

1. Let  $\psi = X(x)Y(y)$  for some functions  $X, Y$ .

$$\begin{aligned}\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} &= 0 \implies \\ X^{(2)}(x)Y(y) + X(x)Y^{(2)}(y) &= 0 \\ \frac{X^{(2)}(x)}{X(x)} &= -\frac{Y^{(2)}(y)}{Y(y)}\end{aligned}$$

Since they are equal, the result of both of these must be independent of  $x$  and  $y$ .

This forms two ordinary differential equations:

$$X^{(2)}(x) - kX(x) = 0 \qquad Y^{(2)}(y) + kY(y) = 0$$

Which leads us to three possible equations dependent on the values of  $k$ :

$$\psi = \begin{cases} (Ax + B)(Cy + D) & \text{if } k = 0 \\ (Ae^{\lambda x} + Be^{-\lambda x})C \sin(\lambda y + \phi) & \text{if } k > 0 \text{ where } k = \lambda^2 \text{ with } 0 \leq \psi < \pi \\ A \sin(\lambda x + \phi)(Ce^{\lambda y} + De^{-\lambda y}) & \text{if } k < 0 \text{ where } k = -\lambda^2 \text{ with } 0 \leq \psi < \pi \end{cases}$$

Note that in cases  $k = 0$  and  $k < 0$ , the function  $Y$  is not cyclic – meaning that the criteria  $\psi(x, 0) = \psi(x, a)$ . So  $k > 0$  must be true.

Using the boundary condition  $\psi(x, 0) = 0$ :

$$\begin{aligned}(Ae^{\lambda x} + Be^{-\lambda x})C \sin(0 + \phi) &= 0 \implies \\ \sin(\phi) &= 0 \implies \\ \psi &= 0\end{aligned}$$

Using the boundary condition  $\psi(x, a) = 0$ :

$$\begin{aligned}(Ae^{\lambda x} + Be^{-\lambda x})C \sin(\lambda a) &= 0 \implies \\ \sin(\lambda a) &= 0 \implies \\ \lambda a &= n\pi \implies \\ \lambda &= \frac{n\pi}{a}\end{aligned}$$

Notice that  $\lim_{x \rightarrow \infty} \psi(x, y) = 0$ . So  $A = 0$ .

Using the principle of superposition and the boundary condition  $\psi(0, y) = \sin\left(\frac{\pi y}{a}\right) + 2 \sin\left(\frac{\pi y}{a}\right)$ :

$$\begin{aligned}\psi &= \sum_{n=0}^{\infty} B_n C_n e^{-\frac{\pi n}{a} x} \sin\left(\frac{\pi n}{a} y\right) \\ \psi(0, y) &= \sum_{n=0}^{\infty} K_n \sin\left(\frac{\pi n}{a} y\right) \\ \sin\left(\frac{\pi y}{a}\right) + 2 \sin\left(\frac{\pi y}{a}\right) &= \sum_{n=0}^{\infty} K_n \sin\left(\frac{\pi n}{a} y\right) \implies \\ K_n &= \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases} \\ \psi(x, y) &= e^{-\frac{\pi x}{a}} \sin\left(\frac{\pi y}{a}\right) + 2e^{-\frac{2\pi x}{a}} \sin\left(\frac{2\pi y}{a}\right)\end{aligned}$$

2.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Using the method of separation of variables:

$$\phi = \begin{cases} (ax + b)(cy + d) \\ (Ae^{\lambda x} + Be^{-\lambda x})C \sin(\lambda y + \theta) \\ A \sin(\lambda x + \theta)(Ce^{\lambda y} + De^{-\lambda y}) \end{cases}$$

Since  $\phi$  is periodic with  $x$ , we can conclude that case 3 is the correct case. Note that since  $x$  is periodic with period  $\pi$  and  $\phi(0, y) = 0$ , we can derive that  $\theta = 0$  and  $\exists n \in \mathbb{N}. \lambda = n$ . By the principle of superposition, we can therefore form:

$$\phi = \sum_{n=1}^{\infty} A_n \sin(nx)(C_n e^{ny} + D_n e^{-ny})$$

Noting the constraint  $\phi(x, 0) = 0$ :

$$C_n + D_n = 0 \implies D_n = -C_n$$

We now have the expression:

$$\phi = \sum_{n=1}^{\infty} A_n C_n \sin(nx)(e^{ny} - e^{-ny})$$

Replacing  $A_n C_n$  with  $K_n$ , equating with  $\phi(x, b)$  and then using orthogonality gives:

$$\begin{aligned} \phi(x, y) &= \sum_{n=1}^{\infty} K_n \sin(nx)(e^{ny} - e^{-ny}) \\ \int_0^{\pi} x(\pi - x) \sin(nx) dx &= K_n \int_0^{\pi} \sin^2(nx)(e^{nb} - e^{-nb}) dx \\ K_n &= \frac{\left[ \frac{x^2}{n} \cos(nx) - \frac{2x}{n^2} \sin(nx) - \frac{2}{n^2} \cos(nx) - \frac{\pi x}{n} \cos(nx) + \frac{\pi}{n^2} \sin(nx) \right]_0^{\pi}}{(e^{nb} - e^{-nb}) \left[ \frac{1}{2}x - \frac{1}{4n} \sin(2nx) \right]_0^{\pi}} \\ K_n &= \frac{\frac{\pi^2}{n} \cos(n\pi) - \frac{2}{n^2} \cos(n\pi) - \frac{\pi^2}{n} \cos(n\pi) + \frac{2}{n^2}}{(e^{nb} - e^{-nb}) \frac{1}{2}\pi} \\ K_n &= \frac{\frac{2}{n^2}(1 - (-1)^n)}{(e^{nb} - e^{-nb}) \frac{1}{2}\pi} \\ K_n &= \frac{4(1 - (-1)^n)}{n^2 \pi (e^{nb} - e^{-nb})} \end{aligned}$$

So:

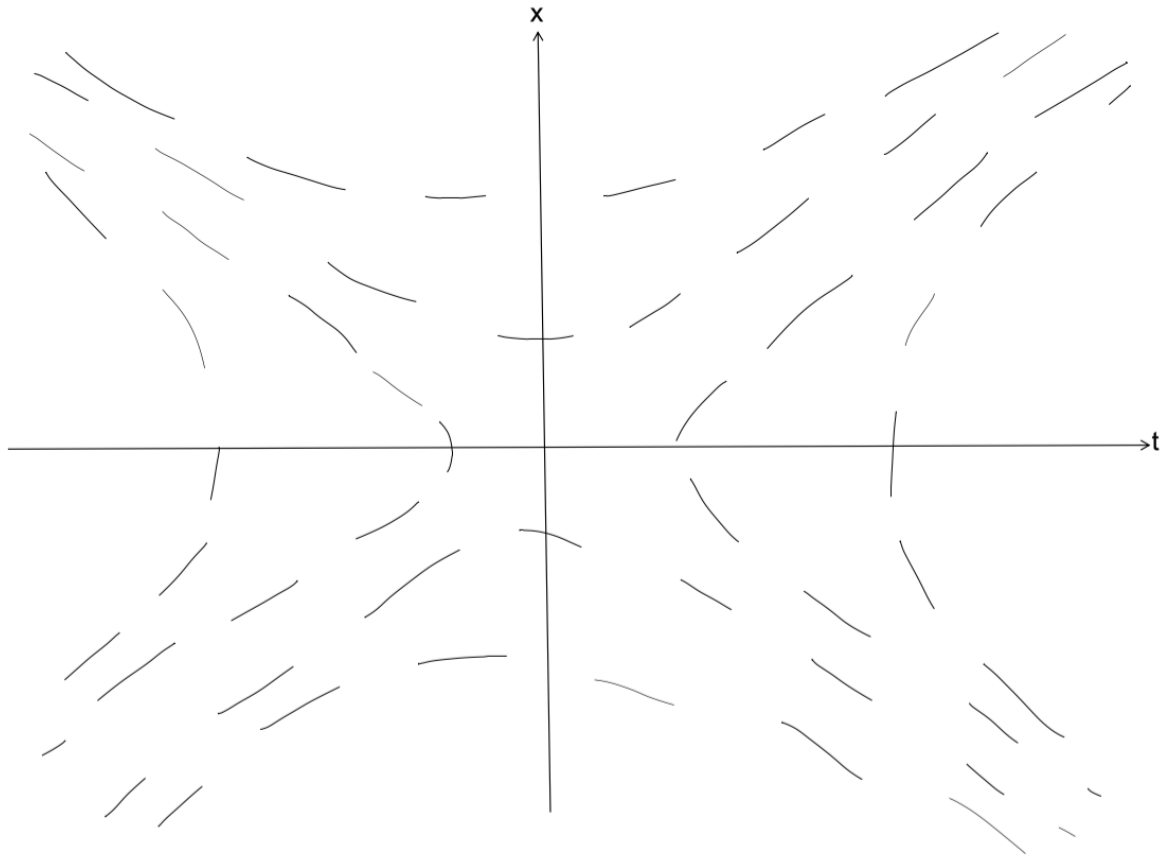
$$\phi(x, y) = \sum_{n=1}^{\infty} \frac{4(1 - (-1)^n)}{n^2 \pi (e^{nb} - e^{-nb})} \sin(nx)(e^{ny} - e^{-ny})$$

3.

$$\begin{aligned} y(x, t) &= f(x - ct) + g(x + ct) \implies \\ \frac{\partial^2 y}{\partial t^2} &= c^2 f^{(2)}(x - ct) + c^2 g^{(2)}(x + ct) \\ \frac{\partial^2 y}{\partial x^2} &= f^{(2)}(x - ct) + g^{(2)}(x + ct) \\ \frac{\partial^2 y}{\partial x^2} &= \frac{1}{c^2} (c^2 f^{(2)}(x - ct) + c^2 g^{(2)}(x + ct)) \\ \frac{\partial^2 y}{\partial x^2} &= \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \text{ as required} \end{aligned}$$

Since this expression holds for all  $f, g$ , consider the case  $f(z) = g(z) = \frac{1}{2(1+z^2)}$ . This gives the solution:

$$y(x, t) = \frac{1}{2(1 + (x + ct)^2)} + \frac{1}{2(1 + (x - ct)^2)}$$



4. Since  $y$  solves the wave equation, the solution is of the form  $y = f(x + ct) + g(x - ct)$ .

Let  $f_n(u) = g_n(u) = A \sin\left(\frac{2n\pi u}{L}\right) + B \cos\left(\frac{2n\pi u}{L}\right)$ . Since  $f(x, 0)$  is an even function, the sin terms must all

be zero. Using the principle of superposition (and knowing that the mean value of  $y(x, 0)$  is  $\frac{v}{2}$ )

$$\begin{aligned}
 y &= \frac{v}{2} + \sum_{n=1}^{\infty} A \cos\left(\frac{2n\pi}{L}(x+ct)\right) + A \cos\left(\frac{2n\pi}{L}(x-ct)\right) \\
 y &= \frac{v}{2} + \sum_{n=1}^{\infty} 2A \cos\left(\frac{2n\pi x}{L}\right) \cos\left(\frac{2n\pi ct}{L}\right) \\
 y(x, 0) &= \frac{v}{2} + \sum_{n=1}^{\infty} 2A \cos\left(\frac{2n\pi x}{L}\right) \\
 \int_0^L y(x, 0) \cos\left(\frac{2n\pi x}{L}\right) dx &= \int_0^L 2A \cos^2\left(\frac{2n\pi x}{L}\right) dx \\
 \int_{\frac{L}{4}}^{\frac{3L}{4}} v \cos\left(\frac{2n\pi x}{L}\right) dx &= A \int_0^L \cos\left(\frac{4n\pi x}{L}\right) + 1 dx \\
 \frac{Lv}{2n\pi} \left[ \sin\left(\frac{2n\pi x}{L}\right) \right]_{\frac{L}{4}}^{\frac{3L}{4}} &= \frac{AL}{4n\pi} \left[ \sin\left(\frac{4n\pi x}{L}\right) + x \right]_0^L \\
 \frac{Lv}{2n\pi} \left( \sin \frac{3n\pi}{2} - \sin \frac{n\pi}{2} \right) &= \frac{AL^2}{4n\pi} \\
 \frac{Lv(1 - (-1)^n)}{2n\pi} &= \frac{AL^2}{4n\pi} \\
 A &= \frac{2v(1 - (-1)^n)}{L}
 \end{aligned}$$

So the expression for  $y(x, t)$  is:

$$y(x, t) = \frac{v}{2} + \frac{4v}{L} \sum_{n=1}^{\infty} (1 - (-1)^n) \cos\left(\frac{2n\pi x}{L}\right) \cos\left(\frac{2n\pi ct}{L}\right)$$

5.

$$\frac{\partial^2 \Theta}{\partial x^2} = \frac{1}{\kappa} \frac{\partial \Theta}{\partial t}$$

Let  $\Theta = X(x)T(t)$ .

$$\frac{\partial^2 \Theta}{\partial x^2} = X^{(2)}(x)T(t) \qquad \frac{\partial \Theta}{\partial t} = X(x)T^{(1)}(t)$$

Substituting this into the diffusion equation gives:

$$\begin{aligned}
 X^{(2)}(x)T(t) &= \frac{1}{\kappa} X(x)T^{(1)}(t) \\
 \frac{X^{(2)}(x)}{X(x)} &= \frac{T^{(1)}(t)}{\kappa T(t)}
 \end{aligned}$$

Since these functions are equal to each other, they must both be independent of  $x$  and  $t$  and so must be equal to some constant. This leads to several solutions. Since we know that the solution for  $t = 0$  is given by a sum of sin in  $x$ , we know that the constant is negative. Let it be  $-\lambda^2$ .

$$\begin{aligned}
 \frac{X^{(2)}(x)}{X(x)} &= -\lambda^2 \\
 X^{(2)}(x) + \lambda^2 X(x) &= 0 \\
 X(x) &= A \cos(\lambda x) + B \sin(\lambda x)
 \end{aligned}$$

Since we have the boundary condition that for  $\Theta(x, 0)$  is a sum of sin, the  $A$  must be zero.

$$\begin{aligned}\frac{T^{(1)}(t)}{\kappa T(t)} &= -\lambda^2 \\ T^{(1)}(t) &= -\kappa\lambda^2 T(t) \\ T(t) &= Ce^{-\kappa\lambda^2 t}\end{aligned}$$

Letting  $AC = b_n$  and substituting  $\lambda = \frac{n\pi}{l}$  for arbitrary  $n$ ,  $\Theta(x, t) = a \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2\kappa x}{l^2}t}$  is a solution the differential equation

By the principle of superposition:

$$\Theta(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{n^2\pi^2\kappa t}{l^2}}$$

So the required expression is both a solution to the diffusion equation and satisfies the boundary conditions.

6.

$$\begin{aligned}u(r, t) &= \frac{A}{t} e^{-\frac{r^2}{4\kappa t}} \\ \frac{\partial u}{\partial r} &= -\frac{Ar}{2\kappa t^2} e^{-\frac{r^2}{4\kappa t}} \\ r \frac{\partial u}{\partial r} &= -\frac{Ar^2}{2\kappa t^2} e^{-\frac{r^2}{4\kappa t}} \\ \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) &= \left( \frac{Ar^3}{4\kappa^2 t^3} - \frac{Ar}{\kappa t^2} \right) e^{-\frac{r^2}{4\kappa t}} \\ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) &= \left( \frac{Ar^2}{4\kappa^2 t^3} - \frac{A}{\kappa t^2} \right) e^{-\frac{r^2}{4\kappa t}}\end{aligned}$$

$$\begin{aligned}u(r, t) &= \frac{A}{t} e^{-\frac{r^2}{4\kappa t}} \\ \frac{\partial u}{\partial t} &= \left( \frac{Ar^2}{4\kappa t^3} - \frac{A}{t^2} \right) e^{-\frac{r^2}{4\kappa t}} \\ \frac{1}{\kappa} \frac{\partial u}{\partial t} &= \left( \frac{Ar^2}{4\kappa t^3} - \frac{A}{t^2} \right) e^{-\frac{r^2}{4\kappa t}} \\ \frac{1}{\kappa} \frac{\partial u}{\partial t} &= \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right)\end{aligned}$$

So  $u(r, t)$  is a solution to the partial differential equation.

The total number of drunks  $N$  is given by the integral over  $r$  at a given time  $t$  (assuming the drunks don't sober up).

$$\begin{aligned}N &= \int_0^{2\pi} d\theta \int_0^{\infty} \frac{A}{t} e^{-\frac{r^2}{4\kappa t}} r dr \\ &= 2\pi \left[ -2A\kappa e^{-\frac{r^2}{4\kappa t}} \right]_0^{\infty} \\ &= 2\pi \times 2A\kappa \\ &= 4\pi A\kappa\end{aligned}$$

Consider the density of drunks at distance  $R$ . We can find the maximum of this by differentiating with respect to  $t$ .

$$\begin{aligned}u(R, t) &= \frac{A}{t} e^{-\frac{R^2}{4\kappa t}} \\ \frac{\partial u}{\partial t} &= \left( \frac{AR^2}{4\kappa t^3} - \frac{A}{t^2} \right) e^{-\frac{R^2}{4\kappa t}} \\ 0 &= \frac{AR^2}{4\kappa t^3} - \frac{A}{t^2} \\ 0 &= R^2 - 4\kappa t \\ t &= \frac{R^2}{4\kappa}\end{aligned}$$

So the maximum density of drunks at distance  $R$  from the pub happens at  $t = \frac{N}{AR^2\pi}$ . Substituting this into the original equation gives the maximum density of drunks at distance  $R$  is:

$$\begin{aligned}u(R, t)_{\max} &= \frac{4A\kappa}{R^2} e^{-\frac{4\kappa R^2}{4\kappa R^2}} \\ &= \frac{4\pi A\kappa}{R^2\pi} e^{-1} \\ &= \frac{N}{R^2\pi e}\end{aligned}$$

