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Further Graphics Exercise Set I

2. What geometric shape does the following curve describe?

$$(x(t), y(t)) = \left(e^t \cos(t), e^t \sin(t)\right)$$

This planar curve describes an anticlockwise spiral.

3. (a) How do you compute the surface normal $\mathbf{n}$ for a parametric surface?

For a parametric surface with equation $f(u, v) = (x, y, z)$, the surface normal $\mathbf{n}$ is given by:

$$\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}$$

(b) Below you find the parameterisation of a *torus*. Compute the surface normal $\mathbf{n}$ for $u = \frac{1}{3}\pi$, $v = \frac{3}{4}\pi$.

$$s(u, v) = \left(\left(3 + \sqrt{2}\cos(v)\right)\cos(u), \left(3 + \sqrt{2}\cos(v)\right)\sin(u), \sqrt{2}\sin(v)\right)$$

$$\frac{\partial s}{\partial u} = \left(-\left(3 + \sqrt{2}\cos(v)\right)\sin(u), \left(3 + \sqrt{2}\cos(v)\right)\cos(u), 0\right)$$

$$\frac{\partial s}{\partial v} = \left(-\sqrt{2}\sin(v)\cos(u), -\sqrt{2}\sin(v)\sin(u), \sqrt{2}\cos(v)\right)$$

Substitute the values in earlier rather than cross-producting the cos, sin terms

$$\begin{aligned}
\mathbf{n} &= \frac{\partial s}{\partial u} \times \frac{\partial s}{\partial u} \\
&= \begin{pmatrix} -\left(3 + \sqrt{2}\cos(v)\right)\sin(u) \\ \left(3 + \sqrt{2}\cos(v)\right)\cos(u) \\ 0 \end{pmatrix} \times \begin{pmatrix} -\sqrt{2}\sin(v)\cos(u) \\ -\sqrt{2}\sin(v)\sin(u) \\ \sqrt{2}\cos(v) \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{2}\left(3 + \sqrt{2}\cos(v)\right)\cos(u)\cos(v) \\ \sqrt{2}\left(3 + \sqrt{2}\cos(v)\right)\sin(u)\cos(v) \\ \sqrt{2}\left(3 + \sqrt{2}\cos(v)\right)\left(\sin^2(u) + \cos^2(u)\right)\sin(v) \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{2}\left(3 + \sqrt{2}\cos(v)\right)\cos(u)\cos(v) \\ \sqrt{2}\left(3 + \sqrt{2}\cos(v)\right)\sin(u)\cos(v) \\ \sqrt{2}\left(3 + \sqrt{2}\cos(v)\right)\sin(v) \end{pmatrix}
\end{aligned}$$

Substituting in the values for (u, v) gives:

$$\begin{aligned}
\mathbf{n}\left(\frac{\pi}{3}, \frac{3\pi}{4}\right) &= \begin{pmatrix} \sqrt{2}\left(3 + \sqrt{2}\cos\left(\frac{3\pi}{4}\right)\right)\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{3\pi}{4}\right) \\ \sqrt{2}\left(3 + \sqrt{2}\cos\left(\frac{3\pi}{4}\right)\right)\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{3\pi}{4}\right) \\ \sqrt{2}\left(3 + \sqrt{2}\cos\left(\frac{3\pi}{4}\right)\right)\sin\left(\frac{3\pi}{4}\right) \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{2}\,(3 - 1)\,\frac{1}{2}\frac{\sqrt{2}}{2} \\ \sqrt{2}\,(3 - 1)\,\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\ \sqrt{2}\,(3 - 1)\,\frac{1}{2} \end{pmatrix} \\
&= \begin{pmatrix} -1 \\ 1 \\ \sqrt{2} \end{pmatrix}
\end{aligned}$$

5. If we define a curve in $\mathbb{R}^2$ with the implicit function $f(x, y) = x^2y^2$, what problem could there arise when computing the surface normal?

   The surface normal for a function which is implicitly defined is:

   $$\mathbf{n} = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial x} \end{pmatrix}$$

   Which in this case is:

   $$\begin{pmatrix} 2xy^2 \\ 2x^2y \end{pmatrix} = 2xy \begin{pmatrix} y \\ x \end{pmatrix}$$

   By definition, at all points on the surface of the curve: $x^2y^2 = 0$. Therefore x or y is zero. So for all points on the curve, $xy = 0$. So, the normal for any point on the curve is:

   $$0 \cdot \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is not a valid normal!

6. (a) What data structure is used to represent triangle meshes?

   Attributes such as position, u-v maps, normals and colour are stored with each of the vertices

   Triangle meshes can be represented as a graph embedded in 3D space where V is the set of vertices; $E \subseteq V \times V$ is the set of edges; $F \subseteq V \times V \times V$ is the set of faces; and $P \subseteq V \times \mathbb{R}^n$ is a mapping from each vertex to a point in n-dimensional space (where n is 3 in most applications).

   (b) What parts of this data structure pertain to the topology and which to the geometry of a surface?

   The mapping from the vertices to the points in 3D pertains to the topology – it varies when the shape is transformed. The edges and faces pertain to the geometry; they are invariant of transformations.

   This is more like a parametric representation, we order the set and then consider this as a function from indices to points

7. Compared to the triangle mesh, a point set lacks a vital piece of information in that it no longer identifies the neighbouring vertices of any given vertex. Why is such information necessary or useful in the first place and how do point set surfaces work without it?

   The adjacency matrix is essential to know the geometry of the polygon we are using to approximate the surface.

   Point-set surfaces instead approximate the underlying surface. In practice most surfaces are smooth and therefore the approximation-based methods get very similar results to polygon-based surface representations.

   A simple approximation method selects a query point and a fixed distance – all points closer to the query point than this fixed distance are in the "neighbourhood". We then assume that the surface is the plane which best fits all the points in the neighbourhood.

   In reality we train a NN to fit the surface from the point-set representation and then query the NN

14. (a) What is rigging? Briefly describe what kind of structure and data are added to the geometry/surface model.

The bones are a tree – there is usually a root which we define to be "a central point which we will use to move the whole object". Sometimes we just have a root bone which is not connected to anything.

Rigging is the process of attaching points on the surface of an object to the "bones" of the object. This is an essential stage in the process of animation.

We create bones, often storing them in a tree such that a transformation on one will move the others below it in the tree – making animation easier.

We add a (sparse) matrix to the surface model. This matrix contains the weights which define the impact of a transformation on a given bone on a point on the surface.

(b) Assume we have a cylinder with an axis aligned to the x-axis. The cylinder extends from -1 to 1 on the x-axis and has a base radius of 1. We embed two bones inside the cylinder along the x-axis: one extends from -1 to 0, the other from 0 to 1. We define the influence of each bone on each point on the cylinder as the inverse distance of the point to the bone. Assume we transform the second bone with the new transformation T. Determine the concrete weights and give a formula of how the new position x' would be computed from the weights $w_1(x)$, $w_2(x)$, x and T for a point on the surface (a) in the middle and (b) on either end.

They meant a point on the surface of a hollow cylinder. All weights must sum to 1.

Let bone 1 be the bone from -1 to 0 and let bone 2 be the bone from 0 to 1.

i. In the middle

$$w_1(x) = 0.5 \qquad w_2(x) = 0.5$$

$$x' = 0.5 \cdot x + 0.5 \cdot \mathbf{T}x$$

ii. On either end.

For $x = (-1, 0)$:

$$w_1(x) = 1 \qquad w_2(x) = 0 \tag{1}$$

$$x' = 1 \cdot x + 0 \cdot \mathbf{T}x = x$$

For $x = (1, 0)$:

$$w_1(x) = 0 \qquad w_2(x) = 1$$

$$x' = 0 \cdot x + 1 \cdot \mathbf{T}x = \mathbf{T}x$$

15. Thinking about fundamental properties

(a) What is the advantage of representing rigid transformations with dual quaternions for blending.

Dual quaternions can be used to perform shortest-path transformations. This allows us to blend transformations while ensuring the resulting transformation is still valid; without distortions.

(b) Briefly explain one fundamental disadvantage of using quaternion based shortest-path blending for rotations as compared to linear blend skinning.

Linear-blend skinning on linearly interpolates points. Therefore, transforming nearby points with the same transformations is guaranteed to leave the transformed points close to each other. Therefore a continuous surface will remain continuous after transformation.

However, quaternions find the shortest path. This path will change suddenly. Therefore there exists weights such that a tiny change in weight will make the same quaternion interpolate onto a completely different path.

Imagine we interpolate between two rotations, θ_1 and θ_2. The shortest-path interpolation is discontinuous at π. The shortest-path at π radians is wildly divergent.

Additionally, quaternions preserve volume and therefore cannot represent volume changes

16. Derivations and deeper understanding

Below we list all required properties of dual quaternions for the rest of the exercise.

A dual quaterion $\hat{\mathbf{q}}$ can be written in the form $\hat{\mathbf{q}} = \mathbf{q}_0 + \epsilon\mathbf{q}_\epsilon$, where $\mathbf{q}_0$ and $\mathbf{q}_\epsilon$ are quaternions and ϵ is the dual unit with the property $\epsilon^2 = 0$. The norm of $\hat{\mathbf{q}}$ is then given by:

$$\|\hat{\mathbf{q}}\| = \|\mathbf{q}_0\| + \epsilon\frac{\langle \mathbf{q}_0, \mathbf{q}_\epsilon\rangle}{\|\mathbf{q}_0\|}$$

Dual quaternions representing rigid transformations can be written in the following form:

$$\hat{\mathbf{q}} = \cos\left(\frac{\hat{\theta}}{2}\right) + \hat{\mathbf{s}}\sin\left(\frac{\hat{\theta}}{2}\right)$$

where $\hat{\theta} = \theta_0 + \epsilon\theta_\epsilon$ and $\hat{\mathbf{s}} = \mathbf{s}_0 + \epsilon\mathbf{s}_\epsilon$. Here, $\mathbf{s}_0$ is the axis of rotation, θ_0 is the rotation angle and θ_ϵ is the amount of translation along $\mathbf{s}_0$. since this is a unit dual quaternion it can be shown that $\langle \mathbf{s}_0, \mathbf{s}_\epsilon\rangle = 0$ and $\langle \mathbf{s}_0, \mathbf{s}_0\rangle = 1$.

The power of a dual quaternion is defined by

$$e^{\hat{\mathbf{q}}} = \cos\left(\|\hat{\mathbf{q}}\|\right) + \frac{\hat{\mathbf{q}}}{\|\hat{\mathbf{q}}\|}\sin\left(\|\hat{\mathbf{q}}\|\right) \qquad \ln\left(\cos\left(\frac{\hat{\theta}}{2}\right) + \hat{\mathbf{s}}\sin\left(\frac{\hat{\theta}}{2}\right)\right) = \hat{\mathbf{s}}\frac{\hat{\theta}}{2}$$

(a) Utilizing the properties above, for a dual quaternion $\hat{\mathbf{q}} = \cos\left(\frac{\hat{\theta}}{2}\right) + \hat{\mathbf{s}}\sin\left(\frac{\hat{\theta}}{2}\right)$, prove that:

$$\hat{\mathbf{q}}^t = \cos\left(\frac{t\hat{\theta}}{2}\right) + \hat{\mathbf{s}}\sin\left(\frac{t\hat{\theta}}{2}\right)$$

$$
\begin{aligned}
\hat{\mathbf{s}}\frac{\hat{\theta}}{2} &= \ln\left(\cos\left(\frac{\hat{\theta}}{2}\right) + \hat{\mathbf{s}}\sin\left(\frac{\hat{\theta}}{2}\right)\right)\\
e^{\hat{\mathbf{s}}\frac{\hat{\theta}}{2}} &= \cos\left(\frac{\hat{\theta}}{2}\right) + \hat{\mathbf{s}}\sin\left(\frac{\hat{\theta}}{2}\right)\\
e^{\hat{\mathbf{s}}\frac{t\hat{\theta}}{2}} &= \cos\left(\frac{t\hat{\theta}}{2}\right) + \hat{\mathbf{s}}\sin\left(\frac{t\hat{\theta}}{2}\right)\\
e^{t\left(\hat{\mathbf{s}}\frac{\hat{\theta}}{2}\right)} &= \cos\left(\frac{t\hat{\theta}}{2}\right) + \hat{\mathbf{s}}\sin\left(\frac{t\hat{\theta}}{2}\right)\\
e^{t\ln\hat{\mathbf{q}}} &= \cos\left(\frac{t\hat{\theta}}{2}\right) + \hat{\mathbf{s}}\sin\left(\frac{t\hat{\theta}}{2}\right)\\
\hat{\mathbf{q}}^t &= \cos\left(\frac{t\hat{\theta}}{2}\right) + \hat{\mathbf{s}}\sin\left(\frac{t\hat{\theta}}{2}\right)
\end{aligned}
$$

(b) How can a single dual quaternion with represent rotations and transformations in the xy-plane?

Chalses' Theorem states that any rigid displacement is equivalent to some screw displacement.

In this case, the dual quaternion would have a screw axis perpendicular to the xy-plane and passing through the midpoint between the center of the original shape and the center of the target with rotation π and displacement 0.

We have a center of rotation and rotate around the center. We just choose the center carefully and then rotate. We choose a vector around which we rotate – so a rotation of π around a vector $\underline{\mathbf{0}} + \begin{pmatrix}1\\0\\0\end{pmatrix}$ is the same as rotating the object with coordinates $\begin{pmatrix}\hat{x}\\\hat{y}\\\hat{z}\end{pmatrix}$ by π and translating it by $\begin{pmatrix}-2\hat{x}\\0\\0\end{pmatrix}$.