1 2009 Paper 1 Question 6

https://www.cl.cam.ac.uk/teaching/exams/pastpapers/y2009p1q6.pdf

(a) State the defining properties of a min-heap. Show how to convert between the tree and the (zero-based) array representation of a min-heap.

Every element in a min heap is smaller than both its children.

There are two possibilities when converting between the tree-based and array-based representation of the heap:

- The tree is almost-full
- The tree is not almost-full

In the first case where the tree is almost-full, all we must do is a breadth-first traversal of the tree and store the output into an array.

However, if the tree is not almost-full then conversion to a heap-based representation is more complicated. All we can do is place the elements in the tree into an array and then heapify the array.

So we do traverse the tree and place every node we encounter into an array (either a pre-order dfs for minimum space complexity or a breadth-first search to preserve as much ordering as possible and minimise the number of swaps we have to do when calling heapify). Now we have an array containing the elements of the tree. This has taken $\Theta(n)$ time.

We now apply the algorithm to turn an unsorted array into a heap. This involves recursively "merging" heaps inside the array and adding another element at the same time. Initially these heaps are the leaf nodes and one parent. However as the parents become the roots of heaps, they are merged again etc. This leads to a $\Theta(n)$ algorithm to make a heap from an unsorted array.

We can merge two heaps in $\lg n$ time. However, when we construct the larger heap from the two smaller heaps, most of the heaps we merge are very small leading to an overall complexity of $\Theta(n)$.

We can merge two heaps with an unsorted element at the top by comparing the roots of the two heaps, and while the smaller is less than the root of the new heap, we make the smaller the roots the root of the new heap, placing the previous root into the heap that root came from and call the algorithm again.

In both cases: the time complexity for conversion is linear.

(b) "An array sorted in ascending order is always a min-heap." True or false? If false, offer a counter-example; otherwise, prove the correctness of this statement with respect to the defining properties of a min-heap you listed in response to part (a).

The statement is true.

For the array to be a heap: every element must be smaller than both its children.

Every element in an array sorted in ascending order is smaller than all the element after it. If we represent a min-heap as an array, then to satisfy the heap property; means that for all n, the element at index n must be smaller than the element at index $2n + 1$ and $2n + 2$. Since the array is sorted, we know that every element is smaller than every element after it – including that at index $2n + 1$ and $2n + 2$. So the heap property.

So any array sorted in ascending order is a min-heap.

(c) The array

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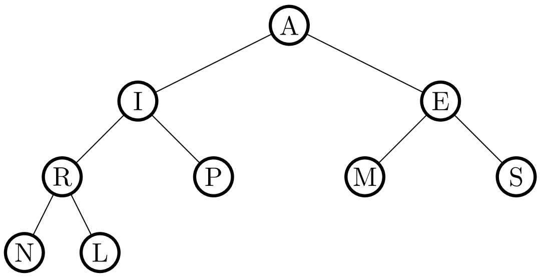
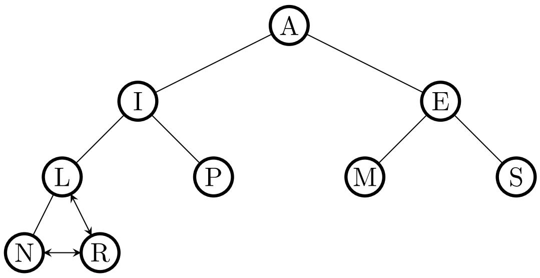
is not a min-heap. Why? Redraw it as a binary tree and turn it into a heap using the $O(n)$ `heapify()` procedure normally used as part of heapsort. Draw the intermediate stages as you go along and add any necessary explanations so that a reader can follow what you are doing and why.

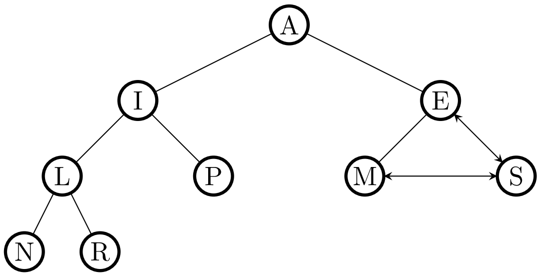
Note that the node R is greater than both of it's children. This violates the heap property and so the array is not a heap.



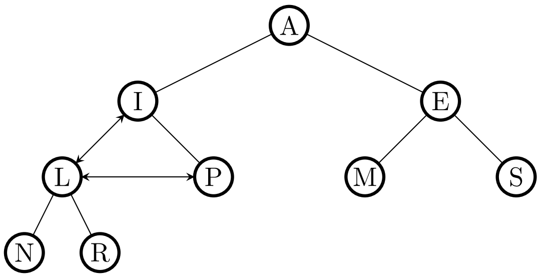
I will first represent the unedited array as a tree. In practice this heap can remain in the array representation when the heapification takes place – no conversion is actually done, the tree-based representation is just for visualisation.



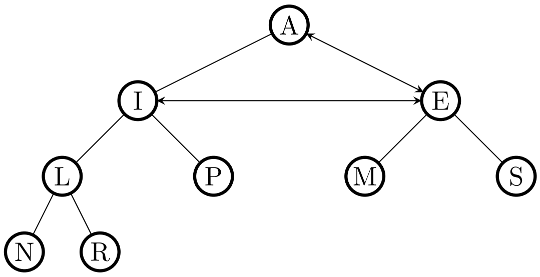
Firstly we must compare the nodes N and L to each other. Then compare the smaller of those to R – in this case L is the smaller and L is smaller than R so we must swap L with R.



Next we compare M to S and the smaller of those nodes to E. The smaller node is M but $E \leq M$ and so no swap is made.



Now we compare L to P and the smaller of those to I. In this case, L is the smaller node – but $I \leq L$ and so we do not make any swaps.

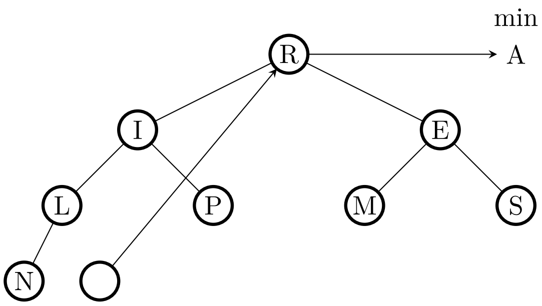


Next we must compare I to E and the smaller of those to A. $E \leq I$ but $A \leq E$ and so no swaps are made. We have now made the tree into a heap.

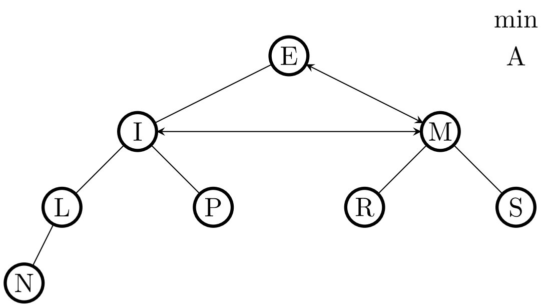
(d) Perform `extractMin()` on the min-heap you produced in part (c). As before, draw the intermediate stages and add explanations as necessary.

To extract the minimum element we must remove the root and put the bottom-most element in it's place; then recursively swap it with it's smallest child until the smallest child is greater than it or it is now a leaf node.

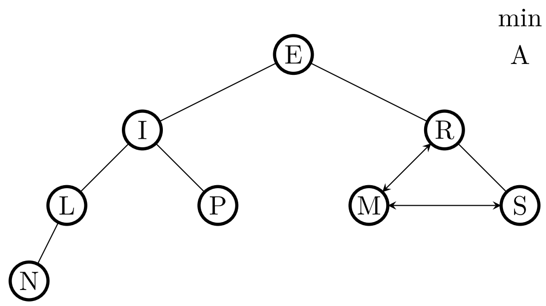
Start by removing A and placing R as the root.



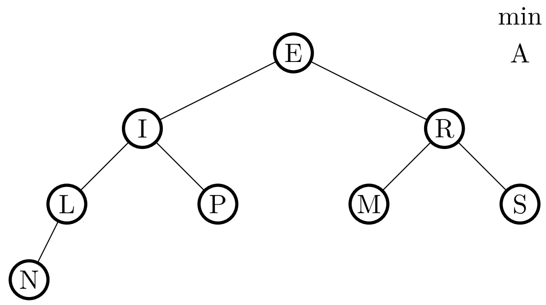
Next, we must recursively compare the two children of R; then compare R to the smallest of those. If the smaller child is smaller than R then we should swap R and the smaller child. We should stop, however when the smaller child is greater than R or R is a leaf node and so has no children.



We have compared E to I, seen that E is the smaller and so next compared E to R. This comparison showed that E was smaller than R and so we swapped E and R.



We compared M to S, noticed that $M \leq S$ and so compare M to R. This comparison showed that $M < R$ and so we swapped them.



Now we note that M is a leaf node – so the `popmin()` terminates; the minheap properties have been preserved and A has been removed from the heap.

(e) What is the asymptotic running time of the heapsort algorithm on an array of length n that is already sorted in ascending order? Justify your answer.

Heapsort forms a max-heap, then pops max and puts the largest element in the end, reducing the size of the array which retains the heap-property by 1 and increases the size of the sorted section of the array by 1. This is repeated until the heap is size 0 – meaning that the rest of the array is now sorted in ascending order.

However, heapsort has no checks or guarantees of a lower cost if the list is already sorted in ascending order. Building the heap takes $\Theta(n)$ time – it always does. Then removal of elements takes $\Theta(\lg n)$ time as usual. Since there are n elements which must be removed; the time complexity is $\Theta(n \lg n)$ – as is normal.

(f) What is the asymptotic running time of the heapsort algorithm on an array of length n that is already sorted in *descending* order? Justify your answer.

$\Theta(n \lg n)$. Creation of the heap takes the usual amount of time – $\Theta(n)$. Although the array which we start with is sorted in descending order and so fulfils the max-heap properties, heapsort does not check this and so we have no efficiency gain. The final heap is reverse sorted: all heapsort must now do is to repeatedly remove the largest element and place it at the end. However, heapsort does not check whether the array is ordered and in fact (the usual implementation) will be marginally worse because of it – when removing the largest element we swap it with the last element. If the heap we start with is sorted, then the element we swap it with is the smallest element. This means that each removal will by guarantee have to go to the bottom of the array – $\Theta(n \lg n)$. So the overall complexity is $\Theta(n \lg n)$.

2 2007 Paper 10 Question 10

https://www.cl.cam.ac.uk/teaching/exams/pastpapers/y2007p10q10.pdf

(a) Give a clear description of an efficient algorithm for finding the i^{th} smallest element of an n-element vector. Write some pseudocode for the algorithm and discuss its time complexity. Compare it with other plausible ways of achieving the same result. [Notes: Use zero-based indexing. You may assume for simplicity that all the elements of the vector are different.]

I will describe a standard Quickselect algorithm. This is expected $O(n)$ however has a worst-case of $\Theta(n^2)$ – this worst case can be eliminated by using a median-of-medians algorithm – the median-of-medians being selected by a Quickselect using a median-of-medians.

`Quickselect(arr, i)` should select a random element in the array. This is the pivot. Split the array into all elements which are smaller than the pivot and those which are larger than the pivot (since elements are distinct we know that the pivot will not be in either of these).

If the size of the list of elements smaller than the pivot is greater than i then we should return `Quickselect(left, i)`. If the size of the list of elements less than the pivot is equal to i then we should return the pivot. Else we should return `Quickselect(right, i - left.size - 1)`.

```
def quickselect(arr, i):
        pivot = median_of_medians(arr)
        left = []
        right = []
        for each in arr:
                if each < pivot:
                        left.append(each)
                elif each > pivot:
                        right.append(each)
        if len(left) > i:
                return quickselect(left, i)
        elif len(left) == i:
                return pivot
        else:
                return quickselect(right, i - len(left) - 1)
```

In the average case this forms the recurrence relation $T(n) = T\left(\frac{n}{2}\right) + \Theta(n)$ – which has the solution $T(n) \in \Theta(n)$. And so Quickselect has an average-case complexity of $\Theta(n)$. Quickselect can have a guaranteed $\Theta(n)$ complexity if using a median-of-medians method to select the pivot.

The most obvious other solution to select the i^{th} element in an array would be to fully sort the array and then select the i^{th} element. However, this requires $\Omega(n \lg n)$ time and has the additional effect of sorting the array – in some cases we may not want this. For example if the array holds two values – say x and y and is sorted by x but you wish to select the i^{th} element by y coordinate then sorting is unsuitable.

Conversely, if we intend to sort the array later or will select more than $\lg n$ of the elements in the array (say if we select $\sqrt{n}$ elements then $\sqrt{n}\Theta(n)$ selections has a complexity of $\Theta(n^{\frac{3}{2}}$ but sorting still has the complexity $\Theta(n \lg n)$) then sorting is more efficient.

(b) Give a clear description of an efficient algorithm for finding the k smallest elements of a very large n-element vector. Compare its running time with that of other plausible ways of achieving the same result, including that of applying k times your solution for part (a). [Note that in part (a) the result of the function consists of one element, whereas here it consists of k elements. As above, you may assume for simplicity that all the elements of the vector are different.

We can select the k smallest elements in a vector in $O(n)$ average time.

Case 1: k is the size of the vector. In this case we return the whole vector.

Case 2: k is not the size of the vector.

In this case we apply an adapted version of quicksort searching for k which appends all the elements which are guaranteed to be less than the pivot to a secondary list – when calling quickselect and realising that the current pivot is too small (or equal to k), rather than discarding the resulting left list, append it to an output list.

Applying `quickselect(i)` for all $i < k$ will be $\Theta(nk)$ time on average since we do a $\Theta(n)$ time operation k times.

Another naïve algorithm would be sorting the list fully and then selecting the first k elements. However whatever algorithm is used: (comparison) sorting is $\Omega(n \lg n)$ and so has a worse complexity than the algorithm I described (and non-comparison sorts [radixsort or bucketsort] are not suitable since the elements are distinct and we know nothing about their distribution).

Here is a python implementation for the modified-quickselect algorithm.

```
def selectk(lst: list, k: int):
    pivot = median_of_medians(lst)
    left = [element for element in lst if element <= pivot]
    right = [element for element in lst if element > pivot]
    if len(left) > k:
        return selectk(left, k)
    elif len(left) == k:
        return left
    else:
        return left + selectk(right, k - len(left))
```

(c) Give an optimal algorithm for solving part (b) for k = 1. Give the worst-case number of comparisons performed by your algorithm as a function of n. [Note: exact number of comparisons, not just asymptotic complexity.]

Set the maximum element to the first element in the vector. For each element in the list, check whether it is smaller than the current smallest element. If it is then set the current smallest element equal to it.

This algorithm has a worst, best and average case number of comparisons of $n - 1$.

```
def minimum(searchlist):
        assert len(searchlist) >= 1
        minimumfound = searchlist[0]
        for element in searchlist:
                if element < minimumfound:
                        minimumfound = element
        return minimumfound
```

(d) Same as part (c), but for k = 2.

Compare every even-numbered element in the array to its neighbour. Keep track of which largest element lost to which smallest element. If any of the larger elements have elements which they are larger than, disregard them. Repeat this recursively. It will take $n - 1$ comparisons to find the smallest element and the list of elements which directly lost to it.

At the end you will have the smallest element and a list of the elements directly which lost to it. The list of elements which lost to it will have size $\lceil \lg n \rceil$. The only element which was smaller than the second smallest element was the smallest element, so we are guaranteed that the second smallest element is in the list behind the smallest element. If the original list is not a perfect power of 2 then it is possible for the "list of potential second largest elements" to be smaller than $\lceil \lg n \rceil$ – however it will be no larger than $\lceil \lg n \rceil$.

Then you can reapply the algorithm from (c) to it and pass through the list of length $\leq \lceil \lg n \rceil$ once (making $\leq \lceil \lg n \rceil - 1$ comparisons to find the second smallest element.

So in total we make a guaranteed $n + \lceil \lg n \rceil - 2$ comparisons in the worst case. This is optimal.

Here is a python implementation of this algorithm:

```python
def select_two(lst: list):
    winners = lst.copy()
    # winners contains the elements which are undefeated
    losers = [[] for _ in range(len(winners))]
    # at position i losers contains the elements which
    # lost directly to the element at position i in winners
    while len(winners) != 1:
        for i in range(len(winners) // 2):
            loser = i + 1 if winners[i] > winners[i + 1] else i
            losers.pop(loser)
            # remove the list of elements which lost to the loser
            # in this comparison
            losers[i].append(winners.pop(loser))
            # add the loser to the winners list
    return winners[0], max(losers[0])
```