5. On sets

5.1 Basic exercises

1. Prove that $\subseteq$ is a partial order, that is, it is:

   (a) reflexive: $\forall$ sets A, $A \subseteq A$

   We shall prove that every element in A is also in A.

$$\begin{aligned} \forall a \in A : a \in A \iff \\ A \subseteq A \text{ as required} \end{aligned} \tag{1}$$

   (b) transistive: $\forall$ sets A, B, C. $(A \subseteq B \wedge B \subseteq C) \implies A \subseteq C$

   We shall prove that every element in A must be in B. Since every element in B is in C: every element in A is also in C.

   Assume $A \subseteq B \wedge B \subseteq C$

$$\begin{gathered} \text{by assumption: } A \subseteq B \iff \\ \forall a \in A : a \in B \\ \text{by assumption: } B \subseteq C \iff \\ \forall b \in B : b \in C \\ \therefore \forall a \in A : a \in B \implies \\ \forall a \in A : a \in C \implies \\ A \subseteq C \end{gathered} \tag{2}$$

   (c) antisymmetric: $\forall$ sets A, B. $(A \subseteq B \wedge B \subseteq A) \iff A = B$

   I shall prove that every $a \in A$ is also in B and every $b \in B$ is also in A. This implies that A and B contain the same elements and hence are the same set.

$$\begin{gathered} A \subseteq B \iff \\ \forall a \in A : a \in B \\ B \subseteq A \iff \\ \forall b \in B : b \in A \\ \forall a \in A : a \in B \wedge \forall b \in B : b \in A \iff \\ A = B \end{gathered} \tag{3}$$

2. Prove the following statements:

   (a) $\forall$ sets A. $\emptyset \subseteq A$

   By definition if S is a set:

$$S \subseteq A \iff \forall s \in S : s \in A \tag{4}$$

   For $\emptyset$ this is vacuously true.

$$\begin{gathered} (\emptyset \subseteq A \iff \forall s \in \emptyset : s \in A) \iff \\ (\emptyset \subseteq A \iff \text{true}) \iff \\ \emptyset \subseteq A \text{ as required} \end{gathered} \tag{5}$$

(b) $\forall$ sets A. $(\forall x : x \notin A) \iff A = \emptyset$
TODO

3. Find the union, and intersection of:

(a) $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$

$$\{1, 2, 3, 4, 5\} \cup \{-1, 1, 3, 5, 7\} = \{-1, 1, 2, 3, 4, 5, 7\} \tag{6}$$

$$\{1, 2, 3, 4, 5\} \cap \{-1, 1, 3, 5, 7\} = \{1, 3, 5\} \tag{7}$$

(b) $\{x \in \mathbb{R} : x > 7\}$ and $\{x \in \mathbb{N} : x > 5\}$

$$\begin{aligned} &\{x \in \mathbb{R} : x > 7\} \cup \{x \in \mathbb{N} : x > 5\} \\ =&\{x \in \mathbb{R} : x > 7 \vee x \in \{6, 7\}\} \end{aligned} \tag{8}$$

$$\begin{aligned} &\{x \in \mathbb{R} : x > 7\} \cap \{x \in \mathbb{N} : x > 5\} \\ =&\{x \in \mathbb{N} : x > 7\} \end{aligned} \tag{9}$$

4. Find the Cartesian product and disjoint union of $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$.
The Cartesian product of two sets S and T is $\{x : \forall s \in S, \forall t \in T : x = (s, t)\}$
For the sets $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$ this is equal to:

$$\begin{aligned} &\{(1, -1), (1, 1), (1, 3), (1, 5), (1, 7), (2, -1), (2, 1), (2, 3), (2, 5), (2, 7), (3, -1), (3, 1), (3, 3), \\ &(3, 5), (3, 7), (4, -1), (4, 1), (4, 3), (4, 5), (4, 7), (5, -1), (5, 1), (5, 3), (5, 5), (5, 7)\} \end{aligned} \tag{10}$$

5. Let $I = \{2, 3, 4, 5\}$ and for each $i \in I$, let $A_i = \{i, i + 1, i - 1, 2 \cdot i\}$.

(a) List the elements of all sets A_i for $i \in I$

$$\begin{aligned} A_2 &= \{1, 2, 3, 4\} \\ A_3 &= \{2, 3, 4, 6\} \\ A_4 &= \{3, 4, 5, 8\} \\ A_5 &= \{4, 5, 6, 10\} \end{aligned} \tag{11}$$

(b) Let $\{A_i : i \in I\}$ stand for $\{A_2, A_3, A_4, A_5\}$. Find $\bigcup\{A_i : i \in I\}$ and $\bigcap\{A_i : i \in I\}$.

$$\bigcup\{A_i : i \in I\}\{1, 2, 3, 4, 5, 6, 8, 10\} \tag{12}$$

$$\bigcap\{A_i : i \in I\}\{4\} \tag{13}$$

6. Let U be a set. For all $A, B \in \mathcal{P}(A)$, prove that:

(a) $A^\mathsf{c} = B \iff (A \cup B = U \wedge A \cap B = \emptyset)$

(b) Double complement elimination: $(A^\mathsf{c})^\mathsf{c} = A$

(c) The De-Morgan laws: $(A \cup B)^\mathsf{c} = A^\mathsf{c} \cap B^\mathsf{c}$ and $(A \cap B)^\mathsf{c} = A^\mathsf{c} \cup B^\mathsf{c}$

5.2 Core exercises

1. Prove that for all sets U and subsets $A, B \subseteq U$:
   (a) $\forall X : A \subseteq X \wedge B \subseteq X \iff (A \cup B) \subseteq X)$
   (b) $\forall Y : Y \subseteq A \wedge Y \subseteq B \iff Y \subseteq (A \cap B)$
2. Either prove or disprove that, for all sets A and B,
   (a) $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$
   (b) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$
   (c) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$
   (d) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$
   (e) $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$
3. Let U be a set. For all $A, B \in \mathcal{P}(U)$ prove that the following statements are equivalent.
   (a) $A \cup B = B$
   (b) $A \subseteq B$
   (c) $A \cap B = A$
   (d) $B^{\mathsf{c}} \subseteq A^{\mathsf{c}}$
4. For sets A, B, C, D, prove or disprove at least three of the following statements:
   (a) $(A \subseteq C \wedge B \subseteq D) \implies A \times B \subseteq C \times D$
   (b) $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$
   (c) $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$
   (d) $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
   (e) $(A \times B) \cup (A \times D) \subseteq A \times (B \cup D)$
5. For sets A, B, C, D, prove or disprove at least three of the following statements:
   (a) $(A \subseteq C \wedge B \subseteq D) \implies A \uplus B \subseteq C \uplus D$
   (b) $(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C)$
   (c) $(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C$
   (d) $(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C)$
   (e) $(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C$
6. Prove the following properties of the big unions and intersections of a family of sets $\mathcal{F} \subseteq \mathcal{P}(A)$:
   (a) $\forall U \subseteq A : (\forall X \in \mathcal{F} : X \subseteq U) \iff \bigcup \mathcal{F} \subseteq U$
   (b) $\forall L \subseteq A : (\forall X \in \mathcal{F} : L \subseteq X) \iff L \subseteq \bigcap \mathcal{F}$
7. Let A be a set.
   (a) For a family $\mathcal{F} \subseteq \mathcal{P}(A)$, let $\mathcal{U} \triangleq \{U \subseteq A : \forall S \in \mathcal{F} : S \subseteq U\}$. Prove that $\bigcup \mathcal{F} = \bigcap \mathcal{U}$.
   (b) Analogously, define the family $\mathcal{L} \subseteq \mathcal{P}(A)$ such that $\bigcap \mathcal{F} = \bigcup \mathcal{L}$. Also prove this statement.

5.3 Optional advanced exercises

1. Prove that for all families of sets $\mathcal{F}_1$ and $\mathcal{F}_2$

$$(\bigcup \mathcal{F}_1) \cup (\bigcup \mathcal{F}_2) = \bigcup(\mathcal{F}_1 \cup \mathcal{F}_2) \tag{14}$$

   State and prove tha analogous property for intersections of non-empty families of sets.

2. For a set U, prove that $(\mathcal{P}(U), \subseteq, \cup, \cap, U, \emptyset, (\cdot)^c)$ is a Boolean algebra.