Tutorial 1

1. A source emits one of four possible symbols during each signaling interval. The symbols occur with the probabilities $\rho_0 = 0.4, \quad \rho_1 = 0.3, \quad \rho_2 = 0.2, \quad \rho_3 = 0.1$, which sum to unity as they should. Find the amount of information gained by observing the source emitting each of theses symbols.

2. A source emits one of four symbols s_0, s_1, s_2 and s_3 with probabilities $\frac{1}{3}, \frac{1}{6}, \frac{1}{4}$ and $\frac{1}{4}$ respectively. The successive symbols emitted by the source are statistically independent. Calculate the entropy of the source.

3. Let X represent the outcome of a single roll of a fair die. What is the entropy of X?

4. The sample function of a Gaussian process of zero mean and unit variance is uniformly sampled and then applied to a uniform quantizer having the input-output amplitude characteristic shown in the Figure 1. Calculate the entropy of the quantizer output.

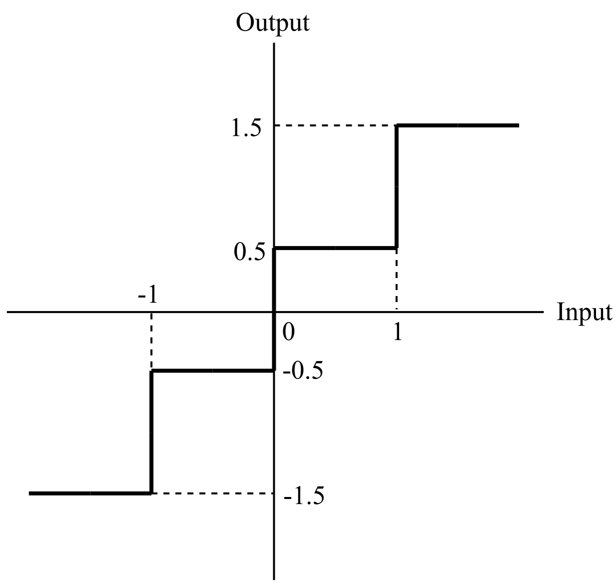

Figure 1: Quantizer

5. Consider a discrete memoryless source with source alphabet $S = \{s_0, s_1, \ldots, s_{K-1}\}$ and source statistics $S = \{p_0, p_1, \ldots, p_{K-1}\}$. The n^{th} extension of this source is another discrete memoryless source with source alphabet $S^{(n)} = \{\sigma_0, \sigma_1, \ldots, \sigma_{M-1}\}$, where $M = K^{(n)}$. Let $P(\sigma_i)$ denotes the probability of σ_i.

   (a) Show that, as expected
   $$\sum_{i=0}^{M-1} P(\sigma_i) = 1$$

   (b) Show that
   $$\sum_{i=0}^{M-1} P(\sigma_i) \log_2 \left(\frac{1}{p_{i_k}} \right) = H(S), \qquad k = 1, 2, \ldots, n$$

where p_{i_k} is the probability of symbol s_{i_k} and $H(S)$ is the entropy of the original source.

(c) Hence, show that

$$\begin{aligned} H\left(S^{(n)}\right) &= \sum_{i=0}^{M-1} P(\sigma_i) \log_2 \left(\frac{1}{P(\sigma_i)} \right) \\ &= nH(S) \end{aligned}$$

6. Consider a discrete memoryless source with source alphabet $S = \{s_0, s_1, s_2\}$ and source statistics $\{0.7, 0.15, 0.15\}$.

   (a) Calculate the entropy of the source.

   (b) Calculate the entropy of the second-order extension of the source.

7. It may come as a surprise, but the number of bits needed to store text is much less than that required to store its spoken equivalent. Can you explain the reason for this statement?

8. Let a discrete random variable X assume values in the set $\{x_1, x_2, \ldots, x_n\}$. Show that the entropy satisfies the inequality

$$H(X) \leqslant \log n$$

   and with equality if, and only if, the probability $p_i = \dfrac{1}{n}$ for all i.

9. Consider a discrete memoryless source whose alphabet consists of K equiprobable symbols.

   (a) Explain why the use of a fixed-length code for the representation of such a source is about as efficient as any code can be.

   (b) what conditions have to be satisfied by K and the codeword length for the coding efficiency to be 100% ?

10. Consider the four codes listed below:

Symbol	Code I	Code II	Code III	Code IV
S_0	0	0	0	00
S_1	10	01	01	01
S_2	110	001	011	10
S_3	1110	0010	110	110
S_4	1111	0011	111	111

   (a) Two of these four codes are prefix codes. Identify them and construct their individual decision trees.

   (b) Apply the Kraft inequality to codes I, II, III, and IV. Discuss your results in light of those obtained in part a.

11. Consider a sequence of letters of the English alphabet with their probabilities of occurrence

Letter	a	i	l	m	n	o	p	y
Probability	0.1	0.1	0.2	0.1	0.1	0.2	0.1	0.1

Compute two different Huffman codes for this alphabet. In one case, move a combined symbol in the coding procedure as high as possible; in the second case, move it as low as possible. Hence, for each of the two codes, find the average codeword length and the variance of the average codeword length over the ensemble of letters. Comment on your results.

12. Consider a discrete memoryless source with alphabet s_0, s_1, s_2 and statistics [0.7,0.15,0.15] for its output.

    (a) Apply the Huffman algorithm to this source. Hence show that the average codeword length of the Huffman code equals 1.3bits/symbol.

    (b) Let the source be extended to order two. Apply the Huffman algorithm to the resulting extended source and show that the average codeword length of the new code equals1.1975 bits/symbols.

    (c) Extend the order of the extended source to three and reapply the Huffman algorithm; hence, calculate the average codeword length.

    (d) Compare the average codeword length calculated in parts b and c with the entropy of the original source.

13. A computer executes four instructions that are designated by the codewords $(00, 01, 10, 11)$. Assuming that the instructions are used independently with probabilities $\left(\frac{1}{2}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}\right)$, calculate the percentage by which the number of bits used for the instructions may be reduced by the use of an optimum source code. Construct a Huffman code to realize the reduction.

14. Consider the following binary sequence $11101001100010110100\ldots$. Use the Lempel-Ziv algorithm to encode this sequence, assuming that the binary symbols 0 and 1 are already in the codebook.

15. Consider the transition probability diagram of a binary symmetric channel. The input binary symbols 0 and 1 occur with equal probability. Find the probabilities of the binary symbols 0 and 1 appearing at the channel output.

16. Repeat the calculation in previous problem, assuming that the binary symbols 0 and 1 occur with probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively.

17. Express the mutual information $I(X;Y)$ in terms of relative entropy

$$D(p(x,y)||p(x)p(y))$$

18. Consider a binary symmetric channel. Let p_0 denote the probability of sending binary symbol $x_0 = 0$ and let $p_1 = 1 - p_0$ denote the probability of sending binary symbol $x_1 = 1$. Let p denote the transition probability of the channel.

    (a) Show that the mutual information between the channel input and channel output is given by
    $$I(X;Y) = H(z) - H(p)$$
    where the entropy functions
    $$\begin{aligned} H(z) &= z\log_2\left(\frac{1}{z}\right) + (1-z)\log_2\left(\frac{1}{1-z}\right) \\ z &= p_0 p + (1-p_0)(1-p) \\ H(p) &= p\log_2\left(\frac{1}{p}\right) + (1-p)\log_2\left(\frac{1}{1-p}\right) \end{aligned}$$

    (b) Show that the value of p_0 that maximizes $I(X;Y)$ is equal to $\frac{1}{2}$.

    (c) Hence, show that the channel capacity equals
    $$C = 1 - H(p)$$

19. Two binary symmetric channels are connected in cascade as shown in Figure 2. Find the overall channel capacity of the cascaded connection, assuming that both channels have same transition probability.

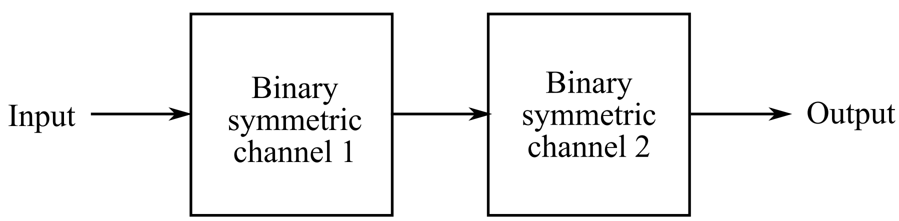

Figure 2: Binary symmetric channel

20. Consider a digital communication system that uses a *repetition code* for the channel for the channel encoding/decoding. In particular, each transmission is repeated n times, where $n = 2m+1$ is an odd integer. The decoder operates as follows. If in a block of n received bits the number of 0s exceeds the number of 1s then the decoder decides in favor of a 0; otherwise, it decides in favor of a 1. An error occurs when $m+1$ or more transmissions out of $n = 2m+1$ are incorrect. Assume a binary symmetric channel.

    (a) For n = 3, show that the average probability of error is given by
    $$P_e = 3p^2(1-p) + p^3$$
    where p is transition probability of the channel.

(b) For n =5, show that the average probability of error is given by

$$P_e = 10p^3(1-p)^2 + 5p^4(1-p) + p^5$$

(c) Hence, for the general case, deduce that the average probability of error is given by

$$P_e = \sum_{i=m+1}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$

21. Let X, Y and Z be three random variables. For each value of the random variable Z, represented by sample z, define

$$A(z) = \sum_x \sum_y p(y)p(z|x,y)$$

Show that the conditional entropy $H(X|Y)$ satisfies the inequality

$$H(X|Y) \leqslant H(z) + \mathbb{E}[\log A]$$

where $\mathbb{E}$ is the expectation operator.

22. Consider two correlated discrete random variables X and Y, each of which takes a value in the set $\{x_i\}_{i=1}^{n}$. Suppose that the value taken by Y is known. The requirement is to guess the value of X. Let P_e denote the probability of error, defined by $P_e = \mathbb{P}[X \neq Y]$

Show that P_e is related to the conditional entropy of X given Y by the inequality,

$$H(X|Y) \leqslant H(P_e) + P_e \log(n-1)$$

The inequality is known as *Fano's inequality.*