

1.

$$\mathbf{f}_1 = \mathbf{e}_1$$

$$\mathbf{f}_2 = \sqrt{2}\mathbf{e}_1 + \sqrt{2}\mathbf{e}_2$$

$$\mathbf{x} = 2\mathbf{f}_1 + \frac{3}{\sqrt{2}}\mathbf{f}_2$$

2. (a)

$$\mathbf{vu}$$

$$= \begin{pmatrix} 1 & 2 & b \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$= 0 \times 1 + 1 \times 2 + 2 \times b$$

$$= 2 + 2b$$

(b)

$$\mathbf{uv}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & b \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & b \\ 2 & 4 & 2b \end{pmatrix}$$

3. • \mathbf{A}^2

This does not exist: A is a 2×3 matrix, so the dimensions are not consistent for matrix multiplication.

• \mathbf{AB}

$$\mathbf{AB}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 4 & 7 & 6 \\ 10 & 10 & 16 & 15 \end{pmatrix}$$

• \mathbf{AC}

This does not exist: A is a 2×3 matrix and C is a 4×2 matrix, so the dimensions are not consistent for matrix multiplication.

• \mathbf{CA}

$$\mathbf{CA}$$

$$= \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 12 & 15 \\ 3 & 6 & 9 \\ 4 & 5 & 6 \\ 9 & 12 & 15 \end{pmatrix}$$

- B^2

This does not exist: B is a 3×4 matrix, so the dimensions are not consistent for matrix multiplication.

- BC

$$\begin{aligned} & \mathbf{BC} \\ &= \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 3 \\ 1 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 3 \\ 9 & 6 \\ 1 & 4 \end{pmatrix} \end{aligned}$$

- CB

This does not exist: C is a 4×2 matrix and B is a 3×4 matrix, so the dimensions are not consistent for matrix multiplication.

- C^2

This does not exist: C is a 4×2 matrix, so the dimensions are not consistent for matrix multiplication.

4. (a) $\forall N, M \in \mathbb{Z}^+$

(b) $\forall N, M \in \mathbb{Z}^+$

5.

$$\begin{aligned} & \mathbf{M}^2 \\ &= \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 2 & 2 \\ 4 & 6 & 1 \\ 2 & 2 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} & \mathbf{MM}^T \\ &= \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 3 & 1 \\ 3 & 10 & 4 \\ 1 & 4 & 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} & \mathbf{M}^T \mathbf{M} \\ &= \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 11 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 2 \end{pmatrix} \end{aligned}$$

6.

$$\mathbf{A} \bullet \mathbf{B} = \mathbf{A} \times \mathbf{B}^T$$

$$\begin{aligned} ((\mathbf{A} \bullet \mathbf{B}) \bullet \mathbf{C})_{ij} &= \sum_k (\mathbf{A} \bullet \mathbf{B})_{ik} (\mathbf{C})_{jk} \\ &= \sum_k \sum_l (\mathbf{A})_{il} (\mathbf{B})_{kl} (\mathbf{C})_{jk} \end{aligned}$$

$$\begin{aligned} (\mathbf{A} \bullet (\mathbf{B} \bullet \mathbf{C}))_{ij} &= \sum_k (\mathbf{A})_{ik} (\mathbf{B} \bullet \mathbf{C})_{jk} \\ &= \sum_k \sum_l (\mathbf{A})_{ik} (\mathbf{B})_{jl} (\mathbf{C})_{kl} \\ &= \sum_k \sum_l (\mathbf{A})_{il} (\mathbf{C})_{kl} (\mathbf{B})_{jk} \end{aligned}$$

Since the expressions are different, bullet multiplication is not associative.

7. (a) Consider an arbitrary symmetric matrix \mathbf{A} and an arbitrary antisymmetric matrix \mathbf{B} which have the same dimensionality (so they can be multiplied).

$$\begin{aligned} ab_{ij} &= \sum a_{ik} b_{kj} \\ Tr(\mathbf{AB}) &= \sum ab_{ij} \\ Tr(\mathbf{AB}) &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} \\ 2Tr(\mathbf{AB}) &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} + \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} \\ 2Tr(\mathbf{AB}) &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} + \sum_{i=1}^n \sum_{k=1}^n a_{ki} b_{ki} \text{ since } \forall i, k \in \mathbb{N}. a_{ik} = a_{ki} \\ 2Tr(\mathbf{AB}) &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} - \sum_{i=1}^n \sum_{k=1}^n a_{ki} b_{ik} \text{ since } \forall i, k \in \mathbb{N}. b_{ik} = -b_{ki} \\ 2Tr(\mathbf{AB}) &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} - \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} \text{ by renaming variables in the second sum} \\ 2Tr(\mathbf{AB}) &= 0 \\ Tr(\mathbf{AB}) &= 0 \end{aligned}$$

Since \mathbf{A} and \mathbf{B} were arbitrary, this holds for all pairs of symmetric and antisymmetric matrices of the same dimensionality.

- (b) Consider an arbitrary antisymmetric $N \times N$ matrix \mathbf{B} and an arbitrary $N \times 1$ column vector \mathbf{x} .

$$\begin{aligned}
\mathbf{x}^T \mathbf{A} \mathbf{x} &= \sum_{i=1}^n \sum_{k=1}^n x_{1i}^T a_{ik} x_{k1} \\
2\mathbf{x}^T \mathbf{A} \mathbf{x} &= \sum_{i=1}^n \sum_{k=1}^n x_{1i}^T a_{ik} x_{k1} + \sum_{i=1}^n \sum_{k=1}^n x_{1i}^T a_{ik} x_{k1} \\
2\mathbf{x}^T \mathbf{A} \mathbf{x} &= \sum_{i=1}^n \sum_{k=1}^n x_{1i}^T a_{ik} x_{k1} - \sum_{i=1}^n \sum_{k=1}^n x_{k1} a_{ki} x_{1i}^T \text{ by rearranging and since } \forall i, k \in \mathbb{N}. b_{ik} = -b_{ki} \\
2\mathbf{x}^T \mathbf{A} \mathbf{x} &= \sum_{i=1}^n \sum_{k=1}^n x_{1i}^T a_{ik} x_{k1} - \sum_{i=1}^n \sum_{k=1}^n x_{1k}^T a_{ki} x_{i1} \text{ by the definition of transpose} \\
2\mathbf{x}^T \mathbf{A} \mathbf{x} &= \sum_{i=1}^n \sum_{k=1}^n x_{1i}^T a_{ik} x_{k1} - \sum_{i=1}^n \sum_{k=1}^n x_{1i}^T a_{ik} x_{k1} \text{ renaming variables in the second sum} \\
2\mathbf{x}^T \mathbf{A} \mathbf{x} &= 0 \\
\mathbf{x}^T \mathbf{A} \mathbf{x} &= 0
\end{aligned}$$

8. a_{ji}

I will prove associativity for multiplying a vector by the result of a matrix multiplication: $\mathbf{A}(\mathbf{B}\mathbf{x}) = (\mathbf{A}\mathbf{B})\mathbf{x}$

$$\begin{aligned}
((ab)x)_i &= \sum_{j=1}^n (ab)_{ij} x_j \\
&= \sum_{j=1}^n \left(\sum_{k=1}^n a_{ik} b_{kj} \right) x_j \\
&= \sum_{j=1}^n \sum_{k=1}^n a_{ik} b_{kj} x_j \\
&= \sum_{k=1}^n a_{ik} \sum_{j=1}^n b_{kj} x_j \\
&= \sum_{k=1}^n a_{ik} (bx)_k \\
&= (a(bx))_i
\end{aligned} \tag{1}$$

(a) $\mathbf{A}\mathbf{B}\mathbf{x}$

(b) $\mathbf{B}\mathbf{x}$

(c) $\mathbf{B}\mathbf{A}^T \mathbf{x}$

(d) $\mathbf{A}\mathbf{A}^T \mathbf{A}$

9. (a)

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(b) Consider:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

10.

$$\mathbf{M}^3 = -\mathbf{M} \implies \mathbf{M}^{2n} = (-1)^{n+1} \mathbf{M}^2 \wedge \mathbf{M}^{2n+1} = (-1)^n \mathbf{M}$$

$$\begin{aligned} \exp \theta \mathbf{M} &= \mathbf{I} + \sum_{n=1}^{\infty} \frac{(\theta \mathbf{M})^n}{n!} \\ &= \mathbf{I} + \sum_{n=0}^{\infty} \frac{(\theta \mathbf{M})^{2n+1}}{(2n+1)!} + \sum_{n=1}^{\infty} \frac{(\theta \mathbf{M})^{2n}}{(2n)!} \\ &= \mathbf{I} + \mathbf{M} \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} + \mathbf{M}^2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \theta^{2n}}{(2n)!} \\ &= \mathbf{I} + \mathbf{M} \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} + \mathbf{M}^2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1} \theta^{2n}}{(2n)!} + \mathbf{M}^2 \\ &= \mathbf{I} + \mathbf{M} \sin \theta - \mathbf{M}^2 \cos \theta + \mathbf{M}^2 \text{ using the Taylor expansion for } \sin \theta \text{ and } \cos \theta \\ &= \mathbf{I} + \mathbf{M} \sin \theta + \mathbf{M}^2 (1 - \cos \theta) \end{aligned}$$

$$\begin{aligned} \exp \theta_1 \mathbf{M} \exp \theta_2 \mathbf{M} &= (\mathbf{I} + \mathbf{M} \sin \theta_1 + \mathbf{M}^2 (1 - \cos \theta_1)) (\mathbf{I} + \mathbf{M} \sin \theta_2 + \mathbf{M}^2 (1 - \cos \theta_2)) \\ &= \mathbf{I} + \mathbf{M} (\sin \theta_1 + \sin \theta_2) + \mathbf{M}^2 (2 - \cos \theta_1 - \cos \theta_2) + \mathbf{M}^2 \sin \theta_1 \sin \theta_2 \\ &\quad + \mathbf{M}^3 (\sin \theta_1 + \sin \theta_2 - 2 \sin \theta_1 \cos \theta_2) + \mathbf{M}^4 (1 - \cos \theta_1 - \cos \theta_2 + \cos \theta_1 \cos \theta_2) \\ &= \mathbf{I} + \mathbf{M} (\sin \theta_1 + \sin \theta_2) + \mathbf{M}^2 (2 + \sin \theta_1 \sin \theta_2 - \cos \theta_1 - \cos \theta_2) \\ &\quad - \mathbf{M} (\sin \theta_1 + \sin \theta_2 - 2 \sin \theta_1 \cos \theta_2) - \mathbf{M}^4 (1 - \cos \theta_1 - \cos \theta_2 + \cos \theta_1 \cos \theta_2) \\ &= \mathbf{I} + \mathbf{M} (\sin \theta_1 + \sin \theta_2 - \sin \theta_1 - \sin \theta_2 + 2 \sin \theta_1 \cos \theta_2) \\ &\quad + \mathbf{M}^2 (2 + \sin \theta_1 \sin \theta_2 - \cos \theta_1 - \cos \theta_2 - 1 + \cos \theta_1 + \cos \theta_2 - \cos \theta_1 \cos \theta_2) \\ &= \mathbf{I} + \mathbf{M} (2 \sin \theta_1 \cos \theta_2) + \mathbf{M}^2 (1 + \sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2) \\ &= \mathbf{I} + \mathbf{M} \sin(\theta_1 + \theta_2) + \mathbf{M}^2 (1 - \cos(\theta_1 + \theta_2)) \\ &= \exp(\theta_1 + \theta_2) \mathbf{M} \end{aligned}$$

$$\begin{aligned} \exp \theta \mathbf{M} &= \mathbf{I}^T + \mathbf{M}^T \sin \theta + (\mathbf{M}^2)^T (1 - \cos \theta) \\ &= \mathbf{I} - \mathbf{M} \sin \theta + \mathbf{M}^2 (1 - \cos \theta) \\ &= \exp(-\theta) \mathbf{M} \end{aligned}$$

$$\begin{aligned} (\exp \theta \mathbf{M})(\exp \theta \mathbf{M})^T &= \exp \theta \mathbf{M} \exp(-\theta) \mathbf{M} \\ &= \exp(\theta - \theta) \mathbf{M} \\ &= \exp 0 \mathbf{M} \\ &= \mathbf{I} + \mathbf{M} \sin 0 + \mathbf{M}^2 (1 - \cos 0) \\ &= \mathbf{I} \end{aligned}$$

This result will not hold for general \mathbf{M} . \mathbf{M}

Find out what type of matrix \mathbf{M} is.

11.

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$$

Consider arbitrary

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathbf{b} \times \mathbf{x} = \mathbf{B}\mathbf{x}$$

$$\begin{pmatrix} b_2z - b_3y \\ b_3x - b_1z \\ b_1y - b_2x \end{pmatrix} = \begin{pmatrix} B_{11}x + B_{12}y + B_{13}z \\ B_{21}x + B_{22}y + B_{23}z \\ B_{31}x + B_{32}y + B_{33}z \end{pmatrix}$$

Equating coefficients of x , y and z gives:

$$\begin{array}{lll} B_{11} = 0 & B_{12} = -b_3 & B_{13} = b_2 \\ B_{21} = b_3 & B_{22} = 0 & B_{23} = -b_1 \\ B_{31} = -b_2 & B_{32} = b_1 & B_{33} = 0 \end{array}$$

So:

$$\mathbf{B} = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix}$$

$$\begin{aligned} \mathbf{B}^2\mathbf{x} &= \mathbf{B}(\mathbf{B}\mathbf{x}) \text{ using (1)} \\ &= \mathbf{b} \times (\mathbf{b} \times \mathbf{x}) \end{aligned}$$

$$\begin{aligned}
 \mathbf{B}^2 \mathbf{x} &= \begin{pmatrix} -b_2^2 - b_3^2 & b_1 b_2 & b_1 b_3 \\ b_1 b_2 & -b_1^2 - b_3^2 & b_2 b_3 \\ b_1 b_3 & b_2 b_3 & -b_1^2 - b_2^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
 &= \begin{pmatrix} -b_2^2 x - b_3^2 x + b_1 b_2 y + b_1 b_3 z \\ b_1 b_2 x - b_1^2 y - b_3^2 y + b_2 b_3 z \\ b_1 b_3 x + b_2 b_3 y - b_1^2 z - b_2^2 z \end{pmatrix} \\
 &= \begin{pmatrix} (b_1 x + b_2 y + b_3 z) b_1 \\ (b_1 x + b_2 y + b_3 z) b_2 \\ (b_1 x + b_2 y + b_3 z) b_3 \end{pmatrix} - \begin{pmatrix} (b_1^2 + b_2^2 + b_3^2) x \\ (b_1^2 + b_2^2 + b_3^2) y \\ (b_1^2 + b_2^2 + b_3^2) z \end{pmatrix} \\
 &= (\mathbf{b} \cdot \mathbf{x}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{b}) \mathbf{x}
 \end{aligned}$$

Equating the two equations gives the required result:

$$\mathbf{b} \times (\mathbf{b} \times \mathbf{x}) = (\mathbf{b} \cdot \mathbf{x}) \mathbf{b} - (\mathbf{b} \cdot \mathbf{b}) \mathbf{x}$$

12.

$$\begin{aligned}
 \det \mathbf{A} \det \mathbf{B} &= (4 - 6) \times (4 - 4) \\
 &= -2 \times 0 \\
 &= 0
 \end{aligned}$$

$$\begin{aligned}
 \det \mathbf{AB} &= \det \begin{pmatrix} 4 & 8 \\ 10 & 20 \end{pmatrix} \\
 &= 80 - 80 \\
 &= 0 \\
 &= \det \mathbf{A} \det \mathbf{B}
 \end{aligned}$$

$$\begin{aligned}
 \det \mathbf{A}^{-1} &= \det \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \\
 &= 1 - \frac{3}{2} \\
 &= -\frac{1}{2} \\
 &= \frac{1}{\det \mathbf{A}}
 \end{aligned}$$

13.

$$\begin{aligned}
 \begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} &= A_{11}C_{1,1} + A_{12}C_{1,2} + A_{13}C_{1,3} \\
 &= 1 \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 0 \\ 1 & 0 \end{vmatrix} \\
 &= 0 - 4 + 0 \\
 &= -4
 \end{aligned}$$

$$\begin{aligned}
 \begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} &= \begin{vmatrix} -2 & 2 & 3 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} -2 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{vmatrix} \\
 &= -4
 \end{aligned}$$

$$\begin{aligned}
 |\mathbf{A}^T| &= \begin{vmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix}^T \\
 &= \begin{vmatrix} 1 & 3 & 1 \\ 2 & 0 & 0 \\ 3 & 1 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} -2 & 3 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} -2 & 3 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} -2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix} \\
 &= -2 \times 2 \times 1 \\
 &= -4 \\
 &= |\mathbf{A}|
 \end{aligned}$$

14. The columns of a matrix whose determinant is zero are linearly dependent on each other.

The vectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 1 \\ 3 \\ 2 \end{pmatrix} \quad \begin{pmatrix} 4 \\ 0 \\ 1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 0 \\ 3 \\ a \end{pmatrix}$$

are linearly dependent if and only if the determinant of the matrix with these vectors as its columns is zero.

$$\begin{aligned}
 \begin{vmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 2 \\ 4 & 0 & 1 & 3 \\ 2 & 0 & 3 & a \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 3 \\ 2 & 0 & 3 & a \end{vmatrix} \\
 &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 4 & 0 & -3 & 3 \\ 2 & 0 & 1 & a \end{vmatrix} \\
 &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 1 & a \end{vmatrix} \\
 &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & a+1 \end{vmatrix} \\
 &= -3(a+1)
 \end{aligned}$$

So the vectors are linearly dependent for $a = -1$.

15. We can use row operations to conserve the determinant of the matrix:

$$\begin{aligned}
 \begin{vmatrix} a-E & -b & -b & -b & -b \\ -b & a-E & -b & -b & -b \\ -b & -b & a-E & -b & -b \\ -b & -b & -b & a-E & -b \\ -b & -b & -b & -b & a-E \end{vmatrix} &= \begin{vmatrix} a+b-E & 0 & 0 & 0 & -b \\ 0 & a+b-E & 0 & 0 & -b \\ 0 & 0 & a+b-E & 0 & -b \\ 0 & 0 & 0 & a+b-E & -b \\ E-a-b & E-a-b & E-a-b & E-a-b & a-E \end{vmatrix} \\
 &= \begin{vmatrix} a+b-E & 0 & 0 & 0 & -b \\ 0 & a+b-E & 0 & 0 & -b \\ 0 & 0 & a+b-E & 0 & -b \\ 0 & 0 & 0 & a+b-E & -b \\ 0 & 0 & 0 & 0 & a-4b-E \end{vmatrix} \\
 &= (a+b-E)^4(a-4b-E)
 \end{aligned}$$

So the determinant is zero when:

$$\begin{aligned}
 (a+b-E)^4(a-4b-E) &= 0 \\
 E &= a+b \vee E = a-4b
 \end{aligned}$$

16.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 1 & 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} p \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 3 \\ 6 \end{pmatrix}$$

17.

$$\begin{aligned}
 4w &= 8 \\
 w &= 2
 \end{aligned}$$

$$\begin{aligned}
 2z + w &= 4 \\
 2z + 2 &= 2 \\
 2z &= 2 \\
 z &= 1
 \end{aligned}$$

$$\begin{aligned}
 2y + 3z + 5w &= 19 \\
 2y + 3 + 10 &= 19 \\
 2y &= 6 \\
 y &= 3
 \end{aligned}$$

$$\begin{aligned}
 6x + 7y + 3z - w &= -2 \\
 6x + 21 + 3 - 2 &= -2 \\
 6x &= -24 \\
 x &= -4
 \end{aligned}$$

$$x = -4 \qquad y = 3 \qquad z = 1 \qquad w = 2$$

18.

$$\begin{aligned}x + 3y &= 17 \\3x + 9y &= 51 \\(3x + 9y) - (3x + 2y) &= 51 - 9 \\7y &= 42 \\y &= 6\end{aligned}$$

$$\begin{aligned}x + 3y &= 17 \\x + 18 &= 17 \\x &= -1\end{aligned}$$

$$x = -1 \qquad y = 6$$

$$\begin{aligned}\begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 9 \\ 17 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \frac{1}{7} \begin{pmatrix} 3 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 9 \\ 17 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -1 \\ 6 \end{pmatrix}\end{aligned}$$

19.

$$\begin{aligned}\mathbf{A}^{-1} &= \frac{1}{|\mathbf{A}|} \begin{pmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{pmatrix}^T \\ &= \frac{1}{4} \begin{pmatrix} 1 & 1 & 2 \\ -2 & 2 & 0 \\ -1 & -1 & 2 \end{pmatrix}^T \\ &= \frac{1}{4} \begin{pmatrix} 1 & -2 & -1 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\mathbf{Ax} &= \mathbf{y} \\ \mathbf{x} &= \mathbf{A}^{-1}\mathbf{y} \\ \mathbf{x} &= \frac{1}{4} \begin{pmatrix} 1 & -2 & -1 \\ 1 & 2 & -1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \\ &= \frac{1}{4} \begin{pmatrix} a - 2b - c \\ a + 2b - c \\ 2a + 2c \end{pmatrix}\end{aligned}$$

$$\mathbf{e}_1 = \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad \mathbf{e}_2 = \frac{1}{4} \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix} \qquad \mathbf{e}_3 = \frac{1}{4} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

The matrix whose columns are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is the inverse matrix of \mathbf{A} .

20.

$$a_i \cdot b_j = \delta_{ij}$$

Consider $a_i \cdot b_i$:

Case $i = j$:

$$\begin{aligned} a_j \cdot b_i &= 1 \\ &= \frac{[a_1, a_2, a_3]}{[a_1, a_2, a_3]} \\ &= \frac{a_i \cdot a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \\ &= a_j \cdot \frac{a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \end{aligned}$$

Case $i \neq j$:

$$\begin{aligned} a_j \cdot b_i &= 0 \\ &= \frac{a_j \cdot a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \\ &= a_j \cdot \frac{a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \end{aligned}$$

$$\begin{aligned} \forall \mathbf{A}, \mathbf{B} \in \mathbb{R}^{3 \times 3}. \mathbf{AB} = \mathbf{I} &\implies \\ \forall i, j \in \{1, 2, 3\}. a_j \cdot b_i &= a_j \cdot \frac{a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \implies \\ \forall i, j \in \{1, 2, 3\}. b_i &= \frac{a_{(i+1)\%3} \times a_{(i+2)\%3}}{[a_1, a_2, a_3]} \end{aligned}$$

Substituting $i = 1, 2, 3$ into these formulae gives the required result:

$$b_1 = \frac{a_2 \times a_3}{[a_1, a_2, a_3]} \quad b_2 = \frac{a_3 \times a_1}{[a_1, a_2, a_3]} \quad b_3 = \frac{a_1 \times a_2}{[a_1, a_2, a_3]}$$

