Résumé Nombres Complexe :

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Forme algébrique :

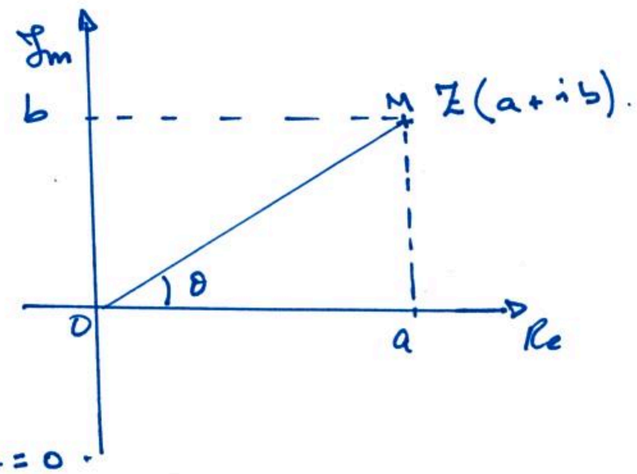
$Z = a + ib \quad (a \in \mathbb{R};\ b \in \mathbb{R};\ i^2 = -1).$

partie réelle

partie imaginaire.

$Re(Z) = a.$

$Im(Z) = b.$



Z est purement imaginaire $\Rightarrow a = 0.$

Z est purement réelle $\Rightarrow b = 0.$

Module et argument d'un nbre complexe :

→ Le module r du nombre complexe Z est la distance OM

$$r = |Z| = \sqrt{a^2 + b^2}$$

→ L'argument du nombre complexe Z est l'angle θ

$$Arg(Z) = \theta.$$

Z peut alors s'écrire sous la forme Trigonométrique :

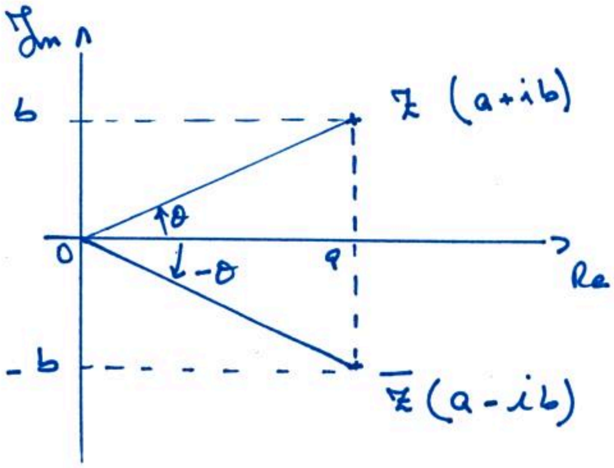
$$Z = r(\cos\theta + i \sin\theta).$$

Ou bien encore sous forme exponentielle

$$Z = r e^{i\theta}.$$

Conjugué d'un nombre complexe :

- Le nbre complexe conjugué de $z = a + ib$ est le complexe $\bar{z} = a - ib$;



$$z\bar{z} = (a+ib)(a-ib)$$
$$= a^2 + b^2$$
$$= |z|^2 = |\bar{z}|^2$$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 = (a_1 + a_2) - i(b_1 + b_2).$$

$z_1 = a_1 + ib_1$

$z_2 = a_2 + ib_2$

$$\overline{z_1 * z_2} = \bar{z}_1 * \bar{z}_2 = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1)$$

Propriétes des nombres complexes :

soit $z = a + ib$ $\quad z' = a' + ib'$

Egalité :

$$z = z' \Rightarrow a = a' \quad \text{et} \quad b = b'$$

Produit :

$$zz' = aa' - bb' + i(ab' + a'b)$$

$$|zz'| = |z| \times |z'|$$

$$\arg(z \cdot z') = \arg z + \arg z'$$

$$zz' = rr'\left(\cos(\theta + \theta') + i(\sin(\theta + \theta'))\right).$$

Quotient :

$$\frac{z}{z'} = \frac{a + ib}{a' + ib'} = \frac{(a+ib)(a'-ib')}{(a'+ib')(a'-ib')} = \frac{aa' + bb' + i(ab' + a'b)}{a'^2 + b'^2}$$

$$\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|}$$

$$\arg\left(\frac{z}{z'}\right) = \arg(z) - \arg(z').$$

$$\frac{z}{z'} = \frac{r}{r'}\left(\cos(\theta - \theta') + i\sin(\theta - \theta')\right).$$

TD N°1 Nombres complexes

Ex 1: Ecrire sous la forme $a+bi$ les nbrs complexes suivants :

$$Z_1 = \frac{2-\sqrt{3}i}{\sqrt{3}-2i}, \quad Z_2 = \frac{(2+i)(3+2i)}{2(2-i)}, \quad Z_3 = \frac{3+4i}{(2+3i)(4+i)}$$

$$Z_4 = \left[\frac{2+i^5}{1+i^{15}}\right]^2$$

→ Multiplier le numerateur et le dénominateur du nbre complexe par le nbre complexe conjugué du dénominateur.

$$Z = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

$$Z_1 = \frac{2-\sqrt{3}i}{\sqrt{3}-2i} = \frac{(2-\sqrt{3}i)(\sqrt{3}+2i)}{(\sqrt{3}-2i)(\sqrt{3}+2i)} = \frac{4}{7}\sqrt{3} + \frac{1}{7}i$$

$$Z_2 = \frac{(2+i)(3+2i)}{2(2-i)} = \frac{(2+i)^2(3+2i)}{2(2-i)(2+i)} = \frac{1}{10} + \frac{18}{10}i$$

$$Z_3 = \frac{3+4i}{(2+3i)(4+i)} = \frac{71}{221} - \frac{22}{221}i$$

$$Z_4 = \left[\frac{2+i^5}{1+i^{15}}\right]^2 = \left(\frac{2+i}{1-i}\right)^2 = -2 + \frac{3}{2}i$$

Ex 2: Déterminer le paramètres α pour que

$$Z = \frac{1+i\alpha}{2\alpha + i(\alpha^2-1)}$$

soit purement imaginaire.

$$Z = \frac{1+i\alpha}{2\alpha + i(\alpha^2-1)} = \frac{(1+i\alpha)\left[2\alpha - i(\alpha^2-1)\right]}{\left[2\alpha + i(\alpha^2-1)\right]\left[2\alpha - i(\alpha^2-1)\right]}$$

$$= \frac{2\alpha(1+i\alpha) - i(1+i\alpha)(\alpha^2-1)}{(2\alpha)^2 + (\alpha^2-1)^2}$$

$$Z = \frac{2\alpha + i2\alpha^2 - i(\alpha^2 - 1 + i\alpha^3 - i\alpha)}{(\alpha^2+1)^2}$$

$$= \frac{2\alpha + i2\alpha^2 - i\alpha^2 + i + \alpha^3 - \alpha}{(\alpha^2+1)^2}$$

$$= \frac{(\alpha + \alpha^3) + i(2 \ast \alpha^2 + 1)}{(\alpha^2+1)^2}$$

$$= \frac{\alpha(1+\alpha^2) + i(1+\alpha^2)}{(1+\alpha^2)^2}$$

$$= \frac{\alpha + i}{\alpha^2 + 1}$$

Z est imaginaire pure pour $\frac{\alpha}{\alpha^2+1} = 0$ soit $\alpha = 0$

pour $\alpha = 0$ $Z = i$

Ex 4 :

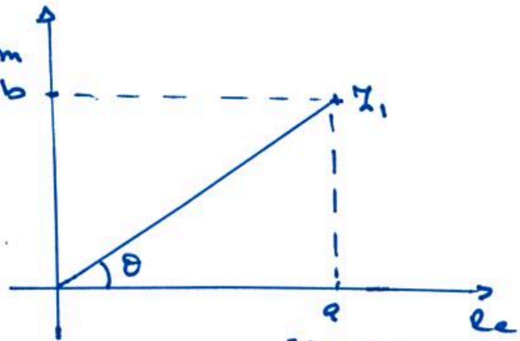
Determiner les modules et les arguments des nombres complexes suivants :

$$Z_1 = 1 + i \qquad Z_2 = \frac{1-i}{1+i} \qquad Z_3 = \frac{1+i\sqrt{3}}{\sqrt{3}+i}$$

$$Z_4 = (1+i)^8 (1 - i\sqrt{3})^{-6}$$

$Z_1 = 1 + i$

$|Z_1| = r_1 = \sqrt{1^2 + 1^2} = \sqrt{2}$

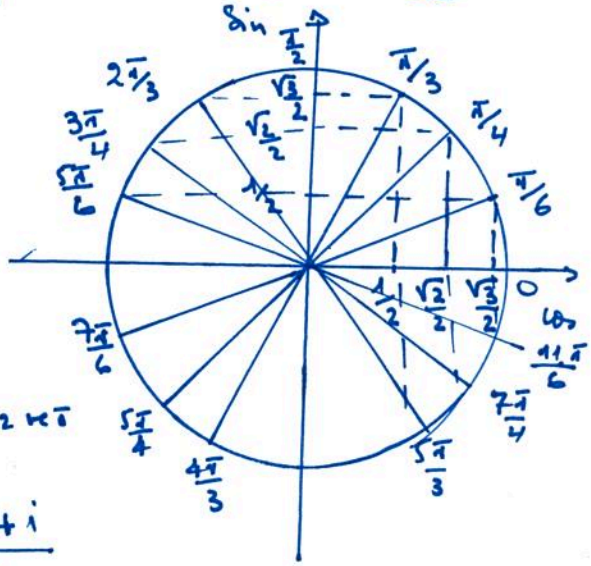


$\cos\theta = \dfrac{a}{\sqrt{a^2+b^2}} \qquad \sin\theta = \dfrac{b}{\sqrt{a^2+b^2}}$

$\theta = \operatorname{arctg}\left(\dfrac{b}{a}\right)$

$tg(\theta) = \dfrac{b}{a}$

$\theta_1 = Arg(Z_1) = \dfrac{\pi}{4} + 2k\pi$



$Z_2 = \dfrac{1-i}{1+i}$

$|Z_2| = r_2 = \dfrac{\sqrt{1^2 + (-1)^2}}{\sqrt{1^2 + (1)^2}} = 1.$

$Arg(Z_2) = \theta_2 = Arg(1-i) - Arg(1+i)$

$= \dfrac{-\pi}{4} - \dfrac{\pi}{4} = \dfrac{-\pi}{2} + 2k\pi$

$Z_3 = \dfrac{1+i\sqrt{3}}{\sqrt{3}+i} = \dfrac{(1+i\sqrt{3})(\sqrt{3}-i)}{4} = \dfrac{\sqrt{3}+i}{2}$

$|Z_3| = r_3 = \dfrac{\sqrt{1^2 + \sqrt{3}^2}}{\sqrt{\sqrt{3}^2 + 1^2}} = \dfrac{\sqrt{4}}{\sqrt{4}} = 1.$

$Arg(Z_3) = \theta_3$

$\cos\theta_3 = \dfrac{\sqrt{3}/2}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} = \dfrac{\sqrt{3}}{2} \qquad \sin\theta_3 = \dfrac{1}{2}$

$\theta_3 = \dfrac{\pi}{6} + 2\pi k.$

$Z_4 = (1+i)^8 (1-i\sqrt{3})^{-6}$

Rappel Formule de Moivre.

$$Z^n = \left(\rho(\cos\theta + i\sin\theta)\right)^n = \rho^n \left(\cos(n\theta) + i \sin(n\theta)\right).$$

$$|Z_4| = r_4 = \frac{\sqrt{2}^8}{\sqrt{1^2+(-\sqrt{3})^2}^6} = \frac{2^4}{2^6} = \frac{1}{4}$$

$\mathrm{Arg}(1+i) = \frac{\pi}{4} + 2k\pi$ $\qquad$ $\mathrm{Arg}(1-i\sqrt{3}) = \frac{-\pi}{3} + 2k\pi$

$\cos(1-i\sqrt{3}) = \frac{1}{2}$ $\qquad$ $\sin(1-i\sqrt{3}) = \frac{-\sqrt{3}}{2}$

$$\mathrm{Arg}(Z_4) = \theta_4 = \frac{8\pi}{4} + \frac{6\pi}{3} + 2k\pi$$

$$= 4\pi + 2k\pi = 2k\pi.$$

Ex : Simplifier les expressions suivantes:

$$Z_1 = \frac{\cos\alpha + i\sin\alpha}{\cos\beta - i\sin\beta} \qquad Z_2 = \frac{(1-i\sqrt{3})(\cos\alpha + i\sin\alpha)}{2(1-i)(\cos\alpha - i\sin\alpha)}$$

$$Z_1 = \frac{\cos\alpha + i\sin\alpha}{\cos\beta - i\sin\beta} = \frac{e^{i\alpha}}{e^{-i\beta}} = e^{i\alpha} \cdot e^{i\beta}$$

$$= (\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)$$

$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$

$$\boxed{Z_1 = \cos(\alpha+\beta) + i\sin(\alpha+\beta)}.$$

sin → $\pi/3$

$\frac{\sqrt{3}}{2}$

$\frac{1}{2}$

Rappel:

$$\boxed{\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}}$$

$$Z_2 = \frac{(1-i\sqrt{3})(\cos\alpha + i\sin\alpha)}{2(1-i)(\cos\alpha - i\sin\alpha)} = \frac{2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(e^{i\alpha}\right)}{\frac{4}{\sqrt{2}}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\left(e^{-i\alpha}\right)}$$

$$= \frac{\sqrt{2}}{2} \frac{\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)(\cos\alpha + i\sin\alpha)}{\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)(\cos\alpha - i\sin\alpha)} = \frac{\sqrt{2}}{2} \frac{e^{-i\pi/3}\, e^{i\alpha}}{e^{-i\frac{\pi}{4}}\, e^{-i\alpha}}$$

$$= \frac{\sqrt{2}}{2} e^{-i\frac{\pi}{3}} e^{i\frac{\pi}{4}} e^{i2\alpha} = \frac{\sqrt{2}}{2} e^{i\left(\frac{-\pi 4}{12} + \frac{\pi 3}{12} + 2\alpha\right)} = \frac{\sqrt{2}}{2} e^{i\left(-\frac{\pi}{12} + 2\alpha\right)}$$

$$z_2 = \frac{\sqrt{2}}{2} e^{i\left(-\frac{\pi}{12} + 2\alpha\right)}$$

$$\boxed{z_2 = \frac{\sqrt{2}}{2} \cos\left(2\alpha - \frac{\pi}{12}\right) + i\frac{\sqrt{2}}{2}\sin\left(2\alpha - \frac{\pi}{12}\right)}$$

Ex 8: Determiner les parties réelles et les parties imaginaires des nombres complexes suivants :

$Z_1 = e^{-1+i\pi/6}$ $\quad Z_2 = e^{2-i}$ $\quad Z_3 = e^{-\frac{i\pi}{2}}$ $\quad Z_4 = e^{(1+i)} e^{(-2+\frac{i\pi}{3})}$

$$Z_1 = e^{-1} e^{i\pi/6}$$

$$= e^{-1}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{1}{e}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$$

$Re(Z_1) = \frac{\sqrt{3}}{2e}$ $\qquad Im(Z_1) = \frac{1}{2e}$

Revenir dessus.

$\to e^2(\cos(-1) + i\sin(-1))$

$= e^2(\cos(1) - i\sin(1))$ $\quad \cos(-\alpha) = \cos\alpha$

$\sin(-\alpha) = -\sin\alpha$

$$Z_2 = e^2 e^{-i} = e^2(\cos 1 - i\sin 1)$$

$Re(Z_2) = e^2\cos 1$ $\qquad Im(Z_2) = -e^2\sin 1.$

$$Z_3 = e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

$$= 0 - i$$

$Re(Z_3) = 0$ $\qquad Im(Z_3) = -1.$

$$Z_4 = e^{(1+i)} e^{(-2+\frac{i\pi}{3})}$$

$$= e^{1+i-2+i\frac{\pi}{3}} = e^{-1+i(1+\frac{\pi}{3})}$$

$$= e^{-1} e^{i(1+\frac{\pi}{3})}$$

$$= e^{-1}\left(\cos(1+\tfrac{\pi}{3}) + i\sin(1+\tfrac{\pi}{3})\right)$$

$Re(Z_4) = \frac{\cos(1+\frac{\pi}{3})}{e}$

$Im(Z_4) = \frac{\sin(1+\frac{\pi}{3})}{e}$

Ex 11 :

a) Résoudre dans $\mathbb{C}$ les équations suivantes :

$$z^2 + z\sqrt{3} + 1 = 0$$

b) 1. $z^2 = -8 + 6i$

2. Déduire les solutions dans $\mathbb{C}$ de l'équation

$$z^2 + (-3+i)z + 4 - 3i = 0 .$$

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a) $z^2 + z\sqrt{3} + 1 = 0$ $\qquad \Delta = b^2 - 4ac .$

$\Delta = \sqrt{3}^2 - 4 = -1$ $\qquad z_1 = \dfrac{-b \pm \sqrt{\Delta}}{2a}$

$\sqrt{\Delta} = i$.

$z_{1,2} = \frac{1}{2}\left(-\sqrt{3} \pm i\right)$.

b) On cherche z sous la forme $z = a + ib$. $\qquad a \in \mathbb{R}$ $b \in \mathbb{R}$

$z^2 = (a+ib)^2 = a^2 - b^2 + 2iab = -8 + 6i$

Par identification

$$\begin{cases} a^2 - b^2 = -8 & \text{--- (1)} \\ 2ab = 6 \end{cases}$$

$$\sqrt{a^2+b^2}^{\,2} = \sqrt{(-8)^2 + 6^2}$$

$a^2 + b^2 = 10$ --- (2)

(1) + (2) $\quad 2a^2 = 2 \qquad a = \pm 1$

$\qquad\qquad\qquad\qquad b = \pm 3$

$z_1 = 1 + i3$

$z_2 = -1 - i3$

$$z^2 + (-3+i)z + 4 - 3i = 0$$

$$\Delta = (-3+i)^2 - 4(4-3i) = -8+6i$$

$$z_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{+3-i+1+3i}{2} = 2+i$$

$$z_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{+3-i-1-3i}{2} = 1-2i$$

Ex 12: Résoudre dans $\mathbb{C}$ l'équation $z^2 + (1-5i)z - 3i - 6 = 0$ sachant que l'une des solution est imaginaire pure.

solution imaginaire pure $z_1 = ia$. $a \in \mathbb{R}$

$$(ia)^2 + (1-5i)(ia) - 3i - 6 = 0$$

$$-a^2 + 5a - 6 + i(a-3) = 0 \Rightarrow \begin{cases} -a^2 + 5a - 6 = 0 \\ a - 3 = 0 \end{cases} \Rightarrow a = 3.$$

$$\boxed{z_1 = 3i}$$

$(z-3i)(z-z_2) = 0$. avec $z_2 = a_2 + ib_2$

$$z^2 - z(a_2 + ib_2) - zi3 + 3i(a_2 + ib_2) = 0$$

$$z^2 - z(a_2 + ib_2 + i3) + 3a_2 i - 3b_2 = 0$$

$$-(a_2 + (b_2+3)i) = 1 - 5i$$

$$-a_2 - (b_2+3)i = 1 - 5i$$

$$\begin{cases} a_2 = -1 \\ b_2 = 2 \end{cases}$$

$$z_2 = -1 + 2i$$

$$S = \{3i, -1+2i\}.$$

Ex 13 :

Linéariser les polynômes trigonométriques suivants :

$P_1(x) = \cos^5 x$.

$P_2(x) = \sin^4 x + x\, 3 \cos^4 x$.

$P_3(x) = \sin x . \cos^4 x$

formule d'Euler : $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ $\qquad \sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$$P_1(x) = \cos^5 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^5 =$$

Ex 07 :

1- Sachant que z appartient au cercle trigonométrique unitaire, $(z = e^{i\theta})$ simplifier l'expression du complexe,

$$Z = \frac{1 - z}{1 + z}$$

2. Simplifier l'expression

$A = (1+z)^n$ avec $z = e^{\frac{2i\pi}{3}}$ $n \in \mathbb{N}$.

$$Z = \frac{[1 - (\cos\theta + i\sin\theta)](1 + \cos\theta - i\sin\theta)}{[1 + (\cos\theta + i\sin\theta)](1 + \cos\theta - i\sin\theta)}$$

$$= \frac{1 + \cancel{\cos\theta} - i\sin\theta - \cancel{\cos\theta} - \cos^2\theta + \cancel{i\sin\theta\cos\theta} - i\sin\theta - \cancel{i\sin\theta\cos\theta} - \sin^2\theta}{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

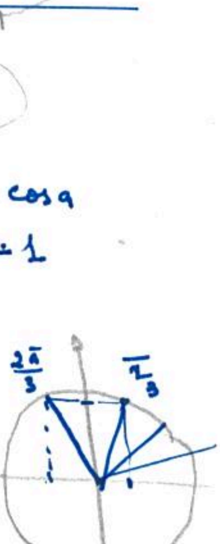
$\cos^2\theta + \sin^2\theta = 1$

$$= \frac{-2i\sin\theta}{2(1+\cos\theta)} = -i\,\frac{\sin\theta}{1+\cos\theta}$$

$\sin 2a = 2 \sin a \cos a$

$\cos 2a = 2\cos^2 a - 1$

$$Z = -i\,\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 + 2\cos^2\frac{\theta}{2} - 1} = -i\,\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = -i\,tg\frac{\theta}{2}.$$



$$A = (1+z)^n = (1 + \cos\theta + i\sin\theta)^n$$

$$= \left(1 + \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^n = \left(1 - \frac{1}{2} + i\sin\frac{2\pi}{3}\right)^n$$

$$= \left(\frac{1}{2} + i\sin\frac{\pi}{3}\right)^n = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^n$$

$$= \left(e^{i\frac{\pi}{3}}\right)^n = e^{in\frac{\pi}{3}}.$$

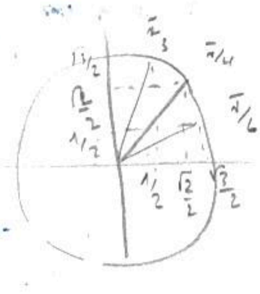
Ex Calculer les racines carrées des nombres complexes :

$z_1 = 2i \qquad z_2 = \frac{1+i}{1-i} \qquad z_3 = 1+i.$

$$z_1 = 2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2e^{i\frac{\pi}{2}}$$

$$\sqrt{z_1} = \sqrt{2}\, e^{i\left(\frac{\pi}{4}+k\pi\right)} = \pm\sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \pm(1+i).$$

$$z_2 = \frac{1+i}{1-i} = \frac{1}{2}(1+i)^2 \Rightarrow \sqrt{z_2} = \pm\frac{\sqrt{2}}{2}(1+i).$$



$$z_3 = 1+i = \frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\sqrt{z_3} = \sqrt[4]{2}\left[\cos\left(\frac{\pi}{8}+k\pi\right) + i\sin\left(\frac{\pi}{8}+k\pi\right)\right]$$

à faire !

Ex Calculer les racines cubiques des nombres suivants :

$z_1 = 1 \qquad z_2 = i \qquad z_3 = 2-2i \qquad z_4 = \frac{i+\sqrt{3}}{i-\sqrt{3}}$

$$z_1 = 1 = e^{2ik\pi} \qquad \sqrt[3]{z_1} = e^{2ik\frac{\pi}{3}} \qquad k=0,1,2$$

$$\sqrt[3]{z_1} = 1, \qquad e^{2i\frac{\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$e^{i\frac{4\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

$$z_2 = i = e^{i\frac{\pi}{2}}$$

$$\sqrt[3]{z_2} = e^{i\pi\left(\frac{1}{6}+\frac{2k}{3}\right)} \qquad k=0,1,2$$

$$\sqrt[3]{z_2} = e^{i\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{i}{2}, \qquad e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}}{2} + \frac{i}{2}, \qquad e^{i\frac{3\pi}{2}} = -i.$$

$$z_3 = 2-2i \Rightarrow 2\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

$$= 2\sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$= 2\sqrt{2}\, e^{-i\frac{\pi}{4}} = 2^{3/2} e^{-i\pi/4}$$

$$\sqrt[3]{z_3} = \sqrt{2}\, e^{i\pi\left(-\frac{1}{12}+\frac{2k}{3}\right)} \qquad k=0,1,2$$

écrire $-\frac{\pi}{12} = \frac{\pi}{6} - \frac{\pi}{4}$

$$\sqrt{2}\, e^{-i\pi/12} = \frac{\sqrt{3}+1}{2} + i\,\frac{1-\sqrt{3}}{2}$$

$$\sqrt{2}\, e^{i\frac{7\pi}{12}} = \qquad \frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$$

$$\sqrt{2}\, e^{i\frac{15\pi}{12}} = \sqrt{2}\, e^{i\frac{5\pi}{4}} = -(1+i)$$

Racines nièmes d'un nombre complexe:

$$z^n = a \qquad a \in \mathbb{C}$$

$$a = r e^{i\alpha} \qquad z = \rho e^{i\theta}$$

$$z^n = a \Leftrightarrow \begin{cases} \rho^n = r \\ n\theta = \alpha + 2k\pi \end{cases} \Leftrightarrow \begin{cases} \rho = \sqrt[n]{r} \\ \theta = \frac{\alpha}{n} + \frac{2k\pi}{n} \end{cases}$$

d'où $z_k = \sqrt[n]{r}\, e^{i\left(\frac{\alpha}{n} + \frac{2k\pi}{n}\right)}$ $\qquad k \in \{0, 1, 2, \ldots, n-1\}$

Rappel

$$\boxed{\begin{aligned} \cos(a+b) &= \cos a \cos b - \sin a \sin b \\ \cos(a-b) &= \cos a \cos b + \sin a \sin b \\ \sin(a+b) &= \cos a \sin b + \sin a \cos b \\ \sin(a-b) &= \cos a \sin b - \sin a \cos b \end{aligned}}$$

$$\sqrt{2}\, e^{i\frac{-\pi}{12}} = \sqrt{2}\, e^{i\left(\frac{\pi}{6} - \frac{\pi}{4}\right)}$$

$$= \sqrt{2}\left[\cos\left(\frac{\pi}{6} - \frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\left[\left(\cos\frac{\pi}{6}\cos\frac{\pi}{4} + \sin\frac{\pi}{6}\sin\frac{\pi}{4}\right) + i\left(\cos\frac{\pi}{6}\sin\frac{\pi}{4} - \sin\frac{\pi}{6}\cos\frac{\pi}{4}\right)\right]$$

$$= \sqrt{2}\left[\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} + i\left(\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} - \frac{1}{2}\frac{\sqrt{2}}{2}\right)\right]$$

$$\boxed{\sqrt{2}\, e^{i\frac{-\pi}{12}} = \frac{\sqrt{3}+1}{2} + i\,\frac{\sqrt{3}-1}{2}}$$

$$\sqrt{2}\, e^{i\frac{7\pi}{12}} = \sqrt{2}\left(\cos\left(\frac{7\pi}{12}\right) + i \sin\left(\frac{7\pi}{12}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)\right)$$

$$= \sqrt{2}\left[\left(\cos\frac{\pi}{4}\cos\frac{\pi}{3} - \sin\frac{\pi}{4}\sin\frac{\pi}{3}\right) + i\left(\cos\frac{\pi}{4}\sin\frac{\pi}{3} + \sin\frac{\pi}{4}\cos\frac{\pi}{3}\right)\right]$$

$$= \sqrt{2}\left[\left(\frac{\sqrt{2}}{2}\frac{1}{2} - \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}\right) + i\left(\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2}\right)\right]$$

$$\boxed{\sqrt{2}\, e^{i\frac{7\pi}{12}} = \frac{1-\sqrt{3}}{2} + i\,\frac{\sqrt{3}+1}{2}}$$

$$Z_4 = \frac{i+\sqrt{3}}{i-\sqrt{3}} = \frac{(\sqrt{3}+i)^2}{4} = \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^2 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{2}$$

$$= e^{i\frac{\pi}{6}}$$

$$\sqrt[3]{Z_4} = e^{i\pi\left(\frac{1}{18}+\frac{2k}{3}\right)} \quad (k=0,1,2)$$