6 On relations

6.1 Basic exercises

1. Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$ and $C = \{x, y, z\}$.
   Let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\} : A \nrightarrow B$
   and $S = \{(b, x), (b, y), (c, y), (d, z)\} : B \nrightarrow C$.

   Draw the internal diagrams of the relations. What is the composition $S \circ R : A \nrightarrow C$?

2. Prove that relational composition is associative and has the identity relation as the neutral element.

   Let $f : A \nrightarrow B$, $g : B \nrightarrow C$, $h : C \nrightarrow D$.

   To prove the associativity of relational composition we must prove for arbitrary sets f, g, h that $\forall\, a \in A, d \in D : a(h \circ (g \circ f))d \iff a((h \circ g) \circ f)d$.

   $$\begin{aligned} a(h \circ (g \circ f))d &\iff \\ \exists\, c \in C : a(g \circ f)c \wedge chd &\iff \\ \exists\, b \in B, c \in C : afb \wedge bgc \wedge chd &\iff \\ \exists\, b \in B : afb \wedge b(h \circ g)d &\iff \\ a((h \circ g) \circ f)d & \end{aligned} \tag{1}$$

   So relational composition is associative as required.

3. For a relation $R : A \nrightarrow B$, let its opposite or dual relation $R^{\mathrm{op}} : B \nrightarrow A$ be defined by:

   $$bR^{\mathrm{op}}a \iff aRb \tag{2}$$

   For $R, S : A \nrightarrow B$ and $T : B \nrightarrow C$, prove that:

   (a) $R \subseteq S \implies R^{\mathrm{op}} \subseteq S^{\mathrm{op}}$

   $$\begin{aligned} &\forall\, a \in A, b \in B : \\ &bR^{\mathrm{op}}a \iff \\ &aRb \implies \\ &aSb \text{ since } R \subseteq S \iff \\ &bS^{\mathrm{op}}a \end{aligned} \tag{3}$$

   $$\begin{aligned} bR^{\mathrm{op}}a \implies bS^{\mathrm{op}}a &\implies \\ R^{\mathrm{op}} \subseteq S^{\mathrm{op}} &\text{ as required} \end{aligned} \tag{4}$$

   (b) $(R \cap S)^{\mathrm{op}} = R^{\mathrm{op}} \cap S^{\mathrm{op}}$

   RHS:

   $$\begin{aligned} R \subseteq S &\iff \\ R \cap S = R &\iff \\ (R \cap S)^{\mathrm{op}} = R^{\mathrm{op}} & \end{aligned} \tag{5}$$

   LHS:

$$\begin{aligned} R^{\mathrm{op}} \subseteq S^{\mathrm{op}} &\iff \\ R^{\mathrm{op}} \cap S^{\mathrm{op}} &= R^{\mathrm{op}} \end{aligned} \tag{6}$$

Combining (5) and (6) gives:

$$(R \cap S)^{\mathrm{op}} = R^{\mathrm{op}} = (R^{\mathrm{op}} \cap S^{\mathrm{op}}) \implies (R \cap S)^{\mathrm{op}} = (R^{\mathrm{op}} \cap S^{\mathrm{op}}) \text{ as required} \tag{7}$$

(c) $(R \cup S)^{\mathrm{op}} = R^{\mathrm{op}} \cup S^{\mathrm{op}}$

RHS:

$$\begin{aligned} R \subseteq S &\iff \\ R \cup S = S &\iff \\ (R \cup S)^{\mathrm{op}} &= S^{\mathrm{op}} \end{aligned} \tag{8}$$

LHS:

$$\begin{aligned} R^{\mathrm{op}} \subseteq S^{\mathrm{op}} &\iff \\ R^{\mathrm{op}} \cup S^{\mathrm{op}} &= S^{\mathrm{op}} \end{aligned} \tag{9}$$

Combining (8) and (9) gives:

$$(R \cup S)^{\mathrm{op}} = S^{\mathrm{op}} = R^{\mathrm{op}} \cup S^{\mathrm{op}} \implies (R \cup S)^{\mathrm{op}} = R^{\mathrm{op}} \cup S^{\mathrm{op}} \tag{10}$$

(d) $(T \circ S)^{\mathrm{op}} = S^{\mathrm{op}} \circ T^{\mathrm{op}}$

LHS:

$$\begin{aligned} & \forall\, a \in A, c \in C : c(S \circ T)^{\mathrm{op}} a \iff \\ a(S \circ T)c &\iff \\ \exists\, b \in B : aTb \wedge bSc &\iff \\ \exists\, b \in B : bT^{\mathrm{op}} a \wedge cS^{\mathrm{op}} b &\iff \\ \exists\, b \in B : cS^{\mathrm{op}} b \wedge bT^{\mathrm{op}} a &\iff \\ c(S^{\mathrm{op}} \circ T^{\mathrm{op}} a & \end{aligned} \tag{11}$$

$$\begin{aligned} & c(S \circ T)^{\mathrm{op}} a \iff c(S^{\mathrm{op}} \circ T^{\mathrm{op}} a \iff \\ & c(S \circ T)^{\mathrm{op}} a = c(S^{\mathrm{op}} \circ T^{\mathrm{op}} a \text{ as required} \end{aligned} \tag{12}$$

6.2 Core exercises

1. Let $R, R' \subseteq A \times B$ and $S, S' \subseteq B \times C$ be two pairs of relations and assume $R \subseteq R'$ and $S \subseteq S'$.
   Prove that $S \circ R \subseteq S' \circ R'$.

$$\begin{aligned} R \subseteq R' &\implies \\ \forall\, a \in A, b \in B : aRb &\implies aR'b \\ S \subseteq S' &\implies \\ \forall\, b \in B, c \in C : bSc &\implies bS'c \end{aligned} \tag{13}$$

$$\begin{aligned} & \forall\, a \in R, c \in S : a(S \circ R)c \iff \\ \exists\, b \in B : aRb \wedge bSc &\implies \\ \exists\, b \in B : aR'b \wedge bS'c \text{ using (13)} &\iff \\ & a(S' \circ R'c) \text{ as required} \end{aligned} \tag{14}$$

2. Let $\mathcal{F} \subseteq \mathcal{P}(A \times B)$ and $\mathcal{G} \subseteq \mathcal{P}(B \times C)$ be two collections of relations from A to B and from B to C, respectively Prove that

$$\left(\bigcup \mathcal{G}\right) \circ \left(\bigcup \mathcal{F}\right) = \bigcup \{S \circ R \mid R \in \mathcal{F}, S \in \mathcal{G}\} : A \nrightarrow C \tag{15}$$

$$\begin{gathered}
\forall\, a \in A, c \in C: \\
a(\left(\bigcup \mathcal{G}\right) \circ \left(\bigcup \mathcal{F}\right))c \iff \\
\exists\, b \in B : \exists\, R \in \mathcal{F} : aRb \land \exists\, S \in \mathcal{G} : bSc \implies \\
\left(\left(\bigcup \mathcal{G}\right) \circ \left(\bigcup \mathcal{F}\right)\right) \subseteq \bigcup \{S \circ R \mid R \in \mathcal{F}, S \in \mathcal{G}\} \iff \\
\left(\bigcup \mathcal{G}\right) \circ \left(\bigcup \mathcal{F}\right) \subseteq \bigcup \{S \circ R \mid R \in \mathcal{F}, S \in \mathcal{G}\}
\end{gathered} \tag{16}$$

$$\begin{gathered}
\forall\, a \in A, c \in C: \\
a \bigcup \{S \circ R \mid R \in \mathcal{F}, S \in \mathcal{G}\} c \iff \\
\exists\, R \in \mathcal{F}, S \in \mathcal{G} : a(S \circ R)b \iff \\
a(\left(\bigcup \mathcal{G}\right) \circ \left(\bigcup \mathcal{F}\right))b \iff \\
\bigcup \{S \circ R \mid R \in \mathcal{F}, S \in \mathcal{G}\} \subseteq \left(\bigcup \mathcal{G}\right) \circ \left(\bigcup \mathcal{F}\right)
\end{gathered} \tag{17}$$

Combining (16) and (17) using the antisymmetry of $\subseteq$ gives:

$$\bigcup \{S \circ R \mid R \in \mathcal{F}, S \in \mathcal{G}\} = \left(\bigcup \mathcal{G}\right) \circ \left(\bigcup \mathcal{F}\right) \text{ as required} \tag{18}$$

Recall that the notation $\{S \circ R : A \nrightarrow C \mid R \in \mathcal{F}, S \in \mathcal{G}\}$ is a common syntactic sugar for the formal definition $\{T \in \mathcal{P}(A \times C) \mid \exists\, R \in \mathcal{F} : \exists\, S \in \mathcal{G} : T = S \circ R\}$. Hence,

$$T \in \{S \circ R \in A \nrightarrow C \mid R \in \mathcal{F}, S \in \mathcal{G}\} \iff \exists\, R \in \mathcal{F} : \exists\, S \in \mathcal{G} : T = S \circ R \tag{19}$$

What happens in the case of big intersections?

3. Suppose R is a relation on a set A. Prove that

   (a) R is reflexive iff $\mathrm{id}_A \subseteq R$

   R is reflexive implies that every element in A is related to itself under R.

   Assume R is reflexive:

$$\begin{gathered}
\forall\, a \in A.aRa \iff \\
\mathrm{id}_A \subseteq R
\end{gathered} \tag{20}$$

   (b) R is symmetric iff $R = R^{\mathrm{op}}$.

   For R to be symmetric, $\forall\, a_1, a_2 \in A, a_1 R a_2 \iff a_2 R a_1$.

   Assume R is symmetric:

$$\begin{gathered}
\forall\, a, b \in A.aRb \iff bRa \iff \\
\forall\, a, b \in A.bR^{\mathrm{op}}a \iff aR^{\mathrm{op}}b \iff \\
\forall\, a, b \in A.aR^{\mathrm{op}}b \iff bR^{\mathrm{op}}a \iff \\
R = R^{\mathrm{op}}
\end{gathered} \tag{21}$$

   (c) R is transitive iff $R \circ R \subseteq R$

Assume that $R \circ R \subseteq R$.

$$\begin{aligned} R \circ R \subseteq R &\iff \\ \forall\, a, c \in A . a(R \circ R)c \implies aRc &\iff \\ \forall\, a, b, c \in A . aRb \wedge bRc \implies aRc & \end{aligned} \tag{22}$$

This is the definition of transistivity and so we are done.

(d) R is antisymmetric iff $R \cap R^{\mathrm{op}} \subseteq \mathrm{id}_A$.

R is antisymmetric if $\forall\, a \in A . aRa$.

Assume $R \cap R^{\mathrm{op}} \subseteq \mathrm{id}_A$.

$$\begin{aligned} R \cap R^{\mathrm{op}} \subseteq \mathrm{id}_a &\iff \\ \forall\, a_1, a_2 \in A . a_1 R a_2 \wedge a_1 R^{\mathrm{op}} a_2 \implies a_1 = a_2 &\iff \\ \forall\, a_1, a_2 \in A . a_1 R a_2 \wedge a_2 R a_2 \implies a_1 = a_2 & \end{aligned} \tag{23}$$

Which is the definition of antisymmetric and so we are done.

4. Let R be an arbitrary relation on a set A, for example, representing an undirected graph. We are interested in constructing the smallest transitive relation (graph) containing R, called the transitive closure of R.

   (a) We define the family of relations which are transitive supersets of R:

   $$\mathcal{T}_R \triangleq \{Q : A \nrightarrow A \mid R \subseteq Q \text{ and } Q \text{ is transitive}\} \tag{24}$$

   R is not necessarily going to be an element of this family, as it might not be transitive. However R is a lower bound for $\mathcal{T}_R$, as it is a subset of every element of the family.

   Prove that the set $\bigcap \mathcal{T}_R$ is the transitive closure for R.

   (b) $\bigcap \mathcal{T}_R$ is the intersection of an infinite number of relations so it's difficult to compute the transitive closure this way. A better approach is to start with R, and keep adding the missing connections until we get a transitive graph. This can be done by repeatedly composing R with itself: after n compositions, all paths of length n in the graph represented by R will have a transitive connection between their endpoints.

   Prove that the (at least once) iterated composition $R^{\circ +} \triangleq R \circ R^{\circ} *$ is the transitive closure for R, i.e. it coincides with the greatest lower bound of $\mathcal{T}_R : R^{\circ +} = \bigcap \mathcal{T}_R$.

   Hint: show that $R^{\circ +}$ is both an element and a lower bound of $\mathcal{T}_R$.

7 On partial functions

7.1 Basic exercises

1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the element of the sets $\mathrm{PFun}(A_i, A_j)$ for $i, j \in \{2, 3\}$.

   Hint: there may be quite a few, so you can think of ways of characterising all of them without giving an explicit listing.

   $$\{\{(1, a), (2, b)\} \vee \{(1, a)\} \vee \{(2, b)\} \vee \varnothing \mid a, b \in A_3\} \tag{25}$$

2. Prove that a relation $R : A \nrightarrow B$ is a partial function iff $R \circ R^{\mathrm{op}} \subseteq \mathrm{id}_B$.

($\Longrightarrow$)

$$\begin{aligned} R \circ R^{\mathrm{op}} \subseteq \ \mathrm{id}_B \Longleftrightarrow \\ \forall\, b_1, b_2 \in B. b_1(R \circ R^{\mathrm{op}})b_2 \Longrightarrow b_1 = b_2 \Longrightarrow \\ \forall\, b_1, b_2 \in B. \exists\, a \in A. b_1 R^{\mathrm{op}} a \wedge aRb_2 \Longrightarrow b_1 = b_2 \Longleftrightarrow \\ \forall\, b_1, b_2 \in B. \exists\, a \in A. aRb_1 \wedge aRb_2 \Longrightarrow b_1 = b_2 \end{aligned} \tag{26}$$

This implies that R is a partial function by definition.

($\Longleftarrow$) Assume R is a partial function and so each a in R is related to at most one b. This means that by assumption $aRb_1 \wedge aRb_2 \Longrightarrow b_1 = b_2$.

$$\begin{aligned} & b_1(R \circ R^{\mathrm{op}})b_2 \Longleftrightarrow \\ & \exists\, a \in A. b_1 R^{\mathrm{op}} a \wedge aRb_2 \Longleftrightarrow \\ & \exists\, a \in A. aR^{\mathrm{op}} b_1 \wedge aRb_2 \Longleftrightarrow \\ & b_1 = b_2 \text{ since } a \text{ is related to at most one } b \in B \Longleftrightarrow \\ & (R \circ R^{\mathrm{op}}) \subseteq \ \mathrm{id}_B \text{ as required} \end{aligned} \tag{27}$$

3. Prove that the identity relation is a partial function, and that the composition of partial functions is a partial function.

   The definition of a partial funtion $R : A \nrightarrow B$ is that each $a \in A$ is related to at most one $b \in B$. This means that $\forall\, a \in A. \forall\, b_1, b_2 \in B. aRb_1 \wedge aRb_2 \Longrightarrow b_1 = b_2$.

   For the identity function, which has domain A and A, by definition $\forall\, a \in A. a \ \mathrm{id}_A\, a \Longrightarrow a = a$.

   Assume $a \ \mathrm{id}_A\, a_1 \wedge a \ \mathrm{id}_A\, a_2$

$$\begin{aligned} & \forall\, a \in A. \forall\, a_1, a_2 \in A. aRa_1 \wedge aRa_2 \Longleftrightarrow \\ & \forall\, a \in A. \forall\, a_1, a_2 \in A. aRa_1 \wedge aRa_2 \Longleftrightarrow \\ & a = a_1 \wedge a = a_2 \Longleftrightarrow \\ & a_1 = a_2 \text{ as required} \end{aligned} \tag{28}$$

   To prove that the composition of partial functions is a partial function we will start with functions $R : A \nrightarrow B$ and $S : B \nrightarrow C$, will compose them and prove that any $a \in A$ is related to only one $c \in C$ under $S \circ R$.

$$\begin{aligned} \forall\, a \in A. \forall\, b_1, b_2 \in B. aRb_1 \wedge aRb_2 \Longrightarrow b_1 = b_2 \wedge \\ \forall\, b \in B. \forall\, c_1, c_2 \in C. bSc_1 \wedge bSc_2 \Longrightarrow c_1 = c_2 \Longleftrightarrow \\ \forall\, a \in A. \forall\, c_1, c_2 \in C. a(S \circ R)c_1 \wedge a(S \circ R)c_2 \Longrightarrow c_1 = c_2 \end{aligned} \tag{29}$$

   This is the expression for a partial function – and so we can conclude that $S \circ R$ is a partial function. Since both R and S were arbitrary we can conclude that for all partial functions, the composition of two partial functions is a partial function.

7.2 Core exercises

1. Show that $(\mathrm{PFun}(A, B), \subseteq)$ is a partial order. What is its least element, if it exists?
2. Let $\mathcal{F} \subseteq \mathrm{PFun}(A, B)$ be a non-empty collection of partial functions from A to B.
   (a) Show that $\bigcap \mathcal{F}$ is a partial function.
   (b) Show that $\bigcup \mathcal{F}$ need not be a partial function by defining two partial functions $f, g : A \rightharpoonup B$ such that $f \cup g : A \nrightarrow B$ is a non-functional relation.
   (c) Let $h : A \rightharpoonup B$ be a partial function. Show that if every element of $\mathcal{F}$ is below h then $\bigcup \mathcal{F}$ is a partial function.

8 On functions

8.1 Basic exercises

1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the sets $\mathrm{Fun}(A_i, A_j)$ for $i, j \in \{2, 3\}$.
2. Prove that the identity partial function is a function, and the composition of functions yields a function
3. Prove or disprove that $(\mathrm{Fun}(A, B), \subseteq)$ is a partial order.
4. Find endofunctions $f, g : A \to A$ such that $f \circ g \neq g \circ f$.

8.2 Core exercises

1. A relation $R : A \nrightarrow B$ is said to be total if $\forall\, a \in A, \exists \in B : aRb$. Prove that this is equivalent to $\mathrm{id}_A \subseteq R^{\mathrm{op}} \circ R$. Conclude that a relation $R : A \nrightarrow B$ is a function iff $R \circ R^{textop} \subseteq \ \mathrm{id}_B$ and $\mathrm{id}_A \subseteq R^{\mathrm{op}} \circ R$.
2. Let $\chi : \mathcal{P}(U) \to (U \Rightarrow [2])$ be the function mapping subsets $S \subseteq U$ to their characteristi functions $\chi_S : U \to [2]$.

   (a) Prove that for all $x \in U$,

   - $\chi_{A \cup B}(x) = (\chi_A(x) \vee \chi_B(x)) = \max(\chi_A(x), \chi_B(x))$
   - $\chi_{A \cap B}(x) = (\chi_A(x) \wedge \chi_B(x)) = \min(\chi_A(x), \chi_B(x))$
   - $\chi_{A^c}(x) = (\chi_A(x)) = (1 - \chi_A(x))$

   (b) For what construction $A?B$ on sets A and B does it hold that

$$\chi_{A?B}(x) = (\chi_A(x) \oplus \chi_B(x)) = (\chi_A(x) +_2 \chi_B(x)) \tag{30}$$

8.3 Optional advanced exercise

Consider a set A together with an element $a \in A$ and an endofunction $f : A \to A$.

Say that a relation $R : \mathbb{N} \nrightarrow A$ is (a, f)-closed whenever

$$R(0, a) \text{ and } \forall\, n \in \mathbb{N}, x \in A : R(n, x) \Rightarrow R(n+1, f(x)) \tag{31}$$

Define the relation $F : \mathbb{N} \nrightarrow A$ as

$$F \triangleq \bigcap \{R : \mathbb{N} \nrightarrow A \mid R \text{ is } (a, f)\text{-closed}\} \tag{32}$$

1. Prove that F is $(a, f$-closed.
2. Prove that F is total, that is: $\forall\, n \in \mathbb{N} : \exists\, y \in A : F(n, y)$.
3. Prove that F is a function $\mathbb{N} \to A$, that is: $\forall\, n \in \mathbb{N}\, \exists! y \in A : F(n, y)$.

   Hint: Proceed by induction. Observe that, in view of the previous item, to show that $\exists! y \in A : F(k, y)$ it suffices to exibit an (a, f)-closed relation R_k such that $\exists! y \in A : R_k(k, y)$. (Why?)

   For instance, as the relation $R_0 = \{(m, y) \in \mathbb{N} \times A \mid m = 0 \Rightarrow y = a\}$ is (a, f)-closed one has that $F(0, y) \Rightarrow R_0(0, y) \Rightarrow y = a$.

4. Show that if h is a function $\mathbb{N} \to A$ with $h(0) = a$ and $\forall\, n \in \mathbb{N} : h(n+1) = f(h(n))$ then $h = F$.

   Thus, for every set A together with an element $a \in A$ and an endofunction $f\colon A \to A$ there exists a unique function $F : \mathbb{N} \to A$, typically said to be inductively defined, satisfying the recurrence relation

$$F(n) = \begin{cases} a & \text{for } n = 0 \\ f(F(n-1)) & \text{for } n \geq 1 \end{cases} \tag{33}$$