Equivariant Filter Theory

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1 Theory

I will be referring to the notations and conventions used in [1].

1.1 Introducing ϕ

We define a right action of a Lie Group $\mathbf{G}$ on a manifold M which is a smooth map $\phi\colon \mathbf{G} \times M \to M$, such that,

$$\phi(Y, \phi(X, \xi)) = \phi(XY, \xi) \tag{1}$$
$$\phi(id, \xi) = \xi \tag{2}$$

for any $X, Y \in \mathbf{G}$ and any $\xi \in M$.
An action ϕ is called transitive if, for any $\xi^1, \xi^2 \in M$, there exists $X \in \mathbf{G}$ such that $\phi(X; \xi^1) = \xi^2$.

1.2 Introducing f

There is a function $f\colon \mathbb{V} \to \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the space of vector fields over M and $\mathbb{V}$ denotes the velocity space, in our case $\mathfrak{se}(3)$. f satisfies the following relation.

$$\dot{\xi} = f_U(\xi) \tag{3}$$

1.3 Defining $d_*\phi$

We define a function $d_*\phi\colon \mathbf{G} \times \mathfrak{X}(M) \to \mathfrak{X}(M)$,

$$d_*\phi(X, f) := d\phi_X \cdot f \circ \phi_X^{-1} \tag{4}$$

This function is a linear map on $\mathfrak{X}(M)$ for a fixed $\mathbf{G}$.

1.4 Problem description

The output measurement is given by the function $h\colon M \to N$, where N is called the output space.
There exists a smooth, transitive, right group action ϕ (as introduced earlier) of $\mathbf{G}$ on M.
A lift is a map $\Lambda\colon M \times \mathbb{V} \to \mathfrak{g}$ satisfying,

$$D_{X|id}\phi(X, \xi)[\Lambda(\xi, U)] = f_U(\xi) \tag{5}$$

1.5 System Equivariance

Does there exist a function $\psi\colon \mathbf{G} \times \mathbb{V} \to \mathbb{V}$ such that,

$$f_{\psi(X,U)} = d_*\phi(X, f_U)$$

Not always. But the system can be extended such that ψ exists. We can take a larger space $V \geq \mathbb{V}$ such that $f\colon V \to \mathfrak{X}(M)$.
We assume that such a ψ exists. In this case we say that the system is **equivariant**.

1.6 Output Equivariance

Does there exist an action $\rho\colon \mathbf{G} \times N \to N$ such that,

$$\rho(X, h(\xi)) = h(\phi(X, \xi))$$

Sometimes. However, the problem we consider will have this property.

1.7 Lift Equivariance

Suppose the system is equivariant, equivariant lift always exists, and can be constructed easily. It satisfies two properties:

1. Lift condition:

$$D_{X|id}\phi(X, \xi)[\Lambda(\xi, U)] = f_U(\xi) \tag{6}$$

2. Lift equivariance:

$$\Lambda(\phi(X, \xi), \psi(X, U)) = Ad_X^{-1}\Lambda(\xi, U) \tag{7}$$

1.8 Observer Architecture

The kinematics of the lifted system are as follows,

$$\xi(t) = \phi(X(t), \xi^\circ) \tag{8}$$

$$\dot{X} = X\Lambda(\phi(X, \xi^\circ), U), \quad \phi(X(0), \xi^\circ) = \xi(0) \tag{9}$$

Define the estimate of the state as $\hat{X}$ which follows,

$$\dot{\hat{X}} = \hat{X}\Lambda(\phi(\hat{X}, \xi^\circ), U) + \Delta\hat{X}, \quad \hat{X}(0) = id \tag{10}$$

where Δ is an innovation term which remains to be chosen.

1.9 Error Dynamics

We can't measure the error directly, so we define the global state error as,

$$e := \phi(\hat{X}^{-1}, \xi) \tag{11}$$

Our aim is to drive $e \to \xi^\circ$.
We define the origin velocity,

$$v^\circ := \psi(\hat{X}^{-1}, v) \in V \tag{12}$$

Now we derive a relation,

$$\begin{aligned}
&e = \phi(\hat{X}^{-1}, \phi(X, \xi^\circ)) = \phi(X\hat{X}^{-1}, \xi^\circ) = \phi_{\xi^\circ}(X\hat{X}^{-1})\\
&\dot{e} = d\phi_{\xi^\circ}(\dot{X}\hat{X}^{-1} - X\hat{X}^{-1}\dot{\hat{X}}\hat{X}^{-1})\\
&\dot{e} = d\phi_{\xi^\circ}(X\Lambda(\xi, U)\hat{X}^{-1} - X\hat{X}^{-1}\hat{X}\Lambda(\hat{\xi}, U)\hat{X}^{-1} - X\hat{X}^{-1}\Delta)\\
&\dot{e} = d\phi_{\xi^\circ}(X\hat{X}^{-1}\hat{X}\Lambda(\xi, U)\hat{X}^{-1} - X\hat{X}^{-1}\hat{X}\Lambda(\hat{\xi}, U)\hat{X}^{-1} - X\hat{X}^{-1}\Delta)\\
&\dot{e} = d\phi_{\xi^\circ}X\hat{X}^{-1}(\hat{X}\Lambda(\xi, U)\hat{X}^{-1} - \hat{X}\Lambda(\hat{\xi}, U)\hat{X}^{-1} - \Delta)\\
&\dot{e} = d\phi_e(\hat{X}(\Lambda(\xi, U) - \Lambda(\hat{\xi}, U))\hat{X}^{-1} - \Delta)\\
&\dot{e} = d\phi_e Ad_{\hat{X}}[\Lambda(\phi(\hat{X}, \phi(\hat{X}^{-1}, \xi)), \psi(\hat{X}, \psi(\hat{X}^{-1}, U)) - \Lambda(\phi(\hat{X}, \xi^\circ), \psi(\hat{X}, \psi(\hat{X}^{-1}, U)))] - d\phi_e\Delta\\
&\dot{e} = d\phi_e[\Lambda(e, v^\circ) - \Lambda(\xi^\circ, v^\circ)] - d\phi_e\Delta
\end{aligned}$$

1.10 Linearisation

We will fix a local coordinate chart $\epsilon\colon U \to \mathbb{R}^m$, where $m = dim(M)$ and U is a neighbourhood of the fixed origin ξ°.

$$\epsilon = \epsilon(e), \quad \epsilon(\xi^\circ) = 0$$

The linearised dynamics of $e(t)$ about the origin is as follows,

$$\dot{e} = d\phi_e[\tilde{\Lambda}_{\xi^\circ}(e, v^\circ)] - d\phi_e\Delta \tag{13}$$

$$\dot{\epsilon} = d\epsilon \cdot d\phi_e[\tilde{\Lambda}_{\xi^\circ}(\epsilon^{-1}(\epsilon), v^\circ)] - d\epsilon \cdot d\phi_e\Delta \tag{14}$$

$$\tag{15}$$

$\tilde{\Lambda}_{\xi^\circ}(\xi^\circ, v^\circ) = 0$, hence we linearise the first term about $\epsilon = 0$

$$\begin{aligned}
d\epsilon \cdot d\phi_e[\tilde{\Lambda}_{\xi^\circ}(\epsilon^{-1}(\epsilon), v^\circ)] &\approx 0 + d\epsilon \cdot D_{e|\xi^\circ} d\phi_e \tilde{\Lambda}_{\xi^\circ}(e, v^\circ) \cdot d\epsilon^{-1}[\epsilon] && (16)\\
&= d\epsilon \cdot D_{e|\xi^\circ} D_{E|id}\phi(E, e)\tilde{\Lambda}_{\xi^\circ}(\xi^\circ, v^\circ) \cdot d\epsilon^{-1}[\epsilon] + && (17)\\
&\quad d\epsilon \cdot D_{E|id}\phi(E, \xi^\circ) \cdot D_{e|\xi^\circ}\tilde{\Lambda}_{\xi^\circ}(e, v^\circ) \cdot d\epsilon^{-1}[\epsilon] && (18)\\
&= d\epsilon \cdot D_{E|id}\phi(E, \xi^\circ) \cdot D_{e|\xi^\circ}\Lambda_{\xi^\circ}(e, v^\circ) \cdot d\epsilon^{-1}[\epsilon] && (19)\\
&= A_t^\circ \epsilon && (20)
\end{aligned}$$

The time variation in the linearisation is solely due to the time dependence of the origin velocity $v^\circ(t)$.

Now we linearise the output,

$$y = h(\xi) = h(\phi(\hat{X}, e)) = h(\phi(\hat{X}, \epsilon^{-1}(\epsilon))) \tag{21}$$

Consider the relation,

$$h(e) = h(\phi(\hat{X}^{-1}, \xi)) = \rho(\hat{X}^{-1}, h(\xi)) \tag{22}$$

We use this relation to use as error in our Filter equations.
We linearise the output error, using a local coordinate chart δ,

$$\delta = \delta(h(e)) \tag{23}$$
$$\approx C^\circ \epsilon \tag{24}$$
$$C^\circ := D_{y|y^\circ}\delta(y) \cdot D_{e|\xi^\circ} h(e) \cdot D_{\epsilon|0}\epsilon^{-1}(\epsilon) \tag{25}$$

The output matrix C° is independent of the state and it is a constant matrix.

1.11 Equivariant Filter

The EqF with Equivariant Output dynamics are thus,

$$\dot{\hat{X}} = \hat{X}\Lambda(\phi(\hat{X}, \xi^\circ), U) + \Delta\hat{X}, \quad \hat{X}(0) = id \tag{26}$$
$$\Delta = D_{E|id}\phi_{\xi^\circ}(E)^\dagger d\epsilon^{-1}\Sigma C^{\circ\intercal} Q^{-1}\delta(\rho_{\hat{X}^{-1}}(h(\xi))) \tag{27}$$
$$\dot{\Sigma} = A_t^\circ\Sigma + \Sigma A_t^{\circ\intercal} + P - \Sigma C^{\circ\intercal} Q^{-1} C^\circ \Sigma, \quad \Sigma(0) = \Sigma_0 \tag{28}$$

References

[1] Robert Mahony, Pieter van Goor, and Tarek Hamel. *Equivariant Filter (EqF).*

[2] Robert Mahony, Jochen Trumpf, and Tarek Hamel. *Equivariant Systems Theory and Observer Design.*