1.

$$\mathbf{f_1} = \mathbf{e_1} \qquad \mathbf{f_2} = \sqrt{2}\mathbf{e_1} + \sqrt{2}\mathbf{e_2}$$

$$\mathbf{x} = 2\mathbf{f_1} + \frac{3}{\sqrt{2}}\mathbf{f_2}$$

2. (a)

$$\begin{aligned}
&\mathbf{vu} \\
=&\begin{pmatrix} 1 & 2 & b \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \\
=&0 \times 1 + 1 \times 2 + 2 \times b \\
=&2 + 2b
\end{aligned}$$

(b)

$$\begin{aligned}
&\mathbf{uv} \\
=&\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & b \end{pmatrix} \\
=&\begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & b \\ 2 & 4 & 2b \end{pmatrix}
\end{aligned}$$

3. • $\mathbf{A}^2$

   This does not exist: A is a 2x3 matrix, so the dimensions are not consistent for matrix multiplication.

   • **AB**

$$\begin{aligned}
&\mathbf{AB} \\
=&\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 2 & 0 \end{pmatrix} \\
=&\begin{pmatrix} 4 & 4 & 7 & 6 \\ 10 & 10 & 16 & 15 \end{pmatrix}
\end{aligned}$$

   • **AC**

   This does not exist: A is a 2x3 matrix and C is a 4x2 matrix, so the dimensions are not consistent for matrix multiplication.

   • **CA**

$$\begin{aligned}
&\mathbf{CA} \\
=&\begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \\
=&\begin{pmatrix} 9 & 12 & 15 \\ 3 & 6 & 9 \\ 4 & 5 & 6 \\ 9 & 12 & 15 \end{pmatrix}
\end{aligned}$$

- $\mathbf{B}^2$

  This does not exist: B is a 3x4 matrix, so the dimensions are not consistent for matrix multiplication.

- **BC**

$$\begin{aligned}&\mathbf{BC}\\ &= \begin{pmatrix}1 & 0 & 1 & 0\\ 0 & 2 & 0 & 3\\ 1 & 0 & 2 & 0\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 0\\ 0 & 1\\ 1 & 2\end{pmatrix}\\ &= \begin{pmatrix}1 & 3\\ 9 & 6\\ 1 & 4\end{pmatrix}\end{aligned}$$

- **CB**

  This does not exist: C is a 4x2 matrix and B is a 3x4 matrix, so the dimensions are not consistent for matrix multiplication.

- $\mathbf{C}^2$

  This does not exist: C is a 4x2 matrix, so the dimensions are not consistent for matrix multiplication.

4. (a) $\forall N, M \in \mathbb{Z}^+$

   (b) $\forall N, M \in \mathbb{Z}^+$

5.

$$\begin{aligned}&\mathbf{M}^2\\ &= \begin{pmatrix}1 & 2 & 0\\ 3 & 0 & 1\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 0\\ 3 & 0 & 1\\ 1 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}7 & 2 & 2\\ 4 & 6 & 1\\ 2 & 2 & 1\end{pmatrix}\end{aligned}$$

$$\begin{aligned}&\mathbf{MM}^T\\ &= \begin{pmatrix}1 & 2 & 0\\ 3 & 0 & 1\\ 1 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 1\\ 2 & 0 & 0\\ 1 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}5 & 3 & 1\\ 3 & 10 & 4\\ 1 & 4 & 2\end{pmatrix}\end{aligned}$$

$$\begin{aligned}&\mathbf{M}^T\mathbf{M}\\ &= \begin{pmatrix}1 & 3 & 1\\ 2 & 0 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 0\\ 3 & 0 & 1\\ 1 & 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}11 & 2 & 4\\ 2 & 4 & 0\\ 4 & 0 & 2\end{pmatrix}\end{aligned}$$

6.
$$\mathbf{A} \bullet \mathbf{B} = \mathbf{A} \times \mathbf{B}^T$$

$$\begin{aligned}
((\mathbf{A} \bullet \mathbf{B}) \bullet \mathbf{C})_{ij} &= \sum_k (\mathbf{A} \bullet \mathbf{B})_{ik} (\mathbf{C})_{jk} \\
&= \sum_k \sum_l (\mathbf{A})_{il} (\mathbf{B})_{kl} (\mathbf{C})_{jk}
\end{aligned}$$

$$\begin{aligned}
(\mathbf{A} \bullet (\mathbf{B} \bullet \mathbf{C}))_{ij} &= \sum_k (\mathbf{A})_{ik} (\mathbf{B} \bullet \mathbf{C})_{jk} \\
&= \sum_k \sum_l (\mathbf{A})_{ik} (\mathbf{B})_{jl} (\mathbf{C})_{kl} \\
&= \sum_k \sum_l (\mathbf{A})_{il} (\mathbf{C})_{kl} (\mathbf{B})_{jk}
\end{aligned}$$

Since the expressions are different, bullet multiplication is not associative.

7. (a) Consider an arbitrary symmetric matrix $\mathbf{A}$ and an arbitrary antisymmetric matrix $\mathbf{B}$ which have the same dimensionality (so they can be multiplied).

$$\begin{aligned}
ab_{ij} &= \sum a_{ik} b_{kj} \\
Tr(\mathbf{AB}) &= \sum ab_{ij} \\
Tr(\mathbf{AB}) &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \\
2Tr(\mathbf{AB}) &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} + \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \\
2Tr(\mathbf{AB}) &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} + \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ki} \text{ since } \forall i, k \in \mathbb{N}. a_{ik} = a_{ki} \\
2Tr(\mathbf{AB}) &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} - \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} b_{ik} \text{ since } \forall i, k \in \mathbb{N}. b_{ik} = -b_{ki} \\
2Tr(\mathbf{AB}) &= \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} - \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} b_{ki} \text{ by renaming variables in the second sum} \\
2Tr(\mathbf{AB}) &= 0 \\
Tr(\mathbf{AB}) &= 0
\end{aligned}$$

Since $\mathbf{A}$ and $\mathbf{B}$ were arbitrary, this holds for all pairs of symmetric and antisymmetric matrices of the same dimensionality.

(b) Consider an arbitrary antisymmetric $N \times N$ matrix $\mathbf{B}$ and an arbitrary $N \times 1$ column vector $\mathbf{x}$.

$$\begin{aligned}
\mathbf{x}^T\mathbf{A}\mathbf{x} &= \sum_{i=1}^{n}\sum_{k=1}^{n} x^T_{1i}a_{ik}x_{k1} \\
2\mathbf{x}^T\mathbf{A}\mathbf{x} &= \sum_{i=1}^{n}\sum_{k=1}^{n} x^T_{1i}a_{ik}x_{k1} + \sum_{i=1}^{n}\sum_{k=1}^{n} x^T_{1i}a_{ik}x_{k1} \\
2\mathbf{x}^T\mathbf{A}\mathbf{x} &= \sum_{i=1}^{n}\sum_{k=1}^{n} x^T_{1i}a_{ik}x_{k1} - \sum_{i=1}^{n}\sum_{k=1}^{n} x_{k1}a_{ki}x^T_{1i} \text{ by rearranging and since } \forall i,k \in \mathbb{N}.b_{ik} = -b_{ki} \\
2\mathbf{x}^T\mathbf{A}\mathbf{x} &= \sum_{i=1}^{n}\sum_{k=1}^{n} x^T_{1i}a_{ik}x_{k1} - \sum_{i=1}^{n}\sum_{k=1}^{n} x^T_{1k}a_{ki}x_{i1} \text{ by the definition of transpose} \\
2\mathbf{x}^T\mathbf{A}\mathbf{x} &= \sum_{i=1}^{n}\sum_{k=1}^{n} x^T_{1i}a_{ik}x_{k1} - \sum_{i=1}^{n}\sum_{k=1}^{n} x^T_{1i}a_{ik}x_{k1} \text{ renaming variables in the second sum} \\
2\mathbf{x}^T\mathbf{A}\mathbf{x} &= 0 \\
\mathbf{x}^T\mathbf{A}\mathbf{x} &= 0
\end{aligned}$$

8. a_{ji}

   I will prove associativity for multiplying a vector by the result of a matrix multiplication: $\mathbf{A}(\mathbf{B}\mathbf{x}) = (\mathbf{A}\mathbf{B})\mathbf{x}$

$$\begin{aligned}
((ab)x)_i &= \sum_{j=1}^{n}(ab)_{ij}x_j \\
&= \sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)x_j \\
&= \sum_{j=1}^{n}\sum_{k=1}^{n} a_{ik}b_{kj}x_j \\
&= \sum_{k=1}^{n} a_{ik}\sum_{j=1}^{n} b_{kj}x_j \\
&= \sum_{k=1}^{n} a_{ik}(bx)_k \\
&= (a(bx))_i
\end{aligned} \tag{1}$$

   (a) $\mathbf{A}\mathbf{B}\mathbf{x}$
   (b) $\mathbf{B}\mathbf{x}$
   (c) $\mathbf{B}\mathbf{A}^T\mathbf{x}$
   (d) $\mathbf{A}\mathbf{A}^T\mathbf{A}$

9. (a)

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   (b) Consider:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

10.

$$\mathbf{M}^3 = -\mathbf{M} \implies \mathbf{M}^{2n} = (-1)^{n+1}\mathbf{M}^2 \wedge \mathbf{M}^{2n+1} = (-1)^n \mathbf{M}$$

$$\begin{aligned}
\exp\theta\mathbf{M} &= \mathbf{I} + \sum_{n=1}^{\infty} \frac{(\theta\mathbf{M})^n}{n!} \\
&= \mathbf{I} + \sum_{n=0}^{\infty} \frac{(\theta\mathbf{M})^{2n+1}}{(2n+1)!} + \sum_{n=1}^{\infty} \frac{(\theta\mathbf{M})^{2n}}{(2n)!} \\
&= \mathbf{I} + \mathbf{M}\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} + \mathbf{M}^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\theta^{2n}}{(2n)!} \\
&= \mathbf{I} + \mathbf{M}\sum_{n=0}^{\infty} \frac{(-1)^n\theta^{2n+1}}{(2n+1)!} + \mathbf{M}^2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\theta^{2n}}{(2n)!} + \mathbf{M}^2 \\
&= \mathbf{I} + \mathbf{M}\sin\theta - \mathbf{M}^2\cos\theta + \mathbf{M}^2 \text{ using the Taylor expansion for } \sin\theta \text{ and } \cos\theta \\
&= \mathbf{I} + \mathbf{M}\sin\theta + \mathbf{M}^2(1-\cos\theta)
\end{aligned}$$

$$\begin{aligned}
\exp\theta_1\mathbf{M}\exp\theta_2\mathbf{M} &= \big(\mathbf{I} + \mathbf{M}\sin\theta_1 + \mathbf{M}^2(1-\cos\theta_1)\big)\big(\mathbf{I} + \mathbf{M}\sin\theta_2 + \mathbf{M}^2(1-\cos\theta_2)\big) \\
&= \mathbf{I} + \mathbf{M}(\sin\theta_1 + \sin\theta_2) + \mathbf{M}^2(2 - \cos\theta_1 - \cos\theta_2) + \mathbf{M}^2\sin\theta_1\sin\theta_2 \\
&\quad + \mathbf{M}^3(\sin\theta_1 + \sin\theta_2 - 2\sin\theta_1\cos\theta_2) + \mathbf{M}^4(1 - \cos\theta_1 - \cos\theta_2 + \cos\theta_1\cos\theta_2) \\
&= \mathbf{I} + \mathbf{M}(\sin\theta_1 + \sin\theta_2) + \mathbf{M}^2(2 + \sin\theta_1\sin\theta_2 - \cos\theta_1 - \cos\theta_2) \\
&\quad - \mathbf{M}(\sin\theta_1 + \sin\theta_2 - 2\sin\theta_1\cos\theta_2) - \mathbf{M}^4(1 - \cos\theta_1 - \cos\theta_2 + \cos\theta_1\cos\theta_2) \\
&= \mathbf{I} + \mathbf{M}(\sin\theta_1 + \sin\theta_2 - \sin\theta_1 - \sin\theta_2 + 2\sin\theta_1\cos\theta_2) \\
&\quad + \mathbf{M}^2(2 + \sin\theta_1\sin\theta_2 - \cos\theta_1 - \cos\theta_2 - 1 + \cos\theta_1 + \cos\theta_2 - \cos\theta_1\cos\theta_2) \\
&= \mathbf{I} + \mathbf{M}(2\sin\theta_1\cos\theta_2) + \mathbf{M}^2(1 + \sin\theta_1\sin\theta_2 - \cos\theta_1\cos\theta_2) \\
&= \mathbf{I} + \mathbf{M}\sin(\theta_1+\theta_2) + \mathbf{M}^2(1 - \cos(\theta_1+\theta_2)) \\
&= \exp(\theta_1+\theta_2)\mathbf{M}
\end{aligned}$$

$$\begin{aligned}
\exp\theta\mathbf{M} &= \mathbf{I}^T + \mathbf{M}^T\sin\theta + \left(\mathbf{M}^2\right)^T(1-\cos\theta) \\
&= \mathbf{I} - \mathbf{M}\sin\theta + \mathbf{M}^2(1-\cos\theta) \\
&= \exp(-\theta)\mathbf{M}
\end{aligned}$$

$$\begin{aligned}
(\exp\theta\mathbf{M})(\exp\theta\mathbf{M})^T &= \exp\theta\mathbf{M}\exp(-\theta)\mathbf{M} \\
&= \exp(\theta-\theta)\mathbf{M} \\
&= \exp 0\mathbf{M} \\
&= \mathbf{I} + \mathbf{M}\sin 0 + \mathbf{M}^2(1-\cos 0) \\
&= \mathbf{I}
\end{aligned}$$

This result will not hold for general **M**. **M**

Find out what type of matrix **M** is.

11.

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

Consider arbitrary

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathbf{b} \times \mathbf{x} = \mathbf{B}\mathbf{x}$$
$$\begin{pmatrix} b_2 z - b_3 y \\ b_3 x - b_1 z \\ b_1 y - b_2 x \end{pmatrix} = \begin{pmatrix} B_{11}x + B_{12}y + B_{13}z \\ B_{21}x + B_{22}y + B_{23}z \\ B_{31}x + B_{32}y + B_{33}z \end{pmatrix}$$

Equating coefficients of x, y and z gives:

$$\begin{aligned} B_{11} &= 0 & B_{12} &= -b_3 & B_{13} &= b_2 \\ B_{21} &= b_3 & B_{22} &= 0 & B_{23} &= -b_1 \\ B_{31} &= -b_2 & B_{32} &= b_1 & B_{33} &= 0 \end{aligned}$$

So:

$$\mathbf{B} = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{B}^2\mathbf{x} &= \mathbf{B}(\mathbf{B}\mathbf{x}) \text{ using (1)} \\ &= \mathbf{b} \times (\mathbf{b} \times \mathbf{x}) \end{aligned}$$

$$\begin{aligned}
\mathbf{B}^2\mathbf{x} &= \begin{pmatrix} -b_2^2 - b_3^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & -b_1^2 - b_3^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & -b_1^2 - b_2^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} -b_2^2 x - b_3^2 x + b_1 b_2 y + b_1 b_3 z \\ b_1 b_2 x - b_1^2 y - b_3^2 y + b_2 b_3 z \\ b_1 b_3 x + b_2 b_3 y - b_1^2 z - b_2^2 z \end{pmatrix} \\
&= \begin{pmatrix} (b_1 x + b_2 y + b_3 z) b_1 \\ (b_1 x + b_2 y + b_3 z) b_2 \\ (b_1 x + b_2 y + b_3 z) b_3 \end{pmatrix} - \begin{pmatrix} (b_1^2 + b_2^2 + b_3^2) x \\ (b_1^2 + b_2^2 + b_3^2) y \\ (b_1^2 + b_2^2 + b_3^2) z \end{pmatrix} \\
&= (\mathbf{b} \cdot \mathbf{x})\mathbf{b} - (\mathbf{b} \cdot \mathbf{b})\mathbf{x}
\end{aligned}$$

Equating the two equations gives the required result:

$$\mathbf{b} \times (\mathbf{b} \times \mathbf{x}) = (\mathbf{b} \cdot \mathbf{x})\mathbf{b} - (\mathbf{b} \cdot \mathbf{b})\mathbf{x}$$

12.

$$\begin{aligned}
\det \mathbf{A} \det \mathbf{B} &= (4 - 6) \times (4 - 4) \\
&= -2 \times 0 \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\det \mathbf{AB} &= \det \begin{pmatrix} 4 & 8 \\ 10 & 20 \end{pmatrix} \\
&= 80 - 80 \\
&= 0 \\
&= \det \mathbf{A} \det \mathbf{B}
\end{aligned}$$

$$\begin{aligned}
\det \mathbf{A}^{-1} &= \det \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \\
&= 1 - \frac{3}{2} \\
&= -\frac{1}{2} \\
&= \frac{1}{\det \mathbf{A}}
\end{aligned}$$

13.

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} &= A_{11} C_{1,1} + A_{12} C_{1,2} + A_{13} C_{1,3} \\
&= 1 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} \\
&= 0 - 4 + 0 \\
&= -4
\end{aligned}$$

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} &= \begin{vmatrix} -2 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{vmatrix} \\
&= -4
\end{aligned}$$

$$\begin{aligned}
|\mathbf{A}^T| &= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix}^T \\
&= \begin{vmatrix} 1 & 3 & 1 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 3 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} \\
&= -2 \times 2 \times 1 \\
&= -4 \\
&= |\mathbf{A}|
\end{aligned}$$

14. The columns of a matrix who's determinant is zero are linearly dependent on each other.

    The vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} 4 \\ 0 \\ 1 \\ 3 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 0 \\ 3 \\ a \end{pmatrix}$$

    are linearly dependent if and only if the determinant of the matrix with these vectors as it's columns is zero.

$$\begin{aligned}
\begin{vmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \\ 4 & 0 & 1 & 3 \\ 2 & 0 & 3 & a \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 3 \\ 2 & 0 & 3 & a \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & -3 & 3 \\ 2 & 0 & 1 & a \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 1 & a \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & a+1 \end{vmatrix} \\
&= -3(a+1)
\end{aligned}$$

    So the vectors are linearly dependent for $a = -1$.

15. We can use row operations to conserve the determinant of the matrix:

$$\begin{aligned}
\begin{vmatrix} a-E & -b & -b & -b & -b \\ -b & a-E & -b & -b & -b \\ -b & -b & a-E & -b & -b \\ -b & -b & -b & a-E & -b \\ -b & -b & -b & -b & a-E \end{vmatrix} &= \begin{vmatrix} a+b-E & 0 & 0 & 0 & -b \\ 0 & a+b-E & 0 & 0 & -b \\ 0 & 0 & a+b-E & 0 & -b \\ 0 & 0 & 0 & a+b-E & -b \\ E-a-b & E-a-b & E-a-b & E-a-b & a-E \end{vmatrix} \\
&= \begin{vmatrix} a+b-E & 0 & 0 & 0 & -b \\ 0 & a+b-E & 0 & 0 & -b \\ 0 & 0 & a+b-E & 0 & -b \\ 0 & 0 & 0 & a+b-E & -b \\ 0 & 0 & 0 & 0 & a-4b-E \end{vmatrix} \\
&= (a+b-E)^4(a-4b-E)
\end{aligned}$$

So the determinant is zero when:

$$\begin{aligned}
(a+b-E)^4(a-4b-E) &= 0 \\
E = a+b &\vee E = a-4b
\end{aligned}$$

16.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 1 & 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} p \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 6 \end{pmatrix}$$

17.

$$\begin{aligned}
4w &= 8 \\
w &= 2
\end{aligned}$$

$$\begin{aligned}
2z + w &= 4 \\
2z + 2 &= 2 \\
2z &= 2 \\
z &= 1
\end{aligned}$$

$$\begin{aligned}
2y + 3z + 5w &= 19 \\
2y + 3 + 10 &= 19 \\
2y &= 6 \\
y &= 3
\end{aligned}$$

$$\begin{aligned}
6x + 7y + 3z - w &= -2 \\
6x + 21 + 3 - 2 &= -2 \\
6x &= -24 \\
x &= -4
\end{aligned}$$

$$x = -4 \qquad y = 3 \qquad z = 1 \qquad w = 2$$

18.

$$\begin{aligned} x + 3y &= 17 \\ 3x + 9y &= 51 \\ (3x + 9y) - (3x + 2y) &= 51 - 9 \\ 7y &= 42 \\ y &= 6 \end{aligned}$$

$$\begin{aligned} x + 3y &= 17 \\ x + 18 &= 17 \\ x &= -1 \end{aligned}$$

$$x = -1 \qquad\qquad y = 6$$

$$\begin{aligned} \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 9 \\ 17 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{7} \begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 9 \\ 17 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 \\ 6 \end{pmatrix} \end{aligned}$$

19.

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{|\mathbf{A}|} \begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix}^T \\ &= \frac{1}{4} \begin{pmatrix} 1 & 1 & 2 \\ -2 & 2 & 0 \\ -1 & -1 & 2 \end{pmatrix}^T \\ &= \frac{1}{4} \begin{pmatrix} 1 & -2 & -1 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \mathbf{Ax} &= \mathbf{y} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{y} \\ \mathbf{x} &= \frac{1}{4} \begin{pmatrix} 1 & -2 & -1 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} a - 2b - c \\ a + 2b - c \\ 2a + 2c \end{pmatrix} \end{aligned}$$

$$\mathbf{e}_1 = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad \mathbf{e}_2 = \frac{1}{4} \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \qquad \mathbf{e}_3 = \frac{1}{4} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

The matrix whose columns are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the inverse matrix of $\mathbf{A}$.

20.

$$a_i \cdot b_j = \delta_{ij}$$

Consider $a_i \cdot b_i$:

Case $i = j$:

$$\begin{aligned} a_j \cdot b_i &= 1 \\ &= \frac{[a_1, a_2, a_3]}{[a_1, a_2, a_3]} \\ &= \frac{a_i \cdot a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \\ &= a_j \cdot \frac{a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \end{aligned}$$

Case $i \neq j$:

$$\begin{aligned} a_j \cdot b_i &= 0 \\ &= \frac{a_j \cdot a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \\ &= a_j \cdot \frac{a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \end{aligned}$$

$$\begin{gathered} \forall \mathbf{A}, \mathbf{B} \in \mathbb{R}^{3\times 3}. \mathbf{AB} = \mathbf{I} \implies \\ \forall i, j \in \{1, 2, 3\}. a_j \cdot b_i = a_j \cdot \frac{a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \implies \\ \forall i, j \in \{1, 2, 3\}. b_i = \frac{a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \end{gathered}$$

Substituting $i = 1, 2, 3$ into these formulae gives the required result:

$$b_1 = \frac{a_2 \times a_3}{[a_1, a_2, a_3]} \qquad b_2 = \frac{a_3 \times a_1}{[a_1, a_2, a_3]} \qquad b_3 = \frac{a_1 \times a_2}{[a_1, a_2, a_3]}$$