Correlation in Random Variables

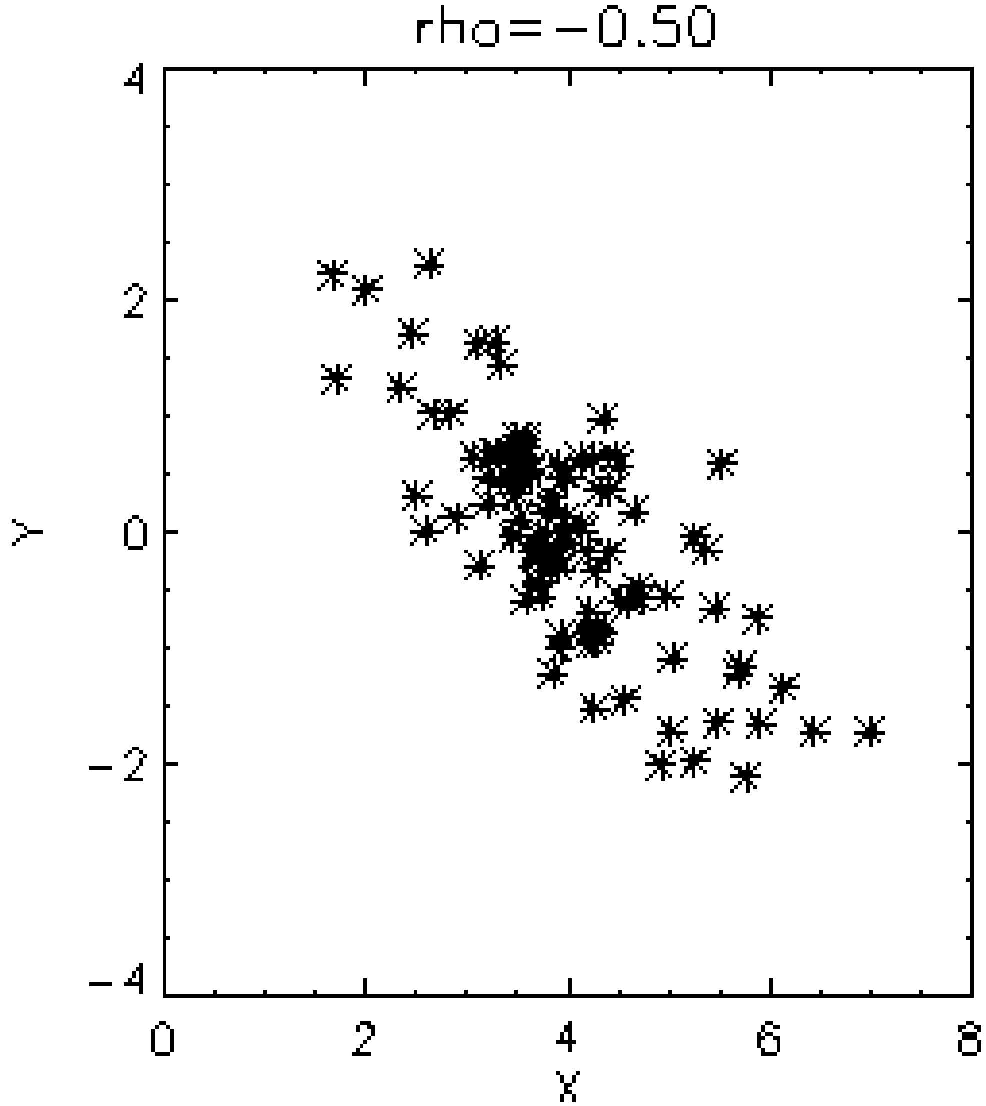
Lecture 11

Spring 2002

Correlation in Random Variables

Suppose that an experiment produces two random variables, X and Y. What can we say about the relationship between them?

One of the best ways to visualize the possible relationship is to plot the (X, Y) pair that is produced by several trials of the experiment. An example of correlated samples is shown at the right



Joint Density Function

The joint behavior of X and Y is fully captured in the joint probability distribution. For a continuous distribution

$$E[X^m Y^n] = \iint_{-\infty}^{\infty} x^m y^n f_{XY}(x,y)dxdy$$

For discrete distributions

$$E[X^m Y^n] = \sum_{x \in \mathcal{S}_x} \sum_{y \in \mathcal{S}_y} x^m y^n P(x,y)$$

Covariance Function

The covariance function is a number that measures the common variation of X and Y. It is defined as

$$
\begin{aligned}
\text{cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E[XY] - E[X]E[Y]
\end{aligned}
$$

The covariance is determined by the difference in $E[XY]$ and $E[X]E[Y]$.

If X and Y were statistically independent then $E[XY]$ would equal $E[X]E[Y]$ and the covariance would be zero.

The covariance of a random variable with itself is equal to its variance. $\texttt{cov}[X, X] = E[(X - E[X])^2] = \texttt{var}[X]$

Correlation Coefficient

The covariance can be normalized to produce what is known as the correlation coefficient, ρ.

$$\rho = \frac{\mathrm{cov(X, Y)}}{\sqrt{\mathrm{var(X)var(Y)}}}$$

The correlation coefficient is bounded by $-1 \leq \rho \leq 1$. It will have value $\rho = 0$ when the covariance is zero and value $\rho = \pm 1$ when X and Y are perfectly correlated or anti-correlated.

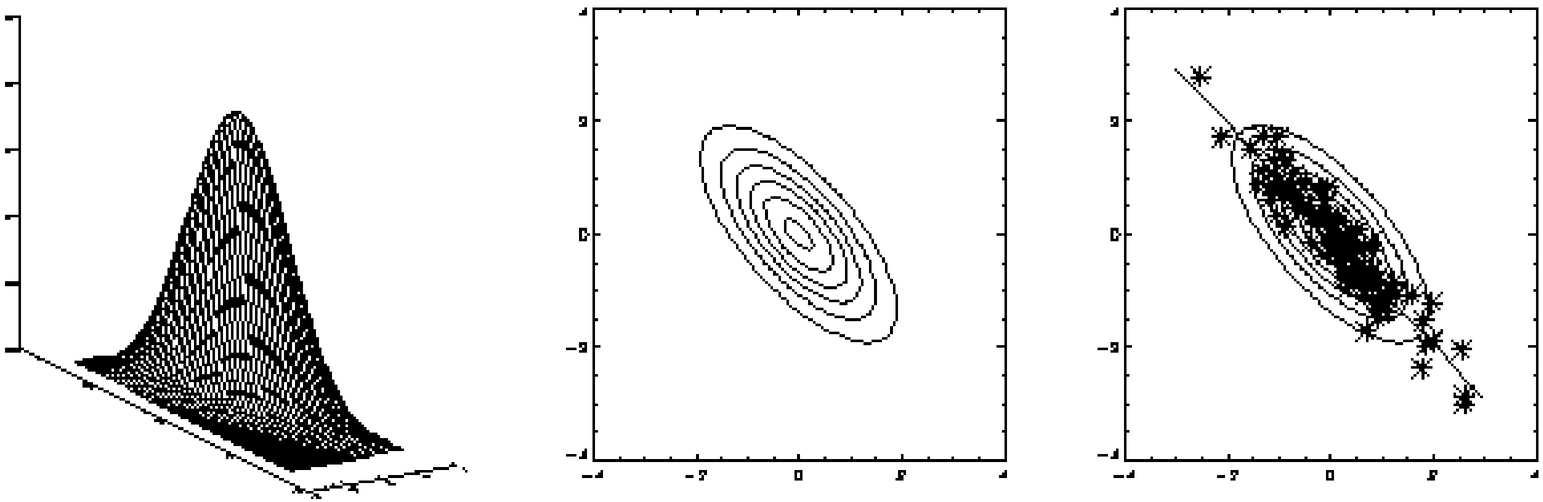
Autocorrelation Function

The autocorrelation function is very similar to the covariance function. It is defined as

$$R(X,Y) = E[XY] = \texttt{cov}(X,Y) + E[X]E[Y]$$

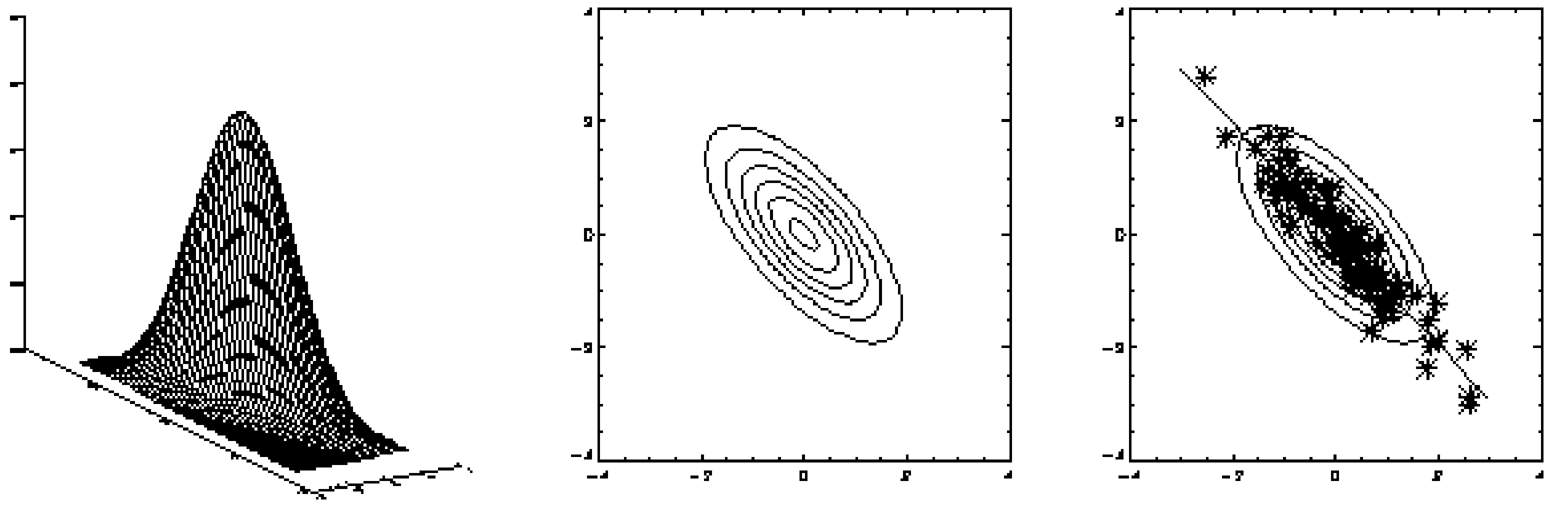
It retains the mean values in the calculation of the value. The random variables are *orthogonal* if $R(X,Y) = 0$.

Normal Distribution



$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2}{2(1-\rho^2)}\right)$$

Normal Distribution



The orientation of the elliptical contours is along the line $y = x$ if $\rho > 0$ and along the line $y = -x$ if $\rho < 0$. The contours are a circle, and the variables are uncorrelated, if $\rho = 0$. The center of the ellipse is (μ_x, μ_y).

Linear Estimation

The task is to construct a rule for the prediction $\hat{Y}$ of Y based on an observation of X.

If the random variables are correlated then this should yield a better result, on the average, than just guessing. We are encouraged to select a linear rule when we note that the sample points tend to fall about a sloping line.

$$\hat{Y} = aX + b$$

where a and b are parameters to be chosen to provide the best results. We would expect a to correspond to the slope and b to the intercept.

Minimize Prediction Error

To find a means of calculating the coefficients from a set of sample points, construct the predictor error

$$\varepsilon = E[(Y - \hat{Y})^2]$$

We want to choose a and b to minimize ε. Therefore, compute the appropriate derivatives and set them to zero.

$$\begin{aligned}
\frac{\partial \varepsilon}{\partial a} &= -2E[(Y - \hat{Y})\frac{\partial \hat{Y}}{\partial a}] = 0 \\
\frac{\partial \varepsilon}{\partial b} &= -2E[(Y - \hat{Y})\frac{\partial \hat{Y}}{\partial b}] = 0
\end{aligned}$$

These can be solved for a and b in terms of the expected values. The expected values can be themselves estimated from the sample set.

Prediction Error Equations

The above conditions on a and b are equivalent to

$$E[(Y - \hat{Y})X] = 0$$
$$E[Y - \hat{Y}] = 0$$

The prediction error $Y - \hat{Y}$ must be orthogonal to X and the expected prediction error must be zero.

Substituting $\hat{Y} = aX + b$ leads to a pair of equations to be solved for a and b.

$$\begin{array}{lcll} E[(Y - aX - b)X] & = & E[XY] - aE[X^2] - bE[X] & = 0 \\ E[Y - aX - b] & = & E[Y] - aE[X] - b & = 0 \end{array}$$

$$\begin{bmatrix} E[X^2] & E[X] \\ E[X] & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} E[XY] \\ E[Y] \end{bmatrix}$$

Prediction Error

$$
\begin{aligned}
a &= \frac{\mathrm{cov(X,Y)}}{\mathrm{var(X)}} \\
b &= E[Y] - \frac{\mathrm{cov(X,Y)}}{\mathrm{var(X)}} E[X]
\end{aligned}
$$

The prediction error with these parameter values is

$$\varepsilon = (1 - \rho^2)\mathrm{var(Y)}$$

When the correlation coefficient $\rho = \pm 1$ the error is zero, meaning that perfect prediction can be made.

When $\rho = 0$ the variance in the prediction is as large as the variation in Y, and the predictor is of no help at all.

For intermediate values of ρ, whether positive or negative, the predictor reduces the error.

Linear Predictor Program

The program `lp.pro` computes the coefficients $[a, b]$ as well as the covariance matrix C and the correlation coefficient, ρ. The covariance matrix is

$$C = \begin{bmatrix} \mathtt{var}[X] & \mathtt{cov}[X,Y] \\ \mathtt{cov}[X,Y] & \mathtt{var}[Y] \end{bmatrix}$$

Usage example:

```
N=100
X=Randomn(seed,N)
Z=Randomn(seed,N)
Y=2*X-1+0.2*Z
p=lp(X,Y,c,rho)
print,'Predictor Coefficients=',p
print,'Covariance matrix'
print,c
print,'Correlation Coefficient=',rho
```

Program lp

```
function lp,X,Y,c,rho,mux,muy
;Compute the linear predictor coefficients such that
;Yhat=aX+b is the minimum mse estimate of Y based on X.

;Shorten the X and Y vectors to the length of the shorter.
n=n_elements(X) < n_elements(Y)
X=(X[0:n-1])[*]
Y=(Y[0:n-1])[*]

;Compute the mean value of each.
mux=total(X)/n
muy=total(Y)/n
```

Continued on next page

Program lp (continued)

```
;Compute the covariance matrix.
V=[[X-mux],[Y-muy]]
C=V##transpose(V)/(n-1)

;Compute the predictor coefficient and constant.
a=c[0,1]/c[0,0]
b=muy-a*mux

;Compute the correlation coefficient
rho=c[0,1]/sqrt(c[0,0]*c[1,1])

Return,[a,b]
END
```

Example

```
Predictor Coefficients=  1.99598     -1.09500
Correlation Coefficient=0.812836
Covariance Matrix
     0.950762      1.89770
      1.89770      5.73295
```

IDL Regress Function

IDL provides a number of routines for the analysis of data. The function REGRESS does multiple linear regression.

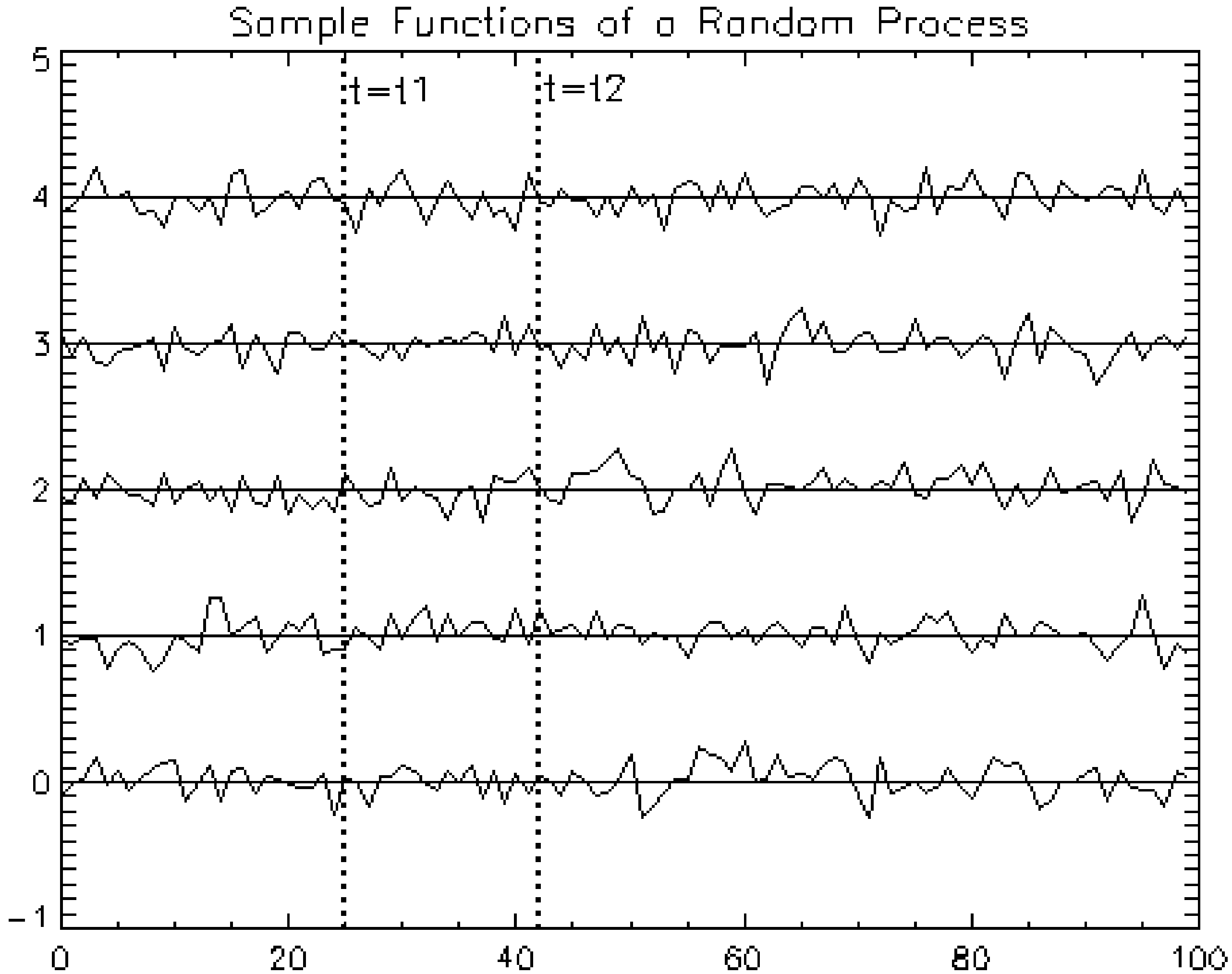
Compute the predictor coefficient a and constant b by

```
a=regress(X,Y,Const=b)
print,[a,b]
1.99598 -1.09500
```

Introduction to Random Processes

Today we will just introduce the basic ideas.

Random Processes



Random Process

- A random variable is a function $X(e)$ that maps the set of experiment outcomes to the set of numbers.

- A *random process* is a rule that maps every outcome e of an experiment to a *function* $X(t,e)$.

- A random process is usually conceived of as a function of time, but there is no reason to not consider random processes that are functions of other independent variables, such as spatial coordinates.

- The function $X(u,v,e)$ would be a function whose value depended on the location (u,v) and the outcome e, and could be used in representing random variations in an image.

Random Process

- The domain of e is the set of outcomes of the experiment. We assume that a probability distribution is known for this set.

- The domain of t is a set, $\mathcal{T}$, of real numbers.

- If $\mathcal{T}$ is the real axis then $X(t, e)$ is a *continuous-time* random process

- If $\mathcal{T}$ is the set of integers then $X(t, e)$ is a *discrete-time* random process

- We will often suppress the display of the variable e and write $X(t)$ for a continuous-time RP and $X[n]$ or X_n for a discrete-time RP.

Random Process

- A RP is a family of functions, $X(t, e)$. Imagine a giant strip chart recording in which each pen is identified with a different e. This family of functions is traditionally called an *ensemble*.

- A single function $X(t, e_k)$ is selected by the outcome e_k. This is just a time function that we could call $X_k(t)$. Different outcomes give us different time functions.

- If t is fixed, say $t = t_{1,}$ then $X(t_1, e)$ is a random variable. Its value depends on the outcome e.

- If both t and e are given then $X(t, e)$ is just a number.

Moments and Averages

$X(t_1, e)$ is a random variable that represents the set of samples across the ensemble at time t_1

If it has a probability density function $f_X(x; t_1)$ then the moments are

$$m_n(t_1) = E[X^n(t_1)] = \int_{-\infty}^{\infty} x^n f_X\,(x; t_1)\,dx$$

The notation $f_X(x; t_1)$ may be necessary because the probability density may depend upon the time the samples are taken.

The mean value is $\mu_X = m_1$, which may be a function of time.

The central moments are

$$E[(X(t_1) - \mu_X(t_1))^n] = \int_{-\infty}^{\infty} (x - \mu_X(t_1))^n\, f_X\,(x; t_1)\,dx$$

Pairs of Samples

The numbers $X(t_1, e)$ and $X(t_2, e)$ are samples from the same time function at different times.

They are a pair of random variables (X_1, X_2).

They have a joint probability density function $f(x_1, x_2; t_1, t_2)$.

From the joint density function one can compute the marginal densities, conditional probabilities and other quantities that may be of interest.