Advanced Robot Control and Learning

Differential Geometry in Robotics

Prof. Sami Haddadin

Advanced Robot Control - Outline

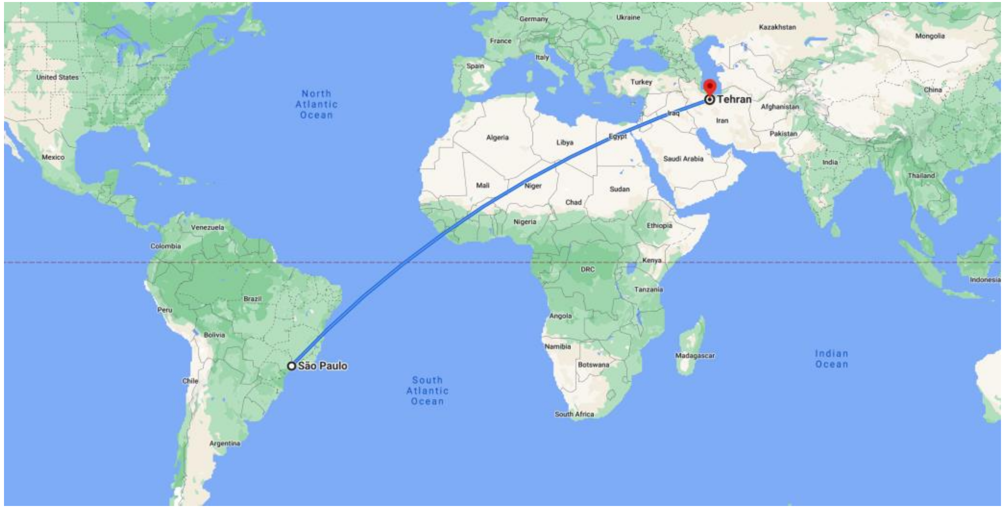
Differential Geometry in Robotics

- Manifolds
- Lie groups, $SO(3)$ and $SE(3)$
- Twists and Wrenches

Task Space Modeling and Control

Chapter 1: Differential Geometry in Robotics

Motivation



- Shortest path on a globe is not the shortest path on a map
- Requires mathematical analysis of curves and surfaces to compute metrics that depend on the geometry of the object

Outline

In order to represent the rigid body dynamics for any reference by the same concise equations, we need to cover the following topics:

1. Manifolds
2. Tangent and Cotangent Space
3. Lie groups and Lie algebra
4. Rotation Group $SO(3)$
5. Euclidian Group $SE(3)$
6. Twists and Wrenches
7. Putting it all together

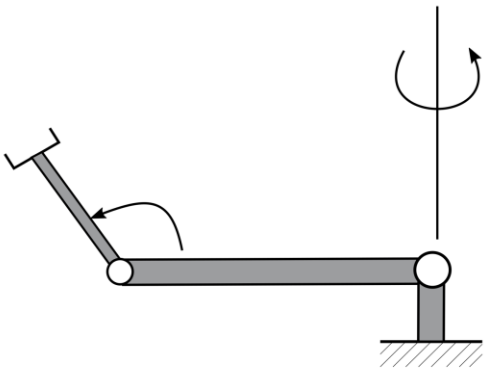
Manifolds

Map: Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. A mapping $f: U \mapsto V$ is a smooth map if all derivatives exist and are continuous.

Homeomorphism: A map f which is bijective and f and $f^{-1}: V \mapsto U$ are continuous. Moreover, f is called a **diffeomorphism** if f and f^{-1} are smooth.

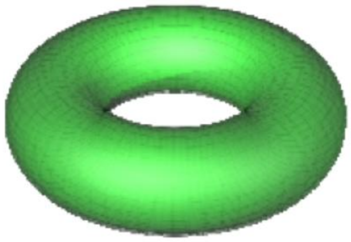
Manifold: A differentiable d-dimensional manifold $\mathcal{M}$ is a topological manifold with differentiable transition maps. The transition maps are compositions of the local coordinate charts (ϕ, U), where $\phi: U \mapsto \mathbb{R}^d$.

Robotics:



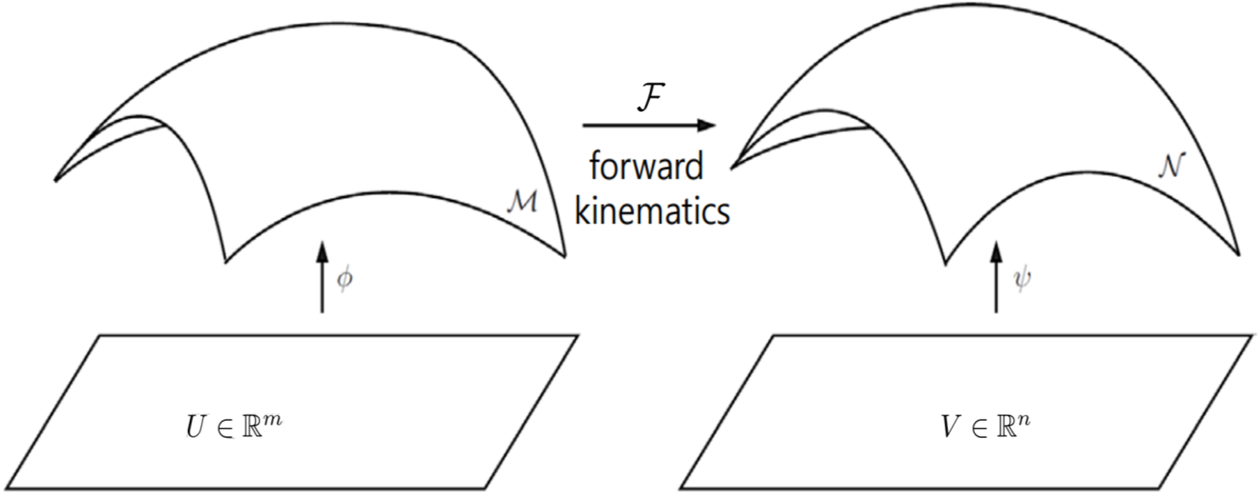
The set of joint angles of a non-planar 2DoF robot q are elements of Q which is a manifold.

The **task-space** of such a robot is also a manifold which has the shape of a torus:



Smooth manifolds

A **smooth manifold** admits a collection of local coordinate charts which cover $\mathcal{M}$ with the property that the transition map $\psi^{-1} \circ \phi$ of two charts (ϕ, U) and (ψ, V) is a diffeomorphism.
The property allows a change of coordinates with $\mathcal{F}: \mathcal{M} \mapsto \mathcal{N}$.



Tangent space

The tangent space of $\mathcal{M}$ at a point p is denoted as $T_p\mathcal{M}$. It is the set of all derivatives, of which elements are called tangent vectors.

The set of derivatives $\left\{\frac{\partial}{\partial x_i}\right\}$ forms a basis for $T_p\mathcal{M}$, such that

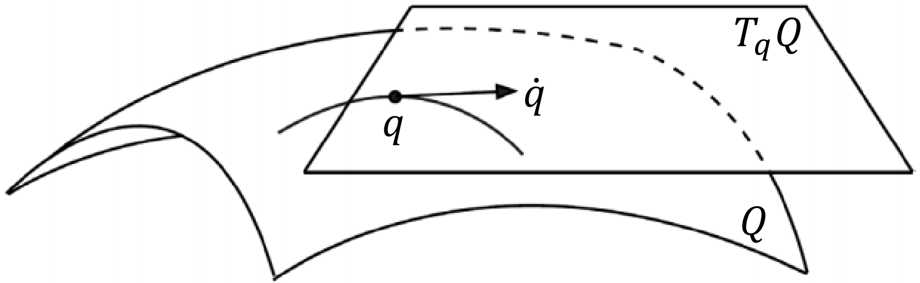
$$X_p = X_1 \frac{\partial}{\partial x_1} + \cdots + X_n \frac{\partial}{\partial x_n}$$

with local coordinates $(x_1, x_2, \dots, x_n)$.

Robotics:

Intuitively, the tangent space of a configuration manifold Q at a point $q \in \mathrm{Q}$, denoted T_qQ, is the set of all possible velocities possible at this configuration.

It is a vector space of the same dimension as Q.



Cotangent space

Given the tangent space $T_p\mathcal{M}$ to a manifold $\mathcal{M}$ at a point p, the cotangent space $T_p^*\mathcal{M}$ at the same point is the dual space of $T_p\mathcal{M}$. It is the set of all linear functions

$$\eta : T_p\mathcal{M} \mapsto \mathbb{R}\,.$$

Robotics:

In a geometric setting for mechanics, generalized forces take values in cotangent spaces. It is dual to a tangent space which represents velocities of the configuration.

For a configuration manifold Q and configuration velocities T_qQ, the dual space T_q^*Q allows a mapping to a scalar power value.

Example: Cotangent space: torque

$$\tau^T\dot{q} = P \qquad \tau = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_i \\ \vdots \\ \tau_n \end{bmatrix}$$

Lie groups

Let $\mathrm{GL}(n)$ be the **general linear group** which is the set of real $n \times n$ nonsingular matrices.

The **Lie group** $\mathbf{G}$ is a differentiable manifold and an algebraic group under matrix multiplication:

- The identity is an element of $\mathbf{G}$
- For every $A \in \mathbf{G}$, its inverse A^{-1}exists and $A^{-1} \in \mathbf{G}$

For example the group of rotation matrices and rigid-body motions are Lie groups.

Robotics:

The set of 3×3 rotation matrices is the **Special Orthogonal group** $\mathrm{SO}(3)$:

$$\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = \mathbb{1}, \mathrm{det} R = 1\}.$$

$\mathrm{SO}(3)$ is a 3-dimensional Lie group and describes all possible orientations of a rigid body in physical space.

The group of matrices that represent a general rigid body motion (translation and rotation) is called the **Special Euclidean group** $\mathrm{SE}(3)$. It is a Lie group as well.

Lie algebra

A **Lie algebra** is a vector space $\mathcal{V}$ equipped with a mapping $[\cdot,\cdot]: \mathcal{V} \times \mathcal{V} \mapsto \mathcal{V}$ called the **Lie bracket.** It is isomorphic to the tangent space of $\mathbf{G}$ at the identity element.

Isomorphism: A map f which is bijective and f and $f^{-1}: V \mapsto U$ are continuous and f^{-1} is also bijective.

Example: The Lie algebra of $\mathrm{GL}(n)$ is the set of all $n \times n$ real matrices, denoted $\mathrm{gl}(n)$, with the Lie bracket

$$[A, B] = AB - BA.$$

Robotics:

The Lie algebra of $SO(3)$ is called $\mathrm{so}(3)$. It is the set of 3×3 skew-symmetric matrices:

$$\mathrm{so}(3) = \{\Omega \in \mathbb{R}^{3\times 3} \mid \Omega + \Omega^T = 0\}.$$

The algebra is a local linearization of its Lie group. This allows us many powerful operations on a vector space instead of a complicated geometric object.

The rotation group

Every configuration of a rigid body that is free to rotate relative to a fixed frame can be identified with a unique $R \in SO(3)$.

The Rotation Group serves as a transformation: Given a point $q_b \in \mathbb{R}^3$in frame B and a rotation matrix R_{ab} w.r.t. to frame A, then

$$q_a = R_{ab} q_b.$$

The skew-symmetric representation of any vector $a = (a_1, a_2, a_3)^T$ will be defined by

$$\hat{a} := \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \in so(3).$$

Matrix exponential

For any Lie group, there exists a unique corresponding Lie algebra, and vice versa. The **matrix exponential** explicitly connects them.

For any real $n \times n$ matrix $\boldsymbol{A} \in \mathrm{gl}(n)$ its matrix exponential is defined by the series

$$e^{\boldsymbol{A}} = I_{n\times n} + \boldsymbol{A} + \frac{1}{2!}\boldsymbol{A}^{\mathbf{2}} + \frac{1}{3!}\boldsymbol{A}^{\mathbf{3}} + \cdots.$$

So that, if $\mathbf{G}$ is a Lie group its algebra $\mathbf{g}$ is defined by

$$\mathbf{g} = \{\boldsymbol{A} \in \mathrm{gl}(n) \mid e^{\mathbf{A}\mathrm{t}} \in \mathbf{G}\ \forall\ \mathrm{t} \in \mathbb{R}\}.$$

Robotics:

The matrix exponential connects $\mathrm{SO}(3)$ and $\mathrm{so}(3)$

$$R(u,\theta) = e^{\hat{u}\theta}$$

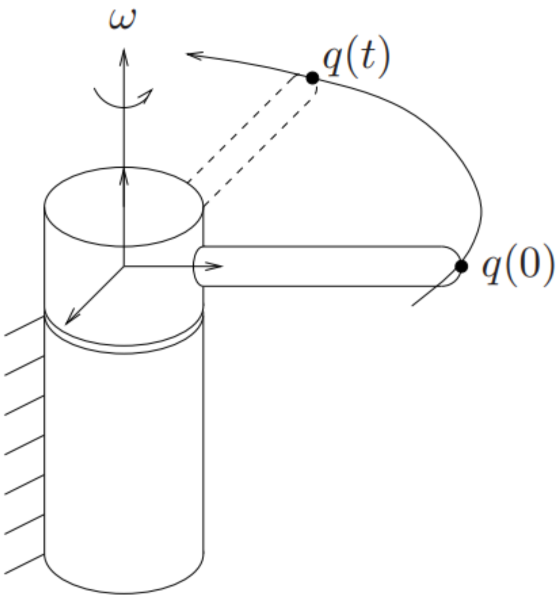
with any scalar θ and rotational axis u, where $\hat{u}\theta \in \mathrm{so}(3)$ (as $\hat{u}$ is skew-symmetric). Any matrix rotation can be represented in such a way.

Rodrigues' formula provides an efficient way to compute the matrix exponential:

$$e^{\widehat{\omega}\theta} = I_{3\times 3} + \widehat{\omega}\sin\theta + \widehat{\omega}^2(1-\cos\theta).$$

Exponential coordinates for rotations

Any orientation can be achieved by rotating a frame with angular velocity $\omega \in \mathbb{R}^3$ for a period of time t $\in \mathbb{R}$.



The time derivative (velocity) of q (joint angle)

$$\dot{q}(t) = \omega \times q(t) = \widehat{\omega} q(t)$$

can be obtain by a straightforward calculation from

$$q(t) = e^{\widehat{\omega} t} q(0).$$

Angular velocities

Given a fixed frame S and a rotating body frame B, let the matrix $R(t) \in SO(3)$ describe the orientation of the body frame w.r.t. the fixed frame at time t.

The columns r_1, r_2, r_3 of $R(t)$ describe the unit axes of B in fixed frame coordinates. At a specific time t, let ω_s be the angular velocity in fixed frame coordinates, then the rate of change of the axes is

$$\dot{r}_i = \omega_s \times r_i, \text{ for } i = 1,2,3.$$

These three equations can be rearranged to

$$\dot{R}(t) = \omega_s \times R(t) := \widehat{\omega}_s R(t).$$

Angular velocities

We drop the time dependency to unclutter the notation and post-multiplying both equations with R^{-1}, such that we get

$$\hat{\omega}_s = \dot{R}R^{-1}.$$

Let $\omega_b = R^{-1}\omega_s = R^T\omega_s$ be the representation of the same angular velocity expressed in body frame coordinates. Then

$$\begin{aligned} \rightarrow \hat{\omega}_b &= \widehat{R^T\omega_s} = R^T\hat{\omega}_s R \\ &= R^T\left(\dot{R}R^{-1}\right)R = R^T\dot{R} \\ &= R^{-1}\dot{R}. \end{aligned}$$

Both matrices $\hat{\omega}_s, \hat{\omega}_b \in so(3)$ are skew-symmetric matrices.

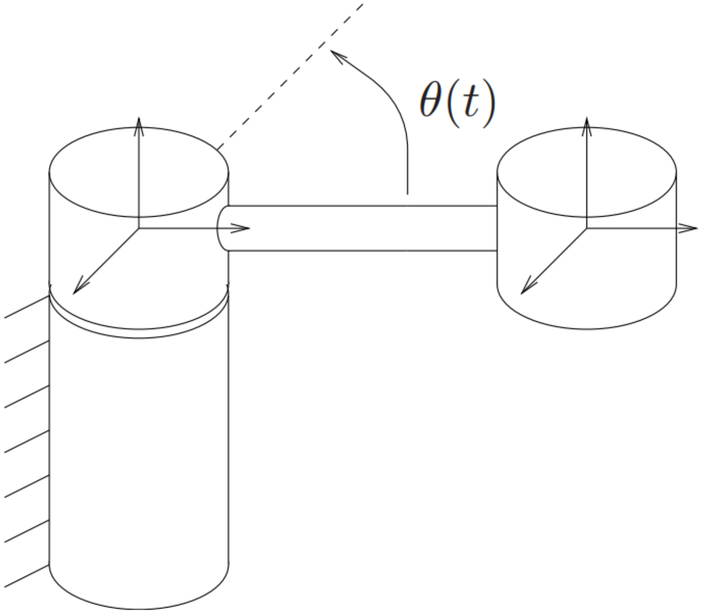
Example: Consider the motion of the manipulator shown with $\theta(t)$ being the angle of rotation about the fixed configuration. Note that $\theta(t)$ is a function of time. The orientation is

$$R(t) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The fixed frame angular velocity is

$$\widehat{\omega}_s = \dot{R}R^T = \begin{bmatrix} -\dot{\theta}\sin\theta & -\dot{\theta}\cos\theta & 0 \\ \dot{\theta}\cos\theta & -\dot{\theta}\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} R^T$$

$$= \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ hence } \omega_s = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}.$$



The Euclidian Group

Let $R \in \mathrm{SO}(3)$ be the orientation and $p \in \mathbb{R}^3$ be the position of a rigid body relative to the fixed frame. The **Special Euclidian Group** $\mathrm{SE}(3)$ consists of matrices of the form

$$\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}.$$

Elements can be written more compactly as

$$\mathrm{SE}(3) = \{(p, R) \mid p \in \mathbb{R}^3, R \in \mathrm{SO}(3)\}.$$

Any $\mathrm{h} \in \mathrm{SE}(3)$ is a rigid body transformation:

1. preserve distance between points: $\|\mathrm{h}p - \mathrm{h}q\| = \|p - q\|$
2. preserve orientation between vectors: $\mathrm{h}(v \times w) = \mathrm{h}v \times \mathrm{h}w$

for all vectors $v, w \in \mathbb{R}^3$ and all points $q, p \in \mathbb{R}^3$.

Twists

The Lie algebra of $\mathrm{SE}(3)$ is denoted $\mathrm{se}(3)$ and it consists of 4×4 matrices

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

where $\hat{\omega} \in \mathrm{so}(3)$ and $v \in \mathbb{R}^3$. Elements of $\mathrm{se}(3)$ are called **twists** and can be written more compactly as

$$\mathrm{se}(3) = \{(\hat{\omega}, v) \mid \hat{\omega} \in \mathrm{so}(3), v \in \mathbb{R}^3\}.$$

Angle axis to twists

Consider a one-link robot, with axis $\omega \in \mathbb{R}^3$, $||\omega|| = 1$ and $q \in \mathbb{R}^3$ being a point on the axis. The velocity of the point $p(t)$ is,

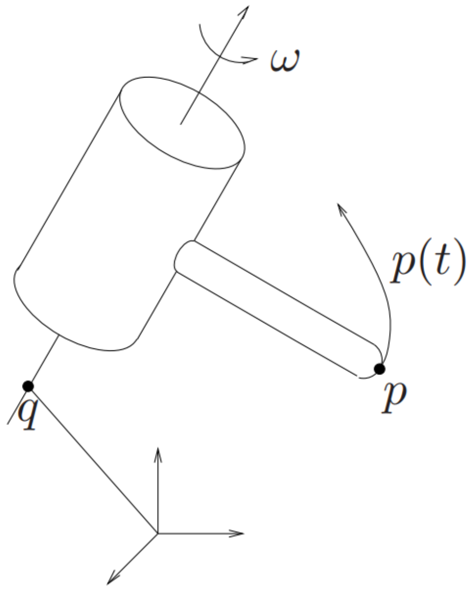
$$\dot{p}(t) = \omega \times (p(t) - q)$$

which can be converted using homogenous coordinates

$$\begin{bmatrix} \dot{p} \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega} & -\omega \times q \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}}_{\hat{\xi}} \begin{bmatrix} p \\ 1 \end{bmatrix}$$

with $v = -\omega \times q$.

⇒ Twists can be seen as the velocity of a general rigid body motion!



Twist velocity

The twist velocity allows us to combine the angular and linear velocities of a moving frame (e.g. end-effector). Let $T_{sb}(\mathrm{t})$ denote the configuration of the moving body frame B to the fixed spatial frame S:

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix},$$

where we dropped the subscripts to unclutter the notation. Analogous to the angular velocity, the velocity of a point q attached to a rigid body in spatial coordinates is

$$q_s = Tq_b \rightarrow \; v_{q_s} = \dot{q}_s = \dot{T}q_b = \dot{T}T^{-1}\, q_s.$$

The velocity of the point in the body frame is given by $v_{q_b} = T^{-1}v_{q_s} = T^{-1}\dot{T}q_b$.

Twist velocity of a point

The matrix $\hat{\mathcal{V}}_s = \dot{T}T^{-1} \in se(3)$ corresponds to the spatial velocity which can be used to find the velocity of a point in spatial coordinates:

$$\hat{\mathcal{V}}_s = \dot{T}T^{-1} = \begin{bmatrix} \dot{R}R^T & -\dot{R}R^T p + p \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \widehat{\omega}_s & v_s \\ 0 & 0 \end{bmatrix}.$$

It has the form of a twist $\mathcal{V}_s = (\omega_s, v_s) = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix}$. Similarly, the twist velocity of the body frame $\hat{\mathcal{V}}_b = T^{-1}\dot{T} \in se(3)$ is given by

$$\hat{\mathcal{V}}_b = T^{-1}\dot{T} = \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \widehat{\omega}_b & v_b \\ 0 & 0 \end{bmatrix}$$

with body twist $\mathcal{V}_b = (\omega_b, v_b)$.

Transforming twist velocities

The two twist velocities are related via the transformation

$$\hat{\mathcal{V}}_s = \dot{T}T^{-1} = T(T^{-1}\dot{T})T^{-1} = T\hat{\mathcal{V}}_b T^{-1},$$

which is the **large adjoint map** $\mathrm{Ad}_T\colon \mathbb{R}^6 \mapsto \mathbb{R}^6$. The transformation allows us to easily change the frame

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ \hat{p}R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_T]\ \mathcal{V}_b = \mathrm{Ad}_T(\mathcal{V}_b)$$

and

$$\mathcal{V}_b = \mathrm{Ad}_{T^{-1}}(\mathcal{V}_s).$$

Recap: Rigid body motions are Lie groups

The Rotation Group $\mathrm{SO}(3)$ and $\mathrm{SE}(3)$ are both Lie groups:

$$\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^T R = I_{3\times 3}, \det R = 1\},$$
$$\mathrm{SE}(3) = \{(p, R) \mid p \in \mathbb{R}^3, R \in \mathrm{SO}(3)\}.$$

Both have unique Lie algebras $\mathrm{so}(3)$ and $\mathrm{se}(3)$, which represent the Tangent space at the identity element,

$$\mathrm{so}(3) = \{\Omega \in \mathbb{R}^{3\times 3} \mid \Omega + \Omega^T = 0\}$$
$$\mathrm{se}(3) = \{(\hat{\omega}, v) \mid \hat{\omega} \in \mathrm{so}(3), v \in \mathbb{R}^3\}.$$

The angular velocity $\hat{\omega} \in \mathrm{so}(3)$ and twist velocity $\hat{\mathcal{V}} \in \mathrm{se}(3)$ are part of these Lie algebras.

Interlude: Adjoint maps

There are two more mappings that play a central role how physical quantities transform under a change of reference frames.

Small adjoint map: Given elements $\boldsymbol{A}, \boldsymbol{B}$ in the Lie algebra $\mathbf{g}$, the small adjoint map is the Lie bracket

$$\mathrm{ad}_{\boldsymbol{A}}(\boldsymbol{B}) = [\boldsymbol{A}, \boldsymbol{B}].$$

Large adjoint map: Let $X \in \mathbf{G}$, the large adjoint map is defined as

$$\mathrm{Ad}_X(\boldsymbol{A}) = \mathrm{X}\mathbf{A}\mathrm{X}^{-1}.$$

Wrenches

The generalized force $\mathcal{F}$ acting at a point consists of a linear component $f \in \mathbb{R}^3$ and an angular component $\tau \in \mathbb{R}^3$, called moment:

$$\mathcal{F} = \begin{bmatrix} \tau \\ f \end{bmatrix}.$$

The pair is referred to as a **wrench.** If A is the inertial frame and $\mathcal{F}_b$is the wrench applied to the body at frame B, then the work can be expressed as

$$W = \int_{t_1}^{t_2} \mathcal{V}_b^T \mathcal{F}_b \, dt = \int_{t_1}^{t_2} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix}^T \begin{bmatrix} \tau_b \\ f_b \end{bmatrix} dt \,.$$

The wrench $\mathcal{F}$ is an element of the dual space to $se(3)$, denoted $se(3)^*$.

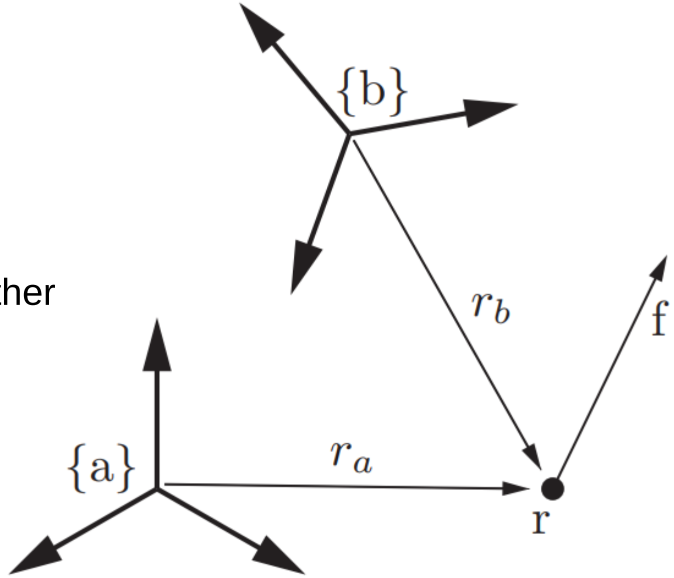
Transforming wrenches

The power generated by a $(\mathcal{F}, \mathcal{V})$ pair must be the same, regardless of the frame:

$$\mathcal{V}_b^T \mathcal{F}_b = \mathcal{V}_a^T \mathcal{F}_a.$$

Since $\mathcal{V}_a = \mathrm{Ad}_{T_{ab}}(\mathcal{V}_b)$ we can easily relate the forces to each other

$$\begin{aligned} \mathcal{V}_b^T \mathcal{F}_b &= \left(\mathrm{Ad}_{T_{ab}}(\mathcal{V}_b)\right)^{\mathrm{T}} \mathcal{F}_a \\ &= \mathcal{V}_b^T \left[\mathrm{Ad}_{T_{ab}}\right]^T \mathcal{F}_a \\ \rightarrow \mathcal{F}_b &= \left[\mathrm{Ad}_{T_{ab}}\right]^T \mathcal{F}_a. \end{aligned}$$



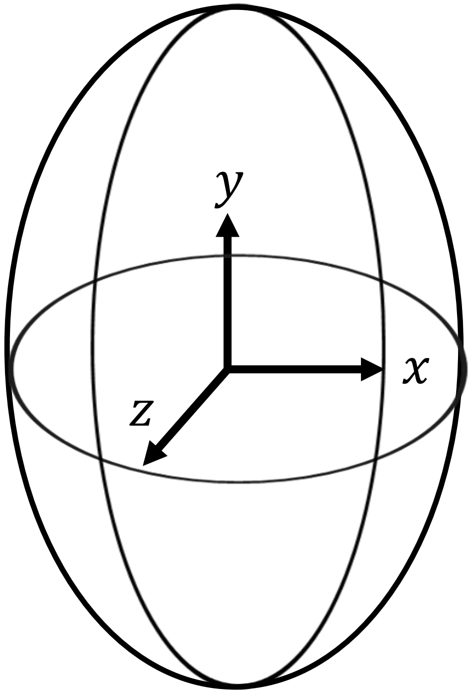
Putting it all together

Example: Consider a single rigid body of mass $\mathfrak{m}$ with a frame x-y-z attached to the body's center of mass. Let $I \in \mathbb{R}^{3\times3}$be the inertia matrix, $f \in \mathbb{R}^3$ and $\tau \in \mathbb{R}^3$ be the external force and moment applied to the body. The familiar equations of motion are

$$
\begin{aligned}
f &= \mathfrak{m}a \\
\tau &= I\alpha + \omega \times I\omega.
\end{aligned}
$$

With:

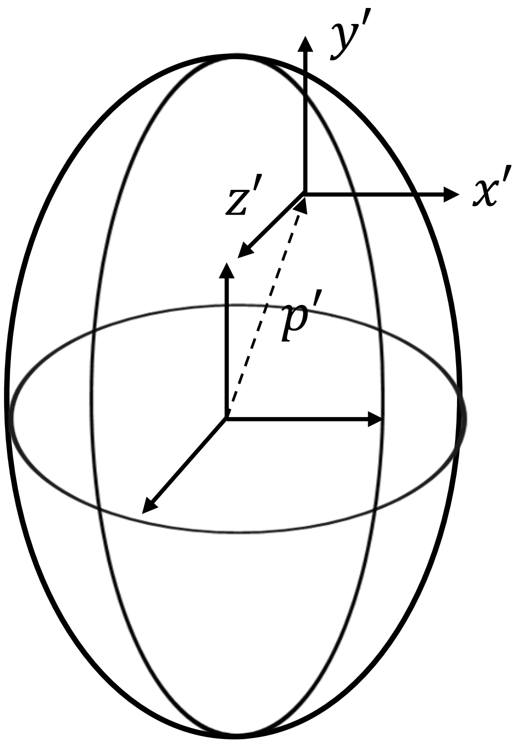
- $a \in \mathbb{R}^3$ being the acceleration of the center of mass,
- $\omega \in \mathbb{R}^3$ being the angular velocity and
- $\alpha \in \mathbb{R}^3$ being the angular acceleration.



Putting it all together

Example cont.: the frame is now attached to another point x'-y'-z' on the body, a velocity v' of the new frame, a vector p' denoting the vector from the new frame to the center of mass and considering identical orientation, the equations become

$$
\begin{aligned}
f' &= -\mathfrak{m}(p' \times \alpha') + \mathfrak{m}a' + \omega' \times \{\mathfrak{m}v' - \mathfrak{m}(p' \times \omega')\} \\
\tau' &= I\alpha' - \mathfrak{m}\big(p' \times (p' \times \alpha)\big) + \mathfrak{m}(p' \times a) \\
&\quad + \omega \times \big\{I\omega' - \mathfrak{m}\big(p' \times (p' \times \omega)\big) + \mathfrak{m}(p' \times v')\big\} \\
&\quad + v' \times \{-\mathfrak{m}(p' \times \alpha') + \mathfrak{m}a'\}.
\end{aligned}
$$



The complexity of the equations increased considerably with respect to the new reference frame! This will be remedied in the next slides.

Putting it all together: version 2

The equations of motion of a single rigid body (from the previous two slides) have a particularly simple form if the body frame B is attached to the center of mass. Again, we denote:

- $\mathcal{V}_b = (\omega_b, v_b) \in se(3)$ the body twist,
- $\mathcal{F}_b = (\tau_b, f_b) \in se(3)^*$ the externally applied wrench,
- $l_b = \mathfrak{m}\, v_b \in \mathbb{R}^3$ the linear momentum with mass $\mathfrak{m}$ and
- $h_b = I_b \omega_b \in \mathbb{R}^3$ the angular momentum with inertia matrix $I_b \in \mathbb{R}^{3\times 3}$.

The six-dimensional **spatial momentum** $\mathcal{P}_b = (h_b, l_b)$ is defined as

$$\mathcal{P}_b = \mathcal{G}_b \mathcal{V}_b \in \mathrm{se}(3)^* \text{ where } \mathcal{G}_b = \begin{bmatrix} I_b & 0 \\ 0 & \mathfrak{m}\mathbb{1} \end{bmatrix} \in \mathbb{R}^{6\times 6} \text{ is a spatial inertia matrix.}$$

Putting it all together

The dynamics equation of the rigid body can be written as

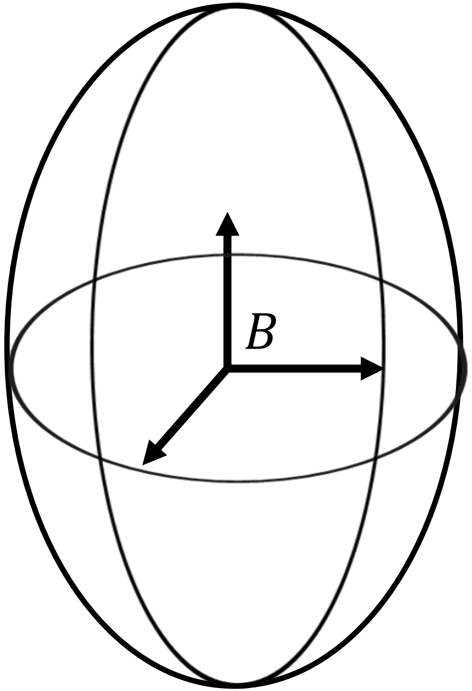
$$\mathcal{G}_b \dot{\mathcal{V}}_b = [\mathrm{ad}_{\mathcal{V}_b}]^T \mathcal{G}_b \mathcal{V}_b + \mathcal{F}_b$$

or

$$\dot{\mathcal{P}}_b = [\mathrm{ad}_{\mathcal{V}_b}]^T \mathcal{P}_b + \mathcal{F}_b,$$

with

$$[\mathrm{ad}_{\mathcal{V}_b}]^T \mathcal{P}_b = \begin{bmatrix} \widehat{\omega}_b & 0 \\ \hat{v}_b & \widehat{\omega}_b \end{bmatrix}^T \begin{bmatrix} h_b \\ l_b \end{bmatrix}.$$

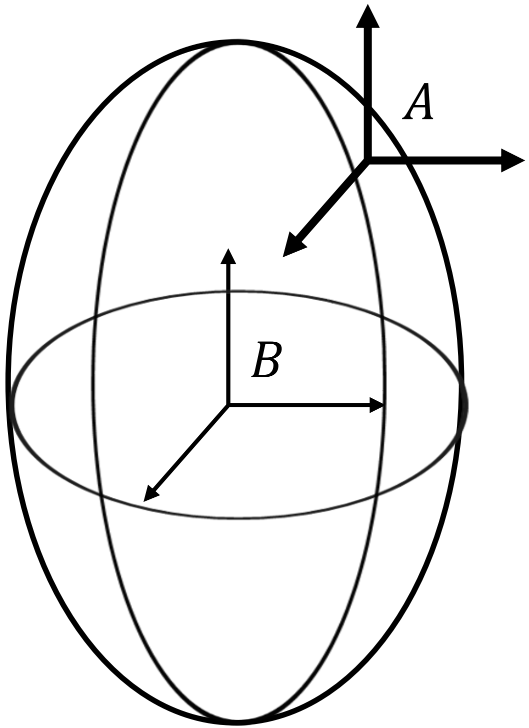


These equations of motion can now easily be transformed to any other reference frame A by applying the transformation rules we introduced.

Putting it all together

The transformation rules to any other reference frame A for rigid body motion are:

$$\begin{aligned}\mathcal{V}_a &= \mathrm{Ad}_{T_{ab}}(\mathcal{V}_b) = [\mathrm{Ad}_{T_{ab}}]\mathcal{V}_b \\ \dot{\mathcal{V}}_a &= \mathrm{Ad}_{T_{ab}}(\dot{\mathcal{V}}_b) \\ \mathcal{F}_a &= [\mathrm{Ad}_{T_{ba}}]^T \mathcal{F}_b.\end{aligned}$$



It can be verified that the dynamic equations expressed in A become

$$\mathcal{G}_a \dot{\mathcal{V}}_a = [\mathrm{ad}_{\mathcal{V}_a}]^T \mathcal{G}_a \mathcal{V}_a + \mathcal{F}_a.$$