

10. (a)

$$\begin{aligned} y \, dx + x \, dy \\ \frac{\partial}{\partial y} (y) &= 1 \\ \frac{\partial}{\partial x} (x) &= 1 \end{aligned} \tag{1}$$

Since $\frac{\partial}{\partial y} (P) = \frac{\partial}{\partial x} (Q)$ the differential is exact.

$$\begin{aligned} y \, dx + x \, dy &= 0 \\ xy &= c \\ y &= \frac{c}{x} \end{aligned} \tag{2}$$

(b)

$$\begin{aligned} y \, dx + x^2 \, dy \\ \frac{\partial}{\partial y} (y) &= 1 \\ \frac{\partial}{\partial x} (x^2) &= 2x \neq 1 \end{aligned} \tag{3}$$

$\frac{\partial}{\partial y} (P) \neq \frac{\partial}{\partial x} (Q)$ and so the differential is not exact.

$$\begin{aligned} \mu(x) &= e^{\int \frac{1}{x^2} \left(\frac{\partial}{\partial y} (y) - \frac{\partial}{\partial x} (x^2) \right) dx} \\ &= e^{\int \frac{1}{x^2} (1 - 2x) dx} \\ &= e^{\int \frac{1}{x^2} - \frac{2}{x} dx} \\ &= e^{-\frac{1}{x} - 2 \ln x} \\ &= x^{-2} e^{-\frac{1}{x}} \end{aligned} \tag{4}$$

$$\begin{aligned} y \, dx + x^2 \, dy &= 0 \\ x^{-2} y e^{-\frac{1}{x}} \, dx + e^{-\frac{1}{x}} \, dy &= 0 \\ y e^{-\frac{1}{x}} &= c \\ y &= c e^{\frac{1}{x}} \end{aligned} \tag{5}$$

(c)

$$\begin{aligned} (x + y) \, dx + (x - y) \, dy \\ \frac{\partial}{\partial y} (x + y) &= 1 \\ \frac{\partial}{\partial x} (x - y) &= 1 \end{aligned} \tag{6}$$

$\frac{\partial}{\partial y} (P) = \frac{\partial}{\partial x} (Q)$ and so the differential is exact.

$$\begin{aligned} (x + y) \, dx + (x - y) \, dy &= 0 \\ (x + y) \, dx + (x - y) \, dy &= 0 \\ \frac{1}{2} x^2 + xy - \frac{1}{2} y^2 &= c \end{aligned} \tag{7}$$

(d)

$$\begin{aligned} & (\cosh x \cos y + \cosh y \cos x) dx - (\sinh x \sin y - \sinh y \sin x) dy \\ & \frac{\partial}{\partial y} (\cosh x \cos y + \cosh y \cos x) = -\cosh x \sin y + \sinh y \cos x \\ & \frac{\partial}{\partial x} (-\sinh x \sin y + \sinh y \sin x) = -\cosh x \sin y + \sinh y \cos x \end{aligned} \quad (8)$$

So $\frac{\partial}{\partial y} (P) = \frac{\partial}{\partial x} (Q)$ and so the differential is exact.

$$\begin{aligned} & (\cosh x \cos y + \cosh y \cos x) dx - (\sinh x \sin y - \sinh y \sin x) dy = 0 \\ & \sinh x \cos y + \cosh y \sin x = c \end{aligned} \quad (9)$$

(e)

$$\begin{aligned} & (\cos x - \sin x) dx + (\sin x + \cos x) dy \\ & \frac{\partial}{\partial y} (\cos x - \sin x) = 0 \\ & \frac{\partial}{\partial x} (\sin x + \cos x) = \cos x - \sin x \end{aligned} \quad (10)$$

So $\frac{\partial}{\partial y} (P) \neq \frac{\partial}{\partial x} (Q)$ and so the differential is not exact.

$$\begin{aligned} \mu(y) &= e^{\int \frac{1}{(\sin x + \cos x)} \left(\frac{\partial}{\partial y} ((\cos x - \sin x)) - \frac{\partial}{\partial x} (\sin x + \cos x) \right) dx} \\ \mu(y) &= e^{\int \frac{1}{(\sin x + \cos x)} (0 - \cos x + \sin x) dx} \\ \mu(y) &= e^{\int 1 dx} \\ \mu(y) &= e^y \end{aligned} \quad (11)$$

$$\begin{aligned} & (\cos x - \sin x) dx + (\sin x + \cos x) dy = 0 \\ & (\cos x - \sin x)e^y dx + (\sin x + \cos x)e^y dy = 0 \\ & (\sin x + \cos x)e^y = c \\ & e^y = \frac{c}{\sin x + \cos x} \\ & y = -\ln A(\sin x + \cos x) \end{aligned} \quad (12)$$

(f)

$$\begin{aligned} & \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx \\ & \frac{\partial}{\partial y} \left(-\frac{y}{x^2 + y^2} \right) = \frac{y^2 - x^2}{x^2 + y^2} \\ & \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = \frac{y^2 - x^2}{x^2 + y^2} \end{aligned} \quad (13)$$

So $\frac{\partial}{\partial y} (P) = \frac{\partial}{\partial x} (Q)$ and so the differential is exact.

$$\begin{aligned} & \frac{x}{x^2 + y^2} dy - \frac{y}{x^2 + y^2} dx = 0 \\ & \arctan \left(\frac{y}{x} \right) = c \\ & \frac{y}{x} = k \\ & y = kx \end{aligned} \quad (14)$$

11. (a)

$$\begin{aligned} H &= U + pV \\ dH &= dU + V dp + p dV \\ dH &= (dU + p dV) + V dp \\ dH &= T dS + V dp \end{aligned} \quad (15)$$

We know this is an exact differential. So it is of the form

$$dH = P dS + Q dp \quad (16)$$

Where $\left(\frac{\partial P}{\partial p}\right)_S = \left(\frac{\partial Q}{\partial S}\right)_p$. Substituting in the actual coefficients of the equation ($P = T$, $Q = V$) gives us:

$$\left(\frac{\partial V}{\partial S}\right)_p = \left(\frac{\partial T}{\partial p}\right)_S \quad (17)$$

As required.

(b)

$$\begin{aligned} dU &= T dS - p dV \\ dU &= T \left(\left(\frac{\partial S}{\partial p}\right)_V dp + \left(\frac{\partial S}{\partial V}\right)_p dV \right) - p dV \\ dU &= T \left(\frac{\partial S}{\partial p}\right)_V dp + \left(T \left(\frac{\partial S}{\partial V}\right)_p - p \right) dV \end{aligned} \quad (18)$$

We know that dU is an exact integral – this implies:

$$\begin{aligned} \frac{\partial}{\partial p} \left(T \left(\frac{\partial S}{\partial V}\right)_p - p \right) &= \frac{\partial}{\partial V} \left(T \left(\frac{\partial S}{\partial p}\right)_V \right) \\ \left(\frac{\partial T}{\partial p}\right)_V \left(\frac{\partial S}{\partial V}\right)_p + T \left(\frac{\partial^2 S}{\partial p \partial V}\right) - 1 &= \left(\frac{\partial T}{\partial V}\right)_p \left(\frac{\partial S}{\partial p}\right)_V + T \left(\frac{\partial^2 S}{\partial p \partial V}\right) \\ \left(\frac{\partial S}{\partial V}\right)_p \left(\frac{\partial T}{\partial p}\right)_V - 1 &= \left(\frac{\partial S}{\partial p}\right)_V \left(\frac{\partial T}{\partial V}\right)_p \\ \left(\frac{\partial S}{\partial V}\right)_p \left(\frac{\partial T}{\partial p}\right)_V - \left(\frac{\partial S}{\partial p}\right)_V \left(\frac{\partial T}{\partial V}\right)_p &= 1 \end{aligned} \quad (19)$$

12.

$$\begin{aligned} G &= U + Vp - ST \\ dG &= dU + \frac{\partial}{\partial p} (Vp)_V dp + \frac{\partial}{\partial V} (Vp)_p dV - \frac{\partial}{\partial S} (ST)_T dS - \frac{\partial}{\partial T} (ST)_S dT \\ dG &= dU + V dp + p dV - T dS - S dT \\ dG &= T dS - p dV + V dp + p dV - T dS - S dT \\ dG &= V dp - S dT \end{aligned} \quad (20)$$

Since we know that dG is an exact differential; we know that the partial derivative of V with respect to T is equal to the partial derivative of $-S$ with respect to p .

$$\begin{aligned}\left(\frac{\partial V}{\partial T}\right)_p &= -\left(\frac{\partial S}{\partial p}\right)_T \\ \left(\frac{\partial S}{\partial p}\right)_T &= -\left(\frac{\partial V}{\partial T}\right)_p\end{aligned}\tag{21}$$

13. (a) Since p is a function of either V and T or V and S ; we can express dp as a function of either also. We can also express any of the variables V, T, S as a function of the other two. I will express S as a function of V and T .

$$dp = \left(\frac{\partial p}{\partial V}\right)_T dV + \left(\frac{\partial p}{\partial T}\right)_V dT\tag{22}$$

$$dp = \left(\frac{\partial p}{\partial V}\right)_S dV + \left(\frac{\partial p}{\partial S}\right)_V dS\tag{23}$$

$$dS = \left(\frac{\partial S}{\partial V}\right)_T dV + \left(\frac{\partial S}{\partial T}\right)_V dT\tag{24}$$

Substitute (24) into (23).

$$\begin{aligned}dp &= \left(\frac{\partial p}{\partial V}\right)_S dV + \left(\frac{\partial p}{\partial S}\right)_V dS \\ dp &= \left(\frac{\partial p}{\partial V}\right)_S dV + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T dV + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial T}\right)_V dT \\ dp &= \left(\frac{\partial p}{\partial V}\right)_S dV + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T dV + \left(\frac{\partial p}{\partial T}\right)_V dT \\ dp &= \left(\left(\frac{\partial p}{\partial V}\right)_S + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T\right) dV + \left(\frac{\partial p}{\partial T}\right)_V dT\end{aligned}\tag{25}$$

Now subtract (22) from (25).

$$\begin{aligned}dp - dp &= \left(\left(\frac{\partial p}{\partial V}\right)_S + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T\right) dV + \left(\frac{\partial p}{\partial T}\right)_V dT - \left(\frac{\partial p}{\partial V}\right)_T dV - \left(\frac{\partial p}{\partial T}\right)_V dT \\ 0 &= \left(\left(\frac{\partial p}{\partial V}\right)_S - \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T\right) dV + \left(\left(\frac{\partial p}{\partial T}\right)_V - \left(\frac{\partial p}{\partial T}\right)_V\right) dT \\ 0 &= \left(\left(\frac{\partial p}{\partial V}\right)_S - \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T\right) dV\end{aligned}\tag{26}$$

So the coefficients of dV must be equal to zero.

$$\begin{aligned}0 &= \left(\frac{\partial p}{\partial V}\right)_S - \left(\frac{\partial p}{\partial V}\right)_T + \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T \\ \left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S &= \left(\frac{\partial p}{\partial S}\right)_V \left(\frac{\partial S}{\partial V}\right)_T \\ \left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S &= \frac{\left(\frac{\partial S}{\partial V}\right)_T}{\left(\frac{\partial S}{\partial p}\right)_V}\end{aligned}\tag{27}$$

(b)

$$\begin{aligned}T dS &= dU + p dV \\ T dS &= \left(\frac{\partial U}{\partial T}\right)_V dT + \left(\left(\frac{\partial U}{\partial V}\right)_T + p\right) dV \\ dS &= \left(\frac{1}{T} \left(\frac{\partial U}{\partial T}\right)_V\right) dT + \left(\frac{1}{T} \left(\frac{\partial U}{\partial V}\right)_T + \frac{p}{T}\right) dV\end{aligned}\tag{28}$$

dS is an exact differential. So the right hand side must also be an exact differential.

$$\begin{aligned}\frac{1}{T} \left(\frac{\partial^2 U}{\partial T \partial V} \right) &= -\frac{1}{T^2} \left(\frac{\partial U}{\partial V} \right)_T + \frac{1}{T} \left(\frac{\partial^2 U}{\partial T \partial V} \right) - \frac{p}{T^2} + \frac{1}{T} \left(\frac{\partial p}{\partial T} \right)_V \\ 0 &= -\frac{1}{T^2} \left(\frac{\partial U}{\partial V} \right)_T - \frac{p}{T^2} + \frac{1}{T} \left(\frac{\partial p}{\partial T} \right)_V \\ \frac{T}{p} \left(\frac{\partial p}{\partial T} \right)_V &= \frac{1}{p} \left(\frac{\partial U}{\partial V} \right)_T + 1\end{aligned}\tag{29}$$

We can form another relation:

$$\begin{aligned}dU &= T dS - p dV \\ dU &= T \left(\frac{\partial S}{\partial T} \right)_V dT + \left(T \left(\frac{\partial S}{\partial V} \right)_T - p \right) dV\end{aligned}\tag{30}$$

Since dU is an exact differential, we also know the right-hand-side of the equation is exact.

$$\begin{aligned}T \left(\frac{\partial^2 S}{\partial T \partial V} \right) &= T \left(\frac{\partial^2 S}{\partial T \partial V} \right) + \left(\frac{\partial S}{\partial V} \right)_T - \left(\frac{\partial p}{\partial T} \right)_V \\ 0 &= \left(\frac{\partial S}{\partial V} \right)_T - \left(\frac{\partial p}{\partial T} \right)_V \\ \left(\frac{\partial p}{\partial T} \right)_V &= \left(\frac{\partial S}{\partial V} \right)_T\end{aligned}\tag{31}$$

Substituting this into (29) gives:

$$\begin{aligned}\frac{T}{p} \left(\frac{\partial p}{\partial T} \right)_V &= \frac{1}{p} \left(\frac{\partial U}{\partial V} \right)_T + 1 \\ \frac{T}{p} \left(\frac{\partial S}{\partial V} \right)_T &= \frac{1}{p} \left(\frac{\partial U}{\partial V} \right)_T + 1\end{aligned}\tag{32}$$

Returning to the original equation gives:

$$\begin{aligned}dU &= T dS - p dV \implies \\ \left(\frac{\partial U}{\partial S} \right)_V &= T\end{aligned}\tag{33}$$

Consider now:

$$\begin{aligned}
 & \left(\frac{\partial \ln p}{\partial \ln V} \right)_T - \left(\frac{\partial \ln p}{\partial \ln V} \right)_S \\
 &= \frac{V}{p} \left(\left(\frac{\partial p}{\partial V} \right)_T - \left(\frac{\partial p}{\partial V} \right)_S \right) \\
 &= \frac{V \left(\frac{\partial S}{\partial V} \right)_T}{p \left(\frac{\partial S}{\partial p} \right)_V} \\
 &= V \frac{\frac{T}{p} \left(\frac{\partial S}{\partial V} \right)_T}{T \left(\frac{\partial S}{\partial p} \right)_V} \\
 &= V \frac{\left(\frac{1}{p} \left(\frac{\partial U}{\partial V} \right)_T + 1 \right)}{\left(\frac{\partial U}{\partial S} \right)_V \left(\frac{\partial S}{\partial p} \right)_V} \\
 &= V \frac{\left(\frac{1}{p} \left(\frac{\partial U}{\partial V} \right)_T + 1 \right)}{\left(\frac{\partial U}{\partial p} \right)_V} \\
 &= V \frac{\left(\frac{1}{p} \left(\frac{\partial U}{\partial V} \right)_T + 1 \right)}{\left(\frac{\partial U}{\partial T} \right)_V \left(\frac{\partial T}{\partial p} \right)_V} \\
 &= V \left(\frac{\partial p}{\partial T} \right)_V \frac{\left(\frac{1}{p} \left(\frac{\partial U}{\partial V} \right)_T + 1 \right)}{\left(\frac{\partial U}{\partial T} \right)_V} \\
 &= \left(\frac{\partial(pV)}{\partial T} \right)_V \frac{\left(\frac{1}{p} \left(\frac{\partial U}{\partial V} \right)_T + 1 \right)}{\left(\frac{\partial U}{\partial T} \right)_V}
 \end{aligned} \tag{34}$$

(c) Note that since pV^γ depends only on S :

$$\begin{aligned}
 & \left(\frac{\partial \ln p}{\partial \ln V} \right)_S = 0 \\
 & \left(\frac{\partial \ln p}{\partial \ln V} \right)_T = \frac{d \ln p}{d \ln V}
 \end{aligned} \tag{35}$$

$$\begin{aligned}
 & U = C_v T \\
 & \left(\frac{\partial U}{\partial T} \right)_V = C_v \\
 & \left(\frac{\partial U}{\partial V} \right)_T = 0 \\
 & pV = RT \\
 & \left(\frac{\partial(pV)}{\partial T} \right)_V = R
 \end{aligned} \tag{36}$$

In the below equation: k is the constant of integration and A is another constant which is equal to $k^{\frac{R}{C_v}}$.

$$\begin{aligned} \left(\frac{\partial \ln p}{\partial \ln V} \right)_T - \left(\frac{\partial \ln p}{\partial \ln V} \right)_S &= \left(\frac{\partial(pV)}{\partial T} \right)_V \frac{\left(\frac{1}{p} \left(\frac{\partial U}{\partial V} \right)_T + 1 \right)}{\left(\frac{\partial U}{\partial T} \right)_V} \\ \frac{d \ln p}{d \ln V} - 0 &= R \frac{(0+1)}{C_v} \\ \frac{d \ln p}{d \ln V} &= \frac{R}{C_v} \\ \ln p &= \frac{R}{C_v} \ln kV \\ \ln p &= \ln k^{\frac{R}{C_v}} V^{-\frac{R}{C_v}} \\ p &= AV^{\frac{R}{C_v}} \\ pV^{-\frac{R}{C_v}} &= A \end{aligned}$$

$$\gamma = -\frac{R}{C_v} \quad (38)$$

(d)

$$\begin{aligned} \gamma &= -\frac{R}{C_v} \\ \gamma &= -\frac{R}{\frac{3}{2}R} \\ \gamma &= -\frac{2}{3} \end{aligned} \quad (39)$$