TTK4115

Lecture 9

Random processes

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This lecture

1. Basic concepts of Random Processes, B&H 2.1-2.4
2. Autocorrelation functions, B&H 2.5
3. Spectral density functions, B&H 2.7-2.8
4. Common random processes, B&H 2.9-2.14

Topic

1. Basic concepts of Random Processes, B&H 2.1-2.4

2. Autocorrelation functions, B&H 2.5

3. Spectral density functions, B&H 2.7-2.8

4. Common random processes, B&H 2.9-2.14

Random Processes

Terminology

Deterministic process Completely predictable.
Random process Uncertain, not completely predictable[1]. *Can be characterized using statistical properties*.

NB!

random process = stochastic process = random signal = random system = stochastic system

[1] But not necessarily completely unpredictable

Distribution Functions

Probability *Distribution* Function

The probability **distribution** function[2] for a continuous random variable X is defined as

$$F_X(\theta) = P(X \leq \theta)$$

Properties

- $F_X(\theta) \to 0$, as $\theta \to -\infty$
- $F_X(\theta) \to 1$, as $\theta \to \infty$
- $F_X(\theta)$ is a nondecreasing function of θ

[2] No: sannsynlighetsfordelingsfunksjon

Distribution Functions

Probability *Density* Function

The probability **density** function[3] for a continuous random variable X is defined as

$$f_X(\theta) = \frac{d}{d\theta} F_X(\theta)$$

Properties

- $F_X(x) = \int_{-\infty}^{x} f_X(\theta) d\theta$
- $P(\theta_1 \leq X \leq \theta_2) = F_X(\theta_2) - F_X(\theta_1) = \int_{\theta_1}^{\theta_2} f_X(\theta) d\theta$
- $\int_{-\infty}^{\infty} f_X(\theta) d\theta = 1$
- $f_X(\theta) \geq 0$

[3] No: sannsynlighetstetthetsfunksjon

Expected value

Expected value

The expected value[4] for a continuous random variable X, written $E(X)$, $\bar{X}$, m_X or μ, is:

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

The expected value of a function $g(X)$:

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

[4] No: forventningsverdi

Moments

The k'th moment

The k'th moment of a continuous random variable X is defined by

$$E(X^k) = \int_{-\infty}^{\infty} x^k f_X(x)\, dx$$

The first moment is equal to the mean $E(X) = m_X$*.*

Central moments

The k'th central moment of a continuous random variable X is:

$$E[(X - m_X)^k] = \int_{-\infty}^{\infty} (x - m_X)^k f_x(x)\, dx$$

Properties:

- The **first** central moment is equal to zero.
- The **second** central moment is called the variance[5], $\mathrm{Var}\, X = E(X^2) - (E(X))^2$
- The square root of the variance is called standard deviation[6], denoted σ_X

[5] No: varians

[6] No: standardavvik

Moments

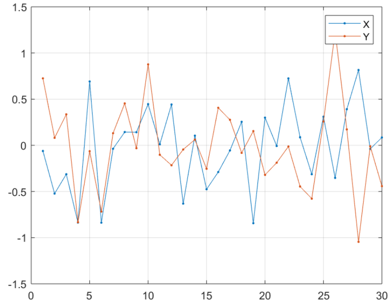
Correlation

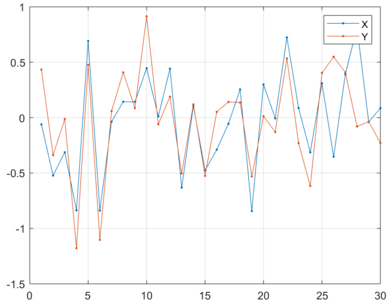
Independence

Independent random variables are said to be uncorrelated

$$E[XY] = E[X] \cdot E[Y]$$

Independent and correlated variables





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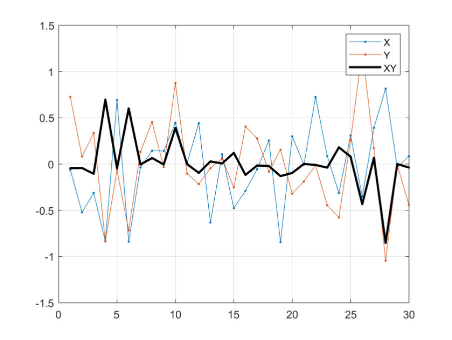

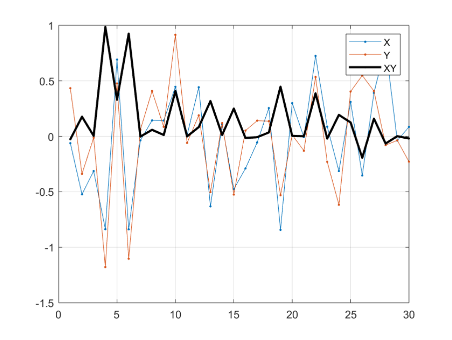

Correlation

Independence

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$$E[XY] = E[X] \cdot E[Y]$$

Covariance

The covariance of X and Y is defined by

$$\mathrm{Cov}(X, Y) = E[(X - m_x)(Y - m_y)]$$

The *correlation coefficient* is defined by

$$\varrho = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Topic

Random Processes

Signals

Deterministic signal *Known* function of time, for example $x(t) = A\sin(\omega_0 t + \varphi)$.

Random signal *Random* function of time. At a given time t, $x(t)$ is the realization (or outcome) of a random variable $X(t)$.

Problem

Can we predict $X(t_2)$ if we know the outcome of $X(t_1)$? Are $X(t_1)$ and $X(t_2)$ correlated, for $t_1 \neq t_2$?

Tools

Random/Stochastic models characterize to what extent signal values at different time instants are correlated. The models assume the form of

- Correlation functions
- Spectral density functions
- Transfer functions
- State space models

Random Processes - Typical examples

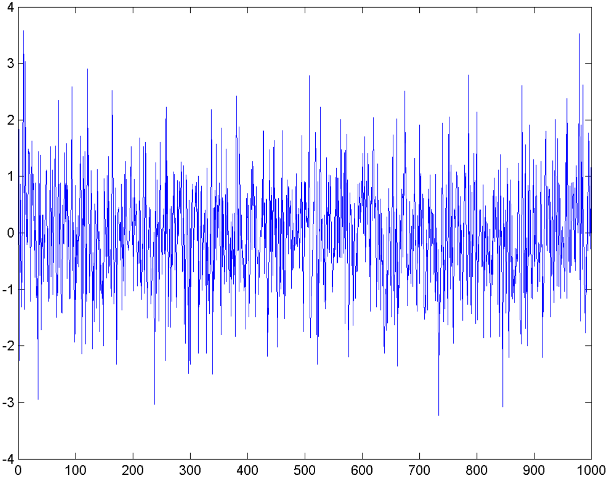


Figure: High-frequency measurement noise

Random Processes - Typical examples

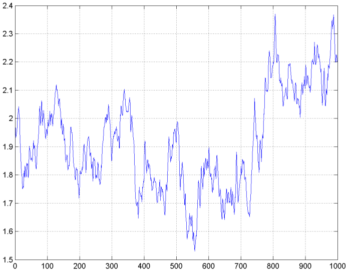


Figure: Process disturbance; e.g wind force, or variations in raw materials.

Random Processes - Typical examples

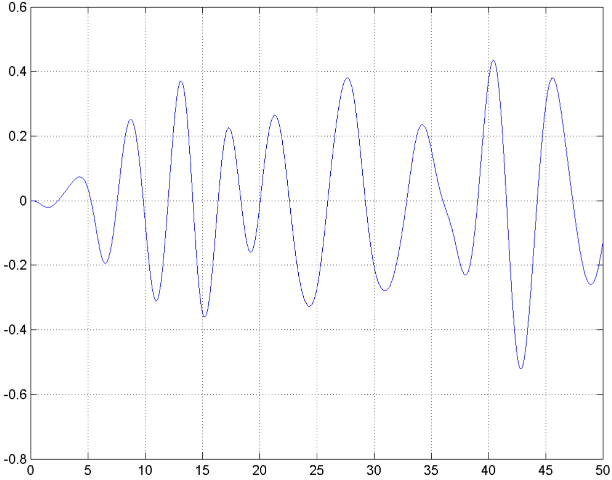


Figure: Irregular waves

Random Processes - Most general characterization

Joint probability density function

$$f_{X(t_1),X(t_2),X(t_3),\ldots}(x_1, x_2, x_3, \ldots)$$

Describes the probability that $X(t_1) = x_1$, $X(t_2) = x_2$, ..., simultaneously.

The Joint probability density function is not practical

Considers a large (possibly infinite) number of time instants $t_1, t_2, t_3, \ldots.$ and is generally too complicated for practical use.

Autocorrelation function

Autocorrelation/Autocovariance

The autocorrelation function for a random process $X(t)$ is defined by

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

The autocovariance function for a random process $X(t)$ is defined by

$$C_X(t_1, t_2) = E[(X(t_1) - m_x(t_1))(X(t_2) - m_x(t_2))]$$

where $m_x(t)$ is the mean of $X(t)$.

These quantities are equal if the process has zero mean.

Stationary random processes

The simultaneous joint probability density function can be shifted in time if the statistical properties of the process does not depend on time. If the process is stationary, then

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

reduces to

$$R_X(\tau) = E[X(t)X(t-\tau)] = E[X(t)X(t+\tau)]$$

Autocorrelation functions

Autocorrelation function

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)]$$

Properties of autocorrelation functions for stationary processes

- $R_X(0)$ is the mean square value of $X(t)$.
- $R_X(0)$ is the variance of $X(t)$, if $X(t)$ has zero mean.
- R_X is symmetric; $R_X(\tau) = R_X(-\tau)$
- $|R_X(\tau)| \leq R_X(0)$ for all τ.
- No periodic components in $X(t)$ if and only if $\lim_{\tau\to\infty} R_X(\tau) = 0$. Why?

Realization of a stationary random process

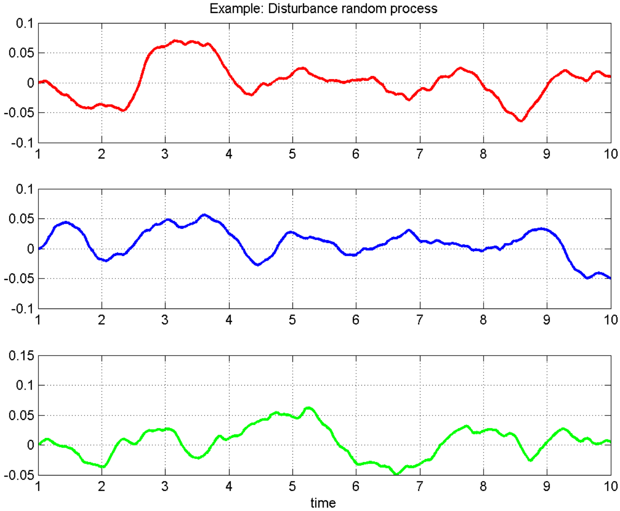
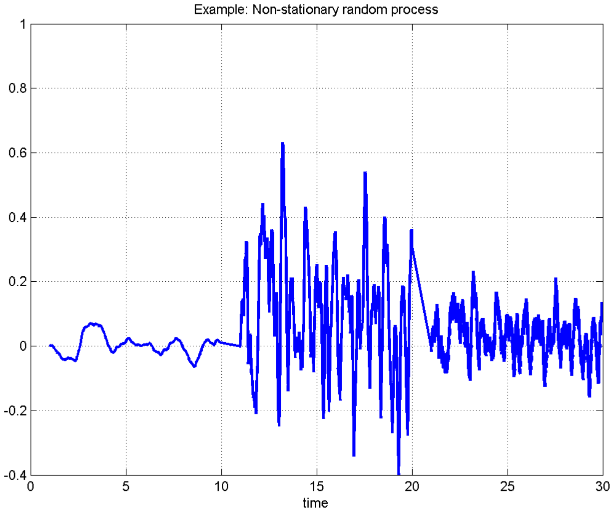


Figure: What happens if you run the same experiment many times? Same process - different realizations.

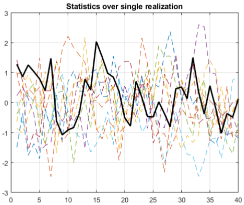
Realization of a non-stationary random process

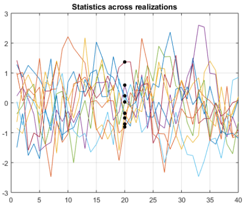


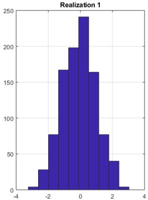
Ergodicity

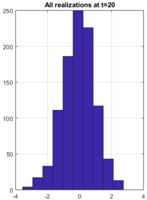
A random process is *ergodic* if averaging over (infinite) time can be used to compute expectations. Then, a single (infinite) realization is enough to capture all statistical properties.

$$R_X(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)x(t+\tau)dt$$









Topic

Power spectral density function

Power spectral density

The power spectral density[7] is related to the autocorrelation function by the Fourier transform $\mathcal{F}$:

$$S_X(j\omega) = \mathcal{F}\left[R_X(\tau)\right] = \int_{-\infty}^{\infty} R_X(\tau) e^{-j\omega\tau} d\tau$$

Autocorrelation function

The autocorrelation function can be found from the power spectral density function by the inverse Fourier transform $\mathcal{F}^{-1}$:

$$R_X(\tau) = \mathcal{F}^{-1}(S_X(j\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(j\omega) e^{j\omega\tau} d\omega$$

Hence, they are equivalent (same information)

[7] No: spektraltetthetsfunksjonen

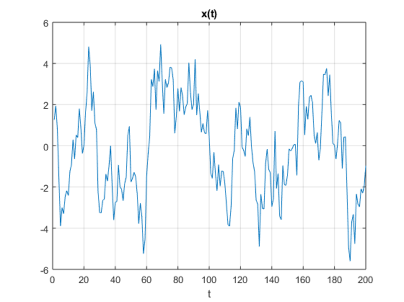
Power spectral density function

Example 2.9: Gauss-Markov process

Consider a random process $X(t)$ whose autocorrelation function is given by:

$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$$

Example realization ($\sigma = 2$, $\beta = 0.2$)



Power spectral density function

Example 2.9: Gauss-Markov process

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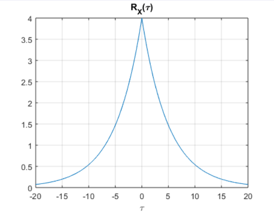
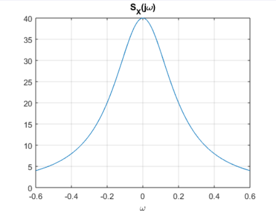
Spectral density function

$$S_X(j\omega) = \mathcal{F}[R_X(\tau)] = \frac{2\sigma^2\beta}{\omega^2 + \beta^2} = \frac{\sigma^2}{j\omega + \beta} + \frac{\sigma^2}{-j\omega + \beta}$$

Autocorrelation and spectral density ($\sigma = 2$, $\beta = 0.2$)





Power spectral density function

Example 2.9: Gauss-Markov process

Consider a random process $X(t)$ whose autocorrelation function is given by:

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Estimating mean square value from spectral function

$$R_X(0) = E[X^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(j\omega)d\omega$$

For the example process $X(t)$:

$$E[X^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sigma^2\beta}{\omega^2+\beta^2}d\omega = \frac{\sigma^2\beta}{\pi}\left[\frac{1}{\beta}\tan^{-1}\frac{\omega}{\beta}\right]_{-\infty}^{\infty} = \sigma^2$$

Periodogram and power spectral density

Estimating $S_X(j\omega)$ from data

Suppose $X_T(t)$ is a (finite) truncation of the random signal $X(t)$ at time T.
For a given realization $x_T(t)$ of $X_T(t)$ we define the periodogram

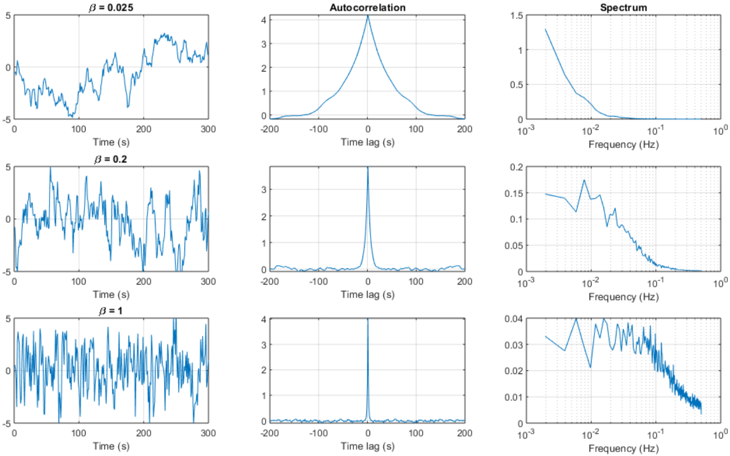
$$\frac{1}{T}\left|\mathcal{F}(x_T(t))\right|^2$$

For an ergodic process, we can show

$$E\left(\frac{1}{T}\left|\mathcal{F}(x_T(t))\right|^2\right) \to S_X(j\omega), \text{ as } T \to \infty$$

Practical estimation of $S_X(j\omega)$ from sampled data therefore often uses a combination of FFT (Fast Fourier Transform), window functions and averaging (see DSP course).

Example



β = 0.025
Autocorrelation
Spectrum
Time (s)
Time lag (s)
Frequency (Hz)
β = 0.2
β = 1

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Normal or Gaussian Random Variables

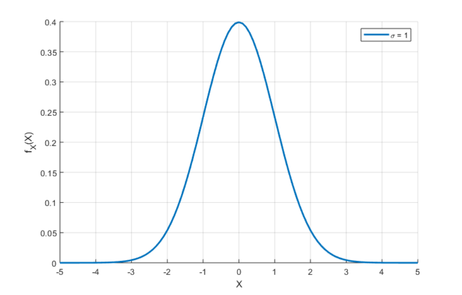
Gaussian distribution

The random variable X (scalar or vector) is called *Gaussian* or *normal* if its probability density function is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2\sigma^2}(x - m_X)^2\right]$$

Gaussian distribution

Probability density functions of Gaussian distributions with mean 0:



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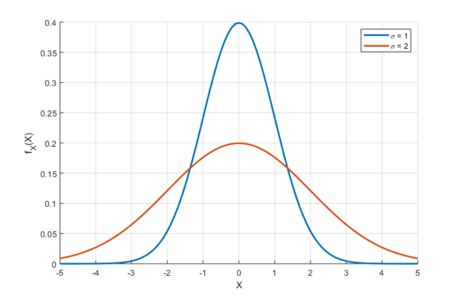

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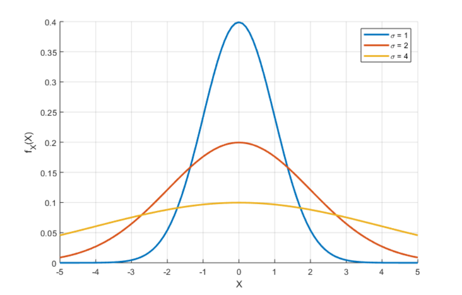

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Notation

The notation $X \sim N(m_x, \sigma^2)$ means that X is normally distributed with mean m_X and variance σ^2.

Multivariable Gaussian

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{C}|^{1/2}} \exp\left[-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{m}_X)\boldsymbol{C}^{-1}(\boldsymbol{x} - \boldsymbol{m}_X)\right]$$

The covariance matrix is defined by its elements

$$\boldsymbol{C}_{ij} = E(X_i - m_{X,i})(X_j - m_{X,j})$$

Gaussian distributions

- It is in many cases sufficient to know the means and covariances[8] of a random signal.
- If additional information is sought, a more explicit probability distribution must be found.
- The normal (Gaussian) distribution represents a wide variety of random processes found in nature and technology.
- The multivariable Gaussian probability distribution is completely specified by the mean vector and covariance matrix.

[8] Understood in the generalized sense including variances on the diagonal.

Gauss-Markov process

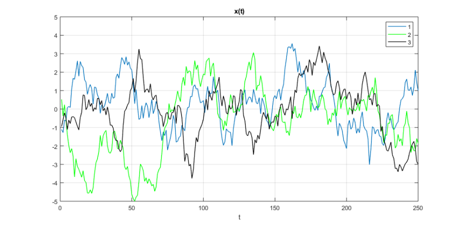
Gauss-Markov processes are Gaussian - i.e. their PDF is a normal distribution.
The process has an exponential autocorrelation function:

$$R_X(\tau) = \sigma^2 e^{-\beta|\tau|}$$

Simulating a Gauss-Markov process

To simulate the next value x_{t+1} of a Gauss-Markov process with parameters β and σ and time step dt:

$$x_{t+1} = f x_t + \sqrt{1 - f^2} \cdot N(0, \sigma^2)$$
$$f = e^{-\beta dt}$$



White noise

White noise

White noise[9] is a stationary random process having a constant spectral density function:

$$S_{wn}(j\omega) = A$$

Autocorrelation function

The corresponding autocorrelation function is

$$R_{wn}(\tau) = A\delta(\tau)$$

where $\delta(\cdot)$ is the *Dirac* function

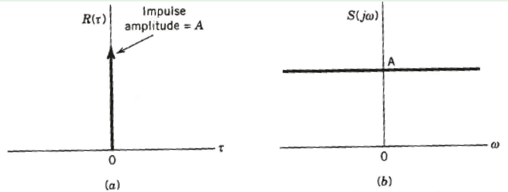
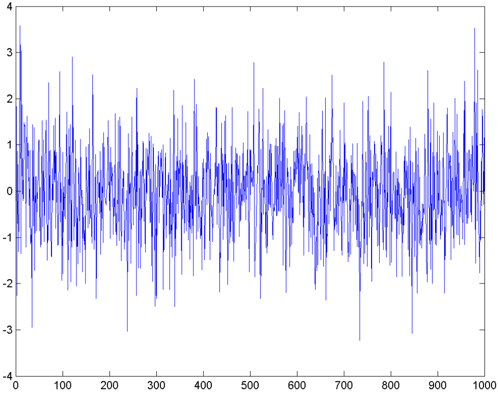


Figure 2.12 White noise. (*a*) Autocorrelation function. (*b*) Spectral density function.

[9] No: hvit støy

Example of white noise realization



White noise

Mathematical abstraction

- $X(t_1)$ and $X(t_2)$ are uncorrelated for all $t_1 \neq t_2$.
- Assuming zero-mean, the variance of the process is $R_X(0) = A\delta(0) \rightarrow \infty$. Must be interpreted and handled carefully.
- Assuming zero-mean, the power of the process is infinite (infinite area under the graph of $S_X(j\omega)$. Must be interpreted and handled carefully.

Useful because:

- Mathematically simple
- Many practical processes can be approximated close as white noise, or more generally *filtered* white noise.

Gaussian white noise

The white noise signal $\mathbbm{n}(t)$ with variance $\mathcal{C}_n = \delta(0)q$ can be construed as having the probability distribution

$$f_{\mathbbm{n}}(n) = \lim_{h\rightarrow\infty} \frac{1}{\sqrt{2\pi hq}} \mathrm{Exp}\left[-\frac{(n-m_n)^2}{2qh}\right]$$

Continuous time Gaussian white noise therefore takes on values in the interval $(-\infty, \infty)$ with equal probability.

Band-limited white noise

Definition

$$S_X(j\omega) = \begin{cases} A, & |\omega| \leq 2\pi W \\ 0, & |\omega| > 2\pi W \end{cases}$$

where W is the cut-off frequency.

Can show:

$$R_X(\tau) = 2WA\frac{\sin(2\pi W\tau)}{2\pi W\tau}$$

Properties

- "Almost uncorrelated" $X(t)$ and $X(t+\tau)$ for large $W\tau$.
- Uncorrelated for $\tau = 1/2W, 2/2W, 3/3W$, etc.
- Finite variance and finite power, hence practically more useful than white noise (e.g. in simulations)
- Ideal low-pass filtering of white noise (non-causal filter)

Wiener process ("random walk" or Brownian motion)

Integrated "white" noise

$$X(t) = \int_0^t U(\tau)d\tau$$

where $U(t)$ is normally distributed delta-correlated ("white") process, i.e. $U(t) \sim N(0, \sigma^2)$.

Properties

At time t, the Wiener process has the following probability density function:

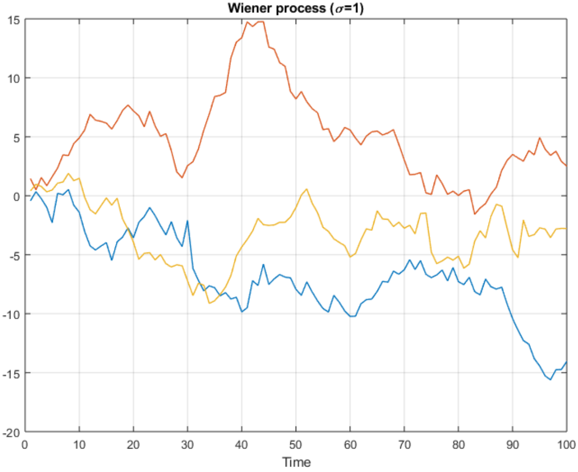
$$f_X(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/(2t)}$$

A normal distribution with $E[X(t)] = 0$ and $E[X^2(t)] = t$.

Applications

- Modelling of slowly random varying disturbances.
- Modelling of slowly random varying parameters and variables.
- Modelling of the accumulated effect of several independent random events, e.g. accumulated win/loss of a series of independent games.

Example of realizations of the same Wiener process



Summary

Key concepts

- Characterization of random processes:
  - Expectation, variance, moments, central moments
  - Probability distribution and probability density functions
  - Joint probability density function
  - Autocorrelation functions
  - Power spectral density functions
- Stationary random process: Statistical properties do not change with time.
- Ergodic random process: Statistical properties can be computed using time averaging from a single (infinite) time series.

Summary

Some common random processes

- Gauss-Markov process (exponential autocorrelation function)
- White noise (no correlations for $\tau \neq 0$)
- Band-limited white noise (finite variance and power)
- Wiener process (integrated white noise)