5. On sets

5.1 Basic exercises

1. Prove that $\subseteq$ is a partial order, that is, it is:

   (a) reflexive: $\forall$ sets A, $A \subseteq A$

   I shall prove that every element in A is also in A.

$$\begin{gathered} \forall a \in A : a \in A \iff \\ A \subseteq A \end{gathered} \tag{1}$$

   (b) transistive: $\forall$ sets A, B, C. $(A \subseteq B \wedge B \subseteq C) \implies A \subseteq C$

   Assume $A \subseteq B \wedge B \subseteq C$

$$\begin{gathered} \text{Take an arbitrary } a \in A \\ \text{By assumption } a \in A \implies a \in B \\ \text{By assumption } a \in B \implies a \in C \\ \text{So: } \forall a \in A : a \in C \iff \\ A \subseteq C \end{gathered} \tag{2}$$

   (c) antisymmetric: $\forall$ sets A, B. $(A \subseteq B \wedge B \subseteq A) \iff A = B$

   $\subseteq$ is antisymmetric.

   So $A \subseteq B \wedge B \subseteq A \iff A = B$ as required.

2. Prove the following statements:

   (a) $\forall$ sets A. $\emptyset \subseteq A$

   By definition if S is a set:

$$S \subseteq A \iff \forall s \in S : s \in A \tag{3}$$

   For $S = \emptyset$ this is vacuously true.

$$\begin{gathered} (\emptyset \subseteq A \iff \forall s \in \emptyset : s \in A) \iff \\ (\emptyset \subseteq A \iff \text{true}) \iff \\ \emptyset \subseteq A \text{ as required} \end{gathered} \tag{4}$$

   (b) $\forall$ sets A. $(\forall x : x \notin A) \iff A = \emptyset$

   By definition $\forall x : x \notin \emptyset$.

$$\begin{aligned} A = \emptyset \wedge \forall x : x \notin \emptyset \text{ by definition} \iff \\ \forall x : x \notin A \text{ as required} \end{aligned} \tag{5}$$

3. Find the union, and intersection of:

   (a) $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$

$$\begin{aligned} &\{1, 2, 3, 4, 5\} \cup \{-1, 1, 3, 5, 7\} \\ =&\{-1, 1, 2, 3, 4, 5, 7\} \end{aligned} \tag{6}$$

$$\begin{aligned} &\{1, 2, 3, 4, 5\} \cap \{-1, 1, 3, 5, 7\} \\ =&\{1, 3, 5\} \end{aligned} \tag{7}$$

(b) $\{x \in \mathbb{R} | x > 7\}$ and $\{x \in \mathbb{N} : x > 5\}$

$$\begin{aligned} &\{x \in \mathbb{R} : x > 7\} \cup \{x \in \mathbb{N} : x > 5\} \\ =&\{x \in \mathbb{R} : x > 7 \vee x \in \{6, 7\}\} \end{aligned} \tag{8}$$

$$\begin{aligned} &\{x \in \mathbb{R} : x > 7\} \cap \{x \in \mathbb{N} : x > 5\} \\ =&\{x \in \mathbb{N} : x > 7\} \end{aligned} \tag{9}$$

4. Find the Cartesian product and disjoint union of $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$.
   The Cartesian product of two sets S and T is $\{x : \forall s \in S, \forall t \in T : x = (s, t)\}$
   For the sets $\{1, 2, 3, 4, 5\}$ and $\{-1, 1, 3, 5, 7\}$ this is equal to:
   $$\begin{aligned} &\{(1, -1), (1, 1), (1, 3), (1, 5), (1, 7), (2, -1), (2, 1), (2, 3), (2, 5), (2, 7), (3, -1), (3, 1), (3, 3), \\ &(3, 5), (3, 7), (4, -1), (4, 1), (4, 3), (4, 5), (4, 7), (5, -1), (5, 1), (5, 3), (5, 5), (5, 7)\} \end{aligned} \tag{10}$$

5. Let $I = \{2, 3, 4, 5\}$ and for each $i \in I$, let $A_i = \{i, i + 1, i - 1, 2 \cdot i\}$.

   (a) List the elements of all sets A_i for $i \in I$

   $$\begin{aligned} A_2 &= \{1, 2, 3, 4\} \\ A_3 &= \{2, 3, 4, 6\} \\ A_4 &= \{3, 4, 5, 8\} \\ A_5 &= \{4, 5, 6, 10\} \end{aligned} \tag{11}$$

   (b) Let $\{A_i | i \in I\}$ stand for $\{A_2, A_3, A_4, A_5\}$. Find $\bigcup\{A_i | i \in I\}$ and $\bigcap\{A_i | i \in I\}$.

   $$\bigcup\{A_i : i \in I\} = \{1, 2, 3, 4, 5, 6, 8, 10\} \tag{12}$$

   $$\bigcap\{A_i : i \in I\} = \{4\} \tag{13}$$

6. Let U be a set. For all $A, B \in \mathcal{P}(U)$, prove that:

   (a) $A^c = B \iff (A \cup B = U \wedge A \cap B = \emptyset)$

   $$\begin{aligned} &A^c = B \iff \\ (\forall b \in U : b \notin A \iff b \in B) \iff& \\ &A \cup B = \forall u \in U : u \in A \vee u \notin A \iff \\ &A \cup B = U \end{aligned} \tag{14}$$

   (b) Double complement elimination: $(A^c)^c = A$

   $$\begin{aligned} A^c &\triangleq \{u | u \in U \wedge u \notin A\} \\ (A^c)^c &= \{u' | u' \in U \wedge u' \notin \{u | u \in U \wedge u \notin A\}\} \\ &= \{u' | u' \in U \wedge \overline{(u' \in U \wedge u' \notin A)}\} \\ &= \{u' | u' \in U \wedge (u' \notin U \vee u' \in A)\} \\ &= \{u' | (u' \in U \wedge u' \notin U) \vee (u' \in U \wedge u' \in A)\} \\ &= \{u' | u' \in U \wedge u' \in A\} \\ &= \{u' | u' \in A\} \text{ (Since } A \subseteq U : u' \in A \implies u' \in U) \\ &= A \end{aligned} \tag{15}$$

(c) The De-Morgan laws: $(A \cup B)^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}}$ and $(A \cap B)^{\mathsf{c}} = A^{\mathsf{c}} \cup B^{\mathsf{c}}$

$$\begin{aligned}(A \cup B)^{\mathsf{c}} &= \{x | x \notin A \cup B\} \\ &= \{x | x \notin A \wedge x \notin B\} \\ &= \{x | x \notin A\} \cap \{x | x \notin B\} \\ &= A^{\mathsf{c}} \cap B^{\mathsf{c}}\end{aligned} \tag{16}$$

$$\begin{aligned}(A \cap B)^{\mathsf{c}} &= \{x | x \notin A \cap B\} \\ &= \{x | x \notin A \vee x \notin B\} \\ &= \{x | x \notin A\} \cup \{x | x \notin B\} \\ &= A^{\mathsf{c}} \cap B^{\mathsf{c}}\end{aligned} \tag{17}$$

5.2 Core exercises

1. Prove that for all sets U and subsets $A, B \subseteq U$:

(a) $\forall X : A \subseteq X \wedge B \subseteq X \iff (A \cup B) \subseteq X$

$$\begin{gathered}\forall X : A \subseteq X \wedge B \subseteq X \iff \\ \forall a \in A : a \in X \wedge \forall b \in B : b \in X \iff \\ \forall x \in A \cup B : x \in X \iff \\ A \cup B \subseteq X\end{gathered} \tag{18}$$

(b) $\forall Y : Y \subseteq A \wedge Y \subseteq B \iff Y \subseteq (A \cap B)$

$$\begin{gathered}\forall Y : Y \subseteq A \wedge Y \subseteq B \iff \\ \forall y \in Y : y \in A \wedge y \in B \iff \\ \forall y \in Y : y \in A \cap B \iff \\ Y \subseteq A \cap B\end{gathered} \tag{19}$$

2. Either prove or disprove that, for all sets A and B,

(a) $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$

$$\begin{gathered}\mathcal{P}(A) \triangleq \{a | a \subseteq A\} \\ A \subseteq B \iff \\ (\forall a : (a \subseteq A) \implies (a \subseteq B)) \iff \\ \{a | a \subseteq A\} \subseteq \{b | b \subseteq B\} \iff \\ \mathcal{P}(A) \subseteq \mathcal{P}(B)\end{gathered} \tag{20}$$

(b) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$

Disproof by counter-example:

$$\begin{gathered}\text{Let } A = \{0\} \wedge B = \{1\} \iff \\ \mathcal{P}(A \cup B) = \mathcal{P}(\{0, 1\}) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} \wedge \\ \mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{0\}\} \cup \{\emptyset, \{1\}\} = \{\emptyset, \{0\}, \{1\}\} \\ \text{In this case: } \mathcal{P}(A \cup B) \nsubseteq \mathcal{P}(A) \cup \mathcal{P}(B)\end{gathered} \tag{21}$$

(c) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

$$\begin{aligned}\mathcal{P}(S) &\triangleq \{s|s \subseteq S\} \implies \\ \mathcal{P}(A) \cup \mathcal{P}(B) &= \{x|x \subseteq A \vee x \subseteq B\} \\ \mathcal{P}(A \cup B) &= \{x|x \subseteq A \cup B\} \\ \forall x : x \subseteq A \vee x \subseteq B &\implies x \subseteq A \cup B \implies \\ \{x|x \subseteq A \vee x \subseteq B\} &\subseteq \{x|x \subseteq A \cup B\} \iff \\ \mathcal{P}(A) \cup \mathcal{P}(B) &\subseteq \mathcal{P}(A \cup B)\end{aligned} \tag{22}$$

(d) $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$

$$\begin{aligned}\mathcal{P}(A) \cup \mathcal{P}(B) &= \{a|a \subseteq A\} \cup \{b|b \subseteq B\} \\ &= \{a|a \subseteq A \wedge a \subseteq B\} \\ &= \{a|a \subseteq A \cap B\} \\ &= \mathcal{P}(A \cap B)\end{aligned} \tag{23}$$

$$\begin{aligned}\text{From (23): } \mathcal{P}(A) \cup \mathcal{P}(B) &= \mathcal{P}(A \cap B) \iff \\ \mathcal{P}(A \cap B) &\subseteq \mathcal{P}(A) \cap \mathcal{P}(B)\end{aligned} \tag{24}$$

(e) $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$

$$\begin{aligned}\text{From (23): } \mathcal{P}(A) \cup \mathcal{P}(B) &= \mathcal{P}(A \cap B) \iff \\ \mathcal{P}(A) \cap \mathcal{P}(B) &\subseteq \mathcal{P}(A \cap B)\end{aligned} \tag{25}$$

3. Let U be a set. For all $A, B \in \mathcal{P}(U)$ prove that the following statements are equivalent.

(a) $A \cup B = B$ (b) $A \subseteq B$ (c) $A \cap B = A$ (d) $B^{\mathsf{c}} \subseteq A^{\mathsf{c}}$

We will show that all the statements are equivalent to $A \subseteq B$.

$$\begin{aligned}A \cup B = B &\iff \\ (\forall x \in A \cup B \iff x \in B) &\iff \\ (\forall x : (x \in A \vee x \in B) \iff x \in B) &\iff \\ \forall x : x \in A \implies x \in B &\iff \\ \forall x \in A : x \in B &\iff \\ A \subseteq B\end{aligned} \tag{26}$$

$$\text{Trivially: } A \subseteq B \iff A \subseteq B \tag{27}$$

$$\begin{aligned}A \cap B = A &\iff \\ \forall x(x \in A \cap B \implies x \in A) &\iff \\ \forall x : ((x \in A \wedge x \in B) \iff x \in A) &\iff \\ \forall x : (x \in A \implies x \in B) &\iff \\ A \subseteq B\end{aligned} \tag{28}$$

$$\begin{aligned}
B^{\mathsf{c}} &\subseteq A^{\mathsf{c}} \iff \\
\forall x \notin B &\implies x \notin A \iff \\
\forall x \in A &\implies x \in B \iff \\
A &\subseteq B
\end{aligned} \tag{29}$$

4. For sets A, B, C, D, prove or disprove at least three of the following statements:

(a) $(A \subseteq C \land B \subseteq D) \implies (A \times B \subseteq C \times D)$

$$\begin{aligned}
\text{Assume: } & A \subseteq C \land B \subseteq D \iff \\
& (a \in A \implies a \in C) \land (b \in B \implies b \in D) \\
& A \times B = \{s | \exists a \in A \land \exists b \in B . s = (a, b)\} \implies \\
& A \times B \subseteq \{s | \exists a \in B \land \exists b \in D . s = (a, b)\} \iff \\
& A \times B \subseteq C \times D
\end{aligned} \tag{30}$$

(b) $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$

Proof by counterexample:

$$\begin{aligned}
\text{Let: } & A = \emptyset, B = \{1\}, C = \{2\}, D = \{3\} \\
\text{So: } (A \cup C) \times (B \cup D) &= \{2\} \times \{1, 3\} \\
&= \{(2, 1), (2, 3)\} \\
\text{And: } (A \times B) \cup (C \times D) &= (\emptyset \times \{1\}) \cup (\{2\} \times \{3\}) \\
&= \emptyset \cup \{(2, 3)\} \\
&= \{(2, 3)\} \\
\text{So in this case: } & (A \cup C) \times (B \cup D) \nsubseteq (A \times B) \cup (C \times D)
\end{aligned} \tag{31}$$

(c) $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$

I will prove distributivity of $\times$ and $\cup$ to use in this and subsequent proofs.

$$\begin{aligned}
A \times (B \cup C) =& \{s | \exists a \in A, \exists x \in B \cup C . s = (a, x)\} \\
=& \{s | \exists a \in A, \exists x . (x \in B \cup x \in C) . s = (a, x)\} \\
=& \{(\exists a \in A \land \exists x \in B) \cup (\exists a \in A \land \exists x \in C) . s = (a, x)\} \\
=& \{(\exists a \in A \land \exists x \in B) . s = (a, x)\} \cup \{(\exists a \in A \land \exists x \in C) . s = (a, x)\} \\
=& (A \times B) \cup (A \times C)
\end{aligned} \tag{32}$$

$$\begin{aligned}
(A \times C) \cup (B \times D) &\subseteq (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D) \\
&\subseteq (A \times (C \cup D)) \cup (B \times (C \cup D)) \\
&\subseteq (A \cup B) \times (C \cup D) \text{ as required}
\end{aligned} \tag{33}$$

(d) $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

$$\begin{aligned}
& A \times (B \cup C) = (A \times B) \cup (A \times C) \implies \text{ using (32)} \\
& A \times (B \cup C) \subseteq (A \times B) \cup (A \times C) \text{ using the antisymmetry of } \subseteq
\end{aligned} \tag{34}$$

(e) $(A \times B) \cup (A \times D) \subseteq A \times (B \cup D)$

$$\begin{aligned}&(A \times B) \cup (A \times D) = A \times (B \cup D) \implies \text{ using (32)}\\&(A \times B) \cup (A \times D) \subseteq A \times (B \cup D) \text{ using the antisymmetry of } \subseteq\end{aligned} \tag{35}$$

5. For sets A, B, C, D, prove or disprove at least three of the following statements:

(a) $(A \subseteq C \land B \subseteq D) \implies A \uplus B \subseteq C \uplus D$

$$\begin{aligned}&\text{Assume: } A \subseteq C \land B \subseteq D \implies\\&a \in A \implies a \in C \land b \in B \implies b \in D\\&x \in (A \uplus B) \iff (\exists a \in A.x = (1,a)) \lor (\exists b \in B.x = (2,b)) \implies\\&x \in (A \uplus B) \implies (\exists a \in C.x = (1,a)) \lor (\exists b \in D.x = (2,b)) \iff\\&x \in (A \uplus B) \implies x \in (C \uplus D) \iff\\&A \uplus B \subseteq C \uplus D\end{aligned} \tag{36}$$

(b) $(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C)$

I will prove the distributivity of $\uplus$ and $\cup$.

$$\begin{aligned}x \in (A \cup B) \uplus C \iff& (\exists a \in A \cup B.x = (1,a)) \lor (\exists c \in C.x = (2,c)) \iff\\x \in (A \cup B) \uplus C \iff& (\exists a \in A.x = (1,a)) \lor (\exists b \in B.x = (1,b)) \lor (\exists c \in C.x = (2,c)) \iff\\x \in (A \cup B) \uplus C \iff& ((\exists a \in A.x = (1,a)) \lor (\exists c \in C.x = (2,c)))\lor\\&((\exists b \in B.x = (1,b)) \lor (\exists c \in C.x = (2,c))) \iff\\x \in (A \cup B) \uplus C \iff& x \in (A \uplus C) \lor x \in (B \uplus C) \iff\\x \in (A \cup B) \uplus C \iff& x \in ((A \uplus C) \cup (B \uplus C)) \iff\\(A \cup B) \uplus C =& (A \uplus C) \cup (B \uplus C)\end{aligned} \tag{37}$$

$$\begin{aligned}&(A \cup B) \uplus C = (A \uplus C) \cup (B \uplus C) \text{ using (37)} \iff\\&(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C) \text{ using the antisymmetry of } \subseteq\end{aligned} \tag{38}$$

(c) $(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C$

$$\begin{aligned}&(A \uplus C) \cup (B \uplus C) = (A \cup B) \uplus C \text{ using (37)} \iff\\&(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C \text{ using the antisymmetry of } \subseteq\end{aligned} \tag{39}$$

(d) $(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C)$

I will prove the distributivity of $\uplus$ and $\cap$.

$$\begin{aligned}x \in (A \cap B) \uplus C \iff& (\exists a \in A \cap B.x = (1,a)) \lor (\exists c \in C.x = (2,c)) \iff\\x \in (A \cap B) \uplus C \iff& ((\exists a \in A.x = (1,a)) \land (\exists b \in B.x = (1,b))) \lor (\exists c \in C.x = (2,c)) \iff\\x \in (A \cap B) \uplus C \iff& ((\exists a \in A.x = (1,a)) \lor (\exists c \in C.x = (2,c)))\land\\&((\exists b \in B.x = (1,b)) \lor (\exists c \in C.x = (2,c))) \iff\\x \in (A \cap B) \uplus C \iff& x \in (A \uplus C) \cap (B \uplus C) \iff\\(A \cap B) \uplus C =& (A \uplus C) \cap (B \uplus C)\end{aligned} \tag{40}$$

$$\begin{aligned}&(A \cap B) \uplus C = (A \uplus C) \cap (B \uplus C) \text{ using (40)} \iff\\&(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C) \text{ using the antisymmetry of } \subseteq\end{aligned} \tag{41}$$

(e) $(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C$

$$\begin{aligned}(A \uplus C) \cap (B \uplus C) = (A \cap B) \uplus C \text{ using (40)} \iff \\ (A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C \text{ using the antisymmetry of } \subseteq\end{aligned} \tag{42}$$

6. Prove the following properties of the big unions and intersections of a family of sets $\mathcal{F} \subseteq \mathcal{P}(A)$:

(a) $\forall U \subseteq A.(\forall X \in \mathcal{F}.X \subseteq U) \iff \bigcup \mathcal{F} \subseteq U$

$$\begin{aligned}&\forall U \subseteq A. \bigcup \mathcal{F} \subseteq U \iff \\ &\forall U \subseteq A.\nexists X \in \mathcal{F}.X \nsubseteq U \iff \\ &\forall U \subseteq A.\forall X \in \mathcal{F}.X \subseteq U\end{aligned} \tag{43}$$

(b) $\forall L \subseteq A.(\forall X \in \mathcal{F}.L \subseteq X) \iff L \subseteq \bigcap \mathcal{F}$

$$\begin{aligned}&\forall U \subseteq A.L \subseteq \mathcal{F} \iff \\ &\forall U \subseteq A.\nexists X \in \mathcal{F}.L \nsubseteq X \iff \\ &\forall U \subseteq A.\forall X \in \mathcal{F}.L \subseteq X\end{aligned} \tag{44}$$

7. Let A be a set.

(a) For a family $\mathcal{F} \subseteq \mathcal{P}(A)$, let $\mathcal{U} \triangleq \{U \subseteq A | \forall S \in \mathcal{F}.S \subseteq U\}$. Prove that $\bigcup \mathcal{F} = \bigcap \mathcal{U}$.

$$\begin{aligned}\mathcal{U} \triangleq \{U \subseteq A | \forall S \in \mathcal{F}.S \subseteq U\} \iff \\ \mathcal{U} = \{U \subseteq A | \bigcup \mathcal{F} \subseteq U\} \text{ using (43)} \iff \\ \bigcup \mathcal{F} \subseteq \bigcup \mathcal{F} \implies \bigcup \mathcal{F} \in \mathcal{U} \iff \\ \forall U \in \mathcal{U}. \bigcup \mathcal{F} \subseteq U \wedge \mathcal{F} \in U \iff \\ \bigcap \mathcal{U} = \bigcup \mathcal{F}\end{aligned} \tag{45}$$

(b) Analogously, define the family $\mathcal{L} \subseteq \mathcal{P}(A)$ such that $\bigcap \mathcal{F} = \bigcup \mathcal{L}$. Also prove this statement.

$$\mathcal{L} \triangleq \{L \subseteq A | \forall S \in \mathcal{F} L \subseteq S\}$$

$$\begin{aligned}\mathcal{L} \triangleq \{L \subseteq A | \forall S \in \mathcal{F} L \subseteq S\} \iff \\ \mathcal{L} = \{L \subseteq A | L \subseteq \bigcup \mathcal{F}\} \text{ using (44)} \iff \\ \bigcup \mathcal{F} \in \mathcal{L} \wedge \forall L \in \mathcal{L} : L \subseteq \mathcal{F} \iff \\ \bigcup \mathcal{L} = \bigcap \mathcal{F}\end{aligned} \tag{46}$$

5.3 Optional advanced exercises

1. Prove that for all families of sets $\mathcal{F}_1$ and $\mathcal{F}_2$

$$(\bigcup \mathcal{F}_1) \cup (\bigcup \mathcal{F}_2) = \bigcup (\mathcal{F}_1 \cup \mathcal{F}_2) \tag{47}$$

$$\begin{aligned}
(\bigcup \mathcal{F}_1) \cup (\bigcup \mathcal{F}_2) &= \{x | \exists S_1 \in \mathcal{F}_1 . x \in S_1\} \cup \{x | \exists S_2 \in \mathcal{F}_2 . x \in S_2\} \iff \\
(\bigcup \mathcal{F}_1) \cup (\bigcup \mathcal{F}_2) &= \{x | \exists S_1 \in \mathcal{F}_1 \vee \exists S_2 \in \mathcal{F}_2 . x \in S_1 \vee x \in S_2\} \iff \\
(\bigcup \mathcal{F}_1) \cup (\bigcup \mathcal{F}_2) &= \{x | \exists S \in \mathcal{F}_1 \cup \mathcal{F}_2 x \in S\} \iff \\
(\bigcup \mathcal{F}_1) \cup (\bigcup \mathcal{F}_2) &= \bigcup (\mathcal{F}_1 \cup \mathcal{F}_2)
\end{aligned} \tag{48}$$

State and prove the analogous property for intersections of non-empty families of sets.

$$(\bigcap \mathcal{F}_1) \cap (\bigcap \mathcal{F}_2) = \bigcap (\mathcal{F}_1 \cap \mathcal{F}_2)$$

$$\begin{aligned}
(\bigcap \mathcal{F}_1) \cap (\bigcap \mathcal{F}_2) &= \{x | \forall S_1 \in \mathcal{F}_1 . x \in S_1\} \cap \{x | \forall S_2 \in \mathcal{F}_2 . x \in S_2\} \iff \\
(\bigcap \mathcal{F}_1) \cap (\bigcap \mathcal{F}_2) &= \{x | \forall S_1 \in \mathcal{F}_1, \forall S_2 \in \mathcal{F}_2 . x \in S_1 \wedge x \in S_2\} \iff \\
(\bigcap \mathcal{F}_1) \cap (\bigcap \mathcal{F}_2) &= \{x | \forall S \in (\mathcal{F}_1 \cap \mathcal{F}_2) . x \in S\} \iff \\
(\bigcap \mathcal{F}_1) \cap (\bigcap \mathcal{F}_2) &= \bigcap (\mathcal{F}_1 \cap \mathcal{F}_2)
\end{aligned} \tag{49}$$

2. For a set U, prove that $(\mathcal{P}(U), \subseteq, \cup, \cap, U, \emptyset, (\cdot)^c)$ is a Boolean algebra.