6. For the function $f(x, y, z) = \ln\left(x^2 + y^2\right) + z$, find ∇f.

$$\nabla f = \left(\frac{2x}{x^2 + y^2}, \frac{2y}{x^2 + y^2}, 1\right) \tag{1}$$

(a) At the point $(3, -4, 0)$ the normal to the cylinder is trivially in the direction $\begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix}$. The magnitude of this vector is 5 and so $\hat{n}$ is $\begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix}$. Using this we can work out the value of f at this point.

$$\begin{aligned}
(\nabla f)(3, -4, 4) \cdot \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} &= \begin{pmatrix} \frac{6}{9+16} \\ -\frac{8}{9+16} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} \\
(\nabla f)(3, -4, 4) \cdot \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} &= \frac{6}{25} \times \frac{3}{5} + \frac{8}{25} \times \frac{4}{5} \\
(\nabla f)(3, -4, 4) \cdot \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} &= \frac{18}{125} + \frac{32}{125} \\
(\nabla f)(3, -4, 4) \cdot \begin{pmatrix} \frac{3}{5} \\ -\frac{4}{5} \\ 0 \end{pmatrix} &= \frac{2}{5}
\end{aligned} \tag{2}$$

(b)

$$\hat{\mathbf{m}} = \frac{\mathbf{m}}{|\mathbf{m}|} = \frac{\mathbf{m}}{\sqrt{5}} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \tag{3}$$

$$\begin{aligned}
(\nabla f)(3, -4, 4) \cdot \mathbf{m} &= \begin{pmatrix} \frac{6}{9+16} \\ -\frac{8}{9+16} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \\ 0 \end{pmatrix} \\
(\nabla f)(3, -4, 4) \cdot \mathbf{m} &= \frac{6}{25} \times \frac{1}{\sqrt{5}} - \frac{8}{25} \times \frac{2}{\sqrt{5}} \\
(\nabla f)(3, -4, 4) \cdot \mathbf{m} &= \frac{6}{25\sqrt{5}} - \frac{16}{25\sqrt{5}} \\
(\nabla f)(3, -4, 4) \cdot \mathbf{m} &= -\frac{2\sqrt{5}}{25}
\end{aligned} \tag{4}$$

7.

$$\begin{aligned}
f &= xz + z^2 - xy^2 \\
\nabla f &= \left(z - y^2, -2xy, x + 2z\right) \\
(\nabla f)(1, 1, 2) &= (1, -2, 5)
\end{aligned} \tag{5}$$

So a normal vector to the surface at the point (1, 1, 2) is $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$.

Hence the equation of the tangent plane at this point is:

$$\begin{aligned}
\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \\
x - 2y + 5z &= 9
\end{aligned} \tag{6}$$

8.

$$
\begin{aligned}
f &= 3x^2y\sin\left(\frac{\pi x}{2}\right) - z \\
\nabla f &= \left(6xy\sin\left(\frac{\pi x}{2}\right) + \frac{3\pi x^2 y}{2}\cos\left(\frac{\pi x}{2}\right), 3x^2\sin\left(\frac{\pi x}{2}\right), -1\right)
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
z(1,1) &= 3\sin\left(\frac{\pi}{2}\right) \\
z(1,1) &= 3
\end{aligned} \tag{8}
$$

$$
(\nabla f)(1,1,3) = (6,3,-1) \tag{9}
$$

So the equation of the plane which is tangent to the surface at the point (1, 1, 3) is:

$$
\begin{aligned}
\begin{pmatrix}6\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}x\\y\\z\end{pmatrix} &= \begin{pmatrix}6\\3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\3\end{pmatrix} \\
6x + 3y - z &= 6
\end{aligned} \tag{10}
$$

At the point with $x = 1, y = \frac{1}{2}$, $z = \frac{3}{2}$.

$$
(\nabla f)\left(1, \frac{1}{2}, \frac{3}{2}\right) = (3,3,-1) \tag{11}
$$

Consider now $-n$. This is also a normal to the plane but facing in the increasing z direction (I assume that we are placing the marble on the plane from above – not suspending it below the plane). Now the normal is $\begin{pmatrix}-3\\-3\\1\end{pmatrix}$. so if we place a marble on the plane then it will roll South-West.

9. Consider the substitution:

$$
\begin{aligned}
x &= a\cos\theta \\
\mathrm{d}x &= -a\sin\theta\,\mathrm{d}\theta \\
y &= a\sin\theta \\
\mathrm{d}y &= a\cos\theta\,\mathrm{d}\theta
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
&\int_\Gamma [P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y] \\
=&\int_\Gamma [-x^2y\,\mathrm{d}x + xy^2\,\mathrm{d}y] \\
=&\int_0^\pi [a^4\cos^2\theta\sin^2\theta\,\mathrm{d}\theta + a^4\cos^2\theta\sin^2\theta\,\mathrm{d}\theta] \\
=&a^4\int_0^\pi 2\cos^2\theta\sin^2\theta\,\mathrm{d}\theta \\
=&a^4\int_0^\pi \frac{1}{2}\sin^2 2\theta\,\mathrm{d}\theta \\
=&a^4\int_0^\pi \frac{1}{4} - \frac{1}{4}\cos 4\theta\,\mathrm{d}\theta \\
=&a^4\left[\frac{1}{4}\theta - \frac{1}{16}\sin 4\theta\right]_0^\pi \\
=&\frac{\pi a^4}{4}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
&\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y \\
=&\int\int y^2 + x^2 \,\mathrm{d}x\,\mathrm{d}y \\
=&\int_0^\pi \int_0^a r^2 \times r \,\mathrm{d}r\,\mathrm{d}\theta \\
=&\int_0^\pi \int_0^a r^3 \,\mathrm{d}r\,\mathrm{d}\theta \\
=&\int_0^\pi \left[\frac{1}{4}r^4\right]_0^a \mathrm{d}\theta \\
=&\int_0^\pi \frac{a^4}{4}\,\mathrm{d}\theta \\
=&\frac{\pi a^4}{4} \\
=&\int_\Gamma [P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y] \text{ as required}
\end{aligned} \tag{14}
$$

10.

$$
\int_\Gamma [f(y)\,\mathrm{d}x + x\cos y\,\mathrm{d}y] = 0 \tag{15}
$$

The integrand of an exact differential along any closed contour is equal to zero. So if $\int_\Gamma [f(y)\,\mathrm{d}x + x\cos y\,\mathrm{d}y]$ is an exact differential, then it will be equal to zero for all closed contours Γ.

For the integrand to be an exact differential:

$$
\begin{aligned}
\left(\frac{\partial}{\partial y}\right) f(y) &= \left(\frac{\partial}{\partial x}\right) x\cos y \\
\left(\frac{\partial}{\partial y}\right) f(y) &= \cos y \\
f(y) &= \int \cos y\,\mathrm{d}y \\
f(y) &= \sin y + c
\end{aligned} \tag{16}
$$

11. (i) For the purposes of this question I will assume the curve is *closed* since the statement is untrue if it is not.

If $\mathbf{F} = -\nabla(\Phi)$ for some Φ, then $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{x} = 0$ for all closed curves C.

$$
\begin{aligned}
&\mathbf{F} = \mathbf{c}\times\mathbf{v} \\
&\mathbf{F} = -\mathbf{c}\left(-\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t}\right) \\
&\mathbf{F} = -\left(\nabla(-\mathbf{x}(t))\right) \\
&\mathbf{F} = -\nabla\Phi
\end{aligned} \tag{17}
$$

This is a necessary and sufficient condition for the $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{x}$ to be equal to zero for all closed curves C.

So the integral $\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{x} = 0$ for all closed curves C.

The integral is equal to the work done. So the work done is equal to 0 for all closed curves C.

(ii) (a)

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} \\ &= \int_0^\pi (y, -x, -1) \cdot (-\sin t, \cos t, 1)\, \mathrm{d}t \\ &= \int_0^\pi (\sin t, -\cos t, -1) \cdot (-\sin t, \cos t, 1)\, \mathrm{d}t \\ &= \int_0^\pi -\sin^2 t - \cos^2 t - 1\, \mathrm{d}t \\ &= \int_0^\pi -1 - 1\, \mathrm{d}t \\ &= \int_0^\pi -2\, \mathrm{d}t \\ &= -2\pi \end{aligned} \tag{18}$$

(b)

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} \\ &= \int_0^\pi (x, y, 0) \cdot (-\sin t, \cos t, 1)\, \mathrm{d}t \\ &= \int_0^\pi -\sin t \cos t + \sin t \cos t\, \mathrm{d}t \\ &= \int_0^\pi 0\, \mathrm{d}t \\ &= 0 \end{aligned} \tag{19}$$

12. A condition satisfied by all conservative vector fields $\mathbf{F} = (P(x,y), Q(x,y))$ is that $\mathbf{F}$ is an exact differential. So:

$$\left(\frac{\partial}{\partial y}\right) P(x,y) = \left(\frac{\partial}{\partial x}\right) Q(x,y) \tag{20}$$

(a)

$$P = x^2 y + y,\ Q = xy^2 + x \tag{21}$$

$$\begin{aligned} \left(\frac{\partial}{\partial y}\right) P(x,y) &= x^2 + 1 \\ \left(\frac{\partial}{\partial x}\right) Q(x,y) &= y^2 + 1 \\ \left(\frac{\partial}{\partial y}\right) P(x,y) &\neq \left(\frac{\partial}{\partial x}\right) Q(x,y) \end{aligned} \tag{22}$$

So $\mathbf{F}$ is **not** a conservative vector field

(b)

$$P = ye^{xy} + 2x + y,\ Q = xe^{xy} + x \tag{23}$$

$$\begin{aligned} \left(\frac{\partial}{\partial y}\right) P(x,y) &= xye^{xy} + e^{xy} + 1 \\ \left(\frac{\partial}{\partial x}\right) Q(x,y) &= xye^{xy} + e^{xy} + 1 \\ \left(\frac{\partial}{\partial y}\right) P(x,y) &= \left(\frac{\partial}{\partial x}\right) Q(x,y) \end{aligned} \tag{24}$$

So $\mathbf{F}$ is a conservative vector field

13. (i)

$$(0,0,0) \longrightarrow (0,0,1)$$
$$x = y = 0, z = t$$
$$\mathrm{d}\mathbf{x} = (0,0,1)$$

$$\begin{aligned}
&\int \mathbf{F} \cdot \mathrm{d}\mathbf{x} \\
=&\int_0^1 (4x^3z + 2x, z^2 - 2y, x^4 + 2yz) \cdot (0,0,1)\,\mathrm{d}t \\
=&\int_0^1 x^4 + 2yz\,\mathrm{d}t \\
=&\int_0^1 0\,\mathrm{d}t \\
=&0
\end{aligned} \tag{25}$$

$$(0,0,1) \longrightarrow (0,1,1)$$
$$x = 0, y = t, z = 1$$
$$\mathrm{d}\mathbf{x} = (0,1,0)$$

$$\begin{aligned}
&\int \mathbf{F} \cdot \mathrm{d}\mathbf{x} \\
=&\int_0^1 (4x^3z + 2x, z^2 - 2y, x^4 + 2yz) \cdot (0,1,0)\,\mathrm{d}t \\
=&\int_0^1 z^2 - 2y\,\mathrm{d}t \\
=&\int_0^1 1 - 2t\,\mathrm{d}t \\
=&0
\end{aligned} \tag{26}$$

$$(0,1,1) \longrightarrow (1,1,1)$$
$$x = t, y = z = 1$$
$$\mathrm{d}\mathbf{x} = (1,0,0)$$

$$\begin{aligned}
&\int \mathbf{F} \cdot \mathrm{d}\mathbf{x} \\
=&\int_0^1 (4x^3z + 2x, z^2 - 2y, x^4 + 2yz) \cdot (1,0,0)\,\mathrm{d}t \\
=&\int_0^1 4x^3z + 2x\,\mathrm{d}t \\
=&\int_0^1 4t^3 + 2t\,\mathrm{d}t \\
=&[t^4 + t^2]_0^1 \\
=&2
\end{aligned} \tag{27}$$

$$0 + 0 + 2 = 2 \tag{28}$$

So the line integral along the sequence of straight line paths is 2.

(ii)

$$\begin{aligned} x = y = z &= t \\ \mathrm{d}\mathbf{x} &= (1,1,1) \end{aligned} \tag{29}$$

$$\begin{aligned} &\int \mathbf{F}\cdot \mathrm{d}\mathbf{x} \\ =&\int_0^1 (4x^3z + 2x, z^2 - 2y, x^4 + 2yz)\cdot(1,1,1)\,\mathrm{d}t \\ =&\int_0^1 4x^3z + 2x + z^2 - 2y + x^4 + 2yz\,\mathrm{d}t \\ =&\int_0^1 4t^4 + 2t + t^2 - 2t + t^4 + 2t^2\,\mathrm{d}t \\ =&\int_0^1 5t^4 + 3t^2\,\mathrm{d}t \\ =&[t^5 + t^3]_0^1 \\ =&2 \end{aligned} \tag{30}$$

A function $f(x,y,z)$ such that $\mathbf{F} = \nabla f$ is:

$$f(x,y,z) = x^4z + x^2 + yz^2 - y^2 + c \tag{31}$$

14. To integrate C, I will make the substitution:

$$\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \\ z &= 0 \\ \mathrm{d}\mathbf{x} &= (-\sin\theta, \cos\theta, 0)\,\mathrm{d}\theta \end{aligned} \tag{32}$$

$$\begin{aligned} &\int_C \mathbf{E}\cdot\mathrm{d}\mathbf{x} \\ =&\int_0^{2\pi} (-ye^{-2t}, xe^{-2t}, 0)\cdot(-\sin\theta,\cos\theta,0)\,\mathrm{d}\theta \\ =&\int_0^{2\pi} (-\sin\theta e^{-2t}, \cos\theta e^{-2t}, 0)\cdot(-\sin\theta,\cos\theta,0)\,\mathrm{d}\theta \\ =&\int_0^{2\pi} \sin^2\theta e^{-2t} + \cos^2\theta e^{-2t}\,\mathrm{d}\theta \\ =&\int_0^{2\pi} (\sin^2\theta + \cos^2\theta)e^{-2t}\,\mathrm{d}\theta \\ =&\int_0^{2\pi} e^{-2t}\,\mathrm{d}\theta \\ =&2\pi e^{-2t} \end{aligned} \tag{33}$$

$$
\begin{aligned}
&-\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B}\cdot \mathrm{d}\mathbf{S} \\
=&-\frac{\mathrm{d}}{\mathrm{d}t}\int_S (0,0,e^{-2t})\cdot \hat{n}\,\mathrm{d}S \\
=&-\frac{\mathrm{d}}{\mathrm{d}t}\int_S (0,0,e^{-2t})\cdot (0,0,1)\,\mathrm{d}S \\
=&-\frac{\mathrm{d}}{\mathrm{d}t}\int_S e^{-2t}\,\mathrm{d}S \\
=&-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\int_0^1 re^{-2t}\,\mathrm{d}r\,\mathrm{d}\theta \\
=&-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\left[\frac{1}{2}r^2e^{-2t}\right]_0^1\mathrm{d}\theta \\
=&-\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi}\frac{1}{2}e^{-2t}\,\mathrm{d}\theta \\
=&-\frac{\mathrm{d}}{\mathrm{d}t}\left(2\pi\times\frac{1}{2}e^{-2t}\right) \\
=&-\frac{\mathrm{d}}{\mathrm{d}t}\left(\pi e^{-2t}\right) \\
=&-\left(-2\pi e^{-2t}\right) \\
=&2\pi e^{-2t} \\
=&\int_C \mathbf{E}\cdot \mathrm{d}\mathbf{x} \text{ as required}
\end{aligned} \tag{34}
$$