Mid-Semester Exam

EE328 - Digital Communications

Examination: February 22nd, 2019, 11:00 AM to 1:00 PM. Maximum score: 35
Roll number: ______________ Name: ____________________________

Notes:

(1) **Write your answers below the questions in the question paper directly and return it!**

(2) This is a closed book exam. No notes, cheat sheets and other documents are permitted.

(3) No doubts in the questions will be entertained. Write your assumptions and then solve the problem.

(4) You may use a scientific calculator if needed.

(5) Notation: $I_A(t) = 1$ if $t \in A$, $\quad Q(x) = \int_x^\infty \frac{e^{-z^2/2}}{\sqrt{2\pi}}$.

Problem 1 (4 points)

You are given three correlators that use the three real signals $s_1(t)$, $s_2(t)$ and $s_3(t)$. If the input to the i-th correlator is $x(t)$, the output is $\int_{-\infty}^{\infty} x(t)s_i(t)dt$, $i = 1, 2, 3$. A real white Gaussian noise process with power spectral density $N_0/2$ is fed to these correlators to obtain the output random vector $\mathbf{n} = [n_1, n_2, n_3]^T$.

(a) Specify the conditions on $s_1(t), s_2(t), s_3(t)$ for $\mathbf{n}$ to be a jointly Gaussian random vector. [1]

> Solution
>
> They will always be jointly Gaussian for square-integrable $s_i(t)$.

(b) Specify the conditions on $s_1(t), s_2(t), s_3(t)$ for which the covariance matrix of $\mathbf{n}$ is diagonal. [1]

> Solution
>
> The three signals are orthogonal.

(c) What choice of $s_1(t), s_2(t), s_3(t)$ would make the covariance matrix of $\mathbf{n}$ non-invertible? [1]

Solution

Linearly dependent signals.

(d) Find the covariance matrix of $\mathbf{n}$ for $s_1(t) = -s_2(t) = \text{rect}(t)$, $s_3(t) = t\text{rect}(t)$. Recall that $\text{rect}(t) = I_{[-0.5,0.5]}$. [1]

Solution

PROBLEM 2 (5 points)

Consider the use of regular PAM-4 modulation in a real AWGN channel. For the input PAM symbol x, the output y is given by

$$y = x + w$$

where w is independent of x and is a zero-mean real-valued unit variance Gaussian random variable. Assume that the transmitter sends the four symbols with equal probabilities and that the receiver performs minimum distance decoding.

(a) Assume that the symbol energy is E_s. Find the probability of symbol error P_e. [2]

Solution

First, we need to normalize the symbols to have energy E_s. If the symbols are $-3a, -a, a, 3a$, then you can find that normalizing gives us $a = \sqrt{E_s}/5$. Now, the symbol error probability can be easily found using the $Q(d_{min}/2\sigma)$ formula:

$$P_{e|3a} = P_{e|-3a} = Q(\sqrt{E_s/5})(1 \text{ point})$$
$$P_{e|a} = P_{e|-a} = 2Q(\sqrt{E_s/5})(1 \text{ point})$$

Simplifying, we get

$$P_e = \frac{3}{2}Q(\sqrt{E_s/5})1 \text{ point}$$

(b) Suppose that the PAM-4 constellation points are (in order from left to right) a_1, a_2, a_3, a_4. Evaluate the partial derivative of P_e with respect to a_3 in terms of E_s. [1]

Solution

In this case, we assume (by symmetry) that the points a_1, a_2, a_3 and a_4 are respectively $-b, -a, a, b$, where a, b are positive real numbers. It can be shown that this assumption doesn't affect optimality.
In this condition, it can be shown that

$$P_e = Q((b-a)/2) + \frac{1}{2}Q(a)$$

Now, differentiating with respect to a, we see that the derivative comes out to be

$$-\frac{1}{2\sqrt{2\pi}}e^{-a^2/2} - \frac{1}{2\sqrt{2\pi}}\left(\frac{-a}{b} - 1\right)e^{-(b-a)^2/8}$$

Here, substituting $b = 3a$, we observe that it does not go to zero.

(c) Evaluate the partial derivative of E_s with respect to a_3. [1]

Solution

We know that $4E_s = a_1^2 + a_2^2 + a_3^2 + a_4^2$.
One can look at this in two ways. If the constellation points are assumed as a_1, a_2, a_3 and a_4 independently, then the derivative with respect to a_3 will be $a_3/2$. However, if we enforce the constraint that $a_3 = -a_2$ (by symmetry), we get a_3. Both are honoured.

(d) Based on the above, comment about the optimality of the regularly spaced PAM-4 constellation from a minimum symbol error rate perspective. Would a different four point one-dimensional constellation produce a lower symbol-error rate for the same E_s? [1]

Solution

From (b) and (c), since the derivative is non-zero, the regular PAM-4 is not optimal (in fact, the optimal constellation has a_2 and a_3 closer than a_1 and a_2.

Problem 3 (6 points)

Consider a digital communication system which can use the following functions for signaling:

$$\psi_1(t) = \cos\left(\frac{2\pi t}{T}\right) I_{[0,T]}(t)$$
$$\psi_2(t) = \cos\left(\frac{2\pi t}{T} + \phi\right) I_{[0,T]}(t)$$

where $I_A(t)$ is a function whose value is 1 for $t \in A$.

(a) What is the energy per symbol for each of these waveforms if each waveform is to represent a symbol? [1]

Solution

$T/2$

(b) Provide an orthonormal basis to represent $\psi_1(t)$ and $\psi_2(t)$. [1]

Solution

The easiest basis is $\sqrt{2/T}\cos(2\pi t/T)$, $\sqrt{2/T}\sin(2\pi t/T)$ for $\phi \neq 0, \pi$, and just $\sqrt{2/T}\sin(2\pi t/T)$ for $\phi = 0$ or π. 0.5 points less for not handling both cases.

(c) What is the number of dimensions in this system? [1]

Solution

2 if $\phi \neq 0, \pi$, else 1. Half point less if only one specified.

(d) Suppose that $\phi = \pi/6$. Your friend implements a binary signaling scheme wherein he sends $\psi_1(t)$ to convey bit zero, and $\psi_2(t)$ to send bit one. Assuming that the signal passes through a real channel which introduces AWGN with variance σ^2, find the BER in terms of the $Q(\cdot)$ function. [2]

Solution

Representing the signals in the above basis, we get the signals to be the vectors $[\sqrt{T/2}, 0]^T$ and $\sqrt{T/2}[\cos(\pi/6), \sin(\pi/6)]^T$. The distance between these is $(2-\sqrt{3})\sqrt{T/2}$. So the answer is $Q((2-\sqrt{3})\sqrt{T/2}/2\sigma)$.

(e) For the same energy usage as the previous system, what is the lowest BER that you can achieve? Outline the strategy and find the lowest BER. [1]

Solution

The best strategy is to use BPSK with any one of the above signals. The BER is $Q(\frac{\sqrt{2T}}{2\sigma})$

PROBLEM 4 (3 points)

Consider a communication system that uses QPSK modulation. The energy per symbol is E_s.

(a) Draw a gray coded constellation representing the transmit symbols, clearly indicating the bits to symbol mapping. [1]

Solution

Any valid constellation diagram will get full marks. Non-gray coded gets zero.

(b) The transmitter uses the carrier $\cos(2\pi f_c t)$, but the receiver uses the carrier $(-\sin(2\pi f_c t))$. What is the BER achieved when this constellation is used over a noiseless system? [1]

(c) Now, sometimes the receiver uses the carrier $\cos(2\pi f_c t)$, and at other times, uses the carrier $-\sin(2\pi f_c t)$. Suppose that we wish to detect which of the carriers is in operation at the receiver. The transmitter and receiver pre-agree to send a known symbol periodically to allow the transmitter to configure itself properly. How many symbols do you need to send from the transmitter over a noiseless system to determine which of these is in operation? [1]

Solution

One symbol is enough.

PROBLEM 5 (4 points)

Consider the problem of estimating an unknown quantity A from a communication system:

$$y_1 = Ab_1 + w_1$$
$$y_2 = Ab_2 + w_2$$

Here, y_1 and y_2 are known at the receiver, $b_1 = 1, b_2 = -1$ and w_1 and w_2 are zero mean jointly Gaussian random variables, each with variance σ^2, and mutual correlation ρ. It is given that $-1 < \rho < 1$.

(a) Specify the distribution of the random vector $[y_1, y_2]^T$ (conditioned on A being known). [1]

Solution

Jointly Gaussian random vector with mean $[A, -A]^T$ and covariance matrix

$$\sigma^2 \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.$$

(b) Find the maximum likelihood estimate of A from y_1 and y_2. [2]

Solution

For this, we consider the alternate random vector

$$\mathbf{y} = \begin{pmatrix} y_1 \\ -y_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} w_1 \\ -w_2 \end{pmatrix}$$

Here, the change is that this new random vector $\mathbf{y}$ has covariance matrix

$$\mathbf{C} = \sigma^2 \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}.$$

We can then see that this matrix can be diagonalized as follows:

$$\mathbf{C} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sigma^2(1-\rho) & 0 \\ 0 & \sigma^2(1+\rho) \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} = \mathbf{U}\Lambda\mathbf{U}^T$$

Thus, if we now consider the random vector $(\mathbf{U}\Lambda^{1/2})^{-1}\mathbf{y}$, we find that its mean is $[\sqrt{2}A/\sigma\sqrt{1-\rho}, 0]^T$ and covariance is identity. Thus, it is easy to identify that the best estimate is $(y_1 - y_2)/2$.

(c) What is the mean-squared error achieved by the ML estimator? [1]

Solution

$\sigma^2(1-\rho)/2$. Zero for incorrect answers.

PROBLEM 6 (6 points)

Consider the following baseband signals used for signaling at $1/T$ symbols per second. Assume that $f_2 > f_1$, and that f_1T and f_2T are very large positive integers.

$$\begin{aligned} s_1(t) &= \cos(2\pi f_1 t)I_{[0,T]} \\ s_2(t) &= -\cos(2\pi f_1 t)I_{[0,T]} \\ s_3(t) &= \cos(2\pi f_2 t)I_{[0,T]} \\ s_4(t) &= -\cos(2\pi f_2 t)I_{[0,T]} \end{aligned}$$

(a) Depict the symbols on a constellation. What is the average signal energy? [2]

Solution

Biorthogonal, $T/2$.

(b) Provide a gray code for this constellation. [1]

Solution

Just like QPSK.

(c) How many nearest neighbours does each symbol have? [1]

Solution

2

(d) Find the BER and symbol error rate for this constellation. [2]

Solution

$Q(\sqrt{E_s/N_0})$, where $E_s = T/2$.

PROBLEM 7 (7 points)

Suppose that $s(t)$ is a signal whose Fourier transform $S(f)$ is square-integrable and satisfies $S(f) = 0 \ \forall\ |f| \geqslant W$. Let $WT = 0.5$. We know that we can reconstruct $s(t)$ directly using just its samples $s(kT), k = 0, \pm 1, \pm 2$ etc.

(a) Provide the equation to reconstruct $s(t)$ using $s(kT)$ values. [1]

Solution

$$s(t) = \sum_{k=-\infty}^{\infty} s(kT)\mathrm{sinc}(t/T - k)$$

(b) Suppose that the actual value of $s(0)$ is 1. However, due to some errors, that sample is lost and replaced with $\tilde{s}(0) = 0$. Find the energy of the reconstruction error, defined as

$$\int_{-\infty}^{\infty} |s(t) - \tilde{s}(t)|^2 dt$$

[1]

Solution

The difference signal is $\mathrm{sinc}(t/T)$. The energy in this signal can be found using Parseval's theorem to be T.

(c) Suppose that the sample $s(0)$ is not known, but it can be modelled as a random variable with a known distribution $P_{s_0}(\cdot)$. Choose a value of $\tilde{s}(0)$ such that the reconstruction in part (b) has the minimum average error energy. Explain. [2]

Solution

The answer is just the mean times T, since the mean minimizes the squared error.

(d) Evaluate

$$\int_{-\infty}^{\infty} \operatorname{sinc}(t)\operatorname{sinc}(t-0.5)dt \quad [1]$$

Solution

$2/\pi$

(e) Now suppose that $s(0) = 1$, and that $s(kT) = 0$ for $k \neq 0$. However, we are only allowed to reconstruct our $s(t)$ as

$$\tilde{s}(t) = \tilde{s}_1 \operatorname{sinc}(t/T - 0.5)$$

Specify the best choice of the number $\tilde{s}_1$ that minimizes the energy of the reconstruction error and compute the resulting error energy. [2]

Solution

$2/\pi, T(1 - 4/\pi^2)$
