Given a distortion constraint

$$\sigma_q^2 \leq D^* \longrightarrow \mathrm{I}$$

find the decision boundaries, reconstruction levels, and binary codes that minimize the rate given by: $R = \sum_{j=1}^{L} \ell_j \int_{b_{j-1}}^{b_j} f_X(x)\,dx$

while satisfying I.

Given a rate constraint

$$R \leq R^* \longrightarrow \text{II}$$

find the decision boundaries, reconstruction levels, and binary codes that minimize the distortion given by:

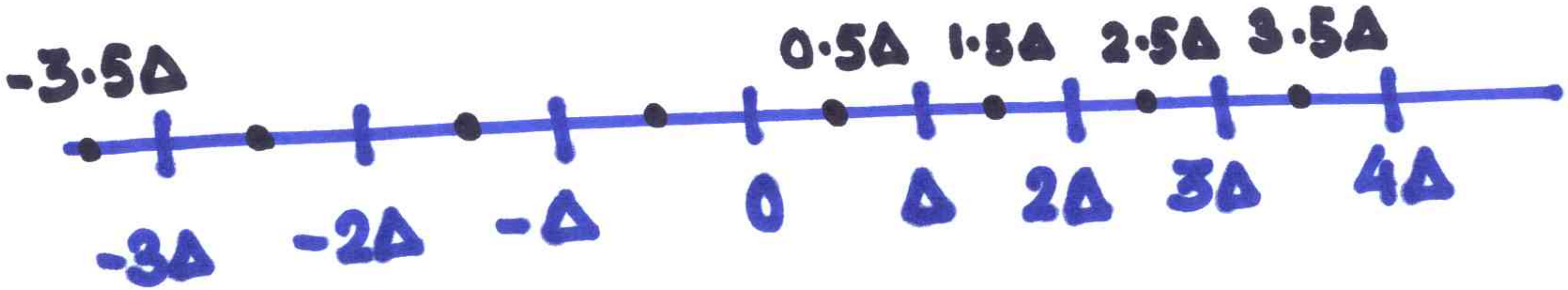
$$\sigma_q^2 = \sum_{j=1}^{L} \int_{b_{j-1}}^{b_j} (x - y_j)^2 f_X(x)\, dx$$

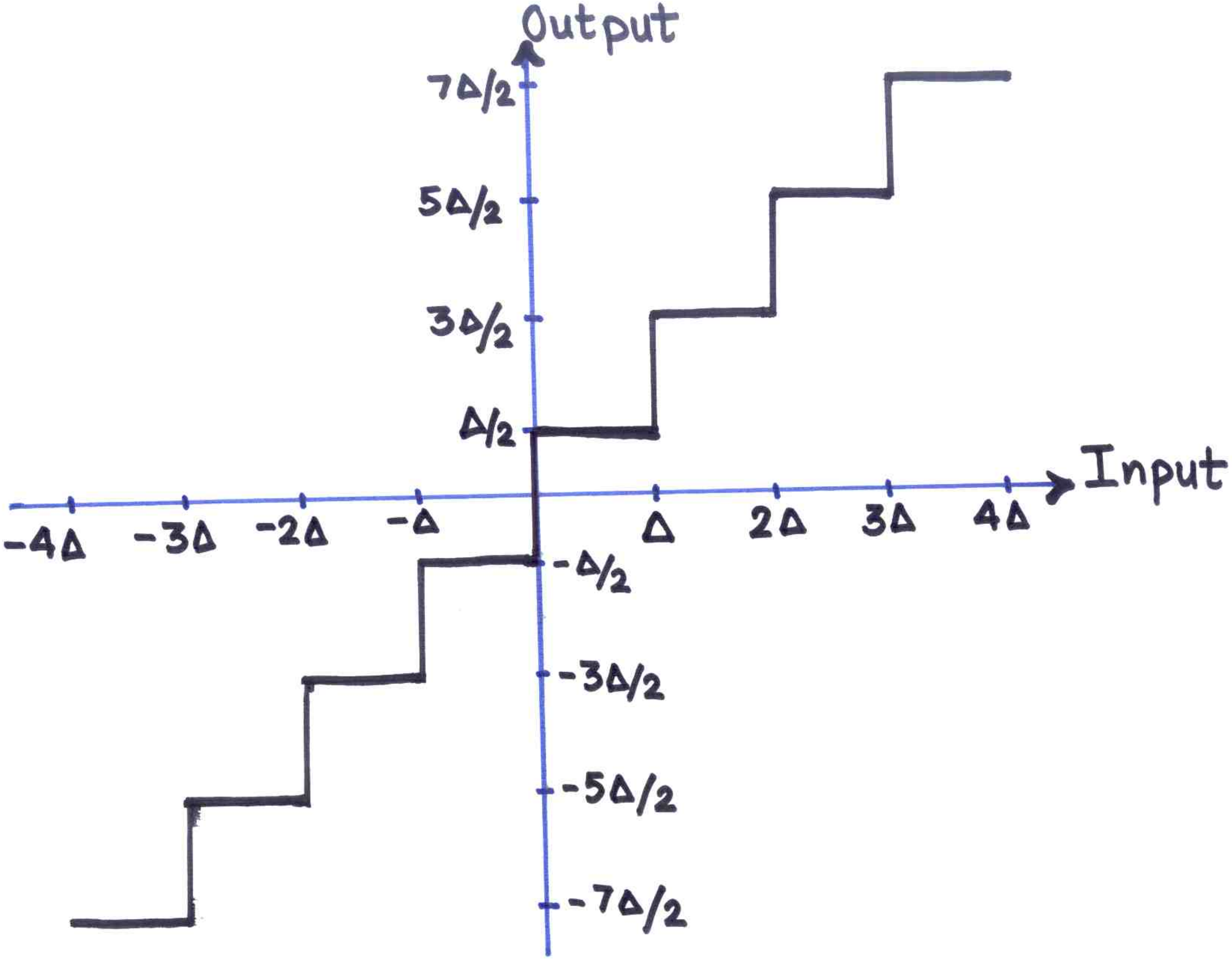
while satisfying Eqn. II

Uniform Quantizer

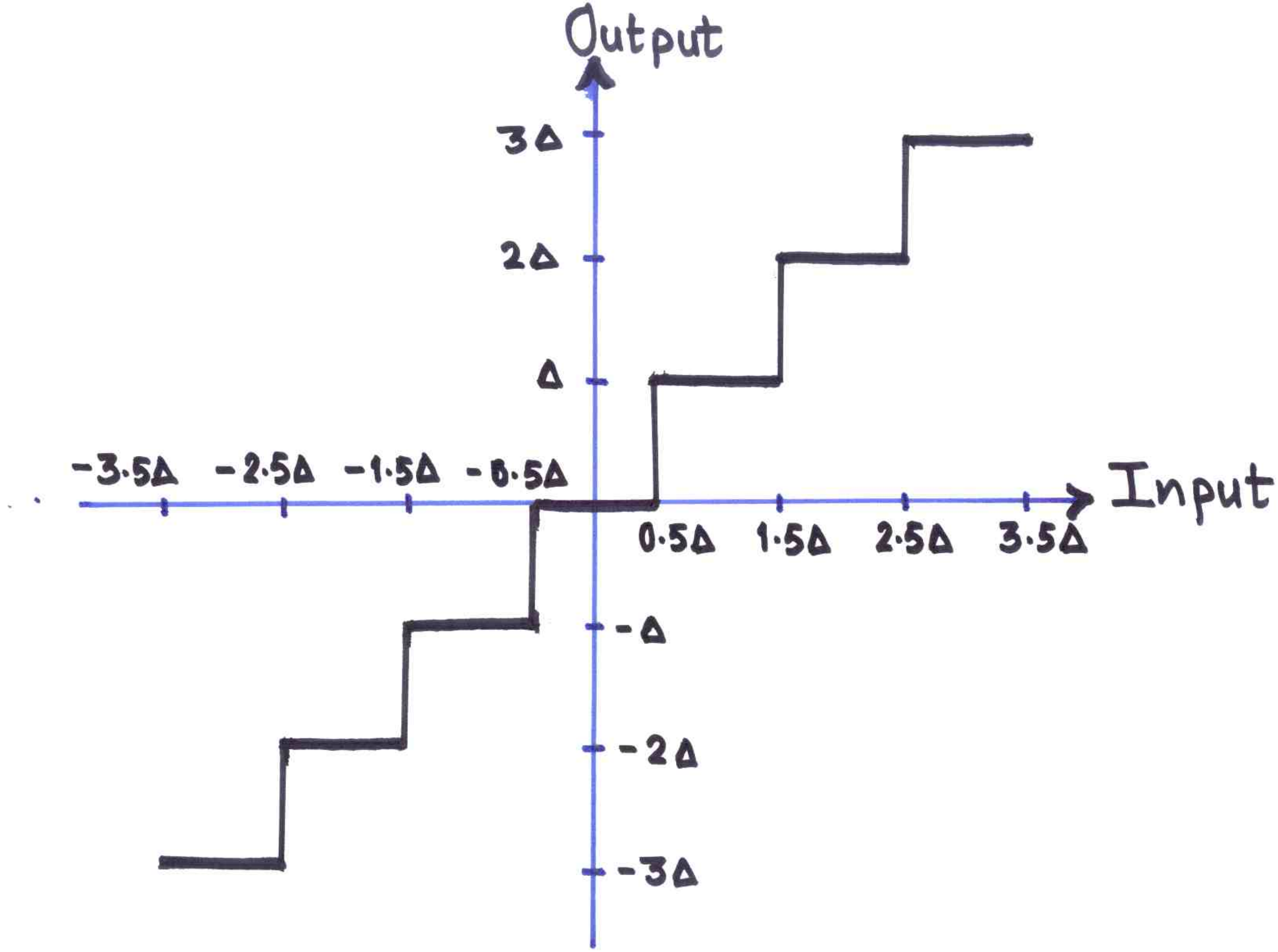
Decision Boundaries (Step size = Δ)

Reconstruction Levels

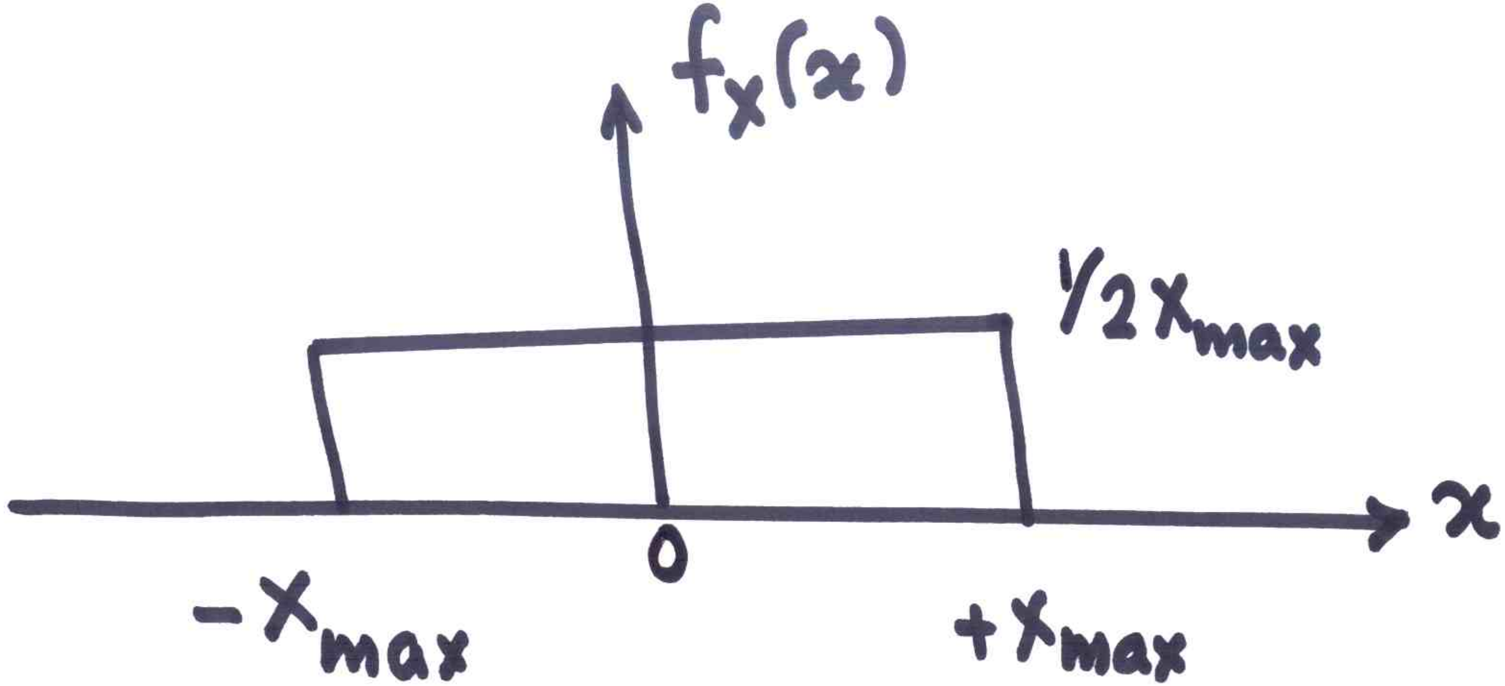


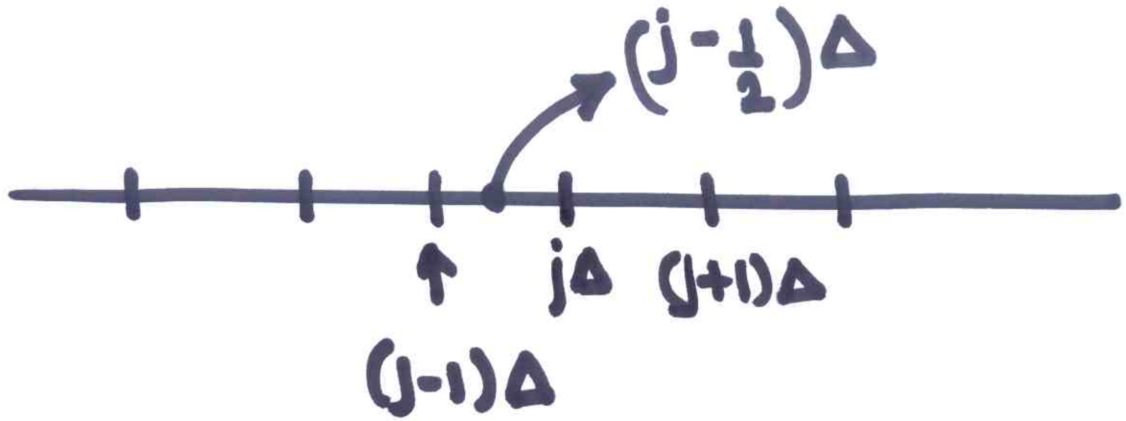


MIDRISE QUANTIZER



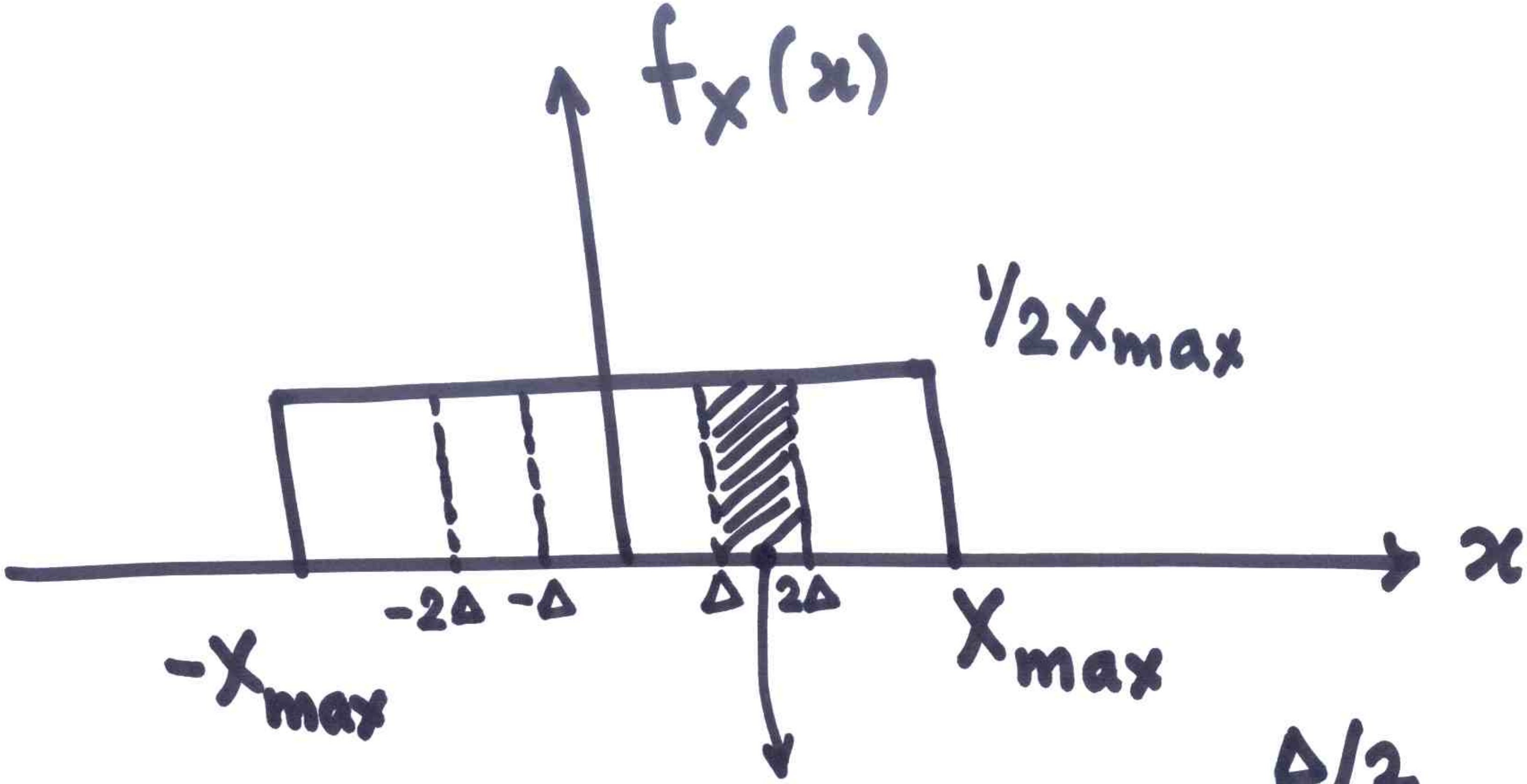
MIDTREAD QUANTIZER


$f_X(x)$
$1/2x_{max}$
x
0
$-x_{max}$
$+x_{max}$



$$\sigma_q^2 = \int_{-\infty}^{\infty} \left(x - Q(x)\right)^2 f_X(x)\,dx$$

$$= 2 \sum_{j=1}^{L/2} \underbrace{\int_{(j-1)\Delta}^{j\Delta} \left(x - \left(\frac{2j-1}{2}\right)\Delta\right)^2 \frac{1}{2X_{max}}\,dx}$$



$$\sigma_q^2 = 2\,\frac{L}{2}\int_{-\Delta/2}^{\Delta/2} q^2\,\frac{1}{2X_{max}}\,dq = \int_{-\Delta/2}^{\Delta/2} q^2\,\frac{1}{\Delta}\,dq$$

$$\Delta = \frac{2X_{max}}{L}$$

$$\sigma_q^2 = \int_{-\Delta/2}^{\Delta/2} q^2 \frac{1}{\Delta} \, dq = \frac{\Delta^2}{12}$$

$$\sigma_x^2 = \frac{(2X_{max})^2}{12} = \frac{\Delta^2 L^2}{12}$$

$$(SNR)_q \, (dB) = 10 \log_{10} \left[\frac{\sigma_x^2}{\sigma_q^2} \right]$$

$$= 10 \log_{10} L^2$$

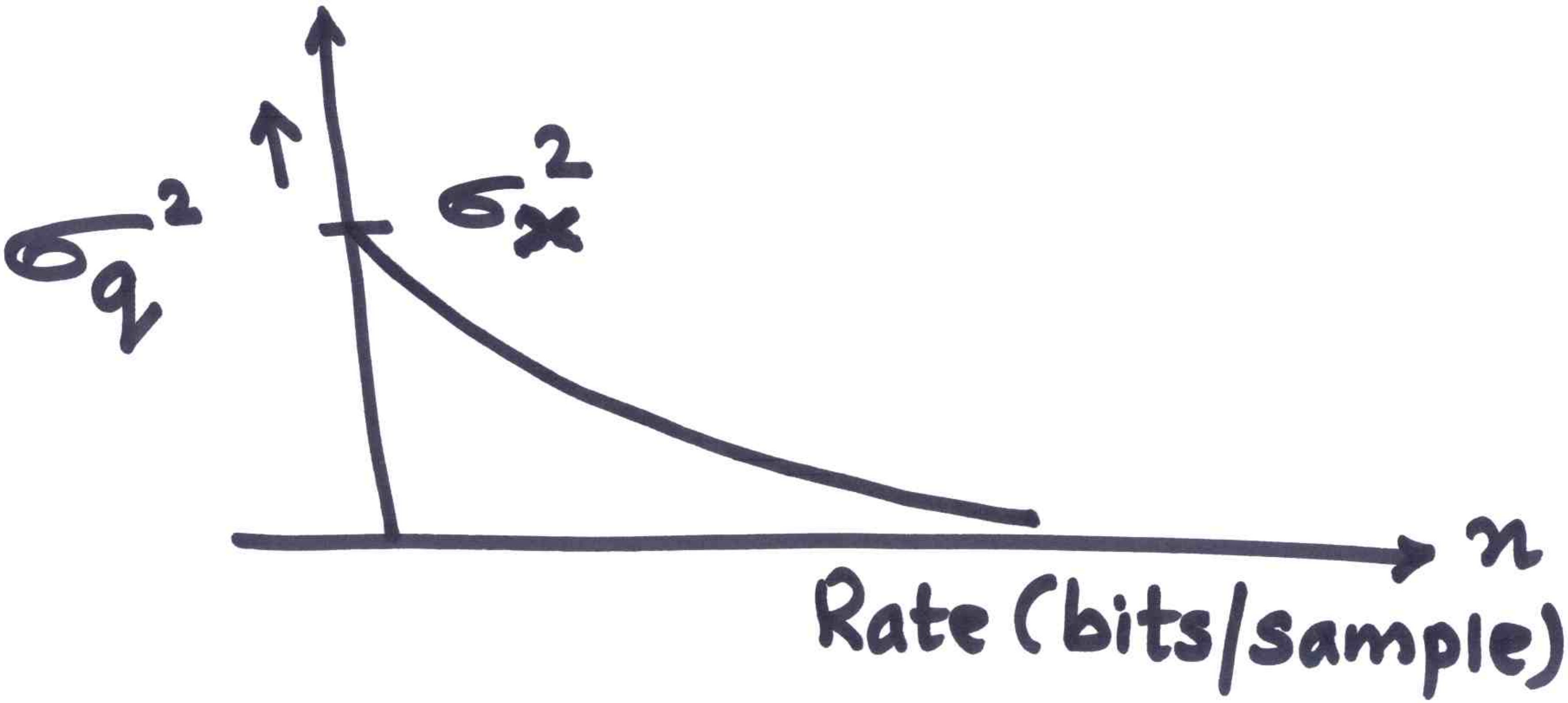
$$L = 2^n$$

$$(SNR)_q = 10\log_{10}(2^n)^2$$

$$= 20\log_{10}2^n$$

$$= 6.02\,n\ \text{dB}$$

$$\sigma_q^2 = \frac{\Delta^2}{12} = \frac{4X_{mam}^2}{12L^2} = \frac{X_{max}^2}{3}\, 2^{-2n}$$



Example: $[-100, 100]$

$[-1, 1]$: 0.95 (Probability)

$\{[-100, -1), (1, 100]\}$ 0.05

$$\Delta = \frac{200}{8} = 25$$

$[-1, 0) \longrightarrow -12.5$ $[0, 1) \rightarrow +12.5$

$[-1, 1]$

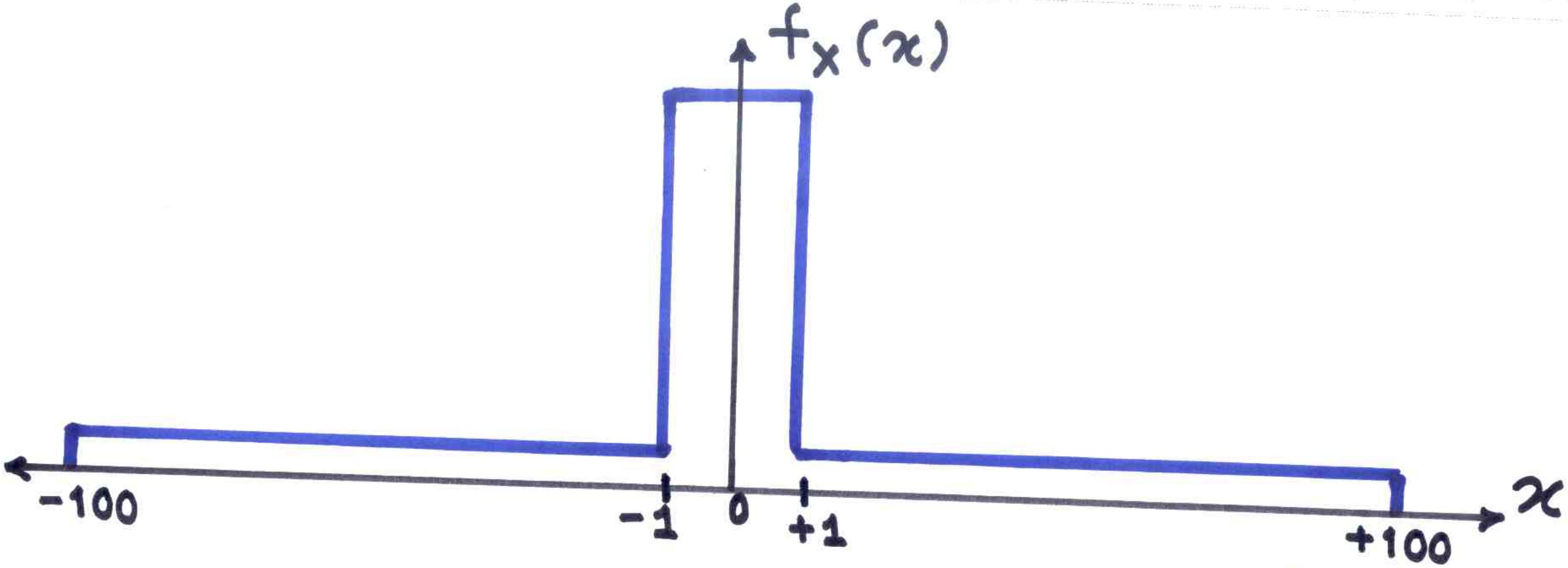
$\Delta = 0.3$

Reconstruction levels:

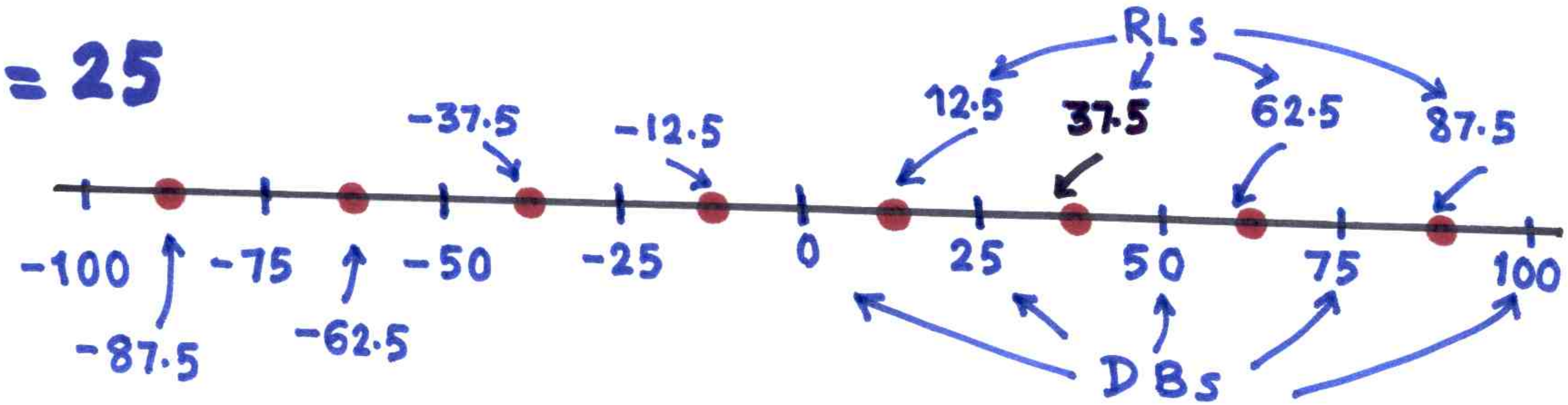
$-1.05, -0.75, -0.45, -0.15, 0.15, 0.45, 0.75, 1.05$

98.95

σ_q^2



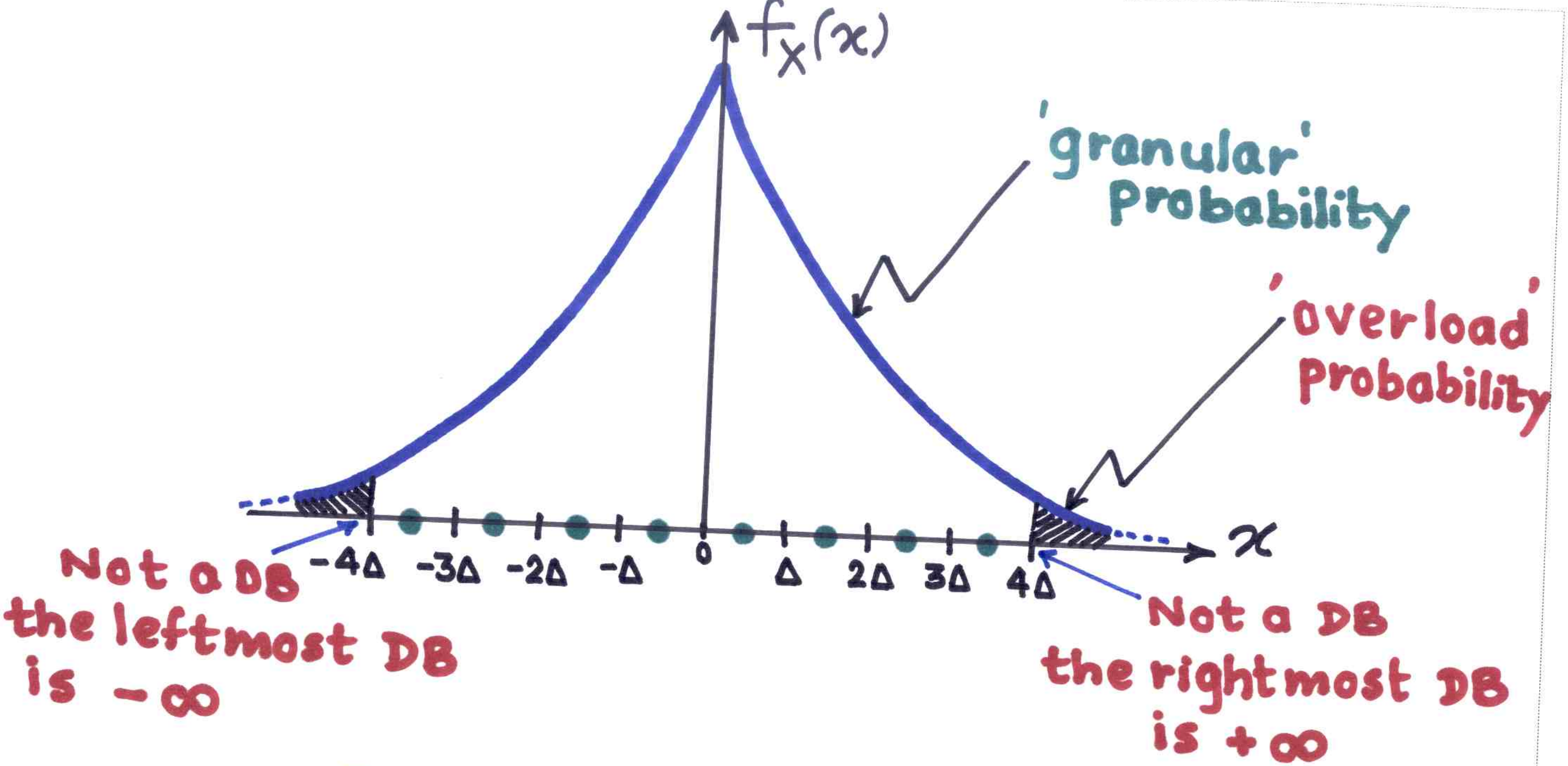
$$\Delta = \frac{200}{8} = 25$$



$$\Delta = 0.3$$

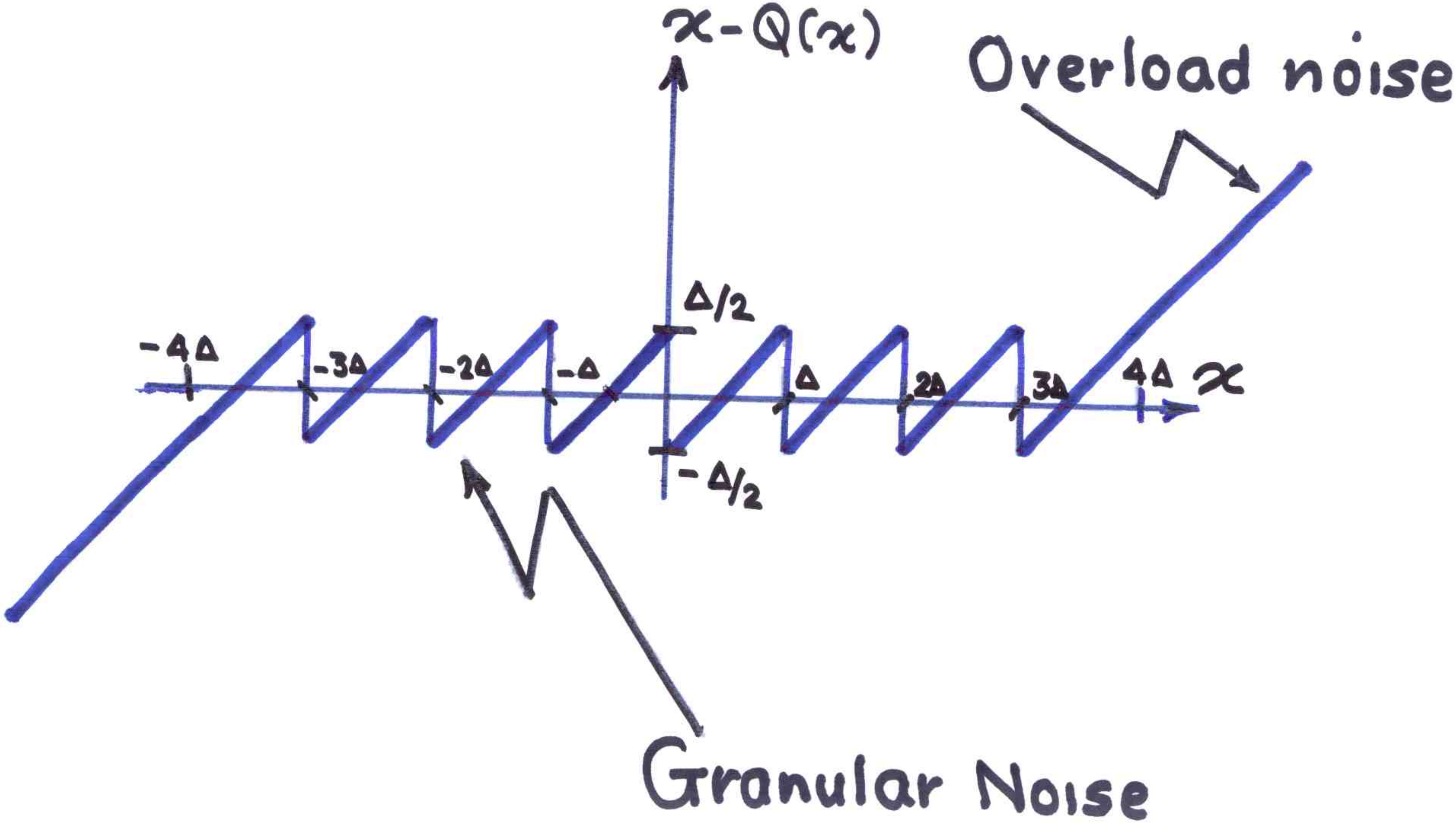
-100 -0.9 -0.6 -0.3 $\overleftarrow{DB}$ $\overrightarrow{DB}$: 0 0.3 0.6 0.9 100

-1.05 -0.75 -0.45 -0.15 $\overleftarrow{RL}$ $\overrightarrow{RL}$: 0.15 0.45 0.75 1.05



Granular and Overload probability for a 3-bit uniform quantizer

Quantization error for a uniform midrise Quantizer



Uniform Quantizer for Nonuniform PDF

Δ : Given L

msqe $\rightarrow$ minimization

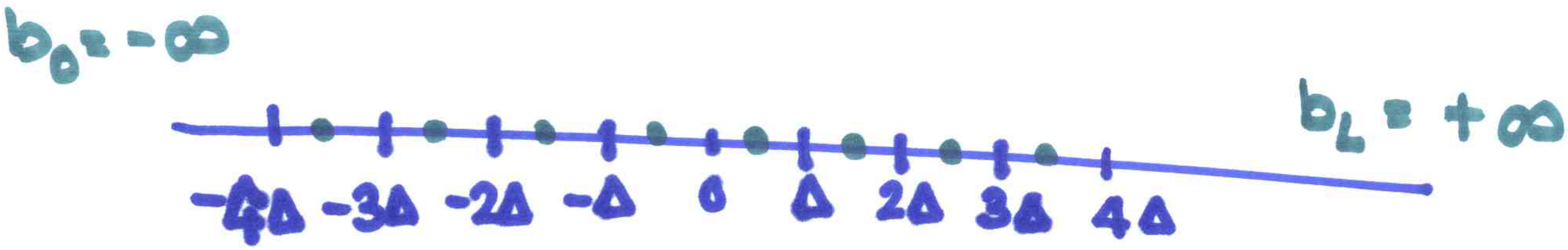
$$\sigma_q^2 = f(\Delta)$$

$$\sigma_q^2 = \sum_{j=1}^{L} \int_{b_{j-1}}^{b_j} (x - y_j)^2 f_X(x)\,dx$$

- pdf symmetric
- uniform midrise $Q(\cdot)$

$$b_j \longrightarrow n \times \Delta$$

$L = 8$



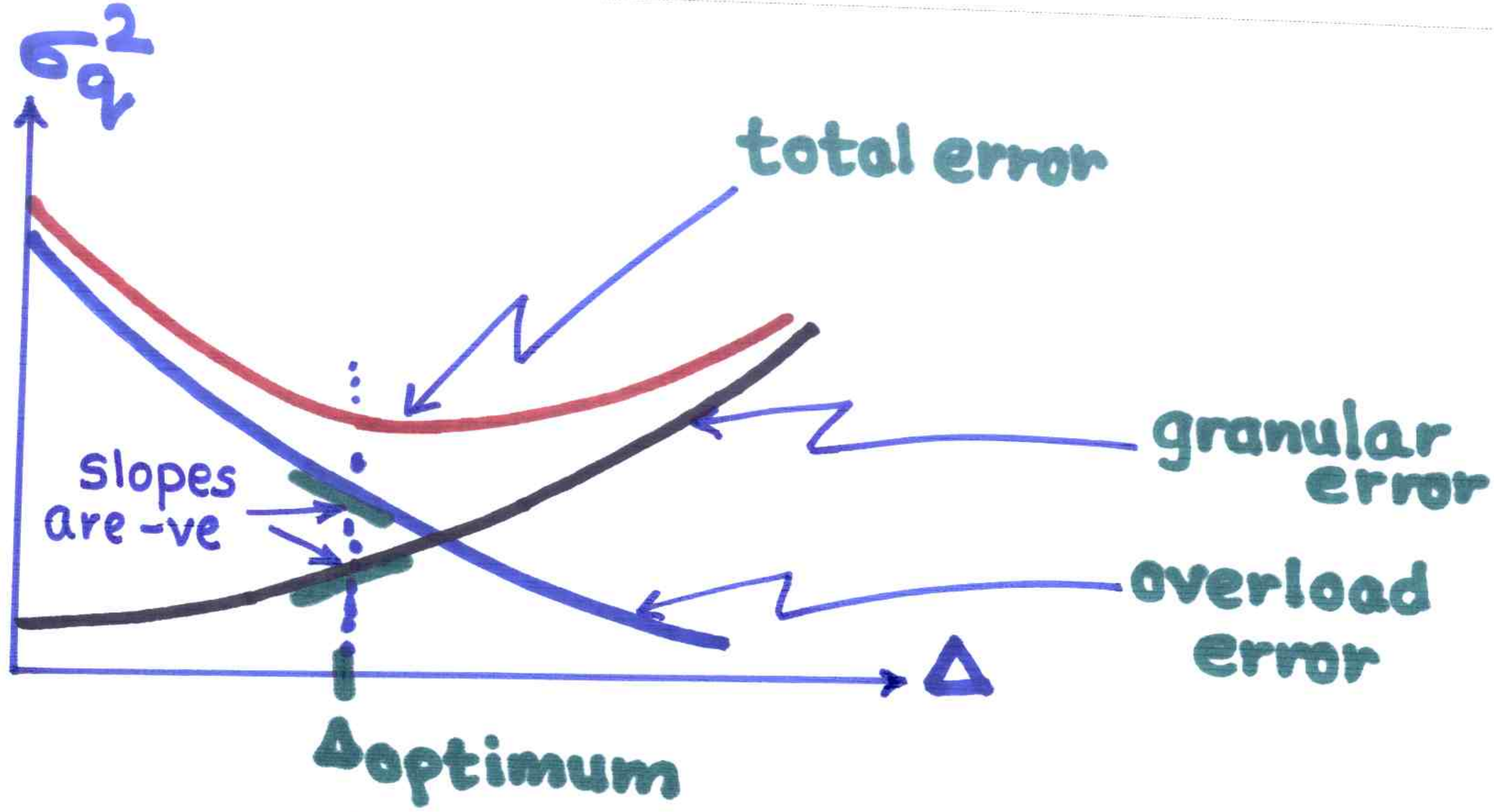
$$\sigma_q^2 = \sum_{j=1}^{L} \int_{b_{j-1}}^{b_j} (x - y_j)^2 f_X(x)\, dx$$

$$y_j : [(j-1)\Delta, j\Delta] : \frac{(2j-1)\Delta}{2}$$

$$\sigma_q^2 = \left[2 \sum_{k=1}^{L/2} \int_{(k-1)\Delta}^{k\Delta} \left\{ x - \frac{(2k-1)}{2}\Delta \right\}^2 f_X(x)\,dx \right.$$

$$\left. + 2 \int_{\frac{L}{2}\Delta}^{\infty} \left(x - \left(\frac{L-1}{2}\right)\Delta \right)^2 f_X(x)\,dx \right]$$

$$\frac{\partial \sigma_q^2}{\partial \Delta} = 0$$



$$\min\{f_1(x)+f_2(x)\}$$

$$\Rightarrow \frac{d}{dx}f_1(x)+\frac{d}{dx}f_2(x)=0$$

$$\Rightarrow \frac{d}{dx}f_1(x)=-\frac{d}{dx}f_2(x)$$

Leibnitz's rule states that if $a(x)$ and $b(x)$ are monotonic, then

$$\frac{\partial}{\partial x}\left[\int_{a(x)}^{b(x)} \phi(\alpha, x)\, d\alpha\right] = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}\phi(\alpha, x)\, d\alpha$$

$$+\phi\big(\alpha = b(x); x\big)\frac{\partial}{\partial x} b(x)$$

$$-\phi\big(\alpha = a(x); x\big)\frac{\partial}{\partial x} a(x)$$

$$\int_{(k-1)\Delta}^{k\Delta} \left(x - \frac{(2k-1)}{2}\Delta\right)^2 f_X(x)\,dx$$

$$\frac{\partial}{\partial\Delta}\left[\int_{(k-1)\Delta}^{k\Delta}\left(x-\frac{(2k-1)}{2}\Delta\right)^2 f_X(x)\,dx\right]$$

1^{st} term

$$= -(2k-1)\int_{(k-1)\Delta}^{k\Delta}\left\{x-\frac{(2k-1)}{2}\Delta\right\} f_X(x)\,dx$$

$$+\left\{k\Delta - \frac{(2k-1)}{2}\Delta\right\}^2 f_X(k\Delta)\cdot k - \left\{(k-1)\Delta - \frac{(2k-1)}{2}\Delta\right\}^2 f_X((k-1)\Delta) \times (k-1)$$

$$= 1^{st}\text{ term} + \left(\frac{\Delta}{2}\right)^2 f_X(k\Delta)\cdot k - \left(\frac{\Delta}{2}\right)^2 f_X((k-1)\Delta)\cdot(k-1)$$

$$\int_{k\Delta}^{(k+1)\Delta}\left(x-\frac{(2k+1)}{2}\Delta\right)^2 f_X(x)\,dx$$

$$\frac{\partial}{\partial\Delta}\left[\int_{k\Delta}^{(k+1)\Delta}\left(x-\frac{(2k+1)}{2}\Delta\right)^2 f_X(x)\,dx\right]$$

1st term

$$= -(2k+1)\int_{k\Delta}^{(k+1)\Delta}\left\{x-\frac{(2k+1)}{2}\Delta\right\} f_X(x)\,dx$$

$$+\left\{(k+1)\Delta-\frac{(2k+1)}{2}\Delta\right\}^2 f_X\big((k+1)\Delta\big)\times(k+1)$$

$$-\left\{k\Delta-\frac{(2k+1)}{2}\Delta\right\}^2 f_X(k\Delta)\times k$$

$$= 1^{st}\,\text{term} + \left(\frac{\Delta}{2}\right)^2 f_X\big((k+1)\Delta\big)\times(k+1) - \left(\frac{\Delta}{2}\right)^2 f_X(k\Delta)\times k$$

$$\frac{\partial \sigma_q^2}{\partial \Delta} = -2\sum_{k=1}^{L/2} (2k-1) \int_{(k-1)\Delta}^{k\Delta} \left(x - \frac{(2k-1)}{2}\Delta\right) f_X(x)\,dx$$

$$- 2(L-1) \int_{\frac{L}{2}\Delta}^{\infty} \left(x - \frac{(L-1)}{2}\Delta\right) f_X(x)\,dx$$

$$= 0$$

TABLE: OPTIMUM Δ, $(SNR)_q$ FOR UNIFORM QUANTIZATION

Alphabet Size	Uniform		Gaussian		Laplacian	
	Step Size	$(SNR)_q$	Step Size	$(SNR)_q$	Step Size	$(SNR)_q$
2	1.732	6.02	1.5960	4.40	1.414	3.00
4	0.866	12.04	0.9957	9.24	1.0873	7.05
6	0.577	15.58	0.7334	12.18	0.8707	9.56
8	0.433	18.06	0.5860	14.27	0.7309	11.39
10	0.346	20.02	0.4908	15.90	0.6334	12.81
12	0.289	21.60	0.4238	17.25	0.5613	13.98
14	0.247	22.94	0.3739	18.37	0.5055	14.98
16	0.217	24.08	0.3352	19.36	0.4609	15.84
32	0.108	30.10	0.1881	24.56	0.2799	20.46

J. Max, “Quantizing for Minimum Distortion”, IRE Trans. on Information Theory, IT-6, pages 7-12, Jan. 1960.

A. K. Jain, “Fundamentals of Image Processing”, Prentice Hall, 1989.