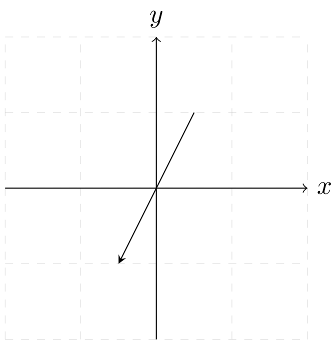
4.

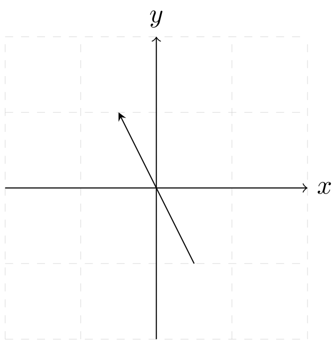
$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(x(y^2+2y-1)\right) \\ &= y^2+2y-1 \\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(x(y^2+2y-1)\right) \\ &= 2xy+2x \end{aligned} \tag{1}$$

$$\nabla(f) = (y^2+2y-1, 2xy+2x) \tag{2}$$

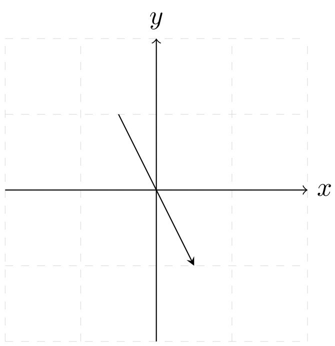
(a) At $(-1,0)$, $\nabla(f) = (-1,-2)$



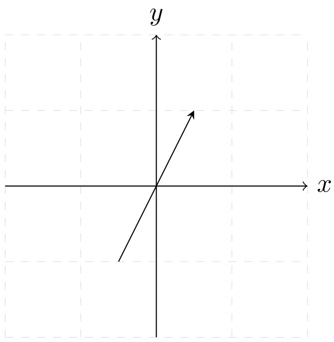
(b) At $(1,0)$, $\nabla(f) = (-1,2)$



(c) At $(-1,1)$, $\nabla(f) = (2,-4)$



(d) At $(-1,1)$, $\nabla(f) = (2,4)$



5.

$$\begin{aligned}
T &= 2\pi\left(\frac{\ell}{g}\right)^{\frac{1}{2}}\\
\ln T &= \ln\left(2\pi\left(\frac{\ell}{g}\right)^{\frac{1}{2}}\right)\\
\ln T &= \ln 2\pi + \frac{1}{2}\ln\left(\frac{\ell}{g}\right)\\
\ln T &= \ln 2\pi + \frac{1}{2}\ln\ell - \frac{1}{2}\ln g\\
\frac{\mathrm{d}T}{T} &= \frac{\mathrm{d}\ell}{2\ell} - \frac{\mathrm{d}g}{2g}\\
\frac{\mathrm{d}g}{g} &= \frac{\mathrm{d}\ell}{\ell} - \frac{2\mathrm{d}T}{T}
\end{aligned} \tag{3}$$

(a)

$$\begin{aligned}
\frac{\mathrm{d}g}{g} &= \frac{\mathrm{d}\ell}{\ell} - \frac{2\mathrm{d}T}{T}\\
\frac{\mathrm{d}g}{g} &= 0.001 + 0\\
\frac{\mathrm{d}g}{g} &= 0.001
\end{aligned} \tag{4}$$

So a 0.1% error in the measurement of ℓ will result in a 0.1% error in g.

(b)

$$\begin{aligned}
\frac{\mathrm{d}g}{g} &= \frac{\mathrm{d}\ell}{\ell} - \frac{2\mathrm{d}T}{T}\\
\frac{\mathrm{d}g}{g} &= 0 + 0.002\\
\frac{\mathrm{d}g}{g} &= 0.002
\end{aligned} \tag{5}$$

So a 0.1% error in the measurement of T will result in a 0.2% error in g.

6. Here are some formulae I will use for the following question:

$$\begin{aligned}
x &= r\cos\phi\\
\left(\frac{\partial x}{\partial r}\right)_\phi &= \cos\phi\\
\left(\frac{\partial x}{\partial \phi}\right)_r &= -r\sin\phi
\end{aligned} \tag{6}$$

$$\begin{aligned}
y &= r\sin\phi \\
\left(\frac{\partial y}{\partial r}\right)_\phi &= \sin\phi \\
\left(\frac{\partial y}{\partial \phi}\right)_r &= r\cos\phi
\end{aligned} \tag{7}$$

$$\begin{aligned}
r^2 &= x^2 + y^2 \\
2r\left(\frac{\partial r}{\partial x}\right)_y &= 2x \\
\left(\frac{\partial r}{\partial x}\right)_y &= \frac{2r\cos\phi}{2r} \\
\left(\frac{\partial r}{\partial x}\right)_y &= \cos\phi \\
2r\left(\frac{\partial r}{\partial y}\right)_x &= 2y \\
\left(\frac{\partial r}{\partial y}\right)_x &= \frac{2r\sin\phi}{2r} \\
\left(\frac{\partial r}{\partial y}\right)_x &= \sin\phi
\end{aligned} \tag{8}$$

$$\begin{aligned}
\phi &= \arctan\left(\frac{x}{y}\right) \\
\left(\frac{\partial \phi}{\partial x}\right)_y &= \frac{y}{x^2+y^2} \\
\left(\frac{\partial \phi}{\partial x}\right)_y &= \frac{r\sin\phi}{r^2} \\
\left(\frac{\partial \phi}{\partial x}\right)_y &= \frac{\sin\phi}{r} \\
\left(\frac{\partial \phi}{\partial y}\right)_x &= -\frac{x}{x^2+y^2} \\
\left(\frac{\partial \phi}{\partial y}\right)_x &= -\frac{r\cos\phi}{r^2} \\
\left(\frac{\partial \phi}{\partial y}\right)_x &= -\frac{\cos\phi}{r}
\end{aligned} \tag{9}$$

Derivation of expressions by differentiating with the chain rule:

$$\begin{aligned}
f(x,y) &= e^{-xy} \\
\left(\frac{\partial f}{\partial x}\right)_y &= -ye^{-xy}
\end{aligned} \tag{10}$$

$$\begin{aligned}
f(x,y) &= e^{-xy} \\
\left(\frac{\partial f}{\partial y}\right)_x &= -xe^{-xy}
\end{aligned} \tag{11}$$

$$
\begin{aligned}
f(x,y) &= e^{-xy} \\
\left(\frac{\partial f}{\partial r}\right)_\phi &= -xe^{-xy}\left(\frac{\partial y}{\partial r}\right)_\phi - ye^{-xy}\left(\frac{\partial x}{\partial r}\right)_\phi \\
\left(\frac{\partial f}{\partial r}\right)_\phi &= -xe^{-xy}\sin\phi - ye^{-xy}\cos\phi \\
\left(\frac{\partial f}{\partial r}\right)_\phi &= -r\cos\phi\sin\phi e^{-r^2\sin\phi\cos\phi} - r\cos\phi\sin\phi e^{-r^2\sin\phi\cos\phi} \\
\left(\frac{\partial f}{\partial r}\right)_\phi &= -2r\cos\phi\sin\phi e^{-r^2\sin\phi\cos\phi} \\
\left(\frac{\partial f}{\partial r}\right)_\phi &= -r\sin 2\phi e^{-\frac{1}{2}r^2\sin 2\phi}
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
f(x,y) &= e^{-xy} \\
\left(\frac{\partial f}{\partial \phi}\right)_r &= -xe^{-xy}\left(\frac{\partial y}{\partial \phi}\right)_r - ye^{-xy}\left(\frac{\partial x}{\partial \phi}\right)_r \\
\left(\frac{\partial f}{\partial \phi}\right)_r &= -r\cos\phi e^{-\frac{1}{2}r^2\sin 2\phi} r\cos\phi + r\sin\phi e^{-\frac{1}{2}r^2\sin 2\phi} r\sin\phi \\
\left(\frac{\partial f}{\partial \phi}\right)_r &= -r^2\cos^2\phi e^{-\frac{1}{2}r^2\sin 2\phi} + r^2\sin^2\phi e^{-\frac{1}{2}r^2\sin 2\phi} \\
\left(\frac{\partial f}{\partial \phi}\right)_r &= -r^2\cos 2\phi e^{-\frac{1}{2}r^2\sin 2\phi}
\end{aligned} \tag{13}
$$

(b) Derivation of expressions by expression in polar coordinates followed by differentiation.

$$
\begin{aligned}
f(r,\phi) &= e^{-\frac{1}{2}r^2\sin 2\phi} \\
\left(\frac{\partial f}{\partial x}\right)_y &= \left(\frac{\partial f}{\partial r}\right)_\phi\left(\frac{\partial r}{\partial x}\right)_y - \frac{\sin\phi}{r}\left(\frac{\partial f}{\partial \phi}\right)_r\left(\frac{\partial \phi}{\partial x}\right)_y \\
\left(\frac{\partial f}{\partial x}\right)_y &= \left(-r\sin 2\phi e^{-\frac{1}{2}{}^2\sin 2\phi}\right)\cos\phi - \left(-r^2\cos 2\phi e^{-\frac{1}{2}r^2\sin 2\phi}\right)\left(\frac{\sin\phi}{r}\right) \\
\left(\frac{\partial f}{\partial x}\right)_y &= \left(r\cos 2\phi\sin\phi - r\sin 2\phi\cos\phi\right)e^{-\frac{1}{2}{}^2\sin 2\phi} \\
\left(\frac{\partial f}{\partial x}\right)_y &= r\left(2\cos^2\phi\sin\phi - \sin\phi - 2\sin\phi\cos^2\phi\right)e^{-\frac{1}{2}{}^2\sin 2\phi} \\
\left(\frac{\partial f}{\partial x}\right)_y &= -r\sin\phi e^{-\frac{1}{2}{}^2\sin 2\phi} \\
\left(\frac{\partial f}{\partial x}\right)_y &= -ye^{-xy}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
&f(r,\phi) = e^{-\frac{1}{2}r^2\sin 2\phi}\\
&\left(\frac{\partial f}{\partial y}\right)_x = \left(\frac{\partial f}{\partial r}\right)_\phi \left(\frac{\partial r}{\partial x}\right)_y + \left(\frac{\partial f}{\partial \phi}\right)_r \left(\frac{\partial \phi}{\partial x}\right)_y\\
&\left(\frac{\partial f}{\partial y}\right)_x = \left(-r\sin 2\phi e^{-\frac{1}{2}r^2\sin 2\phi}\right)\sin\phi + \left(-r^2\cos 2\phi e^{-\frac{1}{2}r^2\sin 2\phi}\right)\left(\frac{cos\phi}{r}\right)\\
&\left(\frac{\partial f}{\partial y}\right)_x = \left(-r\sin 2\phi\sin\phi - r\cos 2\phi\cos\phi\right)e^{-\frac{1}{2}r^2\sin 2\phi}\\
&\left(\frac{\partial f}{\partial y}\right)_x = -r\left(\sin 2\phi\sin\phi + \cos 2\phi\cos\phi\right)e^{-\frac{1}{2}r^2\sin 2\phi}\\
&\left(\frac{\partial f}{\partial y}\right)_x = -r\left(2\sin^\phi\cos\phi + \cos\phi - 2\sin^2\phi\cos\phi\right)e^{-\frac{1}{2}r^2\sin 2\phi}\\
&\left(\frac{\partial f}{\partial y}\right)_x = -r\cos\phi e^{-\frac{1}{2}r^2\sin 2\phi}\\
&\left(\frac{\partial f}{\partial y}\right)_x = -xe^{-xy}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
f(r,\phi) &= e^{-\frac{1}{2}r^2\sin 2\phi}\\
\left(\frac{\partial f}{\partial r}\right)_\phi &= -r\sin 2\phi e^{-\frac{1}{2}r^2\sin 2\phi}\\
\left(\frac{\partial f}{\partial r}\right)_\phi &= -r\sin 2\phi e^{-\frac{1}{2}r^2\sin 2\phi}
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
f(r,\phi) &= e^{-\frac{1}{2}r^2\sin 2\phi}\\
\left(\frac{\partial f}{\partial \phi}\right)_r &= -r\sin 2\phi e^{-\frac{1}{2}r^2\sin 2\phi}\\
\left(\frac{\partial f}{\partial \phi}\right)_r &= -r^2\cos 2\phi e^{-\frac{1}{2}r^2\sin 2\phi}
\end{aligned}
\tag{17}
$$

7.

$$
\begin{aligned}
xyz + x^3 + y^4 + z^5 &= 0\\
yz\left(\frac{\partial x}{\partial y}\right)_z + xz + 3x^2\left(\frac{\partial x}{\partial y}\right)_z + 4y^3 &= 0\\
(yz+3x^2)\left(\frac{\partial x}{\partial y}\right)_z &= -xz - 4y^3\\
\left(\frac{\partial x}{\partial y}\right)_z &= -\frac{xz+4y^3}{yz+3x^2}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
xyz + x^3 + y^4 + z^5 &= 0\\
xz\left(\frac{\partial y}{\partial z}\right)_x + xy + 4y^3\left(\frac{\partial y}{\partial z}\right)_x + 5z^4 &= 0\\
(xz+4y^3)\left(\frac{\partial y}{\partial z}\right)_x &= -xy - 5z^4\\
\left(\frac{\partial y}{\partial z}\right)_x &= -\frac{xy+5z^4}{xz+4y^3}
\end{aligned}
\tag{19}
$$

$$\begin{aligned} xyz + x^3 + y^4 + z^5 &= 0 \\ xy\left(\frac{\partial z}{\partial x}\right)_y + yz + 3x^2 + 5z^4\left(\frac{\partial z}{\partial x}\right)_y &= 0 \\ (xy + 5z^4)\left(\frac{\partial z}{\partial x}\right)_y &= -yz - 3x^2 \\ \left(\frac{\partial z}{\partial x}\right)_y &= -\frac{yz + 3x^2}{xy + 5z^4} \end{aligned} \tag{20}$$

$$\begin{aligned} &\left(\frac{\partial x}{\partial y}\right)_z \times \left(\frac{\partial y}{\partial z}\right)_x \times \left(\frac{\partial z}{\partial x}\right)_y \\ =& -\frac{xz + 4y^3}{yz + 3x^2} \times -\frac{xy + 5z^4}{xz + 4y^3} \times -\frac{yz + 3x^2}{xy + 5z^4} \\ =& -\frac{(xz + 4y^3)(xy + 5z^4)(yz + 3x^2)}{(yz + 3x^2)(xz + 4y^3)(xy + 5z^4)} \\ =& -\frac{(xz + 4y^3)(xy + 5z^4)(yz + 3x^2)}{(xz + 4y^3)(xy + 5z^4)(yz + 3x^2)} \\ =& -1 \end{aligned} \tag{21}$$

8.

$$\begin{aligned} \left(p + \frac{a}{V^2}\right)(V - b) &= RT \\ p &= \frac{RT}{V - b} - \frac{a}{V^2} \\ \left(\frac{\partial p}{\partial V}\right)_T &= -\frac{RT}{(V - b)^2} + \frac{2a}{V^3} \\ \left(\frac{\partial p}{\partial V}\right)_T &= -\frac{RTV^3}{V^3(V - b)^2} + \frac{2a(V - b)^2}{V^3(V - b)^2} \\ \left(\frac{\partial p}{\partial V}\right)_T &= \frac{-RTV^3 + 2a(V - b)^2}{V^3(V - b)^2} \end{aligned} \tag{22}$$

$$\begin{aligned} \left(p + \frac{a}{V^2}\right)(V - b) &= RT \\ \left(\left(p + \frac{a}{V^2}\right) - \frac{2a}{V^3}(V - b)\right)\left(\frac{\partial V}{\partial T}\right)_p &= R \\ \left(\frac{RT}{V - b} - \frac{2a}{V^2} + \frac{2ab}{V^3}\right)\left(\frac{\partial V}{\partial T}\right)_p &= R \\ (RTV^3 - 2aV(V - b) + 2ab(V - b))\left(\frac{\partial V}{\partial T}\right)_p &= RV^3(V - b) \\ (RTV^3 - 2a(V - b)^2)\left(\frac{\partial V}{\partial T}\right)_p &= RV^3(V - b) \\ \left(\frac{\partial V}{\partial T}\right)_p &= \frac{RV^3(V - b)}{RTV^3 - 2a(V - b)^2} \end{aligned} \tag{23}$$

$$\begin{aligned}
RT &= \left(p + \frac{a}{V^2}\right)(V-b) \\
R\left(\frac{\partial T}{\partial p}\right)_V &= (V-b) \\
\left(\frac{\partial T}{\partial p}\right)_V &= \frac{(V-b)}{R}
\end{aligned} \tag{24}$$

$$\begin{aligned}
&\left(\frac{\partial p}{\partial V}\right)_T \cdot \left(\frac{\partial V}{\partial T}\right)_p \cdot \left(\frac{\partial T}{\partial p}\right)_V \\
=&\frac{-RTV^3 + 2a(V-b)^2}{V^3(V-b)^2} \cdot \frac{RV^3(V-b)}{RTV^3 - 2a(V-b)^2} \times \frac{(V-b)}{R} \\
=&\frac{RV^3(V-b)^2(-RTV^3 + 2a(V-b)^2)}{RV^3(V-b)^2(RTV^3 - 2a(V-b)^2)} \\
=&-1
\end{aligned} \tag{25}$$

9. If u and v are rotated axis of x and y then for some constant angle θ:

$$\begin{aligned}
&\text{Let } x = u\cos\theta + v\sin\theta \\
&\left(\frac{\partial x}{\partial u}\right)_v = \cos\theta \\
&\left(\frac{\partial x}{\partial v}\right)_u = \sin\theta \\
&\text{Let } y = u\sin\theta - v\cos\theta \\
&\left(\frac{\partial y}{\partial u}\right)_v = \sin\theta \\
&\left(\frac{\partial y}{\partial v}\right)_u = -\cos\theta
\end{aligned} \tag{26}$$

$$\begin{aligned}
\left(\frac{\partial f}{\partial u}\right)_v &= \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v \\
\left(\frac{\partial^2 f}{\partial u^2}\right)_v &= \left(\frac{\partial^2 f}{\partial x^2}\right)_y \left(\frac{\partial x}{\partial u}\right)_v^2 + \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial^2 x}{\partial u^2}\right)_v + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \left(\frac{\partial y}{\partial u}\right)_v^2 + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial^2 y}{\partial u^2}\right)_v \\
\left(\frac{\partial^2 f}{\partial u^2}\right)_v &= \left(\frac{\partial^2 f}{\partial x^2}\right)_y \cos^2\theta + 0\left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \sin^2\theta + 0\left(\frac{\partial f}{\partial y}\right)_x \\
\left(\frac{\partial^2 f}{\partial u^2}\right)_v &= \left(\frac{\partial^2 f}{\partial x^2}\right)_y \cos^2\theta + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \sin^2\theta
\end{aligned} \tag{27}$$

$$\begin{aligned}
\left(\frac{\partial f}{\partial v}\right)_u &= \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u \\
\left(\frac{\partial^2 f}{\partial v^2}\right)_u &= \left(\frac{\partial^2 f}{\partial x^2}\right)_y \left(\frac{\partial x}{\partial v}\right)_u^2 + \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial^2 x}{\partial v^2}\right)_u + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \left(\frac{\partial y}{\partial v}\right)_u^2 + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial^2 y}{\partial v^2}\right)_u \\
\left(\frac{\partial^2 f}{\partial v^2}\right)_u &= \left(\frac{\partial^2 f}{\partial x^2}\right)_y \sin^2\theta + 0\left(\frac{\partial f}{\partial x}\right)_y + \left(\frac{\partial^2 f}{\partial y^2}\right)_x (-\cos\theta)^2 + 0\left(\frac{\partial f}{\partial y}\right)_x \\
\left(\frac{\partial^2 f}{\partial v^2}\right)_u &= \left(\frac{\partial^2 f}{\partial x^2}\right)_y \sin^2\theta + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \cos^2\theta
\end{aligned} \tag{28}$$

$$\left(\frac{\partial^2 f}{\partial u^2}\right)_v + \left(\frac{\partial^2 f}{\partial v^2}\right)_u = \left(\frac{\partial^2 f}{\partial x^2}\right)_y \cos^2\theta + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \sin^2\theta + \left(\frac{\partial^2 f}{\partial x^2}\right)_y \sin^2\theta + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \cos^2\theta$$
$$\left(\frac{\partial^2 f}{\partial u^2}\right)_v + \left(\frac{\partial^2 f}{\partial v^2}\right)_u = \left(\frac{\partial^2 f}{\partial x^2}\right)_y \left(\sin^2\theta + \cos^2\theta\right) + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \left(\sin^2\theta + \cos^2\theta\right)$$
$$\left(\frac{\partial^2 f}{\partial u^2}\right)_v + \left(\frac{\partial^2 f}{\partial v^2}\right)_u = \left(\frac{\partial^2 f}{\partial x^2}\right)_y + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \text{ as required} \tag{29}$$