15. (a)

$$\begin{aligned}
&\int_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}\\
=&\int_S \frac{1}{a}\begin{pmatrix}\alpha x^3\\ \beta y^3\\ \gamma z^3\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}\mathrm{d}S\\
=&\frac{1}{a}\int_S \alpha x^4+\beta y^4+\gamma z^4\,\mathrm{d}S\\
=&\frac{(\alpha+\beta+\gamma)}{a}\int_S z^4\,\mathrm{d}S \text{ by symmetry}\\
=&\frac{(\alpha+\beta+\gamma)}{a}\int_0^{2\pi}\int_0^{\pi} a^2z^4\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
=&a(\alpha+\beta+\gamma)\int_0^{2\pi}\int_0^{\pi} a^4\cos^4\theta\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\\
=&a^5(\alpha+\beta+\gamma)\int_0^{2\pi}\mathrm{d}\phi\left[-\frac{1}{5}\cos^5\theta\right]_0^{\pi}\\
=&a^5(\alpha+\beta+\gamma)\times 2\pi\times\frac{2}{5}\\
=&\frac{4\pi a^5(\alpha+\beta+\gamma)}{5}
\end{aligned}$$

(b) The integral over the curved surface of the cylinder:

$$\begin{aligned}
&\int_S \mathbf{F}\cdot \mathrm{d}\mathbf{S}\\
=&\int_{-h}^{h}\int_0^{2\pi}\begin{pmatrix}\alpha x^3\\ \beta y^3\\ \gamma z^3\end{pmatrix}\cdot\frac{1}{\sqrt{4x^2+4y^2}}\begin{pmatrix}2x\\ 2y\\ 0\end{pmatrix}a\,\mathrm{d}\theta\,\mathrm{d}z\\
=&\int_{-h}^{h}\int_0^{2\pi}\begin{pmatrix}\alpha x^3\\ \beta y^3\\ \gamma z^3\end{pmatrix}\cdot\frac{2a}{\sqrt{4a^2}}\begin{pmatrix}x\\ y\\ 0\end{pmatrix}\mathrm{d}\theta\,\mathrm{d}z\\
=&\int_0^{2h}\mathrm{d}z\int_0^{2\pi}\begin{pmatrix}\alpha x^3\\ \beta y^3\\ \gamma z^3\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ 0\end{pmatrix}\mathrm{d}\theta\\
=&2h\int_0^{2\pi}\alpha x^4+\beta y^4\,\mathrm{d}\theta\\
=&(\alpha+\beta)2h\int_0^{2\pi}x^4\,\mathrm{d}\theta \text{ by rotational symmetry}\\
=&(\alpha+\beta)2a^4h\int_0^{2\pi}\cos^4\theta\,\mathrm{d}\theta\\
=&(\alpha+\beta)2a^4h\int_0^{2\pi}\frac{1}{8}\cos 4\theta+\frac{1}{2}\cos 2\theta+\frac{3}{8}\,\mathrm{d}\theta\\
=&(\alpha+\beta)2a^4h\left[\frac{1}{32}\sin 4\theta+\frac{1}{4}\sin 2\theta+\frac{3}{8}\theta\right]_0^{2\pi}\\
=&(\alpha+\beta)2a^4h\times\frac{3\pi}{4}\\
=&(\alpha+\beta)\frac{3\pi a^4h}{2}
\end{aligned}$$

The integral over the ends of the cylinder:

$$
\begin{aligned}
&\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} \\
=&\int_0^{2\pi} \begin{pmatrix} \alpha x^3 \\ \beta y^3 \\ \gamma h^3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} a \,\mathrm{d}\theta + \int_0^{2\pi} \begin{pmatrix} \alpha x^3 \\ \beta y^3 \\ -\gamma h^3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} a \,\mathrm{d}\theta \\
=&\int_0^{2\pi} a\gamma h^3 \,\mathrm{d}\theta + \int_0^{2\pi} a\gamma h^3 \,\mathrm{d}\theta \\
=&2a\gamma h^3 \int_0^{2\pi} \mathrm{d}\theta \\
=&4a\gamma\pi h^3
\end{aligned}
$$

So the total integral over the surface of the cylinder is equal to

$$(\alpha + \beta)\frac{3\pi a^4 h}{2} + 4a\gamma\pi h^3 = ah\pi(\frac{3}{2}(\alpha + \beta)a^3 + 4\gamma h^2)$$

16. (a) Note that I am fully aware of the divergence theorem – however 18.(a) implied that we should not use it for this part of the question.

Consider the opposite faces of the cube. These have the same limits of integration and so I will integrate them together and then sum the results at the end to get the total integral over the surface of the cube.

$$\mathbf{F} = \begin{pmatrix} x^2 + ay^2 \\ 3xy \\ 6z \end{pmatrix}$$

Integrating over the faces which are perpendicular to the $x$ axis gives:

$$
\begin{aligned}
&\int_0^1 \int_0^1 \begin{pmatrix} 1^2 + ay^2 \\ 3(1)y \\ 6z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} (0)^2 + ay^2 \\ 3(-1)y \\ 6z \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \mathrm{d}y \,\mathrm{d}x \\
=&\int_0^1 \int_0^1 ay^2 + 1 - ay^2 \,\mathrm{d}y \,\mathrm{d}z \\
=&\int_0^1 \int_0^1 1 \,\mathrm{d}y \,\mathrm{d}z \\
=&1
\end{aligned}
$$

Integrating over the faces which are perpendicular to the $y$ axis gives:

$$
\begin{aligned}
&\int_0^1 \int_0^1 \begin{pmatrix} x^2 + a(1)^2 \\ 3x(1) \\ 6z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} x^2 + a(0)^2 \\ 3x(0) \\ 6z \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \mathrm{d}x \,\mathrm{d}z \\
=&\int_0^1 \int_0^1 3x \,\mathrm{d}x \,\mathrm{d}z \\
=&\frac{3}{2}
\end{aligned}
$$

Integrating over the faces which are perpendicular to the $z$ axis gives:

$$\begin{aligned}
&\int_0^1 \int_0^1 \begin{pmatrix} x^2 + ay^2 \\ 3xy \\ 6(1) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} x^2 + ay^2 \\ 3xy \\ 6(0) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{d}x \, \mathrm{d}y \\
=&\int_0^1 \int_0^1 6 \, \mathrm{d}x \, \mathrm{d}y \\
=&6
\end{aligned}$$

So the sum of the integrals over the faces of the cube is equal to:

$$1 + \frac{3}{2} + 6 = 8\frac{1}{2}$$

(b)

$$\begin{aligned}
&\iiint f \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
=&\int_0^1 bx + 6 \, \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \\
=&\left[\frac{1}{2}bx^2 + 6x\right]_0^1 \times 1 \times 1 \\
=&\frac{1}{2}b + 6
\end{aligned}$$

These integrals have the same value whenever:

$$\begin{aligned}
\frac{1}{2}b + 6 &= 8\frac{1}{2} \\
\frac{1}{2}b &= \frac{5}{2} \\
b &= 5
\end{aligned}$$

So these two integrals have the same value when $b = 5$; equality is not dependant on the value of $a$.

17.

$$\begin{aligned}
&\int_S \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S \\
=&\int_S \frac{1}{r} \times \frac{\mathcal{Q}}{4\pi\epsilon_0 r^3} \mathbf{x} \cdot \mathbf{x} \, \mathrm{d}S \\
=&\frac{\mathcal{Q}}{4\pi\epsilon_0 r^4} \int_S \mathbf{x}^2 \, \mathrm{d}S \\
=&\frac{\mathcal{Q}}{4\pi\epsilon_0 r^4} \int_S |\mathbf{x}|^2 \, \mathrm{d}S \\
=&\frac{\mathcal{Q}}{4\pi\epsilon_0 r^4} \int_S r^2 \, \mathrm{d}S \\
=&\frac{\mathcal{Q}}{4\pi\epsilon_0 r^2} \int_S 1 \, \mathrm{d}S \\
=&\frac{\mathcal{Q}}{4\pi\epsilon_0 r^2} \times 4\pi r^2 \\
=&\frac{\mathcal{Q}}{\epsilon_0} \text{ as required}
\end{aligned}$$

18. (a) Using the divergence theorem:

$$\begin{aligned}\int_S \mathbf{F} \cdot \mathrm{d}S &= \int_V (\nabla \cdot \mathbf{F}) \, \mathrm{d}V \\ &= \int_0^1 \int_0^1 \int_0^1 2x + 3x + 6 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= \int_0^1 5x + 6 \, \mathrm{d}x \int_0^1 \mathrm{d}y \int_0^1 \mathrm{d}z \\ &= \frac{5}{2} + 6 \\ &= 8\frac{1}{2}\end{aligned}$$

(b)

$$\begin{aligned}&\nabla \times \mathbf{E} \\ &= \left(0 - 0, 0 - 0, e^{-2t} + e^{-2t}\right) \\ &= \left(0, 0, 2e^{-2t}\right)\end{aligned}$$

$$\begin{aligned}&-\frac{\partial \mathbf{B}}{\partial t} \\ &= -\left(0, 0, -2e^{-2t}\right) \\ &= \left(0, 0, 2e^{-2t}\right)\end{aligned}$$

So $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ as required.

Taking Stokes Theorem and substituting $\mathbf{E} = \mathbf{F}$ gives:

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} &= \int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}S \\ \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{x} &= \int_S (\nabla \times \mathbf{E}) \cdot \mathrm{d}S \\ \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{x} &= -\int_S \left(\frac{\partial \mathbf{B}}{\partial t}\right) \cdot \mathrm{d}S \\ \int_C \mathbf{E} \cdot \mathrm{d}\mathbf{x} &= -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}S\end{aligned}$$

Which gives the required result.

19. The integral of the curl of any vector field over the volume of an object is equal to the integral of that field over the surface of the object.

$$\int_S \mathbf{F} \cdot \mathrm{d}S = \int_V (\nabla \cdot \mathbf{F}) \, \mathrm{d}V$$

Note that the surface can be closed by adding the circle $x^2 + y^2 = 1, z = 0$.

So by Stoke's Theorem:

$$\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \int_V \nabla \times \mathbf{F} \, \mathrm{d}V - \int_{S_c} \mathbf{F} \cdot \mathbf{n}_c \, \mathrm{d}S$$

Where $S_c$ is the surface of the circle and $n_c$ is the normal to the circle.

Using the Divergence Theorem:

$$\begin{aligned}
&\int_V (\nabla \times \mathbf{F})\,\mathrm{d}V \\
=&\int_0^1 \int_0^{2\pi} \int_0^{1-z} (\nabla \times \mathbf{F})\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
=&\int_0^1 \int_0^{2\pi} \int_0^{1-z} 3x^2 + 3y^2 + 6z\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z \\
=&\int_0^{2\pi} \mathrm{d}\theta \int_0^1 \int_0^{1-z} 3r^2 + 6z\,\mathrm{d}r\,\mathrm{d}z \\
=&2\pi \int_0^1 3(1-z)^2 + 6z\,\mathrm{d}z \\
=&2\pi \int_0^1 3z^2 + 3\,\mathrm{d}z \\
=&2\pi \left[z^3 + 3z\right]_0^1 \\
=&8\pi
\end{aligned}$$

$$\begin{aligned}
&\int_0^{2\pi} \begin{pmatrix} x^3 + 3y + 0^2 \\ y^3 \\ x^2 + y^2 + 3 \times 0^2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \mathrm{d}S \\
=&\int_0^{2\pi} -(x^2 + y^2)\,\mathrm{d}S \\
=&\int_0^{2\pi} -1\,\mathrm{d}S \\
=&-2\pi
\end{aligned}$$

So the integral across the surface is equal to:

$$8\pi - -2\pi = 10\pi$$

20. Stoke's Theorem states that:

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

I will integrate across the surface of the hemisphere, and then across the bounding curve and show that Stoke's Theorem holds.

$$\begin{aligned}
&\int_S (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{S} \\
=&\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \begin{pmatrix} 0-0 \\ 0-0 \\ -1-1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
=&\int_0^{2\pi} \mathrm{d}\phi \int_0^{\frac{\pi}{2}} -2z\sin\theta\,\mathrm{d}\theta \\
=&\int_0^{2\pi} \mathrm{d}\phi \int_0^{\frac{\pi}{2}} -2\sin\theta\cos\theta\,\mathrm{d}\theta \\
=&2\pi \left[\cos^2\theta\right]_0^{\frac{\pi}{2}} \\
=&-2\pi
\end{aligned}$$

$$\begin{aligned}
&\int_C \mathbf{F}\cdot\mathrm{d}\mathbf{x}\\
=&\int_0^{2\pi}\begin{pmatrix}\sin\theta\\-\cos\theta\\0\end{pmatrix}\cdot\begin{pmatrix}-\sin\theta\\\cos\theta\\0\end{pmatrix}\mathrm{d}\theta\\
=&\int_0^{2\pi}-\sin^2\theta-\cos^2\theta\,\mathrm{d}\theta\\
=&\int_0^{2\pi}-1\,\mathrm{d}\theta\\
=&-2\pi
\end{aligned}$$

So for the hemispherical surface $r = 1, z \geq 0$ and the vector field $\mathbf{A}(\mathbf{x}) = (y, -x, z)$, Stoke's Theorem holds.