10. (a)

$$\begin{aligned} y \,\mathrm{d}x + x \,\mathrm{d}y \\ \frac{\partial}{\partial y}(y) = 1 \\ \frac{\partial}{\partial x}(x) = 1 \end{aligned} \tag{1}$$

Since $\frac{\partial}{\partial y}(P) = \frac{\partial}{\partial x}(Q)$ the differential is exact.

$$\begin{aligned} y \,\mathrm{d}x + x \,\mathrm{d}y = 0 \\ xy = c \\ y = \frac{c}{x} \end{aligned} \tag{2}$$

(b)

$$\begin{aligned} y \,\mathrm{d}x + x^2 \,\mathrm{d}y \\ \frac{\partial}{\partial y}(y) = 1 \\ \frac{\partial}{\partial x}\left(x^2\right) = 2x \neq 1 \end{aligned} \tag{3}$$

$\frac{\partial}{\partial y}(P) \neq \frac{\partial}{\partial x}(Q)$ and so the differential is not exact.

$$\begin{aligned} \mu(x) &= e^{\int \frac{1}{x^2}\left(\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial x}\left(x^2\right)\right)\mathrm{d}x} \\ &= e^{\int \frac{1}{x^2}(1-2x)\mathrm{d}x} \\ &= e^{\int \frac{1}{x^2} - \frac{2}{x}\mathrm{d}x} \\ &= e^{-\frac{1}{x} - 2\ln x} \\ &= x^{-2}e^{-\frac{1}{x}} \end{aligned} \tag{4}$$

$$\begin{aligned} y \,\mathrm{d}x + x^2 \,\mathrm{d}y = 0 \\ x^{-2}ye^{-\frac{1}{x}} \,\mathrm{d}x + e^{-\frac{1}{x}} \,\mathrm{d}y = 0 \\ ye^{-\frac{1}{x}} = c \\ y = ce^{\frac{1}{x}} \end{aligned} \tag{5}$$

(c)

$$\begin{aligned} (x+y) \,\mathrm{d}x + (x-y) \,\mathrm{d}y \\ \frac{\partial}{\partial y}(x+y) = 1 \\ \frac{\partial}{\partial x}(x-y) = 1 \end{aligned} \tag{6}$$

$\frac{\partial}{\partial y}(P) = \frac{\partial}{\partial x}(Q)$ and so the differential is exact.

$$\begin{aligned} (x+y) \,\mathrm{d}x + (x-y) \,\mathrm{d}y = 0 \\ (x+y) \,\mathrm{d}x + (x-y) \,\mathrm{d}y = 0 \\ \frac{1}{2}x^2 + xy - \frac{1}{2}y^2 = c \end{aligned} \tag{7}$$

(d)

$$\begin{aligned}
&(\cosh x \cos y + \cosh y \cos x)\,\mathrm{d}x - (\sinh x \sin y - \sinh y \sin x)\,\mathrm{d}y \\
&\frac{\partial}{\partial y}(\cosh x \cos y + \cosh y \cos x) = -\cosh x \sin y + \sinh y \cos x \\
&\frac{\partial}{\partial x}(-\sinh x \sin y + \sinh y \sin x) = -\cosh x \sin y + \sinh y \cos x
\end{aligned} \tag{8}$$

So $\frac{\partial}{\partial y}(P) = \frac{\partial}{\partial x}(Q)$ and so the differential is exact.

$$\begin{aligned}
(\cosh x \cos y + \cosh y \cos x)\,\mathrm{d}x - (\sinh x \sin y - \sinh y \sin x)\,\mathrm{d}y &= 0 \\
\sinh x \cos y + \cosh y \sin x &= c
\end{aligned} \tag{9}$$

(e)

$$\begin{aligned}
&(\cos x - \sin x)\,\mathrm{d}x + (\sin x + \cos x)\,\mathrm{d}y \\
&\frac{\partial}{\partial y}(\cos x - \sin x) = 0 \\
&\frac{\partial}{\partial x}(\sin x + \cos x) = \cos x - \sin x
\end{aligned} \tag{10}$$

So $\frac{\partial}{\partial y}(P) \neq \frac{\partial}{\partial x}(Q)$ and so the differential is not exact.

$$\begin{aligned}
\mu(y) &= e^{\int \frac{1}{(\sin x + \cos x)}\left(\frac{\partial}{\partial y}((\cos x - \sin x)) - \frac{\partial}{\partial x}(\sin x + \cos x)\right)\mathrm{d}x} \\
\mu(y) &= e^{\int \frac{1}{(\sin x + \cos x)}(0 - \cos x + \sin x)\mathrm{d}x} \\
\mu(y) &= e^{\int 1 \mathrm{d}x} \\
\mu(y) &= e^{y}
\end{aligned} \tag{11}$$

$$\begin{aligned}
(\cos x - \sin x)\,\mathrm{d}x + (\sin x + \cos x)\,\mathrm{d}y &= 0 \\
(\cos x - \sin x)e^{y}\,\mathrm{d}x + (\sin x + \cos x)e^{y}\,\mathrm{d}y &= 0 \\
(\sin x + \cos x)e^{y} &= c \\
e^{y} &= \frac{c}{\sin x + \cos x} \\
y &= -\ln A(\sin x + \cos x)
\end{aligned} \tag{12}$$

(f)

$$\begin{aligned}
&\frac{x}{x^2+y^2}\,\mathrm{d}y - \frac{y}{x^2+y^2}\,\mathrm{d}x \\
&\frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right) = \frac{y^2-x^2}{x^2+y^2} \\
&\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{y^2-x^2}{x^2+y^2}
\end{aligned} \tag{13}$$

So $\frac{\partial}{\partial y}(P) = \frac{\partial}{\partial x}(Q)$ and so the differential is exact.

$$\begin{aligned}
\frac{x}{x^2+y^2}\,\mathrm{d}y - \frac{y}{x^2+y^2}\,\mathrm{d}x &= 0 \\
\arctan\left(\frac{y}{x}\right) &= c \\
\frac{y}{x} &= k \\
y &= kx
\end{aligned} \tag{14}$$

11. (a)

$$\begin{aligned} H &= U + pV \\ \mathrm{d}H &= \mathrm{d}U + V\,\mathrm{d}p + p\,\mathrm{d}V \\ \mathrm{d}H &= (\mathrm{d}U + p\,\mathrm{d}V) + V\,\mathrm{d}p \\ \mathrm{d}H &= T\,\mathrm{d}S + V\,\mathrm{d}p \end{aligned} \tag{15}$$

We know this is an exact differential. So it is of the form

$$\mathrm{d}H = P\,\mathrm{d}S + Q\,\mathrm{d}p \tag{16}$$

Where $\left(\frac{\partial P}{\partial p}\right)_S = \left(\frac{\partial Q}{\partial S}\right)_p$. Substituting in the actual coefficients of the equation ($P = T$, $Q = V$) gives us:

$$\left(\frac{\partial V}{\partial S}\right)_p = \left(\frac{\partial T}{\partial p}\right)_S \tag{17}$$

As required.

(b)

$$\begin{aligned} \mathrm{d}U &= T\,\mathrm{d}S - p\,\mathrm{d}V \\ \mathrm{d}U &= T\left(\left(\frac{\partial S}{\partial p}\right)_V \mathrm{d}p + \left(\frac{\partial S}{\partial V}\right)_p \mathrm{d}V\right) - p\,\mathrm{d}V \\ \mathrm{d}U &= T\left(\frac{\partial S}{\partial p}\right)_V \mathrm{d}p + \left(T\left(\frac{\partial S}{\partial V}\right)_p - p\right)\mathrm{d}V \end{aligned} \tag{18}$$

We know that $\mathrm{d}U$ is an exact integral – this implies:

$$\begin{aligned} \frac{\partial}{\partial p}\left(T\left(\frac{\partial S}{\partial V}\right)_p - p\right) &= \frac{\partial}{\partial V}\left(T\left(\frac{\partial S}{\partial p}\right)_V\right) \\ \left(\frac{\partial T}{\partial p}\right)_V\left(\frac{\partial S}{\partial V}\right)_p + T\left(\frac{\partial^2 S}{\partial p \partial V}\right) - 1 &= \left(\frac{\partial T}{\partial V}\right)_p\left(\frac{\partial S}{\partial p}\right)_V + T\left(\frac{\partial^2 S}{\partial p \partial V}\right) \\ \left(\frac{\partial S}{\partial V}\right)_p\left(\frac{\partial T}{\partial p}\right)_V - 1 &= \left(\frac{\partial S}{\partial p}\right)_V\left(\frac{\partial T}{\partial V}\right)_p \\ \left(\frac{\partial S}{\partial V}\right)_p\left(\frac{\partial T}{\partial p}\right)_V - \left(\frac{\partial S}{\partial p}\right)_V\left(\frac{\partial T}{\partial V}\right)_p &= 1 \end{aligned} \tag{19}$$

12.

$$\begin{aligned} G &= U + Vp - ST \\ \mathrm{d}G &= \mathrm{d}U + \frac{\partial}{\partial p}(Vp)_V\,\mathrm{d}p + \frac{\partial}{\partial V}(Vp)_p\,\mathrm{d}V - \frac{\partial}{\partial S}(ST)_T\,\mathrm{d}S - \frac{\partial}{\partial T}(ST)_S\,\mathrm{d}T \\ \mathrm{d}G &= \mathrm{d}U + V\,\mathrm{d}p + p\,\mathrm{d}V - T\,\mathrm{d}S - S\,\mathrm{d}T \\ \mathrm{d}G &= T\,\mathrm{d}S - p\,\mathrm{d}V + V\,\mathrm{d}p + p\,\mathrm{d}V - T\,\mathrm{d}S - S\,\mathrm{d}T \\ \mathrm{d}G &= V\,\mathrm{d}p - S\,\mathrm{d}T \end{aligned} \tag{20}$$

Since we know that $\mathrm{d}G$ is an exact differential; we know that the partial derivative of V with respect to T is equal to the partial derivative of $-S$ with respect to p.

$$\begin{aligned}\left(\frac{\partial V}{\partial T}\right)_p &= -\left(\frac{\partial S}{\partial p}\right)_T \\ \left(\frac{\partial S}{\partial p}\right)_T &= -\left(\frac{\partial V}{\partial T}\right)_p\end{aligned} \tag{21}$$

13. (a) Since p is a function of either V and T or V and S; we can express $\mathrm{d}p$ as a function of either also. We can also express any of the variables V, T, S as a function of the other two. I will express S as a function of V and T.

$$\mathrm{d}p = \left(\frac{\partial p}{\partial V}\right)_T \mathrm{d}V + \left(\frac{\partial p}{\partial T}\right)_V \mathrm{d}T \tag{22}$$

$$\mathrm{d}p = \left(\frac{\partial p}{\partial V}\right)_S \mathrm{d}V + \left(\frac{\partial p}{\partial S}\right)_V \mathrm{d}S \tag{23}$$

$$\mathrm{d}S = \left(\frac{\partial S}{\partial V}\right)_T \mathrm{d}V + \left(\frac{\partial S}{\partial T}\right)_V \mathrm{d}T \tag{24}$$

Substitute (24) into (23).

$$\begin{aligned}\mathrm{d}p &= \left(\frac{\partial p}{\partial V}\right)_S \mathrm{d}V + \left(\frac{\partial p}{\partial S}\right)_V \mathrm{d}S \\ \mathrm{d}p &= \left(\frac{\partial p}{\partial V}\right)_S \mathrm{d}V + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T \mathrm{d}V + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial T}\right)_V \mathrm{d}T \\ \mathrm{d}p &= \left(\frac{\partial p}{\partial V}\right)_S \mathrm{d}V + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T \mathrm{d}V + \left(\frac{\partial p}{\partial T}\right)_V \mathrm{d}T \\ \mathrm{d}p &= \left(\left(\frac{\partial p}{\partial V}\right)_S + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T\right) \mathrm{d}V + \left(\frac{\partial p}{\partial T}\right)_V \mathrm{d}T\end{aligned} \tag{25}$$

Now subtract (22) from (25).

$$\begin{aligned}\mathrm{d}p - \mathrm{d}p &= \left(\left(\frac{\partial p}{\partial V}\right)_S + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T\right) \mathrm{d}V + \left(\frac{\partial p}{\partial T}\right)_V \mathrm{d}T - \left(\frac{\partial p}{\partial V}\right)_T \mathrm{d}V - \left(\frac{\partial p}{\partial T}\right)_V \mathrm{d}T \\ 0 &= \left(\left(\frac{\partial p}{\partial V}\right)_S - \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T\right) \mathrm{d}V + \left(\left(\frac{\partial p}{\partial T}\right)_V - \left(\frac{\partial p}{\partial T}\right)_V\right) \mathrm{d}T \\ 0 &= \left(\left(\frac{\partial p}{\partial V}\right)_S - \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T\right) \mathrm{d}V\end{aligned} \tag{26}$$

So the coefficients of $\mathrm{d}V$ must be equal to zero.

$$\begin{aligned}0 &= \left(\frac{\partial p}{\partial V}\right)_S - \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T \\ \left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S &= \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T \\ \left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S &= \frac{\left(\frac{\partial S}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial p}\right)_V}\end{aligned} \tag{27}$$

(b)

$$\begin{aligned}T\,\mathrm{d}S &= \mathrm{d}U + p\,\mathrm{d}V \\ T\,\mathrm{d}S &= \left(\frac{\partial U}{\partial T}\right)_V \mathrm{d}T + \left(\left(\frac{\partial U}{\partial V}\right)_T + p\right) \mathrm{d}V \\ \mathrm{d}S &= \left(\frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V\right) \mathrm{d}T + \left(\frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{p}{T}\right) \mathrm{d}V\end{aligned} \tag{28}$$

dS is an exact differential. So the right hand side must also be an exact differential.

$$\begin{aligned}\frac{1}{T}\left(\frac{\partial^2 U}{\partial T \partial V}\right) &= -\frac{1}{T^2}\left(\frac{\partial U}{\partial V}\right)_T + \frac{1}{T}\left(\frac{\partial^2 U}{\partial T \partial V}\right) - \frac{p}{T^2} + \frac{1}{T}\left(\frac{\partial p}{\partial T}\right)_V \\ 0 &= -\frac{1}{T^2}\left(\frac{\partial U}{\partial V}\right)_T - \frac{p}{T^2} + \frac{1}{T}\left(\frac{\partial p}{\partial T}\right)_V \\ \frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_V &= \frac{1}{p}\left(\frac{\partial U}{\partial V}\right)_T + 1\end{aligned} \tag{29}$$

We can form another relation:

$$\begin{aligned}\mathrm{d}U &= T\,\mathrm{d}S - p\,\mathrm{d}V \\ \mathrm{d}U &= T\left(\frac{\partial S}{\partial T}\right)_V \mathrm{d}T + \left(T\left(\frac{\partial S}{\partial V}\right)_T - p\right)\mathrm{d}V\end{aligned} \tag{30}$$

Since dU is an exact differential, we also know the right-hand-side of the equation is exact.

$$\begin{aligned}T\left(\frac{\partial^2 S}{\partial T \partial V}\right) &= T\left(\frac{\partial^2 S}{\partial T \partial V}\right) + \left(\frac{\partial S}{\partial V}\right)_T - \left(\frac{\partial p}{\partial T}\right)_V \\ 0 &= \left(\frac{\partial S}{\partial V}\right)_T - \left(\frac{\partial p}{\partial T}\right)_V \\ \left(\frac{\partial p}{\partial T}\right)_V &= \left(\frac{\partial S}{\partial V}\right)_T\end{aligned} \tag{31}$$

Substituting this into (29) gives:

$$\begin{aligned}\frac{T}{p}\left(\frac{\partial p}{\partial T}\right)_V &= \frac{1}{p}\left(\frac{\partial U}{\partial V}\right)_T + 1 \\ \frac{T}{p}\left(\frac{\partial S}{\partial V}\right)_T &= \frac{1}{p}\left(\frac{\partial U}{\partial V}\right)_T + 1\end{aligned} \tag{32}$$

Returning to the original equation gives:

$$\begin{aligned}\mathrm{d}U &= T\,\mathrm{d}S - p\,\mathrm{d}V \implies \\ \left(\frac{\partial U}{\partial S}\right)_V &= T\end{aligned} \tag{33}$$

Consider now:

$$\begin{aligned}
&\left(\frac{\partial \ln p}{\partial \ln V}\right)_T - \left(\frac{\partial \ln p}{\partial \ln V}\right)_S \\
=&\frac{V}{p}\left(\left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S\right) \\
=&\frac{V\left(\frac{\partial S}{\partial V}\right)_T}{p\left(\frac{\partial S}{\partial p}\right)_V} \\
=&V\frac{\frac{T}{p}\left(\frac{\partial S}{\partial V}\right)_T}{T\left(\frac{\partial S}{\partial p}\right)_V} \\
=&V\frac{\left(\frac{1}{p}\left(\frac{\partial U}{\partial V}\right)_T + 1\right)}{\left(\frac{\partial U}{\partial S}\right)_V\left(\frac{\partial S}{\partial p}\right)_V} \\
=&V\frac{\left(\frac{1}{p}\left(\frac{\partial U}{\partial V}\right)_T + 1\right)}{\left(\frac{\partial U}{\partial p}\right)_V} \\
=&V\frac{\left(\frac{1}{p}\left(\frac{\partial U}{\partial V}\right)_T + 1\right)}{\left(\frac{\partial U}{\partial T}\right)_V\left(\frac{\partial T}{\partial p}\right)_V} \\
=&V\left(\frac{\partial p}{\partial T}\right)_V\frac{\left(\frac{1}{p}\left(\frac{\partial U}{\partial V}\right)_T + 1\right)}{\left(\frac{\partial U}{\partial T}\right)_V} \\
=&\left(\frac{\partial (pV)}{\partial T}\right)_V\frac{\left(\frac{1}{p}\left(\frac{\partial U}{\partial V}\right)_T + 1\right)}{\left(\frac{\partial U}{\partial T}\right)_V}
\end{aligned} \tag{34}$$

(c) Note that since pV^γ depends only on S:

$$\begin{aligned}
\left(\frac{\partial \ln p}{\partial \ln V}\right)_S &= 0 \\
\left(\frac{\partial \ln p}{\partial \ln V}\right)_T &= \frac{\mathrm{d}\ln p}{\mathrm{d}\ln V}
\end{aligned} \tag{35}$$

$$\begin{aligned}
U &= C_v T \\
\left(\frac{\partial U}{\partial T}\right)_V &= C_v \\
\left(\frac{\partial U}{\partial V}\right)_T &= 0 \\
pV &= RT \\
\left(\frac{\partial (pV)}{\partial T}\right)_V &= R
\end{aligned} \tag{36}$$

In the below equation: k is the constant of integration and A is another constant which is equal to $k^{\frac{R}{C_v}}$.

$$\begin{aligned}
\left(\frac{\partial \ln p}{\partial \ln V}\right)_T - \left(\frac{\partial \ln p}{\partial \ln V}\right)_S &= \left(\frac{\partial (pV)}{\partial T}\right)_V \frac{\left(\frac{1}{p}\left(\frac{\partial U}{\partial V}\right)_T + 1\right)}{\left(\frac{\partial U}{\partial T}\right)_V} \\
\frac{\mathrm{d} \ln p}{\mathrm{d} \ln V} - 0 &= R\frac{(0+1)}{C_v} \\
\frac{\mathrm{d} \ln p}{\mathrm{d} \ln V} &= \frac{R}{C_v} \\
\ln p &= \frac{R}{C_v} \ln kV \\
\ln p &= \ln k^{\frac{R}{C_v}} V^{-\frac{R}{C_v}} \\
p &= AV^{\frac{R}{C_v}} \\
pV^{-\frac{R}{C_v}} &= A
\end{aligned} \tag{37}$$

$$\gamma = -\frac{R}{C_v} \tag{38}$$

(d)

$$\begin{aligned}
\gamma &= -\frac{R}{C_v} \\
\gamma &= -\frac{R}{\frac{3}{2}R} \\
\gamma &= -\frac{2}{3}
\end{aligned} \tag{39}$$