线性代数B1

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Exercise 1: 用升阶法计算类Vandermonde行列式

计算行列式 $D_n = \begin{vmatrix} a+x_1 & a+x_1^2 & \cdots & a+x_1^{n-1} & a+x_1^n \\ a+x_2 & a+x_2^2 & \cdots & a+x_2^{n-1} & a+x_2^n \\ \vdots & \vdots & & \vdots & \vdots \\ a+x_n & a+x_n^2 & \cdots & a+x_n^{n-1} & a+x_n^n \end{vmatrix}$

Solution: $\Longrightarrow$

1. $a=0$时，我们有

$$D_n = \prod_{j=1}^{n} x_j \prod_{1\le i<j\le n} (x_j - x_i) \tag{1}$$

2. $a \neq 0$时，我们有

$$\begin{aligned} D_n &= \frac{1}{a}\begin{vmatrix} a & a & a & \cdots & a & a \\ 0 & a+x_1 & a+x_1^2 & \cdots & a+x_1^{n-1} & a+x_1^n \\ 0 & a+x_2 & a+x_2^2 & \cdots & a+x_2^{n-1} & a+x_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & a+x_n & a+x_n^2 & \cdots & a+x_n^{n-1} & a+x_n^n \end{vmatrix} \\ &= \frac{1}{a}\begin{vmatrix} a & a & a & \cdots & a & a \\ -a & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\ -a & x_2 & x_2^2 & \cdots & x_2^{n-1} & x_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ -a & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\ -a & x_2 & x_2^2 & \cdots & x_2^{n-1} & x_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{vmatrix} \end{aligned} \tag{2}$$

$$
\begin{aligned}
D_n &= \begin{vmatrix} -a & 1 & 1 & \cdots & 1 & 1 \\ -a & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\ -a & x_2 & x_2^2 & \cdots & x_2^{n-1} & x_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ -a & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{vmatrix} + \begin{vmatrix} a+1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\ 0 & x_2 & x_2^2 & \cdots & x_2^{n-1} & x_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{vmatrix} \\
&= -a \times \begin{vmatrix} 1 & 1^0 & 1^2 & \cdots & 1^{n-1} & 1^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} & x_2^n \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{vmatrix} + (a+1) \times \begin{vmatrix} x_1 & x_1^2 & \cdots & x_1^{n-1} & x_1^n \\ x_2 & x_2^2 & \cdots & x_2^{n-1} & x_2^n \\ \vdots & \vdots & & \vdots & \vdots \\ x_n & x_n^2 & \cdots & x_n^{n-1} & x_n^n \end{vmatrix} \\
&= -a \prod_{0 \le i < j \le n} (x_j - x_i) + (a+1) \prod_{j=1}^{n} x_j \prod_{1 \le i < j \le n} (x_j - x_i)
\end{aligned} \tag{3}
$$

其中$x_0 = 1$

又$a = 0$时，(3)式退化为(1)式。故

$$
D_n = -a \prod_{0 \le i < j \le n} (x_j - x_i) + (a+1) \prod_{j=1}^{n} x_j \prod_{1 \le i < j \le n} (x_j - x_i)
$$

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Exercise 2: 利用n+1阶Vandermonde行列式计算n阶类Vandermonde行列式

计算行列式 $D_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{n-2} & x_2^{n-2} & x_3^{n-2} & \cdots & x_n^{n-2} \\ x_1^n & x_2^n & x_3^n & \cdots & x_n^n \end{vmatrix}$

Solution: $\Longrightarrow$

我们考虑n+1阶Vandermonde行列式$f(x)$

$$f(x) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_1 & x_2 & \cdots & x_n & x \\ \vdots & \vdots & \vdots & \vdots & \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} & x^{n-2} \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} & x^{n-1} \\ x_1^n & x_2^n & \cdots & x_n^n & x^n \end{vmatrix}$$

我们有

$$f(x) = (x - x_1)(x - x_2)\dots(x - x_n) \prod_{1 \le i < j \le n} (x_j - x_i)$$

可以看出D_n是$f(x)$中x^{n-1}的余子式$M_{n,n+1}$，我们有

$$D_n = (-1)^{n+n+1} A_{n,n+1} = -A_{n,n+1}$$

由$f(x)$表达式知，x_{n-1}的系数为

$$A_{n,n+1} = -(x_1 + x_2 + \cdots + x_n) \prod_{1 \le i < j \le n} (x_j - x_i)$$

故

$$D_n = (x_1 + x_2 + \cdots + x_n) \prod_{1 \le i < j \le n} (x_j - x_i)$$

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Exercise 3: 将高阶行列式降阶求解

计算行列式 $D_n = \begin{vmatrix} 1+x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & 1+x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & 1+x_ny_n \end{vmatrix}$

Solution: $\Longrightarrow$

Lemma 1. $\mathbf{A} = \mathbf{A}_{n\times m}, \mathbf{B} = \mathbf{B}_{m\times n}$，则$det(\mathbf{I}_n - \mathbf{AB}) = det(\mathbf{I}_m - \mathbf{BA})$

证明. 由

$$\begin{pmatrix} \mathbf{I}_m & \mathbf{O} \\ \mathbf{A} & \mathbf{I}_n - \mathbf{AB} \end{pmatrix} \begin{pmatrix} \mathbf{I}_m & -\mathbf{B} \\ \mathbf{O} & \mathbf{I}_n \end{pmatrix} \begin{pmatrix} \mathbf{I}_m & \mathbf{O} \\ \mathbf{A} & \mathbf{I}_n \end{pmatrix} = \begin{pmatrix} \mathbf{I}_m - \mathbf{BA} & -\mathbf{B} \\ \mathbf{O} & \mathbf{I}_n \end{pmatrix}$$

两边取行列式，得到

$$det(\mathbf{I}_n - \mathbf{AB}) = det(\mathbf{I}_m - \mathbf{BA})$$

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由

$$D_n = det(\mathbf{I}_n + \alpha\beta)$$

其中

$$\alpha = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},\ \beta = (y_1, y_2, \ldots, y_n)$$

根据引理1我们有

$$D_n = det(\mathbf{I}_n + \alpha\beta) = det(\mathbf{I}_1 + \beta\alpha) = 1 + \sum_{i=1}^{n} x_iy_i$$

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Exercise 4: 向量组线性无关与行列式不为0的等价关系1

证明: n维列向量$(\alpha_1, \alpha_2, \ldots, \alpha_n)$线性无关$\Longleftrightarrow D = \begin{vmatrix} \alpha_1^T\alpha_1 & \alpha_1^T\alpha_2 & \cdots & \alpha_1^T\alpha_n \\ \alpha_2^T\alpha_1 & \alpha_2^T\alpha_2 & \cdots & \alpha_2^T\alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^T\alpha_1 & \alpha_n^T\alpha_2 & \cdots & \alpha_n^T\alpha_n \end{vmatrix} \neq 0$

Solution: $\Longrightarrow$

证明. 令

$$\mathbf{A} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

则

$$\mathbf{A^T A} = \begin{pmatrix} \alpha_1^T \\ \alpha_2^T \\ \vdots \\ \alpha_n^T \end{pmatrix} (\alpha_1, \alpha_2, \ldots, \alpha_n) = \begin{pmatrix} \alpha_1^T\alpha_1 & \alpha_1^T\alpha_2 & \cdots & \alpha_1^T\alpha_n \\ \alpha_2^T\alpha_1 & \alpha_2^T\alpha_2 & \cdots & \alpha_2^T\alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^T\alpha_1 & \alpha_n^T\alpha_2 & \cdots & \alpha_n^T\alpha_n \end{pmatrix}$$

则$D = |\mathbf{A}^T\mathbf{A}| = |\mathbf{A}|^2 \neq 0 \Longleftrightarrow |\mathbf{A}| \neq 0 \Longleftrightarrow \alpha_1, \alpha_2, \ldots, \alpha_n$ 线性无关 ■

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Exercise 5: 向量组线性无关与行列式不为0的等价关系2

已知n维列向量$(\alpha_1, \alpha_2, \ldots, \alpha_s)$, $(2 \le s \le n)$线性无关,$k_1, k_2, \ldots, k_{s-1}$是任意$s-1$个数，证明:向量组$\alpha_1 + k_1\alpha_2, \alpha_2 + k_2\alpha_3, \ldots, \alpha_{s-1} + k_{s-1}\alpha_s, \alpha_s$也线性无关。

Solution: $\Longrightarrow$

证明. 记矩阵

$$\mathbf{A} = (\alpha_1, \alpha_2, \ldots, \alpha_n),\ \mathbf{B} = (\alpha_1 + k_1\alpha_2, \alpha_2 + k_2\alpha_3, \ldots, \alpha_{s-1} + k_{s-1}\alpha_s, \alpha_s)$$

则

$$\mathbf{B} = \mathbf{A}\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ k_1 & 1 & 0 & \cdots & 0 \\ 0 & k_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{s-1} & 1 \end{pmatrix} = \mathbf{AC}$$

则

$$|\mathbf{B}| = |\mathbf{AC}| = |\mathbf{A}| \neq 0$$

即$\alpha_1 + k_1\alpha_2, \alpha_2 + k_2\alpha_3, \ldots, \alpha_{s-1} + k_{s-1}\alpha_s, \alpha_s$也线性无关。 ■

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Exercise 6: 向量组之间的线性关系

设n维列向量$\alpha_1, \alpha_2, \ldots, \alpha_s$线性无关, 且$\beta_1, \beta_2, \ldots, \beta_t$可以由$\alpha_1, \alpha_2, \ldots, \alpha_s$线性表示，即存在$\mathbf{C}_{s\times t}$，使得

$$(\beta_1, \beta_2, \ldots, \beta_t) = (\alpha_1, \alpha_2, \ldots, \alpha_s)\mathbf{C}$$

证明：$r(\mathbf{B}) = r(\mathbf{C})$

Solution: $\Longrightarrow$

证明. 由$\mathbf{B} = \mathbf{AC}$, 我们有

$$r(\mathbf{B}) \leq r(\mathbf{C})$$

又$r(\mathbf{A}) = s$，故存在可逆矩阵$\mathbf{P}, \mathbf{Q}$使得

$$\mathbf{A} = \mathbf{P}\begin{pmatrix}\mathbf{I}_s \\ \mathbf{O}\end{pmatrix}\mathbf{Q}$$

令

$$\mathbf{D} = \mathbf{Q}^{-1}(\mathbf{I}_s, \mathbf{O})\mathbf{P}^{-1}$$

我们有

$$\mathbf{DB} = \mathbf{Q}^{-1}(\mathbf{I}_s, \mathbf{O})\mathbf{P}^{-1}\mathbf{P}\begin{pmatrix}\mathbf{I}_s \\ \mathbf{O}\end{pmatrix}\mathbf{C} = \mathbf{C}$$

故

$$r(\mathbf{C}) \leq r(\mathbf{B})$$

故

$$r(\mathbf{C}) = r(\mathbf{B})$$

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Exercise 7: 基础解系

已知F^5中的向量$\alpha = (1, 2, 3, 4, 5), \beta = (1, -1, 1, -1, 1), \gamma = (1, 2, 4, 8, 16)$，求一个齐次线性方程组，使$\alpha, \beta, \gamma$组成这个方程组的基础解系。

Solution: $\Longrightarrow$

α, β, γ是所求方程组

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0$$

的解。即

$$\begin{cases} a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 = 0 \\ a_1 - a_2 + a_3 - a_4 + a_5 = 0 \\ a_1 + 2a_2 + 4a_3 + 8a_4 + 16a_5 = 0 \end{cases} \tag{4}$$

(4)是关于$a_1, a_2, \ldots, a_5$的齐次线性方程组，解集的维数$dimV = n - rank(A) = 2$。基础解系是α_1, α_2，其中

$$\alpha_1 = (6, 1, -4, 1, 0), \alpha_2 = (16, 6, -11, 0, 1)$$

以α_1, α_2中各分量为系数构造齐次线性方程组

$$\begin{cases} 6x_1 + x_2 - x_3 + x_4 = 0 \\ 16x_1 + 6x_2 - 11x_3 + x_5 = 0 \end{cases} \tag{5}$$

故α, β, γ是(5)的线性无关的解。又(5)的解集的维数$dimV = n - rank(A) = 2$。故α, β, γ为(5)的基础解系。(5)即是我们所求的齐次线性方程组。 ∎

Exercise 8: 秩与同解方程组

设$\mathbf{A}$是n阶矩阵，证明：$r(\mathbf{A}^n) = r(\mathbf{A}^{n+1})$

Solution: $\Longrightarrow$

我们只需要证明$\mathbf{A}^n\mathbf{x} = \mathbf{0}$与$\mathbf{A}^{n+1}\mathbf{x} = \mathbf{0}$同解即可。
注：$\mathbf{A}^n\mathbf{x} = \mathbf{0}$与$\mathbf{A}^{n+1}\mathbf{x} = \mathbf{0}$同解可以推出$r(\mathbf{A}^n) = r(\mathbf{A}^{n+1})$，但$r(\mathbf{A}^n) = r(\mathbf{A}^{n+1})$不能推出$\mathbf{A}^n\mathbf{x} = \mathbf{0}$与$\mathbf{A}^{n+1}\mathbf{x} = \mathbf{0}$同解

证明. 设α是$\mathbf{A}^n\mathbf{x} = \mathbf{0}$的解，则$\mathbf{A}^{n+1}\alpha = \mathbf{A}(\mathbf{A}^n\alpha) = \mathbf{0}$，即$\alpha$也是$\mathbf{A}^{n+1}\mathbf{x} = \mathbf{0}$的解。
设β是$\mathbf{A}^{n+1}\mathbf{x} = \mathbf{0}$的解，即$\mathbf{A}^{n+1}\beta = \mathbf{0}$。假设$\mathbf{A}^n\beta \neq \mathbf{0}$，我们有$\beta, \mathbf{A}\beta, \ldots, \mathbf{A}^n\beta$线性无关，下面我们来证明。
设

$$k_0\beta + k_1\mathbf{A}\beta + \cdots + k_n\mathbf{A}^n\beta = \mathbf{0}$$

则

$$k_0\mathbf{A}^n\beta + k_1\mathbf{A}^{n+1}\beta + \cdots + k_n\mathbf{A}^{2n}\beta = \mathbf{0}$$

又$\mathbf{A}^n\beta \neq \mathbf{0}, \mathbf{A}^{n+1}\beta = \mathbf{0}$，故$k_0 = 0$。同理，我们有$k_1 = \cdots = k_n = 0$。
又$\beta, \mathbf{A}\beta, \ldots, \mathbf{A}^n\beta$是$n+1$个$n$维向量，故$\beta, \mathbf{A}\beta, \ldots, \mathbf{A}^n\beta$线性相关，与假设矛盾。故$\mathbf{A}^n\beta = \mathbf{0}$
故$\mathbf{A}^n\mathbf{x} = \mathbf{0}$与$\mathbf{A}^{n+1}\mathbf{x} = \mathbf{0}$同解，$r(\mathbf{A}^n) = r(\mathbf{A}^{n+1})$。 ■

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Exercise 9: 线性方程组有解问题

设$\mathbf{A}$是$m \times n$矩阵，$\mathbf{b}$是m维向量，求证：线性方程组$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$必有解。

Solution: $\Longrightarrow$

我们只需要证明$rank(\mathbf{A}^T\mathbf{A}, \mathbf{A}^T\mathbf{b}) = rank(\mathbf{A}^T\mathbf{A})$即可。

证明. 令$\mathbf{A}^T\mathbf{b} = \beta$，设$\mathbf{A}^T = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, $b = (b_1, b_2, \ldots, b_m)^T$, 则

$$\beta = \mathbf{A}^T\mathbf{b} = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_m\alpha_m$$

即β为$\alpha_1, \alpha_2, \ldots, \alpha_m$的线性组合。
设$\mathbf{A}^T\mathbf{A} = (\beta_1, \beta_2, \ldots, \beta_n)$, $\mathbf{A} = (a_{ij})_{n\times n}$, 我们有

$$\mathbf{A}^T\mathbf{A} = (\beta_1, \beta_2, \ldots, \beta_n) = (\alpha_1, \alpha_2, \ldots, \alpha_m)\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

故$\beta_i = \sum\limits_{j=1}^{m} a_{ji}\alpha_j$，即$\beta_i$也是$\alpha_1, \alpha_2, \ldots, \alpha_m$的线性组合。故$(\beta_1, \beta_2, \ldots, \beta_n, \beta)$均可由$\alpha_1, \alpha_2, \ldots, \alpha_m$线性表示。又

$$rank(\mathbf{A}^T\mathbf{A}) = rank(\mathbf{A}) = rank(\mathbf{A}^T)$$

故$(\beta_1, \beta_2, \ldots, \beta_n)$与$(\alpha_1, \alpha_2, \ldots, \alpha_m)$等价，故$\beta$可由$\beta_1, \beta_2, \ldots, \beta_n$表示，故

$$rank(\mathbf{A}^T\mathbf{A}, \mathbf{A}^T\mathbf{b}) = rank(\mathbf{A}^T\mathbf{A})$$

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Exercise 10: $\mathbf{AB}=\mathbf{O}$ & $\mathbf{Ax}=\mathbf{0}$

设$\mathbf{A}$是$m\times n$矩阵，秩为m；$\mathbf{B}$是$n\times(n-m)$矩阵，秩为$n-m$；且$\mathbf{AB}=\mathbf{O}$。α是满足条件$\mathbf{A}\alpha=\mathbf{0}$的一个$n$维列向量，证明：存在唯一的一个$n-m$维列向量$\beta$使得$\alpha=\mathbf{B}\beta$

Solution: $\Longrightarrow$

证明. 设

$$\mathbf{B}=(\eta_1,\eta_2,\ldots,\eta_{n-m})$$

由$r(\mathbf{B})=n-m$, 我们有$\eta_1,\eta_2,\ldots,\eta_{n-m}$线性无关。
由$\mathbf{AB}=\mathbf{O}$，我们有

$$\mathbf{A}\eta_i=0$$

故$\eta_1,\eta_2,\ldots,\eta_{n-m}$是$\mathbf{Ax}=\mathbf{0}$的$n-m$个线性无关的解向量。
由$r(\mathbf{A})=m$，我们有

$$dimV=n-m$$

故$\eta_1,\eta_2,\ldots,\eta_{n-m}$是$\mathbf{Ax}=\mathbf{0}$的基础解系。
由$\mathbf{A}\alpha=0$，我们有

$$\alpha=(\eta_1,\eta_2,\ldots,\eta_{n-m})(k_1,k_2,\ldots,k_{n-m})^T=\mathbf{B}\beta$$

下面我们证明β的唯一性：假设存在β_1,β_2使得$\mathbf{B}\beta_1=\alpha,\mathbf{B}\beta_2=\alpha$，我们有

$$\mathbf{B}(\beta_1-\beta_2)=\mathbf{0}$$

又$r(\mathbf{B})=n-m$，故$\mathbf{Bx}=\mathbf{0}$只有零解，故$\beta_1=\beta_2$ ■

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[1] The template is from Prof. Jie Wang, USTC