Candidate 2031B

15T. a) i) $\dfrac{\ln(2+x)}{2-x}$

$$= \frac{\ln 2}{2}\left(\frac{\ln(1+\frac{x}{2})}{1-\frac{x}{2}}\right)$$

$$\approx \frac{\ln 2}{2}\left(\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{24}-\frac{x^4}{64}\right)\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\ldots\right)$$

$$\approx \frac{\ln 2}{2}\left(\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{24}+\frac{x^2}{4}-\frac{x^3}{16}+\frac{x^3}{8}+O(x^4)\right)$$

$$= \frac{\ln 2}{2}\left(\frac{x}{2}+\frac{x^2}{8}+\frac{5x^3}{48}\right)+O(x^4)$$

So the first 3 terms are

$$\left(\frac{\ln 2}{4}\right)x+\left(\frac{\ln 2}{16}\right)x^2+\left(\frac{5\ln 2}{96}\right)x^3$$

ii) $f(x) = \arctan(x)$

$f(0) = 0$

$$f'(x) = \frac{1}{1+x^2}$$

$f'(0) = 1$

$$f^{(2)}(x) = -\frac{2x}{(1+x^2)^2}$$

$f^{(2)}(0) = 0$

$$f^{(3)}(x) = -\frac{2}{(1+x^2)^2}+\frac{8x^2}{(1+x^2)^3}$$

$f^{(3)}(0) = -2$

$$f^{(4)}(x) = \frac{8x}{(1+x^2)^3} + \frac{16x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4}$$

$$f^{(4)}(0) = 0$$

$$f^{(5)}(x) = \frac{24}{(1+x^2)^3} - \frac{144x^2}{(1+x^2)^4} - \frac{144x^2}{(1+x^2)^4} + \frac{388x^4}{(1+x^2)^5}$$

$$f^{(5)}(0) = 24$$

~~So the So arctan~~ $\cancel{x \approx x - \frac{x^3}{3} + \frac{x^5}{5} \dots}$

So $\arctan x \approx x - \frac{x^3}{3} + \frac{x^5}{5}$

iii) $\ln(\cosh x)$

$$\approx \ln(e^x + e^{-x}) - \ln 2$$

$$\approx \ln\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots + 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} \dots\right) - \ln 2$$

$$\approx \ln\left(2 + x^2 + \frac{x^4}{12} + \frac{x^6}{360}\right) - \ln 2$$

$$\approx \ln\left(1 + \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}\right)$$

$$\approx \left(\frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}\right) - \frac{1}{2}\left(\frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}\right)^2 + \frac{1}{3}\left(\frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720}\right) \dots$$

$$\approx \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^6}{720} - \frac{x^4}{8} - \frac{x^6}{96} + \frac{x^6}{24} + \dots O(x^8)$$

$$\approx \frac{x^2}{2} - \frac{x^4}{12} + \frac{47x^6}{1440} + O(x^6)$$

b) i)

$$\frac{dg}{d\ell} = \frac{1}{\frac{d\ell}{dg}}$$

$$\frac{d^2g}{d\ell^2} = -\frac{\left(\frac{d^2\ell}{dg^2}\right)}{\left(\frac{d\ell}{dg}\right)^2}$$

$$\frac{d^3g}{d\ell^3} = \frac{2\left(\frac{d^2\ell}{dg^2}\right)^2 \cancel{d\ell}}{\left(\frac{d\ell}{dg}\right)^3} - \frac{\left(\frac{d^3\ell}{dg^3}\right)}{\left(\frac{d\ell}{dg}\right)^2}$$

ii)

$$b_1 = \frac{1}{a_1}$$

$$b_2 = -\frac{a_2}{a_1^2}$$

$$b_3 = \frac{2a_2^2}{a_1^3} - \frac{a_3}{a_1^2}$$

Candidate 2031B.

16X. a) i) $P(s=0) = 1 - \frac{r}{R}$

ii) $p(s) = 0$ if the delay $= 0$.

else

$p(s)\,ds = P(s < \text{delay} < s + ds)$

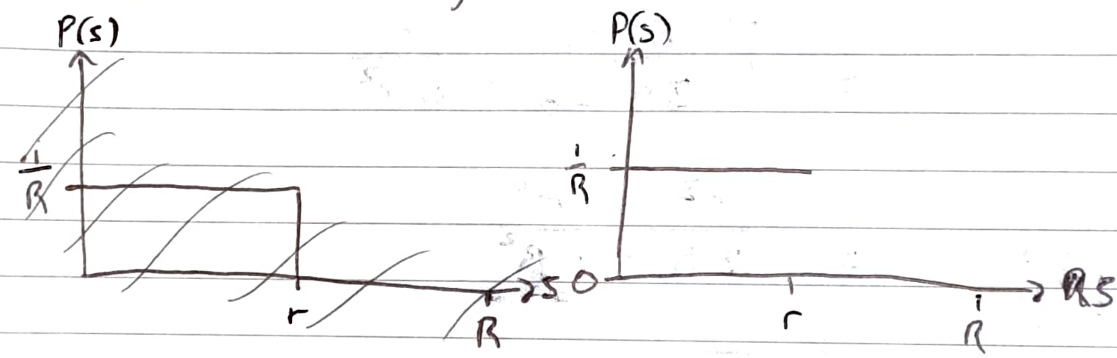
$p(s)\,ds = \frac{ds}{R}$

$p(s) = \frac{1}{R}$

so

$$p(s) = \begin{cases} \frac{1}{R}, & 0 < s \le r \\ 0, & r < s \le R \end{cases}$$

iii)



iv) $P(S=0) + P(0 < s \le r)$

$= 1 - \frac{r}{R} + \int_0^r P(s)\,ds$

$= 1 - \frac{r}{R} + \int_0^r \frac{1}{R}\,ds$

$= 1 - \frac{r}{R} + \frac{r}{R}$

$= 1$ as required

v)

$\mu = \cancel{\$E(\delta)}\ E(s)$

$= \int_0^r \frac{\cancel{x}s}{R}\,ds + 0 \times \left(1 - \frac{r}{R}\right)$

$= \left[\frac{s^2}{2R}\right]_0^r$

$= \frac{r^2}{2R}$

vi)

$\sigma^2 = \cancel{\$}\ E(s^2) - \mu^2$

$= \int_0^r \frac{s^2}{R}\,ds + 0^2 \times \left(1 - \frac{r}{R}\right) - \left(\frac{r^2}{2R}\right)^2$

$= \left[\frac{s^3}{3R}\right]_0^r - \frac{r^4}{4R^2}$

$= \frac{r^3}{3R} - \frac{r^4}{4R^2}$

$\sigma = \sqrt{\sigma^2}$

$= \sqrt{\frac{r^3}{3R} - \frac{r^4}{4R^2}}$

b) i) The probability that $T_1 = 0$ is the probability $T_0 < r$ and $S \geq T_0$

$$P(T_1 = 0) = \begin{cases} 0 & \text{if } T_0 \geq r \\ \frac{r - T_0}{R} & \text{otherwise} \end{cases}$$

ii)

$P(0 < T_1 < T_0)$ is the probability T_0 ~~≥ R~~ or $S < T_0$

$$P(0 < T_1 < T_0) = \begin{cases} \frac{1}{2} & \text{if } T_0 > r \\ \frac{T_0}{R} & \text{otherwise} \end{cases}$$

iii) $P(T_1 \geq T_0)$ is the probability that B stops for a period longer than or equal to the period that A stops for.

$$P(T \geq T_0) = \begin{cases} \frac{1}{2} & \text{if } T_0 > r \\ \frac{R - r}{R} & \text{otherwise} \end{cases}$$

So if $T_0 > r$

$$P(T_1 = 0) + P(0 < T_1 < T_0) + P(T_0 \leq T_1)$$

$$= 0 + \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

if $T_0 \leq r$

$$P(T_1 = 0) + P(0 < T_1 < T_0) + P(T_0 \leq T_1)$$

$$= \frac{r - T_0}{R} + \frac{T_0}{R} + \frac{R - r}{R}$$

$$= 1$$

Candidate 2031B

1772. a) i) $y^{(n)} = 0$ Let a_i be ~~an~~ arbitrary constants

$$y^{(2)} = 0$$

$$y^{(1)} = a_0$$

$$y = a_0 x + a_1$$

ii) Let b_i be ~~an~~ arbitrary constants

$$y^{(n)} = 0 \Rightarrow$$

$$y = \sum_{i=0}^{n-1} b_i x^i$$

b) $y^{(5)} - y^{(1)} = 0$

$$\lambda^5 - \lambda = 0$$

$$\lambda(\lambda-1)(\lambda+1)(\lambda^2+1) = 0$$

$$\lambda = 0 \vee \lambda = 1 \vee \lambda = -1 \vee \lambda = \pm i$$

So the solutions are of the form

$$y = C + Ae^{x} + Be^{-x} + D\cos x + E\sin x$$

So the complementary function is

$$y = A + Be^{x} + Ce^{-x} + D\cos x + E\sin x$$

ii) Try a particular integral of the form

$\cancel{pq+}\ y = px^2 + qx$ (note we multiplied by x since there is already a constant in the complimentary function.

$y^{(5)} = 0$

$y^{(1)} = 2px + q$

$y^{(5)} - y^{(1)} = x$

$-2px - q = x$

by equating coefficients

$-2p = 1$ $\quad$ $-q = 0$

$p = -\frac{1}{2}$ $\quad$ $q = 0$

So the particular integral is

$$y = -\frac{1}{2}x^2$$

iii) So overall, the general solution is:

$\cancel{y = PI + CF}$ $y = CF + PI$

$$y = A + Be^{x} + Ce^{-x} + D\cos x + E\sin x - \frac{1}{2}x^2$$

iv)

$y(0) = 1$

$y(0) = A + B + C + D$

① $1 = A + B + C + D$

$y^{(1)}(0) = 0$

$y^{(1)}(0) = Be^{0} - Ce^{0} - D\sin x + E\cos x - \cancel{x}0$

② $\cancel{y}\ 0 = B - C + E$

$y^{(2)}(0) = 0$

$y^{(2)} = Be^{x} + Ce^{-x} - D\cos x - E\sin x - 1$

③ $0 = B + C - D - 1$

$y^{(3)}(0) = 0$

$y^{(3)} = Be^{x} - Ce^{-x} + D\sin x - E\cos x$

④ $0 = B - C - E$

$y^{(4)}(0) = 0$

$y^{(4)} = Be^{x} + Ce^{-x} + D\cos x + E\sin x$

⑤ $0 = B + C + D$

4 4 $① = ① \Rightarrow$

$② = ④ \Rightarrow$

$B - C + E = B - C - E \Rightarrow$

$E = 0$

$①: ⑤ - ③$

$0 = 2D + 1$

$D = -\frac{1}{2}$

$① - ⑤$

$1 = A + B + C + D - (B + C + D)$

$1 = A$

$②: ② + ③$

$0 = 2B + E - D - 1$

$0 = 2B - \frac{1}{2}$

$B = \frac{1}{4}$

$②\ 0 = B - C + E$

$B = C$

$C = \frac{1}{4}$

$y = 1 + \frac{1}{4}e^{x} + \frac{1}{4}e^{-x} - \frac{1}{2}\cos x - \frac{1}{2}x$

20R. a) i) H is conservative if and only if

$$\nabla \times H = 0$$

ii) Yes there is.

$$\nabla \times \left(G + \begin{pmatrix} f(x) \\ g(y) \\ h(z) \end{pmatrix} \right) = \nabla \times G$$

for all functions f, g, h. (not just constants)
So G has a lot of freedom

iii) $G = \frac{1}{3}z^3 i + \frac{1}{3}x^3 j + \frac{1}{3}y^3 k$

$F = \nabla \times G$

$$= \begin{pmatrix} \frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z} \\ \frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x} \\ \frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} y^2 \\ z^2 \\ x^2 \end{pmatrix}$$

b) i) $$\int_S \nabla \times F \cdot dS = \oint_C F \cdot dr$$

For all ~~fo~~ vector fields F where C is the closed curve bounding the open surface S.

IE if the surface S is ~~this~~ this hemispherical shell then C is the circle at the bottom bounding it.

ii)

$$\cancel{\iint \nabla\times F\cdot dS}$$

$$= \cancel{\oint F\cdot dr}$$

$$\iint_S F\cdot dS$$

$$= \iint \nabla\times G\cdot dS$$

$$= \oint_C G\cdot dr$$ by stoke's theorem

The outward normal points in the direction $\begin{pmatrix}0\\0\\-1\end{pmatrix}$

~~not~~ ~~a~~ The curve is given by the ellipse

$x = a\cos\theta$, $y = b\sin\theta$, $z=0$

~~dA~~ So $r = \begin{pmatrix} a\cos\theta \\ b\sin\theta \\ 0 \end{pmatrix}$

$$dr = \begin{pmatrix} -a\sin\theta \\ b\cos\theta \\ 0 \end{pmatrix} d\theta$$

$$\oint_C G \cdot dr$$

$$= \int_0^{2\pi} -\frac{a}{3} z^3 \overset{\sin}{\cancel{\cos}}\theta + \frac{b}{\cancel{4}3} x^3 \cos\theta + 0 \, d\theta$$

$$= \frac{a^3 b}{3} \int_0^{2\pi} \cos^4\theta \, d\theta$$

$$= \frac{a^3 b}{3} \int_0^{2\pi} \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8} \, d\theta$$

$$= \frac{a^3 b}{3} \left[\frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta \right]_0^{2\pi}$$

$$= \frac{a^3 b}{3} \times \frac{6\pi}{8}$$

$$= \frac{a^3 b \pi}{4}$$

c) The exact same since the curve bounding ~~redo the above calculations with~~ the ellipsoid is unchanged.

We would apply Stoke's theorem and then set r to the same thing (since the bounding curve is unchanged). The problem would now be identical.

Candidate 2031B

13T a) $\nabla \cdot v_1 = 0$

$\nabla \cdot v_2 = 2y + 6y + 2y$

$= 10y$

b) $I_1 \int_C V_1 \cdot dr$

$I_1 = \int_C a \times r \cdot dr$

$= 0$

$$I_2 = \int_0^{\frac{5\pi}{2}} \begin{pmatrix} 2xy \\ x^2+3y^2+z^2 \\ 2yz \end{pmatrix} \cdot \begin{pmatrix} -b\sin\theta \\ b\cos\theta \\ c \end{pmatrix} d\theta$$

$$= \int_0^{\frac{5\pi}{2}} \begin{pmatrix} 2b^2\sin\theta\cos\theta \\ b^2(\sin^2\theta+\cos^2\theta)+2b^2\sin^2\theta+c^2\theta^2 \\ 2b\sin\theta\, c\theta \end{pmatrix} \cdot \begin{pmatrix} -b\sin\theta \\ b\cos\theta \\ c \end{pmatrix} d\theta$$

$$= \int_0^{\frac{5\pi}{2}} -2b^3\sin^2\theta\cos\theta + b^3\cos\theta + 2b^3\sin^2\theta\cos\theta + c^2\theta^2 b\cos\theta + 2bc^2\sin\theta\,\theta \; d\theta$$

$$= \int_0^{\frac{5\pi}{2}} b^3\cos\theta + bc^2\theta^2\cos\theta + 2bc^2\theta\sin\theta \; d\theta$$

$$= \Big[b^3\sin\theta + bc^2\theta^2\sin\theta \Big]_0^{\frac{5\pi}{2}}$$

$$= b^3\sin\frac{5\pi}{2} + bc^2\left(\frac{5\pi}{2}\right)^2\sin\left(\frac{5\pi}{2}\right)^2 - 0$$

$$= b^3 + bc^2 \times \frac{25\pi^2}{4}$$

$$= \frac{25\pi^2 bc^2}{4} + b^3$$

c) A vector field is conservative if and only if its curl is zero.

$$\nabla \times v_2 = \begin{pmatrix} 2y\,2z - 2z \\ 0 - 0 \\ 2x - 2x \end{pmatrix} = \underline{0}$$

So the vector field v_2 is conservative.

$$\cancel{\nabla \times V_1 = \nabla \times (a \times r)}$$

$$\cancel{= \nabla \times \begin{pmatrix} a_y c\theta - \\ \\ \end{pmatrix}}$$

$$\nabla \times v_1 = \nabla \times (a \times r)$$

$$= \nabla \times \left(\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right)$$

$$= \nabla \times \begin{pmatrix} a_y z - a_z y \\ a_z x - a_x z \\ a_x y - a_y x \end{pmatrix}$$

$$= \begin{pmatrix} a_x + a_x \\ a_y + a_y \\ a_z + a_z \end{pmatrix}$$

$$= 2a \neq 0$$

So v_1 is not conservative.

d) $V_2 = \nabla(x^2y + y^3 + yz^2)$

So $\phi = x^2y + y^3 + yz^2$

$$\phi\left(\frac{5\pi}{2}\right) = b^3\cos^2\frac{5\pi}{2}\sin\frac{5\pi}{2} + b^3\sin^3\frac{5\pi}{3} + b\sin\frac{5\pi}{3}\times c^2\times\left(\frac{5\pi}{2}\right)^2$$

$$= 0 + b^3 + bc^2\times\frac{25\pi^2}{4}$$

$$= \frac{25\pi^2bc^2}{4} + b^3$$

~~which agrees with the line integral~~

$$\phi(0) = b^3\cos^2 0\sin 0 + b^3\sin 0 + b\sin 0\times c^2\times 0^2$$

$$= 0$$

$$\phi\left(\frac{5\pi}{2}\right) - \phi(0) = \frac{25\pi^2bc^2}{4} + b^3$$

which agrees with the line integral and is the expected result.