Data-driven linear and nonlinear control

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***Abstract*—Based on the fundamental lemma of behavioral system theory, which allows to view a system as a set of trajectories, researchers have developed numerous methods to achieve representation and control of systems. In recent years, data-driven control has proven itself to be a reliable and practical method both for linear and nonlinear systems. This survey presents the main concepts for data-driven control both in linear and nonlinear scenarios.**

***Index Terms*—Nonlinear control, Linear control, Data-driven control, Behavioral system theory**

I. Introduction

CONTROL engineering focuses on achieving a desirable behavior of any type of system.

In the past decade, given the more complicated and hard-to-identify systems, the concept of learning from data has become prominent and a promising method to bypass the identification procedure. That is where data-driven control shines, and most of the current research focuses on achieving control of a system purely based on measured data. The backbone theory for these methods is the behavioral approach to systems.

The theory behind behavioral systems was conceived by Jan Camiel Willems (18 September 1939 – 31 August 2013), where he presented the notion of behavior, *which consists of the set of time trajectories that are declared possible by the model of a dynamical system* ([1]). This concept allows for a view of the dynamical system without a representation.

This theory can be applied to the development of data-driven control, whose main attraction is the possibility of generating an optimal control strategy only based on data, thus bypassing the model identification step.

In this paper we will explore the main application and methods for both linear and nonlinear data-driven control.

We start by introducing the relevant notation in I-A based on behavioral system theory, followed by a self contained review of behavioral system theory in section II. Then, we start in section III with data-driven linear control, exploring its representations and important results in III-A, feedback control in III-B, linear quadratic regulator (LQR) in III-C and implementations in III-D. In a similar form in section IV, we start by reviewing the representations and control implementations in IV-A, followed by robustness in IV-B, and ending the section with a review on some implementations an expansions in IV-C. In section V, we give our final remarks and conclude this survey paper.

Manuscript received Month Day, 2023; revised Month, 2023.

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A. Notation

The notation used throughout the paper is mostly based on [2] and [3] and summarized in this subsection.

$\mathbb{N} := \{1, 2, \dots\}$ is the set of natural numbers.

$\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of integer numbers.

We define $(\mathbb{R}^q)^{\mathscr{T}}$ as the space of signals $w : \mathscr{T} \longrightarrow \mathbb{R}^q$, where $\mathscr{T}$ is the time axis which can be continuous or discrete. If w is not restricted to an interval, we simply refer to it by its symbol (w).

We refer to the restriction of w to the interval $[1, L]$ as $w|_L := (w(1), \dots, w(L))$.

The concatenation of trajectories is represented as $w = w_p \wedge w_f$.

The unit shift operator is represented as σ, where $(\sigma w)(t) := w(t+1)$.

The discrete-time dynamical system with q variables is represented as $\mathscr{B} \subset (\mathbb{R}^q)^{\mathbb{N}}$. Furthermore, we can also refer to the restriction of $\mathscr{B}$ to the interval $[1, L]$ as $\mathscr{B}|_L := \{w|_L | w \in \mathscr{B}\}$.

The set of Linear Time-Invariant (LTI) systems with q variables is shown as $\mathscr{L}^q$.

$\mathscr{M}$ is a model class to which a dynamical system $\mathscr{B}$ belongs to (such as $\mathscr{L}^q$).

For a given system $\mathscr{B}$, its number of inputs is $m(\mathscr{B})$, the lag is $l(\mathscr{B})$ and the order is $n(\mathscr{B})$. Additionally, we consider the complexity $c(\mathscr{B}) := (m(\mathscr{B}), l(\mathscr{B}), n(\mathscr{B}))$ where the set of bounded complexity is $\mathscr{L}^q_c := \{\mathscr{B} \in \mathscr{L}^q | c(\mathscr{B}) \leq c\}$.

The Hankel matrix with L block rows constructed from w is $\mathscr{H}_L(w)$.

The set of N trajectories of a dynamical system $\mathscr{B}$ is $\mathscr{W}_d := \{w_d^1, \dots, w_d^N\}$ with $w_d^i \in (\mathbb{R}^q)^{\mathscr{T}_i}$ and $w_d^i \in \mathscr{B}|_{\mathscr{T}^i} \forall i = 1, \dots, N$.

The Hankel matrix is generalized to the mosaic-Hankel matrix when dealing with a set of time series $\mathscr{W}_d$ as $\mathscr{H}_L(\mathscr{W}_d) := \begin{bmatrix} \mathscr{H}_L(w_d^1) & \dots & \mathscr{H}_L(w_d^N) \end{bmatrix}$.

The time series of length L of inputs of an input/state representation of a LTI system is presented as $U := \begin{bmatrix} u(0) & u(1) & \dots & u(L-1) \end{bmatrix}$. Respectively, the time series of length $L+1$ of states is $X := \begin{bmatrix} x(0) & x(1) & \dots & x(L) \end{bmatrix}$.

We use indices in time series ($U_{i,t,N}, i \in \mathbb{Z}, t, N \in \mathbb{N}$) to indicate the Hankel matrix with i denoting the time of the first sample, t the number of samples per column and N the number of samples per row.

We use X_- to refer to the predecessor state $X := \begin{bmatrix} x(0) & x(1) & \dots & x(L-1) \end{bmatrix}$ and X_+ to refer to the successor state σX_-.

$\|.\|_F$ represents the Frobenius norm.

$\otimes$ represents the Kronecker product.

We abbreviate $V^T U V$ with $[*]^T U V$.

II. Behavioral Systems Theory

In this section, we review the main results of behavioral systems theory.

Lemma 1: Persistently exciting For a signal to be *persistently exciting of order* L, the hankel matrix $\mathscr{H}_L(u_d)$ must be full rank (rank$(\mathscr{H}_L(u_d)) = mL$).

Lemma 2: The fundamental lemma Consider a LTI controllable system $\mathscr{B} \in \mathscr{L}^q$. Given a data trajectory $w_d = (u_d, y_d) \in \mathscr{B}|_T$, a trajectory of $\mathscr{B}$ and u_d persistently exciting of order $L + n(\mathscr{B})$, then:

$$\text{image } \mathscr{H}_L(w_d) = \mathscr{B}|_L \tag{1}$$

Equation 1 symbolizes the fact that any L-samples trajectory w of $\mathscr{B}$ can be written as a linear combination of the columns of $\mathscr{H}_L(w_d)$ and for $g \in \mathbb{R}^{T-L+1}$, any linear combination $\mathscr{H}_L(w_d)g$ is also a trajectory of $\mathscr{B}$.

The importance of this results lies in the fact that it presents all possible trajectories of a controllable LTI system, only using collected data, hence acting as a non-parametric representation of the system. This makes it a core concept for system identification and data-driven analysis.

One case where we can derive identifiability is considering input/state representation of a LTI system $\mathscr{G}$ where the output is the state:

$$\mathscr{G} = \{(u, x) \,|\sigma x = Ax + Bu\} \tag{2}$$

Noting that $l(\mathscr{G}) = 1$, we can use Lemma 2 to conclude Corollary 2 of [4].

Corollary 1: Consider a trajectory $(u_d, x_d) \in \mathscr{G}_T$ of a system equal to the one in 2 with m as the input u dimension and n as the state x dimension, with (A, B) controllable. We can then say that if u_d is persistently exciting of order $n + L$ then:

$$rank \begin{bmatrix} \mathscr{H}_L(u_d) \\ x_d(1) \ldots x_d(T-L+1) \end{bmatrix} = mL + n \tag{3}$$

This rank condition is used extensively in the literature to test if the trajectory data is persistently exciting.

The results shown depend on an input/output partitioning of the variables. We will now see that some assumptions, such as controllability, can be relaxed and that we can extend the result to deal with a set of trajectories.

Definition 1: Identifiability - [2] (Definition 4) The system $\mathscr{B} \in \mathscr{M}$ is identifiable from data if

$$\hat{\mathscr{B}} \in \mathscr{M} \wedge \mathscr{W}_d \subset \hat{\mathscr{B}} \Longrightarrow \hat{\mathscr{B}} = \mathscr{B} \tag{4}$$

In [5], a condition for identifiability of a LTI system $\mathscr{B} \in \mathscr{L}^q$ from the data as a set of trajectories $\mathscr{W}_d$ if and only if

$$rank \mathscr{H}_{l(\mathscr{B})+1}(\mathscr{W}_d) = m(\mathscr{B})(\mathscr{B}) + 1) + n(\mathscr{B}) \tag{5}$$

Furthermore in [5], the following corollary provided a non-parametric representation of $\mathscr{B}|_L$.

Corollary 2: - [5] (Corollary 19) If B is a LTI system,

$$\text{image } \mathscr{H}_L(\mathscr{W}_d) \subseteq \mathscr{B}|_L \, \forall L \in \{l(\mathscr{B}) + 1, \ldots, L_{max}\} \tag{6}$$

And, for $L \geq l(\mathscr{B})$, image $\mathscr{H}_L(\mathscr{W}_d) = \mathscr{B}|_L$ if and only if

$$rank \mathscr{H}_L(\mathscr{W}_d) = m(\mathscr{B})L + n(\mathscr{B}) \tag{7}$$

The condition 7 is named in [2] as the *generalized persistency of excitation*.

The corollary 2 acts as a more general alternative to the fundamental lemma, by giving conditions that if respected ensure that the image of the Hankel matrix constructed from data generates the restricted behavior. However, it does not have the same restrictions as 2, since it does not requires a given input/output partitioning, nor controllability of the system and allows for multiple trajectories. Additionally, the mosaic-Hankel matrix includes special matrix structures as special cases such as the Hankel matrix, Page matrix and trajectory matrix ([2]. The two latter are mainly advantageous for direct data-driven control, whereas the Hankel matrix is specially effective for system identification.

We can now conclude that under generalized persistency of excitation condition (7),

$$\mathscr{B}|_L = \text{image } \mathscr{H}_L(\mathscr{W}_d). \tag{8}$$

Under 7, condition 8 is valid for any multivariable LTI system. Additionally, this condition allows to use a very useful approach to solve data-driven analysis, based on the ability to represent any input/output trajectory of a system as a linear combination of collected input/output data ([3]),so

$$w \in \mathscr{B}|_L \Longleftrightarrow w = \mathscr{H}_L(\mathscr{W}_d)g \tag{9}$$

has a solution g.

Throughout the paper, we will see how these results are used and adapted to solve different data-driven control problems, both for linear and nonlinear systems.

III. Data-driven linear control

In this section we will explore the main approaches to achieve a controller for a linear system based on data-driven control methods and the theory on section II. During this section we consider a controllable system $\mathscr{G}$ with a input/state/output ([2]) where $\Pi \in \mathbb{R}^{q \times q}$ is a permutation matrix, $x \in (\mathbb{R}^n)^{\mathbb{N}}$ and $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathbb{R}^{(n+p)\times(n+m)}$:

$$\{\Pi\left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] | \sigma x = Ax + Bu, \, y = Cx + Du\} \tag{10}$$

A. Linear controllers representation

Equation 9 allows us to get data-dependent representation of the open-loop and closed-loop dynamics of the system ([3]).

We consider a persistently input sequence of order $n + 1$. From 3 we have that

$$rank \begin{bmatrix} U \\ X_- \end{bmatrix} = n + m. \tag{11}$$

This condition is always possible to verify as long as we have access to the state of the system.

Considering X and U time series of length $L + 1$ and L, we have that

$$X_+ = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U \\ X_- \end{bmatrix}. \tag{12}$$

The following theorem gives a data-based representation of a linear system based on 9.

Theorem 1: - [3] (Theorem 1) Assuming 11 holds, the system 10 has the following equivalent representation:

$$X_+ = X_+ \begin{bmatrix} U \\ X_- \end{bmatrix}^\dagger \begin{bmatrix} U \\ X_- \end{bmatrix} \tag{13}$$

Since

$$\begin{bmatrix} B & A \end{bmatrix} = X_+ \begin{bmatrix} U \\ X_- \end{bmatrix}^\dagger \tag{14}$$

(from 12) where † represents the right inverse since condition 11 ensures 12 can be solved for (B, A).

We can see that theorem 1 serves as practical application of 9 and serves to identify a state-space model from data (indirect data-driven control). Specifically, we want to find $(\hat{B}, \hat{A})$ for a given (U, X), such that we satisfy 12. The right hand-side of 14 is the minimizer of the least square problem:

$$\underset{B,A}{min} \left\| X_+ - \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U \\ X_- \end{bmatrix} \right\|_F = X_+ \begin{bmatrix} U \\ X_- \end{bmatrix}^\dagger \tag{15}$$

Which is the same as saying that $\begin{bmatrix} \hat{B} & \hat{A} \end{bmatrix} = X_+ \begin{bmatrix} U \\ X_- \end{bmatrix}^\dagger$. So, theorem 1 is the solution of a least-square problem ([3]).

We can now use condition 9 to get a parametrization of system 10.

Theorem 2: - [3] (Theorem 2) Consider 11 to be satisfied and system 10 in closed loop with a state feedback $u = Kx$. Then , the system has the following equivalent representation:

$$X_+ = X_+ G_K X_- \tag{16}$$

Where G_K is a $(L \times n)$ that satisfies:

$$\begin{bmatrix} K \\ I_n \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} G_K \tag{17}$$

and $u = UG_K X_-$.

The big improvement of theorem 2 is that contrary to 1, it allows to parameterize K through data as seen in 17. This allows for a design procedure that avoids the identification step of a parametric model of the system ([3]), allowing a change from indirect to direct data-driven control.

B. Feedback Controllers

Theorem 2 shows us that a closed-loop system like 10 under state-feedback $u = Kx$ will fulfill $(A + BK)X_+ G_K$, where G_K satisfies 17 ([3]).

To ensure stability we want to choose G_K, so that $X_+ G_K$ is stable in the Lyapunov sense. The next theorem shows that this problem can be solved with a linear matrix inequality (LMI).

Theorem 3: - [3] (Theorem 3) If condition 11 holds, then any matrix Q that satisfies

$$\begin{bmatrix} X_- Q & X_+ Q \\ Q^T X_+^T & X_- Q \end{bmatrix} \succ 0 \tag{18}$$

is such that

$$K = UQ(X_- Q)^{-1} \tag{19}$$

stabilizes system 10. It is also true that if a gain K stabilizes the system, then it can be written as seen in 19 with Q a solution to 18.

We now explore the extension of the previous results to single-input-single-output (SISO) systems. Start by considering a non-minimal (order $2n$) SISO system $\mathscr{S}$ as in 10,with $D = 0$, in left difference operator representation ([3] - equation 56):

$$\begin{aligned} &y(k) + a_n y(k1) + \dots + a_2 y(kn+1) + a_1 y(kn) \\ &\quad = b_n u(k1) + \dots + b_2 u(kn+1) + b_1 u(kn). \end{aligned} \tag{20}$$

We can reduce the output measurement case to the state measurement by letting

$$\begin{aligned} \chi(k) := col(&y(kn), y(kn+1), \dots, y(k1), \\ &u(kn), u(kn+1), \dots, u(k1)), \end{aligned} \tag{21}$$

from III-B, we obtain the state-space system below - the full explicit system can be found in [3] equation 58.

$$\chi_+ = \mathcal{A}\chi_- + \mathcal{B}U \tag{22}$$

Considering the matrix in 11 for a $L \geq 2n + 1$, if this matrix is full-row rank, then we can apply the analysis made previously for the system 22. This matrix can be presented as

$$\begin{bmatrix} U \\ \hat{X}_- \end{bmatrix} = \begin{bmatrix} u_d(0) & \dots & u_d(L-1) \\ \chi_d(0) & \dots & \chi_d(L-1) \end{bmatrix}. \tag{23}$$

If u is persistently exciting of order $2n+1$, we can conclude the following result:

Lemma 3: - [3] (Lemma 3)

$$\begin{bmatrix} U \\ \hat{X}_- \end{bmatrix} = \begin{bmatrix} U \\ Y_{-n,n,L} \\ U_{-n,-n,L} \end{bmatrix} \tag{24}$$

holds and for the given condition for u,

$$rank \begin{bmatrix} U \\ \hat{X}_- \end{bmatrix} = 2n + 1 \tag{25}$$

We can now express the SISO system via data similar to what we did in the input/state approach. Exactly as before, equation 9 9 grants that

$$\begin{bmatrix} u \\ \chi \end{bmatrix} = \begin{bmatrix} U \\ \hat{X}_- \end{bmatrix} g \tag{26}$$

has a solution g. So, we can replace the right side in 22 for $\hat{X}_+$ given 26:

$$\chi_+ = \hat{X}_+ g \tag{27}$$

It is trivial to now repeat the steps for theorem 1 and get the following representation for a SISO system ([3] - Theorem 7):

$$\begin{aligned} \chi_+ &= \hat{X}_+ + \begin{bmatrix} U \\ \hat{X}_- \end{bmatrix}^\dagger \begin{bmatrix} U \\ \chi_- \end{bmatrix} \\ y &= e_n^T \hat{X}_+ \begin{bmatrix} U \\ \hat{X}_- \end{bmatrix}^\dagger \begin{bmatrix} 0_{1\times 2n} \\ \hat{I}_{2n} \end{bmatrix} \chi_- \end{aligned} \tag{28}$$

with e_n the nth versor of $\mathbb{R}^{2n}$.

We can now reach similar results as theorem 2 and theorem 3 but for feedback controllers:

Theorem 4: - [3] - Equations 71 and 72 The closed-loop system has the following equivalent representation:

$$\chi_+ = \hat{X}_+ G_{\mathcal{K}} \chi_- \tag{29}$$

where $G_{\mathcal{K}}$ is a $L \times 2n$ matrix such that

$$\begin{bmatrix} \mathcal{K} \\ I_{2n} \end{bmatrix} = \begin{bmatrix} U \\ \hat{X}_- \end{bmatrix} G_{\mathcal{K}} \tag{30}$$

Theorem 5: - [3] (Equations 74 and 75) Any matrix $\mathcal{Q}$ satisfying

$$\begin{bmatrix} \hat{X}_- \mathcal{Q} & \hat{X}_+ \mathcal{Q} \\ \mathcal{Q}^T \hat{X}_+^T & \hat{X}_- \mathcal{Q} \end{bmatrix} \succ 0 \tag{31}$$

is such that

$$\mathcal{K} = U \mathcal{Q} (\hat{X}_- \mathcal{Q})^{-1} \tag{32}$$

stabilizes system 22. It is also true that if a gain $\mathcal{K}$ stabilizes the system, then it can be written as seen in 32 with $\mathcal{Q}$ a solution to 31.

C. Linear-Quadratic Regulator (LQR)

We consider a controllable input/state system as in 2 with $(u, x) \in (R^{m+n})^{\mathbb{N}}$, where $u(t) \in \mathbb{R}^m$ and $x(t) \in \mathbb{R}^n$ are available as measurements.

Let us consider the base control problem using state feedback $u = Kx$ for infinite-horizon LQR optimal control ([2]):

$$\begin{aligned} \text{minimize over } u \ & \sum_{t=1}^{\infty} \|x(t)\|_Q^2 + \|u(t)\|_R^2 \\ \text{subject to } & \sigma x = Ax + Bu \end{aligned} \tag{33}$$

Where we have $Q \succeq 0, R \succ 0$ and $(Q^{1/2}, A)$ observable.

We see in [6], that the optimal controller for the LQR problem (with anticipated solution $u = Kx$) can be found by solving the following program:

$$\begin{aligned} \underset{P \succeq I, K}{\text{minimize}} \ & \text{trace } (QP) + \text{trace } (K^T R K P) \\ \text{subject to } & (A + BK) P (A + BK)^T - P + I \preceq 0 \end{aligned} \tag{34}$$

Using the relation seen in 12 and 17, we can achieve a parametrization of the closed-loop matrix $A + BK$ ([6]):

$$A + BK = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} \overset{(17)}{=} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U \\ X_- \end{bmatrix} G_K \overset{(12)}{=} X_+ G_K \tag{35}$$

We can now replace $A + BK$ in 34 by $X_+ G_K$ leading to the formulation

$$\begin{aligned} \underset{P \succeq I, K, G_K}{\text{minimize}} \ & \text{trace } (QP) + \text{trace } (K^T R K P) \\ \text{subject to } & X_+ G_K P (X_+ G_K)^T - P + I \preceq 0 \\ & \begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} G_K \end{aligned} \tag{36}$$

In [6], it showed that adding an *orthogonality constraint*

$$(I - \begin{bmatrix} U \\ X_- \end{bmatrix}^{\dagger} \begin{bmatrix} U \\ X_- \end{bmatrix}) G_K = 0, \tag{37}$$

guarantees the formulation to represent a direct version of the certainty-equivalence problem LQR ([6] - Lemma 3.1). This allows for a representation of a regularized data-driven LQR problem by putting the constraint 37 in the objective ([6] - Theorem 3.2):

$$\begin{aligned} \underset{P \succeq I, K, G_K}{\text{minimize}} \ & \text{trace } (QP) + \text{trace } (K^T R K P) \\ & + \lambda \left\| (I - \begin{bmatrix} U \\ X_- \end{bmatrix}^{\dagger} \begin{bmatrix} U \\ X_- \end{bmatrix}) G_K \right\| \\ \text{subject to } & X_+ G_K P (X_+ G_K)^T - P + I \preceq 0 \\ & \begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U \\ X_- \end{bmatrix} G_K \end{aligned} \tag{38}$$

We can also note that considering disturbances can be done by considering a time-series D and replacing X_+ in the previous equations by $X_+ - D$.

In equation 18 of [6], an additional regularization formulation is suggested by using a design parameter $\rho > 0$ that accounts for the term $G_K P G_K^T$ instead of just G_K, which is denoted as robust LQR. It is conclude in theorems 4.1 and 4.2 of [6], that the robust LQR is more sensitive to noisy data but easier to stabilize.

D. Linear Implementations

In this subsection we will briefly explore some different applications of the previous theory as well as some extensions, such as Linear Parameter-Varying (LPV) systems [7], positive stabilization [8], Virtual Reference Feedback Tuning (VRFT) [9] and Correlation-based Tuning with Guaranteed Stability (CbT-GS) [10].

1) LPV Systems: In nonlinear plants where the dynamic behavior is assumed to be linear, but dependent on measurable signals, the system is usually model as a LPV system ([7]). The system considered is a SISO LPV system with the same representation as seen in 22 but with an additional set p of measurable scheduling variables. Two methods are proposed, one to deal with both noiseless data ([7] - Proposition 1) and one to deal with noisy data ([7] - Proposition 2).

2) Positive Stabilization: A specific constraint of positive systems is that the states have intrinsically non-negative values ([8]). The approach for this system is similar to what was showed in previous sections only with a few added constraints to ensure the positive states.

Theorem 5 of [8] shows that when 11 holds, there exists a feedback control law $u = Kx$ such that the closed-loop system 16 is positive stable if and only if there exists a diagonal positive definite matrix $Z0$ and Y such that:

$$\begin{aligned} \begin{bmatrix} -Z & Y^T X_+^T \\ X_+ Y & -Z \end{bmatrix} \prec 0, \ X_+ Y \geq 0, \ X_- Y = Z, \\ K = U G_K, \ G_K = Y Z^{-1} \end{aligned} \tag{39}$$

Note how this is exactly the same conclusion reached in 18 with $Q = Y$ and inverted relation.

3) VRFT design: In [9], it is proposed a direct design method of feedback controllers for an unknown plant. We assume a LTI SISO dynamical system with a unknown transfer function $P(z)$. The data is collected during on experiment on the plant.

VRFT aims to address the design of a pre-filter of the data, in order to generate a controller that minimizes a set cost function and to deal with data affected by noise.

In VRFT, a reference model $M(z)$ is selected by the user, making it a model reference control problem, advantageous over other methods where the control specifications are given empirically or in a very simple manner ([9]).

4) CbT-GS reference control: In [10], a data-driven controller non iterative correlation-based tuning method is presented. It includes a set of constraints that ensures closed-loop stability. To ensure stability, the constraint are implemented using the discrete Fourier transform of auto- and cross-correlation functions ([10]). Only one experiment is needed to approximate the model (as in VRFT) and the correlation criterion is shown to be a sufficient condition for closed-loop stability with the effectiveness of CbT-GS shown in a practical application on a torsional setup ([10] - Figure 6).

IV. Data-driven nonlinear control

In this section we will explore how the data-driven approach can be extended to the realm of nonlinear control. We start by exploring how to represent nonlinear systems, moving into the design and types of nonlinear controllers, and finishing with recent developments to the area.

A. Nonlinear representations and controllers

We will review a representation method for nonlinear systems using the results achieved on the previous section.

We can start by considering a smooth nonlinear system ([3]) with a known equilibrium pair $(\bar{x}, \bar{u})$ in the form

$$X_+ = f(X_-, U) \tag{40}$$

that can be rewritten as

$$\delta X_+ = A\delta X_- + B\delta U + D \tag{41}$$

where $\delta x := x - \bar{x}$, $\delta u := u - \bar{u}$, D represents higher-order terms which go to zero faster than δx and δu and

$$A := \left.\frac{\partial f}{\partial x}\right|_{(x,u)=(\bar{x},\bar{u})}, B := \left.\frac{\partial f}{\partial u}\right|_{(x,u)=(\bar{x},\bar{u})}. \tag{42}$$

We can now reach a similar result to 3 for nonlinear systems:

Theorem 6: - [3] (Theorem 6) Assuming $\begin{bmatrix} U \\ X_- \end{bmatrix}$ and X_+ are full rank and that for some $\gamma > 0$, $DD^T \preceq \gamma X_+ X_+^T$ in the system in 40 with the equilibrium pair $(\bar{x}, \bar{u})$.

Then, any solution (Q, α) to

$$\begin{gathered} \begin{bmatrix} X_-Q - \alpha X_+ X_+^T & X_+Q \\ Q^T X_+^T & X_-Q \end{bmatrix} \succ 0 \\ \begin{bmatrix} I_L & Q \\ Q^T & X_-Q \end{bmatrix} \succ 0 \end{gathered} \tag{43}$$

such that $\gamma < \alpha^2/(4+2\alpha)$ returns a stabilizing state-feedback gain $K = UQ(X_-Q)^{-1}$ that locally stabilizes the equilibirum pair $(\bar{x}, \bar{u})$.

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We can get a more general representation of a nonlinear model class by considering a kernel of a nonlinear operator. We define a nonlinear time-invariant (NTI) $\mathscr{N}$ as ([11]):

$$\mathscr{N} := \{w \,|R(w, \sigma w, \ldots, \sigma^l w) = 0\}, \tag{44}$$

where $R : \mathbb{R}^{q(l+1)} \longrightarrow \mathbb{R}^g$ and g is the number of equations in the representation. To achieve the model class representation we consider a special case of $q = 2$ ([11]) with

$$\begin{gathered} R(w, \sigma w, \ldots, \sigma^l w) = f(\mathscr{X}(w)) - \sigma^l y \\ \text{with } \mathscr{X}(w) := vec(w, \sigma w, \ldots, \sigma^{l-1} w, \sigma^l u). \end{gathered} \tag{45}$$

This allows us to reach a nonlinear difference to define the model class ([11] - Equation 6):

$$\sigma^l y = f(x) = f(u, y, \sigma u, \sigma y, \ldots \sigma^{l-1} u, \sigma^{l-1} y, \sigma^l u) \tag{46}$$

Since u is an input and y an output this is an input/output representation of a nonlinear SISO system. In [11], $f(x)$ is considered to be an n_x-variate polynomial such that $f(x) = \theta^T \phi(x)$, where ϕ defines the model structure (vector of basis functions), and once it is specified a particular model can be specified with θ ([11] - Equation 7):

$$\mathscr{N}(\theta) := \left\{ w = \begin{bmatrix} u \\ y \end{bmatrix} \,|\sigma^l y = \theta^T \phi(\mathscr{X}(w)) \right\}. \tag{47}$$

Furthermore, in [11] it is shown that it is possible to consider different forms of the nonlinear terms of phi to represent different classes of nonlinear systems such as Hammerstein system, Finite-lag Volterra, bilinear and generalized bilinear.

An expansion beyond polynomial dynamics was proposed in [12] for rational systems using a sum-of-squares (SOS) based controller. It is shown in section III of [12] that with open-loop matrices A, B, P, polynomial basis matrices Z, H and known noise bounding matrices matrices Q_w, R_w, S_w, the set

$$\mathscr{M} := \left\{ \begin{bmatrix} A & B & P \end{bmatrix} \middle| [*]^T \begin{bmatrix} \bar{Q}_w & \bar{S}_w \\ \bar{S}_w^T & \bar{R}_w \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \begin{bmatrix} A & B \end{bmatrix}^T \\ \left(I_n \otimes P\right)^T \end{bmatrix} \\ I \end{bmatrix} \succeq 0 \right\}$$

where

$$\begin{bmatrix} \bar{Q}_w & \bar{S}_w \\ \bar{S}_w^T & \bar{R}_w \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -\begin{bmatrix} Z(X) \\ H(X,U) \end{bmatrix} \\ \hat{Z}_p(X, \dot{X}) \end{bmatrix} & 0 \\ \dot{X} & B_w \end{bmatrix} \begin{bmatrix} Q_w & S_w \\ S_w^T & R_w \end{bmatrix} [*]^T \tag{48}$$

which depends on the noise bound and the measured data with $\dot{X}, Z(X), H(X,U), \hat{Z}_p(X, \dot{X})$ is an equivalent parametrization of a NTI rational system ([12] - Theorem 4) and similar to previous sections a state-feedback controller is achieved in section III-b of [12] via a SOS program.

Other results for similar representations and controllers were also achieved, from similar results to the ones present for SISO feedback linearizable systems ([13]) to other implementations such as for flat nonlinear systems ([14]), representation through state-dependent representation ([15], or using nonlinearity cancellation design methods ([16].

B. Nonlinear robustness

We will now explore how to deal with noisy data when designing nonlinear systems. For this we now consider a NTI system similar to 40, but with an additional noise and disturbance term E_+ ([17]):

$$X_+ = f(X_-, U) + E \tag{49}$$

where E is bounded in magnitude such that $e \in E \implies \|e\| \leq \epsilon$ for a chosen $\epsilon > 0$.

In [18] the result in 43 is extended for noisy data. We consider a slightly different representation proposed in [18] for a polynomial system:

$$\dot{x}_+ = AZ_- + B\bar{U} + E \tag{50}$$

where $Z(x)$ are monomials in x of $f(X_-, U)$ and $\bar{U} = W(x)U$ with $W(x)$ as the monomials in u of $f(X_-, U)$. Additionally, [18] defines $\hat{Z}(x)$ has a lower degree $Z(x)$ to deal with computational issues in a SOS program such that $Z(x) = H(x)\hat{Z}(x)$, with $H(x)$ a polynomial matrix.

We can now conclude the following result:

Theorem 7: - [18] - Theorem 1

Assume that $EE^T \preceq RR^T$ for some known $R \in \mathbb{R}^{n\times L}$.

For given matrices $Y(x), P \succ 0$ that satisfy $Z_-Y(x) = H(x)P$, we can represent the colosed-loop dynamics as $\dot{x} = (X_+ - F\hat{U})Y(x)P^{-1}\hat{Z}(x)$ ([18] - Equation 6), where $F := \begin{bmatrix} B & E \end{bmatrix}$ and $\hat{U} := \begin{bmatrix} \bar{U} - W(x)U \\ I_L \end{bmatrix}$.

Assume that $BV^T \preceq R_B R_B^T$ for some known $R_B \in \mathbb{R}^{n\times q}$. Then it holds that $FF^T \preceq R_F R_F^T$, with $R_F = R + R_B$.

Then, for $P \in \mathbb{R}^{p\times p}$, $L \times p$ matrix polynomial $Y(x)$, SOS matrix $\epsilon_1(x)$ and $\epsilon_2 > 0$ such that

$$\begin{gathered} Z_- Y(x) = H(x)P, \\ \begin{bmatrix} \mathscr{Y}_F(x) - \epsilon_1(x) I_p & Y(x)^T X_+^T \hat{U}^T \\ \hat{U} Y(x) & \epsilon_2 I_{q+L} \end{bmatrix} \\ \text{a } p+q+L \text{ SOS polynomial matrix where} \\ \mathscr{Y}_F(x) := -\frac{\partial \hat{Z}(x)}{\partial x} X_+ Y(x) - Y(x)^T X_+^T \frac{\partial \hat{Z}(x)^T}{\partial x} \\ -\epsilon_2 \frac{\partial \hat{Z}(x)}{\partial x} R_F R_F^T \frac{\partial \hat{Z}(x)^T}{\partial x}, \end{gathered} \tag{51}$$

then the state-feedback controller that stabilizes the polynomial system is

$$u = UY(x)P^{-1}\hat{Z}(x). \tag{52}$$

Regarding rational systems, the representation showed in 48, already considers the possibility of noise.

C. Nonlinear implementations

Similar to what was done for data-driven linear control, in this subsection, we will explore implementations and techniques of the previous theory as well as some extensions such as LTI embedding techniques for generalized bilinear systems ([11]), control of an inverted pendulum with friction ([15]), control of a bipedal robot through feedback linearization ([19]), control of a Hydraulic servo actuator ([20]), control of a water tank system ([17]) and control of a double inverted pendulum ([13]).

1) LTI Embedding for bilinear systems: In [11], it is showed that the behavior of a nonlinear system is included in the behavior of a LTI system. This means we can get set an additional nonlinear constraint to an embedding system such that it coincided with the nonlinear system. The results show better performance for true noisy data .

2) Inverted pendulum control with friction: Using state-dependent representation of the nonlinear system, it is showed in [15] that such representation can be used for the design of a robust nonlinear online data-driven controller which is computationally similar in complexity to a linear control instead of using SOS optimization. In Remark 2 of [15], it is stated that the quality of the data has a grand effect in the amount of noise a system will be able to tolerate, which is shown for the pendulum example by tweaking the noise and realizing that the number of necessary samples increases with noise and that for open-loop unstable systems, the explosion of trajectories is unavoidable.

3) Bipedal robot control: In [19], a 5-degree of freedom under-actuated walking robot (AMBER-3M) is used to test data-driven control feedback linearization method for the SISO nonlinear system and compared to a PD controller.

It is concluded that although the data-driven method was more sensitive to noise due to being a model-free controller, the tracking performance was superior to the PD controller ([19] - Figure 4).

4) Hydraulic servo actuator control: Hydraulic servo actuators (HSA) are very complex nonlinear systems and in [20] a event-triggered data-driven controller of the HSA is considered.

Considering only measured input/output data, an output feedback event-triggered adaptive dynamic programming controller was designed and validated for the control of a HSA which revealed better results that a standard ADP-based control algorithm.

5) Water tank control: In [17], we consider an online direct approach to control a nonlinear system. To test the approach, a water tank system is controlled and the approach is compared to a offline direct design and to a PI linear controller.

The results in Figure 3 of [17] show that the online approach has better tracking performance, showing the main advantage has the ability to use the incoming measurements during operation to improve the performance.

6) Double inverted pendulum control: In [13], the extension of the fundamental lemma to data-driven MIMO feedback linearizable nonlinear systems is shown by using a set of known basis functions.

As a practical example, a fully-actuated double inverted pendulum is considered. It is showed in Figure 1 of [13] that the results are good if the offline data is sufficient. Additionally, it is shown that the more data is collected the less computationally heavy the controller becomes.

V. Conclusion

In this paper we reviewed the main developments in recent year in the topic of data-driven control. This was done by first reviewing the basics of behavioral system theory, which lies as the foundation of data-driven methods. In both linear and nonlinear control we started by reviewing the most important results that allow for the representation of the two types of systems. Then, we reviewed how to find gains to achieve reliable controllers. Finally, for both linear and nonlinear controllers we presented some of the recent developments and implementations.

It becomes clear that even though the fundamental lemma dates back to eighteen yeas ago, recently there has been a rise in extending the result to all different types of systems in an effort to prove the benefits of data-driven methods. Future work is still necessary to fully generalize the online direct data-driven approach to all types of systems with reliability and robustness to noisy data.

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