TTK4115

Lecture 8

Canonical decompositions & Minimal realizations

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This lecture

1. Canonical decompositions

2. Minimal realizations

3. Next 4 weeks

Topic

1. Canonical decompositions

2. Minimal realizations

3. Next 4 weeks

Zero state equivalence

Zero-state equivalence

If the system:

$$\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$$

has the same transfer function as the system:

$$\{\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}}\}$$

they are *zero-state equivalent*.

Canonical decomposition

- The above matrices may have different dimensions...
- ... but the transfer matrices are the same.
- Some information must be thrown away, that is not related to the transfer function!

Canonical decompositions

Example:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Are one of these states **uncontrollable**?

Controllability matrix

$$\mathcal{C} = \begin{bmatrix} \bar{\mathbf{b}} & \bar{\mathbf{A}}\bar{\mathbf{b}} \end{bmatrix} = \begin{bmatrix} \bar{b}_c & \bar{a}_c \bar{b}_c \\ 0 & 0 \end{bmatrix}$$

Canonical decompositions

Example:

$$
\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u
$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}
$$

$\bar{x}_2$ is uncontrollable.

Transfer matrix:

$$
\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix} \right)^{-1} \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u(s)
$$

Canonical decompositions

Example:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

$\bar{x}_2$ is uncontrollable.

Transfer matrix:

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \frac{1}{s-\bar{a}_c} & \frac{\bar{a}_{12}}{(s-\bar{a}_c)(s-\bar{a}_{\mathrm{nc}})} \\ 0 & \frac{1}{s-\bar{a}_{\mathrm{nc}}} \end{bmatrix} \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u(s)$$

Canonical decompositions

Example:

$$
\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_c & \bar{a}_{12} \\ 0 & \bar{a}_{nc} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_c \\ 0 \end{bmatrix} u
$$

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}
$$

$\bar{x}_2$ is uncontrollable.

Transfer matrix:

$$
\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} \bar{c}_{11} & \bar{c}_{1n} \\ \bar{c}_{21} & \bar{c}_{2n} \end{bmatrix} \begin{bmatrix} \frac{\bar{b}_c}{s-\bar{a}_c} \\ 0 \end{bmatrix} u(s) = \frac{\bar{b}_c}{s-\bar{a}_c} \begin{bmatrix} \bar{c}_{11} \\ \bar{c}_{21} \end{bmatrix} u(s)
$$

All information about the uncontrollable state is gone!

Canonical decompositions

General case: Controllability

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Notation

c: Controllable	$\bar{c}$: Uncontrollable
o: Observable	$\bar{o}$: Unobservable

Canonical decompositions

General case: Controllability

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Controllability matrix

$$\mathcal{C} = \begin{bmatrix} \bar{\mathbf{B}}_c & \bar{\mathbf{A}}_c\bar{\mathbf{B}}_c & \dots & \bar{\mathbf{A}}_c^{n-1}\bar{\mathbf{B}}_c \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}, \quad \rho(\mathcal{C}) = n_1 < n$$

Transform (theorem 6.6): $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$, $\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$..

$$\mathbf{T} = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_{n_1} & \cdots & \mathbf{q}_n \end{bmatrix}$$

Use all n_1 linearly independent columns of $\mathcal{C}$, then fill in the rest so that $\mathbf{T}$ is invertible.

Canonical decompositions

General case: Controllability

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Transform (theorem 6.6): $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$, $\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$..

$$\mathbf{T} = \begin{bmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_{n_1} & \cdots & \mathbf{q}_n \end{bmatrix}$$

Use all n_1 linearly independent columns of $\mathcal{C}$, then fill in the rest so that $\mathbf{T}$ is invertible.

Canonical decompositions

General case: Controllability

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{\mathbf{x}}}_c \\ \dot{\bar{\mathbf{x}}}_{\bar{c}} \end{bmatrix} &= \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c \\ \bar{\mathbf{x}}_{\bar{c}} \end{bmatrix} + \mathbf{D}\mathbf{u}
\end{aligned}
$$

Transfer matrix

$$
\begin{aligned}
\mathbf{G}(s) &= \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{\bar{c}} \end{bmatrix} \left(s \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ \mathbf{0} & \mathbb{I} \end{bmatrix} - \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}} \end{bmatrix} \right)^{-1} \begin{bmatrix} \bar{\mathbf{B}}_c \\ \mathbf{0} \end{bmatrix} + \mathbf{D} \\
&= \bar{\mathbf{C}}_c \left(s\mathbb{I} - \bar{\mathbf{A}}_c \right)^{-1} \bar{\mathbf{B}}_c + \mathbf{D}
\end{aligned}
$$

Canonical decompositions

Example:

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -0.5 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

Transform:

$$
\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \implies \mathbf{T} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}
$$

Transformed system:

$$
\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{AT} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, \quad \bar{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\mathbf{C}} = \mathbf{CT} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}
$$

Canonical decompositions

Example:

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} \bar{a}_o & 0 \\ \bar{a}_{21} & \bar{a}_{no} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_1 \\ \bar{b}_n \end{bmatrix} u \\
y_1 &= \begin{bmatrix} \bar{c}_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}
\end{aligned}
$$

Are one of these states **unobservable**?

Observability matrix:

$$
\mathcal{O} = \begin{bmatrix} \bar{c}_1 & 0 \\ \bar{c}_1 \, a_o & 0 \end{bmatrix}
$$

Canonical decompositions

Example:

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} \bar{a}_o & 0 \\ \bar{a}_{21} & \bar{a}_{no} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_1 \\ \bar{b}_n \end{bmatrix} u \\
y_1 &= \begin{bmatrix} \bar{c}_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}
\end{aligned}
$$

$\bar{x}_2$ is **unobservable**.

Transfer matrix:

$$
y_1(s) = \begin{bmatrix} \bar{c}_1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \bar{a}_o & 0 \\ \bar{a}_{21} & \bar{a}_{no} \end{bmatrix} \right)^{-1} \begin{bmatrix} \bar{b}_1 \\ \bar{b}_n \end{bmatrix} u(s)
$$

Canonical decompositions

Example:

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} \bar{a}_o & 0 \\ \bar{a}_{21} & \bar{a}_{no} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \bar{b}_1 \\ \bar{b}_n \end{bmatrix} u$$

$$y_1 = \begin{bmatrix} \bar{c}_1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

$\bar{x}_2$ is **unobservable**.

Transfer matrix:

$$y_1(s) = \frac{\bar{c}_1 \bar{b}_1}{s - \bar{a}_o} u(s)$$

No information about the unobservable state remains..

Canonical decompositions

General case: Observability

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{\mathbf{x}}}_o \\ \dot{\bar{\mathbf{x}}}_{\bar{o}} \end{bmatrix} &= \begin{bmatrix} \bar{\mathbf{A}}_o & \mathbf{0} \\ \bar{\mathbf{A}}_{12} & \bar{\mathbf{A}}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_o \\ \bar{\mathbf{B}}_{\bar{o}} \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= \begin{bmatrix} \bar{\mathbf{C}}_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}
\end{aligned}
$$

Notation

c: Controllable $\bar{c}$: Uncontrollable
o: Observable $\bar{o}$: Unobservable

General case: Observability

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_o \\ \dot{\bar{\mathbf{x}}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_o & \mathbf{0} \\ \bar{\mathbf{A}}_{12} & \bar{\mathbf{A}}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_o \\ \bar{\mathbf{B}}_{\bar{o}} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Observability matrix

$$\mathcal{O} = \begin{bmatrix} \bar{\mathbf{C}}_o & \mathbf{0} \\ \bar{\mathbf{C}}_o \bar{\mathbf{A}}_o & \mathbf{0} \end{bmatrix}, \quad \rho(\mathcal{O}) = n_2 < n$$

Transform (theorem 6.O6): $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$, $\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$..

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{p}_1 \\ \vdots \\ \mathbf{p}_{n_2} \\ \vdots \\ \mathbf{p}_n \end{bmatrix}$$

Use all n_2 linearly independent **rows** of $\mathcal{O}$, then fill in the rest so that $\mathbf{T}^{-1}$ is invertible.

Canonical decompositions

Example:

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
$$

Transform:

$$
\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \implies \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ \color{red}{0} & \color{red}{0} & \color{red}{1} \end{bmatrix}
$$

Transformed system:

$$
\bar{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} 0 & 1 & \color{red}{0} \\ 1 & 1 & \color{red}{0} \\ 1 & 0 & 1 \end{bmatrix}, \quad \bar{\mathbf{B}} = \mathbf{T}^{-1}\mathbf{B} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \quad \bar{\mathbf{C}} = \mathbf{C}\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
$$

Canonical decompositions

General case: Observability

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{\mathbf{x}}}_o \\ \dot{\bar{\mathbf{x}}}_{\bar{o}} \end{bmatrix} &= \begin{bmatrix} \bar{\mathbf{A}}_o & \mathbf{0} \\ \bar{\mathbf{A}}_{12} & \bar{\mathbf{A}}_{\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_o \\ \bar{\mathbf{B}}_{\bar{o}} \end{bmatrix} \mathbf{u} \\
\mathbf{y} &= \begin{bmatrix} \bar{\mathbf{C}}_o & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_o \\ \bar{\mathbf{x}}_{\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}
\end{aligned}
$$

Transfer matrix

$$
\mathbf{G}(s) = \bar{\mathbf{C}}_o \left(s\mathbb{I} - \bar{\mathbf{A}}_o \right)^{-1} \bar{\mathbf{B}}_o + \mathbf{D}
$$

Canonical decompositions

General case

$$
\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}
$$

$$
\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}
$$

Notation

c: Controllable $\quad$ $\bar{c}$: Uncontrollable

o: Observable $\quad$ $\bar{o}$: Unobservable

Kalman decomposition

General case

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Transfer matrix

$$\mathbf{y}(s) = \left[\bar{\mathbf{C}}_{co}(s\mathbb{I} - \bar{\mathbf{A}}_{co})^{-1}\bar{\mathbf{B}}_{co} + \mathbf{D}\right]\mathbf{u}(s)$$

Kalman Decomposition Theorem (theorem 6.7)

Every state-space equation can be transformed into the form above.

Kalman decomposition

General case

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Transfer matrix

$$\mathbf{y}(s) = \left[\bar{\mathbf{C}}_{co}(s\mathbb{I} - \bar{\mathbf{A}}_{co})^{-1}\bar{\mathbf{B}}_{co} + \mathbf{D}\right]\mathbf{u}(s)$$

The **same** transfer matrix

$$\mathbf{y}(s) = \left[\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{u}(s)$$

Kalman decomposition

General case

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_{co} \\ \dot{\bar{\mathbf{x}}}_{c\bar{o}} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}o} \\ \dot{\bar{\mathbf{x}}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{A}}_{co} & \mathbf{0} & \bar{\mathbf{A}}_{13} & \mathbf{0} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{c\bar{o}} & \bar{\mathbf{A}}_{23} & \bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}}_{43} & \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_{co} \\ \bar{\mathbf{B}}_{c\bar{o}} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \bar{\mathbf{C}}_{co} & \mathbf{0} & \bar{\mathbf{C}}_{\bar{c}o} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_{co} \\ \bar{\mathbf{x}}_{c\bar{o}} \\ \bar{\mathbf{x}}_{\bar{c}o} \\ \bar{\mathbf{x}}_{\bar{c}\bar{o}} \end{bmatrix} + \mathbf{D}\mathbf{u}$$

Zero state equivalent system

$$\begin{aligned} \dot{\bar{\mathbf{x}}}_{co} &= \bar{\mathbf{A}}_{co}\bar{\mathbf{x}}_{co} + \bar{\mathbf{B}}_{co}\mathbf{u} \\ \mathbf{y} &= \bar{\mathbf{C}}_{co}\bar{\mathbf{x}}_{co} + \mathbf{D}\mathbf{u} \end{aligned}$$

Canonical decompositions

Implications

- Transfer matrices do not contain any information about the unobservable and uncontrollable parts of the system.
- This explains why transfer matrices may have lower order than the original system.
- Realizations of transfer functions can only produce the controllable and observable subsystem.
- We should consider the unobservable and uncontrollable subsystems also: are they stable?

Eigenvalues

Characteristic polynomial

$$
\begin{aligned}
\Delta(\lambda) &= \left|\lambda\mathbb{I} - \bar{\mathbf{A}}\right| = \begin{vmatrix} \lambda\mathbb{I} - \bar{\mathbf{A}}_{co} & \mathbf{0} & -\bar{\mathbf{A}}_{13} & \mathbf{0} \\ -\bar{\mathbf{A}}_{21} & \lambda\mathbb{I} - \bar{\mathbf{A}}_{c\bar{o}} & -\bar{\mathbf{A}}_{23} & -\bar{\mathbf{A}}_{24} \\ \mathbf{0} & \mathbf{0} & \lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\bar{\mathbf{A}}_{43} & \lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{vmatrix} \\
&= \begin{vmatrix} \lambda\mathbb{I} - \bar{\mathbf{A}}_{co} & \mathbf{0} \\ -\bar{\mathbf{A}}_{21} & \lambda\mathbb{I} - \bar{\mathbf{A}}_{c\bar{o}} \end{vmatrix} \begin{vmatrix} \lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}o} & \mathbf{0} \\ -\bar{\mathbf{A}}_{43} & \lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}\bar{o}} \end{vmatrix} \\
&= \left|\lambda\mathbb{I} - \bar{\mathbf{A}}_{co}\right| \underbrace{\left|\lambda\mathbb{I} - \bar{\mathbf{A}}_{c\bar{o}}\right|\left|\lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}o}\right|\left|\lambda\mathbb{I} - \bar{\mathbf{A}}_{\bar{c}\bar{o}}\right|}_{\text{Not present in } \mathbf{G}(s)} = 0
\end{aligned}
$$

Note

The transfer matrix does not tell the full story. **Check the eigenvalues of the full state space model.**

Topic

Minimal realizations

Minimal realizations

- We have seen that unobservable and uncontrollable subsystems are removed when going **to** the Laplace plane.
- There are infinitely many realizations of a proper rational transfer matrix $\mathbf{G}(s)$.
- By choosing a *minimal* realization, we do not create redundant unobservable and uncontrollable states.
- The resulting state space model will have the same dimensions as:

$$\begin{aligned} \dot{\bar{\mathbf{x}}}_{co} &= \bar{\mathbf{A}}_{co}\bar{\mathbf{x}}_{co} + \bar{\mathbf{B}}_{co}\mathbf{u} \\ \mathbf{y} &= \bar{\mathbf{C}}_{co}\bar{\mathbf{x}}_{co} + \mathbf{D}\mathbf{u} \end{aligned}$$

- which is *minimal*

Minimal realizations

Coprime fractions: SISO case

A state equation $\{\mathbf{A}, \mathbf{b}, \mathbf{c}, d\}$ is a *minimal realization* of a proper rational function $\hat{g}(s)$ if and only if:

- The pair $\{\mathbf{A}, \mathbf{b}\}$ is controllable.
- The pair $\{\mathbf{A}, \mathbf{c}\}$ is observable.
- $n = \dim(\mathbf{A}) = \deg(\hat{g}(s))$
- where $\hat{g}(s) = \frac{N(s)}{D(s)}$, and $N(s)$ and $D(s)$ do not have any common factors.
- I.e.: They are **coprime**, and $\frac{N(s)}{D(s)}$ is a **coprime fraction**.

Minimal realizations

Example

$$\hat{g}(s) = \frac{N(s)}{D(s)} = \frac{s^2 - 1}{4(s^3 - 1)} = \frac{(s-1)(1+s)}{4(s-1)\left(1+s+s^2\right)} = \underbrace{\frac{\cancel{(s-1)}(1+s)}{4\cancel{(s-1)}\left(1+s+s^2\right)}}_{\text{Coprime fraction}}$$

Minimal realizations

Coprimeness and minimal realizations

- If the transfer function is a coprime fraction, we only need to check the dimensions of **A** to verify whether the realized system is *minimal*: $\dim(\mathbf{A}) = \deg(\hat{g}(s))$
- This implies that the system is controllable and observable.
- If a fraction is coprime, every root of $D(s)$ is a root of $\hat{g}(s)$.
- The eigenvalues of the minimal realization are the poles of $\hat{g}(s)$.
- All minimal realizations are equivalent, and relate via an equivalence transform $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$.

Topic

1. Canonical decompositions

2. Minimal realizations

3. Next 4 weeks

Upcoming subjects

Deterministic systems

The material we have covered in the first 8 weeks, from *Linear system theory and design* by Chi-Tsong Chen has focused *mostly* on deterministic systems.

- Mathematical models and model parameters have been assumed to be exact
- Model inputs $\mathbf{u}(t)$ have been assumed to be exact
- Measurements $\mathbf{y}(t)$ have mostly been assumed to be exact

... except in some cases where we have included model uncertainty, disturbances and measurement noise

Stochastic systems

In real systems, model uncertainty, disturbances and measurement noise are often important enough that they require proper treatment:

- Uncertainties must be modelled according to their statistical properties, and represented as *random processes*.
- Systems affected by stochastic disturbances or random input values are *stochastic systems*
- State estimation in stochastic systems needs to take their random properties into account (e.g. the *Kalman filter*)

These are the subjects of the next 4 weeks, and the material is covered by *Introduction to random signals and applied Kalman filtering* by Brown & Hwang.

Upcoming subjects

Coming subjects

- Characterization of random signals in terms of expectation, variance, autocorrelation, power spectrum, correlation/covariance
- Random processes: systems with random inputs or initial values
- Mean and (co)variances of random state-space systems
- Optimal estimation of random process: Kalman filter in continuous and discrete time
- Noise shaping for Kalman filter systems