5. On Sets

5.1 Basic Exercises

1. Prove that $\subseteq$ is a partial order, that is, it is:

   (a) reflexive: $\forall$ sets A, $A \subseteq A$
   I shall prove that every element in A is also in A.

$$\begin{aligned} \forall\, a \in A : a \in A &\iff \\ A \subseteq A& \end{aligned} \tag{1}$$

   (b) transistive: $\forall$ sets A, B, C. $(A \subseteq B \wedge B \subseteq C) \implies A \subseteq C$
   Assume $A \subseteq B \wedge B \subseteq C$

$$\begin{aligned} \text{Take an arbitrary } a \in A& \\ \text{By assumption } a \in A \implies a \in B& \\ \text{By assumption } a \in B \implies a \in C& \\ a \in A \implies a \in C &\implies \\ \forall\, a \in A : a \in C &\iff \\ A \subseteq C& \end{aligned} \tag{2}$$

   (c) antisymmetric: $\forall$ sets A, B. $(A \subseteq B \wedge B \subseteq A) \iff A = B$

$$\begin{aligned} A \subseteq B \wedge B \subseteq A &\iff \\ \forall\, x.(x \in A \implies x \in B) \wedge (x \in B \implies x \in A) &\iff \\ \forall\, x.x \in A \iff x \in B &\iff \\ A = B& \end{aligned} \tag{3}$$

6. Let U be a set. For all $A, B \in \mathcal{P}(U)$, prove that:

   (a) $A^c = B \iff (A \cup B = U \wedge A \cap B = \varnothing)$

$$\begin{aligned} A^c = B &\iff \\ (\forall\, b \in U : b \notin A \iff b \in B) &\iff \\ A \cup B = \{u \mid \forall\, u \in U. u \in A \vee u \in B\} &\iff \\ = \{u \mid \forall\, u \in U. u \in A \vee u \notin A\} &\iff \\ = \{u \mid \forall\, u \in U\} &\iff \\ = U& \end{aligned} \tag{4}$$

$$\begin{aligned} A^c = B &\iff \\ (\forall\, b \in U : b \notin A \iff b \in B) &\iff \\ A \cap B = \{x \mid x \in A \wedge x \in B\}& \\ = \{x \mid x \in A \wedge x \notin A\}& \\ = \varnothing& \end{aligned} \tag{5}$$

   (b) Double complement elimination: $(A^c)^c = A$

or using (a): enough to show $A^c \cup A = \mathcal{U}$ and $A^c \cap A = \emptyset$

$$
\begin{aligned}
A^{\mathsf{c}} &\triangleq \{u \mid u \in U \wedge u \notin A\} \\
(A^{\mathsf{c}})^{\mathsf{c}} &= \{u' \mid u' \in U \wedge u' \notin \{u \mid u \in U \wedge u \notin A\}\} \\
&= \{u' \mid u' \in U \wedge \overline{(u' \in U \wedge u' \notin A)}\} \\
&= \{u' \mid u' \in U \wedge (u' \notin U \vee u' \in A)\} \\
&= \{u' \mid (u' \in U \wedge u' \notin U) \vee (u' \in U \wedge u' \in A)\} \\
&= \{u' \mid u' \in U \wedge u' \in A\} \\
&= \{u' \mid u' \in A\} \text{ (Since } A \subseteq U : u' \in A \implies u' \in U) \\
&= A
\end{aligned}
\tag{6}
$$

(c) The De-Morgan laws: $(A \cup B)^{\mathsf{c}} = A^{\mathsf{c}} \cap B^{\mathsf{c}}$ and $(A \cap B)^{\mathsf{c}} = A^{\mathsf{c}} \cup B^{\mathsf{c}}$

$$
\begin{aligned}
(A \cup B)^{\mathsf{c}} &= \{x \mid x \in U.x \notin A \cup B\} \\
&= \{x \mid x \in U.x \notin A \wedge x \notin B\} \\
&= \{x \mid x \in U.x \notin A\} \cap \{x \mid x \in U.x \notin B\} \\
&= A^{\mathsf{c}} \cap B^{\mathsf{c}}
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
(A \cap B)^{\mathsf{c}} &= \{x \mid x \in U.x \notin A \cap B\} \\
&= \{x \mid x \in U.x \notin A \vee x \notin B\} \\
&= \{x \mid x \in U.x \notin A\} \cup \{x \mid x \in U.x \notin B\} \\
&= A^{\mathsf{c}} \cap B^{\mathsf{c}}
\end{aligned}
\tag{8}
$$

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5.2 Core Exercises

2. Either prove or disprove that, for all sets A and B,

(a) $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$

$$
\begin{gathered}
\mathcal{P}(A) \triangleq \{a \mid a \subseteq A\} \\
A \subseteq B \iff \\
a \subseteq A \implies a \subseteq B \iff \\
\forall\, x \in \{a \mid a \subseteq A\} \implies x \in \{b \mid b \subseteq B\} \iff \\
\forall\, x \in \mathcal{P}(A) \implies x \in \mathcal{P}(B) \iff \\
\mathcal{P}(A) \subseteq \mathcal{P}(B)
\end{gathered}
\tag{9}
$$

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(b) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$
Disproof by counter-example:

$$
\begin{gathered}
\text{Let } A = \{0\} \wedge B = \{1\} \iff \\
\mathcal{P}(A \cup B) = \mathcal{P}(\{0,1\}) = \{\varnothing, \{0\}, \{1\}, \{0,1\}\} \wedge \\
\mathcal{P}(A) \cup \mathcal{P}(B) = \{\varnothing, \{0\}\} \cup \{\varnothing, \{1\}\} = \{\varnothing, \{0\}, \{1\}\} \\
\text{In this case: } \mathcal{P}(A \cup B) \nsubseteq \mathcal{P}(A) \cup \mathcal{P}(B)
\end{gathered}
\tag{10}
$$

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(c) $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

$$\begin{aligned}
\mathcal{P}(S) \triangleq \{s \mid s \subseteq S\} \implies \\
\mathcal{P}(A) \cup \mathcal{P}(B) = \{x \mid x \subseteq A \vee x \subseteq B\} \\
\mathcal{P}(A \cup B) = \{x \mid x \subseteq A \cup B\} \\
\forall\, x : x \subseteq A \vee x \subseteq B \implies x \subseteq A \cup B \implies \\
\{x \mid x \subseteq A \vee x \subseteq B\} \subseteq \{x \mid x \subseteq A \cup B\} \iff \\
\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)
\end{aligned} \tag{11}$$

4. For sets A, B, C, D, prove or disprove at least three of the following statements:

   (a) $(A \subseteq C \wedge B \subseteq D) \implies (A \times B \subseteq C \times D)$

$$\begin{aligned}
\text{Assume: } & A \subseteq C \wedge B \subseteq D \iff \\
& (a \in A \implies a \in C) \wedge (b \in B \implies b \in D) \\
& A \times B = \{s \mid \exists\, a \in A \wedge \exists\, b \in B.s = (a, b)\} \implies \\
& A \times B \subseteq \{s \mid \exists\, a \in B \wedge \exists\, b \in D.s = (a, b)\} \iff \\
& A \times B \subseteq C \times D
\end{aligned} \tag{12}$$

   (b) $(A \cup C) \times (B \cup D) \subseteq (A \times B) \cup (C \times D)$

   Disproof by counterexample:

$$\begin{aligned}
\text{Let: } & A = \varnothing, B = \{1\}, C = \{2\}, D = \{3\} \\
\text{So: } (A \cup C) \times (B \cup D) &= \{2\} \times \{1, 3\} \\
&= \{(2, 1), (2, 3)\} \\
\text{And: } (A \times B) \cup (C \times D) &= (\varnothing \times \{1\}) \cup (\{2\} \times \{3\}) \\
&= \varnothing \cup \{(2, 3)\} \\
&= \{(2, 3)\} \\
\text{So in this case: } & (A \cup C) \times (B \cup D) \nsubseteq (A \times B) \cup (C \times D)
\end{aligned} \tag{13}$$

   (c) $(A \times C) \cup (B \times D) \subseteq (A \cup B) \times (C \cup D)$

   I will prove distributivity of $\times$ and $\cup$ to use in this and subsequent proofs.

$$\begin{aligned}
A \times (B \cup C) =& \{s \mid \exists\, a \in A, \exists\, x \in B \cup C.s = (a, x)\} \\
=& \{s \mid \exists\, a \in A, \exists\, x.(x \in B \cup x \in C).s = (a, x)\} \\
=& \{(\exists\, a \in A \wedge \exists\, x \in B) \cup (\exists\, a \in A \wedge \exists\, x \in C).s = (a, x)\} \\
=& \{(\exists\, a \in A \wedge \exists\, x \in B).s = (a, x)\} \cup \{(\exists\, a \in A \wedge \exists\, x \in C).s = (a, x)\} \\
=& (A \times B) \cup (A \times C)
\end{aligned} \tag{14}$$

$$\begin{aligned}
(A \times C) \cup (B \times D) &\subseteq (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D) \\
&\subseteq (A \times (C \cup D)) \cup (B \times (C \cup D)) \\
&\subseteq (A \cup B) \times (C \cup D) \text{ as required}
\end{aligned} \tag{15}$$

5. For sets A, B, C, D, prove or disprove at least three of the following statements:

   (a) $(A \subseteq C \wedge B \subseteq D) \implies A \uplus B \subseteq C \uplus D$

$$
\begin{aligned}
&\text{Assume: } A \subseteq C \wedge B \subseteq D \implies \\
&a \in A \implies a \in C \wedge b \in B \implies b \in D \\
&\quad x \in (A \uplus B) \iff (\exists\, a \in A.x = (1, a)) \vee (\exists\, b \in B.x = (2, b)) \implies \\
&\quad x \in (A \uplus B) \implies (\exists\, a \in C.x = (1, a)) \vee (\exists\, b \in D.x = (2, b)) \iff \\
&\quad x \in (A \uplus B) \implies x \in (C \uplus D) \iff \\
&\qquad A \uplus B \subseteq C \uplus D
\end{aligned} \tag{16}
$$

(b) $(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C)$

I will prove the distributivity of $\uplus$ and $\cup$.

$$
\begin{aligned}
x \in (A \cup B) \uplus C \iff& (\exists\, a \in A \cup B.x = (1, a)) \vee (\exists\, c \in C.x = (2, c)) \iff \\
x \in (A \cup B) \uplus C \iff& (\exists\, a \in A.x = (1, a)) \vee (\exists\, b \in B.x = (1, b)) \vee (\exists\, c \in C.x = (2, c)) \iff \\
x \in (A \cup B) \uplus C \iff& ((\exists\, a \in A.x = (1, a)) \vee (\exists\, c \in C.x = (2, c))) \vee \\
& ((\exists\, b \in B.x = (1, b)) \vee (\exists\, c \in C.x = (2, c))) \iff \\
x \in (A \cup B) \uplus C \iff& x \in (A \uplus C) \vee x \in (B \uplus C) \iff \\
x \in (A \cup B) \uplus C \iff& x \in ((A \uplus C) \cup (B \uplus C)) \iff \\
(A \cup B) \uplus C =& (A \uplus C) \cup (B \uplus C)
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&(A \cup B) \uplus C = (A \uplus C) \cup (B \uplus C) \text{ using (17)} \implies \\
&(A \cup B) \uplus C \subseteq (A \uplus C) \cup (B \uplus C) \text{ using the antisymmetry of } \subseteq
\end{aligned} \tag{18}
$$

(c) $(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C$

$$
\begin{aligned}
&(A \uplus C) \cup (B \uplus C) = (A \cup B) \uplus C \text{ using (17)} \implies \\
&(A \uplus C) \cup (B \uplus C) \subseteq (A \cup B) \uplus C \text{ using the antisymmetry of } \subseteq
\end{aligned} \tag{19}
$$

(d) $(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C)$

I will prove the distributivity of $\uplus$ and $\cap$.

$$
\begin{aligned}
x \in (A \cap B) \uplus C \iff& (\exists\, a \in A \cap B.x = (1, a)) \vee (\exists\, c \in C.x = (2, c)) \iff \\
x \in (A \cap B) \uplus C \iff& ((\exists\, a \in A.x = (1, a)) \wedge (\exists\, b \in B.x = (1, b))) \vee (\exists\, c \in C.x = (2, c)) \iff \\
x \in (A \cap B) \uplus C \iff& ((\exists\, a \in A.x = (1, a)) \vee (\exists\, c \in C.x = (2, c))) \wedge \\
& ((\exists\, b \in B.x = (1, b)) \vee (\exists\, c \in C.x = (2, c))) \iff \\
x \in (A \cap B) \uplus C \iff& x \in (A \uplus C) \cap (B \uplus C) \iff \\
(A \cap B) \uplus C =& (A \uplus C) \cap (B \uplus C)
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&(A \cap B) \uplus C = (A \uplus C) \cap (B \uplus C) \text{ using (20)} \implies \\
&(A \cap B) \uplus C \subseteq (A \uplus C) \cap (B \uplus C) \text{ using the antisymmetry of } \subseteq
\end{aligned} \tag{21}
$$

(e) $(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C$

$$
\begin{aligned}
&(A \uplus C) \cap (B \uplus C) = (A \cap B) \uplus C \text{ using (20)} \implies \\
&(A \uplus C) \cap (B \uplus C) \subseteq (A \cap B) \uplus C \text{ using the antisymmetry of } \subseteq
\end{aligned} \tag{22}
$$

6. Let A be a set.

   (a) For a family $\mathcal{F} \subseteq \mathcal{P}(A)$, let $\mathcal{U} \triangleq \{U \subseteq A \mid \forall\, S \in \mathcal{F}. S \subseteq U\}$. Prove that $\bigcup \mathcal{F} = \bigcap \mathcal{U}$.

$$\begin{aligned}
\mathcal{U} &\triangleq \{U \subseteq A \mid \forall\, S \in \mathcal{F}. S \subseteq U\} \iff \\
\mathcal{U} &= \{U \subseteq A \mid \bigcup \mathcal{F} \subseteq U\} \text{ using (??)} \iff \\
\bigcup \mathcal{F} &\subseteq \bigcup \mathcal{F} \implies \bigcup \mathcal{F} \in \mathcal{U} \iff \\
\forall\, U \in \mathcal{U}. \bigcup \mathcal{F} &\subseteq U \wedge \mathcal{F} \in U \iff \\
\bigcap \mathcal{U} &= \bigcup \mathcal{F}
\end{aligned} \tag{23}$$

   (b) Analogously, define the family $\mathcal{L} \subseteq \mathcal{P}(A)$ such that $\bigcap \mathcal{F} = \bigcup \mathcal{L}$. Also prove this statement.

$$\mathcal{L} \triangleq \{L \subseteq A \mid \forall\, S \in \mathcal{F} L \subseteq S\}$$

$$\begin{aligned}
\mathcal{L} &\triangleq \{L \subseteq A \mid \forall\, S \in \mathcal{F} L \subseteq S\} \iff \\
\mathcal{L} &= \{L \subseteq A \mid L \subseteq \bigcup \mathcal{F}\} \text{ using (??)} \iff \\
\bigcup \mathcal{F} &\in \mathcal{L} \wedge \forall\, L \in \mathcal{L} : L \subseteq \mathcal{F} \iff \\
\bigcup \mathcal{L} &= \bigcap \mathcal{F}
\end{aligned} \tag{24}$$

6. On Relations

6.1 Basic Exercises

2. Prove that relational composition is associative and has the identity relation as the neutral element.

   Let $f : A \nrightarrow B$, $g : B \nrightarrow C$, $h : C \nrightarrow D$.

   To prove the associativity of relational composition we must prove for arbitrary sets f, g, h that $\forall\, a \in A, d \in D : a(h \circ (g \circ f))d \iff a((h \circ g) \circ f)d$.

$$\begin{aligned}
a(h \circ (g \circ f))d &\iff \\
\exists\, c \in C : a(g \circ f)c \wedge chd &\iff \\
\exists\, b \in B, c \in C : afb \wedge bgc \wedge chd &\iff \\
\exists\, b \in B : afb \wedge b(h \circ g)d &\iff \\
a((h \circ g) \circ f)d &
\end{aligned} \tag{25}$$

   So relational composition is associative as required.

3. For a relation $R : A \nrightarrow B$, let its opposite or dual relation $R^{\mathrm{op}} : B \nrightarrow A$ be defined by:

$$bR^{\mathrm{op}}a \iff aRb \tag{26}$$

   For $R, S : A \nrightarrow B$ and $T : B \nrightarrow C$, prove that:

(a) $R \subseteq S \Longrightarrow R^{\mathrm{op}} \subseteq S^{\mathrm{op}}$

$$\begin{aligned} &\forall\, a \in A, b \in B : \\ &bR^{\mathrm{op}}a \Longleftrightarrow \\ &aRb \Longrightarrow \\ &aSb \text{ since } R \subseteq S \Longleftrightarrow \\ &bS^{\mathrm{op}}a \end{aligned} \tag{27}$$

$$\begin{aligned} bR^{\mathrm{op}}a \Longrightarrow bS^{\mathrm{op}}a \Longrightarrow \\ R^{\mathrm{op}} \subseteq S^{\mathrm{op}} \text{ as required} \end{aligned} \tag{28}$$

✓

(b) $(R \cap S)^{\mathrm{op}} = R^{\mathrm{op}} \cap S^{\mathrm{op}}$

RHS:

$$\begin{aligned} R \subseteq S \Longleftrightarrow \\ R \cap S = R \Longleftrightarrow \\ (R \cap S)^{\mathrm{op}} = R^{\mathrm{op}} \end{aligned} \tag{29}$$

We don't know that $R \subseteq S$. That was only assumed in (a)

LHS:

$$\begin{aligned} R^{\mathrm{op}} \subseteq S^{\mathrm{op}} \Longleftrightarrow \\ R^{\mathrm{op}} \cap S^{\mathrm{op}} = R^{\mathrm{op}} \end{aligned} \tag{30}$$

Combining (29) and (30) gives:

$$(R \cap S)^{\mathrm{op}} = R^{\mathrm{op}} = (R^{\mathrm{op}} \cap S^{\mathrm{op}}) \Longrightarrow (R \cap S)^{\mathrm{op}} = (R^{\mathrm{op}} \cap S^{\mathrm{op}}) \text{ as required} \tag{31}$$

(c) $(R \cup S)^{\mathrm{op}} = R^{\mathrm{op}} \cup S^{\mathrm{op}}$

RHS:

$$\begin{aligned} R \subseteq S \Longleftrightarrow \\ R \cup S = S \Longleftrightarrow \\ (R \cup S)^{\mathrm{op}} = S^{\mathrm{op}} \end{aligned} \tag{32}$$

Same comment as above.

LHS:

$$\begin{aligned} R^{\mathrm{op}} \subseteq S^{\mathrm{op}} \Longleftrightarrow \\ R^{\mathrm{op}} \cup S^{\mathrm{op}} = S^{\mathrm{op}} \end{aligned} \tag{33}$$

Combining (32) and (33) gives:

$$(R \cup S)^{\mathrm{op}} = S^{\mathrm{op}} = R^{\mathrm{op}} \cup S^{\mathrm{op}} \Longrightarrow (R \cup S)^{\mathrm{op}} = R^{\mathrm{op}} \cup S^{\mathrm{op}} \tag{34}$$

(d) $(T \circ S)^{\mathrm{op}} = S^{\mathrm{op}} \circ T^{\mathrm{op}}$

LHS:

$$\begin{aligned} \forall\, a \in A, c \in C : c(S \circ T)^{\mathrm{op}}a \Longleftrightarrow \\ a(S \circ T)c \Longleftrightarrow \\ \exists\, b \in B : aTb \wedge bSc \Longleftrightarrow \\ \exists\, b \in B : bT^{\mathrm{op}}a \wedge cS^{\mathrm{op}}b \Longleftrightarrow \\ \exists\, b \in B : cS^{\mathrm{op}}b \wedge bT^{\mathrm{op}}a \Longleftrightarrow \\ c(S^{\mathrm{op}} \circ T^{\mathrm{op}})a \end{aligned} \tag{35}$$

$$\begin{aligned} c(S \circ T)^{\mathrm{op}}a \Longleftrightarrow c(S^{\mathrm{op}} \circ T^{\mathrm{op}})a \Longleftrightarrow \\ c(S \circ T)^{\mathrm{op}}a = c(S^{\mathrm{op}} \circ T^{\mathrm{op}})a \text{ as required} \end{aligned} \tag{36}$$

✓

6.2 Core Exercises

1. Let $R, R' \subseteq A \times B$ and $S, S' \subseteq B \times C$ be two pairs of relations and assume $R \subseteq R'$ and $S \subseteq S'$.
   Prove that $S \circ R \subseteq S' \circ R'$.

$$\begin{aligned} R \subseteq R' &\implies \\ \forall\, a \in A, b \in B : aRb &\implies aR'b \\ S \subseteq S' &\implies \\ \forall\, b \in B, c \in C : bSc &\implies bS'c \end{aligned} \tag{37}$$

$$\begin{aligned} &\forall\, a \in R, c \in S : \\ &\quad a(S \circ R)c \iff \\ &\exists\, b \in B : aRb \wedge bSc \implies \\ &\exists\, b \in B : aR'b \wedge bS'c \text{ using (37)} \iff \\ &\quad a(S' \circ R'c) \text{ as required} \end{aligned} \tag{38}$$

3. Suppose R is a relation on a set A. Prove that

   (a) R is reflexive iff $\mathrm{id}_A \subseteq R$

   R is reflexive implies that every element in A is related to itself under R.

   Assume R is reflexive:

$$\begin{aligned} \forall\, a \in A . aRa &\iff \\ \mathrm{id}_A \subseteq R& \end{aligned} \tag{39}$$

   (b) R is symmetric iff $R = R^{\mathrm{op}}$.

   For R to be symmetric, $\forall\, a_1, a_2 \in A, a_1 R a_2 \iff a_2 R a_1$.

   Assume R is symmetric:

$$\begin{aligned} &\forall\, a, b \in A . aRb \iff bRa \iff \\ &\forall\, a, b \in A . bR^{\mathrm{op}}a \iff aR^{\mathrm{op}}b \iff \\ &\forall\, a, b \in A . aR^{\mathrm{op}}b \iff bR^{\mathrm{op}}a \iff \\ &\qquad R = R^{\mathrm{op}} \end{aligned} \tag{40}$$

   (c) R is transitive iff $R \circ R \subseteq R$

   Assume that $R \circ R \subseteq R$.

$$\begin{aligned} &R \circ R \subseteq R \iff \\ &\forall\, a, c \in A . a(R \circ R)c \implies aRc \iff \\ &\forall\, a, b, c \in A . aRb \wedge bRc \implies aRc \end{aligned} \tag{41}$$

   This is the definition of transi~~s~~tivity and so we are done.

   (d) R is antisymmetric iff $R \cap R^{\mathrm{op}} \subseteq \mathrm{id}_A$.

   R is antisymmetric if $\forall\, a \in A . aRa$.

   Assume $R \cap R^{\mathrm{op}} \subseteq \mathrm{id}_A$.

$$\begin{aligned} &R \cap R^{\mathrm{op}} \subseteq \mathrm{id}_a \iff \\ &\forall\, a_1, a_2 \in A . a_1 R a_2 \wedge a_1 R^{\mathrm{op}} a_2 \implies a_1 = a_2 \iff \\ &\forall\, a_1, a_2 \in A . a_1 R a_2 \wedge a_2 R a_2 \implies a_1 = a_2 \end{aligned} \tag{42}$$

   Which is the definition of antisymmetric and so we are done.

7. On Partial Functions

7.1 Basic Exercises

2. Prove that a relation $R : A \nrightarrow B$ is a partial function iff $R \circ R^{\mathrm{op}} \subseteq \mathrm{id}_B$.

   $(\Longrightarrow)$

$$\begin{aligned} R \circ R^{\mathrm{op}} \subseteq \mathrm{id}_B \iff \\ \forall\, b_1, b_2 \in B. b_1(R \circ R^{\mathrm{op}})b_2 \Longrightarrow b_1 = b_2 \Longrightarrow \\ \forall\, b_1, b_2 \in B. \exists\, a \in A. b_1 R^{\mathrm{op}} a \wedge aRb_2 \Longrightarrow b_1 = b_2 \iff \\ \forall\, b_1, b_2 \in B. \exists\, a \in A. aRb_1 \wedge aRb_2 \Longrightarrow b_1 = b_2 \end{aligned} \tag{43}$$

   This implies that R is a partial function by definition.

   $(\Longleftarrow)$ Assume R is a partial function and so each a in R is related to at most one b. This means that by assumption $aRb_1 \wedge aRb_2 \Longrightarrow b_1 = b_2$.

$$\begin{aligned} b_1(R \circ R^{\mathrm{op}})b_2 \iff \\ \exists\, a \in A. b_1 R^{\mathrm{op}} a \wedge aRb_2 \iff \\ \exists\, a \in A. aR^{\mathrm{op}} b_1 \wedge aRb_2 \iff \\ b_1 = b_2 \text{ since } a \text{ is related to at most one } b \in B \iff \\ (R \circ R^{\mathrm{op}}) \subseteq \mathrm{id}_B \text{ as required} \end{aligned} \tag{44}$$

7.2 Core Exercises

1. Let $\mathcal{F} \subseteq \mathrm{PFun}(A, B)$ be a non-empty collection of partial functions from A to B.

   (a) Show that $\bigcap \mathcal{F}$ is a partial function.

$$\begin{aligned} x = \bigcap \mathcal{F} \iff \\ \forall\, s \in \mathcal{F}. x \subseteq s \end{aligned} \tag{45}$$

   is this actually true?
   Can't x be a subset of $\cap \mathcal{F}$?

   Since any subset of a partial function is itself a partial function; we know that x is a partial function. Since $x = \bigcap \mathcal{F}$, this means that $\bigcap \mathcal{F}$ is itself a partial function.

   (b) Show that $\bigcup \mathcal{F}$ need not be a partial function by defining two partial functions $f, g : A \rightharpoonup B$ such that $f \cup g : A \nrightarrow B$ is a non-functional relation.

   Let $A = \{1\}$ and $B = \{1, 2\}$
   Let $fA \rightharpoonup B = \{(1, 1)\}$ and $g : A \rightharpoonup B = \{(1, 2)\}$.

   So both f, g are partial functions.

   However, if $\mathcal{F} = \{f, g\}$ then $\bigcup \mathcal{F} = \{(1, 1), (1, 2)\}$ which is not a partial function. So by counterexample: $\bigcup \mathcal{F}$ need not be a partial function.

   (c) Let $h : A \rightharpoonup B$ be a partial function. Show that if every element of $\mathcal{F}$ is below h then $\bigcup \mathcal{F}$ is a partial function.

$$\begin{aligned} \mathcal{F} \subseteq \{x \mid x \subseteq h\} \iff \\ \forall\, x \in F. x \subseteq h \iff \\ \bigcup \mathcal{F} \subseteq h \end{aligned} \tag{46}$$

   Any subset of a partial function is itself a partial function. This means that $\bigcup \mathcal{F}$ must be a partial function as required.

8. On Functions

8.1 Basic Exercises

1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the sets $\text{Fun}(A_i, A_j)$ for $i, j \in \{2, 3\}$.

   {{(1, a), (2, b)}, {(1, a), (2, c)}, {(1, b), (2, a)}, {(1, b), (2, c)}, {(1, c), (2, a)}, {(1, c), (2, b)}} ??

8.2 Core Exercises

1. Let $A_2 = \{1, 2\}$ and $A_3 = \{a, b, c\}$. List the elements of the sets $\text{Fun}(A_i, A_j)$ for $i, j \in \{2, 3\}$. → This is the same question as above.

$$\begin{aligned}\text{Fun}(A_i, A_j) = &\{\{(1, a), (2, b)\}, \{(1, a), (2, c)\}, \{(1, b), (2, a)\}, \\ &\{(a, b), (2, c)\}, \{(1, c), (2, a)\}, \{(1, c), (2, b)\}\}\end{aligned} \tag{47}$$

X

Sorry for the (marginally) late submission – I only just noticed that it was due.

(8.2.1 asks for a different thing)