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Using operators to solve second order ODE's: consider

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = e^{2x} \tag{1}$$

You can convert this into an operator.

This means this is of the form:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 5\frac{\mathrm{d}}{\mathrm{d}x} + 6\right) y = e^{2x} \tag{2}$$

Factorise this:

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - 3\right)\left(\frac{\mathrm{d}}{\mathrm{d}x} - 2\right) y = e^{2x} \tag{3}$$

If you expand this, you do not operate the operator on the constants – you multiply it. So when you expand this for example you do not ie you do not write $\frac{\mathrm{d}}{\mathrm{d}x} \times x = 2\frac{\mathrm{d}}{\mathrm{d}x} \neq 0$.

You now set part of htis expression to be equal to $f(x)$ – say z. Now you have reduced the order of the differential equation. You solve this as a first order differential equation. Then you get an expression for z which you can solve for z. Then you solve the remaining system as another linear differential equation.

Let $z = \left(\frac{\mathrm{d}}{\mathrm{d}x} - 2\right) y$.

$$\begin{aligned}
\left(\frac{\mathrm{d}}{\mathrm{d}x} - 3\right) z &= e^{2x} \\
\frac{\mathrm{d}z}{\mathrm{d}x} - 3z &= e^{2x} \\
e^{-3x}\frac{\mathrm{d}z}{\mathrm{d}x} - 3ze^{-3x} &= e^{-x} \\
ze^{-3x} &= -e^{-x} + c \\
z &= ce^{3x} - e^{2x} \\
\left(\frac{\mathrm{d}}{\mathrm{d}x} - 2\right) y &= ce^{3x} - e^{2x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} - 2y &= ce^{3x} - e^{2x} \\
e^{-2x}\frac{\mathrm{d}y}{\mathrm{d}x} - 2ye^{-2x} &= ce^{x} - 1 \\
ye^{-2x} &= ce^{x} - x + d \\
y &= ce^{3x} + de^{2x} - xe^{2x}
\end{aligned} \tag{4}$$

This method allows you to solve equations without ever having to make a particular integral – and allows you to solve potentially far more difficult operators. For example one of the differentila operators can be a function of x and be near-impossible to be solvable via normal methods.

The definition of ∇ is:

$$\underline{\nabla} =: \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial y} \tag{5}$$

If this acts on a scalar function f, then it is the gradient – this is why it is called grad. This represents the normal to any given surface. ∇f is the normal to any part of that surface.

If $\underline{\nabla} \times \underline{v}$ is equal to zero then it is conserative.

$$\underline{\nabla} \cdot \underline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \tag{6}$$

This is a scalar.

the cross product is equal to the cross product of the nabla operator as a vector and the vector v. This gives a vector quantity.

$$\underline{\nabla} \times \underline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) \underline{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \underline{j} + \left(\frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x} \right) \underline{k} \tag{7}$$

$$\nabla^2 f = \nabla(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{8}$$

This is known as the laplacian and is used in laplaces equation.