Quiz 1, EE302, Control Systems, Weightage: 10%, 3rd Feb 2020

Note:
(a) Attempt all 7 questions: total marks = 40 marks.
(b) If you use a well-known result, you need not prove the same. However, you must <u>mention</u> the result you are using at the appropriate place.
(c) If a method has been asked to be used, then marks are awarded only upon using that particular method.
(d) This question paper contains two pages. Please turn overleaf.

Ques num. 1: (3 marks) <u>**Use**</u> initial value theorem to get $\boldsymbol{f(0^+)}$ and $\boldsymbol{f'(0^+)}$ for $\boldsymbol{F(s)} = \frac{s^2}{s^2-4s+9}$.

Ques num. 2: (2 marks) <u>**Use**</u> final value theorem to obtain $\boldsymbol{f(\infty)}$ for $\boldsymbol{F(s)}$ given by:

(a) $$\frac{2s-7}{s(s^2+0.1s+4)}$$ (b) $$\frac{2s+7}{s(s^2+4)}$$

Ques num. 3: (10 marks) Consider the first order transfer function $\boldsymbol{G(s)} = \frac{k(s-z)}{s-p}$ with $\boldsymbol{p < 0}$ and $\boldsymbol{k \neq 0}$.

(a) Find the step response in terms of $\boldsymbol{k, z, p}$.

(b) Show that the step response of $\boldsymbol{G(s)}$ can be written as
$$y(t) = y(\infty) - (y(\infty) - y(0^+))e^{pt}.$$

(c) Show that when $\boldsymbol{z > 0}$, there exists some $\boldsymbol{t_1 \in (0, \infty)}$ such that $\boldsymbol{y(t_1) = 0}$.

Ques num. 4: (5 marks) Consider $\boldsymbol{G(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ and its response for the unit step input (with $\boldsymbol{0 < \zeta < 1}$). Derive a formula for the % over-shoot.

Ques num. 5: (10 marks) Find inverse Laplace transform to get $\boldsymbol{y_1(t)}$ and $\boldsymbol{y_2(t)}$ for:

(a) $$Y_1(s) = \frac{16}{(s+4)^2}$$ (b) $$Y_2(s) = \frac{256}{(s^2+4s+16)^2}$$

Ques num. 6: (3 marks) A system with input $u(t)$ and output $y(t)$ is governed by the relation:

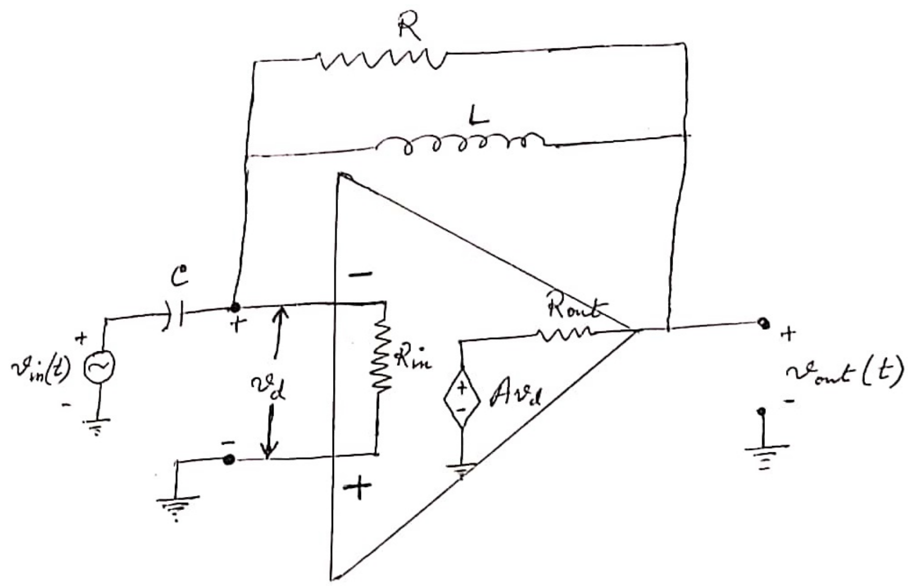
$$\frac{d}{dt}u + u^2 = \frac{d}{dt}y - \frac{d^2}{dt^2}y.$$

(a) Is the system linear? Give a brief reason.

(b) Is the system time-invariant? Give a brief reason.

(c) Is the system causal? Give a brief reason.

Ques num. 7: (7 marks) For the op-amp circuit shown in figure below, find the transfer function $\frac{\hat{V}_{\text{out}}(s)}{\hat{V}_{\text{in}}(s)}$ from input signal $v_{\text{in}}(t)$ to output signal $v_{\text{out}}(t)$.



Q.2 (a) $\dfrac{2s-7}{s(s^2+0.1s+4)}$

FVT $\Rightarrow$ $\lim_{s\to 0} sF(s) = \dfrac{-7}{4} = f(\infty)$

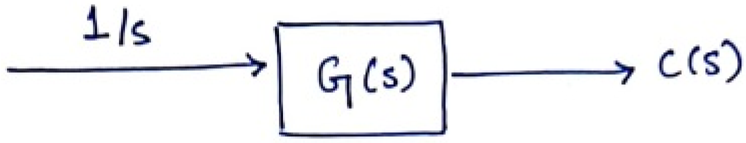
(b) $\dfrac{2s+7}{s(s^2+4)}$

We see it is of the form of sin ωt or cos ωt Laplace Transform.

For these functions, FVT does not apply.

EE302 : QUIZ 1 SOLUTION

Ques. 3



$$G(s) = k\frac{(s-z)}{s-p} \qquad p<0,\ k\neq 0$$

a) $$k\frac{(s-z)}{s-p} = G(s)$$

Step response of $G(s) = y(s) = \frac{1}{s}G(s)$

$$= \frac{k(s-z)}{s(s-p)}$$

$$y(s) = \frac{k(s-z)}{s(s-p)} = \frac{A}{s} + \frac{B}{s-p}$$ (where A, B are constants)

$$A = zk/p \qquad B = \frac{k}{p}(p-z)$$

Taking laplace inverse,

$$y(t) = \frac{zk}{p} + \frac{k(p-z)}{p}e^{pt}$$

b) $$y(\infty) = \frac{zk}{p}$$ (as $e^{pt} \to 0$ as $t\to\infty$ because $p<0$)

$$y(0^+) = \frac{zk}{p} + \frac{k(p-z)}{p} = k$$

Hence, it can be easily shown that

$$y(t) = y(\infty) - (y(\infty) - y(0^+))e^{pt}$$

c) Given $z>0,\ p<0$

Case 1: $k>0$

$$y(0^+) = k > 0$$

$$y(\infty) = \frac{zk}{p} < 0$$

$$y'(t) = k(p-z)e^{pt} < 0$$

As $y(t)$ is monotonous decreasing function, there exists $t_1 \in (0,\infty)$ where $y(t_1)=0$

Case 2: $k<0$

$y(0^+)<0 \qquad y(\infty)>0$

$$y'(t) = k(p-z)e^{pt} > 0$$

As $y(t)$ is montonous increasing function, there exists $t_1 \in (0,\infty)$ where $y(t_1)=0$

Quiz-1 (EE 302)

Q.4 Consider $G(s) = \dfrac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$ and its responce for the unit step input (with $0 < \xi < 1$). Derive a formula for the % overshoot. [5 marks]

Answer: For the G(s), its step response will be,

$$C(s) = \frac{1}{s} \cdot G(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

with $0 < \xi < 1$ (underdamped case),

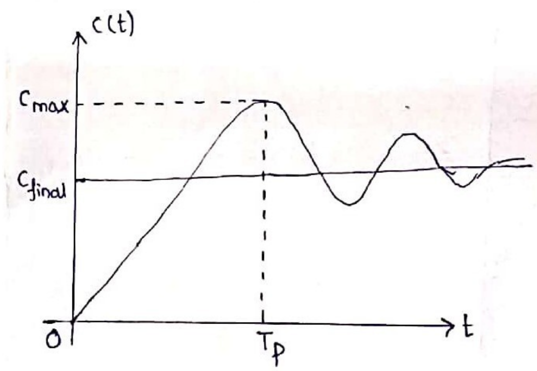
$$C(s) = \frac{1}{s} - \frac{(s + \xi\omega_n) + \frac{\xi}{\sqrt{1-\xi^2}}\omega_n\sqrt{1-\xi^2}}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)}$$

Taking the inverse laplace transform,

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_n\sqrt{1-\xi^2}\,t + \frac{\xi}{\sqrt{1-\xi^2}}\sin\omega_n\sqrt{1-\xi^2}\,t\right]$$

$$= 1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\cos(\omega_n\sqrt{1-\xi^2}\,t - \theta) \quad \text{--- (1)}$$

where, $\theta = \tan^{-1}(\xi/\sqrt{1-\xi^2})$



* Evaluation of % overshoot:

$$\boxed{\%\,OS = \frac{C_{max} - C_{final}}{C_{final}} \times 100}$$

here, $\boxed{C_{final} = 1}$

(since unit step response)

now, $\boxed{C_{max} = C(T_P)}$ i.e. c(t) at the peak time.

* Obtaining T_P (peak time)

→ 'T_P' is obtained by differentiating c(t) in eq^n-(1) and finding the first zero crossing after $t=0$.

Hence, $$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin(\omega_n\sqrt{1-\xi^2}\,t)$$

setting $\dot{c}(t) = 0 \Rightarrow \omega_n\sqrt{1-\xi^2}\,t = n\pi \Rightarrow t = \dfrac{n\pi}{\omega_n\sqrt{1-\xi^2}}$

[P.T.O]

$$\therefore \boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$ (kept $n=1$, as the first peak will occur at the peak time 'T_p')

now, $$C_{max} = C(T_p) = 1 - e^{-(\xi\pi/\sqrt{1-\xi^2})}\left[\cos\pi + \frac{\xi}{\sqrt{1-\xi^2}}\sin\pi\right]$$

$$= 1 + e^{-(\xi\pi/\sqrt{1-\xi^2})}$$

$$\therefore \boxed{C_{max} = 1 + e^{-(\xi\pi/\sqrt{1-\xi^2})}}$$

Therefore, $$\%\,OS = \frac{C_{max} - C_{final}}{C_{final}} \times 100$$

$$= \frac{1 + e^{-(\xi\pi/\sqrt{1-\xi^2})} - 1}{1} \times 100$$

$$= e^{-(\xi\pi/\sqrt{1-\xi^2})} \times 100$$

$$\therefore \boxed{\%\ \text{overshoot} = e^{-(\xi\pi/\sqrt{1-\xi^2})} \times 100}$$

5

a) $Y_1(s) = \frac{16}{(s+4)^2}$

we know $L[t.u(t)] = \frac{1}{s^2}$

& $L[e^{at} f(t)] = F(s-a)$

$\Rightarrow L^{-1}\left[\frac{16}{(s+4)^2}\right] = 16.e^{-4t}.t\,u(t)$

b) $Y_2(s) = \frac{256}{(s^2+4s+16)^2} = \frac{256}{[(s+4)^2+12]^2}$

Now consider

$$L^{-1}\left[\frac{256}{(s^2+12)^2}\right] = L^{-1}\left[\frac{1}{\sqrt{12}\,s} \cdot \frac{2(128)\sqrt{12}\,s}{(s^2+12)^2}\right]$$

$$= L^{-1}\left[\frac{128}{\cancel{12}\sqrt{12}} \cdot \int_0^t \tau.\sin(\sqrt{12}\tau).d\tau\right]$$

$$= \frac{128}{12\sqrt{12}}\left[\sin u - u\cos u + c\right]_0^{\sqrt{12}t}$$

$$= \frac{128}{12\sqrt{12}}\left[\sin(\sqrt{12}t) - \sqrt{12}t.\cos(\sqrt{12}t)\right]$$

$$\Rightarrow L^{-1}[Y_2(s)] = \frac{128}{12\sqrt{12}} e^{-4t}.\left[\sin\sqrt{12}t - \sqrt{12}t.\cos(\sqrt{12}t)\right]$$

(6) $\frac{d}{dt}u + u^2 = \frac{d}{dt}y - \frac{d^2}{dt^2}y$

(a) A system to become Linear It should satisfy the both Homogeneity and superposition.

⇒ because of the square term (u^2) present in the Equation it becomes non linear.

proof

$$a_1\frac{d}{dt}u_1 + a_1u_1^2 = a_1\frac{d}{dt}y_1 - \frac{d^2}{dt^2}y_1 a_1$$

$$a_2\frac{d}{dt}u_2 + a_2u_2^2 = a_2\frac{d}{dt}y_2 - \frac{d^2}{dt^2}y_2(a_2)$$

$$\left[a_1\frac{d}{dt}u_1 + a_2\frac{d}{dt}u_2\right] + \left[a_1u_1^2 + a_2u_2^2\right] = \left[a_1\frac{d}{dt}u_1 + a_2\frac{d}{dt}u_2\right] + \left[a_1u_1 + a_2u_2\right]^2$$

$$\left[a_1\frac{d}{dt}u_1 + a_2\frac{d}{dt}u_2\right] + \left[a_1u_1^2 + a_2u_2^2\right] \neq \left[a_1\frac{d}{dt}u_1 + a_2\frac{d}{dt}u_2\right] + \left[a_1u_1^2 + a_2u_2^2 + 2a_1a_2u_1u_2\right]$$

So, it become non linear.

(b) The system is time invarient because of absence of multiplication by function of t.

proof to delay the i/p by k.

$$\frac{d}{dt}u[t-k] + u^2[t-k] = \frac{d}{dt}y(t,k) - \frac{d^2}{dt^2}y[t,k]$$

(2) Replace 'n' by "n-k" throughout the given equation

Therefore $\frac{d}{dt}u[t-k] + u^2[t-k] = \frac{d}{dt}y[t-k] - \frac{d^2}{dt^2}y[t-k]$

(3) here $y[n,k] = y[n-k]$,

hence proved.

(c) ⇒ The system is causal. because the output $y(t)$ of the system depends only on the present value of $u(t)$.

$$\frac{-V_d - V_{in}}{1/sC} - \frac{V_d}{R_{in}} + \frac{-V_d - V_{out}}{R \| sL} = 0 \quad \text{——— (1)}$$

$$\frac{V_{out} + V_d}{R \| sL} + \frac{V_{out} - AV_d}{R_o} = 0 \quad \text{——— (2)}$$

$$\therefore V_{out}\left[\frac{1}{R\|sL} + \frac{1}{R_o}\right] = V_d\left[\frac{A}{R_o} - \frac{1}{R\|sL}\right]$$

$$\therefore V_d = \frac{V_{out}\left[\frac{1}{R\|sL} + \frac{1}{R_o}\right]}{\left[\frac{A}{R_o} - \frac{1}{R\|sL}\right]} = V_{out}\underbrace{\left[\frac{R_o(R+sL) + RsL}{ARLs - R_o(R+sL)}\right]}_{R_T} = V_{out} \cdot R_T$$

from (1):

$$\frac{V_{out}}{R\|sL} + \left[sC + \frac{1}{R_{in}} + \frac{1}{R\|sL}\right]V_d = -V_{in}(sC)$$

$$\therefore \frac{V_{out}}{R\|sL} + \left[sC + \frac{1}{R_{in}} + \frac{1}{R\|sL}\right]V_{out} \cdot R_T = -V_{in}(sC)$$

$$\therefore V_{out}\left[R_T\left(sC + \frac{1}{R_{in}} + \frac{1}{R\|sL}\right) + \frac{1}{R\|sL}\right] = -V_{in}(sC)$$

$$\therefore \frac{V_{out}(s)}{V_{in}(s)} = \frac{-sC}{R_T \cdot \left(sC + \frac{1}{R_{in}} + \frac{1}{R\|sL}\right) + \frac{1}{R\|sL}}, \text{ where } R_T = \frac{R_o(R+sL) + sRL}{ARLs - R_o(R+sL)}$$

for ideal opamp: $A \to \infty$, $R_o \to 0$ & $R_{in} \to \infty$

$$= -sC\left[\frac{RLs}{R+sL}\right]$$

$$= \frac{-s^2RLC}{R+sL}$$

$$= -\frac{s^2LC}{1+sL/R}$$