# Lab1: Receiver Positioning

## Satellite Navigation Laboratory, Summer Semester 2022

Institute for Communications and Navigation, Technical University of Munich

## Contents

**1 Introduction** 1

**2 Coordinate System** 2
2.1 Conventional Terrestrial Reference System (CTRS) . . . . . . . . . . . . . . . 2
2.2 Conversion to a Local Coordinates . . . . . . . . . . . . . . . . . . . . . . . . 3

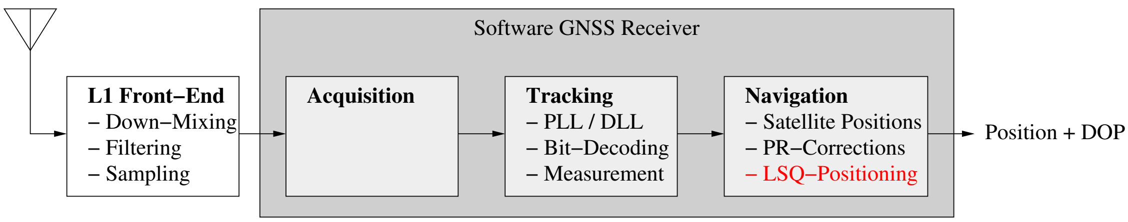
**3 Position Estimation with a Least-Squares Method** 3

**4 Dilution of Precision (DOP)** 5

**5 Homework** 7

**6 Lab Tasks*** 8

## 1 Introduction




In this lab session, we will discuss an iterative approach to estimate the user position and velocity with pseudorange and Doppler measurements, using a least-squares approach with a linearized model.

# 2 Coordinate System

## 2.1 Conventional Terrestrial Reference System (CTRS)

Standard GPS refers to the world geodetic system 1984 (WGS 84) [1, 2], which is a conventional terrestrial reference system (CTRS) whose the origin is at the center of mass of the Earth, the z-axis is towards the conventional terrestrial pole (CTP, the average pole), and the x-axis is aligned with the reference meridian in the equatorial plane. The WGS 84 defined with this reference system is illustrated in Fig. 1.

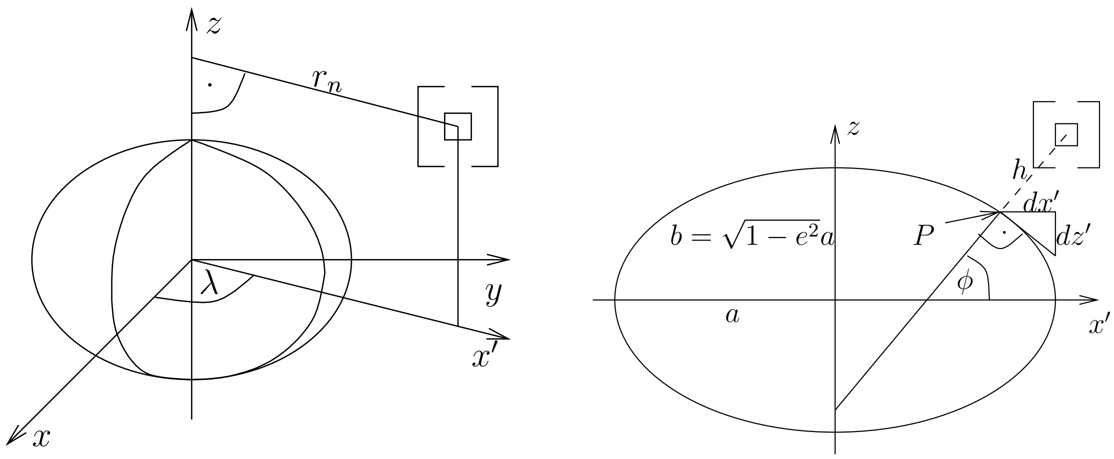



Figure 1: Earth ellipsoid in 3D (left) and the ellipse in the XZ-plane (right)

The semi-major axis of this ellipsoid is $a \approx 6378$km and the inverse of the difference between polar and equatorial diameter in units of the equatorial diameter (flattening $f$) is $1/f \approx 298.26$.

A point in the XZ-plane $P' = [x', z']^T$ can be expressed as function of $\phi$ as

$$P' = \begin{pmatrix} x' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{1+(1-e^2)\tan^2\phi}} \\ \frac{a(1-e^2)\tan\phi}{\sqrt{1+(1-e^2)\tan^2\phi}}. \end{pmatrix}.$$

The coordinates in the 2D plane can be mapped to a Cartesian coordinate in the 3D space $P = [x, y, z]^T$ using longitude $\lambda$ and latitude $\phi$:

$$P = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (x' + h\cos\phi)\cos\lambda \\ (x' + h\cos\phi)\sin\lambda \\ z' + h\sin\phi \end{pmatrix} = \begin{pmatrix} r_n\cos\lambda \\ r_n\sin\lambda \\ z' + h\sin\phi \end{pmatrix},$$

where $r_n = x' + h\cos\phi$ with $h$, the height of the satellite from the user.

Longitude $\lambda$ can be simply retrieved from the Cartesian coordinate:

$$\tan\lambda = \frac{y}{x}.$$

Latitude $\phi$ can be estimated using the following equations:

$$\tan\phi = \frac{z}{p}\left(1 - e^2 \frac{N}{N+h}\right)^{-1}$$
$$p = \sqrt{x^2 + y^2} = (N+h)\cos\phi$$
$$h = \frac{p}{\cos\phi} - N$$

## 2.2 Conversion to a Local Coordinates

The user position $\vec{r}_U$ can be converted to the east-north-up (ENU, local tangent coordinate system) with rotation matrices:

$$\vec{r}_{U,\text{ENU}} = R_X(90^\circ - \phi) R_Z(\lambda + 90^\circ)\vec{r}_U,$$

where the rotation matrices are defined as

$$R_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \quad R_Y = \begin{pmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{pmatrix} \quad R_Z = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

# 3 Position Estimation with a Least-Squares Method

Note that in this lab, we only use pseudorange measurements transmitted from the $k$-th satellites to estimate the receiver (user) positions, without considering carrier-phase residuals.

After correcting ionospheric delay, tropospheric delay, and satellite clock errors, the corrected pseudorange measurements $\rho_c^k$ can be modeled with noise $\eta^k$ as

$$\rho_c^k(\vec{r}_U, c\delta) = \|\vec{r}_U - \vec{r}_c^{\,k}\| + c\delta + \eta^k \tag{1}$$
$$= \sqrt{(x_U - x_c^k)^2 + (y_U - y_c^k)^2 + (z_U - z_c^k)^2} + c\delta + \eta^k, \tag{2}$$

$\vec{r}_U$ User position, $\vec{r}_U = [x_U, y_U, z_U]^T$
$\vec{r}_c^{\,k}$ The corrected position of the $k$-th satellite, $\vec{r}_c^{\,k} = [x_c^k, y_c^k, z_c^k]^T$
$c$ The speed of the light, $\approx 299,792,458\text{m/s}$

The partial derivative of this model of $\vec{r}_U$ and $c\delta$ is

$$\begin{aligned} \nabla\rho_c^k(\vec{r}_U, c\delta) &= \begin{bmatrix} \frac{\partial\rho_c^k(\vec{r}_{U,i}, c\delta_i)}{\partial\vec{r}_U} & \frac{\partial\rho_c^k(\vec{r}_{U,i}, c\delta_i)}{\partial c\delta} \end{bmatrix} \\ &= \begin{bmatrix} \frac{x_U - x_c^k}{\|\vec{r}_U - \vec{r}_c^{\,k}\|} & \frac{y_U - y_c^k}{\|\vec{r}_U - \vec{r}_c^{\,k}\|} & \frac{z_U - z_c^k}{\|\vec{r}_U - \vec{r}_c^{\,k}\|} & 1 \end{bmatrix} \\ &\triangleq \begin{bmatrix} (\vec{e}^{\,k})^T & 1 \end{bmatrix}, \end{aligned}$$

where $\vec{e}^{\,k}$ is the line-of-sight (LoS) vector from the $k$-th satellite to the receiver as shown in Fig. 2:

$$\vec{e}^{\,k} = \frac{\vec{r}_U - \vec{r}_c^{\,k}}{\|\vec{r}_U - \vec{r}_c^{\,k}\|}$$

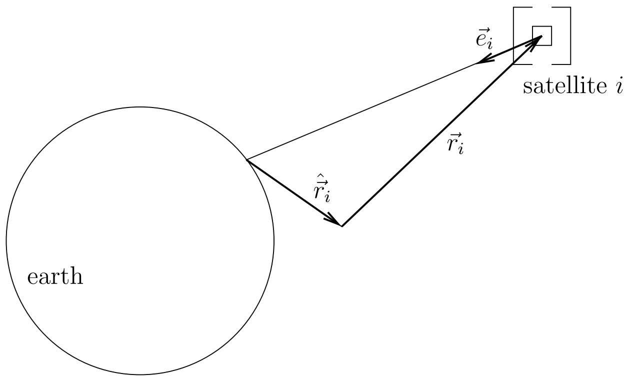



Figure 2: The definition of the line-of-sight (LoS) vector $\vec{e}^k$

Using this Jacobian matrix, the nonlinear model (1) can be linearized using the Taylor expansion around $(\vec{r}_{U,i}, \delta_i)$:

$$\begin{aligned}
\rho_c^k(\vec{r}_U, c\delta) = \rho_c^k(\vec{r}_{U,i} + \Delta\vec{r}_{U,i}, c\delta_i + \Delta\delta) &\approx \rho_c^k(\vec{r}_{U,i}, c\delta_i) + \nabla\rho_c^k(\vec{r}_{U,i}, c\delta_i)\begin{pmatrix}\Delta\vec{r}_{U,i} & c\Delta\delta_i\end{pmatrix} \\
&= \rho_c^k(\vec{r}_{U,i}, c\delta_i) + \begin{pmatrix}\vec{e}^k, & 1\end{pmatrix}^T \begin{pmatrix}\Delta\vec{r}_{U,i} \\ c\Delta\delta\end{pmatrix} \\
&= \rho_c^k(\vec{r}_{U,i}, c\delta_i) + \begin{pmatrix}e_x^k, & e_y^k, & e_z^k, & 1\end{pmatrix}^T \begin{pmatrix}\Delta\vec{r}_{U,i} \\ c\Delta\delta_i\end{pmatrix}.
\end{aligned}$$

Then, the error between the observation $\rho_c^k(\vec{r}_U, c\delta)$ and the value computed with the model $\rho_c^k(\vec{r}_{U,i}, c\delta_i)$ is

$$\begin{aligned}
\Delta\rho_{c,i}^k &= \rho_c^k(\vec{r}_U, c\delta) - \rho_c^k(\vec{r}_{U,i}, c\delta_i) \\
&= \left(\vec{e}^k\right)^T \Delta\vec{r}_{U,i} + c\Delta\delta_i + \eta^k \\
&= \left(\left(\vec{e}^k\right)^T, \quad 1\right)\begin{pmatrix}\Delta\vec{r}_{U,i} \\ c\Delta\delta_i\end{pmatrix} + \eta^k \\
&= \left(\left(\vec{e}^k\right)^T, \quad 1\right)\Delta\vec{\xi}_i + \eta^k,
\end{aligned}$$

where $\Delta\vec{\xi}_i = [\Delta\vec{r}_{U,i}, c\Delta\delta_i]^T$.

With a vector $\Delta\vec{\rho}_c$ includes the observations from all $K$ satellites at the $i$-th iteration,

$$\Delta\vec{\rho}_{c,i} \triangleq \begin{pmatrix}\Delta\rho_{c,i}^1 \\ \Delta\rho_{c,i}^2 \\ \vdots \\ \Delta\rho_{c,i}^K\end{pmatrix} = \begin{pmatrix}\rho_c^1 - \rho_c^1(\vec{r}_{U,i}, c\delta_i) \\ \rho_c^2 - \rho_c^1(\vec{r}_{U,i}, c\delta_i) \\ \vdots \\ \rho_c^K - \rho_c^K(\vec{r}_{U,i}, c\delta_i)\end{pmatrix}, \tag{3}$$

and the geometry matrix $H_i$ with the LoS unit vectors,

$$H_i = \begin{pmatrix} e_x^1 & e_y^1 & e_z^1 & 1 \\ e_x^2 & e_y^2 & e_z^2 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ e_x^K & e_y^K & e_z^K & 1 \end{pmatrix}, \tag{4}$$

we can finally have the following equation that is linear to $\Delta\vec{\xi_i}$:

$$\Delta\vec{\rho}_{c,i} = H_i \Delta\vec{\xi_i}.$$

If $K > 4$, the system is overdetermined, so the least-squares solution $\widehat{\Delta\vec{\xi_i}}$ can be estimated:

$$\widehat{\Delta\vec{\xi_i}} = \arg\min_{\Delta\vec{\xi_i}} \left\| \Delta\vec{\rho}_{c,i} - H_i \Delta\vec{\xi_i} \right\|^2 .$$

When pseudorange measurements have zero mean Gaussian noise with a standard deriviation $\sigma$, the covariance matrix is

$$\widehat{\Sigma} = \sigma^2 \left(H_i^T H_i\right)^{-1} = \sigma^2 C. \tag{5}$$

If $(H_i^T H_i)^{-1}$ is non-singular, the least squares solution can be deterministically computed as

$$\widehat{\Delta\vec{\xi_i}} = (H_i^T H_i)^{-1} H_i^T \Delta\vec{\rho}_i. \tag{6}$$

Using this solution, the user position and the clock offset can be iteratively estimated with Netwon's method:

1. Assume $\vec{r}_{U,i}$ and $c\delta_i$ (initialization)
2. Compute $\Delta\vec{\rho}_i$ (3) and $H_i$ (4) $\Leftarrow$ Corrected satellite positions need to be used (homework 3)
3. Compute the least-square estimate $\widehat{\Delta\vec{\xi_i}} = (H_i^T \Sigma^{-1} H_i)^{-1} H_i^T \Sigma^{-1} \Delta\vec{\rho}_i$
4. Update the state variables $\widehat{\vec{\xi}_{i+1}} = \widehat{\vec{\xi_i}} + \widehat{\Delta\vec{\xi_i}}$
5. Repeat the process from 2 until the stopping criteria is satisfied

In addition to the pseudorange measurements, a GPS receiver obtains the Doppler shift estimates $f_D$ from the phase or frequency tracking process. With these Doppler measurements, we can estimate the relative velocity between the satellite and the receiver in the LoS direction ($\vec{e}^T \cdot \vec{v}$), as well as the clock frequency offset ($\dot{\delta}$):

$$\begin{pmatrix} \hat{\vec{v}} \\ \widehat{c\dot{\delta}} \end{pmatrix} = (H^T H)^{-1} H^T \begin{pmatrix} \dot{\rho}^1 - \left(\vec{e}^{\,1}\right)^T \cdot \vec{v}^{\,1} \\ \dot{\rho}^2 - \left(\vec{e}^{\,2}\right)^T \cdot \vec{v}^{\,2} \\ \vdots \\ \dot{\rho}^K - \left(\vec{e}^{\,K}\right)^T \cdot \vec{v}^{\,K} \end{pmatrix}.$$

## 4 Dilution of Precision (DOP)

The covariance matrix (5) includes the relative geometry between the user and satellites $\left(H^T H\right)^{-1}$), as well as the measurements noise $\sigma^2$ that could be induced by various reasons, such as ionospheric, tropospheric, multipath, and orbital errors. Describing all elements of $C$ in the local east-north-up (ENU) coordinate system, we can define horizontal, vertical, position, time and geometric dilution of precision (DOP) as

$$
\begin{aligned}
\text{HDOP} &= \sqrt{C_{11} + C_{22}} \\
\text{VDOP} &= \sqrt{C_{33}} \\
\text{PDOP} &= \sqrt{C_{11} + C_{22} + C_{33}} \\
\text{TDOP} &= \sqrt{C_{44}} \\
\text{GDOP} &= \sqrt{C_{11} + C_{22} + C_{33} + C_{44}}.
\end{aligned}
$$

## References

[1] National Imagery and Mapping Agency, "Department of Defense World Geodetic System 1984," Tech. Rep., Jan. 2000.

[2] *ICD-GPS-200C, Navstar GPS Space Segment/Navigation User Interfaces*, GPS Navstar JPO.

[3] C. Günther, *Satellite Navigation.* Insitute for Communications and Navigation, Technical University of Munich, 2017.

[4] E. Kaplan and C. J. Hegarty, *Understanding GPS: Principles and Applications, Second Edition.* Artech House, 2006.

# 5 Homework

1. Since range measurements obtained with GPS satellites have different accuracy, it is better to use a weighted least-squares. Derive the solution of the weighted least squares

$$\hat{\xi}_{\mathrm{WLS}} = \arg\min_{\xi} \|\Delta\rho_{c,i} - H_i \Delta\xi\|_W^2$$

with a weight matrix $W = \mathrm{diag}(w_1, w_2, \dots, w_K)$ where $w_k = 1/\sigma_k^2$ is the weight for the measurement obtained with the $k$-th satellite.

2. Determine stopping criteria for Newton's method described in p5.
*Hint: No computation or derivation required.*

3. Satellite positions are described in the Earth-centered, Earth-fixed (ECEF) frame, which rotates with the Earth with the rotation rate $\dot{\Omega}_e = 7.292115147 \cdot 10^{-5}$rad/s) while signals travel from GPS satellites to a receiver. How can we compensate the satellite position error due to this rotation of the reference frame?
*Hint: Use the rotation matrix $R_Z(\theta_z)$*

4. Convert the line-of-sight vector $\vec{e}^{\,k}_{ECEF}$ in the ECEF frame to the local tangential ENU frame $\vec{e}^{\,k}_{ENU}$, and calculate the azimuth $A$ and elevation $E$ of the satellite using the coordinates in the ENU frame.
*Hint: For the frame conversion, use the rotation matrices with longitude $\lambda$ and latitude $\phi$.*

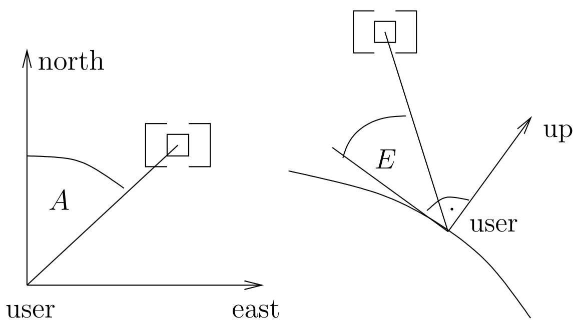



Figure 3: The definition of the azimuth and elevation angles

# 6 Lab Tasks*

*The questions marked with (*) are additional tasks, which will not affect on the grading.

1. Call the function `skyplotStation('ohi2')`, and find two characteristics of the sky-plot. Can the latitude of the receiver be approximately determined with this plot? *Hint: The inclination of the GPS satellites is 55 degrees.*

2. Implement the least squares positioning algorithm (both non-weighted and weighted) in `calcPVT()`. The results will be shown by calling `evalPVT('station*', epochs)`. Set the number of epochs as 2000 to fill out the table, but use a smaller number for testing the code. Interpret the table by comparing the results. (*`stations`: `nya1`[1], `isco`[2], `zimm`[3] ).

| Algorithm | Station | Mean HDOP/VDOP/PDOP | $\sigma_E$ | $\sigma_N$ | $\sigma_U$ |
|---|---|---|---|---|---|
| LS | `nya1` | | | | |
| | `isco` | | | | |
| | `zimm` | | | | |
| WLS | `nya1` | | | | |
| | `isco` | | | | |
| | `zimm` | | | | |

3. Check the results with different initialization of the user position.

| Algorithm | Station | Mean HDOP/VDOP/PDOP | $\sigma_E$ | $\sigma_N$ | $\sigma_U$ |
|---|---|---|---|---|---|
| LS | `nya1` | | | | |
| | `isco` | | | | |
| | `zimm` | | | | |
| WLS | `nya1` | | | | |
| | `isco` | | | | |
| | `zimm` | | | | |

[1] located in Norway (close to North Pole)
[2] located on Cocos island (close to the Equator)
[3] located in Switzerland

4. Add Gaussian noise with $\mu = 5\text{m}, 10\text{m}, 50\text{m}$ to the pseudorange measurements obtained with the first satellite, and fill out the following table.

| Mean | $\sigma_E$ | $\sigma_N$ | $\sigma_U$ |
|---|---|---|---|
| 5m | | | |
| 10m | | | |
| 50m | | | |

   (a) How dose the ranging error affect on the user position estimation?

   (b) How can we detect (and exclude) the faulty satellite?

5. Add error to the line-of-sight vector $e^k$, and check the positioning accuracy. To achieve the meter-level position accuracy, how much accuracy do we need for the line-of-sight vector?

6. When was the best constellation for positioning? Check it with the DOP values, and visualize the satellite positions at that time with a skyplot.

7. (*) Edit `calcAE()` to compute the azimuth and elevation angles of the satellites.

8. (*) Describe equations that describes the relation of ranging rate $\dot{\rho}$ and Doppler frequency measurement $f_D$. Compute the root mean square error of velocity for a day, by extending `calcPVT()`. How does the satellite constellation (geometry) affect on the velocity estimation? Use a station that provides Doppler information (e.g. `nya1`).

| Used satellites | $\sigma_{v,h}$ | $\sigma_{v,v}$ | $\sigma_{v,P}$ |
|---|---|---|---|
| All observable sat. in view | | | |