Ordinary Differential Equations

9. (a)

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y &= 0\\ \lambda^2 - 5\lambda + 6 &= 0\\ (\lambda-2)(\lambda-3) &= 0\\ \lambda = 2 \vee \lambda &= 3\\ y_c &= Ae^{2x} + Be^{3x}\\ y_p &= 0\\ y &= y_c + y_p\\ y &= Ae^{2x} + Be^{3x}\end{aligned} \tag{1}$$

$$\begin{aligned}y(0) &= 0\\ 0 &= Ae^{2\times 0} + Be^{2\times 0}\\ 0 &= A + B\end{aligned} \tag{2}$$

$$\begin{aligned}y'(0) &= 1\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 2Ae^{2x} + 3Be^{3x}\\ \frac{\mathrm{d}y}{\mathrm{d}x}(0) &= 2Ae^{2\times 0} + 3Be^{3\times 0}\\ 1 &= 2A + 3B\\ 1 &= 2(A+B) + B\\ 1 &= B\\ 0 &= A + 1\\ A &= -1\\ y &= -e^{2x} + e^{3x}\end{aligned} \tag{3}$$

(b)

$$\begin{aligned}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} + n^2\right)y &= 0\\ \lambda^2 + n^2 &= 0\\ \lambda &= \pm ni\\ y_c &= P\sin nx + Q\cos nx\\ y_p &= 0\\ y &= y_c + y_p\\ y &= P\sin nx + Q\cos nx\end{aligned} \tag{4}$$

$$\begin{aligned}y(0) &= 0\\ 0 &= P\sin 0 + Q\cos 0\\ 0 &= Q\end{aligned} \tag{5}$$

$$\begin{aligned}y'(0) &= 1\\ \frac{\mathrm{d}y}{\mathrm{d}x}(0) &= nP\cos 0\\ 1 &= nP\\ P &= \frac{1}{n}\\ y &= \frac{1}{n}\sin nx\end{aligned} \tag{6}$$

(c)

$$\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+2\frac{\mathrm{d}}{\mathrm{d}x}+4\right)y&=0\\ \lambda^2+2\lambda+4&=0\\ \lambda&=\frac{-2\pm\sqrt{4-16}}{2}\\ \lambda&=-1\pm\sqrt{3}i\\ y_c&=e^{-x}(P\sin\left(\sqrt{3}x\right)+Q\cos\left(\sqrt{3}x\right))\\ y_p&=0\\ y&=y_p+y_c\\ y&=e^{-x}(P\sin\left(\sqrt{3}x\right)+Q\cos\left(\sqrt{3}x\right))\end{aligned}\tag{7}$$

$$\begin{aligned}y(0)&=0\\ 0&=e^0(P\sin 0+Q\cos 0)\\ 0&=Q\end{aligned}\tag{8}$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=-Pe^{-x}\sin\left(\sqrt{3}x\right)+\sqrt{3}Pe^{-x}\cos\left(\sqrt{3}x\right)\\ \frac{\mathrm{d}y}{\mathrm{d}x}(0)&=-Pe^0\sin 0+e^0\sqrt{3}P\cos(0)\\ 1&=0+\sqrt{3}P\\ P&=\frac{1}{\sqrt{3}}\\ y&=\frac{1}{\sqrt{3}}e^{-x}\sin\left(\sqrt{3}x\right)\end{aligned}\tag{9}$$

(d)

$$\begin{aligned}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+9\right)y&=18\\ \lambda^2+9&=0\\ \lambda&=\pm 3i\\ y_c&=P\sin 3x+Q\cos 3x+y_p\\ y_p&=c\\ 9c&=18\\ c&=2\\ y&=y_c+y_p\\ y&=P\sin 3x+Q\cos 3x+2\end{aligned}\tag{10}$$

$$\begin{aligned}y(0)&=0\\ 0&=P\sin 0+Q\cos 0+2\\ 0&=0+Q+2\\ Q&=-2\end{aligned}\tag{11}$$

$$\begin{aligned} y'(0) &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= 3P\cos 3x - 6\sin 3x \\ \frac{\mathrm{d}y}{\mathrm{d}x}(0) &= 3P\cos 0 - 6\sin 0 \\ 1 &= 3P \\ P &= \frac{1}{3} \\ y &= \frac{1}{3}\sin 3x - 2\cos 3x + 2 \end{aligned} \tag{12}$$

(e)

$$\begin{aligned} \left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 3\frac{\mathrm{d}}{\mathrm{d}x} + 2\right) y &= e^{5x} \\ \lambda^2 - 3\lambda + 2 &= 0 \\ (\lambda - 1)(\lambda - 2) &= 0 \\ \lambda = 1 \vee \lambda &= 2 \end{aligned} \tag{13}$$

$$\begin{aligned} y_c &= Ae^x + Be^{2x} \\ y_p &= ke^{5x} \\ \frac{\mathrm{d}y_p}{\mathrm{d}x} &= 5ke^{5x} \\ \frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} &= 25ke^{5x} \\ \frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y_p}{\mathrm{d}x} + 2y_p &= e^{5x} \\ (25k - 15k + 2k)e^{5x} &= e^{5x} \\ 12k &= 1 \\ k &= \frac{1}{12} \\ y &= y_c + y_p \\ y &= Ae^x + Be^{2x} + \frac{1}{12}e^{5x} \end{aligned} \tag{14}$$

$$\begin{aligned} y(0) &= 0 \\ 0 &= Ae^0 + Be^0 + \frac{1}{12}e^0 \\ 0 &= A + B + \frac{1}{12} \end{aligned} \tag{15}$$

$$
\begin{aligned}
y'(0) &= 1 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= Ae^x + 2Be^{2x} + \frac{5}{12}e^{5x} \\
\frac{\mathrm{d}y}{\mathrm{d}x}(0) &= Ae^0 + 2Be^0 + \frac{5}{12}e^0 \\
1 &= A + 2B + \frac{5}{12} \\
1 &= (A + B + \frac{1}{12}) + B + \frac{4}{12} \\
B &= \frac{2}{3} \\
A &= -\frac{1}{12} - B \\
A &= -\frac{3}{4} \\
y &= -\frac{3}{4}e^x + \frac{2}{3}e^{2x} + \frac{1}{12}e^{5x}
\end{aligned} \tag{16}
$$

(f)

$$
\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \\
\lambda^2 - 2\lambda + 1 = 0 \\
\lambda &= 1 \\
y_c &= (A + Bx)e^x \\
y_p &= 0 \\
y &= y_c + y_p \\
y &= (A + Bx)e^x
\end{aligned} \tag{17}
$$

(g) The complementary function is the same as in the previous question so we can reuse the result and only need to work out the particular integral and the initial conditions.

$$
\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = e^{2x} + e^x \\
y_p &= Pe^{2x} + Qx^2e^x \\
\frac{\mathrm{d}y_p}{\mathrm{d}x} &= 2Pe^{2x} + 2Qxe^x + Qx^2e^x \\
\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} &= 4Pe^{2x} + 2Qe^x + 4Qxe^x + Qx^2e^x
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y_p}{\mathrm{d}x} + y &= e^{2x} + e^x \\
4Pe^{2x} - 4Pe^{2x} + Pe^{2x} + 2Qe^x &= e^{2x} + e^x \\
Pe^{2x} + 2Qe^x &= e^{2x} + e^x \\
P = 1 \wedge Q &= \frac{1}{2} \\
y_p &= e^{2x} + \frac{1}{2}x^2e^x \\
y &= y_c + y_p \\
y &= (A + Bx + \frac{1}{2}x^2)e^x + e^{2x}
\end{aligned} \tag{19}
$$

10.

$$
\begin{aligned}
-iR &= \frac{q}{C} + V \\
i &= -\frac{q}{RC} - \frac{V}{R} \\
-L\frac{\mathrm{d}j}{\mathrm{d}t} &= \frac{q}{C} + V \\
\frac{\mathrm{d}j}{\mathrm{d}t} &= -\frac{q}{LC} - \frac{V}{L} \\
\frac{\mathrm{d}q}{\mathrm{d}t} &= i + j \\
\frac{\mathrm{d}q}{\mathrm{d}t} &= -\frac{q}{RC} - \frac{V}{R} + j \\
\frac{\mathrm{d}^2q}{\mathrm{d}t^2} &= -\frac{1}{RC}\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{1}{R}\frac{\mathrm{d}V}{\mathrm{d}t} + \frac{\mathrm{d}j}{\mathrm{d}t} \\
\frac{\mathrm{d}^2q}{\mathrm{d}t^2} &= -\frac{1}{RC}\frac{\mathrm{d}q}{\mathrm{d}t} - \frac{1}{R}\frac{\mathrm{d}V}{\mathrm{d}t} - \frac{q}{LC} - \frac{V}{L} \\
\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC}q &= -\frac{1}{R}\frac{\mathrm{d}V}{\mathrm{d}t} - \frac{1}{L}V \text{ as required}
\end{aligned} \tag{20}
$$

(a) Since we have initial conditions $Q = 0$ and Q is not changing, we know that both $Q = 0$ and $\frac{\mathrm{d}Q}{\mathrm{d}t} = 0$. This means that the RHS of the equation is now 0.

$$
\begin{aligned}
\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC}q &= 0 \\
L &= 8R^2C \\
\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{8R^2C^2}q &= 0 \\
\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{8R^2C^2} &= 0
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\lambda &= \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{8R^2C^2}}}{2} \\
\lambda &= \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{2R^2C^2}}}{2} \\
\lambda &= \frac{-\frac{1}{RC} \pm \frac{1}{\sqrt{2}RC}}{2} \\
\lambda &= \frac{-2 \pm \sqrt{2}}{4RC} \\
q &= Ae^{\left(\frac{-2+\sqrt{2}}{4RC}\right)t} + Be^{\left(\frac{-2-\sqrt{2}}{4RC}\right)t}
\end{aligned} \tag{22}
$$

$$\begin{aligned}
q(0) &= Q \\
A + B &= Q \\
\frac{\mathrm{d}q}{\mathrm{d}t}(0) &= -\frac{Q}{RC} \\
\left(\frac{-2+\sqrt{2}}{4RC}\right)A + \left(\frac{-2-\sqrt{2}}{4RC}\right)B &= -\frac{Q}{RC} \\
\frac{-2(A+B)}{4RC} + \frac{\sqrt{2}(A-B)}{4RC} &= -\frac{Q}{RC} \\
-\frac{Q}{2RC} + \frac{2\sqrt{2}A}{4RC} - \frac{\sqrt{2}Q}{4RC} &= -\frac{Q}{RC} \\
\frac{\sqrt{2}A}{2RC} &= \frac{(\sqrt{2}-2)Q}{4RC} \\
A &= \frac{(1-\sqrt{2})Q}{2}
\end{aligned} \tag{23}$$

$$\begin{aligned}
A + B &= Q \\
B &= Q - \frac{(1-\sqrt{2})Q}{2} \\
B &= \frac{(1+\sqrt{2})Q}{2}
\end{aligned} \tag{24}$$

$$q = \frac{(1-\sqrt{2})Q}{2}e^{\left(\frac{-2+\sqrt{2}}{4RC}\right)t} + \frac{(1+\sqrt{2})Q}{2}e^{\left(\frac{-2-\sqrt{2}}{4RC}\right)t} \tag{25}$$

The roots to this equation are distinct and real: this is strong dampening. The charge will decrease to 0 exponentially without ever oscillating.

(b)

$$\begin{aligned}
\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC}q &= 0 \\
L &= 4R^2C^2 \\
\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{4R^2C^2}q &= 0 \\
\lambda^2 + \frac{1}{RC}\lambda + \frac{1}{4R^2C^2} &= 0
\end{aligned} \tag{26}$$

$$\begin{aligned}
\lambda &= \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{4R^2C^2}}}{2} \\
\lambda &= -\frac{1}{2RC} \\
q &= (A + Bt)e^{-\frac{t}{2RC}}
\end{aligned} \tag{27}$$

$$\begin{aligned}
q(0) &= Q \\
A &= Q
\end{aligned} \tag{28}$$

$$\begin{aligned}
\frac{\mathrm{d}q}{\mathrm{d}t}(0) &= -\frac{Q}{RC} \\
Be^{-\frac{0}{2RC}} - \frac{Q}{2RC}e^{-\frac{t}{2RC}} &= -\frac{Q}{RC} \\
B - \frac{Q}{2RC} &= -\frac{Q}{RC} \\
B &= -\frac{Q}{2RC}
\end{aligned} \tag{29}$$

$$q = (Q - \frac{Qt}{2RC})e^{-\frac{t}{2RC}} \tag{30}$$

In this equation, the roots to the simultaneous are repeated: this is critial dampening. In this form the charge will decrease to 0 without ever increasing or oscillating – however this decrease will be faster than for strong dampening.

(c)

$$\begin{aligned} \frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{LC}q = 0 \\ L = 2R^2C^2 \\ \frac{\mathrm{d}^2q}{\mathrm{d}t^2} + \frac{1}{RC}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{2R^2C^2}q = 0 \\ \lambda^2 + \frac{1}{RC}\lambda + \frac{1}{2R^2C^2} = 0 \end{aligned} \tag{31}$$

$$\begin{aligned} \lambda &= \frac{-\frac{1}{RC} \pm \sqrt{\frac{1}{R^2C^2} - \frac{4}{2R^2C^2}}}{2} \\ \lambda &= -\frac{1}{2RC} \pm \frac{1}{2RC}i \\ q &= \left(A\sin\left(\frac{t}{2RC}\right) + B\cos\left(\frac{t}{2RC}\right)\right)e^{-\frac{t}{2RC}} \end{aligned} \tag{32}$$

$$\begin{aligned} q(0) = Q \\ B = Q \end{aligned} \tag{33}$$

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}t}(0) &= -\frac{Q}{RC} \\ -\frac{q}{2RC} + \frac{1}{2RC}\left(A\cos\left(\frac{t}{2RC}\right) - Q\sin\left(\frac{t}{2RC}\right)\right)e^{-\frac{t}{2RC}} &= -\frac{Q}{RC} \\ -\frac{Q}{2RC} + \frac{A}{2RC} &= -\frac{Q}{RC} \\ A &= -Q \end{aligned} \tag{34}$$

$$q = \left(-Q\sin\left(\frac{t}{2RC}\right) + Q\cos\left(\frac{t}{2RC}\right)\right)e^{-\frac{t}{2RC}} \tag{35}$$

The roots to the equation in this case are complex: this is weak dampening – in this version the charge will oscillate with constant angular frequency between two exponentially decreasing values.

11. (a)

$$\begin{aligned} y'' - (2+c)y' + (1+c)y = e^{(1+2c)x} \\ \lambda^2 - (2+c)\lambda + (1+c) = 0 \\ (\lambda - 1)(\lambda - (1+c)) = 0 \\ \lambda = 1 \vee \lambda = 1 + c \end{aligned} \tag{36}$$

$$\begin{aligned} y_c &= Pe^x + Qe^{(1+c)x} \\ y_p &= ke^{(1+2c)x} \\ \frac{\mathrm{d}y_p}{\mathrm{d}x} &= k(1+2c)e^{(1+2c)x} \\ \frac{\mathrm{d}^2y_p}{\mathrm{d}x^2} &= k(1+2c)^2e^{(1+2c)x} \end{aligned} \tag{37}$$

$$\begin{aligned} y_p'' + (2+c)y_p' + (1+c)y_p &= e^{(1+2c)x} \\ k((1+2c)^2 - (2+c)(1+2c) + (1+c))e^{(1+2c)x} &= e^{(1+2c)x} \\ k(4c^2 + 4c + 1 - 2c^2 - 5c - 2 + 1 + c) &= 1 \\ 2c^2 k &= 1 \\ k &= \frac{1}{2c^2} \end{aligned} \tag{38}$$

$$\begin{aligned} k &= \frac{1}{2(c+1)(3c+2)} \\ y_p &= \frac{1}{2(c+1)(3c+2)} e^{(1+2c)x} \\ y &= y_c + y_p \\ y &= Pe^x + Qe^{(1+c)x} + \frac{1}{2c^2} e^{(1+2c)x} \end{aligned} \tag{39}$$

Let:

$$\begin{aligned} P &= A - \frac{B}{c} + \frac{1}{2c^2} \\ Q &= B\frac{1}{c} - \frac{1}{c^2} \end{aligned} \tag{40}$$

$$\begin{aligned} y &= Ae^x - B\frac{1}{c}e^x + \frac{1}{2c^2}e^x + B\frac{1}{c}e^{(1+c)x} - \frac{1}{c^2}e^{(1+c)x} + \frac{1}{2c^2}e^{(1+2c)x} \\ y &= Ae^x + B\frac{1}{c}(e^{(1+c)x} - e^x) + \frac{1}{2c^2}(e^{(1+2c)x} - 2e^{(1+c)x} + e^x) \\ y &= Ae^x + B\frac{e^x}{c}(e^{cx} - 1) + \frac{e^x}{2c^2}(e^{2cx} - 2e^{cx} + 1) \text{ as required} \end{aligned} \tag{41}$$

(b) To find the limit as $c \to 0$, I will apply l'hôpitals rule separately for the different parts of the expression.

$$\lim_{c\to 0} f(x,c) = Ae^x + Be^x \lim_{c\to 0}\left(\frac{(e^{cx}-1)}{c}\right) + e^x \lim_{c\to 0}\left(\frac{(e^{2cx} - 2e^{cx} + 1)}{2c^2}\right) \tag{42}$$

$$\begin{aligned} \frac{(e^{cx}-1)}{c} &= \frac{f(c)}{g(c)} \\ \lim_{c\to 0} f(c) &= e^0 - 1 = 0 \\ \lim_{c\to 0} g(c) &= 0 \end{aligned} \tag{43}$$

So we can apply l'hôpitals rule to find the limit as $c \to 0$.

$$\begin{aligned} \lim_{c\to 0} B\frac{e^x}{c}(e^{cx}-1) &= \frac{\frac{\mathrm{d}}{\mathrm{d}c}(e^{cx}-1)}{\frac{\mathrm{d}}{\mathrm{d}c}(c)} \\ &= \frac{xe^{cx}}{1} \\ &= xe^{cx} \\ &= xe^0 \\ &= x \end{aligned} \tag{44}$$

$$\begin{aligned} \frac{(e^{2cx} - 2e^{cx} + 1)}{2c^2} &= \frac{f(c)}{g(c)} \\ \lim_{c\to 0} f(c) &= e^0 - 2e^0 + 1 = 0 \\ \lim_{c\to 0} g(c) &= 2 \cdot 0^2 = 0 \end{aligned} \tag{45}$$

So we can apply l'hôpitals rule to find the limit as $c \to 0$.

$$\begin{aligned}\lim_{c\to 0} B\frac{e^x}{c}(e^{cx}-1) &= \frac{\frac{\mathrm{d}}{\mathrm{d}c}\left(e^{2cx}-2e^{cx}+1\right)}{\frac{\mathrm{d}}{\mathrm{d}c}(2c^2)} \\ &= \frac{2xe^{2cx}-2xe^{cx}}{4c}\end{aligned} \tag{46}$$

Now we must find the limit of this new function as $c \to 0$.

$$\begin{aligned}\frac{2xe^{2cx}-2xe^{cx}}{4c} &= \frac{f(x)}{g(x)} \\ \lim_{c\to 0} f(x) &= 2xe^0 - 2xe^0 = 2x - 2x = 0 \\ \lim_{c\to 0} g(x) &= 4 \times 0 = 0\end{aligned} \tag{47}$$

So we can apply l'hôpitals rule once more.

$$\begin{aligned}\lim_{c\to 0}\frac{2xe^{2cx}-2xe^{cx}}{4c} &= \frac{\frac{\mathrm{d}}{\mathrm{d}c}\left(2xe^{2cx}-2xe^{cx}\right)}{\frac{\mathrm{d}}{\mathrm{d}c}(4c)} \\ &= \frac{4x^2e^{2cx}-2x^2e^{cx}}{4} \\ &= x^2e^0 - \frac{1}{2}x^2e^0 \\ &= \frac{1}{2}x^2\end{aligned} \tag{48}$$

Substituting these results back into the original expression gives:

$$\begin{aligned}\lim_{c\to 0} f(x,c) &= Ae^x + Bxe^x + \frac{1}{2}x^2e^x \\ &= (A + Bx + \frac{1}{2}x^2)e^x\end{aligned} \tag{49}$$

So the solution to the differential equation when $c = 0$ is:

$$y = (A + Bx + \frac{1}{2}x^2)e^x \tag{50}$$

12.

$$\begin{aligned}-\frac{\eta}{2}\frac{\mathrm{d}C}{\mathrm{d}\eta} &= \frac{\mathrm{d}^2C}{\mathrm{d}\eta^2} \\ \frac{1}{\frac{\mathrm{d}C}{\mathrm{d}\eta}}\frac{\mathrm{d}^2C}{\mathrm{d}\eta^2} &= -\frac{\eta}{2} \\ \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\ln\left(\frac{\mathrm{d}C}{\mathrm{d}\eta}\right)\right) &= -\frac{\eta}{2} \text{ as required}\end{aligned} \tag{51}$$

$$\begin{aligned}
\ln\left(\frac{\mathrm{d}C}{\mathrm{d}\eta}\right) &= -\int \frac{\eta}{2} d\eta \\
\ln\left(\frac{\mathrm{d}C}{\mathrm{d}\eta}\right) &= -\frac{\eta^2}{4} + c \\
\frac{\mathrm{d}C}{\mathrm{d}\eta} &= e^{-\frac{\eta^2}{4}+c} \\
\frac{\mathrm{d}C}{\mathrm{d}\eta} &= Ae^{-\frac{\eta^2}{4}} \\
\frac{\mathrm{d}C}{\mathrm{d}\eta}(0) &= Ae^0 \\
\frac{\mathrm{d}C}{\mathrm{d}\eta}(0) &= A \\
C &= B + A\int_0^\eta e^{-\frac{t^2}{4}} dt \\
C(0) &= B + A\int_0^0 e^{-\frac{t^2}{4}} dt \\
C(0) &= B
\end{aligned} \tag{52}$$

So if $\frac{\mathrm{d}C}{\mathrm{d}\eta}(0) = A$ and $C(0) = B$ then $C = B + A\int_0^\eta e^{-\frac{t^2}{4}} dt$ as required.

13. (a)

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= ax \\
\frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t} &= a \\
\ln x &= at + c \\
x &= ke^{at} \\
x(0) &= 2 \\
2 &= ke^0 \\
2 &= k \\
x &= 2e^{at}
\end{aligned} \tag{53}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= ay + bx \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= ay + 2be^{at} \\
\frac{\mathrm{d}y}{\mathrm{d}t} - ay &= 2be^{at} \\
\mu(y) &= e^{\int -a dt} \\
\mu(y) &= e^{-at} \\
e^{-at}\frac{\mathrm{d}y}{\mathrm{d}t} - aye^{-at} &= 2b \\
ye^{-at} &= 2bt + C \\
y &= 2bte^{at} + Ce^{at} \\
y(0) &= 1 \\
1 &= C \\
y &= 2bte^{at} + e^{at}
\end{aligned} \tag{54}$$

(b)

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= x - xy \\
\frac{\mathrm{d}t}{\mathrm{d}x} &= \frac{1}{x - xy} \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= -y + xy \\
\frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}x} &= \frac{-y + xy}{x - xy} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y(x-1)}{x(1-y)}
\end{aligned} \tag{55}$$

And so the coupled differential equations can be transformed into a differential equation of the form $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ as required.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y(x-1)}{x(1-y)} \\
\frac{1-y}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x-1}{x} \\
\left(\frac{1}{y} - 1\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= \left(1 - \frac{1}{x}\right) \\
\ln y - y &= x - \ln x + c \\
e^{\ln y - y} &= e^{x - \ln x + c} \\
ye^{-y} &= e^c \frac{e^x}{x} \\
e^{-c} &= \frac{e^y}{y} \cdot \frac{e^x}{x} \\
A &= \frac{e^y}{y} \cdot \frac{e^x}{x}
\end{aligned} \tag{56}$$

And so $\frac{e^y}{y} \cdot \frac{e^x}{x}$ is independent of t as required.