Eigen Function Expansion

$$u_t = D u_{xx} + f(x,t) \iff u_t - D u_{xx} - f(x,t) = 0$$

$$u(0,t) = u(a,t) = 0 \quad , \quad u(x,0) = 0$$

Solution $\quad u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi}{a} x\right)$ ← depends on BC's

$$u_t(x,t) = \sum_{n=1}^{\infty} \dot{u}_n(t) \sin\left(\frac{n\pi}{a} x\right)$$

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} -u_n(t) \sin\left(\frac{n\pi}{a} x\right) \left(\frac{n\pi}{a}\right)^2$$

we use fact $\gamma_n = \left(\frac{n\pi}{a}\right)^2$

$$\int_0^a \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{a} x\right) dx = 0$$

$$|f(x,t)\rangle = \sum_{n=1}^{\infty} f_n(t) \sin\left(\frac{n\pi}{a} x\right)$$

$$\left\langle \sin\left(\frac{m\pi}{a} x\right) \middle| f(x,t) \right\rangle = f_n(t) \left\langle \sin\left(\frac{m\pi}{a} x\right) \middle| \sin\left(\frac{n\pi}{a} x\right) \right\rangle$$

so $\quad \sum_{n=1}^{\infty} \left(\dot{u}_n(t) + \overset{0}{\overset{\shortparallel}{D}} u_n(t) \gamma_n - f_n(t) \right) \sin\left(\frac{n\pi}{a} x\right)$

$$\frac{\partial u_n}{\partial t} + D\gamma_n u_n = f_n(t) \qquad \text{I.F.} \quad e^{\int D\gamma_n dt} = e^{D\gamma_n t}$$

$$\frac{\partial u_n}{\partial t} e^{D\gamma_n t} + e^{D\gamma_n t} D\gamma_n u_n = e^{D\gamma_n t} f_n(t)$$

$a = e^{D\gamma_n t}$ $\quad b = \frac{\partial u_n}{\partial t}$

$\dot{a} = D\gamma_n e^{D\gamma_n t}$ $\quad \dot{b} = \frac{\partial u_n}{\partial t}$

$$\left[e^{D\gamma_n t} u_n(t) \right]_{t_0=0}^{t'} = \int_{t_0=0}^{t'} e^{D\gamma_n t} f_n(t)\, dt$$

$$\iff e^{D\gamma_n t'} u_n(t') - u_n(0) = \int \iff u_n(t') = u_n(0) e^{-D\gamma_n t'} + \int_0^{t'} e^{-D\gamma_n (t'-t)} f_n(t)\, dt$$

treat g as $g(x,t)$

$$u_t = D u_{xx} + \alpha g$$

$$v_t + w_t = D v_{xx} + D w_{xx} + \alpha g$$

$$v_t = D v_{xx}$$

where $\quad v(0,t) = 0$
$\quad v(1,t) = \cos(t)$

$$\boxed{v(x,t) = \cos(t) x}$$

$$\boxed{u = v + w}$$

want w to have 0 BC's

~~so $w_t = D w_{xx} + \alpha g$~~

~~with $w(0,t) = w(1,t) = 0$~~

$$u(0,t) = \underbrace{v(0,t)}_{0} + \underbrace{w(0,t)}_{0} = 0$$

$$u(1,t) = \underbrace{v(1,t)}_{\cos(t)} + \underbrace{w(1,t)}_{0} = \cos(t)$$

we must have v BC's
$\therefore v(x,t) = \cos(t) x$

$$\boxed{u = \cos(t) x + w(x,t)}$$

we have

$$v_t + w_t = D v_{xx} + D w_{xx} + \alpha g$$

$\therefore$

$$w_t = D w_{xx} + \underbrace{\alpha g + \sin(t) x}_{\cancel{f(x,t)}}$$

$$g_t = D g_{xx} + \beta\mu \qquad g(1,t) - g(0,t) = 0$$

$$g = q + p$$

$$g(0,t) = \underbrace{q(0,t)}_{0} + \underbrace{p(0,t)}_{0} = 0 \qquad \therefore \quad q = x$$

$$g(1,t) = \underbrace{q(1,t)}_{1} + \underbrace{p(1,t)}_{0} = 0$$

so

$$p_t = D p_{xx} + \beta\mu \qquad g([0,1],t) = 0 \qquad g = x + p$$

$$w_t = D w_{xx} + \alpha g + \sin(t)x \qquad w([0,1],t) = 0 \qquad \mu = \cos(t)x + w$$

$$p_t = D p_{xx} + \beta\cos(t)x + \beta w$$
$$w_t = D w_{xx} + \alpha x + \alpha p + \sin(t)x$$

$$\boxed{\begin{aligned} p_t &= D p_{xx} + \beta w + \underbrace{\beta\cos(t)x}_{f(x,t)} \\ w_t &= D w_{xx} + \alpha p + \underbrace{(\alpha + \sin(t))x}_{h(x,t)} \end{aligned}}$$

we have $\quad \gamma_n = n^2\pi^2$

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x) \qquad p(x,t) = \sum_{n=1}^{\infty} p_n(t)\sin(n\pi x)$$

$$w_t = \sum_{n=1}^{\infty} \dot{w}_n(t)\sin(n\pi x) \qquad p_t = \sum_{n=1}^{\infty} \dot{p}_n(t)\sin(n\pi x)$$

$$w_{xx} = -\sum_{n=1}^{\infty} w_n(t)\sin(n\pi x)\gamma_n \qquad p_{xx} = -\sum_{n=1}^{\infty} p_n(t)\sin(n\pi x)\gamma_n$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x) \quad \Longleftrightarrow \quad h(x,t) = \sum_{n=1}^{\infty} h_n(t)\sin(n\pi x)$$

so

$$\sum_{n=1}^{\infty}\left[\dot{p}_n(t) + D\gamma_n p_n(t) - \beta w_n(t) - f_n(t)\right]\sin(n\pi x) = 0$$

$$\sum_{n=1}^{\infty}\left[\dot{w}_n(t) + D\gamma_n w_n(t) - \alpha p_n(t) - h_n(t)\right]\sin(n\pi x) = 0$$

$\frac{\int \beta}{\alpha}$ $\int \frac{\beta}{\alpha}$

$\frac{c_3}{a} - A_n$

$e^{At} = (e^A)^t$

nx A..

$\ddot{x} = 5\dot{x} - 3\dot{y}$
$\ddot{y} = -6\dot{x} - 2\dot{y}$
$\ddot{x} = 5\dot{x} -$

$$\boxed{\begin{aligned} \dot{p}_n + D\gamma_n p_n - \beta w_n - f_n &= 0 \\ \dot{w}_n + D\gamma_n w_n - \alpha p_n - h_n &= 0 \end{aligned}}$$

$\dot{x} = 5x - 3y$
$\dot{y} = -6x + 2y$
$\dot{x} = [5, -3]\begin{bmatrix} x \\ y \end{bmatrix}$

$\therefore$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$\dot{x} = A\underline{x} + \underline{f}(t)$

$\frac{\partial x}{\partial t} = Ax$ x'

$\Leftrightarrow \frac{1}{x}\partial x = A\partial t$

$\int_{x_0}^{x'} \frac{1}{x}\,dx = \int_{t_0=0}^{t'} A\,dt$

$\ln\left(\frac{x'}{x_0}\right) = At'$

$x' = x_0 \exp(At)$

$\begin{bmatrix} x \\ y \end{bmatrix} = e^{At}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$

$\dot{x} = 5x - 3y + f(t)$
$\dot{y} = -6x - 2y + h(t)$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -6 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} f \\ h \end{bmatrix}$$

$\dot{\underline{q}} = A\underline{q} + \underline{c}$

$\dot{\underline{q}} - A\underline{q} = \underline{c}$

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$IF = e^{\int -A\,dt}$
$= e^{-At}$

$e^{-At}\dot{\underline{q}} - e^{-At}A\underline{q} = e^{-At}\underline{c}$

$a = \underline{q}$ $b = e^{-At}$
$\dot{a} = \dot{\underline{q}}$ $\dot{b} = -Ae^{-At}$

$$\left[e^{-At}\underline{q}\right]_0^s = \int_0^s e^{-At}\underline{c}\,dt$$

$\dot{x} = 3x - 4y + 1$
$\dot{y} = 4x - 7y + 10t$ $\Leftrightarrow$ $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 4 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 10t \end{bmatrix}$

So

$$e^{-As}\underline{q}(s) - \underline{q}_0 = \int_0^s e^{-At}\underline{c}\,dt \Leftrightarrow \underline{q}(s) = e^{As}\underline{q}_0 + \int_0^s e^{+A(s-t+s)}\underline{c}(t)\,dt$$

$(s-t)$

$$e^{At} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} e^{4t} & 0 \\ 0 & e^{-5t} \end{bmatrix}\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{4e^{6t}-1}{3e^{5t}} & \frac{-2e^{6t}+2}{3e^{5t}} \\ \frac{2e^{6t}-2}{3e^{5t}} & \frac{-e^{6t}+4}{3e^{5t}} \end{bmatrix}$$

$$e^{A(t+s)}\underline{c}(t) = \frac{1}{3}e^{-5(s-t)}\begin{bmatrix} 4e^{6(s-t)} - 1 + 10t\left(2 - 2e^{6(s-t)}\right) \\ 2e^{6(s-t)} - 2 + 10t\left(4 - e^{6(s-t)}\right) \end{bmatrix}$$

Let $D = \frac{d}{dt}$

$$\dot{x} = 3x - 4y + 1$$
$$\dot{y} = 4x - 7y + 10t$$

$\Leftrightarrow$ $\dot{x} - 3x + 4y = 1$

$\dot{y} - 4x + 7y = 10t$

so

$$(D-3)x + 4y = 1$$
$$-4x + (D+7)y = 10t$$

$-4\,? = (D-3)$

$? = -\frac{1}{4}(D-3)$

$$-\frac{1}{4}(D-3)\left[-4x + (D+7)y\right] = -\frac{1}{4}(D-3)10t$$

$\Leftrightarrow$ $(D-3)x - \frac{1}{4}(D-3)(D+7)y = -\frac{10}{4}(D-3)t$

$$4y + \frac{1}{4}(D-3)(D+7)y = 1 + 2.5(D-3)t$$

$Dt = 1$

$$16y + (D-3)(D+7)y = 4 + 10(D-3)t$$

$\Leftrightarrow$ $16y + (DD + 4D - 21)y = 4 + 10\cancel{D}t - 30t$

$$16y + \ddot{y} + 4\dot{y} - 21y = 14 - 30t$$

so

$$\ddot{y} + 4\dot{y} - 5y = 14 - 30t$$

$m^2 + 4m - 5 = 0$

$(m-1)(m+5) = 0 \therefore m = 1, -5$

so $\ddot{v} + 4\dot{v} - 5v = 0$

when $v = Ae^{t} + Be^{-5t}$

$y = v + p$ where $\ddot{p} + 4\dot{p} - 5p = 14 - 30t$

Deal with one boundary conditions

$p = At + B$

$\dot{p} = A$

$4A - 5At - 5B = 14 - 30t$

$A = +6$

$4A - 5B = 14$

$24 - 5B = 14$, $B = 2$

so

$y = Ae^{t} + Be^{-5t} + 6t + 2$

$\dot{y} = Ae^{t} - 5Be^{-5t} + 6$

$\frac{\dot{y} + 7y - 10t}{4} = x$

$x = \frac{Ae^{t} - 5Be^{-5t} + 6 + 7Ae^{t} + 7e^{-5t}B + 42t + 14}{4}$

$\mathcal{L} \quad \mathcal{L}\mathcal{L} \quad \mathcal{L} \quad \mathcal{L} \quad \mathcal{L} \quad \mathcal{L} \quad \mathcal{L} \quad \mathcal{L} \quad \mathcal{L}$

$-\frac{10t}{4}$ $\qquad$ $-\frac{10t}{4}$

$$x = \frac{8Ae^{t} + 2Be^{-5t} + 42t + 20}{4} = 2Ae^{t} + \frac{1}{2}Be^{-5t} + 10.5t + 5$$

$\mathcal{L}$

$$\dot{p}_n + D\gamma_n p_n - \beta w_n = f_n$$
$$\dot{w}_n + D\gamma_n w_n - \alpha p_n = h_n$$

$$\mathcal{L} \quad (\mathcal{L} + D\gamma_n)p_n - \beta w_n = f_n$$
$$(\mathcal{L} + D\gamma_n)w_n - \alpha p_n = h_n$$
$$\Leftrightarrow \quad -\alpha p_n + (\mathcal{L} + D\gamma_n)w_n = h_n$$

$$-\alpha X = (\mathcal{L} + D\gamma_n) \qquad (\mathcal{L} + D\gamma_n)p_n - \frac{1}{\alpha}(\mathcal{L} + D\gamma_n)^2 w_n = -\frac{1}{\alpha}(\mathcal{L} + D\gamma_n)h_n$$
$$X = -\frac{1}{\alpha}(\mathcal{L} + D\gamma_n) \qquad (\mathcal{L} + D\gamma_n)p_n - \beta w_n = f_n$$

$$\beta w_n - \frac{1}{\alpha}(\mathcal{L} + D\gamma_n)^2 w_n = -\frac{1}{\alpha}(\mathcal{L} + D\gamma_n)h_n - f_n$$

$$\beta w_n - \frac{1}{\alpha}(\mathcal{L}^2 + 2D\gamma_n\mathcal{L} + D^2\gamma_n^2)w_n = -\frac{1}{\alpha}(\mathcal{L} + D\gamma_n)h_n - f_n$$

$\times\alpha$

$$-\alpha\beta w_n + (\ddot{w}_n + 2D\gamma_n\dot{w}_n + D^2\gamma_n^2 w_n) = +\mathcal{L}h_n + D\gamma_n h_n + \alpha f_n$$

$$\ddot{w}_n + \underbrace{2D\gamma_n}_{k}\dot{w}_n + \underbrace{(D^2\gamma_n^2 - \alpha\beta)}_{\xi}w_n = \underbrace{\dot{h}_n + D\gamma_n h_n + \alpha f_n}_{q_n}$$

$k^2 - 4\xi$

$4D^2\gamma_n^2 - 4D^2\gamma_n^2 + 4\alpha\beta$

$$\ddot{w}_n + k\dot{w}_n + \xi w_n = q_n$$

$$m^2 + km + \xi = 0$$

$$m = \frac{-k \pm \sqrt{k^2 - 4\xi}}{2} = \frac{-2D\gamma_n \pm 2\sqrt{\alpha\beta}}{2} = -D\gamma_n \pm \sqrt{\alpha\beta}$$

$m_{+,n}$
$m_{-,n}$

$$\boxed{w_n = Ae^{m_{+,n}t} + Be^{m_{-,n}t} + P.I}$$

z

$$w_n = \underset{z}{\cancel{A}}e^{m_{+,n}t} + Be^{m_{-,n}t} + C_1\cos(t) + C_2\sin(t) + C_3$$

$$f(x,t) = \cancel{\beta\cos(x)}\ \beta\cos(t)x \qquad f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x)$$

$$h(x,t) = (\alpha + \sin(t))x$$

$$2\int_0^1 \sin(n\pi x)(\beta\cos(t)x)\,dx = f_n(t)$$

$$\langle \sin(n\pi x)|(\alpha+\sin(t))x = \sum_{n=1}^{\infty} h_n(t)\langle \sin(n\pi x)|\sin(n\pi x)\rangle$$

So $f_n(t) = 2\beta\cos(t)\int_0^1 x\sin(n\pi x)\,dx$ $\qquad 2\int_0^1 (\alpha+\sin(t))\overset{\sin(n\pi x)}{x}\,dx = h_n(t)$

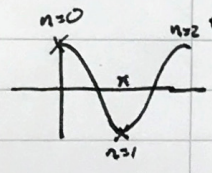
So $h_n(t) = 2(\alpha+\sin(t))\int_0^1 x\sin(n\pi x)\,dx$

with

$$a = x \qquad \dot{b} = \sin(n\pi x)$$

$$\dot{a} = 1 \qquad b = -\frac{\cos(n\pi x)}{n\pi}$$

$$\int_0^1 x\sin(n\pi x)\,dx = \left[-\frac{x\cos(n\pi x)}{n\pi}\right]_0^1 + \frac{1}{n\pi}\underbrace{\int_0^1 \cos(n\pi x)\,dx}_{0}$$



$$= -\frac{\cos(n\pi)}{n\pi} = -\frac{(-1)^n}{n\pi}$$

$\therefore$

$\cos(n\pi) = (-1)^n$ $\qquad$ So $\boxed{f_n(t) = -\frac{2\beta\cos(t)(-1)^n}{n\pi}}$

$$\boxed{h_n(t) = -\frac{2(\alpha+\sin(t))(-1)^n}{n\pi}}$$

$$q_n = \dot{h}_n + D\gamma_n h_n + \alpha f_n \qquad \dot{h}_n(t) = -\frac{2\cos(t)(-1)^n}{n\pi}$$

$$n\pi q_n = -2\cos(t)(-1)^n + D\gamma_n 2(\alpha+\sin(t))(-1)^n - 2\beta\cos(t)(-1)^n\alpha$$

$$\frac{-\frac{1}{2}n\pi q_n}{(-1)^n} = \cos(t) + D\gamma_n(\alpha+\sin(t)) + \alpha\beta\cos(t)$$

$$= (1+\alpha\beta)\cos(t) + D\gamma_n\sin(t) + D\gamma_n\alpha$$

$\Leftrightarrow$

$$q_n = \overbrace{\frac{2(-1)^{n+1}}{n\pi}}^{A_n}\left[(1+\alpha\beta)\cos(t) + D\gamma_n\sin(t) + D\gamma_n\alpha\right]$$

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$$y = C_1\cos(t) + C_2\sin(t) + C_3$$

$$\cancel{\dot{y} = -C_1\cos(t) + C_2 t} \qquad \ddot{y} = -C_1\cos(t) - C_2\sin(t)$$

$$\cancel{\dot{y} = -C_1\cos(t) + C} \qquad y = -\ddot{y} + C_3$$

$$\dot{y} = -C_1\sin(t) + C_2\cos(t) \qquad \dot{y} = -C_1\sin(t) + C_2\cos(t)$$

$$\ddot{y} + \overbrace{2D\gamma_n}^{k}\dot{y} + \overbrace{(D\gamma_n^2 - \alpha\beta)}^{\xi} y = \varepsilon_n$$

$$\ddot{y} + (D^2\gamma_n^2 - \alpha\beta)(C_3 - \ddot{y}) +$$

$$-C_1\cos(t) - C_2\sin(t) - kC_1\sin(t) + kC_2\cos(t) + \xi C_1\cos(t) + \xi C_2\sin(t) + \xi C_3 = \varepsilon_n$$

$$\cos(t)\left[-C_1 + kC_2 + \xi C_1\right] + \sin(t)\left[-C_2 - kC_1 + \xi C_2\right] + \xi C_3 = \varepsilon_n$$

$$\cos(t)\left[C_1(\xi - 1) + C_2 k\right] + \sin(t)\left[C_2(\xi - 1) - kC_1\right] + \xi C_3 = \varepsilon_n$$

$\therefore$ We get

$$\xi C_3 = \frac{2(-1)^{n+1}}{n\pi} D\gamma_n\alpha$$

$$C_1(\xi - 1) + C_2 k = A_n(1 + \alpha\beta)$$

$$C_2(\xi - 1) - C_1 k = A_n D\gamma_n$$

$$\Leftrightarrow \quad \frac{C_2(\xi - 1) - A_n D\gamma_n}{k} = C_1$$

$$\boxed{C_3 = \frac{2(-1)^{n+1} D\gamma_n\alpha}{n\pi(D^2\gamma_n^2 - \alpha\beta)}} = \frac{A_n D\gamma_n\alpha}{\xi}$$

$$\frac{C_2(\xi - 1) - A_n D\gamma_n}{k}(\xi - 1) + C_2 k = A_n(1 + \alpha\beta)$$

$$\Leftrightarrow \quad \left[C_2(\xi - 1) - A_n\gamma_n D\right](\xi - 1) + C_2 k^2 = A_n(1 + \alpha\beta)k$$

$$\Leftrightarrow \quad C_2(\xi - 1)^2 - A_n\gamma_n D(\xi - 1) + C_2 k^2 = A_n(1 + \alpha\beta)k$$

$$\Leftrightarrow \quad C_2\left[(\xi - 1)^2 + k^2\right] - A_n\gamma_n D(\xi - 1) = A_n(1 + \alpha\beta)k$$

$$\Leftrightarrow \quad C_2\left[(\xi - 1)^2 + k^2\right] = A_n\left[\gamma_n D(\xi - 1) + (1 + \alpha\beta)k\right]$$

$$C_2 = \frac{A_n D\gamma_n + C_1 k}{(\xi - 1)}$$

$$\boxed{C_2 = \frac{A_n\left[\gamma_n D(\xi - 1) + (1 + \alpha\beta)k\right]}{\left[(\xi - 1)^2 + k^2\right]}}$$

$$C_1(\xi - 1)^2 + A_n D\gamma_n k + C_1 k^2 = A_n(1 + \alpha\beta)$$

$$C_1\left[(\xi - 1)^2 + k^2\right] = A_n\left[(1 + \alpha\beta)(\xi - 1) - D\gamma_n k\right]$$

$$\boxed{C_1 = \frac{A_n\left[(1 + \alpha\beta)(\xi - 1) - D\gamma_n k\right]}{\left[(\xi - 1)^2 + k^2\right]}}$$

$$\frac{A_n\left[\gamma_n D(\xi - 1)^2 + (1 + \alpha\beta)(\xi - 1)k\right]}{\left[(\xi - 1)^2 + k^2\right]} - A_n D\gamma_n = C_1 k$$

$$\frac{A_n\left[\gamma_n D(\xi - 1)^2 + (1 + \alpha\beta)(\xi - 1)k\right] - A_n\left[D\gamma_n\left((\xi - 1)^2 + k^2\right)\right]}{\left[(\xi - 1)^2 + k^2\right]} = C_1 k$$

$$\frac{A_n\left[(1 + \alpha\beta)(\xi - 1)k - D\gamma_n k^2\right]}{\left[(\xi - 1)^2 + k^2\right]} = C_1 k \Rightarrow C_1 = \frac{A_n\left[(1 + \alpha\beta)(\xi - 1) - D\gamma_n k\right]}{\left[(\xi - 1)^2 + k^2\right]}$$

$$w(x,0) = 0$$
$$p(x,0) = -x$$

$$u(x,t) = \cos(t)x + w(x,t)$$
$$x = u(x,0) = x + \underbrace{w(x,0)}_{0}$$

$$g(x,t) = x + p(x,t)$$
$$0 = g(x,0) = x + \underbrace{p(x,0)}_{-x}$$

$$h_n = -\frac{2(\alpha + \sin(t))(-1)^n}{n\pi}$$

$$\dot{p}_n + D\gamma_n p_n - \beta w_n = f_n$$
$$\dot{w}_n + D\gamma_n w_n - \alpha p_n = h_n$$

$$\frac{1}{\alpha}\left[\dot{w}_n + D\gamma_n w_n - h_n\right] = p_n$$

$$w_n = Z e^{M_{+,n}t} + B e^{M_{-,n}t} + C_1\cos(t) + C_2\sin(t) + C_3$$
$$\dot{w}_n = Z M_{+,n} e^{M_+ t} + B M_- e^{M_- t} + -C_1\sin(t) + C_2\cos(t)$$

$$Z M_+ e^{M_+ t} + B M_- e^{M_- t} - C_1\sin(t) + C_2\cos(t) + D\gamma_n Z e^{M_+ t} + BD\gamma_n e^{M_- t} + D\gamma_n C_1\cos(t)$$
$$+ D\gamma_n C_2\sin(t) + C_3 D\gamma_n + \frac{2(\alpha + \sin(t))(-1)^n}{n\pi} = \alpha p_n$$

$$-A_n(\alpha + \sin(t))$$

$$p_n\alpha = (ZM_+ + D\gamma_n Z)e^{M_+ t} + (BM_- + BD\gamma_n)e^{M_- t}$$
$$+ \sin(t)\left[D\gamma_n C_2 - C_1 - A_n\right] + \cos(t)\left[C_2 + D\gamma_n C_1\right] + C_3 D\gamma_n - \alpha A_n$$

$$p_n = \frac{1}{\alpha}\left[Z(M_+ + D\gamma_n)e^{M_+ t} + B(M_- + D\gamma_n)e^{M_- t} + \sin(t)\left[D\gamma_n C_2 - C_1 - A_n\right] + \cos(t)\left[C_2 + D\gamma_n C_1\right] + C_3 D\gamma_n - \alpha A_n\right]$$

simplifying

$$M_+ + D\gamma_n = \sqrt{\alpha\beta}$$
$$M_- + D\gamma_n = -\sqrt{\alpha\beta}$$

$$D\gamma_n C_2 - C_1 - A_n$$

$$-A_n\left[(\xi-1)^2 + k^2\right]$$

$$\frac{A_n\left[\gamma_n^2 D^2(\xi-1) + D\gamma_n(1+\beta)k\right] - A_n\left[(1+\beta)(\xi-1) - D\gamma_n k\right]}{\left[(\xi-1)^2 + k^2\right]}$$

$$C_3 D\gamma_n \qquad \gamma_n = n^2\pi^2$$

$$\frac{\sqrt{\alpha\beta}}{\alpha} \quad \sqrt{\frac{\beta}{\alpha}} \qquad\qquad \frac{C_3}{\alpha} - A_n$$

$$A_n \frac{D\gamma_n}{(D^2\gamma_n^2 - \alpha\beta)} - A_n$$

$$\cancel{p_n = \frac{1}{\alpha}\left[Z\sqrt{\alpha\beta}\, e^{M_+ t} + B \right.}$$

$$p_n = Z\sqrt{\frac{\beta}{\alpha}}\, e^{M_+ t} - B\sqrt{\frac{\beta}{\alpha}}\, e^{M_- t} + \sin(t)\left[D\gamma_n C_2 - C_1 - A_n\right]\frac{1}{\alpha} + \cos(t)\left[C_2 + D\gamma_n C_1\right]\frac{1}{\alpha} + \frac{C_3 D\gamma_n}{\alpha} - A_n$$

$$w_n(0) = 0$$

$$-x = p(x,0) = \sum_{n=1}^{\infty} p_n(0)\sin(n\pi)$$

$$\frac{1}{\alpha}\cdot\sqrt{\frac{\alpha}{\beta}} = \frac{1}{\sqrt{\alpha\beta}}$$

$$p_n(0) = -2\int_0^1 x\sin(n\pi)\,dx = --\frac{2(-1)^n}{n\pi} = -A_n$$

At $t=0$ we have

$$\cancel{-A_n} = Z\sqrt{\frac{\beta}{\alpha}} - B\sqrt{\frac{\beta}{\alpha}} + \frac{1}{\alpha}\left[C_2 + D\gamma_n C_1\right] + \frac{C_3 D\gamma_n}{\alpha} \cancel{-A_n}$$

(w_n) $\quad 0 = Z + B + C_1 + C_3$

(p_n) $\quad 0 = Z - B + \frac{1}{\sqrt{\alpha\beta}}\left[C_2 + D\gamma_n C_1\right] + \frac{C_3 D\gamma_n}{\sqrt{\alpha\beta}}$

$$0 = 2Z + C_1 + C_3 + \frac{1}{\sqrt{\alpha\beta}}\left[C_2 + D\gamma_n C_1 + C_3 D\gamma_n\right]$$

$$\boxed{Z = -\frac{1}{2}\left[C_1 + C_3 + \frac{1}{\sqrt{\alpha\beta}}\left[C_2 + D\gamma_n C_1 + C_3 D\gamma_n\right]\right]} = -\frac{1}{2}\left[C_1 + C_3 + \frac{1}{\sqrt{\alpha\beta}}\left[C_2 + D\gamma_n(C_1 + C_3)\right]\right]$$

So $\quad 0 = 2B + C_1 + C_3 - \frac{1}{\sqrt{\alpha\beta}}\left[C_2 + D\gamma_n C_1 + C_3 D\gamma_n\right]$

$$\boxed{B = \frac{1}{2}\left[\frac{1}{\sqrt{\alpha\beta}}\left[C_2 + D\gamma_n C_1 + C_3 D\gamma_n\right] - C_1 - C_3\right]} = \frac{1}{2}\left[\frac{1}{\sqrt{\alpha\beta}}\left[C_2 + D\gamma_n(C_1 + C_3)\right] - C_1 - C_3\right]$$

$$\therefore \quad \mu(x,t) = \cos(t)x + \sum_{n=1}^{\infty} w_n(t)\sin(n\pi x) \qquad\qquad M_n = -D\gamma_n \pm \sqrt{\alpha\beta}$$

where $\quad w_n(t) = Z_n e^{M_{+,n} t} + B_n e^{M_{-,n} t} + C_1^{(n)}\cos(t) + C_2^{(n)}\sin(t) + C_3^{(n)}$