9 On bijections

9.1 Basic exercises

1. (a) Define a function that has (i) none, (ii) exactly one, and (iii) more than one retraction.

   (i)

   (ii)

   (iii)

   (b) Define a function that has (i) none, (ii) exactly one, and (iii) more than one section.

   (i)

   (ii)

   (iii)

2. Let n be an integer

   (a) How many sections are there for the absolute-value map $x \mapsto |x| : [-n \dots n] \to [0 \dots n]$?

   (b) How many retractions are there for the exponential map $x \mapsto 2^x : [0 \dots n] \to [0 \dots 2^n]$?

3. Give an example of two sets A and B and a function $f : A \to B$ such that f has a retraction but no section. Explain how you know that f has these properties.

4. Prove that the identity function is a bijection and that the composition of bijections is a bijection.

5. For $F : A \to B$, prove that if there are $g, h : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ h = \mathrm{id}_B$ then $g = h$. Conclude as a corollary that, whenever it exists, the inverse of a function is unique.

9.2 Core exercises

1. We say that two functions $s : A \to B$ and $r : B \to A$ are a section-retraction pair whenever $r \circ s = \mathrm{id}_A$; and that a function $e : B \to B$ is an idempotent whenever $e \circ e = e$. This question demonstrates that section-retraction pairs and idempotents are closely connected: any section-retraction pair gives rise to an idempotent function, and any idempotent function can be split into a section-retraction pair.

   (a) Let $f : C \to D$ and $g : D \to C$ be functions such that $f \circ g \circ f = f$.

   (i) Can you conclude that $f \circ g$ is idempotent? What about $g \circ f$? Justify your answers.

   (ii) Define a map g' using f and g that satisfies both

   $$f \circ g' \circ f = f \text{ and } g' \circ f' \circ g' = g' \tag{1}$$

   (iii) Show that if $s : A \to B$ and $r : B \to A$ are a section-retraction pair then the composite $s \circ r : B \to B$ is idempotent.

   (iv) Show that for every idempotent $e : B \to B$ there exists a set A (called a retract of B) and a section-retraction pair $s : A \to B$ and $r : B \to A$ such that $s \circ r = e$.

10 On equivalence relations

10.1 Basic exercises

1. Prove that the isomorphic relation $\cong$ between sets is an equivalence relation.
2. Prove that the identity relation id_A on a set A is an equivalence relation, and that $A/\mathrm{id}_A \cong A$.
3. Show that, for a positive integer m, the relation $\equiv_m on \mathbb{Z}$ given by
$$x \equiv_m y \iff x \equiv y (\text{mod } m) \tag{2}$$
is an equivalence relation. What are the equivalence classes of this relation?
4. Show that the relation $\equiv$ on $\mathbb{Z} \times \mathbb{Z}^+$ given by
$$(a, b) \equiv (x, y) \iff a \cdot y = x \cdot b \tag{3}$$
is an equivalence relation. What are the equivalence classes of this relation?

10.2 Core exercises

1. Let E_1 and E_2 be two equivalence relations on a set A. Either prove or disprove the following statements
   (a) $E_1 \cup E_2$ is an equivalence relation on A.
   (b) $E_1 \cap E_2$ is an equivalence relation on A.
2. For an equivalence relation E on a set A, show that $[a_1]_E = [a_2]_E$ iff $a_1 E a_2$, where
$$[a]_E = \{x \in A | xEa\}. \tag{4}$$
3. For a function $f : A \to B$ define a relation $\equiv_f$ on A by the rule: for all $a, a' \in A$,
$$a \equiv_f a' \iff f(a) = f(a') \tag{5}$$
4. Show that for every function $f : A \to B$, the relation $\equiv_f$ is an equivalence relation on A.
5. Prove that every equivalence relation E in a set A is equal to $\equiv_q$, where $q : A \twoheadrightarrow A/E$ is the quotient function $q(a) = [a]_E$.
6. Prove that for every surjection $f : A \twoheadrightarrow B$,
$$B \cong (A/ \equiv_f) \tag{6}$$