Q1) Similar to Q4 Problem Sheet 5

Vibrations of square drum skin with air resistance. ($\alpha < \pi c$)

PDE $\quad u_{tt} + 2\alpha u_t = c^2 u_{xx} + c^2 u_{yy} \quad 0 < x < 1$, $0 < y < 1$, $0 < t < \infty$

BC's $\quad \begin{cases} u(0,y,t) = u(1,y,t) = 0 \\ u(x,0,t) = u(x,1,t) = 0 \end{cases} \quad 0 < t < \infty$

IC $\quad \begin{cases} u_t(x,y,0) = 0, \\ u(x,y,0) = xy \end{cases} \quad 0 \le x \le 1 \;\&\; 0 \le y \le 1$

Q2) By Eigenfunction Expansion (Fourier Transform)

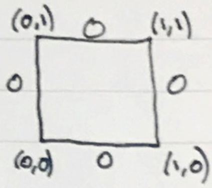
PDE $\quad u_t = D u_{xx} \quad 0 < x < 1 \quad 0 < t < \infty$

BC's $\quad \begin{cases} u(0,t) = 0 \\ u(1,t) = \cos(t) \end{cases} \quad 0 < t < \infty$

IC $\quad u(x,0) = x \quad 0 \le x \le 1$

Animations of the solution to these PDE's are on blackboard :)

Q1)



$$\mu = X(x)Y(y)T(t)$$

$$XY\ddot{T} + 2\alpha XY\dot{T} = c^2\ddot{X}YT + c^2X\ddot{Y}T$$

$\div\, \mu = XTY$

$$\frac{\ddot{T}}{c^2T} + 2\alpha\frac{\dot{T}}{c^2T} = \cancel{c^2}\frac{\ddot{X}}{X} + \cancel{c^2}\frac{\ddot{Y}}{Y} = \lambda$$

$$\frac{\ddot{X}}{X} = \lambda - \frac{\ddot{Y}}{Y} = \lambda_x$$

$$\lambda - \lambda_x = \frac{\ddot{Y}}{Y} = \lambda_y$$

$$\lambda = \lambda_x + \lambda_y$$

$$\frac{\ddot{T}}{c^2T} + 2\alpha\frac{\dot{T}}{c^2T} = \lambda$$

$X(0) = X(1) = 0$
$Y(0) = Y(1) = 0$

$$\frac{\ddot{X}}{X} = \lambda_x \quad , \quad \frac{\ddot{Y}}{Y} = \lambda_y$$

$$\ddot{X} = \lambda_x X \qquad \lambda_x = -k_x^2$$

$$\ddot{x} = -k_x^2 X$$

$\lambda_x = -n_x^2\pi^2$
$k_x = n_x\pi$

$$m^2 = -k_x^2 \iff M = \pm i\sqrt{k_x}$$

$$X(x) = A\cos(k_x x) + B\sin(k_x x) \qquad X(x) = A\sin(k_x x)$$

$$Y(y) = A\sin(k_y y) \qquad \lambda_y = -n_y^2\pi^2$$

$$\lambda = -\pi^2(n_x^2 + n_y^2)$$

$$\frac{\ddot{T}}{T} + 2\alpha\frac{\dot{T}}{T} = \lambda c^2$$

$$\ddot{T} + 2\alpha\dot{T} = \lambda c^2 T$$

$$M^2 + 2\alpha M = \lambda c^2$$

$$M = \frac{-2\alpha \pm \sqrt{4\alpha^2 + 4c^2\lambda}}{2}$$

$$M = -\alpha \pm \sqrt{\alpha^2 + c^2\lambda} \qquad \alpha < \pi c$$

$$\alpha^2 + c^2\lambda < 0$$

$$\alpha^2 < -c^2\lambda \qquad \lambda = -\pi^2(n_x^2 + n_y^2)$$

$$\alpha^2 < c^2\pi^2(n_x^2 + n_y^2)$$

$$\alpha < c\pi\underbrace{(\underbrace{n_x + n_y}_{2})^{0.5}}_{\sqrt{2}}$$

$$\alpha < \pi c < c\pi\sqrt{2}$$

$$M = -\alpha \pm i\sqrt{-\lambda c^2 - \alpha^2}$$

$$M = -\alpha \pm i\underbrace{\sqrt{\pi^2 c^2(n_x^2 + n_y^2) - \alpha^2}}_{Q_{\underline{n}}}$$

$$\underline{n} = (n_x, n_y)$$

$$T(t) = e^{-\alpha t}\left(A\cos(Q_{\underline{n}}t) + B\sin(Q_{\underline{n}}t)\right)$$

$$u_t(x,y,0) = 0 \Rightarrow \dot{T}(0) = 0$$

$$\dot{T}(t) = -\alpha T + e^{-\alpha t}\left(-AQ_{\underline{n}}\sin(Q_{\underline{n}}t) + Q_{\underline{n}}B\cos(Q_{\underline{n}}t)\right)$$

$$T(0) = A$$

$$\dot{T}(0) = -\alpha A + Q_{\underline{n}}B = 0$$

$$\Leftrightarrow \quad B = \frac{\alpha}{Q_n}A$$

$$T(t) = e^{-\alpha t}\left(A\cos(Q_{\underline{n}}t) + \cancel{B}\frac{\alpha}{Q_{\underline{n}}}A\sin(Q_{\underline{n}}t)\right)$$

$X(x)$
$Y(y)$

$$u_{\underline{n}}(x,y,t) = A_{\underline{n}}\sin(\cancel{k}\, n_x\pi x)\sin(n_y\pi y)\, e^{-\alpha t}\left(\cos(Q_{\underline{n}}t) + \frac{\alpha}{Q_{\underline{n}}}\sin(Q_{\underline{n}}t)\right)$$

$$u(x,y,0) = xy$$

Use principle of linear superposition

$\underline{n} = (n_x, n_y)$

$$u(x,y,t) = \sum_{\underline{n}=1}^{\infty} u_{\underline{n}}(x,y,t)$$

$$u(x,y,0) = xy = \sum_{\underline{n}=1}^{\infty} A_{\underline{n}}\sin(n_x\pi x)\sin(n_y\pi y)$$

$$\langle\phi_{\underline{m}}|\phi_{\underline{n}}\rangle = \int_0^1\int_0^1 \phi_{\underline{m}}(x,y)\,\phi_{\underline{n}}(x,y)\,dx\,dy$$

$$\langle \phi_{\underline{m}} | xy \rangle = A_{\underline{m}} \underbrace{\langle \sin(m_x \pi x) \sin(m_y \pi y) | \sin(n_x \pi x) \sin(n_y \pi y) \rangle}_{1/4}$$

$$\langle \phi_{\underline{m}} | xy \rangle = \langle \sin(m_x \pi x) \sin(m_y \pi y) | xy \rangle$$

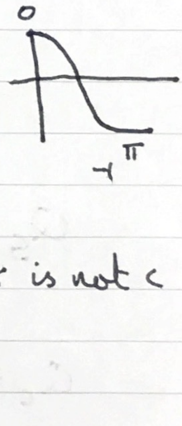
$$= \int_0^1 \int_0^1 \sin(m_x \pi x) \sin(m_y \pi y) \, xy \, dx \, dy$$

$$= \int_0^1 \sin(m_x \pi x) x \, dx \int_0^1 \sin(m_y \pi y) y \, dy$$

$$\int_0^1 \sin(m_x \pi x) x \, dx \qquad u = x \qquad \dot{v} = \sin(m_x \pi x)$$

$$\dot{u} = 1 \qquad v = \frac{-\cos(m_x \pi x)}{m_x \pi}$$

$$= \left[\frac{-x \cos(m_x \pi x)}{m_x \pi} \right]_0^1 \qquad \cos(m_x \pi) = (-1)^{m_x}$$



$$= - \frac{(-1)^{m_x}}{m_x \pi}$$

When α is not $< \pi c$

$$A_{\underline{m}} = \frac{4(-1)^{m_x + m_y}}{\pi^2 m_x m_y}$$

$$u(x, y, t) = \sum_{\underline{n}=1}^{\infty} \frac{4(-1)^{n_x + n_y}}{\pi^2 n_x n_y} \sin(\pi n_x x) \sin(\pi n_y y) e^{-\alpha t} \left(\cos(Q_{\underline{n}} t) + \frac{\alpha}{Q_{\underline{n}}} \sin(Q_{\underline{n}} t) \right)$$

where $Q_{\underline{n}} = \sqrt{\pi^2 c^2 (n_x^2 + n_y^2) - \alpha^2}$

Q1) Step by step Guide

- BC are Homo (Check BC's are 0, if not do a substitution)
- Separation of variable
- $u_n = XYT$
- Linear Super-position

Q2)

$$u(x,t) = w(x,t) + v(x,t)$$

$$0 = u(0,t) = \underbrace{w(0,t)}_{0} + \underbrace{v(0,t)}_{0}$$

$$\cos(t) = u(1,t) = \underbrace{w(1,t)}_{\cos(t)} + \underbrace{v(1,t)}_{0}$$

$$w(x,t) = x\cos(t)$$

$$x = u(x,0) = \underbrace{w(x,0)}_{x} + \underbrace{v(x,0)}_{0}$$

$$u_t = Du_{xx}$$

$$w_t + v_t = \underbrace{Dw_{xx}}_{0} + Dv_{xx}$$

$$\Leftrightarrow -x\sin(t) + v_t = Dv_{xx}$$

Now we get a new PDE

PDE $\quad v_t = Dv_{xx} + x\sin(t)$

BC's $\quad v(0,t) = v(1,t) = 0$

IC's $\quad v(x,0) = 0$

$$V_t = D v_{xx}$$

$$\dot{T}X = D\ddot{X}T \iff \frac{\dot{T}}{DT} = \frac{\ddot{X}}{X} = \lambda$$

$$\ddot{X} = \lambda X$$

as fits boundary conditions

$$X(x) = A\sin(n\pi x)$$

$$2\int_0^1 \sin(n\pi x)\,\phi(x)\,dx$$

$$v(x,t) = \sum_{n=1}^{\infty} \beta_n(t)\sin(n\pi x)$$

$$v_t(x,t) = \sum_{n=1}^{\infty} \dot{\beta}_n(t)\sin(n\pi x)$$

$$v_{xx}(x,t) = \sum_{n=1}^{\infty} -n^2\pi^2\beta_n(t)\sin(n\pi x)$$

$$x\sin(t) = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x)$$

$$\sum_{n=1}^{\infty}\Big(\underbrace{\dot{\beta}_n(t) + Dn^2\pi^2\beta_n(t) - f_n(t)}_{0}\Big)\sin(n\pi x) = 0$$

$$\dot{\beta}_n(t) + Dn^2\pi^2\beta_n(t) = f_n(t)$$

$$IF = e^{\int Dn^2\pi^2\,dt}$$

$$IF = e^{Dn^2\pi^2 t}$$

$$\xi = Dn^2\pi^2$$

$$\dot{\beta}_n(t)e^{\xi t} + \xi\beta_n(t)e^{\xi t} = e^{\xi t}f_n(t)$$

$$\Big[e^{\xi t}\beta_n(t)\Big]_0^t = \int_0^t e^{\xi t}f_n(t)\,dt$$

$$e^{\xi t}\beta_n(t) - \underbrace{\beta_n(0)}_{0} = \int_0^{\bar{t}} e^{\xi t} f_n(t)\,dt$$

$$v(x,0) = 0 = \sum_{n=1}^{\infty} \underbrace{\beta_n(0)}_{0} \sin(n\pi x).$$

where $\xi = Dn^2\pi^2$

$$\beta_n(t) = e^{-\xi t} \int_0^{\bar{t}} e^{\xi t} f_n(t)\,dt$$

$$f_n(t) = 2\int_0^1 x\sin(t)\sin(n\pi x)\,dx = 2\sin(t)\underbrace{\int_0^1 x\sin(n\pi x)\,dx}_{-\frac{(-1)^n}{\pi n}}$$

$$\beta_n(t) = e^{-\xi t}\int_0^{\bar{t}} -\frac{2(-1)^n}{\pi n}\sin(t)e^{\xi t}\,dt$$

$$= -e^{-\xi t}\frac{2(-1)^n}{\pi n}\int_0^{\bar{t}} \sin(t)e^{\xi t}\,dt$$

$$Q_s = \int_0^{\bar{t}} \sin(t)e^{\xi t}\,dt$$

$$Q_c = \int_0^{\bar{t}} \cos(t)e^{\xi t}\,dt$$

$$u = e^{\xi t} \qquad \dot{v} = \sin(t)$$
$$\dot{u} = \xi e^{\xi t} \qquad v = -\cos(t)$$

$$Q_s = \left[-e^{\xi t}\cos(t)\right]_0^t + \xi \underbrace{\int_0^1 e^{\xi t}\cos(t)\,dt}_{Q_c}$$

$$Q_c = \left[e^{\xi t}\sin(t)\right]_0^t - \xi Q_s \qquad u = e^{\xi t} \qquad \dot{v} = \cos(t)$$
$$\dot{u} = \xi e^{\xi t} \qquad v = \sin(t)$$

$$Q_s = -e^{\xi t}\cos(t) + 1 + \xi Q_c \quad , \quad Q_c = e^{\xi t}\sin(t) - \xi Q_s$$

$$Q_s = -e^{\xi t}\cos(t) + 1 + \xi\left(e^{\xi t}\sin(t) - \xi Q_s\right)$$

$$Q_s = -e^{\xi t}\cos(t) + 1 + \xi e^{\xi t}\sin(t) - \xi^2 Q_s$$

$$Q_s = \frac{e^{\xi t}\left(\xi\sin(t) - \cos(t)\right) + 1}{1+\xi^2}$$

$$B_n(t) = -\frac{2(-1)^n}{\pi n(1+\xi^2)}\left(\xi\sin(t) - \cos(t) + e^{-\xi t}\right)$$

$$v(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin(n\pi x) \qquad w(x,t) = x\cos(t)$$

$\therefore$

$$u(x,t) = x\cos(t) - \sum_{n=1}^{\infty}\frac{2(-1)^n}{\pi n(1+\xi^2)}\left[\xi\sin(t) - \cos(t) + e^{-\xi t}\right]\sin(n\pi x)$$

$$\xi = Dn^2\pi^2$$