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INVESTMENT IMPLICATIONS OF THE FRACTAL MARKET HYPOTHESIS

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The Efficient Market Hypothesis (EMH) has been repeatedly demonstrated to be an inferior — or at best incomplete — model of financial market behavior. The Fractal Market Hypothesis (FMH) has been installed as a viable alternative to the EMH. The FMH asserts that markets are stabilized by matching demand and supply of investors' investment horizons while the EMH assumes that the market is at equilibrium. A quantity known as the Hurst exponent determines whether a fractal time series evolves by random walk, a persistent trend or mean reverts. The time dependence of this quantity is explored for two developed market indices and one emerging market index. Another quantity, the fractal dimension of a time series, provides an indicator for the onset of chaos when market participants behave in the same way and breach a given threshold. A relationship is found between these quantities: the larger the change in the fractal dimension before breaching, the larger the rally in the price index after the breach. In addition, breaches are found to occur principally during times when the market is trending.

Keywords: Efficient market hypothesis; fractal market hypothesis; hurst exponent; fractal dimension.

JEL Classifications: C52, G11

1. Introduction

A central tenet of modern portfolio theory (MPT) is the concept of diversification: an assembly of several different assets can achieve a higher rate of return and a lower risk level than any asset in isolation (Markowitz, 1952). MPT has enjoyed remarkable success — it is still in wide use today (2018) — but it has also

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attracted a large and growing critical literature (e.g. Michaud (1989), Elton and Gruber (1997) and Mehdi and Hawley (2013) and references therein). An example of these criticisms is that MPT relies on the statistical independence of underlying asset price changes. This renders predictions of future market movements impossible. Sources of instability and market risk are also assumed to be exogenous under MPT. Were this true, the economic system would converge to a steady-state path, entirely determined by fundamentals and with no associated opportunities for consistent speculative profits in the absence of external price shocks. Empirical evidence, however, shows that prices are not only governed by fundamentals, but also by non-linear market forces and factor interactions which give rise to endogenous fluctuations.

Asset returns are also assumed to be normally distributed, but this omits (or assigns very low probabilities to) large return outliers. This is not an attribute of financial markets: they are characterized by long periods of stasis, punctuated by bursts of activity when volatility escalates — often rapidly and without warning. A consequence of the normal distribution assumption, then, is that these large market changes occur too infrequently to be of concern. Classical financial models, such as the efficient market hypothesis (EMH), embrace the precepts of MPT, so these abrupt market events are omitted from their frameworks.

The EMH with its three varieties (weak, semi-strong and strong) evolved from the MPT (Fama, 1965). Strong form efficiency is considered impossible in the real world (Grossman and Stiglitz, 1980) so only the weak and semi-strong forms of the EMH are empirically viable: both take for granted what Samuelson (1965) proved: that future asset price movements are determined entirely by information not contained in the price series; they *must* follow a random walk (Wilson and Marashdeh, 2007). The literature is, however, replete with evidence that weak and semi-strong forms of efficiency are inaccurate descriptions of financial markets (for example, Jensen (1978), Schwert (2003) and Zunino *et al.* (2008)), so alternative descriptions must be sought.

Two alternatives to *efficient* markets have evolved: the Adaptive (AMH) and Fractal (FMH) market hypotheses. The former offers a biological assessment of financial markets — specifically an evolutionary framework in which markets (and market agents: assets and investors) adapt and evolve dynamically through time. This evolution is fashioned by simple economic principles which, like natural selection, punish the unfit (through extinction) and reward the fit (through survival) as agents compete and adapt — not always optimally (Farmer and Lo, 1999; Farmer, 2002; Lo, 2002, 2004, 2005). Survival is paramount, even if that requires temporarily abandoning profit and utility maximization. Unlike the EMH, the AMH allows for an unstable, dynamic risk/reward relationship in which arbitrage opportunities arise and close depending on prevailing macro and microeconomic conditions which in turn affect the success of investment strategies.

The FMH relaxes the EMH's random walk requirement of asset prices. Hurst (1951, 1956) exploring the annual dependence of water levels on the river Nile noted that these ebbs and flows were not random — as expected — but rather displayed persistence and mean reversion. High levels one year tended to be followed by high levels the next (and vice versa). In other periods, sharp reversions toward the mean were recorded. Hurst's (1956) observations led to the formulation of the Hurst exponent, H, which effectively measures the degree of persistence prevalent in a time series: higher values suggest directional similarity (persistence) and lower values imply directional heterogeneity (reversion to the long-run mean: the further away from the mean, the stronger the tendency to return to it).

The relationship between these competing hypotheses and some of the tests used to determine their validity is summarized in Fig. 1.

The remainder of this paper proceeds as follows. The literature study in Sec. 2 provides a brief overview of salient features of the EMH. The EMH and the less-explored FMH, which addresses some of the former's shortcomings, are also discussed and compared here. Section 3 presents the data used to explore the FMH approach. If market movements are indeed described by fractal geometry, the implications for financial markets are profound. A diminishing fractal dimension, for example, indicates herding behavior until critical values are breached, leading to chaos. This section introduces the theoretical constructs of fractal geometry prevalent in financial time series. The results of the investigation on some global markets are presented in Sec. 4 as well as an empirical discussion on the implications of these results. Section 5 concludes.

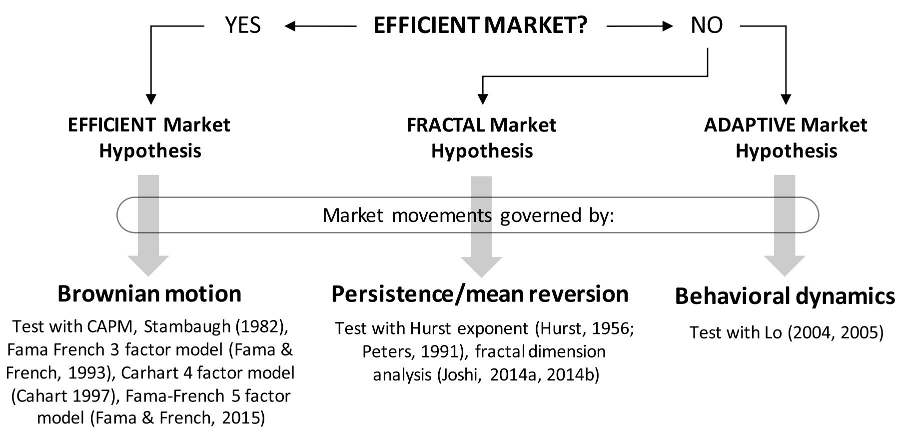


Figure 1. Relationship between efficient, FMH and AMH (Lo, 2012).

2. Literature Survey

The phrase "efficient market", introduced by Fama *et al.* (1965), originally defined a market which received, processed and adapted to new information quickly. A more contemporary definition, which considers rational processing of relevant information, asserts that all available information is reflected in an efficient market's asset prices (Fama, 1991). If the relevant information was free, prices would rise to their "fundamental level", but financial incentives arise if procurement costs are not zero. This is the strong form of the EMH (Grossman and Stiglitz, 1980). The economically realistic, semi-strong version of the EMH, argues that prices reflect information, but only to the point where the marginal costs of collecting the information outweigh the marginal benefits of acting upon it (through expected profits) (Jensen, 1978). The weak form of the EMH suggests that asset prices reflect all past asset price data so technical analysis is of no help in forming investment decisions.

The EMH generates several testable predictions regarding the behavior of asset prices and returns, so much empirical research is devoted to gathering important evidence about the informational efficiency of financial markets and establishing the validity — or otherwise — of the EMH. Some of the more significant assessments are summarized in Table 1.

MPT — which arose from the tenets of EMH — allows for the construction of efficient portfolios (those which generate the highest return possible for a given level of risk) while still maintaining the EMH assertion that *outperforming* the market on a risk-adjusted basis is impossible (Elton and Gruber, 1997).

Far from an orderly system of rational, cooperating investors, financial markets are instead characterized by nonlinear dynamic systems of interacting agents who rapidly process new information. Investors with different investment horizons and

Table 1. EMH predictions and empirical evidence.

Prediction	Empirical evidence	Sources
Asset prices move as random walks over time	Approximately true. However, small positive autocorrelation for short-horizon (daily, weekly and monthly) stock returns Fragile evidence of mean reversion in stock prices at long horizons (3–5 years)	Poterba and Summers (1988); Fama and French (1992); Campbell *et al.* (1997)
New information rapidly incorporated into asset prices	New information usually incorporated rapidly into asset prices, with some exceptions	Chan *et al.* (1996); Fama and French (1998)

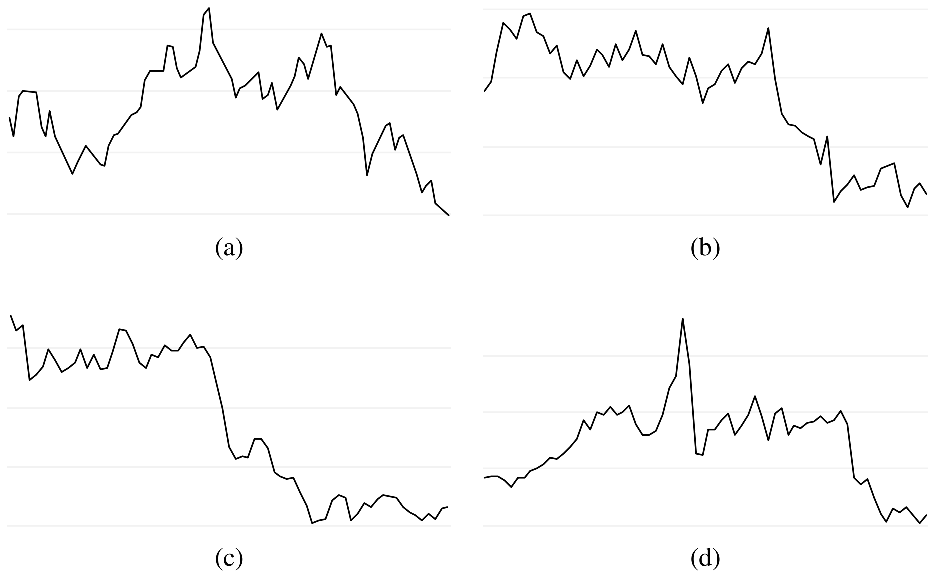
Table 1. (*Continued*)

Prediction	Empirical evidence	Sources
Current information cannot be used to predict future excess returns	Short run, shares with high returns continue to produce high returns (*momentum effects*) Long run, shares with low price-earnings ratios, high book-to-market-value ratios, and other measures of "value" outperform the market (*value effects*) FX market: current forward rate predicts excess returns (it is a biased predictor of future exchange rates)	De Bondt and Thaler (1985); Fama and French (1992); Jegadeesh and Titman (1993); Lakonishok *et al.* (1994); Goodhart (1988)
Technical analysis should provide no useful information	Although technical analysis is in widespread use in financial markets, there is contradictory evidence about whether it can generate excess returns	Levich and Thomas (1993); Osler and Chang (1995); Neely *et al.* (1997); Allen and Karjalainen (1999)
Fund managers cannot systematically outperform the market	Approximately true. Some evidence that fund managers *can* systematically underperform market	Lakonishok *et al.* (1992); Brown and Goetzmann (1995) Kahn and Rudd (1995)
Asset prices remain at levels consistent with economic fundamentals (i.e. they are not misaligned)	At times, asset prices appear to be significantly misaligned, for extended periods	Meese and Rogoff (1983); De Long *et al.* (1990)

holding different market positions employ this information in different ways. Considerable price fluctuations are observed, and these are indistinguishable or "invariant" on different time scales, as illustrated in Fig. 2 which demonstrates this phenomenon for crude oil prices using 70 daily, weekly, monthly and quarterly prices. It is impossible to say which of these with the axes (deliberately, in this case) is unlabeled.

This self-similarity implies market price persistence which would not be observed if returns were indeed independently and identically distributed, as postulated under the EMH. Further evidence of market persistence is shown by prices which deviate from their fundamentals for prolonged periods, and by a greater amount than allowed by the EMH (Carhart, 1997).

These empirical facts have created the need for a more realistic description of market movements than that described by the EMH — a need which was first satisfied by Mandelbrot (1977) who argued that *fractals* (geometric shapes, parts of



Source: Author's calculations.

Figure 2. (a) Daily, (b) weekly, (c) monthly and (d) quarterly crude oil prices measured over 70 periods in each case. Without time-axis labels, these series trace a geometric pattern which appears indistinguishable across different timescales.

which can be identified and isolated, each of which demonstrates a reduced-scale version of the whole) provided such a realistic market risk framework. Prices generated from simulated scenarios based on these fractal models reflect more realistic market activity (Joshi, 2014a; Somalwar, 2016).

The quantification of self-similar structures is non-trivial: an analogy usually invoked in the literature is that of the changing length of a coastline, depending on the ruler used to measure it Feder (1988) and Cajueiro and Tabak (2004a). Differences in estimation arise when line segments (as characterized by a ruler) are used to measure lengths of nested, self-similar structures (Anderson and Noss, 2013). The fractal nature of financial markets has led to the formulation of the FMH which replicates patterns evident in calm markets (predicted by MPT) as well as highly turbulent trading conditions (not predicted by MPT). The FMH and fractal price models may also be calibrated to replicate market price accelerations and collapses, key features of heteroscedastic volatility.

The principal differences between the EMH and the FMH are summarized in Table 2. Note that all the assumptions in the EMH column are false, whilst those in the FMH column are true.

Table 2. Summary of differences between the EMH and the FMH.

EMH	FMH
Return distribution is Normal (Gaussian)	Return distribution is non-Normal (non-Gaussian)
Stationary process (distribution mean does not change)	Non-stationary process (mean of distribution changes)
Returns have no memory (no trends)	Returns have memory (trends)
No repeating patterns at any scale	Many repeating patterns at all scales
Continuously stable at all scales	Possible instabilities at any scale

The FMH assumes that price changes evolve according to *fractional* Brownian motion, a feature quantified by the Hurst exponent. Hurst (1956) explored long-range time series component dependences and formulated the Hurst exponent, H, which records both the level of autocorrelation of a series and estimates the rate at which these autocorrelations diminish as the time delay between pairs of values increases. The range of $H \in [0, 1]$. The EMH is based upon standard Brownian motion processes which assume that prices evolve by random walks, i.e. $H = 0.5$. A natural consequence follows from this framework: forecasting future price movements is impossible because price movements are independent and exhibit no autocorrelation, thus technical analysis provides no assistance to investors. Deviations from $H = 0.5$ indicate autocorrelation which violates a key principle of the EMH. The finite nature of financial time series allows for $H \neq 0.5$, so this possibility must be accounted for (Morales *et al.*, 2012). Table 3 records the differences in time series depending on subranges of H: Figure 3 shows different time series for three sub-regions of H.

The literature exploring the Hurst exponent in finance and its relationship with the EMH is rich. Using daily data from both emerging and developed market indices spanning 10 years (January 1992–December 2002), Cajueiro and Tabak (2004a, b) calculated $H(t)$, the time-varying H. For the emerging markets $H > 0.5$, but the long-term trend was towards $H = 0.5$, indicating increasing efficiency over the observation period. Developed markets' H was not statistically different from 0.5. The results for both markets were confirmed by Di Matteo (2007) who used 32 global market indices and Wang *et al.* (2010) who used daily data to explore the efficiency of Shanghai stock market.

Grech and Mazur (2004) employed H to forecast market crashes. Three such crashes (1929 and 1987 in the US and 1998 in Hong Kong) were investigated using two years of daily data prior to the relevant crash in each case. *Before* each crash, H decreased sharply, an indication of vanishing trends and increasing volatility while *during* each crash, H *increased* significantly, a sign of enhanced inefficiency. Using

Table 3. Characteristics of time series dependency on H.

Range	$H \in [0, 0.5)$ 0.0 — 0.5	$H \approx 0.5$ $\langle 0.5 \rangle$	$H \in (0.5, 1]$ 0.5 — 1.0
Auto-covariance	$< 0 \forall$ lags	$= 0 \forall$ lags	$> 0 \forall$ lags
Behavior	Anti-persistent	Brownian	Persistent
Statistical interpretation	Decrements (increments) more likely to be proceeded by increments (decrements)	Decrements/increments equally likely	Increments (decrements) more likely to be proceeded by increments (decrements)
Character	Reverts to the mean more frequently than a random one	Random motion	Exhibit long-memory and "trends" and "cycles" of varying length
Sources	Barkoulas *et al.* (2000); Kristoufek (2010)	Osborne (1959)	Mandelbrot and Van Ness (1968)

daily data from the Polish stock market, Grech and Pamuła (2008) reached the same conclusions.

Alvarez-Ramirez *et al.* (2008) used daily data spanning 60 years from the S&P 500 and Dow Jones indices and found that H displayed erratic dynamic time

(a) (b) (c)

Source: Author's calculations.

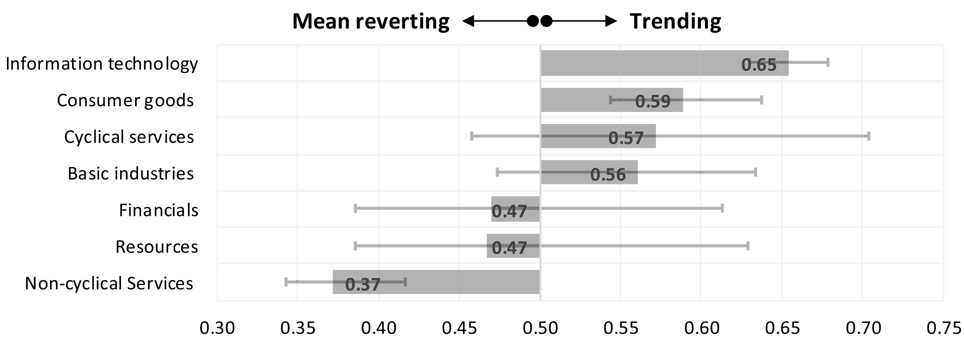
Figure 3. S&P 500 price series for 18-month period in which (a) $0 < H < 0.5$ (mean reverting), (b) $H \approx 0.5$ (Brownian motion) and (c) $0.5 < H < 1.0$ (trending).

dependency. A time-varying evolution of market efficiency was observed with alternating low and high persistent behavior, i.e. $H > 0.5$ in both cases, with different magnitudes.

The consequences for market efficiency of financial crises were explored by Lim *et al.* (2008) who found that the 1997 Asian crisis dramatically reduced the efficiency of global stock markets. Within three years, however, efficiency had recovered to pre-crisis levels. The highest level of market efficiency was recorded during post-crisis periods, followed by pre-crisis periods. During crises, markets exhibit high inefficiency.

Using daily data for 19 months (January 1–July 07), Karangwa (2008) found $H \approx 0.5$ for the JSE. Note that Karangwa's (2008) study concluded before the onset of the 2008 credit crisis, so this event and its aftermath were not included in the analysis. Using monthly data for a longer period (i.e. August 1995–August 2007), Karangwa (2008) found $H = 0.58$. In a more recent study, Ostaszewicz (2012) used two methods (Higuchi and absolute moments) to measure H using JSE price index data for both pre and post 2008 crisis periods and found $H > 0.5$ predominantly in the pre-2008 crisis period and $H < 0.5$ predominantly in the post-2008 crisis period. Chimanga and Mlambo (2014) investigated the fractal nature of the JSE and found $H = 0.61$ using daily data from 2000 to 2010. By sector, the values for the JSE are shown in Fig. 4.

Sarpong *et al.* (2016) found $H = 0.46$ for the JSE using daily data from 1995 to 2015 (thereby embracing the full period investigated by Chimanga and Mlambo, 2014). In addition, Sarpong *et al.* (2016) used the BDS test (Brock *et al.*, 1996) to verify that JSE price index data exhibit non-random chaotic dynamics rather than pure randomness. These results confirm those obtained by Smith (2008) who,



Source: Author's calculations.

Figure 4. Average Hs measured on various JSE sectors over the period 2000–2010. Error bars indicate maximum and minimum values obtained from individual shares within the relevant sector.

using four joint variance ratio tests, rejected the random walk hypothesis on the JSE.

Vamvakaris *et al.* (2017) examined the persistency of the S&P 500 index using daily data from 1996 to 2010 and found that crises affect investors' behavior only temporarily (< six months). In addition, the index exhibited high anti-persistency (an indication of investor "nervousness", $H < 0.5$) prior to periods of high market instability. Considerable fluctuations of H were observed with a roughly annual frequency and amplitude (from peak to trough) of 0.2 to 0.4. No prolonged trends of H were recorded.

3. Data and Methodology

3.1. *Data*

The data used to calibrate the FMH (via the estimation of the Hurst exponent) comprise 22.5 years (July 1995 to December 2017) of daily market index prices for developed (S&P 500, FTSE 100) and emerging market stock exchanges (the JSE). Three years (36 months) of daily index prices were used to determine H^{36}. The data sample was then rolled forward by one month and the next realization of the Hurst exponent calculated, i.e. H^{37}. This was repeated until the latest Hurst exponent in the data sample was calculated, i.e. end of December 2017, using the three years of data from January 2015 to December 2017.

This sample size was selected to include at least three full South African business cycles. This has been shown to be ≈ seven years (Botha, 2004; Thomson and van Vuuren, 2016). In addition, these data embrace a period of non-volatile growth (2003–2008), and considerable turbulence (1998–2000 (the Asian crisis and the dotcom crash) and 2008–2011 (the credit crisis)).

The same indices were used for the fractal dimension, D, analysis to establish whether breaching of a given D led to herding behavior (and a resulting collapse or rally in price). The fractal dimensions of gold and oil prices were investigated over the same period for calibration purposes and to confirm earlier work undertaken by Joshi (2014a, b).

3.2. *Methodology*

Standard Brownian motion describes the trajectory of a financial asset price, S_t, through time by integrating the differential equation (Areerak, 2014):

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{1}$$

where S_t is a financial asset price at time t, dS_t is the infinitesimal change in the asset's price over time dt, μ is the expected rate of return that the asset will earn over dt and σ the expected volatility. dW_t is a Weiner process described by $\varepsilon\sqrt{t}$, where ε is a random number drawn from a standard normal distribution. The solution of this differential equation is

$$S_t = S_0 \exp\left(\mu t - \frac{\sigma^2}{2} t + \sigma W_t\right), \tag{2}$$

where S_0 is the initial asset price. In principle, S_t describes the asset's price trajectory through time, but in practice, many features of financial assets are not captured by this formulation. Cont (2001) assembled a group of stylized statistical facts which describe several financial assets. While not exhaustive, the following list includes the empirical evidence that financial asset returns are characterized by:

(1) insignificant linear autocorrelations (Cont, 2001),
(2) heavy tails and conditional heavy tails (even after adapting returns for volatility clustering) of unconditional return distributions which can be described by power laws or Pareto-like tails with finite tail indices (Horák and Smid, 2009),
(3) asymmetric gains and losses — larger drawdowns than upward movements (Horak and Smid, 2009),
(4) different distributions at different timescales. Known as "aggregational Gaussianity" the return distribution approaches a normal distribution as $t \to \infty$ (Cont, 2001),
(5) a high degree of return variability at all timescales (Di Matteo *et al.*, 2005),
(6) homoscedasticity or volatility clustering: the clustering of high-volatility events and low-volatility events in time (Cont, 2001),
(7) long-range dependence of return data, characterized by the slow decay (as a function of time) of the autocorrelation of absolute returns, often as a power law with exponent $0.2 \leq \beta \leq 0.4$ (Cont, 2001),
(8) negative correlation of the asset's volatility and its returns (Chordia *et al.*, 2008),
(9) higher-than-expected correlation between trading volume and volatility (Blume *et al.*, 1994) and
(10) time scale asymmetry: fine-scale volatility is better predicted than coarse-grained measures rather than the other way around (Di Matteo *et al.*, 2005).

These features are generally *not* captured by standard Brownian motion, which has led to the development of *fractional* Brownian motion. In this formulation, (1) becomes

$$dS_t = S_t(\mu dt + \sigma dZ_t), \tag{3}$$

where $dZ_t = \varepsilon\sqrt{t^{2H}}$ and H ($0 \leq H \leq 1$) is the Hurst parameter. The respective Wiener processes (dW_t in (1) and dZ_t in (3)) have many features in common, but also exhibit strikingly different properties. The Wiener process dZ_t is self-similar in time, while dW_t is self-affine (Mandelbrot, 1977; Feder, 1988). Fractional Brownian Motion, for example, captures dependence among returns. A generalized solution for (3) is

$$S_t = S_0 \exp\left(\mu t - \frac{\sigma^2}{2} t^{2H} + \sigma Z_t\right). \tag{4}$$

If $0 \leq H < 0.5$, changes in S_t are negatively correlated and if $0.5 \leq H < 1$, they are positively correlated. Correlation also increases with H (Shevchenko, 2014).

3.2.1. *Hurst exponent, H*

A variety of methods for estimating H are discussed in the literature, each with associated advantages and drawbacks. Approaches include rescaled-range analysis (proposed by Hurst (1951) himself), wavelet transformations (Simonsen and Hansen, 1998), neural networks (Qian and Rasheed, 2004) and the visibility-graph approach (Lacasa *et al.*, 2009). The most commonly used methodology is rescaled-range analysis, and this will be adopted here as it is also the technique used to determine the fractal dimension, D, also known as the Hausdorff–Besicovitch dimension (Hausdorff, 1919; Manstavičius, 2007).

Hurst (1951) asserted that the variation of fractal time series is related to the horizon over which the time series are assessed by a power law relationship. Starting with a de-meaned time series (to ensure stationarity), define Y_k as the sum of k increments of this series, extending to n increments. The adjusted range (the "distance" the series travels over n time increments) is defined as the difference between the maximum and the minimum of the series:

$$Y_1 Y_2, \ldots, Y_n \quad \text{or} \quad R_n = \max(Y_k) - \min(Y_k), \quad 1 < k < n.$$

If Y is a time series characterized by Gaussian increments (i.e. a random walk), then this range increases with the product of the series' standard deviation and $\sqrt{n}$. Hurst (1951) generalized this relationship to

$$\left(\frac{R}{\sigma}\right)_n = cn^H, \tag{5}$$

where σ is the standard deviation (i.e. the realized volatility) of the stationary time series' n observations and H is the Hurst exponent. Rescaling the series by determining the quotient of the range and σ measures time series that do not exhibit finite variance (or fractals). This method makes no assumption regarding the underlying distribution of increments; only how they scale with time, as measured by H.

The theoretical value of the positive constant, c, is

$$c = \sqrt{\frac{\left(2H \cdot \Gamma\left(\frac{3}{2} - H\right)\right)}{\Gamma\left(\frac{1}{2} + H\right) \cdot \Gamma(2 - 2H)}}, \tag{6}$$

where $\Gamma(\cdot)$ is the Gamma function.

The H exponent captures the degree of persistence in a time series, irrespective of the time scale over which it is measured. For a time series with an observed $H > 0.5$ implies that a large value of the series in one period is likely to be followed by a larger value in a later period (the reverse applies if $H < 0.5$ so such a series is mean reverting). H may be calculated using ordinary least squares regression after taking the logarithm of (5):

$$\ln\left(\frac{R}{\sigma}\right)_n = \ln(c) + H \cdot \ln(n).$$

Using many different increments, n, and regressing $\ln(\frac{R}{\sigma})$ on $\ln(n)$ gives a straight line with $c = \exp(y\text{−intercept})$ — see (6) and $H =$ regression line slope.

Peters (1991) provides the following process for determining H.

Using a time series of $N + 1$ prices $\{P_t\}$, calculate the time series of N returns, $\{X_t\}$ such that $X_t = \ln(P_t/P_{t-1})$. Divide the return time series (length N) into A contiguous subperiods, each of length n (so $A \cdot n = N$). Label each subperiod l_a with $a = 1, 2, 3, \ldots, A$. Label each element in l_a as N_k, where $k = 1, 2, 3, \ldots, n$. For each subperiod, calculate the mean: $e_a = \frac{1}{n}\sum_{k=1}^{n} N_{k,a}$ as shown in Fig. 5.

The time series of cumulative departures from the mean, for each subperiod l_a, are then

$$X_{k,a} = \sum_{i=1}^{k} (N_{i,a} - e_a) \quad \forall\, k = 1, 2, 3, \ldots, n.$$

Define the range as the difference between the maximum and minimum value of $X_{k,a}$ within each subperiod l_a: $R_{l_a} = \max(X_{k,a}) - \min(X_{k,a})$, where $1 < k < n$. The sample standard deviation, σ, for each subperiod l_a is

$$\sigma_{l_a} = \sqrt{\frac{1}{n}\sum_{k=1}^{n} (N_{k,a} - e_a^2)}.$$

A rescaled range, R_{l_a}/σ_{l_a} for each subperiod, l_a, is then determined, the average of which is

$$\left(\frac{R}{\sigma}\right)_n = \frac{1}{A}\sum_{a=1}^{n} \frac{R_{l_a}}{\sigma_{l_a}}.$$

The length n is then increased until there are only two subperiods ($= \frac{N}{2}$).

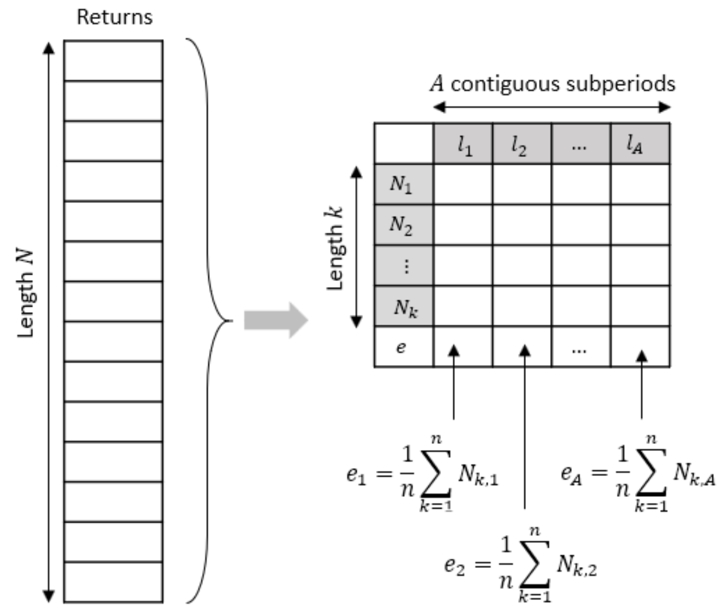
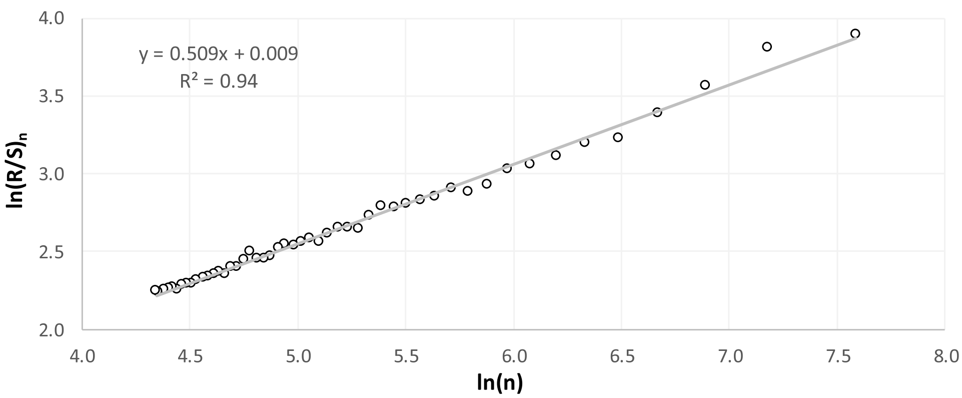


Figure 5. Applying Peters (1991) procedure for measuring e_as.

A least squares regression is performed, with $\ln(n)$ as the independent variable and $\ln(\frac{R}{\sigma})_n$ as the dependent variable. The slope of the regression is H and the y-intercept, c, as shown in Fig. 6 for a single three-year period, as an example. In the subsequent month, this process is followed again using three years of data prior to *that* month, and the next H and c are calculated.



Source: Author's calculations.

Figure 6. Regression results, March 2006–March 2009. $H = 0.509$ and $c = \exp(0.009) = 1.009$.

Lo and MacKinlay (1988, 2001) developed a test statistic to determine the statistical significance of H, i.e. whether the null-hypothesis (that $H = 0.5$) can be rejected or not. Known as the variance ratio test, this tests whether the time series is stationary (the variance of the series remains constant over time) or whether the series is trending (non-stationary). In this latter case, the series variance increases over time and has a unit root (Steffen *et al.*, 2014). No statically significant evidence for stationarity was found in any time series.

The evolution of H was examined using this technique over the two-decade period spanning January 1998 to January 2018. This reveals the characteristic nature of markets over this period: persistence, random walks or mean reversion. The fractal dimension, D, discussed in the next section, and H are related (8) although a different technique (7) is used to measure D in this case as it provides more granular (daily) estimates than (8). When D approaches and breaches a given threshold, the market tends to become chaotic, and given that the market exhibits a level of predictability after the onset of chaos (and the threshold breach), this tendency may be exploited by investors.

3.2.2. *Fractal dimension, D*

Joshi (2014a, b) described the fractal structure of a financial market using the definition of the fractal dimension, D and the rescaled range. The estimation of the time series' fractal dimension rests on the assertion that stock markets are complex adaptive systems — and thus embedded within them is an endogenous tipping point of instability (i.e. no explicit exogenous trigger is required).

Market stability rests on balancing supply and demand (liquidity) and the fractal structure of financial markets optimizes this liquidity. When different investors with many different investment horizons are all active in the market, the market is characterized by a rich fractal structure. Investors with different investment periods focus on different buy and sell signals: traders on technical data and momentum (short horizons) and pension funds on structural fundamentals and valuation (long horizons) for example. Sharp one day sell-offs will be interpreted by traders as a sell signal while pension funds may interpret this event as a buying opportunity. There is ample market liquidity: a large price move is not inevitable (Joshi, 2014a).

If the trader's horizon becomes dominant, however, and liquidity evaporates when sell orders far outweigh the number of buy orders, the fractal structure of the market collapses and violent price corrections become manifest. This is the endogenous tipping point and by monitoring the fractal dimension, discussed below, such thresholds may be monitored and employed as early indicators of market corrections. The lower the fractal dimension, the more unstable the market it measures.

Breaching a fractal dimension threshold of 1.25 triggers market corrections. This empirical limit appears identical across asset classes, geographies and time periods — it is not theoretically derived. It is impossible, however, to ascertain the *magnitude* of the subsequent adjustment or its direction, i.e. the ensuing correction may be > 0 or < 0 (Joshi, 2014b, 2017).

The measurement of D, the fractal dimension, is described by Joshi (2014a, b). If an asset's price is P_i on day i, its one-day log return, r_i, on day i is

$$r_i = \ln\left(\frac{P_i}{P_{i-1}}\right).$$

The scaling factor, n, is used to determine the n-day log return, $R_{i,n}$, on day i:

$$R_{i,n} = \ln\left(\frac{P_i}{P_{i-n}}\right),$$

as well as the scaled return, $N_{i,n}$, on day i:

$$N_{i,n} = \frac{\sum_{i-n}^{i} abs(r_i)}{abs\left(\frac{R_{i,n}}{n}\right)} = \frac{\sum_{i-n}^{i} abs\left(\ln\left(\frac{P_i}{P_{i-1}}\right)\right)}{abs\left(\frac{\ln\left(\frac{P_i}{P_{i-n}}\right)}{n}\right)},$$

and the scaled fractal dimension, $D_{i,n}$, on day i:

$$D_{i,n} = \frac{\ln(N_{i,n})}{\ln(n)} = \frac{\ln\left[\sum_{i-n}^{i} \frac{abs\left(\ln\left(\frac{P_i}{P_{i-1}}\right)\right)}{abs\left(\frac{\ln\left(\frac{P_i}{P_{i-n}}\right)}{n}\right)}\right]}{\ln(n)}. \tag{7}$$

The theoretical relationship between H and D is given by Schepers *et al.* (2002):

$$D = H - 2, \tag{8}$$

but (7) provides a much more granular (daily) estimate of D than (8) since H (in 8) is a *monthly* value, determined using (5).

4. Results and Discussion

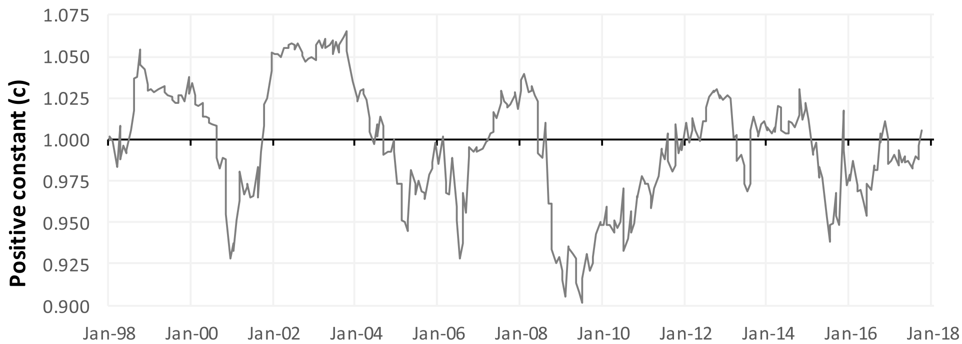
4.1. *Hurst exponent, H*

How H changes over time is useful to market participants: economists to ascertain the nature of the prevailing markets (persistent or mean-reverting), government strategists to establish the economy's current position in the business cycle, long-term investors to exploit market rallies and busts and short-term investors to exploit mean reversion conditions.

The rolling H was explored for three market indices: two in developed markets (US and UK) and one in an emerging market (South Africa). Figure 7(a) shows the results for the S&P 500: Cajueiro and Tabak (2004a, b) found similar results for developed markets ($H \approx 0.5$). Grech and Mazur (2004) found that H decreased sharply before market crashes showing a rapid decrease in trend. This is clearly shown for the September 2001 and September 2008 events — particularly for the latter. After this event, H increases steadily (over three years) from a market dominated by mean reverting to one characterized by random walk prices.

The rolling H for the FTSE 100 is shown in Fig. 8 on the same vertical and timescale as Fig. 7(a). Again, in line with the findings of Cajueiro and Tabak (2004a, b),

(a)



(b)

Source: Author's calculations.

Figure 7. Rolling (a) $H(t)$ and (b) $c(t)$ for the S&P 500 from January 1998–December 2017.

Source: Author's calculations.

Figure 8. Rolling $H(t)$ for the FTSE 100 from January 1998–December 2017 (note the same vertical scale as used for the S&P 500 for comparison).

$H \approx 0.5$. Unlike the results obtained by Grech and Mazur (2004), no sharp decrease of H was observed for the crisis which affected the S&P 500. The events of September 2001 occurred on US soil and so were more damaging to the US economy than the UK economy. The financial crisis of 2008, however, was global in impact and of considerable severity, yet the UK market appears to have been unaffected.

The FTSE 100 exhibits slight persistence ($H > 0.5$) between the time of the onset of the 2008 crisis and early 2012 when the sovereign crisis (which affected several European countries, including the UK albeit not as dramatically) began (Gärtner *et al.*, 2011) — see Fig. 8. At this point, the market changes gradually to become slightly mean reverting and has since followed a random walk since 2014. From 2012, the behavior of H for the FTSE 100 closely resembles that of the S&P 500 over the same period. These developed market results reinforce results obtained previously (e.g. Alvarez-Ramirez *et al.* (2008)).

The JSE All Share index displays behavior significantly different from that of developed market indices (Fig. 9). Until 2006, the JSE trends are strongly unaffected by the "dotcom" crisis in 00 or the events of September 2001. These results confirm and update those found by Karangwa (2008) and Chimanga and Mlambo (2014).

Between 2006 and the start of the 2008 financial crisis, market prices on the JSE evolve by random walk, but changes to a trending market rapidly at the onset of the crisis — the opposite of what is observed in developed markets. This could be because developing markets — in particular South Africa — largely escaped the consequences of the crisis because it occurred in a period to sustained growth for the country and strong fundamentals (Zini, 2008). South African financial institutions were also relatively robust and did not issue credit as freely and loosely

Source: Author's calculations.

Figure 9. Rolling $H(t)$ for the JSE All Share from January 1998–December 2017.

as their global counterparts (Mnyande, 2010). In a trend similar to global markets, JSE prices have become slightly mean-reverting or become random walks since 2012. Smith (2008) also found statistically significant results that $H < 0.5$, but over a shorter horizon and using daily (rather than monthly) data. Sarpong *et al.* (2016) using daily JSE index data spanning 20 years from 1995 to 2015 also found $H < 0.5$ prior to 2012 and $H \approx 0$ after 2012. The South African market was also found to be more "sectorized" or heterogenous with respect to H; different market sectors are characterized by different values of H and these values tend to persist over time.

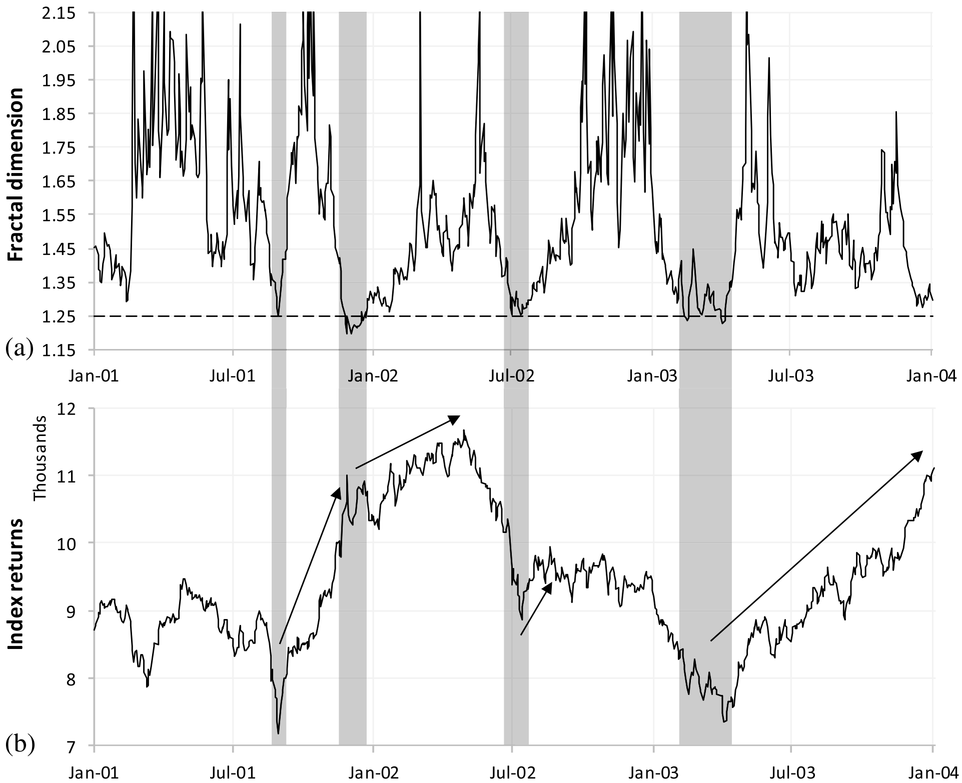
4.2. *Fractal dimension, D*

Analysis of D for the JSE All Share generated interesting results, previously unexplored. The majority (95%) of threshold breaches occur when $H > 0.5$. Only 5% of breaches occur during periods when the market exhibits periods of random walk or mean reversion behavior. This fact alone provides valuable information to market participants, but the percentage change in D — i.e. the rate of change or "speed" of the change of D also provides information about subsequent market movements.

A breach is classified as an event in which $D \rightarrow 1.25$ from "above", i.e. $D > 1.25$. There is no theoretical explanation for why this threshold value is significant. It does appear to be empirically consistent across markets, eras, geographies and asset types. When D breaches 1.25 from "below" (when preceding fractal dimension is < 1.25), this is *not* deemed to be a breach of interest. When threshold breaches were first identified, these occurred primarily during times when the South African market was trending, i.e. between 1998 and 2006 (the same results were obtained for the two developed market indices). Four such prominent breaches are shown in Fig. 10(a). The behavior of the market index over the same period is shown in Fig. 10(b), illustrating the impact of breaches. The shaded area links the timescales on Figs. 10(a) and 10(b) during the four breaches observed during this period.

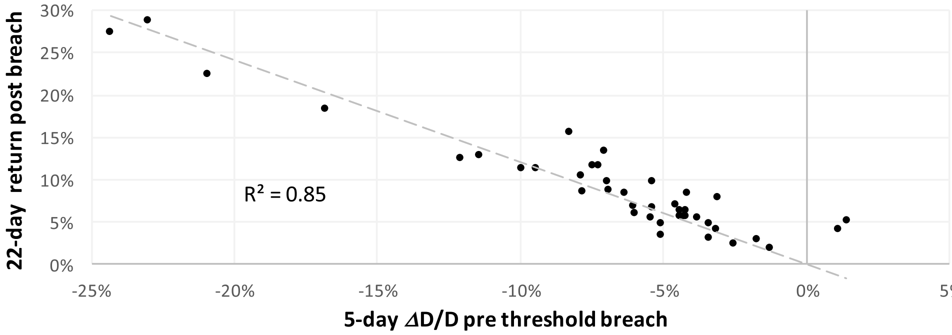
Next, the rate of change of D was determined over one trading week (five days) prior to the breach (over which time D decreases considerably and rapidly, but not instantaneously). One day is too short for a time to capture this time and over two weeks, D has often recovered to pre-breach levels, so one week appears to be an appropriate time to capture a significant, persistent decrease:

$$\frac{D_{t_0} - D_{t-5}}{D_{t-5}},$$



Source: Author's calculations.

Figure 10. (a) Fractal dimension, D over the three-year period between January 2001 and January 2004 showing several breaches (shaded), i.e. when $D \leq 1.25$ and (b) the JSE All Share index over the same period showing the behavior of index prices post breaching.



Source: Author's calculations.

Figure 11. Simple regression of one-month index return *post*-breach ($D \leq 1.25$) against a five-day pre-breach percentage change in fractal dimension ($\Delta D/D$). The period analyzed was July 1995 to December 2017, i.e. the full data sample.

where t_0 is the time D first breaches $D = 1.25$. After t_0 price changes tend to be significant (generally $> 5\%$), sustained and positive. One trading month (22 days) was selected over which to measure index price changes, i.e.

$$\frac{P_{t+22} - P_{t_0}}{P_{t_0}}.$$

Of course, price changes could be measured over shorter or longer periods than one month, and changes in D could be ascertained over shorter or longer periods than one week, but this approach provides a convenient, simple framework to analyze the effect of breaches on asset prices. The results are shown in Fig. 11.

Regression analysis indicates that the larger $\Delta D/D$ over the week prior to a breach, the larger the positive change — over a month — of the index price. $R^2 = 0.85$ indicates a statistically significant result. Similar results were obtained for the developed market indices. All time series used in this analysis were found to be stationary using the Augmented Dickey Fuller test.

The slope of the line is -1.2, so for a 1.0% five-day pre-breach drop in $\Delta D/D$, *ceterus paribus* leads to a 1.2% increase in the post-breach, one-month price series. These results could have significant consequences for investors, and could serve as a complementary tool to support, rationalize and justify investment decisions.

5. Conclusions and Suggestions

This paper examined the fractal properties of developed and developing market indices and examined the evolution of these fractal properties over a two-decade period. The FMH, using empirical evidence, posits that financial time series are self-similar, a feature which arises because of the interaction of investors with different investment horizons and liquidity constraints. The FMH presents a quantitative description of the way financial time series change; so after the testing of observed, empirical properties of financial market prices, forecasts may be formalized. Under the FMH paradigm, liquidity and the heterogeneity of investment horizons are key determinants of market stability, so the FMH embraces potential explanations for the dynamic operation of financial markets, their interaction and inherent instability. During "normal" market conditions, different investor objectives ensure liquidity and orderly price movements, but under stressed conditions, herding behavior dries up liquidity and destabilizes the market through panic selling.

This work also established a relationship between the change in a time series' fractal dimension (before breaching a threshold) and both the magnitude and direction of the subsequent change in the time series. This relationship was found to be prevalent during times of strong price persistence — a feature detectable by elevated Hurst exponents. These results suggest potential investment strategies.

Additional extensions could include more detailed calibration — perhaps by OLS — of the optimal pre-breach period for $\Delta D/D$ and optimal post-breach period for $\Delta P/P$. A comprehensive application of these results to other market indices and asset classes is also needed. Whether the relationship above holds for all asset classes (and, if so, whether the requirement that $H > 0.5$ is a necessary or sufficient prerequisite) also needs to be ascertained.

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