.

Definition 1.9 (*E-chaos*). Let $F : X \mapsto X$, be a continuous map on a **perfect** metric space (X, d). Then the dynamical system $\langle X, F\rangle$ is positively expansive chaotic

(E-chaotic) iff
(E_1): F is topologically transitive,
(E_2): F has dense periodic points (topological regularity), and
(E_3): F is positively expansive.

The one-sided shift dynamical system $\langle \mathcal{A}^{\mathbb{N}}, \sigma \rangle$ on the finite alphabet $\mathcal{A}$ is a paradigmatic example of E-chaos. The two-sided shift dynamical system $\langle \mathcal{A}^{\mathbb{Z}}, \sigma \rangle$ is an example of D-chaos which is not E-chaotic.

1.2. Chaos for cellular automata

In the case of 1-dimensional CA, there have been many attempts of classification according to their asymptotic behavior (see, e.g. [5, 7, 11, 21, 24]), but none of them completely captures the notion of chaos. As an example, Wolfram divides 1-dimensional CA in four classes according to the outcome of a large number of experiments. Wolfram's classification scheme, which does not rely on a precise mathematical definition, has been formalized by Culik and Yu [8] who split CA in three classes of increasing complexity. Unfortunately membership in each of these classes is shown to be undecidable.

In this paper we complete the work initiated in [4, 5, 16], where the authors for the first time apply the definition of chaos given by Devaney and by Knudsen to CA. More precisely:

- In [5] the authors make a detailed analysis of the behavior of the elementary CA based on a particular non-additive rule (rule 180) and prove its chaoticity according to the Devaney's definition of chaos.
- In [16] the authors completely classify 1-dimensional additive CA defined over any alphabet of prime cardinality according to the Devney's definition of chaos.
- In [4] the authors completely characterize topological transitivity for every D-dimensional additive CA over $\mathbb{Z}_m$ ($m \geqslant 2$, and $D \geqslant 1$) and denseness of periodic points for any 1-dimensional additive CA over $\mathbb{Z}_m$ ($m \geqslant 2$).

In this paper we apply both the Devaney's and the Knudsen's definitions of chaos to the class of elementary CA (ECA), i.e., binary 1-dimensional CA with radius 1. To this extent, we introduce the notion of *permutivity* of a map in a certain variable. A boolean map f is permutive in the variable x_i if $f(\dots, x_i, \dots) = 1 - f(\dots, 1 - x_i, \dots)$. In other words, f is permutive in the variable x_i if any change of the value of x_i causes a change of the output produced by f, independently of the values assumed by the other variables. The main results of this paper can be summarized as follows:

(a) Every 1-dimensional CA based on a local rule f which is permutive either in the first (leftmost) or in the last (rightmost) variable is Devaney, and then Knudsen, chaotic.
(b) An ECA based on a local rule f is Devaney chaotic iff f is permutive either in the first (leftmost) or in the last (rightmost) variable (in this case Devaney and Knudsen chaoticity are equivalent).
(c) All the ECA based on a local rule f which is both rightmost and leftmost permutive are expansively chaotic.

(d) There exists a chaotic CA based on a local rule f with radius 1 which is not permutive in any variable.
(e) There exists a chaotic CA defined on a binary set of states based on a local rule f which is not permutive in any variable.

We wish to emphasize that in this paper we propose the first complete classification of the ECA rule space based on a widely accepted rigorous mathematical definition of chaos.

The rest of this paper is organized as follows. In Section 2 we give basic notation and definitions. In Section 3 we classify ECA rule space according to the Devaney's definition of chaos. In Section 4 we discuss the local entropy of ECA rule space. In Section 5 we prove that leftmost and/or rightmost permutivity is not a necessary condition for chaotic behavior of non elementary CA.

2. Notations and definitions

For $m \geqslant 2$, let $\mathscr{A} = \{0, 1, \ldots, m-1\}$ denote the ring of integers modulo m with the usual operations of addition and multiplication modulo m. We call $\mathscr{A}$ the *alphabet* of the CA. Let $f : \mathscr{A}^{2k+1} \to \mathscr{A}$, be any map depending on the $2k+1$ variables $x_{-k}, \ldots, x_k$. We say that k is the *radius* of f. A 1-dimensional CA based on the *local rule* f is the pair $\langle \mathscr{A}^{\mathbb{Z}}, F \rangle$, where

$$\mathscr{A}^{\mathbb{Z}} = \{c : \mathbb{Z} \mapsto \mathscr{A},\ i \to c(i)\}$$

is the *space of configurations* and

$$F : \mathscr{A}^{\mathbb{Z}} \mapsto \mathscr{A}^{\mathbb{Z}}$$

is the global *next state map*, defined as follows. For any configuration $c \in \mathscr{A}^{\mathbb{Z}}$ and for any $i \in \mathbb{Z}$

$$[F(c)](i) = f(c(i-k), \ldots, c(i+k)).$$

Throughout the paper, $F(c) \in \mathscr{A}^{\mathbb{Z}}$ will denote the result of the application of the map F to the configuration $c \in \mathscr{A}^{\mathbb{Z}}$ and $c(i) \in \mathscr{A}$ will denote the ith element of the configuration c. We recursively define $F^n(c)$ by $F^n(c) = F(F^{n-1}(c))$, where $F^0(c) = c$.

In order to specialize the notions of sensitivity and expansivity to the case of D-dimensional CA, we introduce the following distance (known as *Tychonoff distance*) over the space of configurations. For every $a, b \in \mathscr{A}^{\mathbb{Z}}$

$$d(a,b) = \sum_{i=-\infty}^{+\infty} \frac{1}{m^{|i|}} |a(i) - b(i)|,$$

where m is the cardinality of $\mathscr{A}$. It is easy to verify that d is a metric on $\mathscr{A}^{\mathbb{Z}}$ and that the metric topology induced by d coincides with the product topology induced by the discrete topology of $\mathscr{A}$. With this topology, $\mathscr{A}^{\mathbb{Z}}$ is a complete, compact, perfect, and totally disconnected space and F is a (uniformly) continuous map, whatever be the CA local rule inducing this global next state map. Let us recall that the metric topology induced from the Tychonoff distance is the coarsest one with respect to the component-wise convergence of sequences: the sequence of configurations $\{c_n\} \subseteq \mathscr{A}^{\mathbb{Z}}$ is convergent to the configuration $c \in \mathscr{A}^{\mathbb{Z}}$ iff for any $i \in \mathbb{Z}$,

$$\lim_{n\to\infty} c_n(i) = c(i) \in \mathscr{A}.$$

Since the alphabet $\mathscr{A}$ is finite, this component-wise convergence is in a finite number of steps, i.e.,

$$\forall i \in \mathbb{N}\ \exists n_i\colon \forall n \geqslant n_i \quad c_n(i) = c(i).$$

This gives an "empirical" criterion to test if a dynamical evolution converges to an equilibrium point (a similar consideration can be made for cyclic point convergence). During an empirical simulation, let us isolate a finite "window" of observation (for instance all the cells between the site i and the site j, with $i < j$), inside a larger portion of the initial configuration; then apply the local CA rule and look to the dynamical evolution inside the window. If after a finite number of steps we obtain that all the cells of the window reaches an equilibrium point, then we can suggest to be in presence of an equilibrium point configuration.

On the alphabet $\mathscr{A} = \{0, \dots, m-1\}$ it is possible to introduce the *conjugation* operation defined $\forall a \in \mathscr{A} \backslash \{m-1\}$, as $a^* = 1 + a$, and $(m-1)^* = 0$. In this case, it is possible to take into account, for any fixed local rule f, at least three other local rules whose global dynamics cannot be distinguished since they are all mutually isometrically conjugate.

Definition 2.1. Let $f : \mathscr{A}^{2r+1} \mapsto \mathscr{A}$ be a boolean CA local rule, we denote by

1. $f^* : \mathscr{A}^{2r+1} \mapsto \mathscr{A}$ the *conjugate* rule of f, defined by

$$\forall (x_{-r}, \dots, x_0, \dots, x_r) \in \mathscr{A}^{2r+1} \quad f^*(x_{-r}, \dots, x_0, \dots, x_r) = f(x_{-r}^*, \dots, x_0^*, \dots, x_r^*)^*.$$

2. $f^o : \mathscr{A}^{2r+1} \mapsto \mathscr{A}$ the *reflected* rule of f, defined by

$$\forall (x_{-r}, \dots, x_0, \dots, x_r) \in \mathscr{A}^{2r+1} \quad f^o(x_{-r}, \dots, x_0, \dots, x_r) = f(x_r, \dots, x_0, \dots, x_{-r}).$$

3. $f^{o*} : \mathscr{A}^{2r+1} \mapsto \mathscr{A}$ $(= f^{*o})$ the *reflected conjugate* rule of f, defined by

$$\forall (x_{-r}, \dots, x_0, \dots, x_r) \in \mathscr{A}^{2r+1} \quad f^{o*}(x_{-r}, \dots, x_0, \dots, x_r) = f(x_r^*, \dots, x_0^*, \dots, x_{-r}^*)^*.$$

To each rule f we can associate the set of (not necessarily distinct) rules

$$\mathscr{C}(f) = \{f, f^*, f^o, f^{o*}\}.$$

The transformations introduced above give rise to isometrical conjugacy between dynamical systems, as expressed in the following:

Proposition 2.1. *Let $f:\mathscr{A}^{2r+1}\mapsto\mathscr{A}$ be a local rule and f^* its conjugate, the dynamical systems $\langle\mathscr{A}^{\mathbb{Z}},F_f\rangle$ and $\langle\mathscr{A}^{\mathbb{Z}},F_{f^*}\rangle$ are isometrically conjugate. This means that the following diagram commutes*:

$$\begin{tikzcd} \mathscr{A}^{\mathbb{Z}} \arrow[r, "F_f"] \arrow[d, "\phi^*"'] & \mathscr{A}^{\mathbb{Z}} \arrow[d, "\phi^*"] \\ \mathscr{A}^{\mathbb{Z}} \arrow[r, "F_{f^*}"'] & \mathscr{A}^{\mathbb{Z}} \end{tikzcd}$$

where $\phi^*:\mathscr{A}^{\mathbb{Z}}\mapsto\mathscr{A}^{\mathbb{Z}}$ is the surjective isometry

$$c\in\mathscr{A}^{\mathbb{Z}}\to\phi^*(c)=(\ldots,\phi^*_{i-1}(c),\phi^*_i(c),\phi^*_{i+1}(c),\ldots)\in\mathscr{A}^{\mathbb{Z}}$$

defined by

$$\forall i\in\mathbb{Z}\quad \phi^*_i(c)=c(i)^*.$$

An analogous result holds for f^o and f^{o*}, defining the surjective isometries $\phi^o:\mathscr{A}^{\mathbb{Z}}\mapsto\mathscr{A}^{\mathbb{Z}}$ and $\phi^{o*}:\mathscr{A}^{\mathbb{Z}}\mapsto\mathscr{A}^{\mathbb{Z}}$ as follows: $\forall c\in\mathscr{A}^{\mathbb{Z}}$ and $\forall i\in\mathbb{Z}$

$$\phi^o_i(c)=c(-i)\quad\text{and}\quad\phi^{o*}_i(c)=c(-i)^*.$$

The isometrical conjugations between $\langle\mathscr{A}^{\mathbb{Z}},F_f\rangle$ and any of the $\langle\mathscr{A}^{\mathbb{Z}},F_{f^\tau}\rangle$, being τ one of the transformation maps $*$, o, and $o*$, implies that a great number of set theoretic, topological, and metrical qualitative dynamical properties are preserved. We mention, equilibrium and cyclic points [$c\in Per_k(F_f)$ iff $\phi^\tau(c)\in Per_k(F_{f^\tau})$], regularity, transitivity, sensitivity, expansivity, D-chaos, and K-chaos. In particular, the positive motion of initial configuration c is in a one-to-one correspondence with the positive motion of initial state $\phi^\tau(c)$:

$$\forall t\in\mathbb{N},[F_f]^t(c)\xrightarrow{\phi^\tau}[F_{f^\tau}]^t(\phi^\tau(c)). \tag{2.1}$$

Let us remark that there exists a class of CA local rules whose associated global dynamics is quite regular: the set of all its periodic points is a global attractor. We say that a local rule $f:\mathscr{A}^{2k+1}\mapsto\mathscr{A}$, is *trivial* iff there exists a map $g:\mathscr{A}\mapsto\mathscr{A}$ such that $f(x_{-k},\ldots,x_k)=g(x_0)$. Trivial CA (CA based on a trivial local rule) exhibit a very simple behavior. The generic positive orbit of any initial configuration $c=(\ldots,c(i-1),c(i),c(i+1),\ldots)$ is such that

$$\forall t\in\mathbb{N}\quad F_g^t(c)=(\ldots,g^t(c(i-1)),g^t(c(i)),g^t(c(i+1)),\ldots),$$

where, for any fixed $i\in\mathbb{Z}$, the positive orbit $g^t(c(i))$ in the *finite* phase space $\mathscr{A}$ after a finite number of steps necessarily converges to a periodic point $p_i\in\mathscr{A}$ of the DTDS $\langle\mathscr{A},g\rangle$. In conclusion, the DTDS $\langle\mathscr{A}^{\mathbb{Z}},F_g\rangle$ induced from these particular CA has the set of periodic points which is a global attractor. In particular, we have as extreme cases the *null CA rule* $f_0(x_{-k},\dots,x_k)=0$, whose corresponding global dynamics has the null configuration $(\dots,0,0,0,\dots)$ as a one step global attractor, and the *identity CA rule* $f_{id}(x_{-k},\dots,x_k)=x_0$, whose corresponding global dynamics is such that any configuration is an equilibrium point.

We now give the definition of permutive local rule and that of leftmost [resp., rightmost] permutive local rule, respectively.

Definition 2.2 (*Hedlund* [12]). A CA local rule f is *permutive* in x_i, $-k\leqslant i\leqslant k$, iff for any given sequence

$$\overline{x}_{-k},\dots,\overline{x}_{i-1},\overline{x}_{i+1},\dots,\overline{x}_k\in\mathscr{A}^{2k}$$

we have

$$\{f(\overline{x}_{-k},\dots,\overline{x}_{i-1},x_i,\overline{x}_{i+1},\dots,\overline{x}_k)\colon x_i\in\mathscr{A}\}=\mathscr{A}.$$

Definition 2.3. The CA local rule f is said to be *leftmost* [resp., *rightmost*] permutive iff there exists an integer $i\colon -k\leqslant i\leqslant 0$ [resp., $i\colon 0\leqslant i\leqslant k$] such that
1. $i\neq 0$,
2. f is permutive in the ith variable, and
3. f does not depend on x_j, $j<i$, [resp., $j>i$].

We recall the definition of additive local rule.

Definition 2.4. A CA local rule f is said to be *additive* iff there exist $2k+1$ elements $\lambda_i\in\mathscr{A}$ such that for any $(x_{-k},\dots,x_k)\in\mathscr{A}^{2k+1}$

$$f(x_{-k},\dots,x_k)=\left(\sum_{i=-k}^{k}\lambda_i x_i\right)\bmod m. \tag{2.2}$$

The following can be easily proved.

Proposition 2.2. *Let $\mathscr{A}$ be an alphabet of prime cardinality and $\langle\mathscr{A}^{\mathbb{Z}},F\rangle$ be the dynamical system induced from a non trivial additive CA local rule. Then, $\langle\mathscr{A}^{\mathbb{Z}},F\rangle$ is either leftmost or rightmost* (*and in particular cases both*) *permutive.*

Definition 2.5. A 1-dimensional CA based on a local rule $f:\mathscr{A}^{2k+1}\mapsto\mathscr{A}$, is an elementary CA (ECA) iff $k=1$ and $\mathscr{A}=\{0,1\}$.

We enumerate the $2^{2^3}=256$ different ECA as follows. The ECA based on the local rule f is associated with the natural number n_f, where

$$n_f=f(0,0,0)\cdot 2^0+f(0,0,1)\cdot 2^1+\cdots+f(1,1,0)\cdot 2^6+f(1,1,1)\cdot 2^7.$$

In the case of ECA, a rule $f:\{0,1\}^3\mapsto\{0,1\}$ is leftmost permutive iff

$$\forall x_0,x_1\colon\ f(0,x_0,x_1)\neq f(1,x_0,x_1).$$

Similarly, it is rightmost permutive iff

$$\forall x_{-1}, x_0: \quad f(x_{-1}, x_0, 0) \neq f(x_{-1}, x_0, 1).$$

Finally, an ECA local rule f is additive iff there exist three constants $a, b, c \in \{0, 1\}$ such that

$$f(x_{-1}, x_0, x_1) = a\,x_{-1} \oplus b\,x_0 \oplus c\,x_1$$

(where $\oplus$ is the usual "xor" binary boolean operation).

3. Chaotic elementary cellular automata

In this section we analyze global dynamics induced by ECA local rules with respect to both Knudsen's and Devaney's definitions of chaos.

3.1. Leftmost or rightmost permutive CA: D-chaos

We recall the following result due to one of the authors.

Theorem 3.1 (Favati et al. [16]). *Let $\langle \mathscr{A}^{\mathbb{Z}}, F \rangle$ be any leftmost and/or rightmost permutive 1-dimensional CA defined on a finite alphabet $\mathscr{A}$. Then $\langle \mathscr{A}^{\mathbb{Z}}, F \rangle$ is topologically transitive (K-chaotic).*

We now prove that Leftmost [Rightmost] Permutive 1-dimensional CA have dense periodic points. To this extent we need some preliminary definitions and lemmas. We say that a configuration $x \in \mathscr{A}^{\mathbb{Z}}$ is spatially periodic iff there exists $s \in \mathbb{N}$ such that $\sigma^s(x) = x$.

Lemma 3.1. *Let $\langle \mathscr{A}^{\mathbb{Z}}, F \rangle$ be a surjective CA. Every predecessor according to F of a spatially periodic configuration is spatially periodic*

Proof. Let $x, y \in \mathscr{A}^{\mathbb{Z}}$ be such that $F(x) = y$ and $\sigma^s(y) = y$ for some $s \in \mathbb{N}$. For every $i \in \mathbb{Z}$ we have

$$F(\sigma^{is}(x)) = \sigma^{is}(F(x)) = \sigma^{is}(y) = y.$$

Assume that x is not spatially periodic. Then there exist infinitely many predecessors of y according to F namely, $\sigma^{is}(x)$, $i \in \mathbb{Z}$. Since every 1-dimensional surjective CA have a finite number of predecessors (see [12]), we have a contradiction. $\square$

We now give the definition of *Right* [*Left*] CA.

Definition 3.1. Let $\langle \mathscr{A}^{\mathbb{Z}}, F \rangle$ be a CA based on the local rule $f(x_{-r}, \ldots, x_r)$. F is a Right [Left] CA iff f does not depend on $x_{-r}, \ldots, x_0$ [$x_0, \ldots, x_r$].

We have the following Lemma.

Lemma 3.2. *Let $\langle \mathscr{A}^{\mathbb{Z}}, F\rangle$ be a Right* [*Left*] *CA. Then $G = I - F$ is surjective.*

Proof. Since F is a Right [Left] CA, we have that $I - F$ is Leftmost [Rightmost] permutive and then surjective. □

In the next theorem we prove that for surjective Right [Left] CA periodic configurations are also spatially periodic.

Theorem 3.2. *Let $\langle \mathscr{A}^{\mathbb{Z}}, F\rangle$ be a Right* [*Left*] *CA* (*non necessarily surjective*). *Then for every $x \in \mathscr{A}^{\mathbb{Z}}$ we have*

$$(\exists t \in \mathbb{N}\colon F^t(x) = x) \;\Rightarrow\; (\exists s \in \mathbb{N}\colon \sigma^s(x) = x).$$

Proof. If x is periodic for F, i.e., $F^n(x) = x$, then x is a predecessor of the all-zero configuration $(\ldots, 0, 0, 0, \ldots)$ according to $G = I - F^n$. Since F is a Right [Left] CA then F^n is again a Right [Left] CA and then, from Lemma 3.2, we have that $G = I - F^n$ is surjective for every $n \in \mathbb{N}$. From Lemma 3.1 we conclude that x is spatially periodic. □

Corollary 3.1. *Let $\langle \mathscr{A}^{\mathbb{Z}}, F\rangle$ be a* (*non necessarily surjective*) *CA. Let $n \in \mathbb{Z}$ be such that $G = \sigma^n F$ is a Right* [*Left*] *CA global map. Every periodic configuration for G is periodic also for F, i.e.,*

$$(\exists t \in \mathbb{N}\colon G^t(x) = x) \;\Rightarrow\; (\exists t' \in \mathbb{N}\colon F^{t'}(x) = x).$$

Proof. Let x be such that $G^t(x) = x$. From Theorem 3.2 we have that there exists an $s \in \mathbb{N}$ such that $\sigma^s(x) = x$. We have

$$x = G^t(x) = G^{ts}(x) = (\sigma^n F)^{ts}(x) = \sigma^{nts} F^{ts}(x) = F^{ts} \sigma^{nts}(x) = F^{ts}(x). \qquad \square$$

We are now ready to prove the main result of this section.

Theorem 3.3. *Leftmost* [*Rightmost*] *permutive* 1*-dimensional CA have dense periodic points.*

Proof. Assume without loss of generality that F is Rightmost permutive. Let $s \in \mathbb{Z}$ be such that $G_s = \sigma^s F$ is a Right CA. We now prove that G_s has dense periodic orbits. Let

$$w = (w_{-k} \cdots w_0 \cdots w_k) \in \mathscr{A}^{2k+1}$$

be any finite configuration of length $2k+1$. Let $V_0 \in \mathscr{A}_m^{\mathbb{Z}}$ be the following configuration

$$V_0 = \cdots \alpha_2 \alpha_1 w_{-k} \cdots w_0 \cdots w_k \beta_1 \beta_2 \cdots,$$

where w is centered at the origin of the lattice, i.e., $V_0(0)=w_0$. Since G_s is a rightmost permutive CA then, in view of Theorem 3.1, it is transitive and there exist $n\in\mathbb{N}$ and

$$W_0 = \cdots \alpha_2'\alpha_1' w_{-k}\cdots w_0 \cdots w_k \beta_1'\beta_2' \cdots$$

such that

$$G^n(V_0)=W_0.$$

Let

$$W_1 = \cdots \alpha_2'\alpha_1' w_{-k}\cdots w_0 \cdots w_k \beta_1\beta_2' \cdots.$$

Since G_s is a Right and Rightmost permutive CA one can find suitable β_i'', $i\geqslant 2$, such that

$$G_s^n(\cdots \alpha_3\alpha_2\alpha_1 w_{-k}\cdots w_0 \cdots w_k \beta_1\beta_2''\beta_3'' \cdots)=W_1.$$

Let

$$V_1 = \cdots \alpha_3\alpha_2\alpha_1' w_{-k}\cdots w_0 \cdots w_k \beta_1\beta_2''\beta_3'' \cdots.$$

Since G_s is a Right CA, we have

$$G_s^n(V_1) = \cdots \alpha_3''\alpha_2''\alpha_1' w_{-k}\cdots w_0 \cdots w_k \beta_1\beta_2'\beta_3' \cdots,$$

for some α_i'', $i\geqslant 2$.

By repeating the above procedure we are able to construct a sequence of pairs of configurations (V_i,W_i) such that $G^n(V_i)=W_i$ and $V_i(j)=W_i(j)$ for $j=-i-k,\dots,k+i$ and $i=1,2,\dots$. Since $\mathscr{A}^{\mathbb{Z}}$ is a complete space we have

$$\lim_{i\to\infty} W_i = \lim_{i\to\infty} V_i = W \quad \text{and} \quad G^n(W)=W.$$

Since w can be arbitrarily chosen, we conclude that G has dense periodic orbits. Finally, from Corollary 3.1 we conclude that F has dense periodic points. $\square$

Taking into account (1.6), we can summarize the above results with the following chain of implications:

$$\boxed{\text{L or R Permutive}} \Rightarrow \boxed{\text{D-Chaos}} \Rightarrow \boxed{\text{K-Chaos}} \Rightarrow \boxed{\text{Surjective}} \tag{3.1}$$

3.2. *Leftmost or rightmost permutive ECA: topological chaos*

Since boolean CA are based on the alphabet $\{0,1\}$ which has prime cardinality, from Theorem 3.1 we have the following corollary.

Corollary 3.2. *All the leftmost and/or rightmost permutive ECA are D-chaotic.*

We now prove that if an ECA is neither leftmost nor rightmost permutive, then it is not surjective and then not topologically transitive. Let $\mathscr{A}=\{0,\dots,m-1\}$ be any finite alphabet. Let f be any local rule of radius $k\geqslant 0$ defined on $\mathscr{A}$. We define $f_n:\mathscr{A}^{n+2k}\mapsto\mathscr{A}^n$, as follows. For every $c\in\mathscr{A}^{n+2k}$

$$[f_n(c)](i)=f(c(i),\dots,c(i+2k)),\quad 1\leqslant i\leqslant n.$$

We denote by $f_n^{-1}(a)$ the set of the predecessors of $a\in\mathscr{A}^n$ according to the map f_n and by $\#(f_n^{-1}(a))$ its cardinality. We say that a finite configuration c_n of length n is *circular* iff $c_n(1)=c_n(n-1)$ and $c_n(2)=c_n(n)$. We recall the following result due to Hedlund.

Theorem 3.4 (Hedlund [12]). *Let $\langle\mathscr{A}^Z,F\rangle$ be the CA based on the local rule $f:\mathscr{A}^{2k+1}\mapsto\mathscr{A}$, with radius $k\geqslant 0$. Then the two following statements are equivalent*:
1. *F is surjective.*
2. *For every $n\geqslant 1$, and for every $a\in\mathscr{A}^n$, $\#(f_n^{-1}(a))=m^{2k}$.*

Let $f:\mathscr{A}^{2k+1}\mapsto\mathscr{A}$, be any local rule with radius $k\geqslant 0$. we say that f is *balanced* if for any $t\in\mathscr{A}$ $\#(f_1^{-1}(t))=m^{2k}$. We have the following result.

Theorem 3.5. *Let $\langle\{0,1\}^{\mathbb{Z}},F\rangle$ be a non trivial ECA based on the local rule $f:\{0,1\}^3\mapsto\{0,1\}$. If F is surjective, then f is either leftmost or rightmost permutive.*

Proof. Assume that f is a balanced local rule which is neither leftmost nor rightmost permutive. Then there exist two finite configurations c_1 and c_2 of length 2 such that $f_1(0c_1)=f_1(1c_1)=a$ and $f_1(c_20)=f_1(c_21)=b$, where $a,b\in\{0,1\}$. Consider the ECA for which $c_1=c_2$. We have that $\#(f_2^{-1}(ab))\geqslant 4$. Since f is balanced, one can readily verify that $\#(f_2^{-1}(ab))>4$, and then, by Theorem 3.4, we have that F is not surjective. Consider now the case $c_1\neq c_2$. There are only 4 balanced non trivial ECA which are neither leftmost nor rightmost permutive for which $c_1\neq c_2$. They are ECA 43, 113, 142, and 212. For these ECA, it is easy to check that $\#(f_3^{-1}(010))=3$. Again, from Theorem 3.4, we have that F is not surjective.

The next corollary is a direct consequence of (3.1) and Theorem 3.5.

Corollary 3.3. *Let $\langle\{0,1\}^{\mathbb{Z}},F\rangle$ be an ECA based on the local rule f. Then, the following statements are equivalent.*
1. *f is either leftmost or rightmost permutive (or both).*
2. *F is Devaney-chaotic.*
3. *F is Knudsen-chaotic.*
4. *F is surjective and non-trivial.*

Table 1
Leftmost (L), rightmost (R), and central (C) permutive ECA. Rules of the kind x^a are additive. Rules of the kind y^b are of the form $1+x^a$ (1+additive)

C	204^a, 51^b
L	240^a, 15^b, 30, 45, 75, 120, 135, 180, 210, 225
R	170^a, 85^b, 86, 89, 101, 106, 149, 154, 166, 169
LC	60^a, 195^b
CR	102^a, 153^b
LR	90^a, 165^b
LCR	150^a, 105^b

All this can be summarized by the following scheme.

$$\boxed{\text{L and/or R Permutive}} \overset{ECA}{\Longleftrightarrow} \boxed{\text{D-Chaos}} \overset{ECA}{\Longleftrightarrow} \boxed{\text{Non-trivial Surjective}} \tag{3.2}$$

In Table 1 we collect all left and/or rightmost permutive ECA local rules with information also on their central permutivity.

3.3. Leftmost and rightmost permutive CA: E-chaos

For any fixed *initial configuration* $c \in \mathscr{A}^{\mathbb{Z}}$, the CA evolves through a sequence of configurations by the iteration of the global function. The *positive orbit* (*motion*) starting from c is the sequence of configurations $\gamma_c : \mathbb{N} \mapsto \mathscr{A}^{\mathbb{Z}}$ associating with any time step $t \in \mathbb{N}$ the configuration at time t, $\gamma_c(t) \in \mathscr{A}^{\mathbb{Z}}$, obtained by the t-times iteration of the global function:

$$\forall t \in \mathbb{N} \quad \gamma_c(t) := F^t(c). \tag{3.3}$$

The *space-time pattern* of initial configuration c can be represented by a bi-infinite figure, where for the sake of simplicity we set $c^t(j) := [F^t(c)](j)$:

$$\left|\begin{array}{c|ccccccc|l}
t=0 & \dots & c(-2) & c(-1) & c(0) & c(1) & c(2) & \dots & =c \\
t=1 & \dots & c^1(-2) & c^1(-1) & c^1(0) & c^1(1) & c^1(2) & \dots & =F(c) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
t & \dots & c^t(-2) & c^t(-1) & c^t(0) & c^t(1) & c^t(2) & \dots & =F^t(c) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}\right|$$

If we use the (one-sided) sequential notation to denote the positive orbit of initial configuration c:

$$\gamma_c \equiv (c, F(c), F^2(c), \dots, F^t(c), \dots) \in [\mathscr{A}^{\mathbb{Z}}]^{\mathbb{N}}$$

then, the positive orbit of initial state $F(c)$ is represented by the left-shifted (one-sided) sequence

$$\gamma_{F(c)} \equiv (F(c), F^2(c), F^3(c), \ldots, F^{t+1}(c), \ldots) \in [\mathscr{A}^{\mathbb{Z}}]^{\mathbb{N}}.$$

In this way we obtain the following commutative diagram:

$$\begin{tikzcd} c \in \mathscr{A}^{\mathbb{Z}} \arrow[r, "F"] \arrow[d, "\phi^F"'] & \mathscr{A}^{\mathbb{Z}} \ni F(c) \arrow[d, "\phi^F"] \\ \gamma_c \in [\mathscr{A}^{\mathbb{Z}}]^{\mathbb{N}} \arrow[r, "\sigma"'] & [\mathscr{A}^{\mathbb{Z}}]^{\mathbb{N}} \ni \gamma_{F(c)} \end{tikzcd}$$

The map Φ^F, associating to any configuration c the orbit starting from c, is one-to-one but not onto; the map σ is the (one-sided) left-shift map on the alphabet of *infinite* cardinality $[\mathscr{A}^{\mathbb{Z}}]$. Since $\Phi^F(\mathscr{A}^{\mathbb{Z}})$ is shift-invariant, as a conclusion we can say that any DTDS $\langle \mathscr{A}^{\mathbb{Z}}, F \rangle$ (where the next state map F is not necessarily induced by some CA local rule) is topologically conjugated to the subshift $\langle \Phi^F(\mathscr{A}^{\mathbb{Z}}), \sigma \rangle$ based on an infinite alphabet.

We now construct families of topological semi-conjugations, based on finite alphabets, considering suitable "windows of observation" of the dynamics produced by a fixed next state map F.

Let $i, j \in \mathbb{Z}$, with $i \leqslant j$. The set of sites $[i, j] = (i, i+1, \ldots, j)$ is the *window* of observation of extreme cells i and j. For any configuration $c \in \mathscr{A}^{\mathbb{Z}}$ we define its (i, j)-*segment* (or *block*) as the portion of this configuration between the sites i and j: $[c]_{i,j} = (c(i), c(i+1), \ldots, c(j)) \in \mathscr{A}^{j-i+1}$. This segment is a word of length $j - i + 1$ based on the alphabet $\mathscr{A}$.

We can now introduce the map $\Phi^F_{i,j} : \mathscr{A}^{\mathbb{Z}} \mapsto [\mathscr{A}^{j-i+1}]^{\mathbb{N}}$ associating with any configuration $c \in \mathscr{A}^{\mathbb{Z}}$ the one-sided sequence on the *finite* alphabet $[\mathscr{A}^{j-i+1}]$:

$$\Phi^F_{i,j}(c) = [\gamma_c]_{i,j} \equiv ([c]_{i,j}, [F(c)]_{i,j}, [F^2(c)]_{i,j}, \ldots, [F^t(c)]_{i,j}, \ldots) \in [\mathscr{A}^{j-i+1}]^{\mathbb{N}}. \quad (3.4)$$

This sequence can be represented by the space-time pattern, restricted to the window of observation $[i, j]$:

$$\left| \begin{array}{c|ccc|l} t=0 & c(i) & \ldots & c(j) & = [c]_{i,j} \\ t=1 & c^1(i) & \ldots & c^1(j) & = [F(c)]_{i,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t & c^t(i) & \ldots & c^t(j) & = [F^t(c)]_{i,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array} \right|$$

In this way, we have the commutative diagram:

$$\begin{array}{ccc} c \in \mathscr{A}^{\mathbb{Z}} & \xrightarrow{F} & \mathscr{A}^{\mathbb{Z}} \ni F(c) \\ \Phi^F_{i,j} \Big\downarrow & & \Big\downarrow \Phi^F_{i,j} \\ [\gamma_c]_{i,j} \in [\mathscr{A}^{j-i+1}]^{\mathbb{N}} & \xrightarrow[\sigma]{} & [\mathscr{A}^{j-i+1}]^{\mathbb{N}} \ni [\gamma_{F(c)}]_{i,j} \end{array}$$

The two DTDS $\langle \mathscr{A}^{\mathbb{Z}}, F \rangle$ and $\langle \Phi^F_{i,j}(\mathscr{A}^{\mathbb{Z}}), \sigma \rangle$ are now only semi-conjugate since the homomorphism $\Phi^F_{i,j}$ is not one-to-one. It will be interesting to investigate under what conditions for some i and j it is possible to have a conjugation. The following is easily proved.

Proposition 3.1. *Let $\langle \mathscr{A}^{\mathbb{Z}}, F \rangle$ be the DTDS based on the next state map F, non necessarily induced from a 1-dimensional CA local rule. The following are equivalent*:
1. $\langle \mathscr{A}^{\mathbb{Z}}, F \rangle$ *is positively expansive.*
2. $\exists i, j \in \mathbb{Z}$: $\Phi^F_{i,j}(\mathscr{A}^{\mathbb{Z}})$ *is injective.*
3. $\langle \mathscr{A}^{\mathbb{Z}}, F \rangle$ *is topologically conjugate to a one-sided subshift on a finite alphabet.*

In the particular case of CA dynamics we have the following results.

Theorem 3.6. *Let $\langle \mathscr{A}^{\mathbb{Z}}, F_f \rangle$ the DTDS induced from a 1-dimensional CA local rule f. The following are equivalent.*

a. $\langle \mathscr{A}^{\mathbb{Z}}, F_f \rangle$ *is positively expansive.*

b. $\langle \mathscr{A}^{\mathbb{Z}}, F_f \rangle$ *is topologically conjugated to the one-sided subshift $\langle \Phi^F_{0,2k-1}, \sigma \rangle$, which in its turn is topologically conjugated to a one-sided full shift on a finite alphabet* [20].

c. $\langle \mathscr{A}^{\mathbb{Z}}, F_f \rangle$ *is topologically conjugated to a one-sided full shift on a finite alphabet* [18].

In particular the proof of equivalence $a \Leftrightarrow c$ is very simple and self-consistent (for another proof we can quote [15]). All the above results involve properties of the global dynamics of a 1-dimensional CA; the following result gives a very interesting link between the local behavior of a 1-dimensional CA local rule and the global positively expansive dynamics.

Theorem 3.7 (Margara [18]). *Let $\langle \mathscr{A}^{\mathbb{Z}}, F_f \rangle$ be the DTDS based on the 1-dimensional CA local rule f. If f is leftmost* and *rightmost permutive, then $\langle \mathscr{A}^{\mathbb{Z}}, F_f \rangle$ is topologically conjugated to a one-sided full shift on a finite alphabet.*

$$\boxed{\text{L and R Permutive}} \overset{CA}{\Longrightarrow} \boxed{\text{E-Chaos}} \qquad (3.5)$$

4. Rule entropies as indicators of chaos

In [3], one of us [GC] has introduced the *Rule Entropy* (RE) of a CA local rule f. Unlike other forms of entropies for CA proposed in the literature (e.g., [17, 19, 23, 24]), Rule Entropy is a measure solely of the rule structure and not of the initial configuration or the dynamical evolution of the global next state map. Furthermore, it is effectively computable since it only assumes values in a finite set. Informally, it measures how much the initial uncertainty on the values of some cells influences the knowledge about the future configurations of the CA and propagates during its evolution. In other words, the RE expresses the inherent tendency of a CA to increase disorder in presence of uncertainty.

The triple $(\{0,1\}^3, \mathscr{P}(\{0,1\}^3), \mu_c)$ is a probability space, where μ_c is the count probability measure. An ECA local rule f can be viewed as a boolean random variable on the phase space $\{0,1\}^3$. Let $x, y \in \{0,1\}$ be fixed. Let $f^l_{xy} : \{0,1\} \mapsto \{0,1\}$ be the map $f^l_{xy}(t) := f(t,x,y)$. On the probability space defined above, we introduce the partition:

$$\alpha_{f^l_{xy}} := \{A_0, A_1\},$$

where

$$A_0 = (f^l_{xy})^{-1}(0), \qquad A_1 = (f^l_{xy})^{-1}(1),$$

A_0 (resp., A_1) can be seen as the event "a measurement of the random variable f for x, y fixed gives the value 0 (resp., 1)". We have

$$\mu_c(A_1) = \frac{f(0,x,y) + f(1,x,y)}{2}$$

which represents the probability that the application of the rule with x, y fixed, respectively in the second and third position, yields the value 1. Similarly, we have that

$$\mu_c(A_0) = 1 - \frac{f(0,x,y) + f(1,x,y)}{2}.$$

Now we can calculate the entropy of the partition $\alpha_{f^l_{xy}}$ using the canonical definition of entropy for a partition:

$$H(\alpha_{f^l_{xy}}) = \sum_{a\in\{0,1\}} \frac{f(a,x,y)}{2} \log \frac{2}{\sum_{a\in\{0,1\}} f(a,x,y)}$$

$$+ \left(1 - \sum_{a\in\{0,1\}} \frac{f(a,x,y)}{2}\right) \log \frac{1}{1 - \sum_{a\in\{0,1\}} \frac{f(a,x,y)}{2}}.$$

Table 2
Left entropy (E_{l}), right entropy (E_{r}), and central entropy (E_{c}) for leftmost, rightmost, and central permutive ECA

Permutivity	Rule	E_{l}	E_{r}	E_{c}
C	204^a, 51^b	2	2	4
L	240^a, 15^b	6	0	2
R	170^a, 85^b	0	6	2
LC	60^a, 195^b	6	2	6
CR	102^a, 153^b	2	6	6
L	30, 45, 75, 120, 135, 180, 210, 225	6	3.6	4
R	86, 89, 101, 106, 149, 154, 166, 169	3.6	6	4
LR	90^a, 165^b	6	6	2
LCR	150^a, 105^b	6	6	4

For different values of the parameters x and y we obtain different partitions and then different entropies; if we sum all these quantities, we obtain the *left-1 RE*:

$$H_l^{(1)} = \sum_{x,y\in\{0,1\}} \left(\sum_{a\in\{0,1\}} \frac{f(a,x,y)}{2} \right) \log\left(\frac{2}{\sum_{a\in\{0,1\}} f(a,x,y)} \right)$$
$$+ \left(1 - \sum_{a\in\{0,1\}} \frac{f(a,x,y)}{2} \right) \log \frac{1}{1 - \sum_{a\in\{0,1\}} \frac{f(a,x,y)}{2}}.$$

Fixing only a value z we can calculate in a similar way the *left*-2 *RE* $H_l^{(2)}$; and, analogously, we can define *right*-1 and *right*-2 REs. In the sequel, we shall call *left*-*RE* the quantity $E_l = H_l^{(1)} + H_l^{(2)}$ and *right*-*RE* the quantity $E_r = H_r^{(1)} + H_r^{(2)}$, a similar definition can be given for the case of the *central*-*RE*. We say that an ECA local rule f is *independent* on the variable x_{i-1} iff $\forall x_i, x_{i+1} \in \{0,1\}$ we have $f(0,x_i,x_{i+1}) = f(1,x_i,x_{i+1})$. In [3] it has been shown that the RE captures the relationship between permutivity of the rule and, owing to the above Corollary 3.3, the chaotic behavior of the global dynamics.

Proposition 4.1 (Cattaneo et al. [3]). *An ECA local rule f has the maximum left/right rule entropy iff it is left/rightmost permutive. An ECA local rule has zero left/right rule entropy iff it is independent on the left/right variable.*

In Table 2 we collect the rules entropies for the case of all leftmost, rightmost and central permutive ECA.

As shown by the above table, and in agreement with Proposition 4.1, the E-chaos of leftmost *and* rightmost permutive ECA local rules is characterized by the maximum of RE values ($E_{\mathrm{l}} = 6$ and $E_{\mathrm{r}} = 6$); the shift, both leftmost and rightmost permutive, dynamics is characterized by the RE values $(6,0)$ and $(0,6)$. Any intermediate value of the ECA local rule entropies [for instance $(6,2)$, $(2,6)$ and $(6,3.6)$, $(6,3.6)$] can be

Table 3
Additive ECA. Notice that rules 90 and 150 are the unique ECA rules which are both leftmost and rightmost permutive

Permutivity	ECA rule	Chaos
No	0	No
C	204	No
L	240	Right-Shift
R	170	Left-Shift
L[C]	60	Right-Devaney
R[C]	102	Left-Devaney
LR	90	Expansive
LR[C]	150	Expansive

considered as an indicator of a "degree" of chaoticity between the shift-like chaotic dynamics and the "stronger" expansive chaos.

On the contrary, the ECA local rules 204 and 51, whose global dynamics is characterized by the property that all points are fixed (rule 204) or all points are cyclic of period 2 (rule 51), have very small entropy values (2,2).

4.1. Additive and affine ECA

Since for alphabets of prime cardinality nontrivial additive CAs are either leftmost or rightmost permutive (Proposition 2.2), the following is immediate.

Corollary 4.1. *Nontrivial additive CAs defined on an alphabet of prime cardinality are D-chaotic.*

Table 3 collects the main properties of additive ECA with respect to permutivity and corresponding chaos.

We consider now the subclass of ECA based on a local rule of the form $(1 + f) \bmod 2$, where f is an additive local rule, called 1-*affine* in the sequel. Precisely, we are interested to a transformation of ECA rule space $\tau_c : \mathscr{R}(ECA) \mapsto \mathscr{R}(ECA)$ associating with any ECA local rule $f : \{0,1\}^3 \to \{0,1\}$ the corresponding transformed ECA local rule $\tau_c(f) : \{0,1\}^3 \to \{0,1\}$ defined, for any $(x_{-1}, x_0, x_1) \in \{0,1\}^3$, as follows:

$$\tau_c(f)(x_{-1}, x_0, x_1) = 1 - f(x_{-1}, x_0, x_1) = 1 \oplus f(x_{-1}, x_0, x_1). \tag{4.1}$$

In particular, we deal with ECA rules obtained by transformation τ_c applied to additive ECA rules. An additive ECA rule f is necessarily 0-quiescent: $f(0,0,0) = 0$. The null configuration is a fixed point of the global dynamics: $F_f(\underline{0}) = \underline{0}$. This implies that the set $\mathscr{F}_0$ of all configurations $c = (\ldots, 0, 0, 1, *, \ldots, *, 1, 0, 0, \ldots)$ in a background of 0s (or 0-*finite* configurations), is positively invariant (i.e., a trapping subdynamical system): $F_f(\mathscr{F}_0) \subseteq \mathscr{F}_0$. The transformed ECA rule (Table 4) $\tau_c(f)$ is such that $[\tau_c(f)](0,0,0,) = 1$ and thus, any orbit starting from a 0-finite configuration after the first step enter into the set $\mathscr{F}_1$ of all configurations in a background of 1 (or 1-*finite*

Table 4
Additive ECA and corresponding τ_c-transformed ECA

Additive f	0	60	90	102	150	170	204	240
$\tau_c(f)=1\oplus f$	255	195	165	153	105	85	51	15
	$f(1,1,1)=0$				$f(1,1,1)=1$			

configurations). The further dynamical evolution depends in particular from the value $f(1,1,1)$ since

$$[\tau_c(f)](1,1,1)=\begin{cases} 1 & \text{if } f(1,1,1)=0, \\ 0 & \text{if } f(1,1,1)=1. \end{cases} \tag{4.2}$$

Therefore, if $f(1,1,1)=0$ the orbit of $F_{\tau_c f}$ is trapped in $\mathscr{F}_1$ (after one time step); otherwise, it passes alternatively from $\mathscr{F}_0$ to $\mathscr{F}_1$.

5. Permutivity vs. D-chaos for general CA

In this section we discuss the relation between leftmost and/or rightmost permutivity and the Devaney's definition of chaos in the case of non-elementary CA. We prove that leftmost and/or rightmost permutivity are not necessary conditions for having D-chaos both in the case of CA with radius 1 defined over alphabets of cardinality greater than 1 and in the case of binary CA with radius greater than 1.

Let $CA(r,m)$ denote the set of local rules of radius r defined over an alphabet of cardinality m. We have the two following results.

Theorem 5.1. *There exist a chaotic CA based on a local rule $f\in CA(1,4)$ which is not permutive in any input variable.*

Proof. We now give a CA with radius 1 over the alphabet $\mathscr{A}=\{0,1,2,3\}$ which is chaotic in the sense of Devaney but it is neither leftmost nor rightmost permutive. Consider the local rule f defined in Table 5. It takes a little effort to verify that f is not permutive in any variable. In addition, we have $f^2=\sigma$ and then F_f is D-chaotic. □

Theorem 5.2. *There exists a chaotic CA based on a local rule $f\in CA(9,2)$ which is not permutive in any input variable.*

Proof. We now construct a CA which is topologically conjugate to the D-chaotic ECA 90. The conjugacy is given by the injective binary CA based on the local rule h defined by

$$h(x_{-1},x_0,x_1,x_2)=x_0+x_{-1}x_2(1+x_1).$$

Table 5
Definition of local rule $f : \{0,1,2,3\}^3 \mapsto \{0,1,2,3\}$, ($*$ denotes any character)

x,y,z	$f(x,y,z)$	x,y,z	$f(x,y,z)$
$*,0,0$	0	$*,2,0$	0
$*,0,1$	0	$*,2,1$	0
$*,0,2$	1	$*,2,2$	1
$*,0,3$	1	$*,2,3$	1
$*,1,0$	2	$*,3,0$	2
$*,1,1$	2	$*,3,1$	2
$*,1,2$	3	$*,3,2$	3
$*,1,3$	3	$*,3,3$	3

It is easy to verify that $h \circ h = I$, where I is the identity local rule. Let g be the local rule defined by $g = h \circ f_{90} \circ h$. It is easy to verify that g is neither leftmost nor rightmost permutive. Since the binary CA based on the local rule g is topologically conjugate to the ECA 90, it satisfies the same topological properties satisfied by ECA 90 and then it is D-chaotic.

□

6. Conclusions

We have classified elementary cellular automata rule space according to the most popular definitions of chaos given for general discrete time dynamical systems: the Devaney's and the Knudsen's definition of chaos. We wish to emphasize that this is, to our knowledge, the first classification of elementary cellular automata rule space according to a rigorous mathematical definition of chaos. We are currently applying the Devaney's definition of chaos to the case of general cellular automata.

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