

6. For the function $f(x, y, z) = \ln(x^2 + y^2) + z$, find ∇f .

$$\nabla f = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2}, 1 \right) \quad (1)$$

(a) At the point $(3, -4, 0)$ the normal to the cylinder is trivially in the direction $\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$. The magnitude of this vector is 5 and so \hat{n} is $\begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix}$. Using this we can work out the value of f at this point.

$$\begin{aligned} (\nabla f)(3, -4, 4) \cdot \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} &= \begin{pmatrix} \frac{6}{9+16} \\ -\frac{8}{9+16} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} \\ (\nabla f)(3, -4, 4) \cdot \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} &= \frac{6}{25} \times \frac{3}{5} + \frac{8}{25} \times \frac{4}{5} \\ (\nabla f)(3, -4, 4) \cdot \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} &= \frac{18}{125} + \frac{32}{125} \\ (\nabla f)(3, -4, 4) \cdot \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} &= \frac{2}{5} \end{aligned} \quad (2)$$

(b)

$$\hat{\mathbf{m}} = \frac{\mathbf{m}}{|\mathbf{m}|} = \frac{\mathbf{m}}{\sqrt{5}} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \quad (3)$$

$$\begin{aligned} (\nabla f)(3, -4, 4) \cdot \mathbf{m} &= \begin{pmatrix} \frac{6}{9+16} \\ -\frac{8}{9+16} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \\ (\nabla f)(3, -4, 4) \cdot \mathbf{m} &= \frac{6}{25} \times \frac{1}{\sqrt{5}} - \frac{8}{25} \times \frac{2}{\sqrt{5}} \\ (\nabla f)(3, -4, 4) \cdot \mathbf{m} &= \frac{6}{25\sqrt{5}} - \frac{16}{25\sqrt{5}} \\ (\nabla f)(3, -4, 4) \cdot \mathbf{m} &= -\frac{2\sqrt{5}}{25} \end{aligned} \quad (4)$$

7.

$$\begin{aligned} f &= xz + z^2 - xy^2 \\ \nabla f &= (z - y^2, -2xy, x + 2z) \\ (\nabla f)(1, 1, 2) &= (1, -2, 5) \end{aligned} \quad (5)$$

So a normal vector to the surface at the point $(1, 1, 2)$ is $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$.

Hence the equation of the tangent plane at this point is:

$$\begin{aligned} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \\ x - 2y + 5z &= 9 \end{aligned} \quad (6)$$

8.

$$f = 3x^2y \sin\left(\frac{\pi x}{2}\right) - z$$

$$\nabla f = \left(6xy \sin\left(\frac{\pi x}{2}\right) + \frac{3\pi x^2y}{2} \cos\left(\frac{\pi x}{2}\right), 3x^2 \sin\left(\frac{\pi x}{2}\right), -1\right) \quad (7)$$

$$z(1, 1) = 3 \sin\left(\frac{\pi}{2}\right)$$

$$z(1, 1) = 3 \quad (8)$$

$$(\nabla f)(1, 1, 3) = (6, 3, -1) \quad (9)$$

So the equation of the plane which is tangent to the surface at the point $(1, 1, 3)$ is:

$$\begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$6x + 3y - z = 6 \quad (10)$$

At the point with $x = 1, y = \frac{1}{2}, z = \frac{3}{2}$.

$$(\nabla f)\left(1, \frac{1}{2}, \frac{3}{2}\right) = (3, 3, -1) \quad (11)$$

Consider now $-n$. This is also a normal to the plane but facing in the increasing z direction (I assume that we are placing the marble on the plane from above – not suspending it below the plane). Now the normal is $\begin{pmatrix} -3 \\ -3 \\ 1 \end{pmatrix}$. so if we place a marble on the plane then it will roll South-West.

9. Consider the substitution:

$$x = a \cos \theta$$

$$dx = -a \sin \theta d\theta$$

$$y = a \sin \theta$$

$$dy = a \cos \theta d\theta \quad (12)$$

$$\begin{aligned} & \int_{\Gamma} [P(x, y) dx + Q(x, y) dy] \\ &= \int_{\Gamma} [-x^2y dx + xy^2 dy] \\ &= \int_0^{\pi} [a^4 \cos^2 \theta \sin^2 \theta d\theta + a^4 \cos^2 \theta \sin^2 \theta d\theta] \\ &= a^4 \int_0^{\pi} 2 \cos^2 \theta \sin^2 \theta d\theta \\ &= a^4 \int_0^{\pi} \frac{1}{2} \sin^2 2\theta d\theta \\ &= a^4 \int_0^{\pi} \frac{1}{4} - \frac{1}{4} \cos 4\theta d\theta \\ &= a^4 \left[\frac{1}{4} \theta - \frac{1}{16} \sin 4\theta \right]_0^{\pi} \\ &= \frac{\pi a^4}{4} \end{aligned} \quad (13)$$

$$\begin{aligned}
 & \int \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \\
 &= \int \int y^2 + x^2 dx dy \\
 &= \int_0^\pi \int_0^a r^2 \times r dr d\theta \\
 &= \int_0^\pi \int_0^a r^3 dr d\theta \\
 &= \int_0^\pi \left[\frac{1}{4} r^4 \right]_0^a d\theta \\
 &= \int_0^\pi \frac{a^4}{4} d\theta \\
 &= \frac{\pi a^4}{4} \\
 &= \int_\Gamma [P(x, y) dx + Q(x, y) dy] \text{ as required}
 \end{aligned} \tag{14}$$

10.

$$\int_\Gamma [f(y) dx + x \cos y dy] = 0 \tag{15}$$

The integrand of an exact differential along any closed contour is equal to zero. So if $\int_\Gamma [f(y) dx + x \cos y dy]$ is an exact differential, then it will be equal to zero for all closed contours Γ .

For the integrand to be an exact differential:

$$\begin{aligned}
 \left(\frac{\partial}{\partial y} \right) f(y) &= \left(\frac{\partial}{\partial x} \right) x \cos y \\
 \left(\frac{\partial}{\partial y} \right) f(y) &= \cos y \\
 f(y) &= \int \cos y dy \\
 f(y) &= \sin y + c
 \end{aligned} \tag{16}$$

11. (i) For the purposes of this question I will assume the curve is *closed* since the statement is untrue if it is not.

If $\mathbf{F} = -\nabla(\Phi)$ for some Φ , then $\int_C \mathbf{F} \cdot d\mathbf{x} = 0$ for all closed curves C .

$$\begin{aligned}
 \mathbf{F} &= \mathbf{c} \times \mathbf{v} \\
 \mathbf{F} &= -\mathbf{c} \left(-\frac{d\mathbf{x}(t)}{dt} \right) \\
 \mathbf{F} &= -(\nabla(-\mathbf{x}(t))) \\
 \mathbf{F} &= -\nabla\Phi
 \end{aligned} \tag{17}$$

This is a necessary and sufficient condition for the $\int_C \mathbf{F} \cdot d\mathbf{x}$ to be equal to zero for all closed curves C .

So the integral $\int_C \mathbf{F} \cdot d\mathbf{x} = 0$ for all closed curves C .

The integral is equal to the work done. So the work done is equal to 0 for all closed curves C .

(ii) (a)

$$\begin{aligned}
 W &= \int_C \mathbf{F} \cdot d\mathbf{x} \\
 &= \int_0^\pi (y, -x, -1) \cdot (-\sin t, \cos t, 1) dt \\
 &= \int_0^\pi (\sin t, -\cos t, -1) \cdot (-\sin t, \cos t, 1) dt \\
 &= \int_0^\pi -\sin^2 t - \cos^2 t - 1 dt \\
 &= \int_0^\pi -1 - 1 dt \\
 &= \int_0^\pi -2 dt \\
 &= -2\pi
 \end{aligned} \tag{18}$$

(b)

$$\begin{aligned}
 W &= \int_C \mathbf{F} \cdot d\mathbf{x} \\
 &= \int_0^\pi (x, y, 0) \cdot (-\sin t, \cos t, 1) dt \\
 &= \int_0^\pi -\sin t \cos t + \sin t \cos t dt \\
 &= \int_0^\pi 0 dt \\
 &= 0
 \end{aligned} \tag{19}$$

12. A condition satisfied by all conservative vector fields $\mathbf{F} = (P(x, y), Q(x, y))$ is that \mathbf{F} is an exact differential. So:

$$\left(\frac{\partial}{\partial y}\right) P(x, y) = \left(\frac{\partial}{\partial x}\right) Q(x, y) \tag{20}$$

(a)

$$P = x^2 y + y, \quad Q = xy^2 + x \tag{21}$$

$$\left(\frac{\partial}{\partial y}\right) P(x, y) = x^2 + 1$$

$$\left(\frac{\partial}{\partial x}\right) Q(x, y) = y^2 + 1 \tag{22}$$

$$\left(\frac{\partial}{\partial y}\right) P(x, y) \neq \left(\frac{\partial}{\partial x}\right) Q(x, y)$$

So \mathbf{F} is **not** a conservative vector field

(b)

$$P = ye^{xy} + 2x + y, \quad Q = xe^{xy} + x \tag{23}$$

$$\left(\frac{\partial}{\partial y}\right) P(x, y) = xye^{xy} + e^{xy} + 1$$

$$\left(\frac{\partial}{\partial x}\right) Q(x, y) = xye^{xy} + e^{xy} + 1 \tag{24}$$

$$\left(\frac{\partial}{\partial y}\right) P(x, y) = \left(\frac{\partial}{\partial x}\right) Q(x, y)$$

So \mathbf{F} is a conservative vector field

13. (i)

$$\begin{aligned}(0, 0, 0) &\longrightarrow (0, 0, 1) \\ x = y = 0, z &= t \\ d\mathbf{x} &= (0, 0, 1)\end{aligned}$$

$$\begin{aligned}&\int \mathbf{F} \cdot d\mathbf{x} \\&= \int_0^1 (4x^3z + 2x, z^2 - 2y, x^4 + 2yz) \cdot (0, 0, 1) dt \\&= \int_0^1 x^4 + 2yz dt \\&= \int_0^1 0 dt \\&= 0\end{aligned} \tag{25}$$

$$\begin{aligned}(0, 0, 1) &\longrightarrow (0, 1, 1) \\ x = 0, y = t, z &= 1 \\ d\mathbf{x} &= (0, 1, 0)\end{aligned}$$

$$\begin{aligned}&\int \mathbf{F} \cdot d\mathbf{x} \\&= \int_0^1 (4x^3z + 2x, z^2 - 2y, x^4 + 2yz) \cdot (0, 1, 0) dt \\&= \int_0^1 z^2 - 2y dt \\&= \int_0^1 1 - 2t dt \\&= 0\end{aligned} \tag{26}$$

$$\begin{aligned}(0, 1, 1) &\longrightarrow (1, 1, 1) \\ x = t, y = z = 1 & \\ d\mathbf{x} &= (1, 0, 0)\end{aligned}$$

$$\begin{aligned}&\int \mathbf{F} \cdot d\mathbf{x} \\&= \int_0^1 (4x^3z + 2x, z^2 - 2y, x^4 + 2yz) \cdot (1, 0, 0) dt \\&= \int_0^1 4x^3z + 2x dt \\&= \int_0^1 4t^3 + 2t dt \\&= [t^4 + t^2]_0^1 \\&= 2\end{aligned} \tag{27}$$

$$0 + 0 + 2 = 2 \tag{28}$$

So the line integral along the sequence of straight line paths is 2.

(ii)

$$\begin{aligned}x &= y = z = t \\ \mathrm{d}\mathbf{x} &= (1, 1, 1)\end{aligned}\tag{29}$$

$$\begin{aligned}& \int \mathbf{F} \cdot \mathrm{d}\mathbf{x} \\&= \int_0^1 (4x^3z + 2x, z^2 - 2y, x^4 + 2yz) \cdot (1, 1, 1) \, \mathrm{d}t \\&= \int_0^1 4x^3z + 2x + z^2 - 2y + x^4 + 2yz \, \mathrm{d}t \\&= \int_0^1 4t^4 + 2t + t^2 - 2t + t^4 + 2t^2 \, \mathrm{d}t \\&= \int_0^1 5t^4 + 3t^2 \, \mathrm{d}t \\&= [t^5 + t^3]_0^1 \\&= 2\end{aligned}\tag{30}$$

A function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$ is:

$$f(x, y, z) = x^4z + x^2 + yz^2 - y^2 + c\tag{31}$$

14. To integrate C , I will make the substitution:

$$\begin{aligned}x &= \cos \theta \\ y &= \sin \theta \\ z &= 0 \\ \mathrm{d}\mathbf{x} &= (-\sin \theta, \cos \theta, 0) \, \mathrm{d}\theta\end{aligned}\tag{32}$$

$$\begin{aligned}& \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{x} \\&= \int_0^{2\pi} (-ye^{-2t}, xe^{-2t}, 0) \cdot (-\sin \theta, \cos \theta, 0) \, \mathrm{d}\theta \\&= \int_0^{2\pi} (-\sin \theta e^{-2t}, \cos \theta e^{-2t}, 0) \cdot (-\sin \theta, \cos \theta, 0) \, \mathrm{d}\theta \\&= \int_0^{2\pi} \sin^2 \theta e^{-2t} + \cos^2 \theta e^{-2t} \, \mathrm{d}\theta \\&= \int_0^{2\pi} (\sin^2 \theta + \cos^2 \theta) e^{-2t} \, \mathrm{d}\theta \\&= \int_0^{2\pi} e^{-2t} \, \mathrm{d}\theta \\&= 2\pi e^{-2t}\end{aligned}\tag{33}$$

$$\begin{aligned}
 & -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \\
 &= -\frac{d}{dt} \int_S (0, 0, e^{-2t}) \cdot \hat{n} dS \\
 &= -\frac{d}{dt} \int_S (0, 0, e^{-2t}) \cdot (0, 0, 1) dS \\
 &= -\frac{d}{dt} \int_S e^{-2t} dS \\
 &= -\frac{d}{dt} \int_0^{2\pi} \int_0^1 r e^{-2t} dr d\theta \\
 &= -\frac{d}{dt} \int_0^{2\pi} \left[\frac{1}{2} r^2 e^{-2t} \right]_0^1 d\theta \quad (34) \\
 &= -\frac{d}{dt} \int_0^{2\pi} \frac{1}{2} e^{-2t} d\theta \\
 &= -\frac{d}{dt} \left(2\pi \times \frac{1}{2} e^{-2t} \right) \\
 &= -\frac{d}{dt} (\pi e^{-2t}) \\
 &= -(-2\pi e^{-2t}) \\
 &= 2\pi e^{-2t} \\
 &= \int_C \mathbf{E} \cdot d\mathbf{x} \text{ as required}
 \end{aligned}$$