M: Power Series

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1. $\sin x$ about $\frac{\pi}{6}$

Using Taylors Theorem

$$f(x) \simeq f(a) + (x-a)f^{(1)}(a) + (x-a)^2 \times \frac{1}{2} f^{(2)}(a) + \frac{(x-a)^3}{6} f^{(3)}(a)$$

$$f(a) = \sin\frac{\pi}{6} = \frac{1}{2}$$
$$f^{(1)}(a) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$
$$f^{(2)}(a) = -\sin\frac{\pi}{6} = -\frac{1}{2}$$
$$f^{(3)}(a) = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

So: ~~Ell~~

$$\sin\frac{31\pi}{180} \simeq \frac{1}{2} + \frac{\pi}{180} \times \frac{\sqrt{3}}{2} - \left(\frac{\pi}{180}\right)^2 \times \frac{1}{4} - \left(\frac{\pi}{180}\right)^3 \times \frac{\sqrt{3}}{12} + R_n$$

$$R_n = \frac{(x-a)^n}{n!} f^{(n)}(\xi)$$

$$f^{(n)}(\xi) = \sin\xi$$
$$\leq 1$$

$$\therefore |R_n| \leq \frac{(x-a)^n}{n!}$$

$$= \left(\frac{\pi}{180}\right)^4 \times \frac{1}{24}$$

This is small - and so the approximation is precise.

2.a) Using Taylor's Theorem:

$$f(0) \simeq f(0) + x f^{(1)}(0) + \frac{x^2}{2} f^{(2)}(0) + \ldots$$

$$f(x) = \arcsin x.$$

$$f(0) = 0$$

$$f^{(1)}(x) = (1-x^2)^{-\frac{1}{2}}$$
$$f^{(1)}(0) = 1$$

$$f^{(2)}(x) = x(1-x^2)^{-\frac{3}{2}}$$
$$f^{(2)}(0) = 0$$

$$f^{(3)}(x) = (1-x^2)^{-\frac{3}{2}} + 3x^2(1-x^2)^{-\frac{5}{2}}$$
$$f^{(3)}(0) = 1$$

$$f^{(4)}(x) = 9x(1-x^2)^{-\frac{5}{2}} + 15x^3(1-x^2)^{-\frac{7}{2}}$$
$$f^{(4)}(0) = 0$$

$$f^{(5)}(x) = 9(1-x^2)^{-\frac{5}{2}} + 90x^2(1-x^2)^{-\frac{7}{2}} + 105x^4(1-x^2)^{-\frac{9}{2}}$$
$$f^{(5)}(0) = 9$$

So the first three nonzero terms are $f^{(1)}(0)$; $f^{(3)}(0)$; $f^{(5)}(0)$

$$\therefore \arcsin x \simeq x + \frac{x^3}{6} + \frac{9x^5}{120}$$

b) $f(x) = \tan x$

$f(0) = 0$

$f^{(1)}(x) = \sec^2 x$

$f^{(1)}(0) = 1$

$f^{(2)}(x) = 2\sec^2 x \tan x$

$f^{(2)}(0) = 0$

$f^{(3)}(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x$

$f^{(3)}(0) = 2$

$f^{(4)}(x) = 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$

$f^{(4)}(0) = 16$

So: $\tan x \simeq x + \frac{x^3}{3} + \frac{2x^5}{15}$

c) $f(x) = (1+x)^{-\frac{1}{2}}$
$f(0) = 1$
$f^{(1)}(x) = -\frac{1}{2}(1+x)^{-\frac{3}{2}}$
$f^{(1)}(0) = -\frac{1}{2}$
$f^{(2)}(x) = \frac{3}{4}(1+x)^{-\frac{5}{2}}$
$f^{(2)}(0) = \frac{3}{4}$

So the first three non-zero terms of the Taylor Series expansion of $(1+x)^{-\frac{1}{2}}$ about $x=0$ is:

$$1 - \frac{1}{2}x + \frac{3}{8}x^2$$

3.

$$\arctan x = \int \frac{d}{dx}(\arctan x)\,dx$$
$$= \int \frac{1}{1+x^2}\,dx$$

The binomial expansion of $\frac{1}{1+x^2}$ is valid if $|x| \leq 1$.

The binomial expansion of $\frac{1}{1+x^2}$ is

$$\sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\therefore \arctan x = \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx \quad \text{for } |x| \leq 1$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| \leq 1$$

As required.

i) To calculate π to 10 d.p. this way; the last term in the ~~Th~~ expansion of $4\arctan x$ must be less than or equal to 1×10^{-10}.

$$\text{So} \quad \frac{1}{2n-1} \le 2.5\times10^{-11}$$

$$1 \le -2.5\times10^{-11} + 5\times10^{-11}\, n$$

$$2\times10^{10} + \tfrac{1}{2} \le n$$

So we need $2\times10^{10}+1$ terms to calculate π to 10 D.P. this way.

ii)

$$\arctan\tfrac{1}{2} + \arctan\tfrac{1}{3}$$

$$= \arctan \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\times\frac{1}{3}}$$

$$= \arctan \frac{\frac{5}{6}}{\frac{5}{6}}$$

$$= \arctan 1$$

$$= \frac{\pi}{4} \quad \text{as required.}$$

$$\text{So:} \quad \frac{\pi}{4} = \sum_{n=0}^{\infty} (-1)^n\left(\left(\tfrac{1}{2}\right)^n + \left(\tfrac{1}{3}\right)^n\right)$$

For the size of n we will consider: $\left(\frac{1}{3}\right)^n << \left(\frac{1}{2}\right)^n$.

$$\text{So:} \quad \left(\tfrac{1}{2}\right)^n \le 2.5\times10^{-11}$$

$$n \ge \lceil \lg 4\times10^{10} \rceil$$

$n \ge 36$ So we need 37 terms.

iii) $4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$

$= 2 \arctan \dfrac{\frac{1}{5} + \frac{1}{5}}{1 - \frac{1}{25}} - \arctan \frac{1}{239}$

$= 2 \arctan \frac{5}{12} - \arctan \frac{1}{239}$

$= \arctan \dfrac{\frac{5}{12} + \frac{5}{12}}{1 - \frac{25}{144}}$

$= \arctan \dfrac{120}{119} - \arctan \dfrac{1}{239}$

$= \arctan \dfrac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}}$

$= \arctan \left(\dfrac{120 \times 239 - 119}{119 \times 239 + 120} \right)$

$= \arctan \left(\dfrac{120 \times 239 - 119}{120 \times 239 - 119} \right)$

$= \arctan 1$

$= \dfrac{\pi}{4}$

For the n we will consider: $\left(\frac{1}{239}\right)^n << \left(\frac{1}{5}\right)^n$.

So to calculate π to 10 d.p.:

$16 \times \left(\frac{1}{5}\right)^n \leq 1 \times 10^{-10}$

$1 \cdot 6 \times 10^{11} \leq \frac{1}{5}^n$

$\lceil \log_5 1 \cdot 6 \times 10^{11} \rceil \leq n$

$\therefore n \geq 17$

So we need 18 terms (since $n=0$ is a term)

N: Approximation

1. a)

$$\frac{x^3+x}{x+2}$$

$$= \cancel{\tfrac{1}{2}x+}\ \frac{1}{2}x + \frac{x^3-\frac{1}{2}x^2}{x+2}$$

$$= \frac{1}{2}x + O(x)$$

b)

$$\frac{\cos x - 1}{x^2}$$

$$\cancel{\approx}\ \approx \frac{1-\frac{1}{2}x^2+\frac{1}{24}x^4+O(x^6)-1}{x^2}$$

$$= -\frac{1}{2} + \frac{\frac{1}{24}x^4 + O(x^6)}{x^2}$$

$$= -\frac{1}{2} + O(x^2)$$

2.

$$\frac{1+2x+2x^2}{3x+3}$$

$$= \frac{2}{3}x + \frac{1}{3x+3}$$

$$= \frac{2}{3}x + O(x^{-1})$$

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3. $(x^3+x^2+1)^{\frac{1}{3}} - (x^2+x)^{\frac{1}{2}}$

$= x\left(1+\left(\frac{1}{x}+\frac{1}{x^3}\right)\right)^{\frac{1}{3}} - x\left(1+\frac{1}{x}\right)^{\frac{1}{2}}$

$= x\left(1+\frac{1}{3}\left(\frac{1}{x}+\frac{1}{x^3}\right) - \frac{1}{9}\left(\frac{1}{x}+\frac{1}{x^3}\right)^3 + O\left(\frac{1}{x^3}\right) - 1 - \frac{1}{2}\left(\frac{1}{x}\right) + \frac{1}{8}\left(\frac{1}{x}\right)^2 + O\left(\frac{1}{x^3}\right)\right)$

$= x\left(\left(\frac{1}{3}-\frac{1}{2}\right)\left(\frac{1}{x}\right) - \left(\frac{1}{9}+\frac{1}{8}\right)\left(\frac{1}{x^2}\right) + O\left(\frac{1}{x^3}\right)\right)$

$= x\left(-\frac{1}{6x} + \frac{1}{72x^2} + O\left(\frac{1}{x^3}\right)\right)$

$= -\frac{1}{6} + \frac{1}{72x} + O\left(\frac{1}{x^2}\right)$

as $x \to \infty$

as required.

4. $V = L(L+1)(L+2)$

$= L^3+3L^2+2L$

$V^{\frac{1}{3}} = L\left(1+\left(\frac{3}{L^2}+\frac{2}{L^3}\right)\right)$

$A = 2(L(L+1)+L(L+2)+(L+1)(L+2))$

$= 6L^2+12L+4$

$\left(\frac{A}{6}\right)^{\frac{1}{2}} = L\left(1+\frac{2}{L}+\frac{2}{3L^2}\right)^{\frac{1}{2}}$

$V^{\frac{1}{3}} - \left(\frac{A}{6}\right)^{\frac{1}{2}} = L\left(\left(1+\left(\frac{2}{L}+\frac{3}{L^2}\right)\right)^{\frac{1}{3}} - \left(1+\left(\frac{2}{L}+\frac{2}{3L^2}\right)\right)^{\frac{1}{2}}\right)$

$= L\left(1+\frac{1}{3}\left(\frac{2}{L^2}+\frac{3}{L^2}\right) - \frac{1}{9}\left(\frac{2}{L^2}+\frac{3}{L^2}\right)^2 - 1 - \frac{1}{2}\left(\frac{2}{L}+\frac{2}{3L^2}\right) + \frac{1}{8}\left(\frac{2}{L}+\frac{2}{3L^2}\right)^2\right.$

$\left. + O\left(\frac{1}{L^3}\right)\right)$

$= L\left((1-1)+\left(\frac{1}{L}-\frac{1}{L}\right)+\left(\frac{2}{3L^2}-\frac{1}{L^2}-\frac{1}{3L^2}+\frac{1}{2L^2}\right)+O\left(\frac{1}{L^3}\right)\right)$

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$$= L\left(-\frac{1}{6L^2} + O\left(\frac{1}{L^3}\right)\right)$$

$$= -\frac{1}{6L} + O\left(\frac{1}{L^2}\right)$$

as $L \to \infty$

as required.

Q: Newton-Raphson root finding

1. The Taylor series for a function f evaluated at $x+h$ is:

$$f(x+h) = f(x) + hf^{(1)}(x) + \frac{h^2}{2} f^{(2)}(x) + \dots + \frac{h^n}{n!} f^{(n)}(x)$$

Take a first order approximation:

$x_0 = x + h$

$x_0 - x = h$

$$f(x_0) \approx f(x) + hf^{(1)}(x)$$

$$f(x_0) \approx hf^{(1)}(x) \quad \text{since } f(x) = 0$$

$$\frac{f(x_0)}{f^{(1)}(x)} \approx x_0 - x$$

$$x \approx x_0 - \frac{f(x_0)}{f^{(1)}(x)}$$

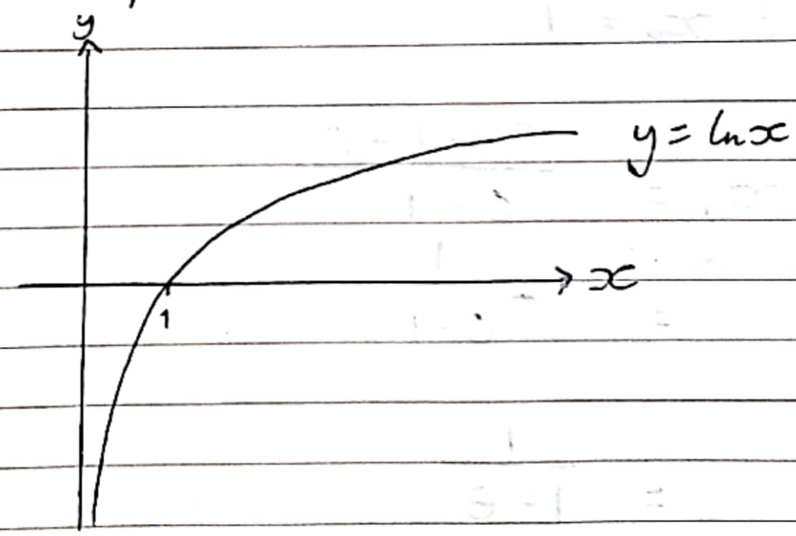
Since x_0 is close to x; in general $f^{(1)}(x_0) \approx f^{(1)}(x)$.

So:

$$x \approx x_0 - \frac{f(x_0)}{f^{(1)}(x_0)}$$

So, in general $x_0 - \frac{f(x_0)}{f^{(1)}(x_0)}$ is a better approximation to x than x_0 when $f(x) = 0$ and x_0 is an approximate solution.

as required.



$\ln x = \epsilon x$

$0 = \epsilon x - \ln x$

So $f(x) = \epsilon x - \ln x$

$$f'(x) = \epsilon - \frac{1}{x}$$

$$x - \frac{f(x)}{f'(x)}$$

$$= x - \frac{\epsilon x - \ln x}{\epsilon - \frac{1}{x}}$$

$$= x - \frac{\epsilon x^2 - x\ln x}{\epsilon x - 1}$$

$$= \frac{\epsilon x - 1 - \epsilon x^2 + x\ln x}{\epsilon x - 1}$$

So $$x_{n+1} = \frac{\epsilon x - 1 - \epsilon x^2 + x\ln x}{\epsilon x - 1}$$

Let $x_0 = 1$

$$\therefore x_1 = \frac{\epsilon - 1 - \epsilon}{\epsilon - 1}$$

$$= -\frac{1}{\epsilon - 1}$$

$$= \frac{1}{1 - \epsilon}$$

$$x_2 = \frac{\epsilon \times \frac{1}{1-\epsilon} - 1 - \frac{\epsilon}{(1-\epsilon)^2} + \frac{1}{1-\epsilon} \ln \frac{1}{1-\epsilon}}{\frac{\epsilon}{1-\epsilon} - 1}$$

$$x_2 = \frac{\epsilon - 1 + \epsilon - \frac{\epsilon}{1-\epsilon} + \ln \frac{1}{1-\epsilon}}{\epsilon - 1 + \epsilon}$$

$$x_2 = \frac{2\epsilon - 1 - \frac{\epsilon}{1-\epsilon} + \ln\frac{1}{1-\epsilon}}{2\epsilon - 1}$$

$$x_2 = 1 + \frac{\epsilon}{(\epsilon-1)(2\epsilon-1)} + \frac{\ln\frac{1}{1-\epsilon}}{2\epsilon-1}$$

$$x_2 \approx 1 + \frac{\epsilon}{(\epsilon-1)(2\epsilon-1)} + \frac{\frac{\epsilon}{1-\epsilon}}{2\epsilon-1}$$

$$= 1 + \frac{\epsilon}{(\epsilon-1)(2\epsilon-1)} + \frac{\epsilon}{(\epsilon-1)(2\epsilon-1)}$$

$$= 1 + \frac{2}{\epsilon-1} - \frac{2}{2\epsilon-1}$$

$$= 1 - \frac{2}{1-\epsilon} + \frac{2}{1-2\epsilon}$$

$$= 1 - 2(1+\epsilon+\epsilon^2+O(\epsilon^3)) + 2(1+2\epsilon+4\epsilon^2+O(\epsilon^3))$$

$$= 1 - 2 - 2\epsilon - 2\epsilon^2 + O(\epsilon^3) + 2 + 4\epsilon + 8\epsilon^2$$

$$= 1 + 2\epsilon + 6\epsilon^2 + O(\epsilon^3)$$