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Fourier Series

21.

$$\begin{aligned}
&\int_{-1}^{1} 1 \times x \,\mathrm{d}x \\
=&\int_{-1}^{1} x \,\mathrm{d}x \\
=&\left[\frac{1}{2}x^2\right]_{-1}^{1} \\
=&\frac{1}{2} - \frac{1}{2} \\
=&0
\end{aligned}$$

So 1 is orthogonal to x on the interval $[-1, 1]$.

$$\begin{aligned}
&\int_{-1}^{1} 1 \times \frac{1}{2}(3x^2 - 1) \,\mathrm{d}x \\
=&\int_{-1}^{1} \frac{1}{2}(3x^2 - 1) \,\mathrm{d}x \\
=&\left[\frac{1}{2}(x^3 - x)\right]_{-1}^{1} \\
=&0 - 0 \\
=&0
\end{aligned}$$

So 1 is orthogonal to $\frac{1}{2}(3x^2 - 1)$ on the interval $[-1, 1]$.

$$\begin{aligned}
&\int_{-1}^{1} 1 \times \frac{1}{2}(5x^3 - 3x) \,\mathrm{d}x \\
=&\int_{-1}^{1} \frac{1}{2}(5x^3 - 3x) \,\mathrm{d}x \\
=&\left[\frac{1}{8}(5x^4 - 6x^2)\right]_{-1}^{1} \\
=&-\frac{1}{8} - -\frac{1}{8} \\
=&0
\end{aligned}$$

So 1 is orthogonal to $\frac{1}{2}(5x^3 - 3x)$ on the interval $[-1, 1]$.

$$\begin{aligned}
&\int_{-1}^{1} x \times \frac{1}{2}(3x^2 - 1) \,\mathrm{d}x \\
=&\int_{-1}^{1} \frac{1}{2}(3x^3 - x) \,\mathrm{d}x \\
=&\left[\frac{1}{8}(3x^4 - 2x^2)\right]_{-1}^{1} \\
=&\frac{1}{8} - \frac{1}{8} \\
=&0
\end{aligned}$$

So x is orthogonal to $\frac{1}{2}(3x^2 - 1)$ on the interval $[-1, 1]$.

$$\begin{aligned}&\int_{-1}^{1} x \times \frac{1}{2}(5x^3 - 3x)\,\mathrm{d}x\\ =&\int_{-1}^{1} \frac{1}{2}(5x^4 - 3x^2)\,\mathrm{d}x\\ =&\left[\frac{1}{2}(x^5 - x^2)\right]_{-1}^{1}\\ =&0 - 0\\ =&0\end{aligned}$$

So x is orthogonal to $\frac{1}{2}(5x^3 - 3x)$ on the interval $[-1, 1]$.

$$\begin{aligned}&\int_{-1}^{1} \frac{1}{2}(3x^2 - 1) \times \frac{1}{2}(5x^3 - 3x)\,\mathrm{d}x\\ =&\int_{-1}^{1} \frac{1}{4}(15x^5 - 14x^3 + 3x)\,\mathrm{d}x\\ =&\left[\frac{1}{8}(5x^6 - 7x^4 + 3x^2)\right]_{-1}^{1}\\ =&\frac{1}{8} - \frac{1}{8}\\ =&0\end{aligned}$$

So $\frac{1}{2}(3x^2 - 1)$ is orthogonal to $\frac{1}{2}(5x^3 - 3x)$ on the interval $[-1, 1]$.

So the functions 1, x, $\frac{1}{2}(3x^2 - 1)$ and $\frac{1}{2}(5x^3 - 3x)$ are orthogonal on the interval $[-1, 1]$.

22.

$$\begin{aligned}&\int_{0}^{a} \sin(mx)\sin(nx)\,\mathrm{d}x\\ =&\frac{1}{2}\int_{0}^{a} \cos((m-n)x) - \cos((m+n)x)\,\mathrm{d}x\\ =&\frac{1}{2}\left[\frac{1}{m-n}\sin((m-n)x) - \frac{1}{m+n}\sin((m+n)x)\right]_{0}^{a}\\ =&\frac{1}{2}\left(\frac{1}{m-n}\sin((m-n)k\pi) - \frac{1}{m+n}\sin((m+n)k\pi) - \frac{1}{m-n}\sin 0 + \frac{1}{m+n}\sin 0\right)\\ =&\frac{1}{2} \times 0\\ =&0\end{aligned}$$

23. For $m \neq n$:

$$\begin{aligned}
&\int_{-T}^{T} \sin\left(\frac{m\pi\theta}{T}\right)\sin\left(\frac{n\pi\theta}{T}\right) \mathrm{d}\theta \\
=&\frac{1}{2}\int_{-T}^{T} \cos\left(\frac{(m-n)\pi\theta}{T}\right) - \cos\left(\frac{(m+n)\pi\theta}{T}\right) \mathrm{d}\theta \\
=&\frac{1}{2}\left[\frac{T}{(m-n)\pi}\sin\left(\frac{(m-n)\pi\theta}{T}\right) - \frac{T}{(m+n)\pi}\sin\left(\frac{(m+n)\pi\theta}{T}\right)\right]_{-T}^{T} \\
=&\frac{1}{2}\left(\frac{T}{(m-n)\pi}\left(\sin\left((m-n)\pi\right) - \sin\left(-(m-n)\pi\right)\right) - \frac{T}{(m+n)\pi}\left(\sin\left((m+n)\pi\right) - \sin\left(-(m+n)\pi\right)\right)\right) \\
=&\frac{1}{2}\left(0\right) \\
=&0
\end{aligned}$$

For $m = n$:

$$\begin{aligned}
&\int_{-T}^{T} \sin\left(\frac{m\pi\theta}{T}\right)\sin\left(\frac{n\pi\theta}{T}\right) \mathrm{d}\theta \\
=&\int_{-T}^{T} \sin^2\left(\frac{n\pi\theta}{T}\right) \mathrm{d}\theta \\
=&\frac{1}{2}\int_{-T}^{T} 1 - \cos\left(\frac{2n\pi\theta}{T}\right) \mathrm{d}\theta \\
=&\frac{1}{2}\left[\theta - \frac{T}{2n\pi}\sin\left(\frac{2n\pi\theta}{T}\right)\right]_{-T}^{T} \\
=&\frac{1}{2}\left(T - 0 - -T + 0\right) \\
=&T
\end{aligned}$$

24.

$$\sin 2\theta = \sin 2\theta$$

$$\cos^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

25. Note that $|x|$ is an even function and the period is ℓ, so the Fourier Series is of the form $a_0 + \sum_{i=1}^{\infty}\cos\left(\frac{i\pi x}{\ell}\right)$.

The constant is given by:

$$\begin{aligned}
a_0 =& \int_{-\ell}^{\ell} |x| \,\mathrm{d}x \\
=&\frac{2}{\ell}\int_{0}^{\ell} x \,\mathrm{d}x \\
=&\frac{2}{\ell}\left[\frac{1}{2}x^2\right]_{0}^{\ell} \\
=&\ell
\end{aligned}$$

The coefficients a_i are given by:

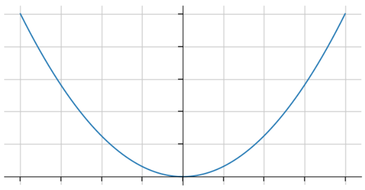
$$\begin{aligned}
a_n &= \frac{1}{\ell}\int_{-\ell}^{\ell} |x| \cos\left(\frac{n\pi x}{\ell}\right) \mathrm{d}x \\
&= \frac{2}{\ell}\int_{0}^{\ell} x \cos\left(\frac{n\pi x}{\ell}\right) \mathrm{d}x \\
&= \frac{2}{\ell}\left[\frac{\ell}{n\pi} x \sin\left(\frac{n\pi x}{\ell}\right) + \left(\frac{\ell}{n\pi}\right)^2 \cos\left(\frac{n\pi x}{\ell}\right)\right]_0^{\ell} \\
&= \frac{2}{\ell}\left(\frac{\ell}{n\pi} x \sin(n\pi) + \left(\frac{\ell}{n\pi}\right)^2 \cos(n\pi) - \frac{\ell}{n\pi} x \sin 0 - \left(\frac{\ell}{n\pi}\right)^2 \cos 0\right) \\
&= \frac{2}{\ell}\left(\frac{\ell}{n\pi}\right)^2 (\cos(n\pi) - 1) \\
a_n &= \begin{cases} 0 & \text{if } n\%2 = 0 \\ -\frac{4\ell}{(n\pi)^2} & \text{if } n\%2 = 1 \end{cases}
\end{aligned}$$

So the Fourier Series for the function that equals $|x|$ when $\ell \leq x \leq \ell$ and is periodic with period 2ℓ is equal to:

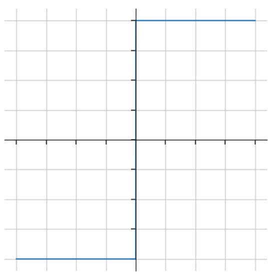
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$f(x) = \frac{\ell}{2} - \frac{4\ell}{\pi^2}\sum_{n=0}^{\infty} \frac{\cos\left(\frac{(2n+1)\pi x}{\ell}\right)}{(2n+1)^2}$$

If the series is integrated then the resulting series will sum to $\int |x| \, \mathrm{d}x$.



If the series is differentiated then the resulting series will sum to $\frac{\mathrm{d}}{\mathrm{d}x}|x|$. This is -1 if $x < 0$ and 1 if $x > 0$.



26. (a)

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \,\mathrm{d}x \\ &= \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left(e^{\pi} - e^{-\pi}\right) \end{aligned}$$

$$\begin{aligned} \int e^x \cos(nx)\,\mathrm{d}x &= e^x\cos(nx) + n\int e^x \sin(nx)\,\mathrm{d}x \\ \int e^x \sin(nx)\,\mathrm{d}x &= e^x\sin(nx) - n\int e^x \cos(nx)\,\mathrm{d}x \\ \int e^x \cos(nx)\,\mathrm{d}x &= e^x\cos(nx) + ne^x\sin(nx) - n^2\int e^x \cos(nx)\,\mathrm{d}x \\ (n^2+1)\int e^x \cos(nx)\,\mathrm{d}x &= e^x\cos(nx) + ne^x\sin(nx) \\ \int e^x \cos(nx)\,\mathrm{d}x &= \frac{1}{n^2+1}(e^x\cos(nx) + ne^x\sin(nx)) \\ \int e^x \sin(nx)\,\mathrm{d}x &= e^x\sin(nx) - ne^x\cos(nx) - n^2\int e^x \sin(nx)\,\mathrm{d}x \\ (n^2+1)\int e^x \sin(nx)\,\mathrm{d}x &= e^x\sin(nx) - ne^x\cos(nx) \\ \int e^x \sin(nx)\,\mathrm{d}x &= \frac{1}{n^2+1}(e^x\sin(nx) - ne^x\cos(nx)) \end{aligned}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos(nx)\,\mathrm{d}x \\ &= \frac{1}{(n^2+1)\pi}\left[e^x\cos(nx) + ne^x\sin(nx)\right]_{-\pi}^{\pi} \\ &= \frac{1}{(n^2+1)\pi}\left(e^{\pi}\cos(n\pi) - e^{-\pi}\cos(n\pi)\right) \\ &= \frac{(-1)^n(e^{\pi} - e^{-\pi})}{(n^2+1)\pi} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin(nx)\,\mathrm{d}x \\ &= \frac{1}{(n^2+1)\pi}\left[e^x\sin(nx) - ne^x\cos(nx)\right]_{-\pi}^{\pi} \\ &= \frac{1}{(n^2+1)\pi}(-ne^{\pi}\cos(n\pi) + ne^{-\pi}\cos(n\pi)) \\ &= \frac{(-1)^{n+1}n(e^{\pi} - e^{-\pi})}{(n^2+1)\pi} \end{aligned}$$

$$f(x) = \frac{e^{\pi} - e^{-\pi}}{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos(nx) - n\sin(nx))\right)$$

When $x = \pi$ and $x = -\pi$, $f(x)$ converges to $\frac{e^{\pi}+e^{-\pi}}{2}$.

(b) Given an arbitrary function f defined for a range $[-a, a]$ where a is a (possibly infinite) real.

Consider the functions defined over the range $[-a, a]$:

$$\begin{aligned} f_e(x) &\triangleq \frac{1}{2}(f(x) + f(-x)) \\ f_o(x) &\triangleq \frac{1}{2}(f(x) - f(-x)) \end{aligned}$$

$$\begin{aligned} f_e(x) &= \frac{1}{2}(f(x) + f(-x)) \\ &= \frac{1}{2}(f(-x) + f(x)) \\ &= f_e(-x) \end{aligned}$$

So $f_e(x)$ is an even function.

$$\begin{aligned} f_o(x) &= \frac{1}{2}(f(x) - f(-x)) \\ &= -\frac{1}{2}(f(-x) - f(x)) \\ &= -f(-x) \end{aligned}$$

So $f_o(x)$ is an odd function.

$$\begin{aligned} f_e(x)f_o(x) &= \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \\ &= f(x) \end{aligned}$$

So given an arbitrary function f we hve constructed an even function f_e and an odd function f_o such that $f = f_e + f_o$.

Hence any function $f(x)$ can be written as the sum of an even function and an odd function.

$$\begin{aligned} \cosh(x) &= \frac{e^x + e^{-x}}{2} \\ &= \frac{e^\pi - e^{-\pi}}{2\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}(\cos(nx))\right) \end{aligned}$$

$$\begin{aligned} \sinh(x) &= \frac{e^x - e^{-x}}{2} \\ &= -\frac{e^\pi - e^{-\pi}}{\pi}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}(n\sin(nx))\right) \end{aligned}$$

27.

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x \\
=&\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} b_n \cos(nx) \sum_{m=1}^{\infty} B_m \sin(mx)\,\mathrm{d}x \\
=&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n B_m \int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x \\
=&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n B_m \delta_{mn}\pi \text{ using (23)} \\
=&\pi\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} b_n B_n
\end{aligned}$$

$$\begin{aligned}
&\int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x \\
=&\frac{a_0^2}{4} + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2)
\end{aligned}$$

28. (a) $f(x) = f(-x) \implies \forall n \in \mathbb{N}.b_n = 0$

$$\begin{aligned}
a_n\cos(nx) + b_n\sin(nx) &= a_n\cos(-nx) + b_n\sin(-nx) \\
a_n\cos(nx) + b_n\sin(nx) &= a_n\cos(nx) + -b_n\sin(nx) \\
a_n = a_n &\wedge b_n = -b_n \\
a_n = a_n &\wedge b_n = 0
\end{aligned}$$

(b) $f(x) = -f(-x) \implies \forall n \in \mathbb{N}.a_n = 0$

$$\begin{aligned}
a_n\cos(nx) + b_n\sin(nx) &= -(a_n\cos(-nx) + b_n\sin(-nx)) \\
a_n\cos(nx) + b_n\sin(nx) &= -a_n\cos(nx) + b_n\sin(nx) \\
a_n = -a_n &\wedge b_n = b_n \\
a_n = 0 &\wedge b_n = 0
\end{aligned}$$

(c) $f(x) = f(\pi - x) \implies \forall n \in \mathbb{N}.(n \equiv_2 1 \implies a_n = b_n = 0)$

$$\begin{aligned}
a_n\cos(nx) + b_n\sin(nx) &= a_n\cos(n\pi - nx) + b_n\sin(n\pi - nx) \\
a_n\cos(nx) + b_n\sin(nx) &= a_n(\cos(n\pi)\cos(nx) - \sin(n\pi)\sin(nx)) + b_n(\cos(n\pi)\sin(nx) - \sin(n\pi)\cos(nx)) \\
a_n\cos(nx) + b_n\sin(nx) &= \cos(n\pi)(a_n\cos(nx) + b_n\sin(nx)) \\
a_n\cos(nx) + b_n\sin(nx) &= (-1)^n(a_n\cos(nx) + b_n\sin(nx))
\end{aligned}$$

So if n is odd then $a_n = b_n = 0$.

(d) $f(x) = -f(\pi - x) \implies \forall n \in \mathbb{N}.(n \equiv_2 0 \implies a_n = b_n = 0)$

$$\begin{aligned}
a_n\cos(nx) + b_n\sin(nx) &= -(a_n\cos(n\pi - nx) + b_n\sin(n\pi - nx)) \\
a_n\cos(nx) + b_n\sin(nx) &= -(a_n(\cos(n\pi)\cos(nx) - \sin(n\pi)\sin(nx)) + b_n(\cos(n\pi)\sin(nx) - \sin(n\pi)\cos(nx))) \\
a_n\cos(nx) + b_n\sin(nx) &= -\cos(n\pi)(a_n\cos(nx) + b_n\sin(nx)) \\
a_n\cos(nx) + b_n\sin(nx) &= (-1)^{n+1}(a_n\cos(nx) + b_n\sin(nx))
\end{aligned}$$

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So if n is even then $a_n = b_n = 0$.

(e) $f(x) = f\left(\frac{\pi}{2} + x\right) \implies \forall n \in \mathbb{N}.(n\%4 \neq 0 \implies a_n = b_n = 0)$

$$
\begin{aligned}
a_n \cos(nx) + b_n \sin(nx) &= a_n \cos\left(\frac{\pi n}{2} + nx\right) + b_n \sin\left(\frac{\pi n}{2} + nx\right) \\
a_n \cos(nx) + b_n \sin(nx) &= a_n \left(\cos\left(\frac{\pi n}{2}\right)\cos(nx) - \sin\left(\frac{\pi n}{2}\right)\sin(nx)\right) + b_n \left(\sin\left(\frac{\pi n}{2}\right)\cos(nx) + \cos\left(\frac{\pi n}{2}\right)\sin(nx)\right) \\
a_n \cos(nx) + b_n \sin(nx) &= \left(a_n \cos\left(\frac{\pi n}{2}\right) + b_n \sin\left(\frac{\pi n}{2}\right)\right)\cos(nx) + \left(b_n \cos\left(\frac{\pi n}{2}\right) - a_n \sin\left(\frac{\pi n}{2}\right)\right)\sin(nx)
\end{aligned}
$$

Equating the coefficients of $\cos(nx)$ and $\sin(nx)$ gives:

$$
\begin{aligned}
a_n &= a_n \cos\left(\frac{\pi n}{2}\right) + b_n \sin\left(\frac{\pi n}{2}\right) \\
b_n &= b_n \cos\left(\frac{\pi n}{2}\right) - a_b \sin\left(\frac{\pi n}{2}\right)
\end{aligned}
$$

Considering the four cases of n % 4:

n % 4

0	1	2	3
$a_n = a_b$ $b_n = b_n$	$a_n = b_n$ $b_n = -a_n$	$a_n = -a_n$ $b_n = -b_n$	$a_n = -b_n$ $b_n = a_n$

$n \equiv_4 1$, $n \equiv_4 2$ and $n \equiv_4 3$ all imply $a_n = -a_n$, $b_n = -b_n$ which implies that $a_n = b_n = 0$.

So if $n\%4 \neq 0$ then $a_n = b_n = 0$.

(f) $f(x) = f\left(\frac{\pi}{2} - x\right) \implies \forall n \in \mathbb{N}.(n\%4 \neq 0 \implies a_n = b_n = 0)$

$$
\begin{aligned}
a_n \cos(nx) + b_n \sin(nx) &= a_n \cos\left(\frac{\pi n}{2} - nx\right) + b_n \sin\left(\frac{\pi n}{2} - nx\right) \\
a_n \cos(nx) + b_n \sin(nx) &= a_n \left(\cos\left(\frac{\pi n}{2}\right)\cos(nx) + \sin\left(\frac{\pi n}{2}\right)\sin(nx)\right) + b_n \left(\cos\left(\frac{\pi n}{2}\right) - b_n \sin\left(\frac{\pi n}{2}\right)\right)\cos(nx) \\
a_n \cos(nx) + b_n \sin(nx) &= \left(a_n \cos\left(\frac{\pi n}{2}\right) - b_n \sin\left(\frac{\pi n}{2}\right)\right)\cos(nx) + a_n \left(\sin\left(\frac{\pi n}{2}\right) + b_n \cos\left(\frac{\pi n}{2}\right)\right)\sin(nx)
\end{aligned}
$$

Equating the coefficients of $\cos(nx)$ and $\sin(nx)$ gives:

$$
\begin{aligned}
a_n &= a_n \cos\left(\frac{\pi n}{2}\right) - b_n \sin\left(\frac{\pi n}{2}\right) \\
b_n &= a_n \sin\left(\frac{\pi n}{2}\right) + b_n \cos\left(\frac{\pi n}{2}\right)
\end{aligned}
$$

Considering the four cases of n % 4:

n % 4

0	1	2	3
$a_n = a_b$ $b_n = b_n$	$a_n = -b_n$ $b_n = a_n$	$a_n = -a_n$ $b_n = -b_n$	$a_n = b_n$ $b_n = -a_n$

$n \equiv_4 1$, $n \equiv_4 2$ and $n \equiv_4 3$ all imply $a_n = -a_n$, $b_n = -b_n$ which implies that $a_n = b_n = 0$.

So if $n\%4 \neq 0$ then $a_n = b_n = 0$.

(g) $f(x) = f(2x) \implies \forall n \in \mathbb{Z}^+ . a_n = b_n = 0$

Note that this restriction is placed on $n \in \mathbb{Z}^+$ – so a_0 is not restricted.

$$a_n \cos(nx) + b_n \sin(nx) = a_n \cos(2nx) + b_n \sin(2nx)$$

For $n \neq 0$, consider $x = \frac{\pi}{4n}$.

$$\begin{aligned} a_n \cos\left(\frac{\pi}{4}\right) + b_n \sin\left(\frac{\pi}{4}\right) &= a_n \cos\left(\frac{\pi}{2}\right) + b_n \sin\left(\frac{\pi}{2}\right) \\ a_n \frac{\sqrt{2}}{2} + b_n \frac{\sqrt{2}}{2} &= 0 + b_n \\ a_n &= b_n(\sqrt{2} - 1) \end{aligned}$$

For $n \neq 0$, consider $x = \frac{3\pi}{4n}$.

$$\begin{aligned} a_n \cos\left(\frac{3\pi}{4}\right) + b_n \sin\left(\frac{3\pi}{4}\right) &= a_n \cos\left(\frac{3\pi}{2}\right) + b_n \sin\left(\frac{3\pi}{2}\right) \\ -a_n \frac{\sqrt{2}}{2} + b_n \frac{\sqrt{2}}{2} &= 0 + -b_n \\ a_n &= b_n(\sqrt{2} + 1) \end{aligned}$$

Equating these gives:

$$\begin{aligned} b_n(\sqrt{2} + 1) &= b_n(\sqrt{2} - 1) \\ b_n &= -b_n \\ b_n &= 0 \end{aligned}$$

$$\begin{aligned} a_n &= b_n(\sqrt{2} - 1) \\ a_n &= 0 \end{aligned}$$

So for $n \neq 0$, $a_n = b_n = 0$.

(h) $f(x) = f(-x) = f\left(\frac{\pi}{2} - x\right) \implies \forall n \in \mathbb{N}. b_n = 0 \wedge (n\%4 \neq 0 \implies a_n = b_n = 0)$

$$\begin{aligned} f(x) = f(-x) = f\left(\frac{\pi}{2} - x\right) &\iff \\ f(x) = f(-x) \wedge f(x) = f\left(\frac{\pi}{2} - x\right) \wedge f(-x) = f\left(\frac{\pi}{2} - x\right) &\iff \\ f(x) = f(-x) \wedge f(x) = f\left(\frac{\pi}{2} - x\right) \wedge f(x) = f\left(\frac{\pi}{2} - x\right) \text{ using } f(x) = f(-x) &\iff \\ f(x) = f(-x) \wedge f(x) = f\left(\frac{\pi}{2} - x\right) & \end{aligned}$$

This reduces the problem down to two problems both of which we have already solved.

So $\forall n \in \mathbb{N}. b_n = 0$ and if $n\%4 \neq 0$ then $a_n = 0$.

29.

$$\sum_{-\infty}^{\infty} c_n e^{inx} = x^2$$
$$\sum_{-\infty}^{\infty} c_n e^{inx} e^{-imx} = x^2 e^{-imx}$$
$$\sum_{-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{inx} e^{-imx}\,\mathrm{d}x = \int_{-\pi}^{\pi} x^2 \cos(mx)\,\mathrm{d}x$$
$$\sum_{-\infty}^{\infty} 2\pi\delta_{mn} c_n = \left[\frac{1}{m}x^2 \sin(mx) + \frac{2}{m^2}x\cos(mx) - \frac{2}{m^3}\sin(mx)\right]_{-\pi}^{\pi}$$
$$2\pi c_m = \left[\frac{2}{m^2}x\cos(mx)\right]_{-\pi}^{\pi}$$
$$2\pi c_m = \frac{4\pi}{m^2}\cos(m\pi)$$
$$c_m = \frac{2}{m^2}\cos(m\pi)$$
$$c_m = (-1)^m \frac{2}{m^2}$$

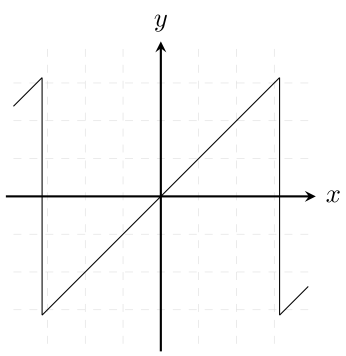
Consider a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,\mathrm{d}x$$
$$a_0 = \frac{2}{3\pi}\left[x^3\right]_0^{\pi}$$
$$a_0 = \frac{2\pi^2}{3}$$

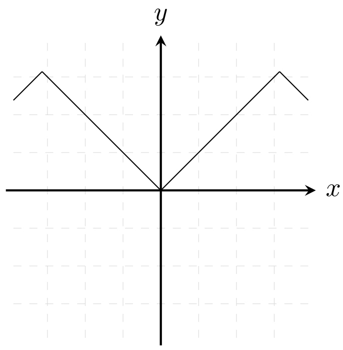
Combining the above results gives:

$$x^2 = \frac{2\pi^2}{3\times 2} + \sum_{-\infty}^{\infty}(-1)^m \frac{2}{m^2}\cos(mx)$$

30. (a) The *odd* function which is periodic with period 2π and equal to $f(x)$ for $0 \le x \le pi$:



(b) The *even* function which is periodic with period 2π and equal to $f(x)$ for $0 \le x \le pi$.



If the function is odd then $\forall n \in \mathbb{N}. a_n = 0$.

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,\mathrm{d}x \\
b_n &= \frac{1}{\pi}\left[-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right]_{-\pi}^{\pi} \\
b_n &= \frac{1}{\pi}\left(-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) - \frac{\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi)\right) \\
b_n &= -\frac{2}{n}\cos(n\pi) \\
b_n &= 2\frac{(-1)^{n+1}}{n}
\end{aligned}$$

So:

$$\begin{aligned}
f(x) &= \sum_{0}^{\infty} 2\frac{(-1)^{n+1}}{n} \\
f(x) &= 2\sum_{0}^{\infty} \frac{(-1)^{n+1}}{n}
\end{aligned}$$

As required.

If the function is even then $\forall n \in \mathbb{N}. b_n = 0$.

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,\mathrm{d}x \\
a_0 &= \frac{2}{\pi}\int_{0}^{\pi} x\,\mathrm{d}x \\
a_0 &= \frac{2}{\pi}\left[\frac{1}{2}x^2\right]_{0}^{\pi} \\
a_0 &= \pi
\end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,\mathrm{d}x \\
a_n &= \frac{2}{\pi}\int_{0}^{\pi} x\cos(nx)\,\mathrm{d}x \\
a_n &= \frac{2}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{x^2}\cos(nx)\right]_0^{\pi} \\
a_n &= \frac{2}{\pi}\left(\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\right) \\
a_n &= \frac{2}{\pi n^2}(\cos(n\pi) - 1) \\
a_n &= \begin{cases} 0 & \text{if } n\%2 = 0 \\ -\frac{4}{\pi n^2} & \text{if } n\%2 = 1 \end{cases}
\end{aligned}
$$

$$\sum_{n=1}^{\infty} a_n = \sum_{k=0}^{\infty} a_{2k+1}$$

So:

$$
\begin{aligned}
f(x) &= \frac{\pi}{2} + \sum_{k=0}^{\infty} -\frac{4}{\pi(2k+1)^2}\cos(2k+1)x \\
f(x) &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}
\end{aligned}
$$

As required.