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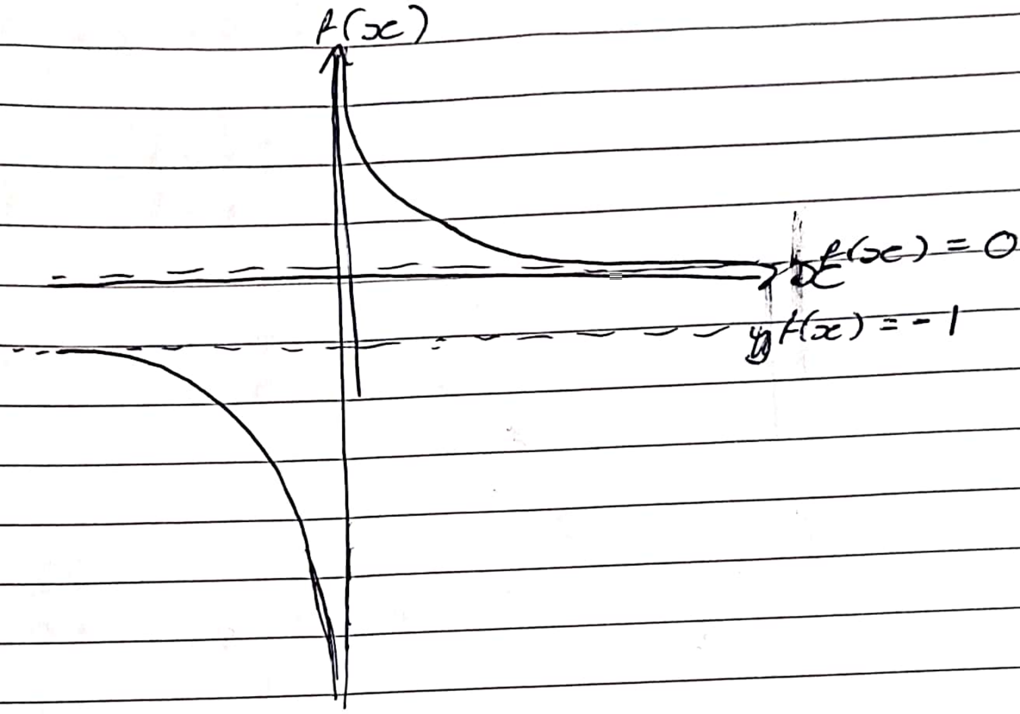
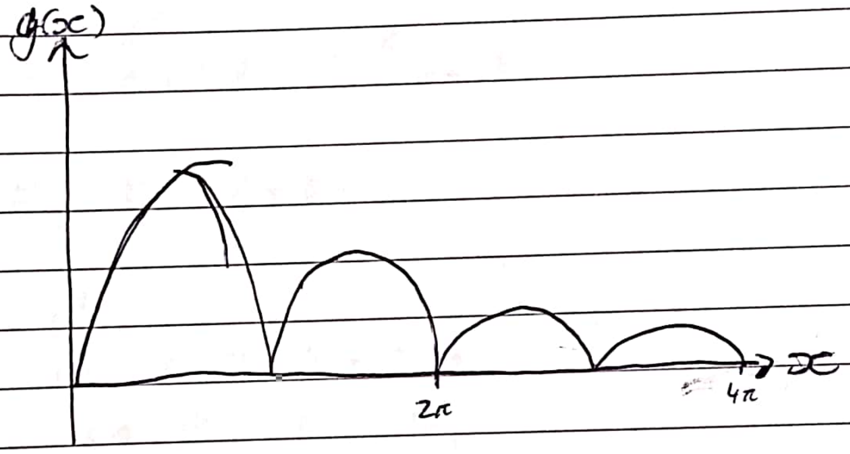
Section A

1. a)



b)



2. $\sin^3\theta - \frac{1}{4}\sin\theta = \sin^2\theta - \frac{1}{4}$

$4\sin^3\theta - 4\sin^2\theta - \sin\theta + 1 = 0$

at $\cancel{\sin}\,\theta = \frac{\pi}{2}$: $\sin\theta = 1$

$(\sin\theta - 1)(4\sin^2\theta - 1) = 0$

So $\sin\theta = 1$

or $4\sin^2\theta = 1$

$\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$4\sin^2\theta = 1 \Rightarrow$

$\sin\theta = \pm\frac{1}{2} \Rightarrow$

$\therefore \theta = \frac{\pi}{6}$ or $\theta = \frac{5\pi}{6}$ or $\theta = -\frac{\pi}{6}$ or $\theta = -\frac{5\pi}{6}$

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3.a) $\frac{d}{dx}(\ln(\ln x))$

$$= \frac{\frac{d}{dx}(\ln x)}{\ln x}$$

$$= \frac{1}{x \ln x}$$

b) $\frac{d}{dx}\left(\int_0^x e^{-y^2} \sin y \, dy\right)$

$$= \frac{d}{dx}(x)\, e^{-x^2} \sin x$$

$$= e^{-x^2} \sin x$$

4. $\frac{dy}{dx} = \cancel{x}2 + 2x$

at $x = 1$; $\frac{dy}{dx} = 4$

the gradient of the normal is the negative inverse of the gradient of the tangent

So $m_{norm} = -\frac{1}{4}$

$$y - 3 = -\frac{1}{4}(x - 1)$$

$$y = -\frac{1}{4}x + \frac{1}{4} + 3$$

$$y = -\frac{1}{4}x + \frac{13}{4}$$

5. a) ~~Let~~ ~~$z=$~~

$$|i-1| = \sqrt{2}$$

$$\arg(i-1) = \frac{3\pi}{4}$$

So $$i-1 = \sqrt{2}\, e^{\frac{3\pi}{4} i}$$

$$\sqrt{i-1} = \sqrt{\sqrt{2}\, e^{\frac{3\pi}{4} i}}$$

$$= 2^{\frac{1}{4}} \times e^{\frac{3\pi i}{8} + k\pi i}$$

$$= 2^{\frac{1}{4}} e^{\frac{3\pi i}{8}} \text{ or } 2^{\frac{1}{4}} \times e^{-\frac{5\pi i}{8}}$$

for $-\pi < \theta \leq \pi$

b) $$(1+i)$$

$$= \sqrt{2}\, e^{\frac{\pi}{4} i}$$

$$(1+i)^{10} = \left(\sqrt{2}\, e^{\frac{\pi}{4} i}\right)^{10}$$

$$= 2^{5} e^{\frac{10\pi}{4} i}$$

$$= 32\, e^{\frac{\pi}{2} i}$$

$$= 32i$$

So $\operatorname{Re}((1+i)^{10}) = 0$

$\operatorname{Im}((1+i)^{10}) = 32$

6. $x^4 - 3x^2 + 2 = 0$

$(x^2 - 2)(x^2 - 1) = 0$

$x^2 = 2$ or $x^2 = 1$

So $x = \sqrt{2}$, $x = -\sqrt{2}$, $x = -1$, $x = 1$ are the solutions

7. $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\sinh x = \dfrac{e^x - e^{-x}}{2}$

LHS

$$\tanh(-x) = \frac{\sinh(-x)}{\cosh(-x)}$$

$$= \frac{\left(\frac{e^{-x} - e^{--x}}{2}\right)}{\left(\frac{e^{-x} + e^{--x}}{2}\right)}$$

$$= \frac{\left(\frac{e^{-x} - e^{x}}{2}\right)}{\left(\frac{e^{x} + e^{-x}}{2}\right)}$$

$$= \frac{-\sinh x}{\cosh x}$$

$$= -\tanh x$$ as required.

8. a) $\displaystyle\int_{-1000}^{1000} \tanh^5 x \, dx$

$\displaystyle = \int_{0}^{1000} \tanh^5 x \, dx + \int_{-1000}^{0} \tanh^5 x \, dx$

$\displaystyle = \int_{0}^{1000} \tanh^5 x \, dx + \int_{0}^{1000} \tanh(-x) \, dx$

$\displaystyle = \int_{0}^{1000} \tanh^5 x + \tanh^5(-x) \, dx$

$\displaystyle = \int_{0}^{1000} \tanh^5 x - \tanh^5 x \, dx$

$\displaystyle = \int_{0}^{1000} 0 \, dx$

$= 0$

So since $\tanh^5 x$ is an odd function:

$\displaystyle\int_{-1000}^{1000} \tanh^5 x \, dx = 0$

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8.a) $\cancel{\int_{-1000}^{1000} \tanh^5 x \, dx = 0}$

~~since tanh x is an odd function.~~

$\cancel{\forall a \in \mathbb{R}: \int_{-a}^{a} \tanh^5 x \, dx = 0}$

b) $$\int \frac{1}{3-2x} \, dx$$

$$= -\frac{1}{2} \ln |3-2x| + C$$

$\cancel{\neq}$

$\cancel{\pm 1 - \frac{1}{2}\ln}$

9, a) $$\sum_{n=-100}^{99} (n+1)^3 = 0$$

$$\cancel{= \sum_{-2k}^{2k-1} (n+1)^3}$$

b) $$\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{3k} = \frac{a(r^n - 1)}{r-1}$$

$$= \frac{a(1-r^n)}{1-r}$$

as $k \to \infty$; $\left(\frac{1}{8}\right)^k \to 0$

$$= \frac{a}{1-r}$$

$$= \frac{1}{1-\frac{1}{8}}$$

$$= \frac{8}{7}$$

10. $f(x) = x^{\frac{1}{x}}$

$$= e^{\frac{\ln x}{x}}$$

$$f'(x) = \frac{d}{dx}\left(\frac{\ln x}{x}\right) e^{\frac{\ln x}{x}}$$

$$= \left(\frac{\frac{x}{x} - \ln x}{x^2}\right) e^{\frac{\ln x}{x}}$$

$$= \frac{1 - \ln x}{x^2} \, e^{\frac{\ln x}{x}}$$

$e^{\frac{\ln x}{x}} \neq 0$

So at $f'(x) = 0$.

$$\frac{1 - \ln x}{x^2} = 0$$

$1 - \ln x = 0$ n $x^2 \neq 0$

~~kE~~ $\ln x = 1$

$$x = e$$

so $f(x)$ has a stationary point at e.

$f(e) = e^{\frac{1}{e}}$