1 Further Graphics Exercise Set II

1. Simple 3D engines often approximate lighting as the superposition/combination of diffuse ambient light and direct illumination from point sources. This is a rather crude (yet efficient) approximation to the full rendering equation. What kind of effects are not captured by this simple approximation, i.e. what kind of visual effects cannot be rendered.

   - Reflections
   - Caustics
   - Soft shadows

   You can't have transparent objects.

   The crude approximation for transparent objects is alpha blending. You render the object behind and then render the first surface with α.

   Fog

   Indirect illumination – requiring illumination from other objects which are nearby ie ambient occlusion

2. If there is only diffuse ambient light, we can easily determine the radiosity of a point on the surface through ambient occlusion, i.e. by measuring how much of the "sky" is visible from that specific point.

   Briefly describe how information about the curvature of a surface can help in this situation.

   In this situation, linear interpolation can provide a good approximation to the surface. However, linearly interpolating over a curved surface will give artefacts. Knowing the curvature of the surface informs us of how much we can interpolate and when we should resample.

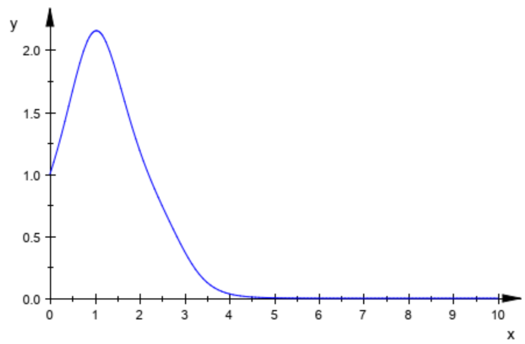
   Under this, we can approximate the light by curvature. If a surface has a high positive curvature (in a deep trench) then the light is very low. If a surface has a zero or negative curvature then we can give it the normal light.

   IE approximate ambient occlusion can be approximated by curvature.

   Ambient occlusion picks up high-frequency features which would otherwise be saturated.

3. Consider the function $f(x)$ with the graph shown below. In the range 0 to 10, it covers an area of $I = \int_0^{10} f(x)\mathrm{d}x \approx 4.329268516$.

$$f(x) = \frac{40}{(1+e^x)\cdot((x-2)^4+4)}$$



(a) Approximate the integral I through uniform sampling at the five points $x_k = 1, 3, 5, 7, 9$.

$$\langle I \rangle = \sum_k \frac{f(x_k)}{p(x_k)}$$

Since there are five uniformly distributed points, the density for all of them is $p = \frac{5}{10} = \frac{1}{2}$. What is the relative error of the number you get?

$$\begin{aligned}\langle I \rangle &= 2 \cdot \sum_k f(x_k) \\ &= 2 \cdot 2.5341479220696312 \\ &= 5.0682958441392625\end{aligned}$$

The relative error is therefore:

$$\frac{|5.0682958441392625 - 4.329268516|}{4.329268516} = 0.1707\ldots$$

(b) According to the idea of importance sampling, you concentrate the evaluation of function values on the region where the values are largest i.e . between 0 and 4 in our case. Sample the function at the points $x = 0.5, 1.5, 2.5, 3.5$ as well as $x = 7$ and compute again the estimator $\langle I \rangle$ for the integral I. The density for the first four points (each covering an interval of length 1) is $p = \frac{1}{1} = 1$ whereas the density for $x = 7$ (which covers an integral of length 6) is $p = \frac{1}{6}$.

$$\begin{aligned}\langle I \rangle &= \sum_k p(x_k) \cdot f(x_k) \\ &= 4.338875262071001\end{aligned}$$

The relative error from this is therefore:

$$\frac{|4.338875262071001 - 4.329268516|}{4.329268516} = 0.00221902292165451$$

5. Write down the directional form of the rendering equation. Briefly explain each of the terms and the integration domain.

$$L_0(\mathbf{x}, \vec{\omega}) = L_e(\mathbf{x}, \vec{\omega}) + \int_{H^2} f_r(\mathbf{x}, \omega_i, \vec{\omega}) \cdot L_i(\mathbf{x}, \omega_i) \cdot \cos\theta_i \mathrm{d}\omega_i$$

- $L_e(\mathbf{x}, \vec{\omega})$ is the light emitted from the point $\mathbf{x}$ in direction $\vec{\omega}$.
- $\int_{H^2} \mathrm{d}\omega_i$ integrates over all the angles in a hemisphere H.
- $L_0(\mathbf{x}, \vec{\omega})$ is the light emitting emitted from $\mathbf{x}$ in the direction $\vec{\omega}$.
- $f_r(\mathbf{x}, \omega_i, \vec{\omega})$ is the proportion of the light incident from direction ω_i which is reflected in the direction $\vec{\omega}$.
- $L_i(\mathbf{x}, \omega_i)$ is the amount of light incident at the point $\mathbf{x}$ from direction ω_i.
- θ_i is the angle between the normal and the incident ray.

6. Assume a scene containing only diffuse surfaces with diffuse reflectance $\rho(x)$. The scene is surrounded by a single distant light source with constant emission $\bar{L}$. Let $V(\mathbf{x}, \vec{\omega})$ be a function returning the visibility of the light source from point $\mathbf{x}$ along direction $\vec{\omega}$.

   Change/simplify the rendering equation you provided in (a) as much as possible to estimate only direct illumination due to the light source.

$$L_0(\mathbf{x}, \vec{\omega}) = \int_{H^2} \frac{\rho(x) \cdot (\vec{\omega} - \vec{\omega}_i)}{\pi} \cdot (\bar{L} \cdot V(\mathbf{x}, \vec{\omega})) \cdot \cos\theta_i \mathrm{d}\omega_i$$

7. Recall the surface area form of the rendering equation:

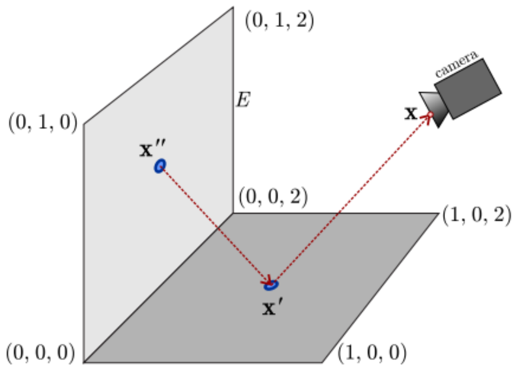
$$L(\mathbf{x}, \mathbf{z}) = L_e(\mathbf{x}, \mathbf{z}) + \int_A f_r(\mathbf{x}, \mathbf{y}, \mathbf{z}) \cdot L(\mathbf{x}, \mathbf{y}) \cdot G(\mathbf{x}, \mathbf{y}) \mathrm{d}A(\mathbf{y})$$

   (a) Provide the mathematical formulation of $G(\mathbf{x}, \mathbf{y})$ and explain its terms.

$$G(\mathbf{x}, \mathbf{y}) = V(x, y) \cdot \cos\theta_x \cdot \cos\theta_y \cdot \frac{1}{|\mathbf{y} - \mathbf{x}|}^2$$

   $\mathbf{G}(x, y)$ is a geometry term. So $\mathbf{G}(x, y)$ contains the angle, the visibility, the cosine between the normal and the cosine between the angle and the normal at the light source.

   (b) Consider the following scene configuration. The rectangle E on the left is an area emitter with $L_e = 1$ and a black BSDF, ie.e. $f_r = 0$. The coordinates of its corners are labeled. The ground plane is a non-emissive surface with $f_r = \frac{1}{\pi}$. Provide pseudocode for a Monte-Carlo estimator of L, given as input a point on the ground plane $\mathbf{x}'$, a point on the camera $\mathbf{x}$ and the desired amount of samples N. You can obtain numbers uniformly in $[0, 1)$ by calling `RAND()`.




```
vector illumination(vector x, vector x'){
        n = ...
        light = 0;
        for (int i = 0; i < n; i++{
                θ = 2 * π * RAND();
                φ = π/2 * RAND();
                y = (sin θ cos φ, sin θ sin φ, cos θ);
                if (intercepts_E(x+λy)){
                        light += L_e * cos θ;
```

```
            }
        }
        return light / n * x/|x| · (0, 1, 0);
    }
```

$$\texttt{return light / n * } \tfrac{\mathbf{x}}{|\mathbf{x}|} \cdot \begin{pmatrix}0\\1\\0\end{pmatrix};$$

8. Consider the following single-sample Monte Carlo estimator $\langle F\rangle$ of an integral F of an arbitrary non-negative integrand $f(x)$ over an arbitrary domain D with volume 1, driven by a random variable $X \sim p(x)$.

$$F = \int_D f(x)\mathrm{d}x \approx \frac{f(X)}{p(X)} = \langle F\rangle, \qquad p(x) = \frac{1}{2}\left(p_{\text{good}}(x) + p_{\text{bad}}(x)\right)$$

The "good" probability density function $p_{\text{good}}(x)$ is proportinoal to $f(x)$ and the "bad" probability density function p_{bad} is zero whenever $f(x)$ is non-zero.

Derive step-by-step the variance of the estimator $\langle F\rangle$ as a function of F.

$$\begin{aligned} p_{\text{good}}(x) &= c \cdot f(x) \implies \\ \int_{-\infty}^{\infty} p_{\text{good}}(x)\mathrm{d}x &= \int_{-\infty}^{\infty} c \cdot f(x)\mathrm{d}x \\ 1 &= c \cdot \int_D f(x)\mathrm{d}x \\ \frac{1}{c} &= \int_D f(x)\mathrm{d}x \end{aligned}$$

Consider choosing a probability from the distribution p as being a binomial choice between p_{good} and p_{bad}

$$\begin{aligned} E\left(\frac{f(X)}{p(X)}\right) &= \sum p(x)\frac{\langle F(x)\rangle}{p(x)} \\ &= p_{\text{good}}(x)\frac{F(x)}{p_{\text{good}}(x)} \\ &= F(x) \end{aligned}$$

$$\begin{aligned} E\left(\langle F\rangle^2\right) &= \sum p(x)\langle F(x)\rangle^2 \\ &= p_{\text{good}}(x) \cdot \left(\frac{F(x)}{p_{\text{good}}(x)}\right)^2 \\ &= 2 \cdot F(x) \end{aligned}$$

$$\begin{aligned} \mathrm{Var}(\langle F\rangle) &= E(\langle F\rangle^2) - E(\langle F\rangle)^2 \\ &= 2 \cdot F^2 - F^2 \\ &= F^2 \end{aligned}$$

9. Consider a spotlight emitting radiant intensity $I = 20\frac{W}{sr}$ confined in a cone of directions with solid angle $\omega = 3sr$. Compute the total emitted radiant flux from this light.

$$\Phi = \int_0^3 20 \mathrm{d}\omega = [20]_0^3 = 60W$$

If you integrate over the square radians then you get the correct answer – even though integrating over a square is nasty, it's like integrating over the total area.