14. Since $a \geq 0$, we know that $x^2 + y^2 + a^2 \geq 0$.

$$
\begin{aligned}
h(x,y) &= \frac{a(x+y)}{x^2+y^2+a^2} \\
\left(\frac{\partial h}{\partial x}\right)_y &= \frac{a((x^2+y^2+a^2) - 2x(x+y))}{(x^2+y^2+a^2)^2} \\
\left(\frac{\partial h}{\partial x}\right)_y &= \frac{a(y^2-2xy-x^2+a^2)}{(x^2+y^2+a^2)^2} \qquad (1) \\
0 &= \frac{a(y^2-2xy-x^2+a^2)}{(x^2+y^2+a^2)^2} \\
0 &= a(y^2-2xy-x^2+a^2) \\
0 &= y^2-2xy-x^2+a^2
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{\partial h}{\partial y}\right)_x &= \frac{a((x^2+y^2+a^2) - 2y(x+y))}{(x^2+y^2+a^2)^2} \\
\left(\frac{\partial h}{\partial y}\right)_x &= \frac{a(x^2-2xy-y^2+a^2)}{(x^2+y^2+a^2)^2} \qquad (2) \\
0 &= a(x^2-2xy-y^2+a^2) \\
0 &= x^2-2xy-y^2+a^2
\end{aligned}
$$

Since at any stationary point, both $\left(\frac{\partial h}{\partial x}\right)_y = 0$ and $\left(\frac{\partial h}{\partial y}\right)_x = 0$, they must be equal to each other.

$$
\begin{aligned}
x^2-2xy-y^2+a^2 &= y^2-2xy-x^2+a^2 \\
x^2-y^2 &= y^2-x^2 \\
2x^2 &= 2y^2 \qquad (3) \\
x^2 &= y^2
\end{aligned}
$$

$$
\begin{aligned}
0 &= x^2-2xy-y^2+a^2 \\
0 &= a^2-2xy+(x^2-y^2 \\
0 &= a^2-2xy \\
xy &= \frac{a^2}{2} \qquad (4) \\
\pm x^2 &= \frac{a^2}{2}
\end{aligned}
$$

Since $a \in \mathbb{R}$, a^2 must be a positive number and so $\pm x^2$ must be positive.
This means that:

$$
\begin{aligned}
x^2 &= \frac{a^2}{2} \\
x &= \pm\frac{a}{\sqrt{2}} \qquad (5) \\
y &= \pm\frac{a}{\sqrt{2}}
\end{aligned}
$$

For a point to be a maximum or a minimum: $h_{xx}h_{yy} > h_{xy}^2$ and $h_{xx} > 0$ and $h_{yy} > 0$.

$$\begin{aligned}
\left(\frac{\partial h}{\partial x}\right)_y &= \frac{a(y^2-2xy-x^2+a^2)}{(x^2+y^2+a^2)^2}\\
\left(\frac{\partial^2 h}{\partial x^2}\right)_y &= \frac{a((-2y-2x)(x^2+y^2+a^2)-4x(y^2-2xy-x^2+a^2))}{(x^2+y^2+a^2)^3}\\
\left(\frac{\partial^2 h}{\partial x^2}\right)_y &= \frac{a(-2x^2y-2x^3-2y^3-2xy^2-2a^2y-2a^2x-4xy^2+8x^2y+4x^3-4a^2x)}{(x^2+y^2+a^2)^3}\\
\left(\frac{\partial^2 h}{\partial x^2}\right)_y &= \frac{a(6x^2y+2x^3-2y^3-6xy^2-2a^2y-6a^2x)}{(x^2+y^2+a^2)^3}\\
\left(\frac{\partial h}{\partial y}\right)_x &= \frac{a(x^2-2xy-y^2+a^2)}{(x^2+y^2+a^2)^2}\\
\left(\frac{\partial^2 h}{\partial y^2}\right)_x &= \frac{a(6xy^2+2y^3-2x^3-6x^2y-2a^2x-6a^2y)}{(x^2+y^2+a^2)^3}\\
\left(\frac{\partial^2 h}{\partial x\partial y}\right) &= \frac{a((2y-2x)(x^2+y^2+a^2)-4y(y^2-2xy-x^2+a^2))}{(x^2+y^2+a^2)^3}\\
\left(\frac{\partial^2 h}{\partial x\partial y}\right) &= \frac{a(6x^2y-2y^3-2x^3+6xy^2-2a^2x-2a^2y)}{(x^2+y^2+a^2)^3}
\end{aligned} \tag{6}$$

There are four stationary points which we must establish the type of: $(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}})$, $(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}})$, $(-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}})$, $(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}})$
At $(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}})$:

$$\begin{aligned}
\left(\frac{\partial^2 h}{\partial x^2}\right)_y &= \frac{a(-\sqrt{2}a^3-3\sqrt{2}a^3)}{(\frac{a^2}{2}+\frac{a^2}{2}+a^2)^3}\\
\left(\frac{\partial^2 h}{\partial x^2}\right)_y &= \frac{-4\sqrt{2}a^4}{8a^6}\\
\left(\frac{\partial^2 h}{\partial x^2}\right)_y &= -\frac{\sqrt{2}}{2a^2}\\
\left(\frac{\partial^2 h}{\partial y^2}\right)_x &= -\frac{\sqrt{2}}{2a^2}\\
\left(\frac{\partial^2 h}{\partial x\partial y}\right) &= 0
\end{aligned} \tag{7}$$

Since both $h_{xx} < 0$ and $h_{yy} < 0$ and $h_{xx}h_{yy} = \frac{2}{4a^4} > 0 = h_{xy}$. This is a maximum.
At $(\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}})$:

$$\begin{aligned}
\left(\frac{\partial^2 h}{\partial x^2}\right)_y &= -\frac{\sqrt{2}}{2a^2}\\
\left(\frac{\partial^2 h}{\partial y^2}\right)_x &= \frac{\sqrt{2}}{2a^2}
\end{aligned} \tag{8}$$

Since h_{xx} and h_{yy} have different signs, we know there cannot be a maxima or minima at this point and so will not investigate further.
At $(-\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}})$:

$$\begin{aligned}
\left(\frac{\partial^2 h}{\partial x^2}\right)_y &= -\frac{\sqrt{2}}{2a^2}\\
\left(\frac{\partial^2 h}{\partial y^2}\right)_x &= \frac{\sqrt{2}}{2a^2}
\end{aligned} \tag{9}$$

Since h_{xx} and h_{yy} have different signs, we know there cannot be a maxima or minima at this point and so will not investigate further.

At $(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}})$:

$$\begin{aligned}\left(\frac{\partial^2 h}{\partial x^2}\right)_y &= \frac{\sqrt{2}}{2a^2}\\ \left(\frac{\partial^2 h}{\partial y^2}\right)_x &= \frac{\sqrt{2}}{2a^2}\\ \left(\frac{\partial^2 h}{\partial x \partial y}\right) &= 0\end{aligned} \tag{10}$$

Since both $h_{xx} > 0$ and $h_{yy} > 0$ and $h_{xx}h_{yy} = \frac{2}{4a^4} > 0 = h_{xy}$. This is a minimum.

So the maximum is at $(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}})$ and the minimum is at $(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}})$.

At the maximum:

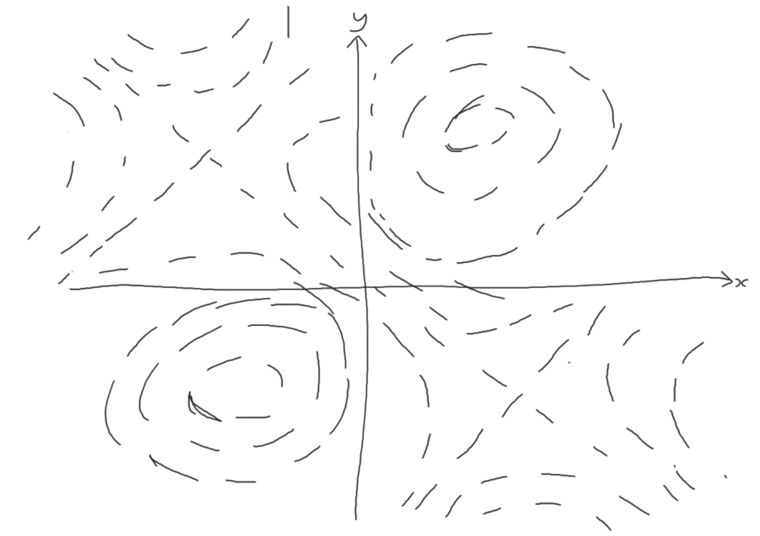
$$\begin{aligned}h(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}) &= \frac{a\left(\frac{a}{\sqrt{2}} + \frac{a}{\sqrt{2}}\right)}{\left(\frac{a}{\sqrt{2}}\right)^2 + \left(\frac{a}{\sqrt{2}}\right)^2 + a^2}\\ h(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}) &= \frac{a\left(\sqrt{2}a\right)}{\frac{a^2}{2} + \frac{a^2}{2} + a^2}\\ h(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}) &= \frac{\sqrt{2}a^2}{2a^2}\\ h(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}) &= \frac{1}{\sqrt{2}}\end{aligned} \tag{11}$$

So the maximum height is $\frac{1}{\sqrt{2}}$.

At the minimum:

$$\begin{aligned}h(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}) &= \frac{a\left(-\frac{a}{\sqrt{2}} - \frac{a}{\sqrt{2}}\right)}{\left(-\frac{a}{\sqrt{2}}\right)^2 + \left(-\frac{a}{\sqrt{2}}\right)^2 + a^2}\\ h(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}) &= \frac{a\left(-\sqrt{2}a\right)}{\frac{a^2}{2} + \frac{a^2}{2} + a^2}\\ h(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}) &= \frac{-\sqrt{2}a^2}{2a^2}\\ h(-\frac{a}{\sqrt{2}}, -\frac{a}{\sqrt{2}}) &= -\frac{1}{\sqrt{2}}\end{aligned} \tag{12}$$

So the minimum height is $-\frac{1}{\sqrt{2}}$.



15. (a)

$$
\begin{aligned}
z &= (x^2 - y^2)e^{-x^2-y^2} \\
\left(\frac{\partial z}{\partial x}\right)_y &= 2xe^{-x^2-y^2} - 2x(x^2-y^2)e^{-x^2-y^2} \\
0 &= 2x(y^2 - x^2 + 1)e^{-x^2-y^2} \\
0 &= x(y^2 + 1 - x^2) \\
\left(\frac{\partial z}{\partial y}\right)_x &= -2ye^{-x^2-y^2} - 2y(x^2-y^2)e^{-x^2-y^2} \\
0 &= -2y(x^2 - y^2 + 1)e^{-x^2-y^2} \\
0 &= y(x^2 - y^2 + 1)
\end{aligned} \tag{13}
$$

Assume that there is a stationary point with $x \neq 0, y \neq 0$.

$$
\begin{aligned}
y^2 - x^2 + 1 &= 0 \\
x^2 - y^2 + 1 &= 0 \\
y^2 - x^2 + 1x^2 - y^2 + 1 &= 0 \\
2 &= 0
\end{aligned} \tag{14}
$$

This is absurd. So there is no stationary point such that $x \neq 0$ and $y \neq 0$.

Take $x = 0$:

$$
\begin{aligned}
0 &= y(x^2 - y^2 + 1) \\
y = 0 &\vee 1 - y^2 = 0 \\
y = 0 &\vee y = 1 \vee y = -1
\end{aligned} \tag{15}
$$

Take $y = 0$:

$$
\begin{aligned}
0 &= x(y^2 - x^2 + 1) \\
x = 0 &\vee 1 - y^2 = 0 \\
x = 0 &\vee x = 1 \vee x = -1
\end{aligned} \tag{16}
$$

So the stationary points are at:
$(0, 0), (0, 1), (0, -1), (-1, 0), (1, 0)$

(b) Moving along the contour $z = 0$ satisfies the equation:

$$\begin{aligned} 0 &= (x^2 - y^2)e^{-x^2-y^2} \\ 0 &= x^2 - y^2 \\ x &= \pm y \end{aligned} \tag{17}$$

Using this we can see that $(0, 0)$ lies on the contour $z = 0$ and so $(0, 0)$ must be a saddle point.

$$\begin{aligned} \left(\frac{\partial z}{\partial x}\right)_y &= 2xe^{-x^2-y^2} - 2x(x^2 - y^2)e^{-x^2-y^2} \\ \left(\frac{\partial z}{\partial x}\right)_y &= (2x - 2x^3 + 2xy^2)e^{-x^2-y^2} \\ \left(\frac{\partial^2 z}{\partial x^2}\right)_y &= (2 - 6x^2 + 2y^2 - 4x^2 + 4x^4 - 4x^2y^2)e^{-x^2-y^2} \\ \left(\frac{\partial^2 z}{\partial x^2}\right)_y &= (2 - 10x^2 + 2y^2 + 4x^4 - 4x^2y^2)e^{-x^2-y^2} \\ \left(\frac{\partial z}{\partial y}\right)_x &= -2ye^{-x^2-y^2} - 2y(x^2 - y^2)e^{-x^2-y^2} \\ \left(\frac{\partial z}{\partial y}\right)_x &= (-2y - 2x^2y + 2y^3)e^{-x^2-y^2} \\ \left(\frac{\partial^2 z}{\partial y^2}\right)_x &= (-2 - 2x^2 + 6y^2 + 4y^2 + 4x^2y^2 - 4y^4)e^{-x^2-y^2} \\ \left(\frac{\partial^2 z}{\partial y^2}\right)_x &= (-2 - 2x^2 + 10y^2 + 4x^2y^2 - 4y^4)e^{-x^2-y^2} \\ \left(\frac{\partial^2 z}{\partial x \partial y}\right) &= 4xy(x^2 - y^2)e^{-x^2-y^2} \end{aligned} \tag{18}$$

At $(1, 0)$:

$$\begin{aligned} \left(\frac{\partial^2 z}{\partial x^2}\right)_y &= -4e^{-1} \\ \left(\frac{\partial^2 z}{\partial y^2}\right) &= -4e^{-1} \\ \left(\frac{\partial^2 z}{\partial x \partial y}\right) &= 0 \end{aligned} \tag{19}$$

So $z_{xx} < 0$, $z_{yy} < 0$ and $z_{xx}z_{yy} = 16e^{-1} > f_{xy}^2$.
So there is a maximum at $(1, 0)$.

At $(-1, 0)$:

$$\begin{aligned} \left(\frac{\partial^2 z}{\partial x^2}\right)_y &= -4e^{-1} \\ \left(\frac{\partial^2 z}{\partial y^2}\right) &= -4e^{-1} \\ \left(\frac{\partial^2 z}{\partial x \partial y}\right) &= 0 \end{aligned} \tag{20}$$

So $z_{xx} < 0$, $z_{yy} < 0$ and $z_{xx}z_{yy} = 16e^{-1} > f_{xy}^2$.
So there is a maximum at $(-1, 0)$.

At $(0,1)$:

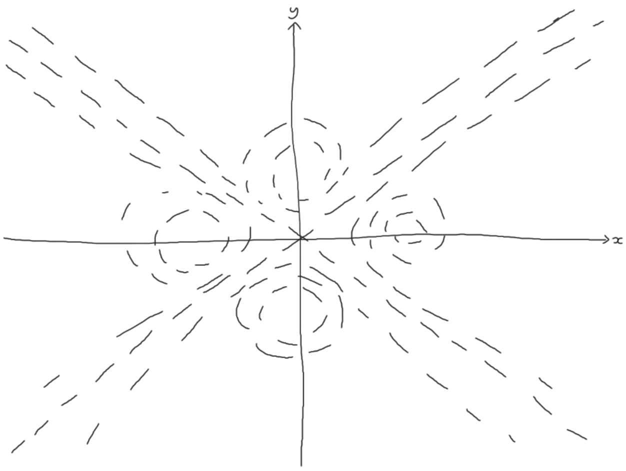
$$\begin{aligned}\left(\frac{\partial^2 z}{\partial x^2}\right)_y &= 4e^{-1}\\ \left(\frac{\partial^2 z}{\partial y^2}\right) &= 4e^{-1}\\ \left(\frac{\partial^2 z}{\partial x \partial y}\right) &= 0\end{aligned} \tag{21}$$

So $z_{xx} > 0$, $z_{yy} > 0$ and $z_{xx}z_{yy} = 16e^{-1} > f_{xy}^2$.
So there is a minimum at $(0,1)$.

At $(0,-1)$:

$$\begin{aligned}\left(\frac{\partial^2 z}{\partial x^2}\right)_y &= 4e^{-1}\\ \left(\frac{\partial^2 z}{\partial y^2}\right) &= 4e^{-1}\\ \left(\frac{\partial^2 z}{\partial x \partial y}\right) &= 0\end{aligned} \tag{22}$$

So $z_{xx} > 0$, $z_{yy} > 0$ and $z_{xx}z_{yy} = 16e^{-1} > f_{xy}^2$.
So there is a minimum at $(0,-1)$.



16. (a)

$$\begin{aligned}f &= \frac{1}{x^2+y^2+1}\\ \left(\frac{\partial f}{\partial x}\right)_y &= \frac{-2x}{(x^2+y^2+1)^2}\\ \left(\frac{\partial^2 f}{\partial x^2}\right)_y &= \frac{2x^2-2y^2-2}{(x^2+y^2+1)^3}\\ \left(\frac{\partial^2 f}{\partial x \partial y}\right) &= \frac{8xy}{(x^2+y^2+1)^3}\\ \left(\frac{\partial f}{\partial y}\right)_x &= \frac{-2y}{(x^2+y^2+1)^2}\\ \left(\frac{\partial^2 f}{\partial x^2}\right)_y &= \frac{2y^2-2x^2-2}{(x^2+y^2+1)^3}\end{aligned} \tag{23}$$

At any stationary point for the function f, $f_x = 0$ and $f_y = 0$:

$$\begin{aligned} \left(\frac{\partial f}{\partial x}\right)_y &= 0 \\ \frac{-2x}{(x^2+y^2+1)^2} &= 0 \\ -2x &= 0 \\ x &= 0 \end{aligned} \tag{24}$$

$$\begin{aligned} \left(\frac{\partial f}{\partial x}\right)_y &= 0 \\ \frac{-2y}{(x^2+y^2+1)^2} &= 0 \\ -2y &= 0 \\ y &= 0 \end{aligned} \tag{25}$$

So the only stationary point for the function f is at $(0, 0)$.
At $(0, 0)$:

$$\begin{aligned} \left(\frac{\partial^2 f}{\partial x^2}\right)_y &= -2 \\ \left(\frac{\partial^2 f}{\partial y^2}\right)_x &= -2 \\ \left(\frac{\partial^2 f}{\partial x \partial y}\right) &= 0 \end{aligned} \tag{26}$$

So $f_{xx} < 0$, $f_{yy} < 0$ and $f_{xx}f_{yy} > f_{xy}^2$.
This means that there is a maximum at $(0, 0)$.

(b)

$$\begin{aligned} f &= \sin x \sin y \\ \left(\frac{\partial f}{\partial x}\right)_y &= \cos x \sin y \\ \left(\frac{\partial^2 f}{\partial x^2}\right)_y &= -\sin x \sin y \\ \left(\frac{\partial f}{\partial y}\right)_x &= \sin x \cos y \\ \left(\frac{\partial^2 f}{\partial y^2}\right)_x &= -\sin x \sin y \\ \left(\frac{\partial^2 f}{\partial x \partial y}\right) &= \cos x \cos y \end{aligned} \tag{27}$$

At any stationary point of the function f, $f_x = f_y = 0$.

$$\begin{aligned} 0 &= \cos x \sin y = \sin x \cos y \\ \cos x &= 0 \wedge \cos y = 0 \vee \sin y = 0 \wedge \sin x = 0 \\ x &= \frac{\pi}{2} \wedge y = \frac{\pi}{2} \end{aligned} \tag{28}$$

So the only stationary point of the function in the range $x, y \in (0, \pi)$ is $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

At $(\frac{\pi}{2}, \frac{\pi}{2})$, the partial derivatives are:

$$\begin{aligned}\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= -1\\ \left(\frac{\partial^2 f}{\partial y^2}\right)_x &= -1\\ \left(\frac{\partial^2 f}{\partial x \partial y}\right) &= 0\end{aligned} \tag{29}$$

So $f_{xx}f_{yy} = 1 > f_{xy}^2$.
So this point is a maximum:
The only stationary point on the curve in the range given is $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ and is a maximum.

(c)

$$\begin{aligned}f &= (xy - y)e^{2x - x^2 - y^2}\\ \left(\frac{\partial f}{\partial x}\right)_y &= y(-2x^2 + 4x - 1)e^{2x - x^2 - y^2}\\ \left(\frac{\partial^2 f}{\partial x^2}\right)_y &= y(4x^3 - 12x^2 + 6x + 2)e^{2x - x^2 - y^2}\\ \left(\frac{\partial^2 f}{\partial x \partial y}\right) &= (4x^2y^2 - 8xy^2 + 2y^2 - 2x^2 + 4x - 1)e^{2x - x^2 - y^2}\\ \left(\frac{\partial f}{\partial y}\right)_x &= (x - 1)e^{2x - x^2 - y^2} - 2y(xy - y)e^{2x - x^2 - y^2}\\ \left(\frac{\partial f}{\partial y}\right)_x &= (x - 1 - 2xy^2 + 2y^2)e^{2x - x^2 - y^2}\\ \left(\frac{\partial^2 f}{\partial y^2}\right)_x &= (4y - 4xy)e^{2x - x^2 - y^2} - 2y(x - 1 - 2xy^2 + 2y^2)e^{2x - x^2 - y^2}\\ \left(\frac{\partial^2 f}{\partial y^2}\right)_x &= y(6 - 6x + 4xy^2 - 4y^2)e^{2x - x^2 - y^2}\end{aligned} \tag{30}$$

At any stationary point, both $f_x = 0$ and $f_y = 0$.

$$\begin{aligned}\left(\frac{\partial f}{\partial x}\right)_y &= 0\\ y(-2x^2 + 4x - 1)e^{2x - x^2 - y^2} &= 0\\ y = 0 \vee 2x^2 - 4x + 1 &= 0\\ y = 0 \vee x = 1 + \frac{\sqrt{2}}{2} \vee x &= 1 - \frac{\sqrt{2}}{2}\end{aligned} \tag{31}$$

$$\begin{aligned}
\left(\frac{\partial f}{\partial y}\right)_x &= 0 \\
(x-1-2xy^2+2y^2)e^{2x-x^2-y^2} &= 0 \\
x-1-2xy^2+2y^2 &= 0 \\
\text{For } y &= 0 \\
x-1 &= 0 \\
x &= 1 \\
\text{For } x &= 1+\frac{\sqrt{2}}{2} \\
1+\frac{\sqrt{2}}{2}-1-2y^2-\sqrt{2}y^2+y^2 &= 0 \\
\sqrt{2}y^2 &= \frac{\sqrt{2}}{2} \\
y &= \pm\frac{\sqrt{2}}{2} \\
\text{For } x &= 1-\frac{\sqrt{2}}{2} \\
1-\frac{\sqrt{2}}{2}-1-2y^2+\sqrt{2}y^2+y^2 &= 0 \\
\sqrt{2}y^2 &= \frac{\sqrt{2}}{2} \\
y &= \pm\frac{\sqrt{2}}{2}
\end{aligned} \tag{32}$$

So there are five stationary points:
$(1,0),(1-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}),(1-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}),(1+\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}),(1+\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})$
At (1, 0):

$$\begin{aligned}
\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= y(4x^3-12x^2+6x+2)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= 0 \\
\left(\frac{\partial^2 f}{\partial y^2}\right)_x &= y(6-6x+4xy^2-4y^2)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial y^2}\right)_x &= 0 \\
\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= (4x^2y^2-8xy^2+2y^2-2x^2+4x-1)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= e \\
\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial f}{\partial y}\right)_x &= 0 < \left(\frac{\partial^2 f}{\partial x \partial y}\right)
\end{aligned} \tag{33}$$

However, since $\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial y}\right)_x = 0$, we cannot claim that they have the opposite side and so cannot tell whether this is a maximum, minimum or a saddle point.

At $(1-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$:

$$\begin{aligned}
\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= y(4x^3 - 12x^2 + 6x + 2)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= 2 \\
\left(\frac{\partial^2 f}{\partial y^2}\right)_x &= y(6 - 6x + 4xy^2 - 4y^2)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial y^2}\right)_x &= 2 \\
\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= (4x^2y^2 - 8xy^2 + 2y^2 - 2x^2 + 4x - 1)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= 0 \\
\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial f}{\partial y}\right)_x &= 4 > \left(\frac{\partial^2 f}{\partial x \partial y}\right)
\end{aligned} \tag{34}$$

Since both $\left(\frac{\partial^2 f}{\partial x^2}\right)_y$ and $\left(\frac{\partial^2 f}{\partial y^2}\right)_x$ are positive at this value, this must be a minimum. So the function has a minimum at $(1-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

At $(1-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$:

$$\begin{aligned}
\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= y(4x^3 - 12x^2 + 6x + 2)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= -2 \\
\left(\frac{\partial^2 f}{\partial y^2}\right)_x &= y(6 - 6x + 4xy^2 - 4y^2)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial y^2}\right)_x &= -2 \\
\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= (4x^2y^2 - 8xy^2 + 2y^2 - 2x^2 + 4x - 1)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= 0 \\
\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial f}{\partial y}\right)_x &= 4 > \left(\frac{\partial^2 f}{\partial x \partial y}\right)
\end{aligned} \tag{35}$$

Since both $\left(\frac{\partial^2 f}{\partial x^2}\right)_y$ and $\left(\frac{\partial^2 f}{\partial y^2}\right)_x$ are negative at this value, this must be a maximum. So the function has a maximum at $(1-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

At $(1+\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$:

$$\begin{aligned}
\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= y(4x^3 - 12x^2 + 6x + 2)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= -2 \\
\left(\frac{\partial^2 f}{\partial y^2}\right)_x &= y(6 - 6x + 4xy^2 - 4y^2)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial y^2}\right)_x &= -2 \\
\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= (4x^2y^2 - 8xy^2 + 2y^2 - 2x^2 + 4x - 1)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= 0 \\
\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial f}{\partial y}\right)_x &= 4 > \left(\frac{\partial^2 f}{\partial x \partial y}\right)
\end{aligned} \tag{36}$$

Since both $\left(\frac{\partial^2 f}{\partial x^2}\right)_y$ and $\left(\frac{\partial^2 f}{\partial y^2}\right)_x$ are negative at this value, this must be a maximum. So the function has a maximum at $(1+\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

At $(1+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$:

$$\begin{aligned}
\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= y(4x^3 - 12x^2 + 6x + 2)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial x^2}\right)_y &= 2 \\
\left(\frac{\partial^2 f}{\partial y^2}\right)_x &= y(6 - 6x + 4xy^2 - 4y^2)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial y^2}\right)_x &= 2 \\
\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= (4x^2y^2 - 8xy^2 + 2y^2 - 2x^2 + 4x - 1)e^{2x-x^2-y^2} \\
\left(\frac{\partial^2 f}{\partial x \partial y}\right) &= 0 \\
\left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial f}{\partial y}\right)_x &= 4 > \left(\frac{\partial^2 f}{\partial x \partial y}\right)
\end{aligned} \tag{37}$$

Since both $\left(\frac{\partial^2 f}{\partial x^2}\right)_y$ and $\left(\frac{\partial^2 f}{\partial y^2}\right)_x$ are positive at this value, this must be a minimum. So the function has a minimum at $(1+\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$.

17. (a)

$$\begin{aligned}
f &= xy^2 \\
\nabla(f) &= (y^2, 2xy)
\end{aligned} \tag{38}$$

$$\begin{aligned}
1 &= x^2 + y^2 \\
0 &= x^2 + y^2 - 1 \\
g &= x^2 + y^2 - 1 \\
\nabla(g) &= (2x, 2y)
\end{aligned} \tag{39}$$

$$\begin{aligned} y^2 &= 2\lambda x \\ 2xy &= 2\lambda y \\ x^2 + y^2 &= 1 \\ x^2 + 2\lambda x &= 1 \\ x^2 + 2\lambda x - 1 &= 0 \end{aligned} \tag{40}$$

$$\begin{aligned} x &= \frac{-2\lambda \pm \sqrt{4\lambda^2 + 4}}{2} \\ x &= -\lambda \pm \sqrt{\lambda^2 + 1} \\ 2\lambda y &= 2(\lambda \pm \sqrt{\lambda^2 + 1})y \\ \lambda &= -\lambda \pm \sqrt{\lambda^2 + 1} \vee y = 0 \\ 2\lambda &= \pm\sqrt{\lambda^2 + 1} \\ 4\lambda^2 &= \lambda^2 + 1 \\ 3\lambda^2 &= 1 \\ \lambda &= \pm\frac{\sqrt{3}}{3} \end{aligned} \tag{41}$$

$$\begin{aligned} 2xy &= 2\lambda y \\ x &= \lambda \\ x &= \pm\frac{\sqrt{3}}{3} \\ y^2 &= 2\lambda x \\ y^2 &= 2\lambda^2 \\ y &= \pm\sqrt{2}\lambda \\ y &= \pm\frac{\sqrt{6}}{3} \end{aligned} \tag{42}$$

At $y = 0$, $x = \pm 1$.

So the function xy^2 subject to the constraint $x^2 + y^2 = 1$ has stationary points at: $(1, 0), (-1, 0), \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right), \left(\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\right), \left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{6}}{3}\right), \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{6}}{3}\right)$

Verification by using the substitution $x = \cos\theta$, $y = \sin\theta$.

$$\begin{aligned} f &= xy^2 \\ f &= \cos\theta - \cos^3\theta \\ \frac{\mathrm{d}f}{\mathrm{d}\theta} &= -\sin\theta + 3\sin\theta\cos^2\theta \\ 0 &= -\sin\theta + 3\sin\theta\cos^2\theta \\ 0 &= \sin\theta(3\cos\theta - 1) \end{aligned} \tag{43}$$

So $\sin\theta = 0$ or $3\cos^2\theta = 1$.

At $\sin\theta = 0$:

$$\begin{aligned} \sin\theta &= 0 \\ \theta &= 0 \vee \theta = \pi \\ y = 0 \wedge x = -1 &\vee y = 0 \wedge x = 1 \end{aligned} \tag{44}$$

These are stationary points we also found using the lagrangian method.

At $3\cos^2\theta = 1$:

$$\begin{aligned} \cos\theta &= \pm\frac{\sqrt{3}}{3} \\ x = \pm\frac{\sqrt{3}}{3}&, y = \pm\frac{\sqrt{6}}{3} \end{aligned} \tag{45}$$

These are the rest of the stationary points we found using the lagrangian method.

So we have found all the same stationary points using the two different methods.

(b)

$$\begin{aligned} f &= e^{-xy} \\ \nabla(f) &= (-ye^{-xy}, -xe^{-xy}) \\ g &= x^2 + y^2 - 1 \\ \nabla(g) &= (2x, 2y) \end{aligned} \tag{46}$$

Using these and the initial constraint we can form three equations and then solve to find the stationary points.

$$\begin{aligned} -ye^{-xy} &= 2\lambda x \\ -xe^{-xy} &= 2\lambda y \\ x^2 + y^2 &= 1 \end{aligned} \tag{47}$$

Note that at $x = 0$, y can be either positive or negative. If we make $x > 0$ for some small value, then the value of $-xy$ will become nonzero. So e^{-xy} will increase if $y < 0$ and decrease if $y > 0$. The inverse argument holds for making $x < 0$. Since a small change in x can increase or decrease e^{-xy} we can conclude that this is not a minimum. So we do not need to consider this case.

$$\begin{aligned} -ye^{-xy} &= 2\lambda x \\ -\frac{e^{-xy}}{2\lambda} &= \frac{x}{y} \\ -xe^{-xy} &= 2\lambda y \\ -\frac{e^{-xy}}{2\lambda} &= \frac{y}{x} \\ \frac{x}{y} &= \frac{y}{x} \\ x^2 &= y^2 \\ x^2 + y^2 &= 1 \\ x^2 + x^2 &= 1 \\ 2x^2 &= 1 \\ x^2 &= \frac{1}{2} \\ x = \pm\frac{\sqrt{2}}{2}&, y = \pm\frac{\sqrt{2}}{2} \end{aligned} \tag{48}$$

So the function e^{-xy} subject to the constraint $x^2 + y^2 = 1$ has stationary points at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

Informally looking at the value of the function at those points, I can conclude that the stationary points at $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ are maxima and those at $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ are minima.
Verification using the substition $x = \cos\theta$ and $y = \sin\theta$:

$$\begin{aligned} f &= e^{-\frac{1}{2}\sin 2\theta} \\ \frac{\mathrm{d}f}{\mathrm{d}\theta} &= -\cos 2\theta e^{-\frac{1}{2}\sin 2\theta} \\ 0 &= -\cos 2\theta e^{-\frac{1}{2}\sin 2\theta} \\ 0 &= -\cos 2\theta \\ \theta &= \frac{\pi}{4} \vee \frac{3\pi}{4} \vee \frac{5\pi}{4} \vee \frac{7\pi}{4} \\ x = \pm\frac{\sqrt{2}}{2}, y &= \pm\frac{\sqrt{2}}{2} \end{aligned} \tag{49}$$

This is the same result as obtained by the lagrangian method.

18.

$$\begin{aligned} V &= 2x \times 2y \times 2z \\ V &= 8xyz \\ \nabla(V) &= (8yz, 8xz, 8xy) \\ g &= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \\ \nabla(g) &= \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right) \end{aligned} \tag{50}$$

We now form the equations shown below and use these to find the stationary points.

$$\begin{aligned} 8yz &= \lambda\frac{2x}{a^2} \\ 8xz &= \lambda\frac{2y}{b^2} \\ 8xy &= \lambda\frac{2z}{c^2} \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \end{aligned} \tag{51}$$

Since we know that x, y, z are distances: they are nonzero and non-negative. This means that we know minmuma will occur whenever $x = 0, y = 0, z = 0$ and can multiply and divide by x, y, z without worrying about zeros.

$$\begin{aligned} y &= \lambda\frac{x}{4a^2z} \\ 8xz &= \lambda^2\frac{x}{2a^2b^2z} \\ 16a^2b^2z^2 &= \lambda^2 \\ \lambda 8x\frac{x}{4a^2z} &= \lambda\frac{2z}{c^2} \\ \frac{2x^2}{a^2z} &= \frac{2z}{c^2} \\ c^2x^2 &= a^2z^2 \\ z^2 &= \frac{c^2x^2}{a^2} \end{aligned} \tag{52}$$

$$\begin{aligned}
16a^2b^2z^2 &= \lambda^2 \\
16a^2b^2\frac{c^2x^2}{a^2} &= \lambda^2 \\
x^2 &= \frac{\lambda^2}{16b^2c^2} \\
z^2 &= \frac{c^2x^2}{a^2} \\
z^2 &= \frac{\lambda^2}{16a^2b^2} \\
y^2 &= \frac{\lambda^2x^2}{16a^4z^2} \\
y^2 &= \frac{\lambda^2}{16a^2c^2}
\end{aligned} \tag{53}$$

Now we can substitute our results into the euqation, work out the value for λ and then substitute it back into the value for x, y, z to work out the maxima.

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} &= 1 \\
\frac{\lambda}{16a^2b^2c^2} + \frac{\lambda}{16a^2b^2c^2} + \frac{\lambda}{16a^2b^2c^2} &= 1
\end{aligned} \tag{54}$$

$$\begin{aligned}
3\lambda^2 &= 16a^2b^2c^2 \\
\lambda^2 &= \frac{16a^2b^2c^2}{3} \\
x^2 &= \frac{\lambda^2}{16b^2c^2} \\
x^2 &= \frac{16a^2b^2c^2}{48b^2c^2} \\
x^2 &= \frac{a^2}{3} \\
y^2 &= \frac{\lambda^2}{16a^2c^2} \\
y^2 &= \frac{16a^2b^2c^2}{48a^2c^2} \\
y &= \frac{b^2}{3} \\
z^2 &= \frac{\lambda^2}{16a^2b^2} \\
z^2 &= \frac{16a^2b^2c^2}{48a^2b^2} \\
z^2 &= \frac{c^2}{3}
\end{aligned} \tag{55}$$

Substituting this back into the equation for Volume gives:

$$\begin{aligned}
V^2 &= 64x^2y^2z^2 \\
V^2 &= \frac{64a^2b^2c^2}{27} \\
V &= \frac{8abc}{\sqrt{27}}
\end{aligned} \tag{56}$$

Note that we took the positive root since we know that volume must be greater than zero.

19. I will solve the general formula first maximising area when $a+b+c=k$ and then will set $k=2$.

$$\begin{aligned}
A &= \sqrt{s(s-a)(s-b)(s-c)} \\
\nabla(A) &= \left(-\frac{\sqrt{s(s-b)(s-c)}}{2\sqrt{s-a}}, -\frac{\sqrt{s(s-a)(s-c)}}{2\sqrt{s-b}}, -\frac{\sqrt{s(s-a)(s-b)}}{2\sqrt{s-c}}\right) \\
s &= \frac{1}{2}(a+b+c) \\
0 &= \frac{1}{2}(a+b+c) - s \\
0 &= a+b+c-2s \\
g &= a+b+c-2s \\
\nabla(g) &= (1,1,1)
\end{aligned} \tag{57}$$

Now we have four equations:

$$\begin{aligned}
\frac{\sqrt{s(s-b)(s-c)}}{\sqrt{s-a}} &= \lambda \\
\frac{\sqrt{s(s-a)(s-c)}}{\sqrt{s-b}} &= \lambda \\
\frac{\sqrt{s(s-a)(s-b)}}{\sqrt{s-c}} &= \lambda \\
a+b+c-2s &= 0
\end{aligned} \tag{58}$$

Equating them will solve the equation.

$$\begin{gathered}
-\frac{\sqrt{s(s-b)(s-c)}}{2\sqrt{s-a}} = -\frac{\sqrt{s(s-a)(s-c)}}{2\sqrt{s-b}} \\
s-b = s-a \\
a = b \\
-\frac{\sqrt{s(s-b)(s-c)}}{2\sqrt{s-a}} = -\frac{\sqrt{s(s-a)(s-b)}}{2\sqrt{s-c}} \\
s-c = s-a \\
a = c
\end{gathered} \tag{59}$$

So the maximum occurs when $a=b=c$. Since all the side lengths are equal, this is an equilateral triangle. So the maximal area of a triangle with total side length $2s$ occurs when $a=b=c$ and the triangle is equilateral.

Since s is arbitrary and we have proved this result for perimeter $=2s$, this result holds for all perimeters.

20.

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \nabla(r) &= \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) \\ 1 &= \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \\ 0 &= \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 \\ g &= \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 \\ \nabla(g) &= \left(\frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2}\right) \\ h &= \ell x + my + nz \\ \nabla(h) &= (\ell, m, n) \end{aligned} \tag{60}$$

$$\begin{aligned} \frac{x}{r} &= \lambda\frac{2x}{a^2}+\mu\ell \\ x &= \frac{\mu r a^2\ell}{a^2-2r\lambda} \\ \frac{y}{r} &= \lambda\frac{2y}{b^2}+\mu m \\ y &= \frac{\mu r b^2 m}{b^2-2r\lambda} \\ \frac{z}{r} &= \lambda\frac{2z}{c^2}+\mu n \\ z &= \frac{\mu r c^2 n}{c^2-2r\lambda} \\ 1 &= \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \\ 0 &= \ell x + my + nz \\ 0 &= \ell\left(\frac{\mu r a^2\ell}{a^2-2r\lambda}\right)+m\left(\frac{\mu r b^2 m}{b^2-2r\lambda}\right)+n\left(\frac{\mu r c^2 n}{c^2-2r\lambda}\right) \\ 0 &= \mu r\left(\frac{\ell^2a^2}{a^2-2r\lambda}+\frac{m^2b^2}{b^2-2r\lambda}+\frac{n^2c^2}{c^2-2r\lambda}\right) \end{aligned} \tag{61}$$

Note that $\mu = 0$ is not a valid solution and that since there are value of r which satisfy the constaints which are > 0, the maximum value of r is nonzero. So:

$$0 = \frac{\ell^2a^2}{a^2-2r\lambda}+\frac{m^2b^2}{b^2-2r\lambda}+\frac{n^2c^2}{c^2-2r\lambda} \tag{62}$$

Consider now $\frac{1}{a^2}$, $\frac{1}{b^2}$ and $\frac{1}{b^2}$:

Derived from the equations above:

$$\begin{aligned}
\frac{1}{a^2} &= \frac{x - \mu r\ell}{2\lambda rx} \\
\frac{1}{b^2} &= \frac{y - \mu rm}{2\lambda ry} \\
\frac{1}{c^2} &= \frac{z - \mu rn}{2\lambda rz} \\
1 &= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \\
1 &= \frac{x^2(x - \mu r\ell)}{2\lambda rx} + \frac{y^2(y - \mu rm)}{2\lambda ry} + \frac{z^2(z - \mu rn)}{2\lambda rz} \\
1 &= \frac{x(x - \mu r\ell)}{2\lambda r} + \frac{y(y - \mu rm)}{2\lambda r} + \frac{z(z - \mu rn)}{2\lambda r} \\
2\lambda r &= x(x - \mu r\ell) + y(y - \mu rm) + z(z - \mu rn) \\
2\lambda r &= x^2 + y^2 + z^2 - \mu r(\ell x + my + nz) \\
2\lambda r &= r^2
\end{aligned} \tag{63}$$

Substitute this into the earlier equation:

$$\begin{aligned}
0 &= \frac{\ell^2 a^2}{a^2 - 2r\lambda} + \frac{m^2 b^2}{b^2 - 2r\lambda} + \frac{n^2 c^2}{c^2 - 2r\lambda} \\
0 &= \frac{\ell^2 a^2}{a^2 - r^2} + \frac{m^2 b^2}{b^2 - r^2} + \frac{n^2 c^2}{c^2 - r^2}
\end{aligned} \tag{64}$$

As required.

Geometrically this problem is:

Find the greatest distance from the origin at which the ellipsoid with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and the plane $\ell x + my + nz = 0$ intersect.

μ is the reciprocal of the magnitude of the vector which we are using as the normal to the plane. IE $\mu = \frac{1}{\sqrt{\ell^2+m^2+n^2}}$.

21. Note that since ln is an increasing function and W is guaranteed to be positive, we can take ln of both sides and maximise this function to find the maximum of W.

 (a)

$$\begin{aligned}
W &= \prod_{s=1}^{N} \frac{(g_s - 1 + n_s)!}{(g_s - 1)! n_s!} \\
W &\approx \prod_{s=1}^{N} \frac{(g_s + n_s)!}{g_s! n_s!} \text{ since } g_s \gg 1 \\
\ln W &= \sum_{s=1}^{N} \ln\left(\frac{(g_s + n_s)!}{g_s! n_s!}\right) \\
\ln W &= \sum_{s=1}^{N} (\ln(g_s + n_s)! - \ln g_s! - \ln n_s!) \\
\nabla(\ln W) &= \sum_{s=1}^{N} (\ln(g_s + n_s) - \ln n_s) \text{ using Stirlings approximation} \\
f &= \left(\sum_{n=1}^{N} n_s E_s\right) - \hat{E} \\
\nabla(f) &= \left(\sum_{s=1}^{N} E_s\right) \\
h &= \left(\sum_{n=1}^{N} n_s\right) - \hat{N} \\
\nabla(h) &= 1
\end{aligned} \tag{65}$$

Let β and $-\mu\beta$ be the lagrangian multipliers for f and h respectively:

$$\begin{aligned}
\ln(g_s + n_s) - \ln n_s &= \beta E_s - \mu\beta \\
\ln\left(\frac{g_s + n_s}{n_s}\right) &= \beta E_s - \mu\beta \\
\frac{g_s + n_s}{n_s} &= e^{\beta(E_s - \mu)} \\
g_s + n_s &= n_s e^{\beta(E_s - \mu)} \\
g_s &= n_s e^{\beta(E_s - \mu)} - n_s \\
g_s &= n_s(e^{\beta(E_s - \mu)} - 1) \\
n_s &= \frac{g_s}{e^{\beta(E_s - \mu)} - 1}
\end{aligned} \tag{66}$$

This is the expression we were required to derive and so we are done.

(b)

$$\begin{aligned}
W &= \prod_{s=1}^{N} \frac{g_s!}{n_s!(g_s - n_s)!} \\
\ln W &= \sum_{s=1}^{N} (\ln g_s! - \ln n_s! - \ln(g_s - n_s)!) \\
\nabla(\ln W) &= \sum_{s=1}^{N} (\ln(g_s - n_s) - \ln n_s) \\
f &= \left(\sum_{s=1}^{N} n_s E_s\right) - \hat{E} \\
\nabla(f) &= \left(\sum_{s=1}^{N} E_s\right) \\
h &= \left(\sum_{s=1}^{N} n_s\right) - \hat{n} \\
\nabla(h) &= 1
\end{aligned} \tag{67}$$

Let β and $-\mu\beta$ be the lagrangian multipliers for f and h respectively:

$$\begin{aligned}
\ln(g_s - n_s) - \ln n_s &= \beta E_s - \mu\beta \\
\frac{g_s - n_s}{n_s} &= e^{\beta(E_s - \mu)} \\
g_s - n_s &= n_s e^{\beta(E_s - \mu)} \\
g_s &= n_s(e^{\beta(E_s - \mu)} + 1) \\
n_s &= \frac{g_s}{e^{\beta(E_s - \mu)} + 1}
\end{aligned} \tag{68}$$

22.

$$n \propto \sqrt{E} e^{-\beta E - \alpha} \tag{69}$$

To find the most probable value, we must find the maximum of n. So we must find the maximum of $\sqrt{E}e^{-\beta E - \alpha}$.

Note also the constraint that $\sum_{s=1}^{N} n_s E_s = E$.

$$\begin{aligned}
n &= \sum_{s=1}^{N} \sqrt{E_s} e^{-\beta E_s - \alpha} \\
\nabla(n) &= \left(\sum_{s=1}^{N} \left(\frac{1}{2\sqrt{E_s}} - \beta\sqrt{E_s}\right) e^{-\beta E_s - \alpha}\right) \\
f &= \sum_{s=1}^{N} n_s E_s - \hat{E} \\
\nabla(f) &= \left(\sum_{s=1}^{N} n_s\right)
\end{aligned} \tag{70}$$

Equating coefficients from these two derivatives gives:

$$\begin{aligned}
\left(\frac{1}{2\sqrt{E_s}} - \beta\sqrt{E_s}\right)e^{-\beta E_s - \alpha} &= \gamma n_s \\
\left(\frac{1}{2\sqrt{E_s}} - \beta\sqrt{E_s}\right)e^{-\beta E_s - \alpha} &= \gamma\sqrt{E_s}e^{-\beta E_s - \alpha} \\
\left(\frac{1}{2\sqrt{E_s}} - (\beta+\gamma)\sqrt{E_s}\right)e^{-\beta E_s - \alpha} &= 0 \\
\frac{1}{2\sqrt{E_s}} - (\beta+\gamma)\sqrt{E_s} &= 0 \\
1 - 2(\beta+\gamma)E_s &= 0 \\
E &= \frac{1}{2(\beta+\gamma)}
\end{aligned} \tag{71}$$

So the most probable kinetic energy E of a particle is $\frac{1}{2(\beta+\gamma)}$ where γ is the Lagrange multiplier.

We know that the total kinetic energy of all the particles in the gas is constant ($\hat{E}$) and that the amount of particles of gas is constant ($\hat{N}$).

Since we know both the total number of particles and the total energy of those particles, we can work out the mean (expected) energy of a particle:

$$\overline{E} = \frac{\hat{E}}{\hat{N}} \tag{72}$$

Where $\overline{E}$ is the expected energy, $\hat{E}$ is the total internal energy (which we have been told is constant) and $\hat{N}$ is the total number of particles (which is also constant).