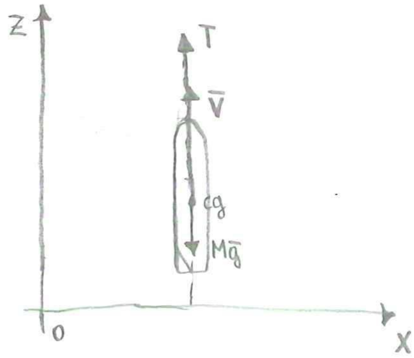
29/01/09 EXAM

Question 1

a) Equations of motion for the vertical flight of a single-stage rocket in an homogeneus gravity field and in vacuum



Equation of Motion

$/\!/ \bar{V}$ $\quad M\dfrac{dV}{dt} = T - Mg$

$\perp \bar{V}$ $\quad MV\dfrac{d\gamma}{dt} = 0$

assumptions

① homogeneus gravity field $\Rightarrow g = g_0$

② vacuum $\Rightarrow D = 0$

from the first : $\dfrac{dV}{dt} = \dfrac{T}{M} - g$

BURNOUT VELOCITY V_b

$dV = \dfrac{T}{M}dt - gdt$ $\qquad T = \dot{m}c$

$dV = c\dfrac{\dot{m}dt}{M} - g_0 dt$ $\qquad \dot{m} = -\dfrac{dM}{dt}$

$dV = -c\dfrac{dM}{M} - g_0 dt$ $\qquad$ if $t_0 = 0$, $V_0 = 0$

integrating

M_0 = initial mass

$\Lambda = \dfrac{M_0}{M_0 - \dot{m}t_b} = \dfrac{M_0}{M_e}$

$$V_b = c\ln\Lambda - g_0 t_b$$

$\psi_0 = \dfrac{T}{M_0 g_0}$

BURNOUT HEIGHT h_b

$$h_b = \int_0^{t_b} V\,dt = \int_0^{t_b} c\ln\frac{M_0}{M}\,dt - \int_0^{t_b} g_0 t\,dt$$

$dt = -\dfrac{dM}{\dot{m}}$

$$h_b = \int_{M_0}^{M_e}\left(-c\ln\frac{M}{M_0}\right)\left(-\frac{dM}{\dot{m}}\right) - \frac{g_0 t_b^2}{2}$$

Using :

- $t_b = \dfrac{M_0 - M_e}{\dot{m}} = \dfrac{cM_0}{T}\left(1 - \dfrac{M_e}{M_0}\right)\dfrac{g_0}{g_0} = \dfrac{I_{sp}}{\psi_0}\left(1 - \dfrac{1}{\Lambda}\right)$ $\qquad I_{sp} = \dfrac{c}{g_0}$

- $\int \ln x\,dx = x\ln x - x$

$$\Rightarrow h_b = \frac{M_0 c}{\dot{m}}\left[\frac{M}{M_0}\ln\frac{M}{M_0} - \frac{M}{M_0}\right]_{M_0}^{M_e} - \frac{g_0 I_{sp}^2}{2\psi_0^2}\left(1 - \frac{1}{\Lambda}\right)^2 =$$

$$= \frac{M_0 c}{\dot{m}}\left[-\frac{1}{\Lambda}\ln\Lambda - \frac{1}{\Lambda} + 1\right] - \frac{g_0 I_{sp}^2}{2\psi_0^2}\left(1 - \frac{1}{\Lambda}\right)^2 =$$

$$= \frac{M_0 c^2}{T}\frac{g_0}{g_0}\left[1 - \frac{1}{\Lambda}(\ln\Lambda + 1)\right] - \frac{\cancel{g_0} c^2}{2g_0^2\psi_0^2}\left(1 - \frac{1}{\Lambda}\right)^2 =$$

$$h_b = \frac{c^2}{g_0\psi_0}\left[1 - \frac{1}{\Lambda}(\ln\Lambda + 1)\right] - \frac{c^2}{2g_0\psi_0^2}\left(1 - \frac{1}{\Lambda}\right)^2 = \frac{c^2}{g_0\psi_0}\left\{\left[1 - \frac{1}{\Lambda}(\ln\Lambda + 1)\right] - \frac{1}{2\psi_0}\left(1 - \frac{1}{\Lambda}\right)^2\right\}$$

b) Compute the TOTAL starting mass of the rocket

$$\dot{m} = \frac{T}{c} = \frac{T}{I_{sp}g_0} = 3\times\frac{3300000\ N}{4022{,}1\ m/s} = 2461{,}4\ \frac{Kg}{s}$$

$$M_p = \frac{\dot{m}\cdot t_b}{3} = 205117\ Kg$$

$$\varepsilon = \frac{M_c}{M_c + M_p} = 0{,}12 \Rightarrow \varepsilon M_c + \varepsilon M_p = M_c \Rightarrow M_c = \frac{\varepsilon M_p}{(1-\varepsilon)} = 27970{,}5\ Kg$$

$$M_{stage} = M_P + M_C = \frac{\varepsilon}{1-\varepsilon} M_P + M_P = \left(\frac{1}{1-\varepsilon}\right) M_P = 233087.5 \text{ Kg}$$

$$M_{TOT} = 3M_{stage} + M_{2nd\ stage} + M_U = 699262.5 \text{ Kg} + 20'000 \text{ Kg} + 25'000 \text{ Kg} = \underline{744'262.5 \text{ Kg}}$$

c) Compute the instantaneous mass and acceleration of the rocket at the following moments:

1. Lift-off $\quad M_1 = M_{TOT} = 744'262.5 \text{ Kg}$

1st equation $\Rightarrow \quad M_{TOT} \frac{dV}{dt} = T - M_{TOT}\, g$

$$a_1 = \frac{dV}{dt} = \frac{T}{M_{TOT}} - g = \frac{9900000 \text{ N}}{744'262.5 \text{ Kg}} - 9.81 \text{ m/s}^2 = 3.49 \text{ m/s}^2$$

2. Before the central CBC throttles down

$M_2 = M_1 - \dot{m}t_2 = M_{tot} - \dot{m}t_2$

$\dot{m} = -\frac{dM}{dt} \Rightarrow dM = -\dot{m}\,dt$

$M_2 - M_1 = -\dot{m}t_2 + t_1$

$M_2 = 744'262.5 \text{ Kg} - 2461.4 \frac{\text{Kg}}{\text{s}} \cdot 50 \text{ s} =$

$M_2 = M_1 - \dot{m}t_2$

$M_2 = 621192.5 \text{ Kg}$

$t_2 = 50 \text{ s}$

$$a_2 = \frac{dV}{dt} = \frac{T}{M_2} - g = \frac{9900000 \text{ N}}{621192.5 \text{ Kg}} - 9.81 \text{ m/s}^2 = 6.13 \text{ m/s}^2$$

3. After the central CBC throttles down

$M_3 = M_2 = 621192.5 \text{ Kg}$

$T_3 = 6'600'000 \text{ N} + 50\% \text{ of } 3'300'000 \text{ N} = 8250000 \text{ N}$

$$a_3 = \frac{dV}{dt} = \frac{T_3}{M_3} - g = \frac{8250000 \text{ Kg}}{621192.5 \text{ Kg}} - 9.81 \text{ m/s}^2 = 3.47 \text{ m/s}^2$$

4. Before the outboard CBCs shut down

$t_4 = 200 \text{ s}$

$$\dot{m}_4 = \frac{T_3}{c} = \frac{T_3}{g_0 I_{sp}} = \frac{8250000 \text{ N}}{4022.1 \text{ m/s}} = 2051.2 \frac{\text{Kg}}{\text{s}}$$

$$M_4 = M_3 - \dot{m}_4 t_4 = 621192.5 \text{ Kg} - 410240 \text{ Kg} = 210952.5 \text{ Kg}$$

$$a_4 = \frac{dV}{dt} = \frac{T_3}{M_4} - g = \frac{8250000 \text{ N}}{210952.5 \text{ Kg}} - g = 29.3 \text{ m/s}^2$$

5 After the central CBC has throttled back up to full power

$$M_5 = M_4 - \underbrace{2M_C}_{\text{Because the external CBCs are jettisoned}} = 210952.5 \text{ Kg} - 55941 \text{ Kg} = 155011.5 \text{ Kg}$$

$T_5 = T' = 3'300'000 \text{ N}$

$$a_5 = \frac{dV}{dt} = \frac{T_5}{M_5} - g = \frac{3'300'000 \text{ N}}{155011.5 \text{ Kg}} - 9.81 \text{ m/s}^2 = 11.5 \text{ m/s}^2$$

6 Burnout of the central CBC

$$\dot m_6 = \frac{T_6}{g_0 I_{sp}} = \frac{3'300'000\ N}{4022.1\ m/s} = 820.5\ \frac{Kg}{s}$$ $T_6 = T' = 3'300'000\ N$

after 50 s $= t_2$ after $t_4 = 200s$ $\dot m_{central} = \frac{50\%\, T'}{g_0 I_{sp}} = \frac{1650\,000\ N}{4022.1\ m/s} = 410.2\ \frac{Kg}{s}$

$M' = \dot m_6 t_2 = 41025\ Kg$ $M'' = \dot m_{central}\, t_4 = 82040\ kg$

$M_P = 205117\ Kg$ $M_{rem} = M_P - M' - M'' = 82052\ Kg$

$M_6 = M_5 - M_{rem} = M_5 - (M_P - M' - M'') = 155011.5\ Kg - 82052\ Kg = 72959.5\ Kg$

Verification

$M_6 = M_C + M_U + M_{2nd stage} = 27970.5\ Kg + 25\,000\ Kg + 20'000\ Kg = 72970\ Kg$

$a_6 = \frac{dV}{dt} = \frac{T_6}{M_6} - g_0 = \frac{3'300'000\ N}{72959.5\ Kg} - 9.81\ m/s^2 = 35.4\ m/s^2$

? d) Compute the velocity and the height of the rocket at the following moments

1. Throttle down of centre CBC $t_2 = 50s$

$V_2 = c \ln \Lambda_2 - g_0 t_2$ $\Lambda_2 = \frac{M_{TOT}}{M_{TOT} - \dot m t_2} = \frac{M_{TOT}}{M_2} = \frac{744'262.5\ Kg}{621\,192.5\ Kg} = 1.2$

$V_2 = g_0 I_{sp} \ln \Lambda_2 - g_0 t_2 = 4022.1\ m/s \cdot \ln(1.2) - 9.81\ m/s^2 \cdot 50s = 733.3\ m/s - 490.5\ m/s = \underline{242.8\ m/s}$

$$h_2 = \frac{c^2}{g_0 \psi_0}\left\{\left[1 - \frac{1}{\Lambda_2}(\ln \Lambda_2 + 1)\right] - \frac{1}{2\psi_0}\left(1 - \frac{1}{\Lambda_2}\right)^2\right\}$$ $\psi_0 = \frac{T}{M_{TOT} g_0} = \frac{9'900'000\ N}{7301215.1\ N} = 1.4$

$h_2 = \frac{(g_0 I_{sp})^2}{g_0 \psi_0} \ldots = \frac{16177288.4\ m^2/s^2}{13.7\ m/s^2}\left\{[1 - 0.83(0.18+1)] - 0.36(1-0.83)^2\right\} =$

$= 1180823.97\ m \{\underbrace{0.0206 - 0.0104}_{0.0102}\} = 12044.4\ m \simeq \underline{12\ Km}$

2. Throttle back up of centre CBC

$t_4 = 200s$

$\Lambda_5 = \frac{M_2}{M_5} = \frac{621192.5\ Kg}{155011.5\ Kg} = 4$

$V_5' = g_0 I_{sp} \ln \Lambda_5 - g_0 (t_4) = 4022.1\ m/s \cdot 1.4 - 9.81\ m/s^2 \cdot 200\ s = 5630.9\ m/s - 1962\ m/s = \underline{3668.9\ m/s}$

$V_5 = V_5' + V_2 = 3911.7\ m/s$

$h_5' = \frac{(g_0 I_{sp})^2}{g_0 \psi_0}\left\{\left[1 - \frac{1}{\Lambda_5}(\ln \Lambda_5 + 1)\right] - \frac{1}{2\psi_0}\left(1 - \frac{1}{\Lambda_5}\right)^2\right\} = 1180823.97\ m\left\{[1 - 0.25(1.4+1)] - 0.36(1 - 0.25)^2\right\} =$

$= 1180823.97\ m \{\underbrace{0.4 - 0.203}_{0.1975}\} = 233213\ m \simeq 233\ Km$ $h_5 = h_5' + h_2 + t_4 \cdot V_2 \simeq 294\ Km$

3 Burnout of centre CBC

$t_{fin} = \frac{M_{rem}}{\dot m_6} = \frac{82052\ Kg}{820.5\ Kg/s} = 100\ s$

$$\Lambda_6 = \frac{M_5}{M_6} = \frac{155011.5\ Kg}{72959.5\ Kg} = 2.1$$

$$V'_6 = g_0 I_{sp} \ln \Lambda_6 - g_0 (t_6) = 4022.1\ m/s \cdot 0.75 - 9.81\ m/s^2 \cdot 100\ s = 3016.6\ m/s - 981\ m/s = 2035.6\ m/s$$

$$V_6 = V'_6 + V_5 = 5936.3\ m/s$$

$$h'_6 = \frac{(g_0 I_{sp})^2}{g_0 \psi_0} \left\{ \left[1 - \frac{1}{\Lambda_6}(\ln \Lambda_6 + 1)\right] - \frac{1}{2\psi_0}\left(1 - \frac{1}{\Lambda_6}\right)^2 \right\} = 1180823.97\ m \left\{ \left[1 - 0.48 \cdot (0.75 + 1)\right] - 0.36\,(1 - 0.48)^2 \right\} =$$

$$= 1180823.97\ m \left\{ \underbrace{0.16 - 0.097}_{0.063} \right\} = 74392\ m \cong 74\ Km$$

$$h_6 = h'_6 + h_5 + t_6 \sin V_5 \cong 759\ Km$$

?e) Compute the maximum height achieved by the combined second stage and payload, assuming the second stage is not ignited

$$t_{coasting} = \frac{V_6}{g_0} = \frac{5817.33\ m/s}{9.81\ m/s^2} = 593\ s$$

$$h'_c = \frac{(V_6)^2}{2 g_0} = 1796109\ m \cong 1'796\ Km$$

$$h_{c\,max} = h'_c + h_6 \cong 2555\ Km$$