DEPARTMENT OF ELECTRICAL ENGINEERING, IIT BOMBAY

MID-SEMESTER EXAMINATION: SPRING SEMESTER JAN-APR 2020

COURSE NUMBER: EE 338 **COURSE NAME**: Digital Signal Processing

LEVEL: Undergraduate Third Year Core Course

Maximum Marks: 40 (~~20 percent weight~~)

Date: 25 February 2020 (Tuesday) **Time**: 11:00 to 13:00 hours (2 hours)

Instructions:

1. This is a **closed book, closed notes** examination.
2. Candidates may use **non-programmable electronic calculators** and regular stationery.
3. Please **write your roll number** on the front page of all answer scripts that you use.
4. Please **show important steps** clearly in your answers.

Q1. (10 marks)

A linear shift invariant system, with impulse response $h[n] = 0.5^{n} u[n]$, is subjected to the input $x[n]$ described by:

$$x[n] = \begin{cases} 3 & \text{when } n = -5 \\ -7 & \text{when } n = +5 \\ \cos(2\pi n/7); & \text{for all n, other than } n = +5, n = -5 \end{cases}$$

Obtain the output $y[n]$ of the system, as a closed form expression in n.

Q2. (10 marks)

A system with input $x[n]$ and output $y[n]$ is described by the Linear Constant Coefficient Difference Equation (LCCDE):

$y[n] = 0.3\, y[n-1] + x[n]$

for $n = 0 \ldots 100$.

$y[-1] = 1$; $x[n] = 0.3^{n} + 0.5^{n}$ for $n = 0, \ldots, 100$

Obtain a closed form expression for $y[n]$, $n = 0, \ldots, 100$.

Q3. (10 marks)

An ideal digital differentiator is described by the frequency response:

$H_D(e^{j\omega}) = j\omega \qquad -\pi < \omega < +\pi$

(a) Obtain the impulse response of the ideal digital differentiator.
(b) Is it causal? Explain clearly.
(c) Is it stable? Explain clearly.

Q4. (10 marks) Two sequences $x_1[n]$ and $x_2[n]$, with respective Discrete Time Fourier Transforms $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ as shown in Fig. Q4-1 and Fig. Q4-2, are multiplied point by point, to obtain the sequence

$x[n] = x_1[n]\, x_2[n]$

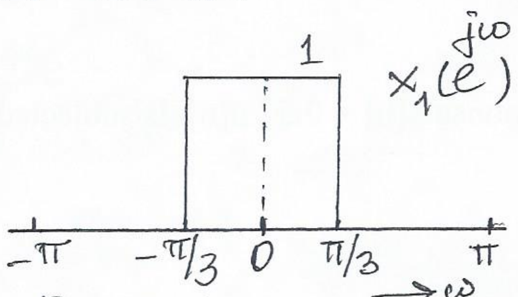


FIG. Q4-1

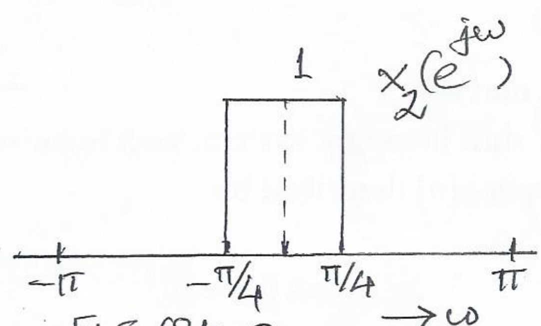


FIG. Q4-2

Obtain, showing the explanation/ working clearly:

(a) The Discrete Time Fourier Transform (DTFT) $X(e^{j\omega})$ of the sequence $x[n]$.
(b) $\Sigma_{\text{over all integer } n}\, x[n]$
(c) $\Sigma_{\text{over all integer } n}\, |x[n]|^2$

$$X_1(e^{j\omega}) \begin{cases} = 1 & 0 \le |\omega| \le \frac{\pi}{3} \\ = 0 & \frac{\pi}{3} < |\omega| \le \pi \end{cases}$$

(End of question paper)

$$X_2(e^{j\omega}) \begin{cases} = 1 & 0 \le |\omega| \le \pi/4 \\ = 0 & \frac{\pi}{4} < |\omega| \le \pi \end{cases}$$

Q1 –

$$x[n] = \cos(2\pi n/7) + (3 - \cos(2\pi(-5)/7))\,\delta[n+5] + (-7 - \cos(2\pi(5)/7))\,\delta[n-5]$$

for all n.

$$= \cos(2\pi n/7) + (3 - \cos(10\pi/7))\,\delta[n+5] + (-7 - \cos(10\pi/7))\,\delta[n-5]$$

The output $y[n]$ is the sum of the output to each of these components.

$$\cos(2\pi n/7) \longrightarrow |H(e^{j\frac{2\pi}{7}})| \cos\left(\frac{2\pi n}{7} + \measuredangle H(e^{j\frac{2\pi}{7}})\right)$$

where $H(e^{j\omega})$ = Discrete Time Fourier Transform of $h[n]$.

or the frequency response of the linear shift invariant system.

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} 0.5^{n} e^{-j\omega n} = \frac{1}{1 - 0.5 e^{-j\omega}}.$$

$$\delta[n+5] \longrightarrow h[n+5]$$

$$\delta[n-5] \longrightarrow h[n-5]$$

Therefore,

$$y[n] = \left|\frac{1}{1 - 0.5 e^{-j2\pi/7}}\right| \cos\left(\frac{2\pi}{7} n + \measuredangle \frac{1}{(1 - 0.5 e^{-j\frac{2\pi}{7}})}\right)$$

$$+ \left(3 - \cos\left(\tfrac{10}{7}\pi\right)\right) 0.5^{n+5} u[n+5]$$

$$+ \left(-7 - \cos\left(\tfrac{10}{7}\pi\right)\right) 0.5^{n-5} u[n-5]$$

We do need to simplify the magnitude and angle.

$$\left| \frac{1}{1-0.5e^{-j\frac{2\pi}{7}}} \right| = \frac{1}{\left|1-0.5e^{-j\frac{2\pi}{7}}\right|}$$

$$= \frac{1}{\left|\left(1-0.5\cos\frac{-2\pi}{7}\right)-j0.5\sin\left(-\frac{2\pi}{7}\right)\right|}$$

$$= \frac{1}{+\sqrt{\left(1-0.5\cos\frac{2\pi}{7}\right)^2+0.5^2\sin^2\left(\frac{2\pi}{7}\right)}}$$

$$= \frac{1}{+\sqrt{1+0.5^2\cos^2\frac{2\pi}{7}-\cos\frac{2\pi}{7}+0.5^2\sin^2\frac{2\pi}{7}}}$$

$$= \frac{1}{\sqrt{1+0.5^2-\cos\frac{2\pi}{7}}}$$

$$= \frac{1}{\sqrt{1.25-\cos\frac{2\pi}{7}}}$$

$$\measuredangle \frac{1}{1-0.5e^{-j\frac{2\pi}{7}}} = \measuredangle \frac{1}{\left(1-0.5\cos\frac{2\pi}{7}\right)+j0.5\sin\frac{2\pi}{7}}$$

$$= -\tan^{-1}\frac{0.5\sin\frac{2\pi}{7}}{1-0.5\cos\frac{2\pi}{7}}$$

Q2 -

If the LCCDE was valid for all n, the system function of the resultant LTI system would be obtained from

$$Y(Z) = 0.3 Z^{-1} Y(Z) + X(Z)$$

$$\Rightarrow \left(1 - 0.3 Z^{-1}\right) Y(Z) = X(Z)$$

$$\Rightarrow \frac{Y(Z)}{X(Z)} = \frac{1}{1 - 0.3 Z^{-1}}$$

with only one system pole at $Z = 0.3$

$x[n] = 0.3^n + 0.5^n$ would then produce a response of the form

$$A_1 . 0.5^n + \left(A_2 + A_3 n\right) 0.3^n$$

as 0.3^n would elicit a resonant response

0.5^n would elicit a forced response

This form continues to hold, for this problem.

$A_1 . 0.5^n$ must satisfy:

$$A_1 . 0.5^n = 0.3 . A_1 \, 0.5^{n-1} + 0.5^n$$

$$\Rightarrow A_1 = 0.3 \times 0.5^{-1} A_1 + 1$$

$$\Rightarrow A_1 = \frac{0.3}{0.5} A_1 + 1 \Rightarrow \left(1 - \frac{3}{5}\right) A_1 = 1$$

$$\Rightarrow A_1 = \frac{1}{1 - \frac{3}{5}} = \frac{5}{2}$$

$A_3 n\, 0.3^n$ must satisfy

$$A_3 n\, 0.3^n = 0.3 A_3 (n-1) 0.3^{n-1} + 0.3^n$$

as this is the component of the solution which is akin to a 'forced response' due to 0.3^n

$$\Rightarrow \quad n \cdot A_3\, 0.3^n = n \cdot A_3 \cdot 0.3^n - A_3 \cdot 0.3^n + 0.3^n$$

$$\Rightarrow \quad A_3 = 1$$

$A_2 \cdot 0.3^n$ essentially comprises the natural response and must satisfy, at $n = 0$:

$$A_2 \cdot 0.3^0 = 0.3\, y[-1] + 0.3^0$$

$$\Rightarrow \quad A_2 = 0.3(1) + 1 = 1.3$$

Accordingly, the closed form expression for $y[n]$, $n = 0, \ldots, 100$ could be

$$y[n] = \frac{5}{2}(0.5^n) + (1.3 + n)\, 0.3^n$$

Note that any other closed form expression equivalent to this one is also acceptable.

Q3 -

(a) Impulse response of ideal digital differentiator

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega n} j\omega \, d\omega$$

$$= \frac{j}{2\pi}\int_{-\pi}^{\pi} \omega . e^{j\omega n} \, d\omega. \quad \text{For } n \neq 0:$$

$$\int_{-\pi}^{\pi} \omega . e^{j\omega n} \, d\omega = \left[\omega . \frac{e^{j\omega n}}{jn} - \int \frac{e^{j\omega n}}{jn} d\omega \right]_{-\pi}^{\pi}$$

(For $n \neq 0$)

$$= \omega . \frac{e^{j\omega n}}{jn}\Big|_{-\pi}^{\pi} - \frac{1}{jn} . \frac{e^{j\omega n}}{jn}\Big|_{-\pi}^{\pi}$$

$$= \frac{\pi e^{j\pi n}}{jn} - \frac{(-\pi) e^{-j\pi n}}{jn} - \frac{1}{(jn)^2}\left(e^{j\pi n} - e^{-j\pi n}\right)$$

$$= \frac{2\pi}{jn}(-1)^n$$

Thus the impulse response

$$= \frac{j}{2\pi} . \frac{2\pi}{jn}(-1)^n = \frac{(-1)^n}{n} \quad \text{for } n \neq 0$$

For $n = 0$, $\int_{-\pi}^{\pi} \omega . e^{j\omega n} \, d\omega = \int_{-\pi}^{\pi} \omega \, d\omega = \frac{\omega^2}{2}\Big|_{-\pi}^{\pi} = 0.$

Q3 - (b) Clearly the impulse response is an odd function of n and is not that of a causal system, as it is nonzero for an infinite number of negative n.

(c) The absolute sum of the impulse response is divergent:

$$2 \sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \right| = 2 \sum_{n=1}^{\infty} \frac{1}{n}$$

Consider $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

Regroup:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>} + \cdots$$

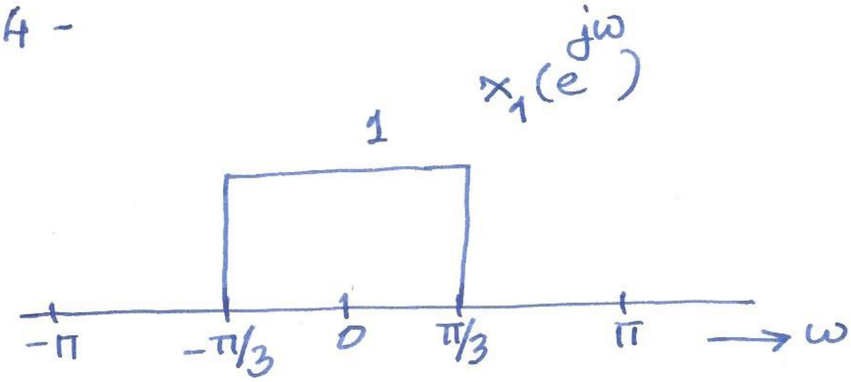
$$\left(\frac{1}{4} + \frac{1}{4}\right) \quad \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \quad \cdots$$

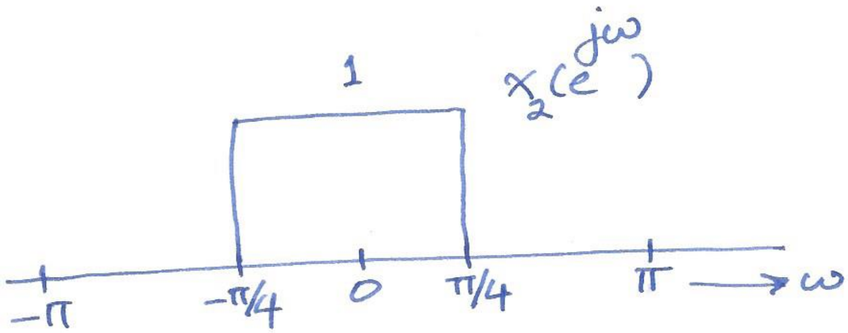
$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

is thus a lower bound on this sum and the lower bound itself is divergent.

Hence the system is unstable.

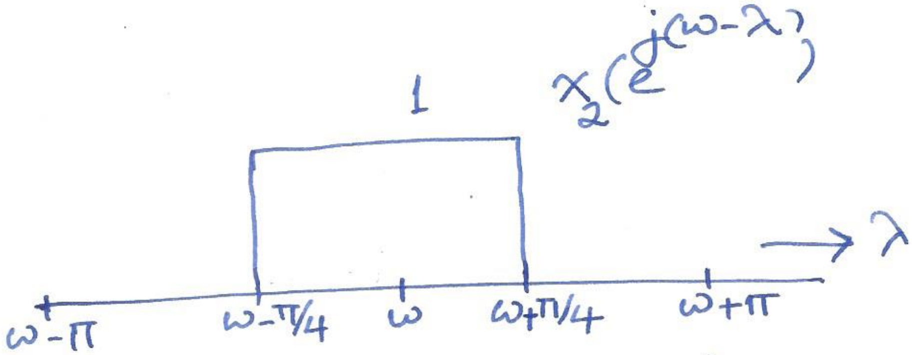
Q4 -

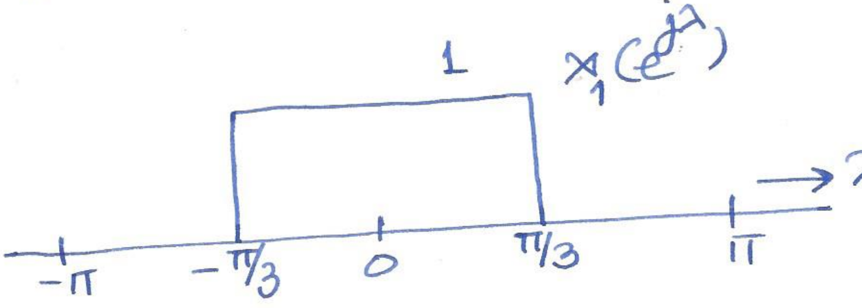




(a)
The Discrete Time Fourier Transform of $x[n]$

$$= X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega-\lambda)})\, d\lambda$$





Critical points:

$$\omega + \pi/4 = -\pi/3 \Rightarrow \omega = -\pi/3 - \pi/4 = -7\pi/12$$

$$\omega - \pi/4 = -\pi/3 \Rightarrow \omega = \pi/4 - \pi/3 = -\pi/12$$

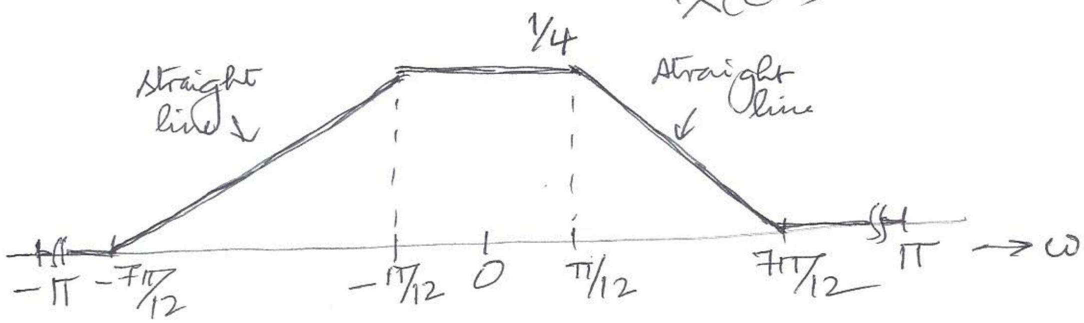
and the negatives of these

From $\omega = -7\pi/12$ to $\omega = -\pi/12$, the convolution rises linearly and at $\omega = -\pi/12$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega-\lambda)})\, d\lambda$$

$$= \frac{1}{2\pi} \cdot 2 \times \pi/4 \times 1 = \frac{1}{4}.$$

Accordingly, $X(e^{j\omega})$ is as follows:

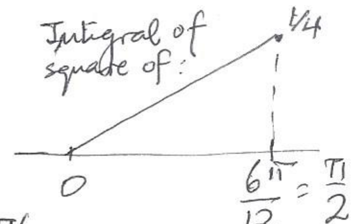


(b) $\sum_{n=-\infty}^{+\infty} x[n] = X(e^{j0}) = \frac{1}{4}.$

(c) $\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega$ from Parseval's Theorem

Now $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega$ is the sum of $\int_{-7\pi/12}^{-\pi/12}$, $\int_{-\pi/12}^{\pi/12}$, $\int_{\pi/12}^{7\pi/12} (\ldots)\, d\omega$

$$\int_{-7\pi/12}^{-\pi/12} |X(e^{j\omega})|^2\, d\omega = \int_{\pi/12}^{7\pi/12} |X(e^{j\omega})|^2\, d\omega =$$ Integral of square of:



which is $\frac{\omega}{\pi/2} \cdot \frac{1}{4} = \frac{\omega}{2\pi}$ from 0 to $\pi/2$

Thus $\int_{-7\pi/12}^{-\pi/12} |X(e^{j\omega})|^2 \, d\omega = \int_{\pi/12}^{7\pi/12} |X(e^{j\omega})|^2 \, d\omega$

$$= \int_0^{\pi/2} \left(\frac{\omega}{2\pi}\right)^2 d\omega = \frac{\omega^3}{3(2\pi)^2}\Bigg|_0^{\pi/2}$$

$$= \frac{(\pi/2)^3}{3\,(2\pi)^2}$$

Hence $\frac{1}{2\pi}\int_{-7\pi/12}^{-\pi/12} |X(e^{j\omega})|^2 \, d\omega + \frac{1}{2\pi}\int_{\pi/12}^{7\pi/12} |X(e^{j\omega})|^2 \, d\omega$

$$= \frac{2}{2\pi} \cdot \frac{(\pi/2)^3}{3\,(2\pi)^2} = \frac{2/2^3}{2 \times 3 \times 2^2}$$

$$= \frac{1}{8 \times 4 \times 3} = \frac{1}{96}$$

$$\frac{1}{2\pi}\int_{-\pi/12}^{\pi/12} |X(e^{j\omega})|^2 \, d\omega = \frac{1}{2\pi}\left(\frac{1}{4}\right)^2 \cdot \frac{2\pi}{12} = \frac{1}{16 \times 12} = \frac{1}{192}$$

Thus $\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$

$$= \frac{1}{96} + \frac{1}{192} = \frac{2+1}{192} = \frac{3}{192}$$