

## Ordinary Differential Equations

9. (a)

$$\begin{aligned}\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0 \\ (\lambda - 2)(\lambda - 3) &= 0 \\ \lambda = 2 \vee \lambda = 3\end{aligned}\tag{1}$$

$$y_c = Ae^{2x} + Be^{3x}$$

$$y_p = 0$$

$$y = y_c + y_p$$

$$y = Ae^{2x} + Be^{3x}$$

$$y(0) = 0$$

$$0 = Ae^{2 \times 0} + Be^{3 \times 0}\tag{2}$$

$$0 = A + B$$

$$y'(0) = 1$$

$$\frac{dy}{dx} = 2Ae^{2x} + 3Be^{3x}$$

$$\frac{dy}{dx}(0) = 2Ae^{2 \times 0} + 3Be^{3 \times 0}$$

$$1 = 2A + 3B\tag{3}$$

$$1 = 2(A + B) + B$$

$$1 = B$$

$$0 = A + 1$$

$$A = -1$$

$$y = -e^{2x} + e^{3x}$$

(b)

$$\left(\frac{d^n}{dx^n} + n^2\right)y = 0$$

$$\lambda^2 + n^2 = 0$$

$$\lambda = \pm ni$$

$$y_c = P \sin nx + Q \cos nx\tag{4}$$

$$y_p = 0$$

$$y = y_c + y_p$$

$$y = P \sin nx + Q \cos nx$$

$$y(0) = 0$$

$$0 = P \sin 0 + Q \cos 0\tag{5}$$

$$0 = Q$$

$$y'(0) = 1$$

$$\frac{dy}{dx}(0) = nP \cos 0$$

$$1 = nP\tag{6}$$

$$P = \frac{1}{n}$$

$$y = \frac{1}{n} \sin nx$$

(c)

$$\begin{aligned}\left(\frac{d^2}{dx^2} + 2\frac{d}{dx} + 4\right)y &= 0 \\ \lambda^2 + 2\lambda + 4 &= 0 \\ \lambda &= \frac{-2 \pm \sqrt{4 - 16}}{2} \\ \lambda &= -1 \pm \sqrt{3}i \\ y_c &= e^{-x}(P \sin(\sqrt{3}x) + Q \cos(\sqrt{3}x)) \\ y_p &= 0 \\ y &= y_p + y_c \\ y &= e^{-x}(P \sin(\sqrt{3}x) + Q \cos(\sqrt{3}x))\end{aligned}\tag{7}$$

$$\begin{aligned}y(0) &= 0 \\ 0 &= e^0(P \sin 0 + Q \cos 0) \\ 0 &= Q\end{aligned}\tag{8}$$

$$\begin{aligned}\frac{dy}{dx} &= -Pe^{-x} \sin(\sqrt{3}x) + \sqrt{3}Pe^{-x} \cos(\sqrt{3}x) \\ \frac{dy}{dx}(0) &= -Pe^0 \sin 0 + e^0 \sqrt{3}P \cos(0) \\ 1 &= 0 + \sqrt{3}P \\ P &= \frac{1}{\sqrt{3}} \\ y &= \frac{1}{\sqrt{3}}e^{-x} \sin(\sqrt{3}x)\end{aligned}\tag{9}$$

(d)

$$\begin{aligned}\left(\frac{d^2}{dx^2} + 9\right)y &= 18 \\ \lambda^2 + 9 &= 0 \\ \lambda &= \pm 3i \\ y_c &= P \sin 3x + Q \cos 3x + y_p \\ y_p &= c \\ 9c &= 18 \\ c &= 2 \\ y &= y_c + y_p \\ y &= P \sin 3x + Q \cos 3x + 2\end{aligned}\tag{10}$$

$$\begin{aligned}y(0) &= 0 \\ 0 &= P \sin 0 + Q \cos 0 + 2 \\ 0 &= 0 + Q + 2 \\ Q &= -2\end{aligned}\tag{11}$$

$$\begin{aligned}
 y'(0) &= 1 \\
 \frac{dy}{dx} &= 3P \cos 3x - 6 \sin 3x \\
 \frac{dy}{dx}(0) &= 3P \cos 0 - 6 \sin 0 \\
 1 &= 3P \\
 P &= \frac{1}{3} \\
 y &= \frac{1}{3} \sin 3x - 2 \cos 3x + 2
 \end{aligned} \tag{12}$$

(e)

$$\begin{aligned}
 \left( \frac{d^2}{dx^2} - 3 \frac{d}{dx} + 2 \right) y &= e^{5x} \\
 \lambda^2 - 3\lambda + 2 &= 0 \\
 (\lambda - 1)(\lambda - 2) &= 0 \\
 \lambda &= 1 \vee \lambda = 2
 \end{aligned} \tag{13}$$

$$\begin{aligned}
 y_c &= Ae^x + Be^{2x} \\
 y_p &= ke^{5x} \\
 \frac{dy_p}{dx} &= 5ke^{5x} \\
 \frac{d^2 y_p}{dx^2} &= 25ke^{5x} \\
 \frac{d^2 y_p}{dx^2} - 3 \frac{dy_p}{dx} + 2y_p &= e^{5x} \\
 (25k - 15k + 2k)e^{5x} &= e^{5x} \\
 12k &= 1 \\
 k &= \frac{1}{12} \\
 y &= y_c + y_p \\
 y &= Ae^x + Be^{2x} + \frac{1}{12}e^{5x}
 \end{aligned} \tag{14}$$

$$\begin{aligned}
 y(0) &= 0 \\
 0 &= Ae^0 + Be^0 + \frac{1}{12}e^0 \\
 0 &= A + B + \frac{1}{12}
 \end{aligned} \tag{15}$$

$$\begin{aligned}
 y'(0) &= 1 \\
 \frac{dy}{dx} &= Ae^x + 2Be^{2x} + \frac{5}{12}e^{5x} \\
 \frac{dy}{dx}(0) &= Ae^0 + 2Be^0 + \frac{5}{12}e^0 \\
 1 &= A + 2B + \frac{5}{12} \\
 1 &= (A + B + \frac{1}{12}) + B + \frac{4}{12} \\
 B &= \frac{2}{3} \\
 A &= -\frac{1}{12} - B \\
 A &= -\frac{3}{4} \\
 y &= -\frac{3}{4}e^x + \frac{2}{3}e^{2x} + \frac{1}{12}e^{5x}
 \end{aligned} \tag{16}$$

(f)

$$\begin{aligned}
 \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y &= 0 \\
 \lambda^2 - 2\lambda + 1 &= 0 \\
 \lambda &= 1 \\
 y_c &= (A + Bx)e^x \\
 y_p &= 0 \\
 y &= y_c + y_p \\
 y &= (A + Bx)e^x
 \end{aligned} \tag{17}$$

(g) The complementary function is the same as in the previous question so we can reuse the result and only need to work out the particular integral and the initial conditions.

$$\begin{aligned}
 \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y &= e^{2x} + e^x \\
 y_p &= Pe^{2x} + Qx^2e^x \\
 \frac{dy_p}{dx} &= 2Pe^{2x} + 2Qxe^x + Qx^2e^x \\
 \frac{d^2y_p}{dx^2} &= 4Pe^{2x} + 2Qe^x + 4Qxe^x + Qx^2e^x \\
 \frac{d^2y_p}{dx^2} - 2\frac{dy_p}{dx} + y &= e^{2x} + e^x \\
 4Pe^{2x} - 4Pe^{2x} + Pe^{2x} + 2Qe^x &= e^{2x} + e^x \\
 Pe^{2x} + 2Qe^x &= e^{2x} + e^x \\
 P = 1 \wedge Q &= \frac{1}{2} \\
 y_p &= e^{2x} + \frac{1}{2}x^2e^x \\
 y &= y_c + y_p \\
 y &= (A + Bx + \frac{1}{2}x^2)e^x + e^{2x}
 \end{aligned} \tag{18}$$

$$P = 1 \wedge Q = \frac{1}{2} \tag{19}$$

$$y_p = e^{2x} + \frac{1}{2}x^2e^x$$

$$y = y_c + y_p$$

$$y = (A + Bx + \frac{1}{2}x^2)e^x + e^{2x}$$

10.

$$\begin{aligned}
 -iR &= \frac{q}{C} + V \\
 i &= -\frac{q}{RC} - \frac{V}{R} \\
 -L \frac{dj}{dt} &= \frac{q}{C} + V \\
 \frac{dj}{dt} &= -\frac{q}{LC} - \frac{V}{L} \\
 \frac{dq}{dt} &= i + j \\
 \frac{dq}{dt} &= -\frac{q}{RC} - \frac{V}{R} + j \\
 \frac{d^2q}{dt^2} &= -\frac{1}{RC} \frac{dq}{dt} - \frac{1}{R} \frac{dV}{dt} + \frac{dj}{dt} \\
 \frac{d^2q}{dt^2} &= -\frac{1}{RC} \frac{dq}{dt} - \frac{1}{R} \frac{dV}{dt} - \frac{q}{LC} - \frac{V}{L} \\
 \frac{d^2q}{dt^2} + \frac{1}{RC} \frac{dq}{dt} + \frac{1}{LC} q &= -\frac{1}{R} \frac{dV}{dt} - \frac{1}{L} V \text{ as required}
 \end{aligned} \tag{20}$$

- (a) Since we have initial conditions  $Q = 0$  and  $\dot{Q}$  is not changing, we know that both  $\dot{Q} = 0$  and  $\frac{dQ}{dt} = 0$ . This means that the RHS of the equation is now 0.

$$\begin{aligned}
 \frac{d^2q}{dt^2} + \frac{1}{RC} \frac{dq}{dt} + \frac{1}{LC} q &= 0 \\
 L &= 8R^2C \\
 \frac{d^2q}{dt^2} + \frac{1}{RC} \frac{dq}{dt} + \frac{1}{8R^2C^2} q &= 0 \\
 \lambda^2 + \frac{1}{RC} \lambda + \frac{1}{8R^2C^2} &= 0 \\
 \lambda &= \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{8R^2C^2}}}{2} \\
 \lambda &= \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{2R^2C^2}}}{2} \\
 \lambda &= \frac{-\frac{1}{RC} \pm \frac{1}{\sqrt{2}RC}}{2} \\
 \lambda &= \frac{-2 \pm \sqrt{2}}{4RC} \\
 q &= Ae^{\left(\frac{-2+\sqrt{2}}{4RC}\right)t} + Be^{\left(\frac{-2-\sqrt{2}}{4RC}\right)t}
 \end{aligned} \tag{21}$$

(22)

$$\begin{aligned}
 q(0) &= Q \\
 A + B &= Q \\
 \frac{dq}{dt}(0) &= -\frac{Q}{RC} \\
 \left(\frac{-2+\sqrt{2}}{4RC}\right)A + \left(\frac{-2-\sqrt{2}}{4RC}\right)B &= -\frac{Q}{RC} \\
 \frac{-2(A+B)}{4RC} + \frac{\sqrt{2}(A-B)}{4RC} &= -\frac{Q}{RC} \\
 -\frac{Q}{2RC} + \frac{2\sqrt{2}A}{4RC} - \frac{\sqrt{2}Q}{4RC} &= -\frac{Q}{RC} \\
 \frac{\sqrt{2}A}{2RC} &= \frac{(\sqrt{2}-2)Q}{4RC} \\
 A &= \frac{(1-\sqrt{2})Q}{2}
 \end{aligned} \tag{23}$$

$$\begin{aligned}
 A + B &= Q \\
 B &= Q - \frac{(1-\sqrt{2})Q}{2} \\
 B &= \frac{(1+\sqrt{2})Q}{2}
 \end{aligned} \tag{24}$$

$$q = \frac{(1-\sqrt{2})Q}{2}e^{\left(\frac{-2+\sqrt{2}}{4RC}\right)t} + \frac{(1+\sqrt{2})Q}{2}e^{\left(\frac{-2-\sqrt{2}}{4RC}\right)t} \tag{25}$$

The roots to this equation are distinct and real: this is strong dampening. The charge will decrease to 0 exponentially without ever oscillating.

(b)

$$\begin{aligned}
 \frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{1}{LC}q &= 0 \\
 L &= 4R^2C^2 \\
 \frac{d^2q}{dt^2} + \frac{1}{RC}\frac{dq}{dt} + \frac{1}{4R^2C^2}q &= 0 \\
 \lambda^2 + \frac{1}{RC}\lambda + \frac{1}{4R^2C^2} &= 0
 \end{aligned} \tag{26}$$

$$\begin{aligned}
 \lambda &= \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{4R^2C^2}}}{2} \\
 \lambda &= -\frac{1}{2RC} \\
 q &= (A + Bt)e^{-\frac{t}{2RC}}
 \end{aligned} \tag{27}$$

$$\begin{aligned}
 q(0) &= Q \\
 A &= Q
 \end{aligned} \tag{28}$$

$$\begin{aligned}
 \frac{dq}{dt}(0) &= -\frac{Q}{RC} \\
 Be^{-\frac{0}{2RC}} - \frac{Q}{2RC}e^{-\frac{t}{2RC}} &= -\frac{Q}{RC} \\
 B - \frac{Q}{2RC} &= -\frac{Q}{RC} \\
 B &= -\frac{Q}{2RC}
 \end{aligned} \tag{29}$$

$$q = \left(Q - \frac{Qt}{2RC}\right)e^{-\frac{t}{2RC}} \tag{30}$$

In this equation, the roots to the simultaneous are repeated: this is critical dampening. In this form the charge will decrease to 0 without ever increasing or oscillating – however this decrease will be faster than for strong dampening.

(c)

$$\begin{aligned}\frac{d^2q}{dt^2} + \frac{1}{RC} \frac{dq}{dt} + \frac{1}{LC}q &= 0 \\ L &= 2R^2C^2 \\ \frac{d^2q}{dt^2} + \frac{1}{RC} \frac{dq}{dt} + \frac{1}{2R^2C^2}q &= 0 \\ \lambda^2 + \frac{1}{RC}\lambda + \frac{1}{2R^2C^2} &= 0\end{aligned}\tag{31}$$

$$\begin{aligned}\lambda &= \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{2R^2C^2}}}{2} \\ \lambda &= -\frac{1}{2RC} \pm \frac{1}{2RC}i\end{aligned}\tag{32}$$

$$\begin{aligned}q &= \left( A \sin\left(\frac{t}{2RC}\right) + B \cos\left(\frac{t}{2RC}\right) \right) e^{-\frac{t}{2RC}} \\ q(0) &= Q \\ B &= Q\end{aligned}\tag{33}$$

$$\begin{aligned}\frac{dq}{dt}(0) &= -\frac{Q}{RC} \\ -\frac{Q}{2RC} + \frac{1}{2RC} \left( A \cos\left(\frac{t}{2RC}\right) - Q \sin\left(\frac{t}{2RC}\right) \right) e^{-\frac{t}{2RC}} &= -\frac{Q}{RC} \\ -\frac{Q}{2RC} + \frac{A}{2RC} &= -\frac{Q}{RC} \\ A &= -Q\end{aligned}\tag{34}$$

$$q = \left( -Q \sin\left(\frac{t}{2RC}\right) + Q \cos\left(\frac{t}{2RC}\right) \right) e^{-\frac{t}{2RC}}\tag{35}$$

The roots to the equation in this case are complex: this is weak dampening – in this version the charge will oscillate with constant angular frequency between two exponentially decreasing values.

11. (a)

$$\begin{aligned}y'' - (2+c)y' + (1+c)y &= e^{(1+2c)x} \\ \lambda^2 - (2+c)\lambda + (1+c) &= 0 \\ (\lambda-1)(\lambda-(1+c)) &= 0 \\ \lambda &= 1 \vee \lambda = 1+c\end{aligned}\tag{36}$$

$$\begin{aligned}y_c &= Pe^x + Qe^{(1+c)x} \\ y_p &= ke^{(1+2c)x} \\ \frac{dy_p}{dx} &= k(1+2c)e^{(1+2c)x} \\ \frac{d^2y_p}{dx^2} &= k(1+2c)^2e^{(1+2c)x}\end{aligned}\tag{37}$$

$$\begin{aligned}
 y_p'' + (2+c)y_p' + (1+c)y_p &= e^{(1+2c)x} \\
 k((1+2c)^2 - (2+c)(1+2c) + (1+c))e^{(1+2c)x} &= e^{(1+2c)x} \\
 k(4c^2 + 4c + 1 - 2c^2 - 5c - 2 + 1 + c) &= 1 \\
 2c^2k &= 1 \\
 k &= \frac{1}{2c^2}
 \end{aligned} \tag{38}$$

$$\begin{aligned}
 k &= \frac{1}{2(c+1)(3c+2)} \\
 y_p &= \frac{1}{2(c+1)(3c+2)} e^{(1+2c)x} \\
 y &= y_c + y_p \\
 y &= Pe^x + Qe^{(1+c)x} + \frac{1}{2c^2} e^{(1+2c)x}
 \end{aligned} \tag{39}$$

Let:

$$\begin{aligned}
 P &= A - \frac{B}{c} + \frac{1}{2c^2} \\
 Q &= B\frac{1}{c} - \frac{1}{c^2}
 \end{aligned} \tag{40}$$

$$\begin{aligned}
 y &= Ae^x - B\frac{1}{c}e^x + \frac{1}{2c^2}e^x + B\frac{1}{c}e^{(1+c)x} - \frac{1}{c^2}e^{(1+c)x} + \frac{1}{2c^2}e^{(1+2c)x} \\
 y &= Ae^x + B\frac{1}{c}(e^{(1+c)x} - e^x) + \frac{1}{2c^2}(e^{(1+2c)x} - 2e^{(1+c)x} + e^x) \\
 y &= Ae^x + B\frac{e^x}{c}(e^{cx} - 1) + \frac{e^x}{2c^2}(e^{2cx} - 2e^{cx} + 1) \text{ as required}
 \end{aligned} \tag{41}$$

- (b) To find the limit as  $c \rightarrow 0$ , I will apply l'hôpital's rule separately for the different parts of the expression.

$$\lim_{c \rightarrow 0} f(x, c) = Ae^x + Be^x \lim_{c \rightarrow 0} \left( \frac{(e^{cx} - 1)}{c} \right) + e^x \lim_{c \rightarrow 0} \left( \frac{(e^{2cx} - 2e^{cx} + 1)}{2c^2} \right) \tag{42}$$

$$\begin{aligned}
 \frac{(e^{cx} - 1)}{c} &= \frac{f(c)}{g(c)} \\
 \lim_{c \rightarrow 0} f(c) &= e^0 - 1 = 0 \\
 \lim_{c \rightarrow 0} g(c) &= 0
 \end{aligned} \tag{43}$$

So we can apply l'hôpital's rule to find the limit as  $c \rightarrow 0$ .

$$\begin{aligned}
 \lim_{c \rightarrow 0} B\frac{e^x}{c}(e^{cx} - 1) &= \frac{\frac{d}{dc}(e^{cx} - 1)}{\frac{d}{dc}(c)} \\
 &= \frac{xe^{cx}}{1} \\
 &= xe^{cx} \\
 &= xe^0 \\
 &= x
 \end{aligned} \tag{44}$$

$$\begin{aligned}
 \frac{(e^{2cx} - 2e^{cx} + 1)}{2c^2} &= \frac{f(c)}{g(c)} \\
 \lim_{c \rightarrow 0} f(c) &= e^0 - 2e^0 + 1 = 0 \\
 \lim_{c \rightarrow 0} g(c) &= 2 \cdot 0^2 = 0
 \end{aligned} \tag{45}$$



So we can apply l'hôpital's rule to find the limit as  $c \rightarrow 0$ .

$$\begin{aligned}\lim_{c \rightarrow 0} B \frac{e^x}{c} (e^{cx} - 1) &= \frac{\frac{d}{dc}(e^{2cx} - 2e^{cx} + 1)}{\frac{d}{dc}(2c^2)} \\ &= \frac{2xe^{2cx} - 2xe^{cx}}{4c}\end{aligned}\quad (46)$$

Now we must find the limit of this new function as  $c \rightarrow 0$ .

$$\begin{aligned}\frac{2xe^{2cx} - 2xe^{cx}}{4c} &= \frac{f(x)}{g(x)} \\ \lim_{c \rightarrow 0} f(x) &= 2xe^0 - 2xe^0 = 2x - 2x = 0 \\ \lim_{c \rightarrow 0} g(x) &= 4 \times 0 = 0\end{aligned}\quad (47)$$

So we can apply l'hôpital's rule once more.

$$\begin{aligned}\lim_{c \rightarrow 0} \frac{2xe^{2cx} - 2xe^{cx}}{4c} &= \frac{\frac{d}{dc}(2xe^{2cx} - 2xe^{cx})}{\frac{d}{dc}(4c)} \\ &= \frac{4x^2e^{2cx} - 2x^2e^{cx}}{4} \\ &= x^2e^0 - \frac{1}{2}x^2e^0 \\ &= \frac{1}{2}x^2\end{aligned}\quad (48)$$

Substituting these results back into the original expression gives:

$$\begin{aligned}\lim_{c \rightarrow 0} f(x, c) &= Ae^x + Bxe^x + \frac{1}{2}x^2e^x \\ &= (A + Bx + \frac{1}{2}x^2)e^x\end{aligned}\quad (49)$$

So the solution to the differential equation when  $c = 0$  is:

$$y = (A + Bx + \frac{1}{2}x^2)e^x \quad (50)$$

12.

$$\begin{aligned}-\frac{\eta}{2} \frac{dC}{d\eta} &= \frac{d^2C}{d\eta^2} \\ \frac{1}{\frac{dC}{d\eta}} \frac{d^2C}{d\eta^2} &= -\frac{\eta}{2} \\ \frac{d}{d\eta} \left( \ln \left( \frac{dC}{d\eta} \right) \right) &= -\frac{\eta}{2} \text{ as required}\end{aligned}\quad (51)$$

$$\begin{aligned}
 \ln \left( \frac{dC}{d\eta} \right) &= - \int \frac{\eta}{2} d\eta \\
 \ln \left( \frac{dC}{d\eta} \right) &= -\frac{\eta^2}{4} + c \\
 \frac{dC}{d\eta} &= e^{-\frac{\eta^2}{4} + c} \\
 \frac{dC}{d\eta} &= Ae^{-\frac{\eta^2}{4}} \\
 \frac{dC}{d\eta}(0) &= Ae^0 \\
 \frac{dC}{d\eta}(0) &= A \\
 C &= B + A \int_0^\eta e^{-\frac{t^2}{4}} dt \\
 C(0) &= B + A \int_0^0 e^{-\frac{t^2}{4}} dt \\
 C(0) &= B
 \end{aligned} \tag{52}$$

So if  $\frac{dC}{d\eta}(0) = A$  and  $C(0) = B$  then  $C = B + A \int_0^\eta e^{-\frac{t^2}{4}} dt$  as required.

13. (a)

$$\begin{aligned}
 \frac{dx}{dt} &= ax \\
 \frac{1}{x} \frac{dx}{dt} &= a \\
 \ln x &= at + c \\
 x &= ke^{at} \\
 x(0) &= 2 \\
 2 &= ke^0 \\
 2 &= k \\
 x &= 2e^{at}
 \end{aligned} \tag{53}$$

$$\begin{aligned}
 \frac{dy}{dt} &= ay + bx \\
 \frac{dy}{dt} &= ay + 2be^{at} \\
 \frac{dy}{dt} - ay &= 2be^{at} \\
 \mu(y) &= e^{\int -adt} \\
 \mu(y) &= e^{-at} \\
 e^{-at} \frac{dy}{dt} - aye^{-at} &= 2b \\
 ye^{-at} &= 2bt + C \\
 y &= 2bte^{at} + Ce^{at} \\
 y(0) &= 1 \\
 1 &= C \\
 y &= 2bte^{at} + e^{at}
 \end{aligned} \tag{54}$$

(b)

$$\begin{aligned}
 \frac{dx}{dt} &= x - xy \\
 \frac{dt}{dx} &= \frac{1}{x - xy} \\
 \frac{dy}{dt} &= -y + xy \\
 \frac{dy}{dt} \frac{dt}{dx} &= \frac{-y + xy}{x - xy} \\
 \frac{dy}{dx} &= \frac{y(x - 1)}{x(1 - y)}
 \end{aligned} \tag{55}$$

And so the coupled differential equations can be transformed into a differential equation of the form  $\frac{dy}{dx} = f(x, y)$  as required.

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{y(x - 1)}{x(1 - y)} \\
 \frac{1 - y}{y} \frac{dy}{dx} &= \frac{x - 1}{x} \\
 \left(\frac{1}{y} - 1\right) \frac{dy}{dx} &= \left(1 - \frac{1}{x}\right) \\
 \ln y - y &= x - \ln x + c \\
 e^{\ln y - y} &= e^{x - \ln x + c} \\
 ye^{-y} &= e^c \frac{e^x}{x} \\
 e^{-c} &= \frac{e^y}{y} \cdot \frac{e^x}{x} \\
 A &= \frac{e^y}{y} \cdot \frac{e^x}{x}
 \end{aligned} \tag{56}$$

And so  $\frac{e^y}{y} \cdot \frac{e^x}{x}$  is independent of  $t$  as required.

