TTK4115

Lecture 4

State feedback (continued), optimal control

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This lecture

1. Reference feed-forward
2. Integral effect
3. Optimal Control
4. Lunar lander

Topic

1. Reference feed-forward

2. Integral effect

3. Optimal Control

4. Lunar lander

Reference feed-forward

Aim: asymptotic convergence to reference

We want[1] $\mathbf{y}(t) \rightarrow \mathbf{r}$ as $t \rightarrow \infty$.

Implementation

$$\mathbf{u} = \underbrace{\mathbf{K}_r\mathbf{r}}_{\text{Reference feedforward}} - \underbrace{\mathbf{K}\mathbf{x}}_{\text{State Feedback}}$$

Equilibrium conditions

Assuming that feedback results in a stable equilibrium yields the steady-state condition:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{K}_r\mathbf{r} - \mathbf{K}\mathbf{x}) = \mathbf{0} \quad \Rightarrow (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}_\infty = -\mathbf{B}\mathbf{K}_r\mathbf{r}_0 \quad \Rightarrow \mathbf{y}_\infty = [\mathbf{C}(\mathbf{B}\mathbf{K} - \mathbf{A})^{-1}\mathbf{B}]\mathbf{K}_r\mathbf{r}_0$$

Finding $\mathbf{K}_r$

Inversion gives the correct feedforward gain:

$$\mathbf{K}_r = [\mathbf{C}(\mathbf{B}\mathbf{K} - \mathbf{A})^{-1}\mathbf{B}]^{-1}$$

Note that the number of references must be the same as the number of outputs.

[1] The elements in **y** are the variables we wish to control, not the measurement per sé. In state feedback *all* states are assumed known.

A common mistake

Suppose now that pure error feedback is used.:

$$\mathbf{u} = \mathbf{K}(\mathbf{r} - \mathbf{x})$$

This yields the system:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{r}$$

$$\Rightarrow \frac{\hat{\mathbf{x}}(s)}{\hat{\mathbf{r}}(s)} = (s\mathbb{I} - [\mathbf{A} - \mathbf{B}\mathbf{K}])^{-1}\mathbf{B}\mathbf{K}$$

Assume that the reference is constant $\hat{\mathbf{r}}(s) = \mathbf{r}_0/s$. The final-value theorem yields (in general):

$$\mathbf{x}(\infty) = \lim_{s\to 0} \frac{\hat{\mathbf{x}}(s)}{\hat{\mathbf{r}}(s)}\mathbf{r}_0 = -(\mathbf{A} - \mathbf{B}\mathbf{K})^{-1}\mathbf{B}\mathbf{K}\mathbf{r}_0 \neq \mathbf{r}_0$$

Correct approach

Suppose now that state feedback + reference feedforward is used instead:

$$\mathbf{u} = [(\mathbf{BK} - \mathbf{A})^{-1}\mathbf{B}]^{-1}\mathbf{r} - \mathbf{Kx}$$

This yields the system:

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{B}[(\mathbf{BK} - \mathbf{A})^{-1}\mathbf{B}]^{-1}\mathbf{r}$$

$$\Rightarrow \frac{\hat{\mathbf{x}}(s)}{\hat{\mathbf{r}}(s)} = (s\mathbb{I} - [\mathbf{A} - \mathbf{BK}])^{-1}\mathbf{B}[(\mathbf{BK} - \mathbf{A})^{-1}\mathbf{B}]^{-1}$$

Assume that the reference is constant $\hat{\mathbf{r}}(s) = \mathbf{r}_0/s$. The final-value theorem yields:

$$\mathbf{x}(\infty) = \lim_{s \to 0} \frac{\hat{\mathbf{x}}(s)}{\hat{\mathbf{r}}(s)}\mathbf{r}_0 = (\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}[(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}]^{-1}\mathbf{r}_0 = \mathbf{r}_0$$

Topic

Problem

What if we don't know our system perfectly? (We never do)
What about disturbances: **w**?

Solution

Use integral effect!

Integral effect

Plant[2]

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \overbrace{\mathbf{B}\mathbf{w}}^{\text{Disturbance}} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}$$

Integrator state augmentation

$$\mathbf{x}_a = \int_0^t \mathbf{r}(\tau) - \mathbf{C}\mathbf{x}(\tau)\, d\tau$$

Augmented system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ \mathbb{I} \end{bmatrix} \mathbf{r}$$

[2] The disturbance **w** is assumed to act in a way that can be cancelled by the input.

Integral effect

Augmented system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{w} + \begin{bmatrix} \mathbf{0} \\ \mathbb{I} \end{bmatrix} \mathbf{r}$$

State feedback

As before state feedback + reference feedforward[3] is used:

$$\mathbf{u} = -\begin{bmatrix} \mathbf{K} & \mathbf{K}_a \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \mathbf{K}_r \mathbf{r}$$

Closed loop system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & -\mathbf{B}\mathbf{K}_a \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{B}\mathbf{K}_r \\ \mathbb{I} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{w}$$

[3] This is actually optional with integral effect.

Integral effect

Closed loop system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & -\mathbf{BK}_a \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{BK}_r \\ \mathbb{I} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{w}$$

Steady state behavior - with feedforward

Provided that the feedback results in an asymptotically stable augmented system the output will tend towards a constant reference $\mathbf{r}_0$:

$$\dot{\mathbf{x}}_a = \mathbf{0} \quad \Rightarrow \mathbf{Cx}_\infty = \mathbf{r}_0$$

The state vector exhibits the following limit:

$$\dot{\mathbf{x}} = \mathbf{0} \quad \Rightarrow \mathbf{Cx}_\infty = \mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}[\mathbf{K}_a\mathbf{x}_{a,\infty} - \mathbf{K}_r\mathbf{r}_0 - \mathbf{w}_0]$$

If the feedforward is implemented as shown earlier ($\mathbf{K}_r = [\mathbf{C}(\mathbf{BK} - \mathbf{A})^{-1}\mathbf{B}]^{-1}$) we have:

$$\mathbf{Cx}_\infty - \mathbf{r}_0 = \mathbf{C}(\mathbf{A} - \mathbf{BK})^{-1}\mathbf{B}[\mathbf{w}_0 - \mathbf{K}_a\mathbf{x}_{a,\infty}] = \mathbf{0}$$

The integral action thus acts as a disturbance estimator/compensator:

$$\mathbf{K}_a\mathbf{x}_{a,\infty} = \mathbf{w}_0$$

Integral effect

Closed loop system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_a \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & -\mathbf{BK}_a \\ -\mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_a \end{bmatrix} + \begin{bmatrix} \mathbf{BK}_r \\ \mathbb{I} \end{bmatrix} \mathbf{r} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{w}$$

Steady state behavior - without feedforward

Provided that the feedback results in an asymptotically stable augmented system the output will tend towards a constant reference $\mathbf{r}_0$:

$$\dot{\mathbf{x}}_a = \mathbf{0} \quad \Rightarrow \mathbf{Cx}_\infty = \mathbf{r}_0$$

The state vector exhibits the following limit:

$$\dot{\mathbf{x}} = \mathbf{0} \quad \Rightarrow \mathbf{Cx}_\infty = \mathbf{C}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{B}[\mathbf{K}_a\mathbf{x}_{a,\infty} - \mathbf{w}_0]$$

If the feedforward is *not implemented* we have:

$$\mathbf{Cx}_\infty - \mathbf{r}_0 = \mathbf{C}(\mathbf{A}-\mathbf{BK})^{-1}\mathbf{B}[\mathbf{K}_a\mathbf{x}_{a,\infty} - \mathbf{w}_0] - \mathbf{r}_0 = \mathbf{0}$$

The integral action now performs double duty and compensates for the disturbance as well as setting the correct bias[4]:

$$\mathbf{K}_a\mathbf{x}_{a,\infty} = -\mathbf{K}_r\mathbf{r}_0 + \mathbf{w}_0$$

[4] But only asymptotically..

Example

Cruise control

Car model:

$$m\dot{v} = -dv + u + w, \quad y = v$$

Matrices:

$$\mathbf{A} = -\frac{d}{m}, \quad \mathbf{B} = \frac{1}{m}, \quad \mathbf{C} = 1$$

State feedback

Control:

$$u = -kv + k_r r$$

Feedforward gain:

$$k_r = [\mathbf{C}(\mathbf{BK} - \mathbf{A})^{-1}\mathbf{B}]^{-1} = d + k$$

Closed loop dynamics:

$$m\dot{v} = (d + k)(r - v) + w, \quad y = v$$

Stable equilibrium at $y = r$ if $w = 0$.

Example

Cruise control

Car model:

$$m\dot{v} = -dv + u + w, \quad y = v$$

Matrices:

$$\mathbf{A} = -\frac{d}{m}, \quad \mathbf{B} = \frac{1}{m}, \quad \mathbf{C} = 1$$

Augmented state space

$$\begin{bmatrix} \dot{v} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} -\frac{d}{m} & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v \\ x_a \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} w$$

State feedback

Control:

$$u = -kv - k_a x_a + k_r r$$

Closed loop dynamics:

$$\begin{bmatrix} \dot{v} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} -\frac{d+k}{m} & -\frac{k_a}{m} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v \\ x_a \end{bmatrix} + \begin{bmatrix} \frac{k_r}{m} \\ 1 \end{bmatrix} r + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} w$$

Example

Cruise control

Car model:

$$m\dot{v} = -dv + u + w, \quad y = v$$

Matrices:

$$\mathbf{A} = -\frac{d}{m}, \quad \mathbf{B} = \frac{1}{m}, \quad \mathbf{C} = 1$$

State feedback

Control:

$$u = -kv - k_a x_a + k_r r$$

Closed loop dynamics:

$$\begin{bmatrix} \dot{v} \\ \dot{x}_a \end{bmatrix} = \begin{bmatrix} -\frac{d+k}{m} & -\frac{k_a}{m} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v \\ x_a \end{bmatrix} + \begin{bmatrix} \frac{k_r}{m} \\ 1 \end{bmatrix} r + \begin{bmatrix} \frac{1}{m} \\ 0 \end{bmatrix} w$$

Transfer functions

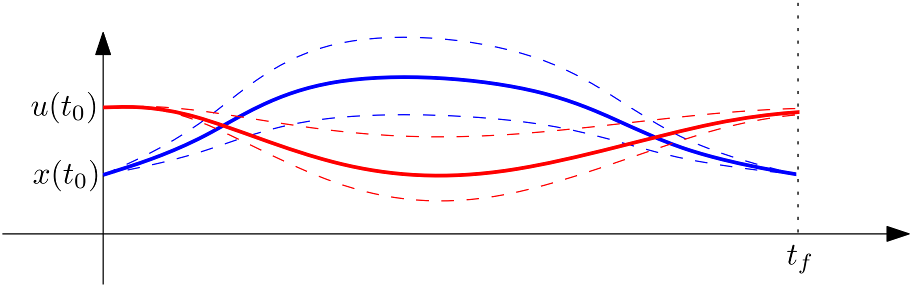
$$\hat{y}(s) = \frac{\overbrace{(d+k)s}^{\text{r.ff.}} - k_a}{ms^2 + (d+k)s - k_a}\hat{r}(s) + \frac{s}{ms^2 + (d+k)s - k_a}\hat{w}(s)$$

- Retaining reference feedforward may yield faster reference tracking.

Topic

Optimal Control

$$J = \int_{t_0}^{t_f} x^T(\tau)Qx(\tau) + u(\tau)^T Ru(\tau)d\tau$$



Optimal control

Optimal control is concerned with finding an input $u(t)$ that minimizes a cost function over some interval of time, subject to system dynamics.

LQR

The **L**inear **Q**uadratic **R**egulator is a special case where the plant is **L**inear and the cost function is **Q**uadratic. The time interval is infinite.

Optimal Control

Finding a minimum

- Finding the control that minimizes the objective function is in general not trivial.
- Analytical solutions are in general very difficult to obtain.
- Discretizing the dynamics and finding the optimum numerically over a finite horizon is a viable method (MPC,NMPC).
- Linear models with a quadratic cost functions are a ”lucky” case.

Optimal Control

Finding a minimum

We want to find the minimum of:

$$J = \int_0^{\infty} \mathbf{y}^{\mathsf{T}}(t)\mathbf{Q}\mathbf{y}(t) + \mathbf{u}^{\mathsf{T}}(t)\mathbf{R}\mathbf{u}(t)dt$$

for the system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

The ”trick” is to rewrite the cost functional on the special form:

$$\underline{J = H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) + \int_0^{\infty} \Lambda(\mathbf{x}(t), \mathbf{u}(t))dt}$$

where:

- $H(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ is not affected by the input, directly or indirectly.
- $\Lambda(\mathbf{x}(t), \mathbf{u}(t))$ has an obvious minimum in terms of $\mathbf{u}(t)$.

Optimal control

Desired form of cost function

$$J = H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) + \int_0^\infty \Lambda(\mathbf{x}(t), \mathbf{u}(t))dt$$

The functional[a] $H(\mathbf{x}(\cdot), \mathbf{u}(\cdot))$ above can be described as a *feedback invariant*. The functional takes the system inputs and states as arguments, but its value depends only on the initial condition $\mathbf{x}(0)$. It is not affected by the input, directly or indirectly.

[a] A functional takes functions as arguments and returns a scalar.

Feedback invariance

The functional:

$$H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) \triangleq -\int_0^\infty \frac{d}{dt}\left[\mathbf{x}^\mathsf{T}(t)\mathbf{S}\mathbf{x}(t)\right] dt$$

happens to be feedback invariant as long as $\mathbf{x}(t) \to 0$ as $t \to \infty$:

$$-\int_0^\infty \frac{d}{dt}\left[\mathbf{x}^\mathsf{T}(t)\mathbf{S}\mathbf{x}(t)\right] dt = -\left[\mathbf{x}^\mathsf{T}(t)\mathbf{S}\mathbf{x}(t)\right]_0^\infty = \mathbf{x}^\mathsf{T}(0)\mathbf{S}\mathbf{x}(0) - \cancel{\mathbf{x}^\mathsf{T}(\infty)\mathbf{S}\mathbf{x}(\infty)}$$

The integrand can be rewritten as follows:

$$\frac{d}{dt}\left[\mathbf{x}^\mathsf{T}\mathbf{S}\mathbf{x}(t)\right] = \dot{\mathbf{x}}^\mathsf{T}\mathbf{S}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{S}\dot{\mathbf{x}} = \mathbf{x}^\mathsf{T}\mathbf{S}[\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}] + [\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}]^\mathsf{T}\mathbf{S}\mathbf{x}$$

Optimal control

Desired form of cost function

$$J = H(\mathbf{x}(\cdot),\mathbf{u}(\cdot)) + \int_0^\infty \Lambda(\mathbf{x}(t),\mathbf{u}(t))dt \quad \left(= \int_0^\infty \mathbf{y}^\mathsf{T}(t)\mathbf{Q}\mathbf{y}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)dt\right)$$

$$H(\mathbf{x}(\cdot),\mathbf{u}(\cdot)) = -\int_0^\infty [\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)]^\mathsf{T}\mathbf{S}\mathbf{x}(t) + \mathbf{x}^\mathsf{T}(t)\mathbf{S}[\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t)]dt$$

Develop by adding and subtracting

$$J = H(\mathbf{x},\mathbf{u}) + \left[\underline{\int_0^\infty \mathbf{y}^\mathsf{T}(t)\mathbf{Q}\mathbf{y}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)dt} - H(\mathbf{x}(\cdot),\mathbf{u}(\cdot))\right]$$

$$= H(\mathbf{x},\mathbf{u}) + \left[\int_0^\infty \mathbf{x}^\mathsf{T}\mathbf{C}^\mathsf{T}\mathbf{Q}\mathbf{C}\mathbf{x} + \mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} + (\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u})^\mathsf{T}\mathbf{S}\mathbf{x} + \mathbf{x}^\mathsf{T}\mathbf{S}(\mathbf{A}\mathbf{x}+\mathbf{B}\mathbf{u})dt\right]$$

$$= H(\mathbf{x},\mathbf{u}) + \left[\int_0^\infty \underbrace{\mathbf{x}^\mathsf{T}(\mathbf{A}^\mathsf{T}\mathbf{S}+\mathbf{S}\mathbf{A}+\mathbf{C}^\mathsf{T}\mathbf{Q}\mathbf{C})\mathbf{x} + \mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} + 2\mathbf{u}^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x}}_{\Lambda(\mathbf{x},\mathbf{u})}\, dt\right]$$

Optimal control

Desired form of cost function

$$J = H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) + \int_0^\infty \Lambda(\mathbf{x}(t), \mathbf{u}(t))dt$$

$$H(\mathbf{x}(\cdot), \mathbf{u}(\cdot)) = -\int_0^\infty [\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)]^\mathsf{T}\mathbf{S}\mathbf{x}(t) + \mathbf{x}^\mathsf{T}(t)\mathbf{S}[\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)]dt$$

$$\Lambda(\mathbf{x}, \mathbf{u}) = \mathbf{x}^\mathsf{T}(\mathbf{A}^\mathsf{T}\mathbf{S} + \mathbf{S}\mathbf{A} + \mathbf{C}^\mathsf{T}\mathbf{Q}\mathbf{C})\mathbf{x} + \mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} + 2\mathbf{u}^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x}$$

Refine $\Lambda(\mathbf{x}, \mathbf{u})$ by "completing the squares" (matrix version):

$$\mathbf{z}^T\mathbf{M}\mathbf{z} - 2\mathbf{b}^T\mathbf{z} = (\mathbf{z} - \mathbf{M}^{-1}\mathbf{b})^T\mathbf{M}(\mathbf{z} - \mathbf{M}^{-1}\mathbf{b}) - \mathbf{b}^T\mathbf{M}^{-1}\mathbf{b}$$

Thus we can rewrite $\mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} + 2\mathbf{u}^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x}$ by choosing $\mathbf{u}$ as $\mathbf{z}$ and $-\mathbf{B}^T\mathbf{S}\mathbf{x}$ as $\mathbf{b}$:

$$\mathbf{u}^\mathsf{T}\mathbf{R}\mathbf{u} + 2\mathbf{u}^\mathsf{T}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x} = (\mathbf{u} + \mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x})^\mathsf{T}\mathbf{R}(\mathbf{u} + \mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x}) - \mathbf{x}^\mathsf{T}\mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S}\mathbf{x}$$

Desired form of cost function

$$J = H(\mathbf{x}(\cdot),\mathbf{u}(\cdot)) + \int_0^\infty \Lambda(\mathbf{x}(t),\mathbf{u}(t))dt$$

$$H(\mathbf{x}(\cdot),\mathbf{u}(\cdot)) = -\int_0^\infty [\mathbf{Ax}(t)+\mathbf{Bu}(t)]^\mathsf{T}\mathbf{Sx}(t) + \mathbf{x}^\mathsf{T}(t)\mathbf{S}[\mathbf{Ax}(t)+\mathbf{Bu}(t)]dt$$

$$\Lambda(\mathbf{x},\mathbf{u}) = \mathbf{x}^\mathsf{T}(\mathbf{A}^\mathsf{T}\mathbf{S}+\mathbf{SA}+\mathbf{C}^\mathsf{T}\mathbf{QC}-\mathbf{SBR}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S})\mathbf{x} + (\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{Sx})^\mathsf{T}\mathbf{R}(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{Sx})$$

Minimization step 1:

Solve CARE:

$$\mathbf{A}^\mathsf{T}\mathbf{S}+\mathbf{SA}+\mathbf{C}^\mathsf{T}\mathbf{QC}-\mathbf{SBR}^{-1}\mathbf{B}^\mathsf{T}\mathbf{S} = 0$$

Minimization step 2:

Choose:

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^\mathsf{T}\mathbf{Sx}$$

The minimum is thus obtained:

$$\Lambda(\mathbf{x}(t),\mathbf{u}(t)) \equiv 0 \quad \Rightarrow J = \mathbf{x}^\mathsf{T}(0)\mathbf{Sx}(0)$$

LQR

Cost functional to be *minimized* w.r.t. $\mathbf{u}(t)$

$$J_{LQR} = \int_0^\infty \mathbf{y}^\mathsf{T}(t)\mathbf{Q}\mathbf{y}(t) + \mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)dt$$

with:

$$\mathbf{Q} > 0,\ \mathbf{Q} = \mathbf{Q}^\mathsf{T}, \quad \mathbf{R} > 0,\ \mathbf{R} = \mathbf{R}^\mathsf{T}$$

Output *energy*

$$\mathbf{y}^\mathsf{T}(t)\mathbf{Q}\mathbf{y}(t)$$

Making this function smaller requires more input energy.

Control *energy*

$$\mathbf{u}^\mathsf{T}(t)\mathbf{R}\mathbf{u}(t)$$

Making this function smaller requires less input energy which leads to higher output energy.

LQR

Minimal solution

The solution that minimizes the cost function for the system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

is a linear feedback:

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$$

where:

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{S}$$

leading to the closed loop system:

$$\dot{\mathbf{x}}(t) = [\mathbf{A} - \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{S}]\mathbf{x}(t)$$

LQR

Algebraic Riccati Equation

The matrix **S** is found by solving:

$$[\mathbf{A}^{\mathsf{T}}\mathbf{S} + \mathbf{S}\mathbf{A}] + \mathbf{C}^{\mathsf{T}}\mathbf{Q}\mathbf{C} - \mathbf{S}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathsf{T}}\mathbf{S} = 0$$

where $\mathbf{S} > 0$ must be **positive definite**.

Example $\dot{x}(t) = -\lambda x(t) + u(t),\ y(t) = x(t)$

The (now scalar) p is found by solving:

$$-2\lambda p + q - \frac{p^2}{r} = 0$$

The solutions are:

$$p = -\lambda r \pm \sqrt{qr + \lambda^2 r^2} \quad : \text{Pick the } \underline{\text{positive}} \text{ solution}$$

The feedback matrix becomes:

$$u(t) = -kx(t) = -\frac{p}{r}x(t) = \left(\lambda - \sqrt{\frac{q}{r} + \lambda^2}\right)x(t)$$

LQR

Closed loop system $\dot{x}(t) = -\lambda x(t) - kx(t)$

State dynamics:

$$\begin{aligned}\dot{x}(t) &= -\left(\lambda - \frac{p}{r}\right) x(t) \\ &= -\left(\lambda - [\lambda - \sqrt{\frac{q}{r} + \lambda^2}]\right) x(t) \\ &= -\left(\sqrt{\frac{q}{r} + \lambda^2}\right) x(t)\end{aligned}$$

Input:

$$u(t) = -kx(t) = -\frac{p}{r}x(t) = \left(\lambda - \sqrt{\frac{q}{r} + \lambda^2}\right) x(t)$$

Note:

There is a direct tradeoff between q and r, (**Q** and **R** in the general case)

Tuning

Tuning is done by selecting the weights **Q** and **R**. We typically choose these as diagonal matrices.

Bryson's Rule (Rule of thumb)

$$\begin{aligned} Q_{ii} &= \frac{1}{\text{maximum acceptable value of } x_i^2} \\ R_{jj} &= \frac{1}{\text{maximum acceptable value of } u_j^2} \end{aligned}$$

Matlab commands

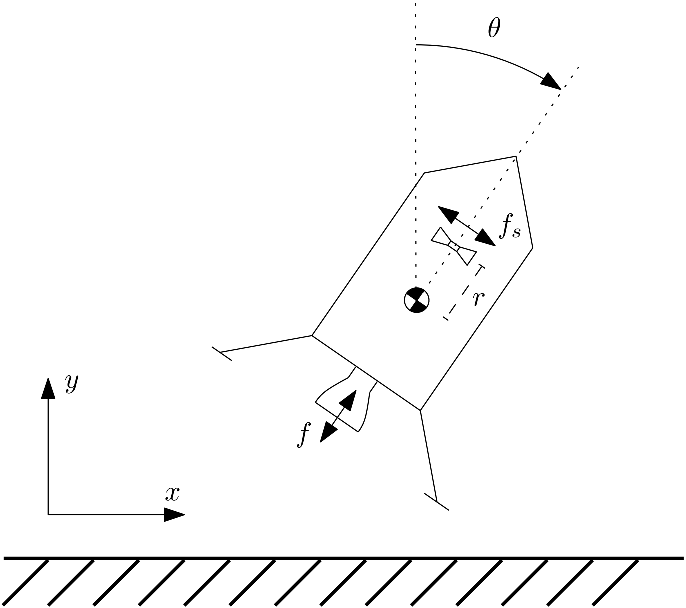
- "SYS = ss(A,B,C,D)"
- "[K,S,E] = lqr(SYS,Q,R,N)"
- "[K,S,E] = lqry(SYS,Q,R,N)"

Remarks

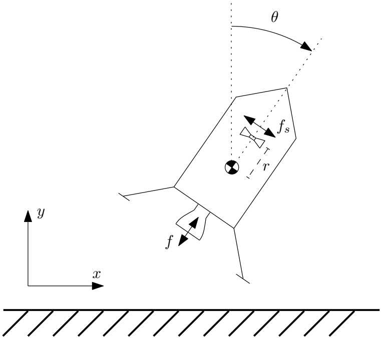
- $\{\mathbf{A}, \mathbf{B}\}$ must be controllable.
- If weighing the outputs, all unstable modes must show up in the output.

Topic

Lunar lander



Lunar lander



Nonlinear model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \\ 0 & r \end{bmatrix} \begin{bmatrix} f \\ f_s \end{bmatrix} - \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix}$$

Lunar lander

Nonlinear model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \\ 0 & r \end{bmatrix} \begin{bmatrix} f \\ f_s \end{bmatrix} - \begin{bmatrix} 0 \\ mg \\ 0 \end{bmatrix}$$

Inputs

$$u_1 = f - mg \quad u_2 = f_s$$

Perturbed nonlinear model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \\ 0 & r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + mg \begin{bmatrix} \sin(\theta) \\ \cos(\theta) - 1 \\ 0 \end{bmatrix}$$

Lunar lander

Perturbed nonlinear model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \sin(\theta) & \cos(\theta) \\ \cos(\theta) & -\sin(\theta) \\ 0 & r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + mg \begin{bmatrix} \sin(\theta) \\ \cos(\theta) - 1 \\ 0 \end{bmatrix}$$

Linearized model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + mg \begin{bmatrix} \theta \\ 0 \\ 0 \end{bmatrix}$$

Lunar lander

Linearized model

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & j \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + mg \begin{bmatrix} \theta \\ 0 \\ 0 \end{bmatrix}$$

State equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{r}{j} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Lunar lander

State equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{r}{j} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Controllability Matrix

$$\mathcal{C} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & \frac{gr}{j} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{r}{j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{m} & 0 & 0 & 0 & \frac{gr}{j} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{r}{j} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Lunar lander

State equation

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & g & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \\ 0 & \frac{r}{j} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Output

$$\mathbf{y} = \mathbf{C}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \\ \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$

Lunar lander

Simulink model

A Simulink model for the lunar lander with accompanying Matlab script is available on Blackboard.

Nonlinearities

The true system is not linear!

Actuator saturation

Thrusters are limited. In this example to:

$$0 \leq f \leq 2mg \qquad -0.1mg \leq f_s \leq 0.1mg$$