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Taking the **differential** (not the derivative) gives:

$$\mathrm{d}\ln y = \frac{\mathrm{d}y}{y} \tag{1}$$

This is the differential of $\ln y$ – not with respect to anything but just taking the differential.

To do S5 we will take differentials of this.

$$\begin{aligned} f(x,y,z) &= xyz + x^3 + y^4 + z^5 \\ \mathrm{d}f &= \mathrm{d}xyz + \mathrm{d}x^3 + \mathrm{d}y^4 + \mathrm{d}z^5 \\ \mathrm{d}f &= yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z + 3x^2\,\mathrm{d}x + 4y^3\,\mathrm{d}y + 5z^4\,\mathrm{d}z \\ \mathrm{d}f &= (yz+3x^2)\,\mathrm{d}x + (xz+4y^3)\,\mathrm{d}y + (xy+5z^4)\,\mathrm{d}z \end{aligned} \tag{2}$$

The partial derivative with some constant kept constant is equal to the normal derivative with the other variable kept constant. It is the exact same. Dividing the differential can give easier methods for finding this:

$$\begin{aligned} \left(\frac{\partial x}{\partial y}\right)_y &\equiv \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)_z \\ 0 &= (yz+3x^2)\frac{\mathrm{d}x}{\mathrm{d}y} + (xz+4y^3) + (xy+5z^4)\left(\frac{\mathrm{d}z}{\mathrm{d}y}\right)_z \\ 0 &= (yz+3x^2)\frac{\mathrm{d}x}{\mathrm{d}y} + (xz+4y^3) \\ \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)_z &= -\frac{xz+4y^3}{yz+3x^2} \\ \left(\frac{\partial x}{\partial y}\right)_z &= -\frac{xz+4y^3}{yz+3x^2} \end{aligned} \tag{3}$$

Note that $\mathrm{d}z\,y$ with z held constant is 0.

Consider a function $f = f(x,y,z) = 0$.

$$\begin{aligned} \mathrm{d}f &= \left(\frac{\partial f}{\partial x}\right)_{y,z}\mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)_{x,z}\mathrm{d}y + \left(\frac{\partial f}{\partial z}\right)_{x,y}\mathrm{d}z \\ 0 &= \left(\frac{\partial f}{\partial x}\right)_{y,z}\mathrm{d}x + \left(\frac{\partial f}{\partial y}\right)_{x,z}\mathrm{d}y + \left(\frac{\partial f}{\partial z}\right)_{x,y}\mathrm{d}z \\ 0 &= \left(\frac{\partial f}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial f}{\partial y}\right)_{x,z}(\mathrm{d}y\,y)_z + \left(\frac{\partial f}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial y}\right)_z \\ 0 &= \left(\frac{\partial f}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial y}\right)_z + \left(\frac{\partial f}{\partial y}\right)_{x,z}\times 1 + \left(\frac{\partial f}{\partial z}\right)_{x,y}\times 0 \\ -\left(\frac{\partial f}{\partial y}\right)_{x,z} &= \left(\frac{\partial f}{\partial x}\right)_{y,z}\left(\frac{\partial x}{\partial y}\right)_z \\ \left(\frac{\partial x}{\partial y}\right)_z &= -\frac{\left(\frac{\partial f}{\partial y}\right)_{x,z}}{\left(\frac{\partial f}{\partial x}\right)_{y,z}} \end{aligned} \tag{4}$$

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For the other two functions:

$$\left(\frac{\partial y}{\partial z}\right)_x = -\frac{\left(\frac{\partial f}{\partial z}\right)_{x,y}}{\left(\frac{\partial f}{\partial y}\right)_{x,z}}$$
$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{\left(\frac{\partial f}{\partial x}\right)_{y,z}}{\left(\frac{\partial f}{\partial z}\right)_{x,y}} \tag{5}$$

Repeating this for other functions gives similar results. All the denominators and numerators of the functions cancel out. This gives:

$$\left(\frac{\partial x}{\partial y}\right)_x \cdot \left(\frac{\partial y}{\partial z}\right)_x \cdot \left(\frac{\partial z}{\partial x}\right)_y = -1 \tag{6}$$

This generalises to functions with higher numbers of variables: if you multiply the partial derivatives of a n variable function together then it is equal to $(-1)^n$.
IE for any arbitrary function f:

$$f = f(x_1, x_2, x_3, x_4, x_5) \tag{7}$$

$$\left(\frac{\partial x_1}{\partial x_2}\right)_{x_3,x_4,x_5} \times \left(\frac{\partial x_2}{\partial x_3}\right)_{x_1,x_4,x_5} \times \left(\frac{\partial x_3}{\partial x_4}\right)_{x_1,x_2,x_5} \times \left(\frac{\partial x_4}{\partial x_5}\right)_{x_1,x_2,x_3} \times \left(\frac{\partial x_5}{\partial x_1}\right)_{x_2,x_3,x_4} = (-1)^5$$
$$\left(\frac{\partial x_1}{\partial x_2}\right)_{x_3,x_4,x_5} \times \left(\frac{\partial x_2}{\partial x_3}\right)_{x_1,x_4,x_5} \times \left(\frac{\partial x_3}{\partial x_4}\right)_{x_1,x_2,x_5} \times \left(\frac{\partial x_4}{\partial x_5}\right)_{x_1,x_2,x_3} \times \left(\frac{\partial x_5}{\partial x_1}\right)_{x_2,x_3,x_4} = -1 \tag{8}$$

Also note that when you partially derive a n variable function then you need to keep $n-2$ variables constant.

Ie if $f = f(x_1 \ldots x_n)$

Then the partial derivative of x_1 with respect to x_2 is:

$$\left(\frac{\partial x_1}{\partial x_2}\right)_{x_3 \ldots x_n} \tag{9}$$

Once you have the first derivative of a function then you can use the operator method to find the second derivative very quickly as shown below.

$$x = u\cos\theta - v\sin\theta$$
$$y = u\sin\theta + v\cos\theta \tag{10}$$

$$\frac{\partial f}{\partial u} = \left(\frac{\partial}{\partial u}\right) f = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial f}{\partial u} = \left(\frac{\partial}{\partial u}\right) f = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$
$$\frac{\partial f}{\partial v}\left(\frac{\partial}{\partial v}\right) f = \frac{\partial f}{\partial x}\frac{\partial u}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$
$$\frac{\partial f}{\partial v} = \left(\frac{\partial}{\partial v}\right) f = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta \tag{11}$$

So we know what the differential operator with respect to u does to a function. This means we can re-apply this to $\frac{\partial f}{\partial u}$ immediately giving us a function for $\frac{\partial^2 f}{\partial u^2}$ without further calculation.

$$
\begin{aligned}
\frac{\partial^2 f}{\partial u^2} &= \left(\frac{\partial}{\partial u}\right)\frac{\partial f}{\partial u} = \frac{\partial^2 f}{\partial x^2}\cos^2\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 f}{\partial x \partial y}\sin\theta\cos\theta \\
\frac{\partial^2 f}{\partial v^2} &= \left(\frac{\partial}{\partial v}\right)\frac{\partial f}{\partial v} = \frac{\partial^2 f}{\partial x^2}\sin^2\theta + \frac{\partial^2 f}{\partial y^2}\cos^2\theta - 2\frac{\partial^2 f}{\partial x \partial y}\sin\theta\cos\theta
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} &= (\sin^2\theta + \cos^2\theta)\frac{\partial^2 f}{\partial x^2} + (\sin^2\theta + \cos^2\theta)\frac{\partial^2 f}{\partial y^2} \\
\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}
\end{aligned} \tag{13}
$$