CS-3510-**Your Section#** Algorithms, Spring 2025 Mid-Term 1 Take-Home
FirstName LastName Collaborator(s):

Problem 1 (10 pts)

For this problem, we explore the issue of *truthfulness* in the Gale-Shapley algorithm for Stable Matching. Show that a participant can improve its outcome by lying about its preferences. Consider $r \in R$. Suppose r prefers p to p', but p and p' are low on r's preference list. Show that it is possible that by switching the order of p and p' on r's preference list, r achieves a better outcome, e.g., is matched with a p'' higher on the preference list than the one if the actual order was used.

Hint: Prove the claim by finding one specific instance of stable matching problem and comparing the stable matching before and after the switching.

Problem 2 (10 pts)

Arrange in increasing order of asymptotic growth. All logs are in base 2.

1. $n^{\frac{5}{3}} \log^2 n$
2. $2^{\sqrt{\log n}}$
3. $\sqrt{n^n}$
4. $\frac{n^2}{\log n}$
5. 2^n

Problem 3 (10 pts)

We say that $T(n)$ is $O(f(n))$ if there exist c and n_0 such that for all $n > n_0$, $T(n) \leq cf(n)$. Use this definition for parts a and b.

1. Prove that $4n^2 + 3n\log n + 6n + 20\log^2 n + 11$ is $O(n^2)$. (You may use, without proof, the fact that $\log n < n$ for $n \geq 1$.)
2. Suppose that $f(n)$ is $O(r(n))$ and $g(n)$ is $O(s(n))$. Let $h(n) = f(n)g(n)$ and $t(n) = r(n)s(n)$. Prove that $h(n)$ is $O(t(n))$.

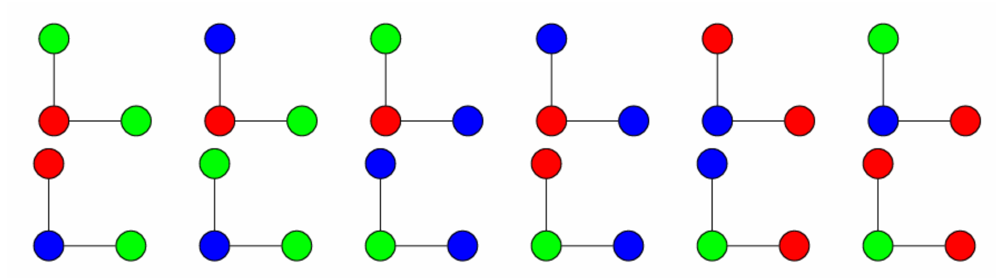
Problem 4 (10 pts)

The *diameter* of an undirected graph is the maximum distance between any pair of vertices. If a graph is not connected, its diameter is infinite. Let G be an n node undirected graph, where n is even. Suppose that every vertex has degree at least n/2. Prove that G has diameter at most 2.

Hint: Proof by contradiction

Problem 5 (10 pts)

Show that there are at least $3 \cdot 2^{n-1}$ ways to properly color vertices of a tree T with n vertices using 3 colors, i.e., to color vertices with three colors such that any two adjacent vertices have distinct colors. Note that it can be shown that there are exactly $3 \cdot 2^{n-1}$ ways to properly color vertices of T with 3 colors but in this problem, to receive full credit, it is enough prove the "at least" part.
For example, there are (at least) $3 \cdot 2^2 = 12$ ways to color a tree with 3 vertices as show below:



Problem 6 (Extra Credit: 10 pts)

Given a directed graph G with n vertices $V = \{1, 2, \cdots, n\}$ and m edges. We say that a vertex j is reachable from i if there is a directed path from i to j. Design an $O(m+n)$-time algorithm (show the pseudo-code) that for any vertex i outputs the smallest label reachable from i. For example, given the following graph you should output 1,2,2,2,1 corresponding to the smallest indices reachable from vertices 1,2,3,4,5 respectively.