1. Let $\psi = X(x)Y(y)$ for some functions X, Y.

$$\begin{aligned}\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} &= 0 \implies \\ X^{(2)}(x)Y(y) + X(x)Y^{(2)}(y) &= 0 \\ \frac{X^{(2)}(x)}{X(x)} &= -\frac{Y^{(2)}(y)}{Y(y)}\end{aligned}$$

Since they are equal, the result of both of these must be independent of x and y.

This forms two ordinary differential equations:

$$X^{(2)}(x) - kX(x) = 0 \qquad\qquad Y^{(2)}(y) + kY(x) = 0$$

Which leads us to three possible equations dependent on the values of k:

$$\psi = \begin{cases}(Ax+B)(Cy+D) & \text{if } k = 0 \\ (Ae^{\lambda x} + Be^{-\lambda x})C\sin(\lambda y + \phi) & \text{if } k > 0 \text{ where } k = \lambda^2 \text{ with } 0 \le \psi < \pi \\ A\sin(\lambda x + \phi)(Ce^{\lambda y} + De^{-\lambda y}) & \text{if } k < 0 \text{ where } k = -\lambda^2 \text{ with } 0 \le \psi < \pi\end{cases}$$

Note that in cases $k = 0$ and $k < 0$, the function Y is not cyclic – meaning that the criteria $\psi(x, 0) = \psi(x, a)$. So $k > 0$ must be true.

Using the boundary condition $\psi(x, 0) = 0$:

$$\begin{aligned}(Ae^{\lambda x} + Be^{-\lambda x})C\sin(0 + \phi) &= 0 \implies \\ \sin(\phi) &= 0 \implies \\ \psi &= 0\end{aligned}$$

Using the boundary condition $\psi(x, a) = 0$:

$$\begin{aligned}(Ae^{\lambda x} + Be^{-\lambda x})C\sin(\lambda a) &= 0 \implies \\ \sin(\lambda a) &= 0 \implies \\ \lambda a &= n\pi \implies \\ \lambda &= \frac{n\pi}{a}\end{aligned}$$

Notice that $\lim_{x\to\infty} \psi(x, y) = 0$. So $A = 0$.

Using the principle of superposition and the boundary condition $\psi(0, y) = \sin\left(\frac{\pi y}{a}\right) + 2\sin\left(\frac{\pi y}{a}\right)$:

$$\begin{aligned}\psi &= \sum_{n=0}^{\infty} B_n C_n e^{-\frac{\pi n}{a}x} \sin\left(\frac{\pi n}{a} y\right) \\ \psi(0, y) &= \sum_{n=0}^{\infty} K_n \sin\left(\frac{\pi n}{a} y\right) \\ \sin\left(\frac{\pi y}{a}\right) + 2\sin\left(\frac{\pi y}{a}\right) &= \sum_{n=0}^{\infty} K_n \sin\left(\frac{\pi n}{a} y\right) \implies \\ K_n &= \begin{cases}1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 0 & \text{otherwise}\end{cases} \\ \psi(x, y) &= e^{-\frac{\pi x}{a}} \sin\left(\frac{\pi y}{a}\right) + 2e^{-\frac{2\pi x}{a}} \sin\left(\frac{2\pi y}{a}\right)\end{aligned}$$

2.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Using the method of separation of variables:

$$\phi = \begin{cases} (ax+b)(cy+d) \\ (Ae^{\lambda x} + Be^{-\lambda x})C\sin(\lambda y + \theta) \\ A\sin(\lambda x + \theta)(Ce^{\lambda y} + De^{-\lambda y}) \end{cases}$$

Since ϕ is periodic with x, we can conclude that case 3 is the correct case. Note that since x is periodic with period π and $\phi(0, y) = 0$, we can derive that $\theta = 0$ and $\exists n \in \mathbb{N}.\lambda = n$. By the principle of superposition, we can therefore form:

$$\phi = \sum_{n=1}^{\infty} A_n \sin(nx)(C_n e^{ny} + D_n e^{-ny})$$

Noting the constraint $\phi(x, 0) = 0$:

$$C_n + D_n = 0 \implies D_n = -C_n$$

We now have the expression:

$$\phi = \sum_{n=1}^{\infty} A_n C_n \sin(nx)(e^{ny} - e^{-ny})$$

Replacing $A_n C_n$ with K_n, equating with $\phi(x, b)$ and then using orthogonality gives:

$$\begin{aligned}
\phi(x,y) &= \sum_{n=1}^{\infty} K_n \sin(nx)(e^{ny} - e^{-ny}) \\
\int_0^{\pi} x(\pi - x)\sin(nx)dx &= K_n \int_0^{\pi} \sin^2(nx)(e^{nb} - e^{-nb})dx \\
K_n &= \frac{\left[\frac{x^2}{n}\cos(nx) - \frac{2x}{n^2}\sin(nx) - \frac{2}{n^2}\cos(nx) - \frac{\pi x}{n}\cos(nx) + \frac{\pi}{n^2}\sin(nx)\right]_0^{\pi}}{(e^{nb} - e^{-nb})\left[\frac{1}{2}x - \frac{1}{4n}\sin(2nx)\right]_0^{\pi}} \\
K_n &= \frac{\frac{\pi^2}{n}\cos(n\pi) - \frac{2}{n^2}\cos(n\pi) - \frac{\pi^2}{n}\cos(n\pi) + \frac{2}{n^2}}{(e^{nb} - e^{-nb})\frac{1}{2}\pi} \\
K_n &= \frac{\frac{2}{n^2}(1 - (-1)^n)}{(e^{nb} - e^{-nb})\frac{1}{2}\pi} \\
K_n &= \frac{4(1 - (-1)^n)}{n^2\pi(e^{nb} - e^{-nb})}
\end{aligned}$$

So:

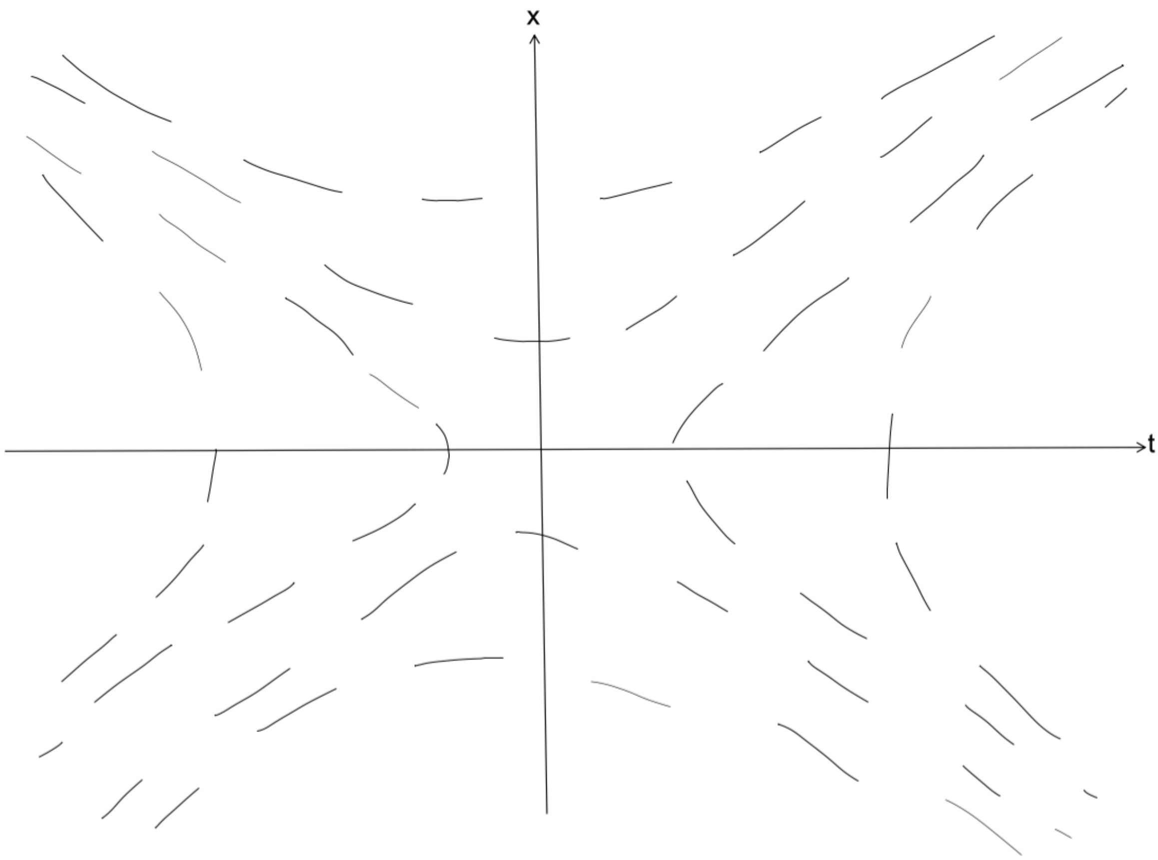
$$\phi(x,y) = \sum_{n=1}^{\infty} \frac{4(1 - (-1)^n)}{n^2\pi(e^{nb} - e^{-nb})} \sin(nx)(e^{ny} - e^{-ny})$$

3.

$$\begin{aligned}
y(x,t) &= f(x - ct) + g(x + ct) \implies \\
\frac{\partial^2 y}{\partial t^2} &= c^2 f^{(2)}(x - ct) + c^2 g^{(2)}(x + ct) \\
\frac{\partial^2 y}{\partial x^2} &= f^{(2)}(x - ct) + g^{(2)}(x + ct) \\
\frac{\partial^2 y}{\partial x^2} &= \frac{1}{c^2}\left(c^2 f^{(2)}(x - ct) + c^2 g^{(2)}(x + ct)\right) \\
\frac{\partial^2 y}{\partial x^2} &= \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \text{ as required}
\end{aligned}$$

Since this expression holds for all f, g, consider the case $f(z) = g(z) = \frac{1}{2(1+z^2)}$. This gives the solution:

$$y(x,t) = \frac{1}{2(1+(x+ct)^2)} + \frac{1}{2(1+(x-ct)^2)}$$



4. Since y solves the wave equation, the solution is of the form $y = f(x+ct) + g(x-ct)$.
Let $f_n(u) = g_n(u) = A\sin\left(\frac{2n\pi u}{L}\right) + B\cos\left(\frac{2n\pi u}{L}\right)$. Since $f(x,0)$ is an even function, the sin terms must all

be zero. Using the principle of superposition (and knowing that the mean value of $y(x,0)$ is $\frac{v}{2}$)

$$\begin{aligned}
y &= \frac{v}{2} + \sum_{n=1}^{\infty} A\cos\left(\frac{2n\pi}{L}(x+ct)\right) + A\cos\left(\frac{2n\pi}{L}(x-ct)\right) \\
y &= \frac{v}{2} + \sum_{n=1}^{\infty} 2A\cos\left(\frac{2n\pi x}{L}\right)\cos\left(\frac{2n\pi ct}{L}\right) \\
y(x,0) &= \frac{v}{2} + \sum_{n=1}^{\infty} 2A\cos\left(\frac{2n\pi x}{L}\right) \\
\int_0^L y(x,0)\cos\left(\frac{2n\pi x}{L}\right)\mathrm{d}x &= \int_0^L 2A\cos^2\left(\frac{2n\pi x}{L}\right)\mathrm{d}x \\
\int_{\frac{L}{4}}^{\frac{3L}{4}} v\cos\left(\frac{2n\pi x}{L}\right)\mathrm{d}x &= A\int_0^L \cos\left(\frac{4n\pi x}{L}\right) + 1\,\mathrm{d}x \\
\frac{Lv}{2n\pi}\left[\sin\left(\frac{2n\pi x}{L}\right)\right]_{\frac{L}{4}}^{\frac{3L}{4}} &= \frac{AL}{4n\pi}\left[\sin\left(\frac{4n\pi x}{L}\right) + x\right]_0^L \\
\frac{Lv}{2n\pi}\left(\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\right) &= \frac{AL^2}{4n\pi} \\
\frac{Lv(1-(-1)^n)}{2n\pi} &= \frac{AL^2}{4n\pi} \\
A &= \frac{2v(1-(-1)^n)}{L}
\end{aligned}$$

So the expression for $y(x,t)$ is:

$$y(x,t) = \frac{v}{2} + \frac{4v}{L}\sum_{n=1}^{\infty}(1-(-1)^n)\cos\left(\frac{2n\pi x}{L}\right)\cos\left(\frac{2n\pi ct}{L}\right)$$

5.

$$\frac{\partial^2\Theta}{\partial x^2} = \frac{1}{\kappa}\frac{\partial\Theta}{\partial t}$$

Let $\Theta = X(x)T(t)$.

$$\frac{\partial^2\Theta}{\partial x^2} = X^{(2)}(x)T(t) \qquad\qquad \frac{\partial\Theta}{\partial t} = X(x)T^{(1)}(t)$$

Substituting this into the diffusion equation gives:

$$\begin{aligned}
X^{(2)}(x)T(t) &= \frac{1}{\kappa}X(x)T^{(1)}(t) \\
\frac{X^{(2)}(x)}{X(x)} &= \frac{T^{(1)}(t)}{\kappa T(t)}
\end{aligned}$$

Since these functions are equal to each other, they must both be independent of x and t and so must be equal to some constant. This leads to several solutions. Since we know that the solution for $t=0$ is given by a sum of sin in x, we know that the constant is negative. Let it be $-\lambda^2$.

$$\begin{aligned}
\frac{X^{(2)}(x)}{X(x)} &= -\lambda^2 \\
X^{(2)}(x) + \lambda^2 X(x) &= 0 \\
X(x) &= A\cos(\lambda x) + B\sin(\lambda x)
\end{aligned}$$

Since we have the boundary condition that for $\Theta(x,0)$ is a sum of sin, the A must be zero.

$$\begin{aligned}\frac{T^{(1)}(t)}{\kappa T(t)} &= -\lambda^2\\ T^{(1)}(t) &= -\kappa\lambda^2 T(t)\\ T(t) &= Ce^{-\kappa\lambda^2 t}\end{aligned}$$

Letting $AC = b_n$ and substituting $\lambda = \frac{n\pi}{l}$ for arbitrary n, $\Theta(x,t) = a\sin\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2\kappa x}{l^2}t}$ is a solution the differential equation
By the principle of superposition:

$$\Theta(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)e^{-\frac{n^2\pi^2\kappa t}{l^2}}$$

So the required expression is both a solution to the diffusion equation and satisfies the boundary conditions.

6.

$$\begin{aligned}u(r,t) &= \frac{A}{t}e^{-\frac{r^2}{4\kappa t}}\\ \frac{\partial u}{\partial r} &= -\frac{Ar}{2\kappa t^2}e^{-\frac{r^2}{4\kappa t}}\\ r\frac{\partial u}{\partial r} &= -\frac{Ar^2}{2\kappa t^2}e^{-\frac{r^2}{4\kappa t}}\\ \frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= \left(\frac{Ar^3}{4\kappa^2 t^3} - \frac{Ar}{\kappa t^2}\right)e^{-\frac{r^2}{4\kappa t}}\\ \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= \left(\frac{Ar^2}{4\kappa^2 t^3} - \frac{A}{\kappa t^2}\right)e^{-\frac{r^2}{4\kappa t}}\end{aligned}$$

$$\begin{aligned}u(r,t) &= \frac{A}{t}e^{-\frac{r^2}{4\kappa t}}\\ \frac{\partial u}{\partial t} &= \left(\frac{Ar^2}{4\kappa t3} - \frac{A}{t^2}\right)e^{-\frac{r^2}{4\kappa t}}\\ \frac{1}{\kappa}\frac{\partial u}{\partial t} &= \left(\frac{Ar^2}{4\kappa t3} - \frac{A}{t^2}\right)e^{-\frac{r^2}{4\kappa t}}\\ \frac{1}{\kappa}\frac{\partial u}{\partial t} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)\end{aligned}$$

So $u(r,t)$ is a solution to the partial differential equation.

The total number of drunks N is given by the integral over r at a given time t (assuming the drunks don't sober up).

$$\begin{aligned}N &= \int_0^{2\pi} \mathrm{d}\theta \int_0^{\infty} \frac{A}{t}e^{-\frac{r^2}{4\kappa t}}r\,\mathrm{d}r\\ &= 2\pi\left[-2A\kappa e^{-\frac{r^2}{4\kappa t}}\right]_0^{\infty}\\ &= 2\pi \times 2A\kappa\\ &= 4\pi A\kappa\end{aligned}$$

Consider the density of drunks at distance R. We can find the maximum of this by differentiating with respect to t.

$$
\begin{aligned}
u(R,t) &= \frac{A}{t}e^{-\frac{R^2}{4\kappa t}} \\
\frac{\partial u}{\partial t} &= \left(\frac{AR^2}{4\kappa t^3} - \frac{A}{t^2}\right)e^{-\frac{R^2}{4\kappa t}} \\
0 &= \frac{AR^2}{4\kappa t^3} - \frac{A}{t^2} \\
0 &= R^2 - 4\kappa t \\
t &= \frac{R^2}{4\kappa}
\end{aligned}
$$

So the maximum density of drunks at distance R from the pub happens at $t = \frac{N}{AR^2\pi}$. Substituting this into the original equation gives the maximum density of drunks at distance R is:

$$
\begin{aligned}
u(R,t)_{\max} &= \frac{4A\kappa}{R^2}e^{-\frac{4\kappa R^2}{4\kappa R^2}} \\
&= \frac{4\pi A\kappa}{R^2\pi}e^{-1} \\
&= \frac{N}{R^2\pi e}
\end{aligned}
$$