Example Analytic Solution Of PDE

Coupled Time Dependence (Parabolic)

$$u(0,t)=0$$

$$u_t = Du_{xx} + \alpha g \qquad u(1,t)=\cos(t)\,,\quad u(x,0)=x$$

$$g_t = Dg_{xx} + \beta u \qquad g(0,t)=0\,,\ g(1,t)=1\,,\ g(x,0)=0$$

~~Let~~ $\cancel{u=z}$ $u = v + w$ $\qquad g = z + p$

$$0 = u(0,t) = \underbrace{v(0,t)}_{0} + \underbrace{w(0,t)}_{0} \qquad 0 = g(0,t) = \underbrace{z(0,t)}_{0} + \underbrace{p(0,t)}_{0}$$

$$\overset{\cos(t)}{\cancel{0}} = u(1,t) = \underbrace{v(1,t)}_{\cos(t)} + \underbrace{w(1,t)}_{0} \qquad 1 = g(1,t) = \underbrace{z(1,t)}_{1} + \underbrace{p(1,t)}_{0}$$

$$v(x,t) = x\cos(t) \qquad z(x,t) = x$$

$$x = u(x,0) = x + \underbrace{w(x,0)}_{0} \qquad 0 = g(x,0) = x + p(x,0)$$

$$p(x,0) = -x$$

so $u = x\cos(t) + w(x,t)$ $\qquad$ so $g = x + p(x,t)$

with $\qquad$ with

$$w_t = Dw_{xx} + \alpha g + x\sin(t) \qquad p_t = Dp_{xx} + \beta u$$

$$w([0,1],t)=0,\ \ w(x,0)=0 \qquad p([0,1],t)=0,\ \ p(x,0)=-x$$

so

$$w_t = Dw_{xx} + \alpha p + x(\alpha + \sin(t))$$

$$p_t = Dp_{xx} + \beta w + \beta x\cos(t)$$

$$w([0,1],t)=0\,,\quad w(x,0)=0\,,\quad p([0,1],t)=0\,,\quad p(x,0)=-x$$

$$\gamma_n = n^2\pi^2$$

$$w^{(x,t)} = \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x) \qquad p(x,t) = \sum_{n=1}^{\infty} p_n(t)\sin(n\pi x)$$

$$w_t(x,t) = \sum_{n=1}^{\infty} \dot{w}_n(t)\sin(n\pi x) \qquad p_t(x,t) = \sum_{n=1}^{\infty} \dot{p}_n(t)\sin(n\pi x)$$

$$w_{xx} = \sum_{n=1}^{\infty}(-n^2\pi^2)w_n(t)\sin(n\pi x), \quad p_{xx} = \sum_{n=1}^{\infty}(-n^2\pi^2)p_n(t)\sin(n\pi x)$$

$$= -\sum_{n=1}^{\infty}\gamma_n w_n(t)\sin(n\pi x)$$

$$(\alpha+\sin(t))x = \sum_{n=1}^{\infty} \cancel{\alpha\sin(n\pi x)}\, f_n(t)\sin(n\pi x) \qquad \beta x\cos(t) = \sum_{n=1}^{\infty} h_n(t)\sin(n\pi x)$$

with $h_n(t) = \cancel{2}\beta\cos(t)\,\xi_n$

$$\langle \sin(m\pi x), (\alpha+\sin(t))x\rangle = \sum_{n=1}^{\infty} f_n(t)\langle\sin(m\pi x),\sin(n\pi x)\rangle$$

$$2(\alpha+\sin(t))\langle\sin(m\pi x),x\rangle = f_m(t)$$

$$a = x \qquad \dot{b} = \sin(m\pi x)$$
$$\dot{a} = 1 \qquad b = -\frac{\cos(m\pi x)}{m\pi}$$

$$\langle\sin(m\pi x),x\rangle = \int_0^1 \sin(m\pi x)\,x\,dx$$

$$= \left[-\frac{x\cos(m\pi x)}{m\pi}\right]_0^1 + \underbrace{\int_0^1 \frac{\cos(m\pi x)}{m\pi}\,dx}_{=0} = -\frac{(-1)^m}{m\pi} = \frac{\xi_m}{2}$$

so

$$f_n(t) = \cancel{2}(\alpha+\sin(t))\,\xi_m, \quad h_n(t) = \cancel{2}\beta\cos(t)\,\xi_n \qquad \xi_m = -\frac{2(-1)^m}{m\pi}$$

$$\sum_{n=1}^{\infty}\left[\dot{w}_n(t) + \gamma_n^D w_n(t) - \alpha p_n(t) - f_n(t)\right]\cancel{x}\sin(n\pi x) = 0$$

$$\sum_{n=1}^{\infty}\left[\dot{p}_n(t) + \gamma_n^D p_n(t) - \beta w_n(t) - h_n(t)\right]\sin(n\pi x) = 0$$

Coupled ODE system

$$\dot{w}_n + D\gamma_n w_n - \alpha p_n = \cancel{2}(\alpha+\sin(t))\,\xi_n$$
$$\dot{p}_n + D\gamma_n p_n - \beta w_n = \cancel{2}\beta\cos(t)\,\xi_n$$

$$-x = p(x,0) = \sum_{n=1}^{\infty} p_n(0)\sin(n\pi x) \Rightarrow \quad p_n(0) = \frac{(-1)^n}{n\pi}$$

$$0 = w(x,0) = \sum_{n=1}^{\infty} w_n(0)\sin(n\pi x) \Rightarrow \quad w_n(0) = 0$$

$$f_n(t) = (\alpha + \sin(t))\,\Xi_n$$
$$\dot{f}_n(t) = \cos(t)\,\Xi_n$$
$$h_n(t) = \beta\cos(t)\,\Xi_n$$

$$(\mathcal{L} + D\gamma_n)w_n - \alpha p_n = f_n \qquad \times\ (\mathcal{L}+D\gamma_n)$$

$$(\mathcal{L} + D\gamma_n)p_n - \beta w_n = h_n \qquad \times\ \alpha$$

$$\cancel{\beta(\mathcal{L}+D\gamma_n)w_n - \alpha\beta p_n = \alpha f_n}$$
$$\cancel{(\mathcal{L}+D\gamma_n)^2 p_n - \beta(\mathcal{L}+D\gamma_n)w_n = (\mathcal{L}+D\gamma_n)h_n}$$

$$-\alpha(\mathcal{L}+D\gamma_n)p_n + (\mathcal{L}+D\gamma_n)^2 w_n = (\mathcal{L}+D\gamma_n)f_n$$
$$\alpha(\mathcal{L}+D\gamma_n)p_n - \alpha\beta w_n = \alpha h_n$$

$$(\mathcal{L}+D\gamma_n)^2 w_n - \alpha\beta w_n = \dot{f}_n + D\gamma_n f_n + \alpha h_n$$

$$(\mathcal{L}^2 + 2D\gamma_n\mathcal{L} + D^2\gamma_n^2)w_n - \alpha\beta w_n = \dot{f}_n + D\gamma_n f_n + \alpha h_n$$

$$\ddot{w}_n + \underbrace{2D\gamma_n}_{K_n}\dot{w}_n + \underbrace{\left[D^2\gamma_n^2 - \alpha\beta\right]}_{Q_n}w_n = \dot{f}_n + D\gamma_n f_n + \alpha h_n$$

$$\ddot{w} + k_n\dot{w}_n + Q_n w_n = \cos(t)\,\Xi_n + D\gamma_n(\alpha + \sin(t))\,\Xi_n + \alpha\cos(t)\,\Xi_n\beta$$
$$= \Xi_n\left[\cos(t)\left[1+\alpha\beta\right] + D\gamma_n(\alpha + \sin(t))\right] = q_n$$
$$= \Xi_n\left[\cos(t)\left[1+\alpha\beta\right] + D\gamma_n\sin(t) + D\gamma_n\alpha\right]$$

$$m^2 + k_n m + Q_n = 0$$
$$m = \frac{-k_n \pm \sqrt{k_n^2 - 4Q_n}}{2} = \frac{-k_n \pm \sqrt{4D^2\gamma_n^2 - 4D^2\gamma_n^2 + \alpha\beta 4}}{2} = -D\gamma_n \pm \sqrt{\alpha\beta}$$

$$w_n = Ae^{m_+t} + Be^{m_-t} + P.I \quad , \quad P.I = C_1\cos(t) + C_2\sin(t) + C_3$$

$$\dot{w}_n = \cancel{Am_+e^{m_+t}} + \cancel{Bm_-e^{m_-t}} - C_1\sin(t) + C_2\cos(t)$$
$$\ddot{w}_n = \cancel{Am_+^2e^{m_+t}} + \cancel{Bm_-^2e^{m_-t}} - C_1\cos(t) - C_2\sin(t)$$

$$-C_1\cos(t) - C_2\sin(t) - k_nC_1\sin(t) + k_nC_2\cos(t) + Q_nC_1\cos(t) + Q_nC_2\sin(t) + Q_nC_3$$
$$= \cancel{C_1\cos(t) + C_2\sin(t) + C_3} \quad q_n$$
$$\cos(t)\left[-C_1 + k_nC_2 + Q_nC_1\right] + \sin(t)\left[-C_2 - k_nC_1 + Q_nC_2\right] + Q_nC_3 = q_n$$

$$q_n = \xi_n \left[\cos(t)\left[1+\alpha\beta\right] + D\gamma_n \sin(t) + D\gamma_n \alpha\right]$$

$$\cos(t)\left[Q_n C_1 + k_n C_2 - C_1\right] + \sin(t)\left[Q_n C_2 - k_n C_1 - C_2\right] + Q_n C_3 = q_n$$

$$\cos(t)\left[(Q_n-1)C_1 + k_n C_2\right] + \sin(t)\left[(Q_n-1)C_2 - k_n C_1\right] + Q_n C_3 = q_n$$

$$Q_n C_3 = \xi_n D\gamma_n \alpha$$

So we get

$$\Leftrightarrow \boxed{C_3 = \frac{\xi_n D\gamma_n \alpha}{Q_n}}$$

$$(Q_n-1)C_1 + k_n C_2 = \xi_n(1+\alpha\beta)$$
$$(Q_n-1)C_2 - k_n C_1 = \xi_n D\gamma_n$$

$$k_n C_2 + (Q_n-1)C_1 = \xi_n(1+\alpha\beta)$$
$$(Q_n-1)C_2 - k_n C_1 = \xi_n D\gamma_n$$

$$\cancel{C_2 = \frac{-k_n \xi_n(1+\alpha\beta) - (Q_n-1)\xi_n D\gamma_n}{(Q_n-1)^2 - k_n^2}}$$

$$\cancel{C_1 = \frac{\xi_n(1+\alpha\beta)(Q_n-1) - (Q_n-1) k_n D\gamma_n \xi_n}{k_n^2 - (Q_n-1)^2}}$$

so

$$\cancel{C_2 = \xi_n \frac{(1+\alpha\beta) - (Q_n-1)D\gamma_n}{(Q_n-1)^2 - k_n^2}}$$

$$\cancel{C_1 = \xi_n(Q_n-1)}$$

$$C_2 = \frac{-\xi_n k_n(1+\alpha\beta) - (Q_n-1)\xi_n D\gamma_n}{-(Q_n-1)^2 - k_n^2} = \xi_n \frac{k_n(1+\alpha\beta) + (Q_n-1)D\gamma_n}{k_n^2 + (Q_n-1)^2}$$

$$C_1 = \frac{\xi_n(1+\alpha\beta)(Q_n-1) - k_n \xi_n D\gamma_n}{+(Q_n-1)^2 + k_n^2} = \xi_n \frac{(1+\alpha\beta)(Q_n-1) - k_n D\gamma_n}{k_n^2 + (Q_n-1)^2}$$

$$\boxed{C_2 = \frac{\xi_n\left[k_n(1+\alpha\beta) + (Q_n-1)D\gamma_n\right]}{k_n^2 + (Q_n-1)^2}, \quad C_1 = \frac{\xi_n\left[(1+\alpha\beta)(Q_n-1) - k_n D\gamma_n\right]}{k_n^2 + (Q_n-1)^2}}$$

$$w_n(0)=0, \quad p_n(0) = -\xi_n \qquad \frac{\sqrt{\alpha\beta}}{\alpha} = \sqrt{\frac{\beta}{\alpha}}$$

$$w_n = Ae^{m_+ t} + Be^{m_- t} + C_1\cos(t) + C_2\sin(t) + C_3$$

$$\frac{1}{\alpha}\left[\dot{w}_n + D\gamma_n w_n - (\alpha+\sin(t))\xi_n\right] = p_n$$

$$\dot{w}_n = Am_+e^{m_+t} + Bm_-e^{m_-t} - C_1\sin(t) + C_2\cos(t)$$

$$\begin{aligned}\alpha p_n = {} & Am_+e^{m_+t} + Bm_-e^{m_-t} - C_1\sin(t) + C_2\cos(t) \\ & + D\gamma_n Ae^{m_+t} + D\gamma_n Be^{m_-t} + D\gamma_n C_2\sin(t) + D\gamma_n C_1\cos(t) + D\gamma_n C_3 \\ & - \sin(t)\xi_n - \alpha\xi_n\end{aligned}$$

$$\begin{aligned}\alpha p_n = {} & Ae^{m_+t}\left[m_+ + D\gamma_n\right] + Be^{m_-t}\left[m_- + D\gamma_n\right] + \sin(t)\left[-\xi_n + D\gamma_n C_2 - C_1\right] + \cos(t)\left[C_2 + D\gamma_n C_1\right] \\ & + D\gamma_n C_3 - \alpha\xi_n\end{aligned}$$

$$\begin{aligned}\alpha p_n = {} & \sqrt{\alpha\beta}\,\cancel{D\gamma_n}e^{m_+t}A - \cancel{\sqrt{\alpha\beta}}B\cancel{e^{m_-t}}\cancel{D\gamma_n}e^{m_-t} + \sin(t)\left[-\xi_n + D\gamma_n C_2 - C_1\right] + \cos(t)\left[C_2 + D\gamma_n C_1\right] \\ & + D\gamma_n C_3 - \alpha\xi_n\end{aligned}$$

$$\begin{aligned}p_n = {} & A\cancel{D\gamma_n}\sqrt{\frac{\beta}{\alpha}}e^{m_+t} - B\cancel{D\gamma_n}\sqrt{\frac{\beta}{\alpha}}e^{m_-t} + \sin(t)\frac{1}{\alpha}\left[-\xi_n + D\gamma_n C_2 - C_1\right] + \cos(t)\frac{1}{\alpha}\left[C_2 + D\gamma_n C_1\right] \\ & + \frac{D\gamma_n C_3}{\alpha} - \xi_n\end{aligned}$$

$$p_n(0) = A\sqrt{\frac{\beta}{\alpha}} - B\sqrt{\frac{\beta}{\alpha}} + \frac{1}{\alpha}\left[C_2 + D\gamma_n C_1\right] + \frac{D\gamma_n C_3}{\alpha} \cancel{-\xi_n} = -\xi_n$$

$$w_n(0) = A + B + C_1 + C_3 = 0$$

$$A\sqrt{\frac{\beta}{\alpha}} + B\sqrt{\frac{\beta}{\alpha}} + C_1\sqrt{\frac{\beta}{\alpha}} + C_3\sqrt{\frac{\beta}{\alpha}} = 0$$

$$A\sqrt{\frac{\beta}{\alpha}} - B\sqrt{\frac{\beta}{\alpha}} + \frac{1}{\alpha}\left[C_2 + D\gamma_n C_1\right] + \frac{D\gamma_n C_3}{\alpha} = 0$$

$$2A\sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{\beta}{\alpha}}\left[C_1 + C_3\right] + \frac{1}{\alpha}\left[C_2 + D\gamma_n(C_1 + C_3)\right] = 0$$

$$2A\sqrt{\frac{\beta}{\alpha}} = -\frac{1}{\alpha}\left[C_2 + D\gamma_n(C_1 + C_3)\right] - \sqrt{\frac{\beta}{\alpha}}\left[C_1 + C_3\right]$$

$$\boxed{A = -\frac{1}{2}\left[\frac{1}{\sqrt{\alpha\beta}}\left[C_2 + D\gamma_n(C_1 + C_3)\right] + C_1 + C_3\right]}$$

$$2B\sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{\beta}{\alpha}}\left[C_1 + C_3\right] - \frac{1}{\alpha}\left[C_2 + D\gamma_n(C_1 + C_3)\right] = 0$$

$$2B\sqrt{\frac{\beta}{\alpha}} = \frac{1}{\alpha}\left[C_2 + D\gamma_n(C_1 + C_3)\right] - \sqrt{\frac{\beta}{\alpha}}\left[C_1 + C_3\right] \iff \boxed{B = \frac{1}{2}\left[\frac{1}{\sqrt{\alpha\beta}}\left[C_2 + D\gamma_n(C_1 + C_3)\right] - (C_1 + C_3)\right]}$$