# 1 1999 Paper 6 Question 10

https://www.cl.cam.ac.uk/teaching/exams/pastpapers/y1999p6q10.pdf

(a) Describe the role of Herbrand models in mechanical theorem proving. What may we infer when a set of clauses has no Herbrand model?

A Herbrand Model is an interpretation such that the clauses hold and the domain of the interpretation is the Herbrand Universe. Formally, a Herbrand Model is a pair $(\mathcal{H}, I)$ where $\mathcal{H}$ is the domain of the variables and $I$ is a mapping from variables and function literals to elements in the Herbrand Universe $\mathcal{H}$ or concrete functions.

The goal of many mechanical theorem provers (such as those using DPLL) is to *find* a Herbrand Model. If a set of clauses has no Herbrand Model; the set of clauses is *unsatisfiable*.

(b) Convert the following problem into clause form. Justify each step you take and explain in what respect the set of clauses is equivalent to the original problem.

$$\exists x\,[P(x) \land Q(x)] \longrightarrow \exists x\,[P(f(x,x)) \lor \forall y Q(y)]$$

To convert to clause form; we must negate then Skolemize and convert to CNF. The clauses are the disjunctions.

For clarity, I will rename the variables such that the new expression is $\alpha$-equivalent and has no duplicate names

$$\exists x\,[P(x) \land Q(x)] \longrightarrow \exists z\,[P(f(z,z)) \lor \forall y Q(y)]$$

Firstly, we must negate the expression

$$\begin{gathered}
\neg\exists x\,[P(x) \land Q(x)] \land \neg\exists z\,[P(f(z,z)) \lor \forall y Q(y)] \\
\forall x \neg\,[P(x) \land Q(x)] \land \forall z \neg\,[P(f(z,z)) \lor \forall y Q(y)] \\
\forall x\,[\neg P(x) \lor \neg Q(x)] \land \forall z\,[\neg P(f(z,z)) \land \neg\forall y Q(y)] \\
\forall x\,[\neg P(x) \lor \neg Q(x)] \land \forall z\,[\neg P(f(z,z)) \land \exists y \neg Q(y)]
\end{gathered}$$

Next we must Skolemize. Note that Skolemization does not preserve meaning or validity – so must be done **after negation**. The first step of Skolemization is to replace all existentially quantified variables by Skolem functions of all universally quantified variables in scope – or Skolem constants if there are no universally quantified variables in scope.

$$\forall x\,[\neg P(x) \lor \neg Q(x)] \land \forall z\,[\neg P(f(z,z)) \land \neg Q(g(x))]$$

The next step of Skolemization is to remove the universal quantifiers

$$(\neg P(x) \lor \neg Q(x)) \land \neg P(f(z,z)) \land \neg Q(g(x))$$

This is already in CNF and therefore we can read clauses off

$$\{\neg P(x) \lor \neg Q(x)\}\ \{\neg P(f(z,z))\}\ \{\neg Q(g(x))\}$$

Skolemization does not preserve meaning or validity of a formula. However, it preserves consistency. The final set of clauses is inconsistent if and only if the original formula was inconsistent.

(c) Describe the Herbrand universe for your clauses.

The Herbrand universe $\mathcal{H}$ for these clauses is described as follows:

$$
\begin{aligned}
H_0 &= \{c, d\} \\
H_{i+1} &= H_i \cup \{f(a,b) \mid a, b \in H_i\} \cup \{g(a) \mid a \in H_i\} \\
\mathcal{H} &= \bigcup_{i \geq 0} H_i
\end{aligned}
$$

The Herbrand Universe can also be defined recursively as follows:

$$\mathcal{H} = \{f(x,y) \mid x, y \in \mathcal{H}\} \cup \{g(x) \mid x \in \mathcal{H}\}$$

(d) Produce a resolution proof from your clauses or give reasons why none exists. There exists no resolution proof for the formula. All to prove a theorem valid, we must produce the empy clause $\square$. To produce this, we must resolve clauses together. However, every predicate is negated so no resolution can be performed. It's therefore impossible to prove the theorem using resolution.

(e) Exhibit a Herbrand model for your clauses or give reasons why none exists.

A Herbrand Model is an interpretation for which the clauses are satisfied.

## 2 1998 Paper 5 Question 10

https://www.cl.cam.ac.uk/teaching/exams/pastpapers/y1998p5q10.pdf

(b) Attempt to prove the above formula using the sequent calculus until either it is proved or the proof cannot be continued.

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$P \to Q, R \Rightarrow R$}
\RightLabel{$(\neg l)$}
\UnaryInfC{$P \to Q, \neg R, R \Rightarrow$}
\AxiomC{$R \Rightarrow P, Q$}
\AxiomC{}
\UnaryInfC{$Q, R \Rightarrow Q$}
\RightLabel{$(\to l)$}
\BinaryInfC{$P \to Q, R \Rightarrow Q$}
\RightLabel{$(\neg l)$}
\UnaryInfC{$P \to Q, \neg Q, R \Rightarrow$}
\RightLabel{$(\vee l)$}
\BinaryInfC{$P \to Q, \neg R \vee \neg Q, R \Rightarrow$}
\RightLabel{$(\neg r)$}
\UnaryInfC{$P \to Q, \neg R \vee \neg Q \Rightarrow \neg R$}
\RightLabel{$(\wedge l)$}
\UnaryInfC{$(P \to Q) \wedge (\neg R \vee \neg Q) \Rightarrow \neg R$}
\RightLabel{$(\to r)$}
\UnaryInfC{$\Rightarrow [(P \to Q) \wedge (\neg R \vee \neg Q)] \to \neg R$}
\end{prooftree}
$$

(c) Design a method for determining whether a prepositional formula is inconsistent. The method should work by examining the formula's disjunctive normal form. Demonstrate your method by applying it to the formula:

$$\neg [(P \wedge Q) \vee (Q \to P)]$$

Informally:

- Convert the formula to DNF
- If the reduction to DNF yields **f** then the formula is inconsistent and unsatisfiable.
- If the reduction yields **t** then the formula is valid
- If the DNF is a disjunction of multiple terms, then the formula is satisfiable but not valid.

Formally:

- Start with a propositional formula $F$.

- Replace all occurrences of $A \leftrightarrow B$ with $(A \to B) \land (B \to A)$:

$$A \leftrightarrow B \Rightarrow (A \to B) \land (B \to A)$$

- Replace all occurrences of $A \leftarrow B$ with $B \to A$

$$A \leftarrow B \Rightarrow B \to A$$

- Replace all occurrences of $A \to B$ with $\neg A \lor B$:

$$A \to B \Rightarrow \neg A \lor B$$

- While there are any negated formulae (ie negations which are not literals); push in a negation using De Morgens Laws. IE while it can be done; , do one one of the following operations.

$$\neg(A \lor B) \Rightarrow \neg A \land \neg B$$
$$\neg(A \land B) \Rightarrow \neg A \lor \neg B$$

  The formula is now in negation normal form

- While there is a double negation, eliminate a double-negation:

$$\neg\neg A \Rightarrow A$$

- While it can be done; apply distributivity to push all $\land$s in.

$$A \land (B \lor C) \Rightarrow A \land B \lor A \land C$$

- Use associativity to remove all brackets
- Remove all conjuncts of the form $A \land \cdots \land \neg A$

After performing these operations, the formula is in DNF. If the formula is non-empty then there exists one conjunct which was not removed – this conjunct is satisfiable. Since Skolemization preserves consistency, we can conclude that the original formula was consistent in this case.

# 3 2005 Paper 5 Question 9

https://www.cl.cam.ac.uk/teaching/exams/pastpapers/y2005p5q9.pdf

(a) In order to prove the following formula by resolution, what set of clauses should be submitted to the prover? Justify your answer briefly.

$$\forall x\,[P(x) \lor Q \to \neg R(x)] \land \forall x\,[(Q \to \neg S(x)) \to (P(x) \land R(x))] \to \forall x S(x)$$

We should Skolemize the formula, then convert to DNF and negate (so the negation is in CNF). The clauses we submit to the prover should be the disjunctions.

In this particular case, this formula uses implication. Therefore, we can simplify it quicker using the identity $\neg(A \to B) \simeq A \land \neg B$.

$$\forall x\,[P(x) \lor Q \to \neg R(x)] \land \forall x\,[(Q \to \neg S(x)) \to (P(x \land R(x)))] \to \forall x S(x)$$

Skolemize the formula. Since there are no existential quantifiers, this involves only dropping the universal quantifiers

$$[P(x) \lor Q \to \neg R(x)] \land [(Q \to \neg S(x)) \to (P(x) \land R(x))] \to S(x)$$

Next negate. I will use the identity $\neg(A \to B) \simeq A \wedge \neg B$

$$
\begin{aligned}
&\simeq [P(x) \vee Q \to \neg R(x)] \wedge [(Q \to \neg S(x)) \to (P(x) \wedge R(x))] \wedge \neg S(x) \\
&\simeq [\neg(P(x) \vee Q) \vee \neg R(x)] \wedge [\neg\,(Q \to \neg S(x)) \wedge (P(x) \wedge R(x))] \wedge \neg S(x) \\
&\simeq [\neg P(x) \wedge Q \vee \neg R(x)] \wedge [\neg\,(\neg Q \vee \neg S(x)) \wedge P(x) \wedge R(x)] \wedge \neg S(x) \\
&\simeq [\neg(P(x) \vee Q) \vee \neg R(x)] \wedge [\neg\,(Q \to \neg S(x)) \wedge (P(x) \wedge R(x))] \wedge \neg S(x) \\
&\simeq [\neg P(x) \wedge \neg Q \vee \neg R(x)] \wedge [Q \wedge S(x) \wedge P(x) \wedge R(x)] \wedge \neg S(x) \\
&\simeq (\neg P(x) \vee \neg R(s)) \wedge (\neg Q \vee \neg R(x)) \wedge Q \wedge S(x) \wedge P(x) \wedge R(x) \wedge \neg S(x) \\
&\simeq \{\neg P(x), \neg R(s)\}\ \{\neg Q, \neg R(x)\}\ \{Q\}\ \{S(x)\}\ \{P(x)\}\ \{R(x)\}\ \{\neg S(x)\}
\end{aligned}
$$

(b) Derive the empty clause using resolution with the following set of clauses, or give convincing reasons why it cannot be derived.

$$\{\neg P(x,x)\}\ \{P(x,f(x))\}\ \{\neg P(x,y), \neg P(y,z), P(x,z)\}$$

Note that variables scope is *only* in their clause. For clarity, I will rename the variables

$$=\{\neg P(v,v)\}\ \{P(w,f(w))\}\ \{\neg P(x,y), \neg P(y,z), P(x,z)\}$$

To apply resolution, we must unify $\neg P(v,v)$ and $P(x,z)$. Using the most general unifier $\sigma = [v/x, v/z]$

$$
\begin{aligned}
&=\{\neg P(v,v)\}\ \{P(w,f(w))\}\ \{\neg P(v,y), \neg P(y,v), P(v,v)\} \\
&=\{P(w,f(w))\}\ \{\neg P(v,y), \neg P(y,v)\}
\end{aligned}
$$

Notice the two terms in the RHS clause are the same under $\alpha$-equivalence. Using idempotence, we can remove one.

$$=\{P(w,f(w))\}\ \{\neg P(y,v)\}$$

Use the unifier $\sigma = [w/y, f(x)/v]$

$$
\begin{aligned}
&=\{P(w,f(w))\}\ \{\neg P(w,f(w))\} \\
&=\square
\end{aligned}
$$

(c) Derive the empty clause using resolution with the following set of clauses or give convincing reasons why it cannot be derived. (Note that $a$ and $b$ are constants.)

$$\{\neg P(a)\}\ \{Q(a)\}\ \{R(b)\}\ \{S(b)\}\ \{\neg Q(x), P(x), \neg R(y), \neg Q(y)\}\ \{\neg S(x), \neg R(x), Q(x)\}$$

Since variables are scoped only within their clause, I rename for clarity

$$=\{\neg P(a)\}\ \{Q(a)\}\ \{R(b)\}\ \{S(b)\}\ \{\neg Q(x), P(x), \neg R(y), \neg Q(y)\}\ \{\neg S(z), \neg R(z), Q(z)\}$$

Using the unifier $\sigma = [a/x]$

$$
\begin{aligned}
&=\{R(b)\}\ \{S(b)\}\ \{\neg P(a)\}\ \{Q(a)\}\ \{\neg Q(a), P(a), \neg R(y), \neg Q(y)\}\ \{\neg S(z), \neg R(z), Q(z)\} \\
&=\{R(b)\}\ \{S(b)\}\ \{\neg P(a)\}\ \{P(a), \neg R(y), \neg Q(y)\}\ \{\neg S(z), \neg R(z), Q(z)\} \\
&=\{R(b)\}\ \{S(b)\}\ \{\neg R(y), \neg Q(y)\}\ \{\neg S(z), \neg R(z), Q(z)\}
\end{aligned}
$$

Using the unifier $\sigma = [b/y, b/z]$

$=\{R(b)\}\ \{S(b)\}\ \{\neg R(b), \neg Q(b)\}\ \{\neg S(b), \neg R(b), Q(b)\}$
$=\{R(b)\}\ \{\neg R(b), \neg Q(b)\}\ \{S(b)\}\ \{\neg S(b), \neg R(b), Q(b)\}$

Using idempotence, we can duplicate the clause $\{R(b)\}$

$=\{R(b)\}\ \{\neg R(b), \neg Q(b)\}\ \{R(b)\}\ \{S(b)\}\ \{\neg S(b), \neg R(b), Q(b)\}$
$=\{\neg Q(b)\}\ \{R(b)\}\ \{\neg R(b), Q(b)\}$
$=\{\neg Q(b)\}\ \{Q(b)\}$
$=\square$