H: revision of calculus

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4. i)

$$y + e^y \sin y = \frac{1}{x}$$

$$x = \frac{1}{y + e^y \sin y}$$

$$\frac{dx}{dy} = \frac{-(1 + e^y \sin y - e^y \cos y)}{(y + e^y \sin y)^2}$$

$$\frac{dy}{dx} = -\frac{(y + e^y \sin y)^2}{1 + e^y \sin y + e^y \cos y}$$

ii)

$$y + e^y \sin y = \frac{1}{x}$$

$$\frac{dy}{dx} + \frac{dy}{dx} e^y \sin y + \frac{dy}{dx} e^y \cos y = -\frac{1}{x^2}$$

$$\frac{dy}{dx}(1 + e^y \sin y - e^y \cos y) = -\frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{1}{x^2(1 + e^y \sin y + e^y \cos y)}$$

$$y + e^y \sin y = \frac{1}{x}$$

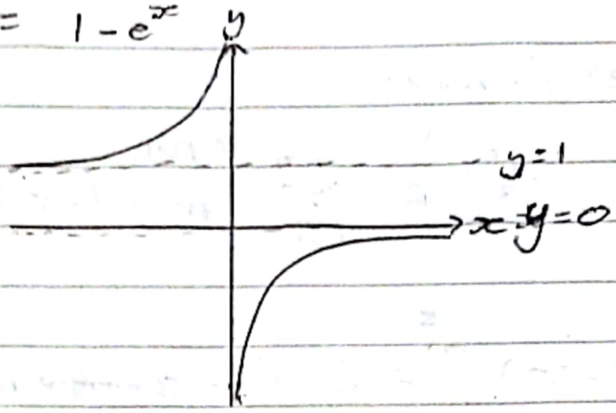
$$\therefore (y + e^y \sin y)^2 = \frac{1}{x^2}$$

sub in

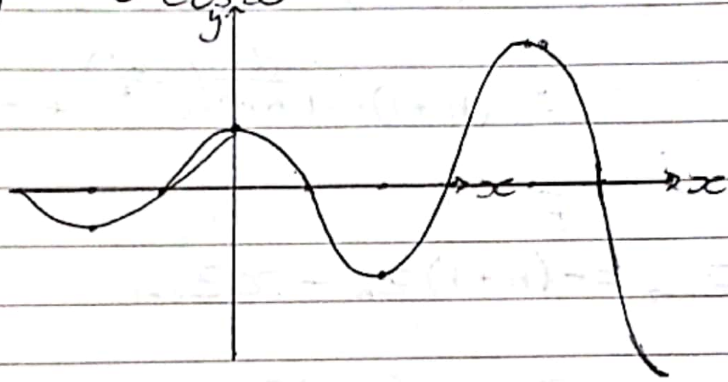
$$\therefore \frac{dy}{dx} = -\frac{(y + e^y \sin y)^2}{(1 + e^y \sin y + e^y \cos y)}$$

Same as i) as required.

e) $y = \dfrac{1}{1-e^{x}}$



f) $y = e^{x}\cos x$



I: Leibnitz's formula

1. $$Z_n = \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right)$$

$$\frac{d}{dx}(Z_n) = Z_{n+1}$$

$$Z_{n+2} = \frac{d^{n+2}}{dx^{n+2}}\left(e^{-\frac{x^2}{2}}\right)$$

$$= \frac{d^{n+1}}{dx^{n+1}}\left(\frac{d}{dx}\left(e^{-\frac{x^2}{2}}\right)\right)$$

$$= \frac{d^{n+1}}{dx^{n+1}}\left(-xe^{-\frac{x^2}{2}}\right)$$

I 1 continued

Using Leibnitz's formula:

$$\frac{d^{n+1}}{dx^{n+1}}\left(-x e^{-\frac{x^2}{2}}\right) = \sum_{m=0}^{n+1} \binom{n+1}{m} (-x)^{(m)} \left(e^{-\frac{x^2}{2}}\right)^{(n+m)}$$

Since $(-x)^{(m)} = 0$ for ~~x (m≥2)~~ $m \geq 2$;

$$= \binom{n+1}{\cancel{n}+1} (-x)^{(1)} \left(e^{-\frac{x^2}{2}}\right)^{(n)} + \binom{n+1}{0} (-x)^{(0)} \left(e^{-\frac{x^2}{2}}\right)^{(n+1)}$$

$$= (n+1) \times -1 \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right) + -x \frac{d^{n+1}}{dx^{n+1}}\left(e^{-\frac{x^2}{2}}\right)$$

$$\therefore \; z_{n+2} = -(n+1) z_n - x z_{n+1}$$

$$z_{n+2} + x z_{n+1} + (n+1) z_n = 0$$

which (since $z_{n+1} = \frac{d}{dx} z_n$) is equivalent to:

$$\frac{d^2 z_n}{dx^2} + x \frac{dz_n}{dx} + (n+1) z_n = 0$$

as required.

So z_n ~~is~~ $= \frac{d^n}{dx^n}\left(e^{-\frac{x^2}{2}}\right)$ is a solution to the differential equation.

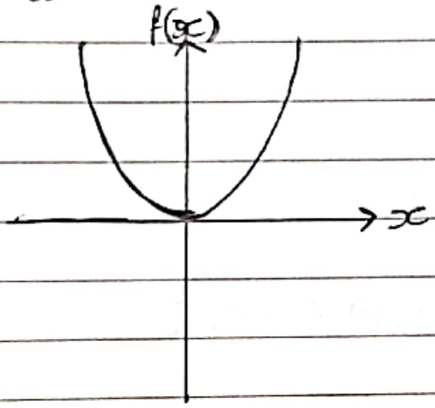
J : Elementary Analysis

1. i) If $\lim\limits_{\delta x \to 0} f(x_0 + \delta x) = f(x_0)$ then $f(x)$ is continuous at $\cancel{x = x_0}$ x_0.

ii) For a function $f(x)$ to be differentiable at x_0; it must be continuous, finite and defined and:

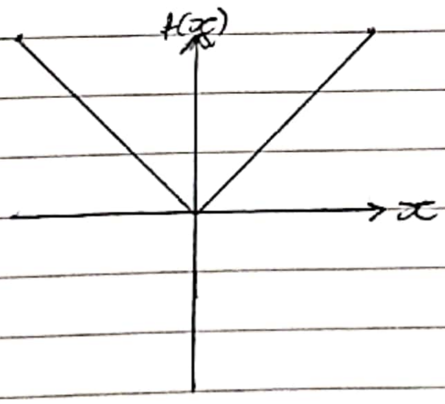
$$\lim_{\delta x \to 0} \left(\frac{f(x) - f(x - \delta x)}{\delta x} \right) = \lim_{\delta x \to 0} \left(\frac{f(x + \delta x) - f(x)}{\delta x} \right)$$

iii)



$f(x) = x^2$

iv)



$f(x) = |x|$

v)

$$f(x) = x \sin\left(\frac{1}{x}\right)$$

$$= \frac{\sin \frac{1}{x}}{\frac{1}{x}}$$

~~If~~ $\lim\limits_{x \to \infty} \sin \frac{1}{x} = 0$

$\lim\limits_{x \to \infty} \frac{1}{x} = 0$

J 1 v continued

So using L'hôpitals rule:

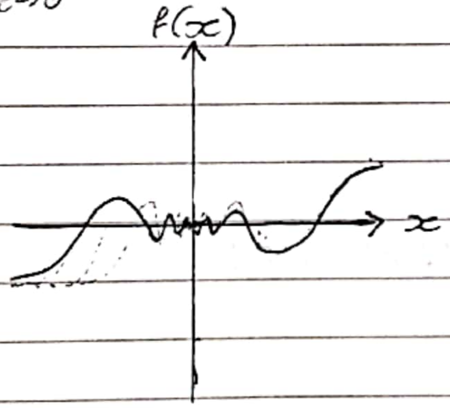
$$\lim_{x \to \infty} \frac{\sin \frac{1}{x}}{\frac{1}{x}} = \lim_{x \to \infty} \frac{-\frac{1}{x^2} \cos \frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \cos \frac{1}{x}$$

$$= 1$$

$$\lim_{x \to -\infty} x \sin \frac{1}{x} = -1$$

$$\cancel{\lim_{x \to 0} x \sin x =}$$



$f(x) = x \sin \frac{1}{x}$

H: Limits

1. a)

$$\lim_{n\to\infty} \frac{5^{n+2} - 7^{n+2}}{5^n - 7^n}$$

take $\displaystyle \lim_{n\to\infty} \left| \frac{5^{n+2} - 7^{n+2}}{5^n - 7^n} - 49 \right|$

$$= \lim_{n\to\infty} \left| - \frac{24 \times 5^n}{5^n - 7^n} \right|$$

$$= 0$$

So: $\displaystyle \lim_{n\to\infty} \frac{5^{n+2} - 7^{n+2}}{5^n - 7^n} = 49$

b)

$$\lim_{n\to\infty} \frac{n + (-1)^n}{n - (-1)^n}$$

~~consider:~~ $\cancel{\lim_{n\to\infty}}$

$$= 1 + \lim_{n\to\infty} \frac{2 \times (-1)^n}{n - (-1)^n}$$

$$= 1 + 0$$

$$= 0$$

c)

$$\lim_{n\to\infty} \frac{n + (-2)^n}{n - (-2)^n}$$

$$= -1 + \lim_{n\to\infty} \frac{2n}{n - (-2)^n}$$

$$= -1 + 0$$

$$= 0$$

d) $\lim\limits_{n\to\infty} \sqrt{n+1} - \sqrt{n}$

$$= \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$= 0$$

2.a) $\lim\limits_{x\to 0^+} x^{\alpha} \ln x$

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-\alpha}}$$

$\lim\limits_{x\to 0^+} \ln x = -\infty$

$\lim\limits_{x\to 0^+} x^{-\alpha} = \infty$

So we can use L'hôpitals rule

$$\therefore \lim_{x\to 0^+} \frac{\ln x}{x^{-\alpha}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\alpha x^{-(\alpha+1)}}$$

$$= \lim_{x\to 0^+} -\frac{x^{\alpha}}{\alpha}$$

$$= 0$$

b) $\lim_{x \to 0^+} x^{-\alpha} = \infty$

$\lim_{x \to 0^+} \ln x = -\infty$

$\therefore \lim_{x \to 0^+} x^{-\alpha} \ln x = -\infty \times \infty$
$= -\infty$

c) $\lim_{x \to 0^+} x^{\alpha} = 0$

$\lim_{x \to 0^+} e^{-x} = 1$

So $\lim_{x \to 0^+} x^{\alpha} e^{-x} = 0 \times 1$
$= 0$

d) $\lim_{x \to 0^+} x^{-\alpha} = \infty$

$\lim_{x \to 0^+} e^{x} = 1$

$\therefore \lim_{x \to 0^+} x^{-\alpha} e^{x} = \infty \times 1$
$= \infty$

e) $\lim_{x \to 0^+} \frac{\sin \alpha x}{x}$

&

$\lim_{x \to 0^+} \sin \alpha x = 0$

$\lim_{x \to 0^+} x = 0$

So we can use L'hôpitals rule

$\therefore \lim_{x \to 0^+} \frac{\sin \alpha x}{x} = \lim_{x \to 0^+} \frac{\alpha \cos \alpha x}{1} = \alpha$

f) $\lim_{x \to 0^+} x\cos\left(\frac{\alpha}{x}\right)$

$\lim_{x \to 0} x = 0$

$\lim_{x \to 0} \cos\left(\frac{\alpha}{x}\right)$ is undefined. It oscillates rapidly.

$\cos\left(\frac{\alpha}{x}\right) \in [-1, 1]$

So $\cos\left(\frac{\alpha}{x}\right)$ is $O(1)$

$\therefore \lim_{x \to 0^+} x\cos\left(\frac{\alpha}{x}\right) = 0 \times O(1)$

$= 0$

L: Convergence of Series

1.d) Using d'Alembert's ratio test

If $\lim\limits_{n \to \infty} \frac{u_{n+1}}{u_n} < 1$ then the series converges.

$$= \lim_{n \to \infty} \frac{\left(\frac{(n+1)^{10}}{(n+1)!}\right)}{\left(\frac{n^{10}}{n!}\right)}$$

$$= \lim_{n \to \infty} \frac{\left(\frac{(n+1)^{9}}{n!}\right)}{\left(\frac{n^{10}}{n!}\right)}$$

$$= \lim_{n \to \infty} \frac{(n+1)^9}{n^{10}}$$

$$= 0$$

So the series converges.

Hence $\sum u_n$ converges.

~~d)~~ So $\sum\limits_{n=1}^{\infty} \frac{n^{10}}{n!}$ converges.

e) Using d'Alembert's ratio test; if $\lim\limits_{n \to \infty} \frac{u_{n+1}}{u_n} > 1$ then the series and the sum of the series ~~converges~~ diverges.

$$\lim_{n \to \infty} \frac{u_{n+1}}{u_n}$$

$$= \lim_{n \to \infty} \frac{\left(\frac{(n+1)!}{10^{n+1}}\right)}{\left(\frac{n!}{10^{n}}\right)}$$

$$= \lim_{n \to \infty} \frac{n}{10}$$

$$= \infty$$

So the sum of the series diverges.

f) Using d'Alembert's ratio test:

$$\lim_{n \to \infty} \frac{\mu_{n+1}}{\mu_n}$$

$$= \lim_{n \to \infty} \frac{3n+2}{4n-3}$$

$$= \frac{3}{4}$$

Since the limit is less than 1; the series converges.

2.a) Consider $\sum |\mu_n|$

$$= \sum \frac{1}{\sqrt{n}}$$

So: if the integral $\lim_{a \to \infty} \int_1^a \frac{1}{\sqrt{n}} \, dn$ ~~con~~ diverges then

$\sum |\mu_n|$ diverges and hence $\sum \mu_n$ is conditionally convergent.

$$\lim_{a \to \infty} \int_1^a \frac{1}{\sqrt{n}} \, dn = \lim_{a \to \infty} \left[2\sqrt{n} \right]_1^a$$

$$= \lim_{a \to \infty} 2\sqrt{a} - 2$$

$$= \infty$$

So: $\sum_{n=1}^{\infty} |\mu_n|$ ~~is~~ diverges.

Hence the series $\sum \mu_n$ ~~absolu~~ is conditionally convergent.

b) Using $|\mu_n| = \left(\frac{2n+5}{3n+1}\right)^n$

Using Cauchy's root test:

if $\lim_{n\to\infty} (|\mu_n|)^{\frac{1}{n}} < 1$ then the series converges

$$\lim_{n\to\infty} (|\mu_n|)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2n+5}{3n+1}$$

$$= \frac{2}{3}$$

So the series $\sum_{n=1}^{\infty} |\mu_n|$ converges.

So $\sum_{n=1}^{\infty} \mu_n$ is absolutely convergent.

c) $|\mu_n| = \frac{1}{(2n-1)^2}$

if $\lim_{a\to\infty} \int_1^{a} \frac{1}{(2n-1)^2}\,dn$ converges then $\sum_{n=1}^{\infty} |\mu_n|$ converges.

$$\lim_{a\to\infty} \int_1^{a} \frac{1}{(2n-1)^2}\,dn = \lim_{a\to\infty} \left[-\frac{1}{2(2n-1)}\right]_1^a$$

$$= \frac{1}{4} - \lim_{a\to\infty} \frac{1}{2(2a-1)}$$

$$= \frac{1}{4} - 0$$

$$= \frac{1}{4}$$

So $\sum_{n=1}^{\infty} |\mu_n|$ is convergent.

Hence $\sum_{n=1}^{\infty} \mu_n$ is absolutely convergent.