SC 635

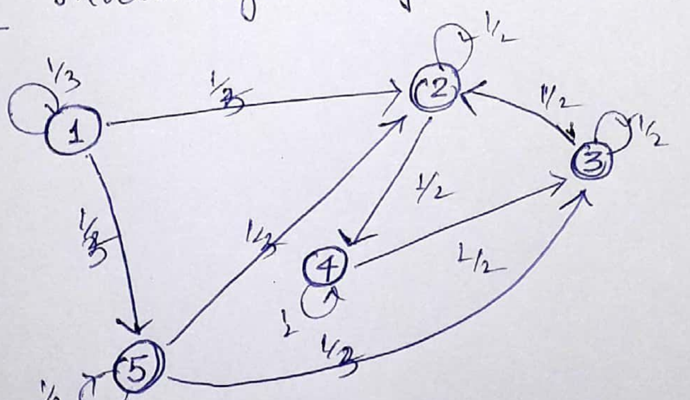
The concepts on 'Graph' studied in class:-

1. Graph - Neighbours, degree, path, cycle, connected, subgraph, ...
2. Digraph - In-degree, Out-degree, tree, spanning tree, strongly connected, globally reachable node, ...
3. Weighted digraph - adjacency matrix, ...

→ Let G be weighted digraph with n nodes, with adjacency matrix A. ~~and~~ for all $i, j \in \{1, \dots, n\}$ and $k \in \mathbb{N}$:

the (i,j) entry of A^k is positive if and only if there exists a path of length k from node i to node j.

~~Matrices,~~ Multi-agent System



$$A = \begin{bmatrix} \cancel{0}\,\frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \cancel{0}\,\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \cancel{0}\,\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \cancel{0}\,\frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & \cancel{0}\,\frac{1}{3} \end{bmatrix}$$

→ There are 5 agents. Agents can be robots, human, ...

→ Each agent has some 'states' associated to itself

→ The states evolve over time. The states are modified based on the states of the neighbours.

→ The states can be position, velocity or orientation in case of a robot, opinion in case of human etc...

$x_i(k+1)$ = ~~$x_i(k)$~~ average of its neighbours at k^{th} instant.

In vector notation,

$$x(k+1) = A\,x(k)$$

A is the adjacency matrix (with self loops possibly).

Then as time evolves,

$$\lim_{k\to\infty} x(k) = \left(\lim_{k\to\infty} A^k\right) x(0).$$

The states of the agents ~~at~~ ~~s~~ will converge or not depends on $\left(\lim_{k\to\infty} A^k\right)$.

~~⇒ Observe that row sum of adjacency matrix = 1~~

<u>Definition</u>: A matrix A is semi-convergent if $\lim_{k\to\infty} A^k$ exists.

and it is convergent if it is semi convergent and

$$\lim_{k\to\infty} A^k = 0_{n\times n}$$

1 → Therefore, we need the adjacency matrix to be ~~converg semi-con~~ semi-convergent.

→ An eigenvalue is <u>simple</u> if it has algebraic and geometric multiplicity equal to 1.

$A = TJT^{-1}$ J: Jordan form.

$$\therefore A^k = TJ^kT^{-1} = T \begin{bmatrix} J_1^k & & \\ & J_2^k & \\ & & \ddots \; J_m^k \end{bmatrix} T^{-1}$$

~~J_i : Jordan block~~

J_i : Jordan block 'i'

→ <u>Spectral radius</u> of A is the maximum norm of the eigenvalues of A

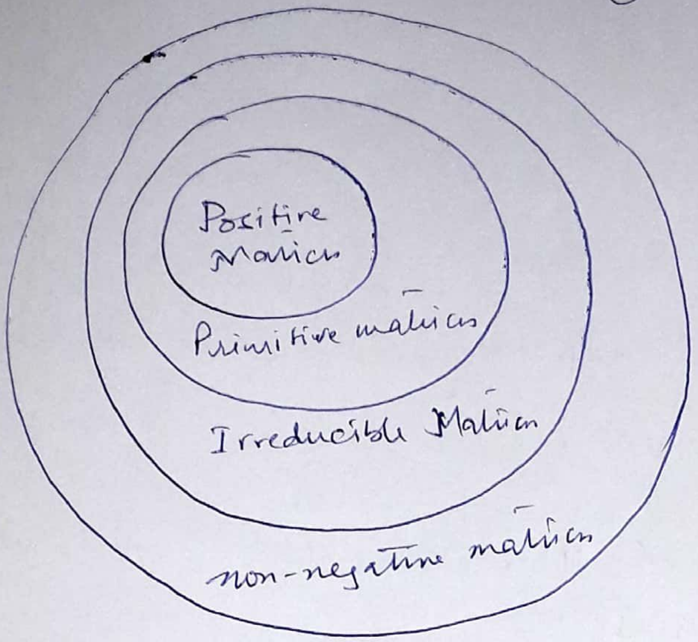
$$\rho(A) = \max\{ |\lambda| \mid \lambda \in \operatorname{spec}(A)\}.$$

→ <u>thm</u>: for a square matrix A,

- A is convergent if and only if $\rho(A) < 1$
- 2 A is semi-convergent if and only if
  - $\rho(A) < 1$ or
  - 1 is a simple eigenvalue and all other eigenvalues have magnitude less than 1.

→ <u>Definition</u>: For $n \geq 2$, $A \in \mathbb{R}^{n\times n}$ is

1. irreducible if $\sum_{k=0}^{n-1} A^k$ is positive.
2. primitive if $\exists\, k \in \mathbb{N}$ such that A^k is positive.



→ Thm (Perron-Frobenius Theorem) : Let $A \in \mathbb{R}^{n \times n}$, $n \geq 2$. ~~If~~

If A is non-negative, then

(1) $\exists$ a real eigenvalue $\lambda \geq |\mu| \geq 0$ for all other eigenvalue of A.

(2) right and left eigenvector v & w of λ can be selected non-negative.

If additionally, A is irreducible, then

(3) λ is strictly positive and simple.

(4) v and w ~~are~~ of λ are unique and positive, up to rescaling

If additionally, A is primitive, then

(3) (5) λ satisfies $\lambda > |\mu|$ $\forall \mu \in \text{spec}(A) \setminus \lambda$.

→ <u>Definition</u> $A \in \mathbb{R}^{n \times n}$ is

(1) non negative if $a_{ij} \geq 0$ $\forall i, j$

(2) row stochastic if non-negative and $A\mathbb{1}_n = \mathbb{1}_n$

(3) column-stochastic if non-negative and $A^T\mathbb{1}_n = \mathbb{1}_n$

(4) doubly-stochastic if it is row and column stochastic.

Gersgorin Disk Theorem. For any $A \in \mathbb{R}^{n \times n}$

$$\text{Spec}(A) \subset \bigcup_{i \in \{1,\dots,n\}} \underbrace{\left\{ z \in \mathbb{C} \;\middle|\; |z - a_{ii}| \le \sum_{\substack{j=1 \\ j \ne i}}^{n} |a_{ij}| \right\}}_{\text{disk of radius } \sum_{j \ne i} |a_{ij}| \text{ centred at } a_{ii}}$$

→ For a row stochastic matrix A

(4)
1. 1 is an eigenvalue.
2. spec(A) is a subset of the unit disk and $\rho(A) = 1$.

From (3) and (4)

If a row stochastic matrix is primitive, then it is ~~semi~~ semi-convergent.

→ Thm: Let A be a non-negative matrix with dominant ~~eigenvalue~~ eigenvalue λ. Let v and w denote ~~the~~ right and left dominant eigenvectors normalized so that $v \geq 0$ and $v^T w = 1$.
If λ is simple and strictly larger in magnitude than all other eigenvalues, then A/λ is semi-convergent &

(5)
$$\lim_{k \to \infty} \frac{A^k}{\lambda^k} = v w^T.$$

→ Thm Let G be a weighted digraph with $n \geq 2$ and with weighted adjacency matrix A. Then,

G is strongly connected iff A is irreducible. $\left(\sum_{k=0}^{n-1} A^k > 0 \right)$

→ Since there is always a path from any node to any other node, hence $\sum_{k=0}^{n-1} A^k > 0$ is ~~#~~ true.

→ Corollary: Let G be a weighted digraph with n nodes and weighted adjacency matrix A and self-loop at each node. Then, G is strongly connected iff A^{n-1} is positive, so that A is primitive

This leads to the result —

→ Thm:

6 G is strongly connected and aperiodic iff A is primitive, that is, $\exists\, k \in \mathbb{N}$ s.t. $A^k > 0$.

Main Result:

Let A be a row-stochastic matrix and let G be its associated graph. The following statements are equivalent:—

1. Eigenvalue 1 is simple and all other eigenvalues μ satisfy $|\mu| < 1$
2. A is semi-convergent and $\lim_{k\to\infty} A^k = \mathbb{1}_n w^T$ for some $w \in \mathbb{R}^n$, $w \geq 0$ & $\mathbb{1}_n^T w = 1$.
3. G contains a globally reachable node and the subgraph of globally reachable nodes is aperiodic

If the above are true, then

4. $w \geq 0$ is the left dominant eigenvector of A and $w_i > 0$ iff node i is globally reachable.
5. $\lim_{k\to\infty} x(k) = (w^T x(0))\, \mathbb{1}_n$
6. if A is doubly ~~sche~~ stochastic, then $w = \frac{1}{n}\mathbb{1}_n$

$\therefore \lim_{k\to\infty} x(k) = \frac{\mathbb{1}_n^T x(0)}{n} = \text{Avg}(x(0))\, \mathbb{1}_n$