Problem 1 (10 pts)

For this problem, we explore the issue of *truthfulness* in the Gale-Shapley algorithm for Stable Matching. Show that a participant can improve its outcome by lying about its preferences. Consider $r \in R$. Suppose r prefers p to p', but p and p' are low on r's preference list. Show that it is possible that by switching the order of p and p' on r's preference list, r achieves a better outcome, e.g., is matched with a p'' higher on the preference list than the one if the actual order was used.

Hint: Prove the claim by finding one specific instance of stable matching problem and comparing the stable matching before and after the switching.

Solution

We can show that it is possible for a receiver r to obtain a more-preferred match by being untruthful, by switching the order of two options that r actually likes less (even though in his true ranking one is better than the other).

Let the two sides be:

- **Residents (receivers)** R: r, s, t
- **Programs (proposers)** P: p, p', q

Assume that the matching is produced by a **program-proposing Gale-Shapley algorithm**.

True Preference Lists

- **Resident r's true preferences:**

$$q > p > p'$$

  So r likes q best; although he prefers p to p', both p and p' are lower than q.

- **Resident s's true preferences:**

$$p > q > p'$$

- **Resident t's true preferences:**

$$p > p' > q$$

- **Programs' preferences:**
  - p: $r > s > t$
  - p': $r > t > s$
  - q: $s > r > t$

Outcome When r Reports Truthfully

We run the **program-proposing Gale-Shapley algorithm** using the true lists.

1. **Round 1:**
   - p proposes to its top choice: r.
   - p' also proposes to its top choice: r.
   - q proposes to its top choice: s.
2. **Resident Decisions:**
   - r receives proposals from p and p'.
   - Using his true order $q > p > p'$, he compares these two. Since $p > p'$ in his true list, he tentatively holds p and rejects p'.
   - s gets q.
   - t gets no proposal in this round.
3. **Round 2:**
   - p', having been rejected by r, now moves to its next choice and proposes to t.
4. **Final Matching:**
   - r is matched with p.
   - s is matched with q.
   - t is matched with p'.

Outcome for r: He gets p, but he would prefer q over p. Since q proposed only to s, q is already taken.

Outcome When r Misreports

Now suppose that r lies about his 2nd and 3rd rankings by switching p and p'. In other words, he submits as his preference list:

$$q > p' > p$$

Even though he actually prefers p to p', he reverses these when reporting.

Now run the algorithm again:

1. **Round 1:**
   - p proposes to its top choice: r.
   - p' proposes to its top choice: r.
   - q proposes to its top choice: s.

2. **Resident Decisions:**
   - r receives proposals from p and p'.
   - Using his reported order $q > p' > p$, he compares the two proposals and holds p' (since $p' > p$ in his reported list), rejecting p.
   - s gets q.
   - t gets nothing yet.
3. **Round 2:**
   - p, having been rejected by r, moves to its next choice and proposes to s.
   - s's true order is $p > q > p'$, so she prefers p over q.
   - s drops q and holds p.
   - q, now rejected by s, proposes to its next choice, which is r.
4. **Round 3:**
   - r is currently holding p' from round 1.
   - Now r receives a proposal from q.
   - Since r's true order is $q > p > p'$, he prefers q over his current match p'.
   - So r drops p' and accepts q.
5. **Round 4:**
   - p', now rejected by r, goes to its next choice and proposes to t.
   - t (true order: $p > p' > q$) accepts p'.
6. **Final Matching:**
   - r is now matched with q.
   - s is matched with p.
   - t is matched with p'.

Conclusion

By simply swapping the order of p and p' in his reported preference list (even though he truly prefers p to p'), resident r can trigger a different chain reaction of proposals.

In the truthful run, he ended up with p (his second choice). However, by lying, he eventually receives a proposal from q, his true first choice.

This proves that in the program-proposing Gale-Shapley algorithm, truth-telling is not necessarily an optimal strategy. A participant can benefit by misreporting their preferences.

Problem 2 (10 pts)

Arrange in increasing order of asymptotic growth. All logs are in base 2.

1. $n^{\frac{5}{3}} \log^2 n$
2. $2^{\sqrt{\log n}}$
3. $\sqrt{n^n}$
4. $\frac{n^2}{\log n}$
5. 2^n

Solution

We want to compare the asymptotic growth of five functions as $n \to \infty$:

$$f_1(n) = n^{\frac{5}{3}}(\log n)^2, \quad f_2(n) = 2^{\sqrt{\log n}}, \quad f_3(n) = \sqrt{n^n} = n^{\frac{n}{2}}, \quad f_4(n) = \frac{n^2}{\log n}, \quad f_5(n) = 2^n.$$

Step 1: Comparing $f_2(n)$ with polynomials

First, compare $f_2(n) = 2^{\sqrt{\log n}}$ to polynomial functions of n.

Note that

$$\log_2\left(2^{\sqrt{\log_2 n}}\right) = \sqrt{\log_2 n}, \quad \log_2\left(n^x\right) = x \log_2 n.$$

As $n \to \infty$, $x \log_2 n$ (linear in $\log_2 n$) will outgrow $\sqrt{\log_2 n}$. Therefore

$$2^{\sqrt{\log n}} = 2^{\sqrt{\log_2 n}} \; < \; n^x \quad \text{for any fixed } x > 0.$$

Because $f_1(n) = n^{5/3}(\log n)^2$ and $f_4(n) = \frac{n^2}{\log n}$ are both polynomial, we can conclude

$$f_2(n) \; < \; f_1(n), \quad f_2(n) \; < \; f_4(n).$$

Step 2: Comparing $f_1(n)$ and $f_4(n)$

Next, compare

$$f_1(n) = n^{\frac{5}{3}}(\log n)^2 \quad \text{and} \quad f_4(n) = \frac{n^2}{\log n}.$$

Examine their ratio:

$$\frac{f_1(n)}{f_4(n)} = \frac{n^{\frac{5}{3}}\,(\log n)^2}{\frac{n^2}{\log n}} = \frac{n^{\frac{5}{3}}(\log n)^3}{n^2} = \frac{(\log n)^3}{n^{\frac{1}{3}}}.$$

Since $n^{\frac{1}{3}}$ outgrows $(\log n)^3$ as $n \to \infty$, we have $\frac{(\log n)^3}{n^{1/3}} \to 0$. Thus

$$f_1(n) \; < \; f_4(n).$$

So far, we have

$$f_2(n) \; < \; f_1(n) \; < \; f_4(n).$$

Step 3: Comparing the exponentials $f_3(n)$ vs. $f_5(n)$

Finally, compare

$$f_3(n) = n^{\frac{n}{2}}, \quad f_5(n) = 2^n.$$

Rewrite them as exponentials:

$$f_3(n) = \exp\Big(\tfrac{n}{2}\ln n\Big), \quad f_5(n) = \exp\big(n\ln 2\big).$$

Compare their exponents:

$$\tfrac{n}{2}\ln n \quad \text{vs.} \quad n\ln 2.$$

Since $\ln n$ grows unbounded, eventually $\frac{1}{2}\ln n$ exceeds $\ln 2$, so

$$\tfrac{n}{2}\ln n \;\gg\; n\ln 2.$$

Hence

$$n^{\frac{n}{2}} \;\gg\; 2^n,$$

meaning $f_3(n)$ grows faster than $f_5(n)$.

Step 4: Final Ordering

Combining all comparisons, the increasing order of asymptotic growth is:

$$\boxed{2^{\sqrt{\log n}} \;<\; n^{\frac{5}{3}}(\log n)^2 \;<\; \frac{n^2}{\log n} \;<\; 2^n \;<\; \sqrt{n^n}\,.}$$

Problem 3 (10 pts)

We say that $T(n)$ is $O(f(n))$ if there exist c and n_0 such that for all $n > n_0$, $T(n) \leq cf(n)$. Use this definition for parts a and b.

1. Prove that $4n^2 + 3n\log n + 6n + 20\log^2 n + 11$ is $O(n^2)$. (You may use, without proof, the fact that $\log n < n$ for $n \geq 1$.)

2. Suppose that $f(n)$ is $O(r(n))$ and $g(n)$ is $O(s(n))$. Let $h(n) = f(n)g(n)$ and $t(n) = r(n)s(n)$. Prove that $h(n)$ is $O(t(n))$.

Solution

(a) Proof that $4n^2 + 3n\log n + 6n + 20\log^2 n + 11$ is $O(n^2)$.

Let

$$T(n) \;=\; 4n^2 \;+\; 3n\log n \;+\; 6n \;+\; 20(\log n)^2 \;+\; 11.$$

We want to prove that $T(n) = O(n^2)$. We bound each term separately:

- $4n^2$ is trivially $\leq 4n^2$.

- Since $\log n < n$ for $n \geq 1$, we have $3n \log n < 3n^2$.
- For $n \geq 1$, $6n < 6n^2$.
- For large n, $(\log n)^2 < n^2$, so $20(\log n)^2 < 20n^2$.
- For $n \geq 1$, $11 \leq 11n^2$.

Summing these bounds, we obtain

$$T(n) \;\leq\; 4n^2 + 3n^2 + 6n^2 + 20n^2 + 11n^2 \;=\; 44n^2.$$

Hence, by the definition of Big-O notation, $T(n)$ is $O(n^2)$.

(b) Proof that $h(n)$ is $O(t(n))$.

Assume $f(n)$ is $O(r(n))$ and $g(n)$ is $O(s(n))$. By definition, there exist constants $c_1, c_2 > 0$ and n_1, n_2 such that

$$f(n) \;\leq\; c_1\, r(n) \quad \text{for all } n > n_1, \quad \text{and} \quad g(n) \;\leq\; c_2\, s(n) \quad \text{for all } n > n_2.$$

Let $n_0 = \max\{n_1,\, n_2\}$. Then for $n > n_0$, we have

$$f(n)\, g(n) \;\leq\; \bigl(c_1\, r(n)\bigr)\,\bigl(c_2\, s(n)\bigr) \;=\; c_1\, c_2\, \bigl(r(n)\, s(n)\bigr).$$

Since $c_1 c_2$ is just a constant, this shows

$$f(n)\, g(n) = O\bigl(r(n)\, s(n)\bigr).$$

Problem 4 (10 pts)

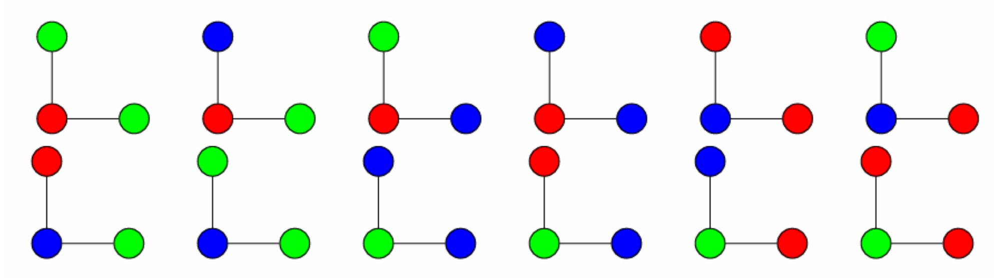
The *diameter* of an undirected graph is the maximum distance between any pair of vertices. If a graph is not connected, its diameter is infinite. Let G be an n node undirected graph, where n is even. Suppose that every vertex has degree at least n/2. Prove that G has diameter at most 2.

Hint: Proof by contradiction

Problem 5 (10 pts)

Show that there are at least $3 \cdot 2^{n-1}$ ways to properly color vertices of a tree T with n vertices using 3 colors, i.e., to color vertices with three colors such that any two adjacent vertices have distinct colors. Note that it can be shown that there are exactly $3 \cdot 2^{n-1}$ ways to properly color vertices of T with 3 colors but in this problem, to receive full credit, it is enough prove the "at least" part.

For example, there are (at least) $3 \cdot 2^2 = 12$ ways to color a tree with 3 vertices as show below:



Problem 6 (Extra Credit: 10 pts)

Given a directed graph G with n vertices $V = \{1, 2, \cdots, n\}$ and m edges. We say that a vertex j is reachable from i if there is a directed path from i to j. Design an $O(m+n)$-time algorithm (show the pseudo-code) that for any vertex i outputs the smallest label reachable from i. For example, given the following graph you should output 1,2,2,2,1 corresponding to the smallest indices reachable from vertices 1,2,3,4,5 respectively.