Ordinary Differential Equations

3. (a)

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x^3}{(y+1)^2} \\ (y+1)^2\frac{\mathrm{d}y}{\mathrm{d}x} &= -x^3 \\ \frac{1}{3}(y+1)^3 &= -\frac{1}{4}x^4 + c \\ (y+1)^3 &= c - \frac{3}{4}x^4 \\ y &= -1 + \sqrt[3]{c - \frac{3}{4}x^4}\end{aligned} \tag{1}$$

(b)

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4y}{x(y-3)} \\ \frac{y-3}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4}{x} \\ (1-\frac{3}{y})\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4}{x} \\ y - 3\ln y &= 4\ln x + c\end{aligned} \tag{2}$$

4. (a)

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy &= 4x \\ \mu(x) &= e^{\int 2x dx} \\ \mu(x) &= e^{x^2} \\ e^{x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 2xe^{x^2}y &= 4xe^{x^2} \\ ye^{x^2} &= 2e^{x^2} + c \\ y &= 3 + ce^{-x^2}\end{aligned} \tag{3}$$

(b)

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + (2-3x^2)x^{-3}y &= 1 \\ \mu(x) &= e^{\int \frac{2}{x^3} - \frac{3}{x} dx} \\ \mu(x) &= e^{-\frac{1}{x^2} - 3\ln x} \\ \mu(x) &= x^{-3}e^{-\frac{1}{x^2}} \\ x^{-3}e^{-\frac{1}{x^2}}\frac{\mathrm{d}y}{\mathrm{d}x} + (2-3x^2)x^{-6}e^{-\frac{1}{x^2}}y &= x^{-3}e^{-\frac{1}{x^2}} \\ x^{-3}e^{-\frac{1}{x^2}}y &= \frac{1}{2}e^{-\frac{1}{x^2}} + c \\ y &= \frac{1}{2}x^3 + cx^3e^{\frac{1}{x^2}}\end{aligned} \tag{4}$$

5.

$$\begin{aligned}
v &= (x+y+1) \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}x} + 1 \\
(x+y+1)^2\frac{\mathrm{d}y}{\mathrm{d}x} + (x+y+1)^2 + x^3 &= 0 \\
(x+y+1)^2(\frac{\mathrm{d}y}{\mathrm{d}x} + 1) + x^3 &= 0 \\
v^2\frac{\mathrm{d}v}{\mathrm{d}x} + x^3 &= 0 \\
\frac{1}{3}v^3 + \frac{1}{4}x^4 &= c \\
v^3 &= c - \frac{3}{4}x^4 \\
(x+y+1)^3 &= c - \frac{3}{4}x^4 \\
x+y+1 &= \sqrt[3]{c - \frac{3}{4}x^4} \\
y &= -x - 1 + \sqrt[3]{c - \frac{3}{4}x^4}
\end{aligned} \tag{5}$$

6. (a)

$$\begin{aligned}
v &= y^{-4} \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= -4y^{-5}\frac{\mathrm{d}y}{\mathrm{d}x} \\
-\frac{y^5}{4}\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} - y &= xy^5 \\
-\frac{y^5}{4}\frac{\mathrm{d}v}{\mathrm{d}x} - y &= xy^5 \\
\frac{\mathrm{d}v}{\mathrm{d}x} + 4y^{-4} &= -4x \\
\frac{\mathrm{d}v}{\mathrm{d}x} + 4v &= -4x \\
\mu(x) &= e^{\int 4dx} \\
\mu(x) &= e^{4x} \\
e^{4x}\frac{\mathrm{d}v}{\mathrm{d}x} + 4ve^{4x} &= -4xe^{4x} \\
ve^{4x} &= -xe^{4x} + \frac{1}{4}e^{4x} + c \\
v &= -x + \frac{1}{4} + ce^{-4x} \\
\frac{1}{y^4} &= -x + \frac{1}{4} + ce^{-4x} \\
y &= \sqrt[4]{\frac{1}{-x + \frac{1}{4} + ce^{-4x}}}
\end{aligned} \tag{6}$$

(b)

$$
\begin{aligned}
v &= \frac{1}{y} \\
\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} \\
-y^2\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} + y &= y^2(\cos x - \sin x) \\
-y^2\frac{\mathrm{d}v}{\mathrm{d}x} + y &= y^2(\cos x - \sin x) \\
\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{1}{y} &= \sin x - \cos x \\
\frac{\mathrm{d}v}{\mathrm{d}x} - v &= \sin x - \cos x \\
\mu(x) &= e^{\int -1 dx} \\
\mu x &= e^{-x} \\
e^{-x}\frac{\mathrm{d}v}{\mathrm{d}x} - ve^{-x} &= \sin x e^{-x} - \cos x e^{-x} \\
ve^{-x} &= -\sin x e^{-x} + c \\
v &= -\sin x + ce^{x} \\
\frac{1}{y} &= -\sin x + ce^{x} \\
y &= \frac{1}{-\sin x + ce^{x}}
\end{aligned} \tag{7}
$$

7.

$$
\begin{aligned}
y &= vx \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= v + x\frac{\mathrm{d}v}{\mathrm{d}x} \\
(y - x)\frac{\mathrm{d}y}{\mathrm{d}x} + (2x + 3y) &= 0 \\
x(vx - x)\frac{\mathrm{d}v}{\mathrm{d}x} + v(xv - x) + 2x + 3vx &= 0 \\
x(vx - x)\frac{\mathrm{d}v}{\mathrm{d}x} + v^2x + 2vx + 2x &= 0 \\
x(v - 1)\frac{\mathrm{d}v}{\mathrm{d}x} + v^2 + 2v + 2 &= 0 \\
x(v - 1)\frac{\mathrm{d}v}{\mathrm{d}x} &= -(v^2 + 2v + 2) \\
\frac{v - 1}{v^2 + 2v + 2}\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{x} \\
\left(\frac{v + 1}{v^2 + 2v + 2} - \frac{2}{(v + 1)^2 + 1}\right)\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{x} \\
\frac{1}{2}\ln\big(v^2 + 2v + 2\big) - 2\arctan(v + 1) &= -\ln Ax \\
\frac{1}{2}\ln\left(\frac{y^2}{x^2} + \frac{2y}{x} + 2\right) - 2\arctan\left(\frac{y}{x} + 1\right) &= -\ln Ax
\end{aligned} \tag{8}
$$

8. (a)

$$\begin{aligned}
v &= \frac{y}{x} \\
y &= vx \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= v + x\frac{\mathrm{d}v}{\mathrm{d}x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y}{x} + \tan\Big(\frac{y}{x}\Big) \\
v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= v + \tan v \\
x\frac{\mathrm{d}v}{\mathrm{d}x} &= \tan v \\
\cot v\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{x} \\
\ln\sin v &= \ln Ax \\
\sin v &= Ax \\
\sin\frac{y}{x} &= Ax \\
y &= x\arcsin Ax
\end{aligned} \tag{9}$$

(b)

$$\begin{aligned}
\ln y &= vx \\
\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= v + x\frac{\mathrm{d}v}{\mathrm{d}x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) \\
(\ln y - x)\frac{\mathrm{d}y}{\mathrm{d}x} - y\ln y &= 0 \\
y(\ln y - x)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) - y\ln y &= 0 \\
y(xv - x)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) - yxv &= 0 \\
(v-1)\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) - v &= 0 \\
v^2 - v + x(v-1)\frac{\mathrm{d}v}{\mathrm{d}x} - v &= 0 \\
x(v-1)\frac{\mathrm{d}v}{\mathrm{d}x} &= -(v^2 - 2v) \\
\frac{v-1}{v(v-2)}\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{x} \\
\left(\frac{1}{2v} - \frac{1}{2(v-2)}\right)\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{x} \\
\frac{1}{2}\ln v + \frac{1}{2}\ln(v-2) &= -\ln Ax \\
\ln v + \ln(v-2) &= -2\ln Ax \\
\ln(v(v-2)) &= \ln\frac{1}{(Ax)^2} \\
v(v-2) &= \frac{1}{(Ax)^2} \\
\frac{\ln y}{x}\left(\frac{\ln y}{x} - 2\right) &= \frac{1}{(Ax)^2}
\end{aligned} \tag{10}$$

(c)

$$
\begin{aligned}
v &= xy \\
y &= \frac{v}{x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{v}{x^2} + \frac{1}{x}\frac{\mathrm{d}v}{\mathrm{d}x} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{y}{x} + \frac{1}{x}\frac{\mathrm{d}v}{\mathrm{d}x} \\
xy\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 + y^2 + x) &= 0 \\
xy\left(-\frac{y}{x} + \frac{1}{x}\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (x^2 + y^2 + x) &= 0 \\
-y^2 + y\frac{\mathrm{d}v}{\mathrm{d}x} + x^2 + y^2 + x &= 0 \\
\frac{v}{x}\frac{\mathrm{d}v}{\mathrm{d}x} + x^2 + x &= 0 \\
v\frac{\mathrm{d}v}{\mathrm{d}x} + x^3 + x^2 &= 0 \\
\frac{1}{2}v^2 + \frac{1}{4}x^4 + \frac{1}{3}x^3 &= c \\
x^2y^2 + \frac{1}{2}x^4 + \frac{2}{3}x^3 &= c \\
y^2 &= cx^2 - \frac{1}{2}x^2 - \frac{2}{3}x \\
y &= \sqrt{cx^2 - \frac{1}{2}x^2 - \frac{2}{3}x}
\end{aligned}
\tag{11}
$$