Chapter 4

Satellite Orbits

The satellite orbit is the succession of positions $\vec{r}^k(t)$ occupied by a satellite as a function of time. It is determined by the forces acting on the satellite, which include gravitational forces exerted by the earth, the moon, the sun and the planets, as well as the radiation pressure, mostly from the sun. Precise orbits are very complex. Fortunately, satellite orbits can be well approximated by Keplerian ellipses and some simple corrections. The present chapter shall develop the mathematical description of these orbits. Some of the content was inspired by the lectures on classical mechanics of Jost [1].

4.1 Historical Introduction

Humans must have been observing stars forever. They discovered the phases of the moon, the eclipsis, and from a certain time on, also the movement of the planets with respect to the background of fixed stars.

In the late 1563, at the age of 17, Tycho Brahe observed:

> *I've studied all available charts of the planets and stars and none of them match the others. There are just as many measurements and methods as there are astronomers and all of them disagree. What's needed is a long term project with the aim of mapping the heavens conducted from a single location over a period of several years.*

He then performed by far the most accurate and extensive measurements series of his time, achieving an accuracy of less than one arc-minute, which is a 21600-th of the full circle, with metallic instrument of a diameter of up to 2.9 meters. Brahe's work was first sponsored by the Danish King Frederik II and after his death by the Emperor Rudolph II. Both had observatories constructed for him. Brahe heard of Johannes Kepler's work and became interested in his theoretical strength. Kepler became Brahe's assistant, and his successor as imperial astronomer in 1601. This gave him free access to Brahe's measurements, and led him to derive, what was to become Kepler's laws.

On July 5th 1687, Newton published his *Philosophiae Naturalis Principia Mathematica.* The Principia combined Kepler's laws of planetary motion with the findings of Galileo about acceleration on earth into what was to become classical mechanics. Newton's mechanics is the starting point for the subsequent discussion.

4.2 Newton's Mechanics

A massive body located in $\vec{x}$, subjected to a force $\vec{F}$ experiences an acceleration $\ddot{\vec{x}}$, given by the equation

$$m_i\ddot{\vec{x}} = \vec{F},$$

with m_i denoting the inertial mass of the body. In the case, of two bodies at the positions $\vec{x}$ and $\vec{X}$, the gravitational force exerted on the mass in $\vec{x}$ is given by

$$\vec{F} = -GM_g m_g \frac{\vec{x} - \vec{X}}{\|\vec{x} - \vec{X}\|^3},$$

with M_g and m_g being the gravitational masses of the bodies. As Mach noted, these quantities effectively describe an interaction strength, whereas m_i describes the reaction of a body by acceleration when subjected to a force. The two definitions are conceptually distinct, as discussed in the late 19th century. In 1899, Baron Roland von Eötvös performed a pendulum experiment for proving the numerical equivalence of inertial and gravitation mass. Today, the numerical agreement is konown to be better than 10^{-12} [2]. Therefore, the assumption

$$m_g = m_i = m$$

is widely accepted in physics[1]. An immediate implication of this assumption is that an acceleration of the coordinate system cannot be distinguished from a gravitational force. This observation was a key for the formulation of the general theory of relativity by Einstein (see Chapter 8).

4.3 Kepler's Laws

Kepler's first law describes the shape of the orbits, as being ellipses. It is a consequence of the conservation of energy and angular momentum. The second law describes the relative velocity in the various locations on the ellipse. It is an expression of the conservation of angular momentum. Kepler's third law finally relates the period of the movement to the size of the ellipse. It is again a consequence of the above conservation laws.

4.3.1 Kepler's 2nd Law

Consider a satellite of mass m, and let M be the mass of the earth. Furthermore, let $\vec{x}$, and $\vec{X}$ denote the respective positions of the satellite and the earth, then the acceleration of the satellite and the earth are given by (see also Figure 4.1):

$$m\ddot{\vec{x}} = -GMm \frac{\vec{x} - \vec{X}}{\|\vec{x} - \vec{X}\|^3},$$

and

$$M\ddot{\vec{R}} = GMm \frac{\vec{x} - \vec{X}}{\|\vec{x} - \vec{X}\|^3}.$$

This can be simplified by introducing the distance vector $\vec{r} = \vec{x} - \vec{X}$, which measures the satellite position from the center of the earth. Taking a weighted difference of the two equations above leads to:

$$\ddot{\vec{r}} = -G\mu \frac{\vec{r}}{r^3}, \tag{4.1}$$

with $\mu = M + m$, and $r = \|\vec{r}\|$. Since the satellite's mass $m \sim 10^3$ [kg] is negligible as compared to the mass of the earth $M \simeq 5.9721910^{24}$ [kg], one has[2]

$$G\mu \simeq GM = 398600.4418 \pm 0.0002 \text{ [km}^3/\text{s}^2].$$

The two body problem with a centrally symmetric force that decays like $1/r^2$ is one of the few

[1]In a strict sense, the important property is that the values of the two types of masses are strictly proportional $m_g = \alpha m_i$. A universal proportionality factor can always be absorbed in G.

[2]The product GM is known to a better accuracy than each of the quantities individually. This is the reason why the product GM is typically used.

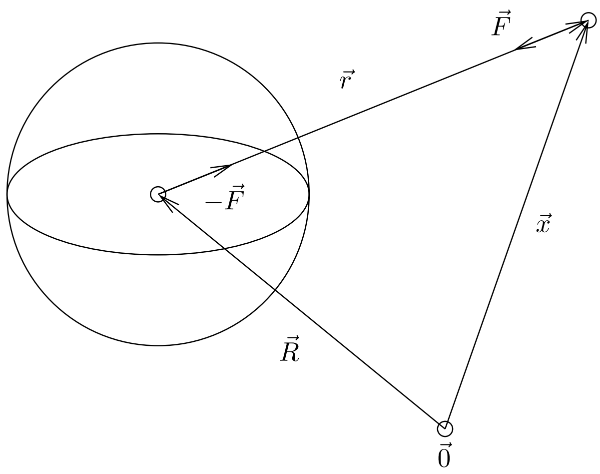

Figure 4.1: The earth exerts a gravitational force $\vec{F} = -GM\vec{r}/\|\vec{r}\|^3$ on the satellite. The satellite exerts an identical force on earth. The sign of the two force vectors are opposite.

problems in classical mechanics that can be solved in closed form. Such closed form solutions rely on conservation laws, which are themselves consequence of symmetries in the equations. In the case of the energy, the symmetry is the absence of an explicit time dependence of the gravitational force. Kepler's second law is an expression of spherical symmetry. The identity

$$\frac{\vec{r}}{r^3} = -\mathrm{grad}\,\frac{1}{r}, \tag{4.2}$$

implies that the gravitational force $\vec{F}$ on the right hand side of Equation (4.1) can be derived from a so called potential

$$V(\vec{r}) = -\frac{GMm}{r}.$$

This leads to the following reformulation of Newton's law:

$$m\ddot{\vec{r}} = -\mathrm{grad}\,V(\vec{r}). \tag{4.3}$$

The introduction of the potential is very useful for deriving conservation laws in mechanics, such as the conservation of angular momentum and energy.

Consider an arbitrary rotation $R_{\vec{a}}(\varphi)$ by the angle φ around the axis $\vec{a}$, as described in Appendix A, then the invariance of the potential under rotations, i.e. the invariance of $1/r$ under rotations implies:

$$\begin{aligned}
\frac{d}{d\varphi}V(R_{\vec{a}}(\varphi)\vec{r})\bigg|_{\varphi=0} &= 0 \\
&= \mathrm{grad}\,V(\vec{r})\cdot\frac{d}{d\varphi}R_{\vec{a}}(\varphi)\vec{r}\bigg|_{\varphi=0}) \\
&= \mathrm{grad}\,V(\vec{r})\cdot(\vec{a}\wedge\vec{r}) \\
&= (\vec{r}\wedge\mathrm{grad}\,V(\vec{r}))\cdot\vec{a},
\end{aligned}$$

(see Appendix A for an introduction into infinitesimal rotations, the vector product and their relation). The above equation must be valid for any rotation axis $\vec{a}$. Together with Netwon's law, i.e. Equation (4.3), this implies that

$$\begin{aligned} \vec{r} \wedge m\ddot{\vec{r}} &= 0 \\ &= \frac{d}{dt}(\vec{r} \wedge m\dot{\vec{r}}), \end{aligned}$$

i.e. that the angular momentum:

$$\vec{L} := \vec{r} \wedge m\dot{\vec{r}}$$

is constant. This law holds in any centrally symmetric potential $V(\vec{r}) = V(r)$. From the properties of the vector product, one concludes that

$$\vec{r} \cdot \vec{L} = \dot{\vec{r}} \cdot \vec{L} = 0,$$

i.e., that both, the position and the velocity vectors, are orthogonal to the angular momentum at all times, which means that the satellite moves in a plane orthogonal to $\vec{L}$. Without loss of generality $\vec{L} = L(0,0,1)$. In this case, the movement of the satellite can be parameterized by the radius r and the true anomaly ν :

$$\vec{r} = r(\cos\nu, \sin\nu, 0), \quad \dot{\vec{r}} = \dot{r}(\cos\nu, \sin\nu, 0) + r(-\sin\nu, \cos\nu, 0)\dot{\nu}. \tag{4.4}$$

The angular momentum becomes

$$\vec{L} = mr^2\dot{\nu}(0,0,1), \tag{4.5}$$

which implies that its norm is given by $L = mr^2\dot{\nu}$. The true anomaly is measured in the mathematically positive sense. It is zero, when the satellite is in its perigee, i.e. closest to the earth. Figure 4.2 implies that the surface covered by the position vector of the satellite $\vec{r}$ is given by

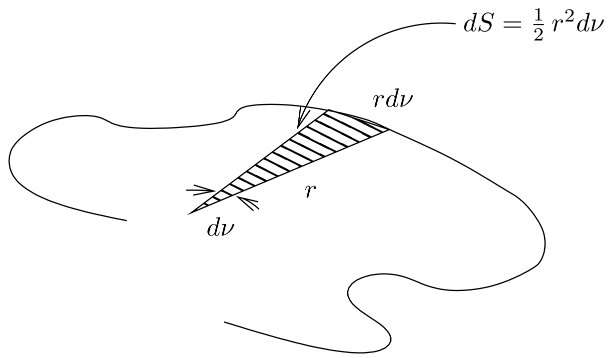

Figure 4.2: The infinitesimal surface element covered by the vector r is given by the surface of the triangle with the sides r and $rd\nu$. Note that the change dr is of second order.

$$dS = \frac{1}{2}r^2\dot{\nu}dt = \frac{L}{2m}dt.$$

Integrating both sides provides the surface $S(t, t_0)$ covered by $\vec{r}$ in the time interval $[t_0, t]$:

$$S(t, t_0) = \frac{L}{2m}(t - t_0) \tag{4.6}$$

and thus

Kepler's 2nd Law: *The position vector of the satellite covers equal surfaces in equal times as the satellite moves along its orbit.*

This statement is valid for an arbitrary spherically symmetric potential. In the case of a $1/r$-potential, Kepler's 1st law - derived below - states that the curve is an ellipse and that the source of the field is in one of its foci. In such a case, Kepler's 2nd law implies that a satellite moves slower at the apogee (farest point) than at the perigee (closest point). This is used in Tundra and Molniya orbits for maintaining the satellite over a geographic region for a prolonged period of time. The Japanese QZSS, which stands for Quasi Zenith Satellite System, takes advantage of such an orbit for augmenting GPS over Japan.

4.3.2 Kepler's 1st Law

The absence of an explicit time dependence in the gravitational potential $V(\vec{r},t) = V(\vec{r})$ implies that:

$$\begin{aligned} \frac{d}{dt}V(\vec{r}) &= \operatorname{grad} V(\vec{r}) \cdot \dot{\vec{r}} \\ &= -m\ddot{\vec{r}} \cdot \dot{\vec{r}} \\ &= -m\frac{d}{dt}\frac{1}{2}\|\dot{\vec{r}}\|^2. \end{aligned}$$

Thus the energy

$$H(\vec{r},\dot{\vec{r}}) = \frac{1}{2}m\|\dot{\vec{r}}\|^2 + V(\vec{r}) = \mathcal{E}.$$

is preserved. The function $H(\vec{r},\dot{\vec{r}})$ is more precisely called Hamiltonian. It plays an important role in deriving the equations of motion of complex dynamical systems.

The conservation of energy can be expressed in polar coordinates, using Equation (4.4) and (4.5):

$$\mathcal{E} = \frac{m}{2}(\dot{r}^2 + r^2\dot{\nu}^2) + V(r) = \frac{m}{2}\left(\dot{r}^2 + \left(\frac{L}{mr}\right)^2\right) + V(r). \tag{4.7}$$

In a next step we shall use the conservation energy and angular momentum for determining the shape of the orbit, i.e. an expression for the radius r as a function of the true anomaly ν. It is convenient to introduce the auxiliary variable

$$s = \frac{1}{r},$$

and to express s in terms of ν. Applying the chain rule to $s(r(t(\nu)))$ implies:

$$\begin{aligned} \frac{ds}{d\nu} &= \frac{ds}{dr} \cdot \frac{dr}{dt} \cdot \frac{dt}{d\nu} \\ &= -s^2 \cdot \left(\frac{2}{m}\left[\mathcal{E} + GMm\,s\right] - \left(\frac{L}{m}\right)^2 s^2\right)^{1/2} \cdot \frac{1}{\frac{L}{m}s^2} \\ &= -\left(\frac{2m\mathcal{E}}{L^2} + \frac{2GMm^2}{L^2}s - s^2\right)^{1/2}. \end{aligned}$$

The first factor was obtained by computing the derivative and the second factor by using the conservation of energy. The third factor was transformed using[3] $dt/d\nu = (d\nu/dt)^{-1}$, as well as the equation for the conservation of angular momentum. The factorization of the quadratic expression inside the square root leads to:

$$\frac{ds}{d\nu} = -\sqrt{(s_1 - s)(s - s_2)} = -\sqrt{-s_1s_2 + (s_1 + s_2)s - s^2}, \tag{4.8}$$

[3]This property is obtained by taking the derivative of $t(\nu(t)) = t$ with respect to t.

with[4]

$$s_1 s_2 = -\frac{2m\mathcal{E}}{L^2}, \quad \text{and} \quad s_1 + s_2 = \frac{2GMm^2}{L^2}.$$

The cross-term in this expression can be eliminated by subtracting the mean-value of the zeros, i.e. by introducing the variable $u = s - (s_1 + s_2)/2$:

$$\frac{du}{d\nu} = -\sqrt{\left(\frac{s_1 - s_2}{2}\right)^2 - u^2}.$$

The normalization $u = v(s_1 - s_2)/2$ finally implies that the differential equation becomes:

$$\frac{dv}{d\nu} = -\sqrt{1 - v^2}.$$

This equation is solved by the method of separation of variables[5]:

$$-\frac{dv}{\sqrt{1 - v^2}} = d\nu$$

and can be integrated in closed form:

$$\arccos v = \nu - \nu_0$$

with ν_0 being an integration constant. Applying the succession of tarnsformations implies:

$$\begin{aligned} s &= \frac{s_1 + s_1}{2} + \frac{s_1 - s_2}{2} \cos(\nu - \nu_0) \\ &= \frac{s_1 + s_2}{2} [1 + e \cos(\nu - \nu_0)], \end{aligned} \tag{4.9}$$

with

$$\begin{aligned} e &= \frac{s_1 - s_2}{s_1 + s_2} \\ &= \sqrt{1 - \frac{4 s_1 s_2}{(s_1 + s_2)^2}} \\ &= \sqrt{1 + \frac{2\mathcal{E}L^2}{(GM)^2 m^3}}. \end{aligned}$$

The solution s and thus $r = 1/s$ is extremum when $\nu - \nu_0 = k\pi$, with k integer. Without loss of generality, $\nu_0 = 0$ is typically chosen. The equation for the orbital radius[6] r becomes:

$$r = \frac{p}{1 + e \cos \nu}, \tag{4.10}$$

with

$$p = 2/(s_1 + s_2) = L^2/(GMm^2), \tag{4.11}$$

and the points on the orbit are described by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{p}{1 + e \cos \nu} \begin{pmatrix} \cos \nu \\ \sin \nu \end{pmatrix}.$$

This describes a conical section with one focal point at the origin. It can be:

[4] The roots s_1 and s_2 actually are the inverse of the radius at the perigee and apogee respectively. This can be verified by using the equation developed in this section.

[5] The method of separation of variables allows to solve differential equations of the form $dv/d\nu = f^{-1}(v)g(\nu)$ by rewriting the equation as $f(v)(dv/d\nu) = g(\nu)$ and by integrating both sides with respect to ν. Substituting the variable ν by v on the left hand side then leads to $\int f(v)dv = \int g(\nu)d\nu$.

[6] By extension, we call the norm $r = \|\vec{r}\|$ "orbital radius," even if the orbit is a general conical sections, which implies that r is time varying.

- An ellipse for $e < 1$. In this case, there is no singularity in r. The satellite is captured by the central body. The total energy is negative: $\mathcal{E} < 0$. The radius is minimal in the perigee ($\nu = 0$) and maximal in the apogee ($\nu = \pi$). Define $a = p/(1-e^2)$, then the ellipse is centered around $(-ae, 0)$, and satisfies the equation

$$\frac{(x+ae)^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{with} \quad b = a\sqrt{1-e^2},$$

  i.e. has the semi-major axis a and the semi-minor axis b.

- A parabola for $e = 1$, in which case there is a single angle $\nu = \pi$ for which the radius becomes infinite. The total energy is 0. The satellite comes from infinity and returns to infinity. "At infinity," the satellite has transformed all its kinetic energy into potential energy. The radius is minimal in the vertex ($\nu = 0$) and infinite for $\nu = \pi$. The parabola has its focus in 0, and its vertex in $(p/2, 0)$. It is described by the equation

$$\frac{y^2}{p^2} = 1 - \frac{2x}{p}.$$

- A hyperbola for $e > 1$. The values of ν such that $\cos\nu > -1/e$ lead to a real radius. The values of ν such that $\cos\nu = -1/e$ describe the asymptotes of a scattering process. The energy is positive - the satellite approaches the central body from infinity with a positive velocity and returns to infinity with that same positive velocity. The radius is again minimal in the vertex ($\nu = 0$). Again define $a = p/(1-e^2)$, then the asymptotes of the hyperbola intersect in $(-ae, 0)$, and the hyperbola is described by the equation:

$$\frac{(x+ae)^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{with} \quad b = a\sqrt{e^2-1}.$$

The first case is the interesting one for earth-bound satellites:

Kepler's 1st law: *The orbit of a satellite is an ellipse with one body in a focal point of the ellipse.*

Equation (4.10) together with the above expression for the semi-minor axis a, resolved for p, describes the radius in the form:

$$r = \frac{a(1-e^2)}{1+e\cos\nu}, \tag{4.12}$$

with e being the eccentricity of the ellipse.

4.3.3 Integrals and Kepler's 3rd Law

The two quantities "angular momentum" and "energy" are constant. They are also called integrals. The angular momentum is related to p by Equation (4.11) and since $p = a(1-e^2)$, this implies that:

$$L^2 = GMm^2a(1-e^2). \tag{4.13}$$

The energy is most easily computed from its value at the perigee. At the perigee, the following holds $\nu = 0$ by definition, $\dot{r} = 0$, and $r = a(1-e)$. Using these results in the expression for the energy (see Equation (4.7)), leads to the result:

$$\mathcal{E} = -\frac{GMm}{2a}. \tag{4.14}$$

Equation (4.6) relates the angular momentum to the surface of the ellipse: $S(t_0, t_0+T) = \frac{L}{2m}T$ with T being the period. The surface of the ellipse is also equal to $S = \pi ab$, with $b = a\sqrt{1-e^2}$ being the semi-major axis. Using Equation (4.13) to express L in terms of a and equating the two expressions for the surface leads to

Kepler's 3rd law: *The orbital period and the semi-major axis of the elliptical orbit are related by:*

$$T = 2\pi\sqrt{\frac{a^3}{GM}}.$$

The satellite navigation systems GPS and Galileo provide the eccentricity e and the square root of the semi-major axis $\sqrt{a}$ in their navigation messages in order to define the shape of the orbit and the orbital period.

4.4 Further Characterization of the Orbit

Kepler's laws state that the orbital movement takes place in a plane, and that the orbit is an ellipse, i.e. they provide $r(\nu)$. A full description of the satellite movement in its orbital plane requires the determination of $(r(t), \nu(t))$, as well as of the argument of the perigee. The former aspect is addressed next. The argument of perigee, and the orientation of the orbital plane, i.e. the orientation of $\vec{L}$, are described in a further section.

4.4.1 Time Dependence

The geometry of the solution $r(\nu)$ is described in Equation (4.12). This expression can be used either in the conservation of angular momentum (Equation (4.5)) or in the conservation of energy and angular momentum (Equation (4.7)).

The first approach leads to the following differential equation for $\nu(t)$:

$$\frac{\dot{\nu}}{(1+e\cos\nu)^2} = \frac{L}{ma^2(1-e^2)^2} = \frac{2\pi}{T}\frac{1}{(1-e^2)^{3/2}},$$

which can again be solved by the method of separation of the variables, and integration

$$\int_0^{\nu} d\nu' \frac{(1-e^2)^{3/2}}{(1+e\cos\nu')^2} = \frac{2\pi}{T}(t-t_0).$$

In this expression t_0 denotes again the time of passage through the perigee $\nu = 0$. The right hand side is called mean anomaly and denoted by M :

$$M = \frac{2\pi}{T}(t-t_0).$$

It grows from 0 to 2π during the first period. The integral can be evaluated in closed form[7]:

$$2\arctan\left(\frac{1-e}{\sqrt{1-e^2}}\tan\left(\frac{\nu}{2}\right)\right) - e\sqrt{1-e^2}\frac{\sin\nu}{1+e\cos\nu} = \frac{2\pi}{T}(t-t_0).$$

This is an equation for t as a function of ν, which can be inverted at least in principle. Although this directly leads to a function $\nu(t)$ this approach is usually not followed, due to the intricate form of the solution.

The alternative is to rewrite Equation (4.7) using the expressions from Equation (4.14) for L and Equation (4.13) for $\mathcal{E}$:

$$\begin{aligned}\dot{r}^2 &= \frac{2}{m}\left(\mathcal{E} - \frac{L^2}{2mr^2} + \frac{GMm}{r}\right)\\ &= \frac{GMa}{r^2}\left(e^2 - \left(1-\frac{r}{a}\right)^2\right).\end{aligned} \tag{4.15}$$

[7]The present result was obtained using Mathematica. It is easily verified by differentiation.

Since $-e < 1-r/a < e$, the variable r can be reparametrized using the eccentric anomaly $E \in [0, 2\pi]$ i.e.:

$$1 - \frac{r}{a} = e \cos E. \tag{4.16}$$

Since r is periodic in ν, with period 2π, this also applies to $\cos E$. Equation (4.16) implies that $dr/a = e \sin E \, dE$. Substituting the above results in Equation (4.15) and taking the square root leads to:

$$(1 - e \cos E) dE = \sqrt{\frac{GM}{a^3}} dt = \frac{2\pi}{T} dt,$$

i.e., the variables can again be separated. In the perigee $1 - r/a = e$, and thus $E = 0$. Integration on both sides implies:

$$E - e \sin E = \frac{2\pi}{T}(t - t_0), \tag{4.17}$$

with integration constants mapped into t_0. The right hand side is again the mean anomaly M, originally introduced in this context. The left hand side has a much simpler form than before, but in a variable E still to be investigated. Equation (4.17) is an expression for $t(E)$, which can be inverted. The solution $E(t)$ is typically computed numerically, e.g. using the Newton algorithm. Since ν is a periodic function of M with period 2π and since $\cos E$ is a periodic function of ν with period 2π this implies that $M - E = -e \sin E$ is a periodic function of M with that same period. Since the function is furthermore odd ($E(-M) = -E(M)$), $M - E$ can be expanded into a Fourier series of the form:

$$E - M = \sum_{k=1}^{\infty} a_k \sin kM,$$

with coefficients:

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} dM (E - M) \sin kM \\ &= \frac{2}{k} \frac{1}{\pi} \int_0^{\pi} dE \cos[k(E - e \sin E)]. \end{aligned}$$

The last line is obtained by partial integration and substitution. The integral is Bessel function $J_k(z)$ of first kind and k-th order:

$$\begin{aligned} J_k(z) &= \frac{1}{\pi} \int_0^{\pi} dE \, \cos(k\,E - z \sin E) \\ &= \left(\frac{z}{2}\right)^k \sum_{l=0}^{\infty} \frac{(-z^2/4)^l}{(k+l)!\, l!}. \end{aligned}$$

Thus, the value of E can be computed using a power series in M:

$$E = M + 2 \sum_{k=1}^{\infty} \frac{1}{k} J_k(ke) \sin(kM). \tag{4.18}$$

In order to understand the meaning of the eccentric anomaly, Equation (4.12) is used to express the cartesian coordinate $x = r \cos \nu$ in the form:

$$x = \frac{1}{e}(p - r) = a(\cos E - e),$$

and $y = \sqrt{r^2 - x^2}$ is used to derive:

$$y = a\sqrt{1 - e^2} \sin E = b \sin E.$$

Thus the eccentric anomaly E is the angle measured from the center of the ellipse of a point obtained from a projection of the endpoint of the vector $\vec{r}$ onto the circle circumscribing the ellipse. The projection is performed parallel to the semi-minor axis (see Figure 4.3). The above equations fully describe the movement of the satellite in cartesian coordinates. Alternatively, these equations can also be used to determine ν by computing the solution in polar coordinates. Using $(x, y) = r(\cos\nu, \sin\nu)$ one obtains

$$\tan\nu = \frac{\sqrt{1-e^2}\sin E}{\cos E - e}.$$

This fully describes the movement of the satellite on its elliptical orbit:

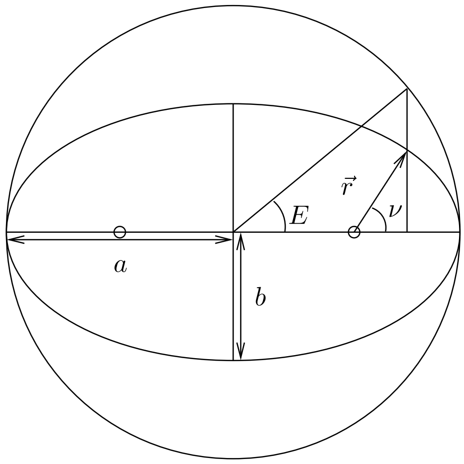

Figure 4.3: The eccentric anomaly E is defined as the angle between the x-axis and the projection of the satellite location onto the circumscribing circle. The expression of the satellite's dynamics in terms of E happens to lead to a closed form solution.

$$\vec{r}(t) = \begin{pmatrix} a\cos E(t) - ae \\ b\sin E(t) \end{pmatrix} = r(E(t)) \begin{pmatrix} \cos(\nu(E(t))) \\ \sin(\nu(E(t))) \\ 0 \end{pmatrix}$$

The orbital plane can itself be arbitrarily oriented in space, which involves two additional degrees of freedom.

4.4.2 Orientation in Space

The rotation axis of the earth provides a natural direction in space, typically called z-axis. It is attached to the center of gravity of the earth, which defines the zero of the coordinate system. The direction of the rotation axis is subject to periodic and non-periodic changes. Thus the convention was made to choose the Conventional Terrestrial Pole (CTP), which is the direction of the pole as determined by the International Earth Rotation Service (IERS)[8] at the epoch 1984.0. The x-axis is chosen to be the intersection of the Greenwich meridian with the equator. The y-axis is chosen in such a manner that the three axis x, y, z define a right-handed orthogonal system (see Figure 4.4). These axes, attached at the center of gravity, define the Conventional Terrestrial Reference System (CTRS) of the World Geodetic System 84 (WGS84). The latter system is used to describe the

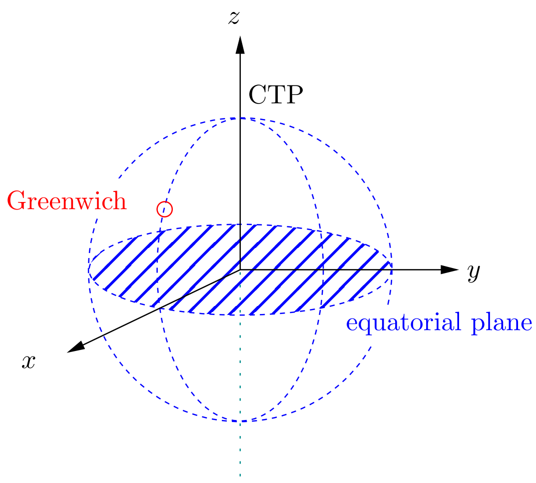

Figure 4.4: The Conventional Terrestrial Frame Reference System (CTRS) is an orthonormal system with its z-axis parallel to the IERS Conventional Terrestrial Pole (CTP) and its x-axis passing through the Greenwich meridian.

orbits of the satellites in GPS. The rotation of the CTRS system induces centrifugal and Coriolis forces and thus makes the system non-inertial.

The orbital plane, i.e. the plane, which contains the orbit, intersects the equatorial plane, i.e. the xy-plane, along the line of nodes. The angle between the two planes is called inclination i. The orbit itself intersects the equatorial plane from south to north in the ascending node and from north to south in the descending node. The angle between the ascending node and the x-axis is called Longitude of the Ascending Node (LAAN). The argument of perigee ω finally defines the direction of the perigee in the orbital plane measured from the direction of the ascending node.

In summary, the orientation of the orbit in space is described by a coordinate transformation from the CTRS into the coordinate system with an x-axis in the direction of the perigee and a z-axis orthogonal to the orbital plane, i.e. parallel to $\vec{L}$. Let $\vec{u}$ be an arbitrary vector in the orbital coordinate system, and $\vec{v}$ be that vector expressed in the CTRS, then the transformation from $\vec{v}$ to $\vec{u}$ involves three successive rotations: a rotation $R_3(\Omega)$ by Ω around the z-axis, a rotation $R_1(i)$ by i around the new x-axis, i.e., the line of nodes, and finally a rotation $R_3(\omega)$ by ω around the new z-axis i.e., $\vec{L}/\|\vec{L}\|$:

$$\vec{u} = R_3(\omega)R_1(i)R_3(\Omega)\vec{v} = R\vec{v}.$$

This is most easily understood for the three orthogonal unit vectors. Since the rotation matrix R does not depend on $\vec{v}$, and since all other vectors can be represented as a linear combination of unit vectors, it also applies to them. The inverse is immediately obtained from:

$$\vec{v} = R_3(-\Omega)R_1(-i)R_3(-\omega)\vec{u}. \tag{4.19}$$

Expressed in the CTRS, the angular momentum becomes

$$\vec{L}_{\text{CTRS}} = L \begin{pmatrix} \sin\Omega\sin i \\ -\cos\Omega\sin i \\ \cos i \end{pmatrix}.$$

[8]Formerly the Bureau International de l'Heure (BIH).

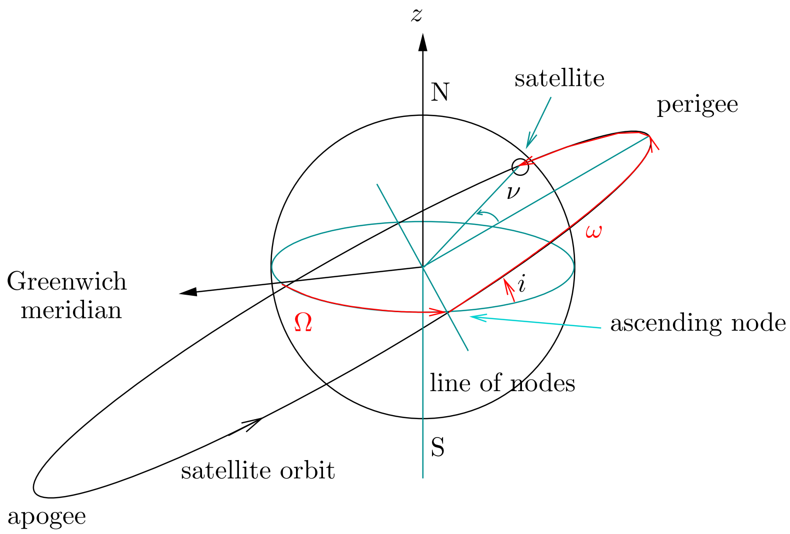

Figure 4.5: The orientation of the orbit in space is determined by three angles: the Longitude of the Ascending Node (LAAN) Ω, the inclination i of the orbital plane with respect to the equatorial plane, and the argument of perigee ω.

For later reference, it is useful to note that

$$R_3(-\omega)\vec{u} = r \begin{pmatrix} \cos(\nu+\omega) \\ \sin(\nu+\omega) \\ 0 \end{pmatrix}., \tag{4.20}$$

with r and ν being both functions of the eccentric anomaly E, and through Equation (4.18) of the mean anomaly M.

4.5 Description of the Orbit in the Navigation Message

The navigation message of GPS contains a total of 37'500 bits. It is transmitted with 50 bits/s in 12.5 minutes. The part concerning clocks and orbits repeats every 30 seconds. The GPS Joint Program Office has published the Interface Control Document, called GPS-ICD [4], which specifies all information relevant to the use of the signal. The Galileo Joint Undertaking has published a similar document for Galileo [5]. The orbital parameters described in Table 4.1 are taken from the ICD of GPS. Figure 4.6 shows an example of the encoding of orbital data in the navigation message. Most of the parameters shown in Table 4.1 have either been introduced previously or are easy to understand. The additional parameters include a number of corrections to the movement of the satellite in a centrally symmetric field (an ideal earth). The most important correction is caused by asymmetries in the earth's gravitational field. The dominant asymmetry is due to its flattening. Furthermore, the earth and the GNSS satellite are not isolated in the universe.

The equatorial bulge, i.e. the ellipsoidal shape of the earth, as well as gravitational anomalies, due to the irregular mass distribution inside the earth cause perturbations to the Kepler orbits. The same applies for the gravitational field of the sun, the moon, and the planets, as well as for the tidal deformations of the earth, and for the radiation pressure. The order of magnitude of the different terms is listed in Table 4.2. The influence of these terms is handled by corrections.

Variable	Meaning
M_0	mean anomaly at reference time (initial value of $M(t)$
Δn	correction to $n = \sqrt{GM/a^3}$
e	eccentricity
$\sqrt{a}$	square root of the semi major axis a
Ω_0	longitude of the ascending node at weekly epoch
$\dot{\Omega}$	correction to Ω_0
i_0	inclination at reference time
$\dot{i}$	correction to i_0
ω	argument of perigee
C_{uc}, C_{us}	amplitudes of the harmonic corrections to the argument of latitude
C_{rc}, C_{rs}	amplitudes of the harmonic corrections to the orbital radius
C_{ic}, C_{is}	amplitudes of the harmonic corrections to the inclination
t_{oe}	reference time for the ephemeris data, i.e. the orbital parameters
IODE	Issue Of Data - Ephemeris is a circular counter to identify the data, e.g. ensuring the consistency of clock and ephemeris data

Table 4.1: Definition of the ephemeris data in GPS

Force	Acceleration [m/s^2]	Orbit error after one day [m]
Centrally symmetric $-GM\vec{r}/r^3$	0.56	
Equatorial Bulge	$5 \cdot 10^{-5}$	10'000
Lunar gravity	$5 \cdot 10^{-6}$	3'000
Solar gravity	$2 \cdot 10^{-6}$	800
Higher order earth gravity	$3 \cdot 10^{-7}$	200
Solar radiation pressure (direct)	$9 \cdot 10^{-8}$	200
Y-Bias	$5 \cdot 10^{-10}$	2
Solid Earth Tides	$1 \cdot 10^{-9}$	0.3

Table 4.2: Strength and impact of major perturbations affecting the orbits of artificial satellites. The meaning of most terms is obvious. The Y-Bias is a radiation pressure terms. [Courtesy XXX, Univ. of Bern, [3]]
.

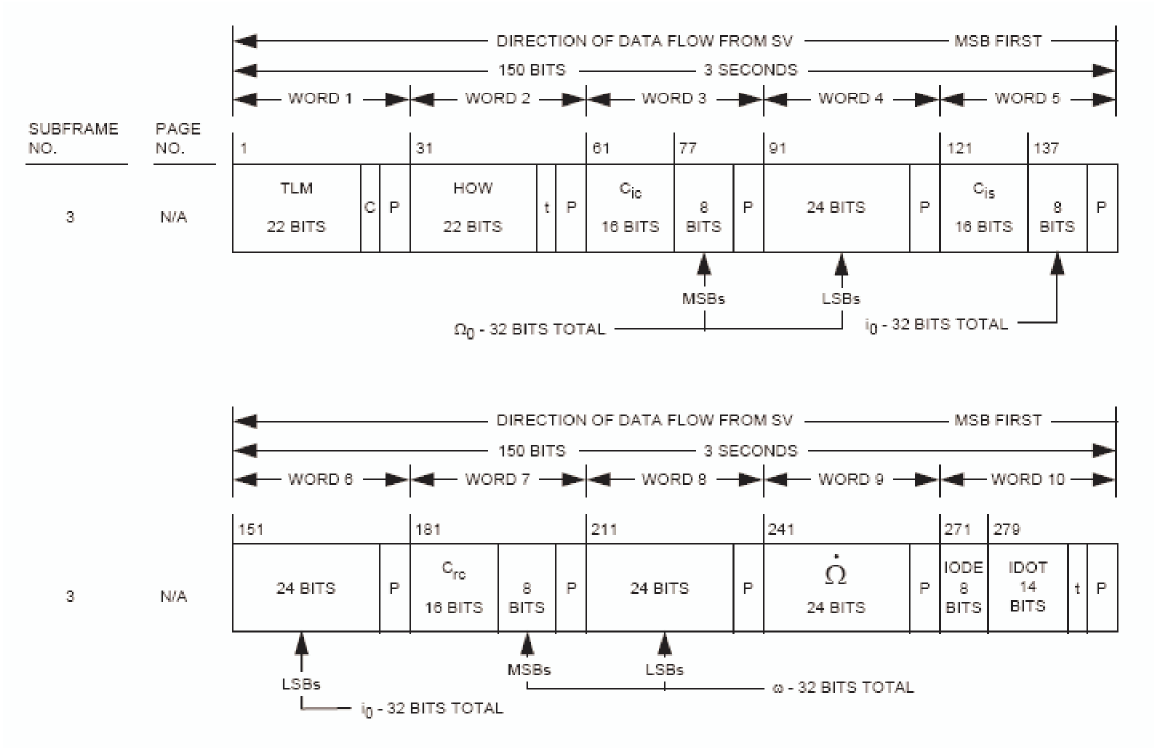

Figure 4.6: Extract from the ICD [4] showing the coding of some orbital parameters in Subframe 3 of the navigation message.

The GPS-ICD advises the user to compute the orbital parameters as described below. Let $t_k = t - t_{oe}$ be the time difference between the current time and the reference time. All quantities that are referenced to this time will carry a lower index k. This is a convention in the ICD. The rate of change of the mean anomaly, i.e. the mean motion $n = 2\pi/T$, which is an angular frequency, is determined by its value from Kepler's 3rd law plus a correction Δn:

$$n = \sqrt{\frac{GM}{a^3}} + \Delta n.$$

The mean anomaly M_k is thus given by:

$$M_k = M_0 + n t_k.$$

This is the basis for computing the eccentric anomaly E_k, using either Newton's algorithm or the power series expansion from Equation (4.18):

$$E_k - e \sin E_k = M_k.$$

It leads to an expression of the position in polar coordinates r_k and ν_k in the orbital reference system:

$$r_k = a(1 - e\cos E_k), \quad \tan \nu_k = \frac{\sqrt{1-e^2}\sin E_k}{\cos E_k - e}.$$

The argument of perigee ω is added to the argument of latitude ν_k, as described in Equation (4.20):

$$\phi_k = \nu_k + \omega.$$

Second harmonic corrections are applied to correct for perturbations of the orbit:

$$\begin{aligned}
\delta u_k &= C_{uc} \cos 2\phi_k + C_{us} \sin 2\phi_k \\
\delta r_k &= C_{rc} \cos 2\phi_k + C_{rs} \sin 2\phi_k \\
\delta i_k &= C_{ic} \cos 2\phi_k + C_{is} \sin 2\phi_k .
\end{aligned}$$

The corrections are of second order. First order corrections to u_k and r_k are included in the Keplerian elements. Furthermore, the equatorial bulge has the same impact on the satellite, while it is in the ascending or descending segment of the orbit. There is thus no contribution of first order to the correction of the inclination.

In a next step, the corrected argument of latitude is computed:

$$u_k = \phi_k + \delta u_k,$$

as well as the corrected radius

$$r_k = a(1 - e \cos E_k) + \delta r_k,$$

and the corrected inclination

$$i_k = i_0 + \dot{i} t_k + \delta i_k.$$

The LAAN Ω_k is directly derived from quantities in the navigation message:

$$\Omega_k = \Omega_0 - \dot{\Omega}_e t_{oe} + (\dot{\Omega} - \dot{\Omega}_e) t_k,$$

with $\dot{\Omega}_e$ being the angular velocity of the earth rotation:

$$\dot{\Omega}_e = 7.92921151467 \cdot 10^{-5} \left[\frac{\text{rad}}{\text{s}} \right], \tag{4.21}$$

and with $\dot{\Omega}$ being the rate of change of the LAAN Ω. The dominant cause for $\dot{\Omega}$ is that the equatorial bulge exercises a torque on the satellite's orbit.

Combining all results derived so far, leads to the final result:

$$\vec{v} = R_3(-\Omega_k) R_1(-i_k) \begin{pmatrix} r_k \cos u_k \\ r_k \sin u_k \\ 0 \end{pmatrix},$$

which has the same form as in the Equations (4.19) and (4.20). Omitting the rate terms or the periodic corrections quickly leads to unacceptable errors. Figure 4.7 shows the size of the corrections.

4.6 Summary

The movement of the satellites is determined by the intricate gravitational field of the earth, as well as by contributions from the sun and the moon. The radiation pressure from the sun must also be taken into account. The associated complexity is difficult to handle in receivers. Thus the movement of the satellites is described by Kepler ellipses and a number of judiciously chosen corrections. This description is sufficient to achieve accuracies at submeter level.

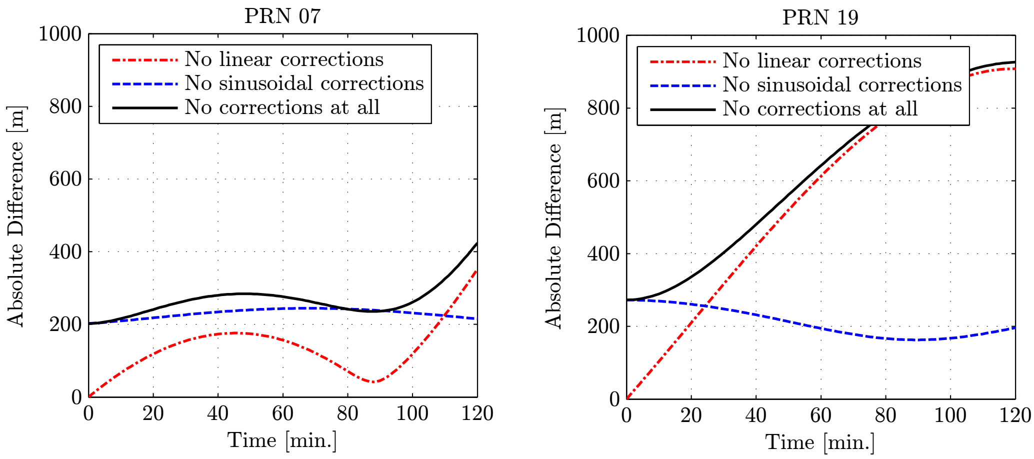

Figure 4.7: Orbital error due to the absence of the linear or sinusoidal term or the absence of both for the satellites PRN07 and PRN19 at 0:00 on January 1st, 2012 [Courtesy K. Giger, TUM, 2012].

Appendix A Rotations and Vector Products

Let $\vec{a}$ be a normalized vector, i.e. $\|\vec{a}\| = 1$ and φ be a scalar, then the rotation $R_{\vec{a}}(\varphi)$ by the angle φ around the axis $\vec{a}$ is characterized by the properties that it leaves the lengths of an arbitrary vector $\vec{r}$ and the rotation axis $\vec{a}$ unchanged, i.e.,

$$\|R_{\vec{a}}(\varphi)\vec{r}\|^2 = \|\vec{r}\|^2, \quad \text{and} \quad R_{\vec{a}}(\varphi)\vec{a} = \vec{a}. \tag{4.22}$$

The first property implies that the matrix is orthonormal[9], i.e.

$$R_{\vec{a}}(\varphi)^T R_{\vec{a}}(\varphi) = \mathbb{1}. \tag{4.23}$$

Define the infinitesimal generator $\Omega_{\vec{a}}$ of the group of rotations around the axis $\vec{a}$ by

$$\Omega_{\vec{a}} = \frac{d}{d\varphi} R_{\vec{a}}(\varphi)\bigg|_{\varphi=0}. \tag{4.24}$$

then taking the derivative of Equation (4.23) wrto φ at $\varphi = 0$ together with the derivative of the second property in (4.22) leads to:

$$\Omega_{\vec{a}} = -\Omega_{\vec{a}}^T \quad \text{and} \quad \Omega_{\vec{a}}\vec{a} = 0.$$

The first properties means that $\Omega_{\vec{a}}$ has the form

$$\Omega_{\vec{a}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{4.25}$$

and the second property implies that

$$\vec{\omega} = \varphi\vec{a},$$

with φ a positive real number, which shall later be interpreted as the angle of rotation. The index $\vec{a}$ on the matrix Ω is usually dropped. Now, let ϵ be the totally antisymmetric tensor with the properties:

$$\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{kji} \quad \text{and} \quad \epsilon_{123} = 1.$$

[9] For a proof, compute $\|R_{\vec{a}}(\varphi)(\alpha\vec{u}+\beta\vec{v})\|^2$ and use the invariance property, as well as the property that the scalar $\vec{u}^T R_{\vec{a}}(\varphi)^T R_{\vec{a}}(\varphi)\vec{v}$ is equal to its own transposition to show that $\vec{u}^T R_{\vec{a}}(\varphi)^T R_{\vec{a}}(\varphi)\vec{v} = \vec{u}^T\vec{v}$ for arbitrary $\vec{u}$ and $\vec{v}$.

The matrices obtained by fixing the first component are:

$$\epsilon_{1..} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \epsilon_{2..} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \epsilon_{3..} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the infinitesimal generator can be written in the form

$$\Omega_{ik} = -\sum_{j=1}^{3} \omega_j \epsilon_{jik}.$$

When applied to an arbitrary vector $\vec{r}$, this leads to

$$(\Omega\vec{r})_i = \sum_k \sum_j \epsilon_{ijk}\omega_j r_k =: (\vec{\omega} \wedge \vec{r})_i, \tag{4.26}$$

i.e., to the definition of the vector product of $\vec{\omega}$ and $\vec{r}$. [10] In general, the i-th component of the vector product of $\vec{x}$ and $\vec{y}$ is given by:

$$(\vec{x} \wedge \vec{y})_i = \epsilon_{ijk}\, x_j\, y_k,$$

with the Einstein summation rule[11] being assumed. The following identity can be easily worked out:

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}.$$

Together with the other properties of ϵ_{ijk} it leads to the following three identities that are often used in the context of the vector product:

$$\begin{aligned} \vec{x} \wedge \vec{y} &= -\vec{y} \wedge \vec{x} && (4.27)\\ (\vec{x} \wedge \vec{y}) \cdot \vec{z} &= (\vec{z} \wedge \vec{x}) \cdot \vec{y} && (4.28)\\ (\vec{x} \wedge \vec{y}) \wedge \vec{z} &= (\vec{x} \cdot \vec{z})\vec{y} - (\vec{y} \cdot \vec{z})\vec{x}. && (4.29) \end{aligned}$$

If capital letters are used to denote the totally antisymmetric matrix associated with a vector, these equations can be rewritten in matrix form:

$$\begin{aligned} X\vec{y} &= -Y\vec{x} && (4.30)\\ \vec{z}^T X\vec{y} &= -\vec{y}^T X\vec{z} && (4.31)\\ -ZX\vec{y} &= (\vec{x}^T\vec{z})\vec{y} - \vec{x}(\vec{z}^T\vec{y}). && (4.32) \end{aligned}$$

The last equation is obtained from:

$$ZX = \vec{x} \otimes \vec{z}^T - \vec{x}^T\vec{z}\,\mathbb{1}$$

which is easily verified in components[12]. It is a matter of circumstances and taste, which notations are preferred. Equation (4.31) for example is trivial, while Equation (4.28) requires some reflection. For the Equations (4.27) and (4.30), the situation is reversed.

[10] The possibility of representing an infinitesimal rotation by a vector $\vec{a}$ and the vector product $\wedge$ is a particularity of 3-dimensional space. In two dimensional space the number of independent components in a totally antisymmetric tensor is $n(n-1)/2 = 1$. The only matrix is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with its square being $-\mathbb{1}$. With the 2-dimensional space mapped to complex number, the multiplication by this matrix becomes the multiplication by $j = \sqrt{-1}$. In 4-dimensional space the number of independent components in a totally antisymmetric tensor is $n(n-1)/2 = 6$, and can not be mapped to a 4-dimensional vector.

[11] This rule states that a summation is performed over all indices that occur twice

[12] $\otimes$ is the tensor product $(\vec{x} \otimes \vec{y})_{ij} = x_i y_j$.

Consider again rotations around an axis $\vec{a}$, then the composition property of rotations, i.e. that a first rotation by ψ followed by a second rotation by φ is the same as a single rotation by $\varphi + \psi$:

$$R_{\vec{a}}(\varphi + \psi) = R_{\vec{a}}(\varphi) R_{\vec{a}}(\psi)$$

and Equation (4.24) lead to the following description of such rotations:

$$\frac{d}{d\varphi} R_{\vec{a}}(\varphi) = \Omega_{\vec{a}} R_{\vec{a}}(\varphi) = R_{\vec{a}}(\varphi)\Omega_{\vec{a}}. \tag{4.33}$$

Together with the property that a rotation by $\varphi = 0$ is the unit matrix:

$$R_{\vec{a}}(0) = \mathbb{1}$$

this implies that

$$R_{\vec{a}}(\varphi) = \exp(\Omega_{\vec{a}}\varphi).$$

Therefore, the pseudovector[13] $\varphi\vec{a}$ and the matrix $R_{\vec{a}}(\varphi)$ can be used interchangeably to represent a rotation. The associated representation of the rotation using vectors reads:

$$R_{\vec{a}}(\varphi)\vec{r} = (\vec{a} \cdot \vec{r})\vec{a} + \vec{a} \wedge \vec{r} \sin\varphi - \vec{a} \wedge (\vec{a} \wedge \vec{r}) \cos\varphi. \tag{4.34}$$

The proof is left as an exercise. This identity shows another facet of the vector product.

Appendix B Ground Tracks

There are different ways for representing satellite orbits, which are all projections of the orbit in three dimensional space into lower dimensional spaces. The ground track is one such projections. It shows the intersection of the line joining the satellite to the center of the earth on a spherical earth. This location can also be described as the location on the surface of the earth where the satellite is in the zenith, i.e. at an elevation $E = 90°$.

Ground track representations, often include the area of coverage of the satellite, i.e., the area where the satellite is seen above a certain predetermined elevation angle on earth. Such representations provide valuable insights for satellite control centers, such as those located in Houston for NASA, or in Oberpfaffenhofen for the German Space Operation Center (GSOC/DLR). The operators in those centers use the ground track and the coverage map to select the ground stations for communicating with the satellite. In system design ground tracks provide an initial understanding of the capability achieved with a certain orbit.

According to the above definition the longitude and latitude of the pierce point through a spherical earth has to be computed. Equations (**??**), (**??**), and (4.19) describe the position of the satellite in space. The resulting normalized vectors can be represented in latitude ϕ and longitude λ, i.e., in 3-dimensional polar coordinate as follows:

$$\begin{pmatrix} \cos\lambda\cos\phi \\ \sin\lambda\cos\phi \\ \sin\phi \end{pmatrix} = R_3(-\Omega)R_2(-i) \begin{pmatrix} \cos(\nu+\omega) \\ \sin(\nu+\omega) \\ 0 \end{pmatrix},$$

or equivalently:

$$\begin{pmatrix} \cos(\lambda-\Omega)\cos\phi \\ \sin(\lambda-\Omega)\cos\phi \\ \sin\phi \end{pmatrix} = \begin{pmatrix} \cos i\cos(\nu+\omega) \\ \sin(\nu+\omega) \\ \sin i\cos(\nu+\omega) \end{pmatrix},$$

[13] $\vec{a}$ has slightly different transformation properties. It stays unchanged under reflection, this is the reason why it is often called a pseudovector rather than a vector.

i.e.,

$$\tan(\lambda - \Omega) = \frac{\tan(\nu + \omega)}{\cos i}$$
$$\sin\phi = \sin i \cos(\nu + \omega).$$

An immediate consequences of the first equation, is that for $i = 0$ the longitude of the point on the ground track λ is equal to the argument of latitude of the satellite ν shifted by a certain amount. This is obvious since the orbit is in the equatorial plane in that case. A more interesting observation is that the latitude swings from plus the inclination to minus the inclination. A more detailed characterization of the orbits needs to consider the relative frequencies of the satellite and earth movement. To the lowest level of approximation, the earth rotation is given by

$$\Omega(t) = \Omega_0 + \dot{\Omega}(t - t_0)$$

The simplest satellite orbit is circular, i.e., $e = 0$. In this case:

$$\nu(t) = E(t) = 2\pi\frac{(t - t_0)}{T} + \omega.$$

Even in this simplest case, there is a wide variety of different orbits[14]. These orbits are symmetric with respect to the equator. The orbits can be closed or not, depending on whether the ratio $2\pi\dot{\Omega}/T$ is rational or not. The geostationary orbit is the simplest orbit. The altitude is chosen such that the period of the satellite orbit equals the period of earth rotation. If in addition the inclination is chosen to be $i = 0$, the ground track becomes a point. In the case $i \neq 0$ the orbit becomes 8-shaped with the node on the equator. Such an orbit type was considered for a European Satellite navigation system. It was abandoned, when Europe decided to build a global system. For global satellite navigation systems, the most interesting orbits are MEO orbits with a closed ground track (see the discussion in Section 2.2.1). GPS and Galileo ground tracks are of this type.

Exercises

1. Central Symmetry and the Ellipse

   - Prove that the angular momentum is constant for forces of the form:

     $$\vec{F} = f\frac{\vec{r}}{\|\vec{r}\|}.$$

     Hint: Compute the derivative and use vector identities!

   - An ellipse is defined by the equation:

     $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

     with a, and b being the semi-major and semi-minor axis. Show that the solution of the two-body problem described by Equation (4.12) is an ellipse. Note that r is measured from $(ae, 0)$.

   - The two points $(ae, 0)$, and $(-ae, 0)$ are called the focal points of the ellipse. Show that the sum of the distances to these points is identical for all points on the ellipse.

2. Rotations and the Vector Product

[14] Explicit computations of ground tracks for generic orbits need to be performed using computers. WXTrack (see [6]) is a shareware programm that generates ground tracks and coverage maps of satellites.

- Use geometric considerations to prove the validity of Equation (4.34) for the representation of an arbitrary rotation $R_{\vec{a}}(\varphi)\vec{r}$ in terms of the vector of position $\vec{r}$, the direction of the rotational axis $\vec{a}$, and the rotation angle φ.
  Hint: Let $\vec{e}_{\vec{r}} = \vec{r}/\|\vec{r}\|$, then $\vec{a}$, $\vec{a} \wedge \vec{e}_{\vec{r}}$, and $\vec{a} \wedge (\vec{a} \wedge \vec{e}_{\vec{r}})$ form an orthonormal system. Make a drawing.
- Compute $(\Omega_{\vec{a}})^n \vec{r}$. *Hint: use the vector identity (4.29).*
  Use the result to derive the right hand side of Equation (4.34).
  Hint: perform the summation and compare with the series expansion of the trigonometric functions. Finally compute the constant term using Equation (4.29).
- Derive the expression for the convolution

$$\epsilon_{ijk}\epsilon_{ilm}$$

  of the totally antisymmetric tensor.
- Verify the vector identities (4.27)-(4.29) at the end of Appendix A using the properties of ϵ_{ijk}.
- Obtain the same result using the matric representation of the vector product.

3. Orbit Determination
   Assume that the satellite S is on a Medium Earth Orbit (MEO) around the earth. The processing center has determined the location and velocity of the satellite at a given instant of time in the CTRF:

$$\vec{r}_S = (-16188.6, 20219.6, 2257.4) \text{ km},$$

   and

$$\dot{\vec{r}}_S = (-2.552, -2.2585, 1.92798) \text{ km/s}.$$

   - Determine the inclination i of the orbital plane.
     Hint: Think about the geometry and use the vector product.
   - Determine the longitude of the ascending node Ω.
   - Determine the eccentricity of the orbit e.
     Hint: Compute the total energy of the satellite and the angular momentum, and relate them to the eccentricity. $GM = 398'600.4415$ km^3/s^2.

Bibliography

[1] R. Jost, *Allgemeine Mechanik,* unpublished Lecture Notes, 1975.

[2] P.G. Roll, R. Krotov, R.H. Dicke, *Annals of Physics,* vo. 26, p. 442, 1964

[3] R. Dach, U. Hugentobler, P. Fridez, M. Meindl, *Bernese GPS Software* Version 50, Astron. Inst., Univ. of Bern., Jan. 2007 `http://www.bernese.unibe.ch/docs/DOCU50.pdf`.

[4] *GPS Interface Control Document*, GPS-ICD200C `http://www.navcen.uscg.gov/pubs/gps/icd200/icd200cw1234.pdf`

[5] *Galileo Open Service Signal-in-Space ICD* `http://www.galileoju.com`

[6] D. Taylor, WXTrack - Satellite Tracking
`http://www.david-taylor.pwp.blueyonder.co.uk/software/wxtrack.htm`

[7] O. Montenbruck, E. Gill, *Satellite Orbits*, Springer Verlag, Berlin, Heidelberg, 2000.