Harry Langford Hjelz

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2. a) The ratio test says that if

$\lim_{n\to\infty} \left|\frac{V_{n+1}}{V_n}\right| < 1$ then the series converges.

The comparison test says that

if $\forall n : U_n \geq 0 \wedge V_n \geq 0$

and $\sum_{n=0}^{\infty} U_n$ converges

then if for all n :

$V_n \leq U_n$ then $\sum_{n=0}^{\infty} V_n$ also

converges.

b) i) $\cancel{\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n\right)}$

$\cancel{\approx \sum_{n=1}^{\infty} \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2}{n}\right)}$

$\cancel{\approx \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{2}{n}\right)^2}$

$\cancel{\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right) \approx \sum_{n=1}^{\infty} \left(\frac{1}{n} - \left(\frac{1}{n}\right)^3 \times \frac{1}{6} + O\left(\frac{1}{n^5}\right)\right)^2}$

$\cancel{\approx \sum_{n=1}^{\infty} \frac{1}{n^2} + O\left(\frac{1}{n^4}\right)}$

~~$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges~~

~~For all~~ ~~$s > 1 : \sum_{n=1}^{\infty} s n^{-s}$ converges~~

~~So~~

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b)ii) $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right) \approx \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{6n^3} + \dots O(n^5)\right)^2$

$$\approx \sum_{n=1}^{\infty} \frac{1}{n^2} + O(n^4)$$

Forall $s<-1$: $\sum_{n=1}^{\infty} n^{-s}$ converges.

So $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Take an arbitrary large constant K :
Such that forall $n \in \mathbb{Z}^+$ $\frac{K}{n^2} > \sin^2\left(\frac{1}{n}\right)$:

$$K \sum_{n=1}^{\infty} \frac{1}{n^2} > \sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$$

$K \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

So $\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$ converges also.

ii) $\sum_{n=1}^{\infty} n^4 e^{-n^2}$

Using the Integral test:

If $\int_1^{\infty} x^4 e^{-x^2}\,dx$ converges then

$\sum_{n=1}^{\infty} n^4 e^{-n^2}$ converges

$$\int_1^{\infty} x^4 e^{-x^2}\,dx = \left[-\tfrac{1}{2}x^3 e^{-x^2}\right]_1^{\infty} + \int_1^{\infty} \tfrac{3}{2}x^2 e^{-x^2}\,dx$$

$$\int x^2 e^{-x^2}\,dx = \left[-\tfrac{x}{2}e^{-x^2}\right]_1^{\infty} + \tfrac{1}{2}\int_1^{\infty} e^{-x^2}\,dx$$

$$\int_1^{\infty} x^4 e^{-x^2}\,dx = e^{-1} + \tfrac{3}{2}e^{-1} + \tfrac{3}{4}\int_1^{\infty} e^{-x^2}\,dx$$

$\int_1^{\infty} e^{-x^2}\,dx$ converges.

So $\int_1^{\infty} x^4 e^{-x^2}\,dx$ also converges

So $\sum_{n=1}^{\infty} n^4 e^{-n^2}$ converges.

c) i) $E(x^3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^3 e^{-\frac{x^2}{2}} \, dx$

$$= \frac{1}{\sqrt{2\pi}} \int u \frac{dv}{dx} \, dx$$

~~where $u = -\frac{1}{2}x^2$~~

where $u = -x^2$, $v = e^{-\frac{x^2}{2}}$

$$E(x^3) = \frac{1}{\sqrt{2\pi}} \left(\left[-x^2 e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 2x e^{-\frac{x^2}{2}} \, dx \right)$$

$$= \frac{1}{\sqrt{2\pi}} \left(0 + 2 \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} \, dx \right)$$

$$= \frac{2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} \, dx$$

$$= \frac{2}{\sqrt{2\pi}} \left[-e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty}$$

$$= \frac{2}{\sqrt{2\pi}} \times 0$$

$$= 0$$

So $E(x^3) = 0$

ii) $$E(x^4) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\left[-x^3 e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} 3x^2 e^{-\frac{x^2}{2}} dx \right)$$

$$= \frac{3}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx$$

Since $\left[-x^3 e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} = 0$

$$= \frac{3}{\sqrt{2\pi}} \left(\left[-x e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx \right)$$

$$= \frac{3}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$$

$$= \frac{3\sqrt{2\pi}}{\sqrt{2\pi}}$$

$$= 3$$