TTK4115

Lecture 3

Discretization, controllability, state feedback

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This lecture

1. Discretization

2. Controllability

   Controllability Gramians

   Eigenvector tests

   Controllability in practice

3. State feedback

   Pole placement

   Controllability indices

Topic

Analog state space model

The continuous state space model is analog[1]:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

To simulate it *as is* would require an analog computer.

Discretization

Discretization is a necessary step for computer simulation. A recursive model is sought:

$$\begin{aligned}\mathbf{x}[k+1] &= \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k]\end{aligned}$$

Many methods are available for obtaining a discretized model. We examine the two most common methods: **Exact** and **Euler** discretization.

Approach

$$\text{TF} \xrightarrow{\text{Realize}} \text{CLTI} \xrightarrow{\text{Discretize}} \text{DLTI} \xrightarrow{\text{Recursion}} \text{Solution}$$

[1] Some systems are discrete by nature, such as financial systems or discrete filters. Most plants will however be continuous as they are based on a physical model.

LTI solution

The exact solution of the LTI system forms the theoretical basis of **exact** discretization.

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Sampling

Time is discretized into intervals of duration T: $t = kT$. Sample index is denoted k.
The state solution from one sample to the next is:

$$\mathbf{x}((k+1)T) = e^{\mathbf{A}T}\mathbf{x}(kT) + \int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Here $\mathbf{x}[k] \triangleq \mathbf{x}(t)|_{t=kT}$ serves as an initial condition
The resulting solution is evaluated at $\mathbf{x}[k+1] \triangleq \mathbf{x}(t)|_{t=(k+1)T}$.

Piecewise constant input

The input is assumed to stay approximately constant between samples:

$$\mathbf{u}[k] \simeq \mathbf{u}(t), \quad kT \leq t < (k+1)T$$

Sampled model

$$\mathbf{x}[k+1] = e^{\mathbf{A}T}\mathbf{x}[k] + \left(\int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}d\tau\right)\mathbf{Bu}[k]$$

Substitution rule

$$\int_{y(a)}^{y(b)} F(x)dx = \int_a^b F(y(x))\frac{dy}{dx}dx$$

Change of variable: $\alpha(\tau) \triangleq (k+1)T - \tau, \quad d\tau = -d\alpha$

Integration limits are simplified:

$$\tau_0 = kT \to \alpha_0 = T, \quad \tau_1 = (k+1)T \to \alpha_1 = 0$$

along with integrand:

$$e^{\mathbf{A}[(k+1)T-\tau]} \to e^{\mathbf{A}\alpha}$$

$$\mathbf{B}_d = \left(\int_{kT}^{(k+1)T} e^{\mathbf{A}[(k+1)T-\tau]}d\tau\right)\mathbf{B} = \left(\int_0^T e^{\mathbf{A}\alpha}d\alpha\right)\mathbf{B}$$

Discretization

Exactly discretized model

$$
\begin{aligned}
\mathbf{x}[k+1] &= \overbrace{e^{\mathbf{A}T}}^{\mathbf{A}_d}\mathbf{x}[k] + \overbrace{\left(\int_0^T e^{\mathbf{A}\alpha}\,d\alpha\right)\mathbf{B}}^{\mathbf{B}_d}\mathbf{u}[k] \\
\mathbf{y}[k] &= \underbrace{\mathbf{C}}_{\mathbf{C}_d}\mathbf{x}[k] + \underbrace{\mathbf{D}}_{\mathbf{D}_d}\mathbf{u}[k]
\end{aligned}
$$

Discrete time system

This model is *exact* under the assumption:

$$\mathbf{u}[k] = \mathbf{u}(t), \quad kT \leq t \leq (k+1)T$$

It is recursive, and very efficient in implementation:

$$
\begin{aligned}
\mathbf{x}[k+1] &= \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] \\
\mathbf{y}[k] &= \mathbf{C}_d\mathbf{x}[k] + \mathbf{D}_d\mathbf{u}[k]
\end{aligned}
$$

Discretization

Euler discretization

Euler's method proceeds via the definition of the derivative[2]:

$$\dot{\mathbf{x}}[k] \approx \frac{\mathbf{x}[k+1] - \mathbf{x}[k]}{T}$$

Thus:

$$\dot{\mathbf{x}}[k] = \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \quad \Rightarrow \mathbf{x}[k+1] = \mathbb{I}\mathbf{x}[k] + T\mathbf{A}\mathbf{x} + T\mathbf{B}\mathbf{u}$$

Stability

Euler's method may be unstable although the underlying plant is stable. This problem gets worse with larger timesteps. A mathematical criterion for first order systems may be stated as:

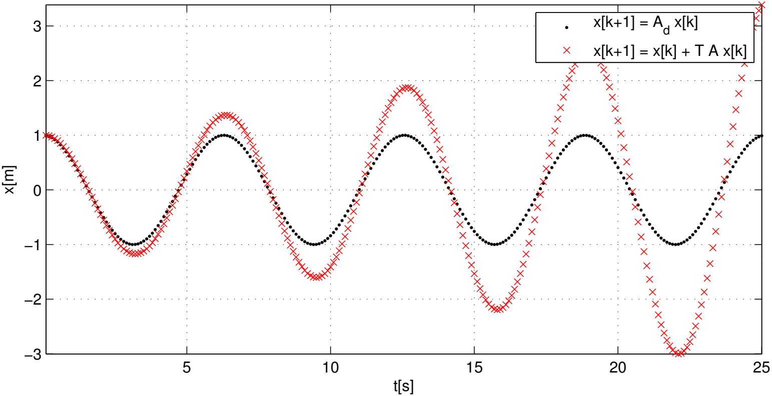
$$|1 + T\lambda| \leq 1$$

Insufficiently stable systems or large timesteps will result in a divergent solution.

[2] $x[k] = x(kT)$

Euler's Method vs Discretization

$$
\begin{aligned}
\mathbf{x}[k+1] &= \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] \\
\mathbf{x}_e[k+1] &= \mathbf{x}_e[k] + T\mathbf{A}\mathbf{x}_e[k] + T\mathbf{B}\mathbf{u}[k]
\end{aligned}
$$



Topic

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Controllability Gramians

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Example

Consider the single input **discrete** system:

$$\mathbf{x}[k+1] = \mathbf{A}\mathbf{x}[k] + \mathbf{b}u[k] \quad \mathbf{A} \in \mathbb{R}^{4\times 4}$$

4 steps forward

Starting point is $\mathbf{x}_0$

$$\begin{aligned}
\mathbf{x}[1] &= \mathbf{A}\mathbf{x}[0] + \mathbf{b}u[0] \\
\mathbf{x}[2] &= \mathbf{A}\mathbf{x}[1] + \mathbf{b}u[1] \\
\mathbf{x}[3] &= \mathbf{A}\mathbf{x}[2] + \mathbf{b}u[2] \\
\mathbf{x}[4] &= \mathbf{A}\mathbf{x}[3] + \mathbf{b}u[3]
\end{aligned}$$

Last step may be written as:

$$\mathbf{x}[4] = \mathbf{A}^4\mathbf{x}[0] + \mathbf{A}^3\mathbf{b}u[0] + \mathbf{A}^2\mathbf{b}u[1] + \mathbf{A}\mathbf{b}u[2] + \mathbf{b}u[3]$$

The k'th step is linear in the initial condition and the sequence of inputs

$$\mathbf{x}[4] = \overbrace{\mathbf{A}^4\mathbf{x}[0]}^{\text{zir}} + \overbrace{\begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix} \begin{bmatrix} u[3] \\ u[2] \\ u[1] \\ u[0] \end{bmatrix}}^{\text{zsr}}$$

Key idea

Iff $\begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}$ has *full rank*, we can choose $\mathbf{x}[4]$ as we like with our inputs.

Controllability matrix

Iff the controllability matrix has full rank: $\text{rank}(\mathcal{C}) = n$

$$\mathcal{C} \triangleq \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \dots & \mathbf{A}^{(n-1)}\mathbf{b} \end{bmatrix}$$

the state can be placed anywhere with the right sequence of inputs.
-**This is controllability**.

Topic

Controllability Gramian

Given an LTI system:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Attempt to place the system at $\mathbf{x}_1$ at $t = t_1$:

$$\mathbf{x}_1 = \mathbf{x}(t_1) = e^{\mathbf{A}t_1}\mathbf{x}_0 + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

We clearly need the proper input signal $\mathbf{u}(t)$ to do this.

If we can find such an input, the system is controllable.

Controllability Gramian: definition

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)}d\tau$$

The existence of a nonsingular Controllability Gramian guarantees that a sufficient $\mathbf{u}(t)$ exists.

Place the system at $\mathbf{x}_1$ at $t = t_1$

$$\mathbf{x}_1 = \mathbf{x}(t_1) = e^{\mathbf{A}t_1}\mathbf{x}_0 + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

We need an input $\mathbf{u}(t)$ to do this.

Educated guess:

$$\mathbf{u}(t) = -\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1-t)}\mathbf{W}_c^{-1}(t_1)\left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right]$$

Result

$$\begin{aligned}
\mathbf{x}_1 = e^{\mathbf{A}t_1}\mathbf{x}_0 - \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\left(\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1-\tau)}\mathbf{W}_c^{-1}(t_1)\left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right]\right)d\tau \\
= e^{\mathbf{A}t_1}\mathbf{x}_0 - \overbrace{\left(\int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1-\tau)}d\tau\right)}^{\mathbf{W}_c(t_1)}\mathbf{W}_c^{-1}(t_1)\left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right] \\
= e^{\mathbf{A}t_1}\mathbf{x}_0 - \left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1\right] = \underline{\mathbf{x}_1}
\end{aligned}$$

The gramian and controllability

Iff $\mathbf{W}_c(t)$ is invertible, the system is controllable.

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)} d\tau$$

If $\mathbf{W}_c(t)$ is singular for t, a nonzero vector $\mathbf{v}$ must exist such that

$$\mathbf{v}^\mathsf{T}\mathbf{W}_c(t)\mathbf{v} = \int_0^t \overbrace{\mathbf{v}^\mathsf{T} e^{\mathbf{A}(t-\tau)}\mathbf{B}}^{\mathbf{0}}\,\overbrace{\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)}\mathbf{v}}^{\mathbf{0}}\, d\tau = \mathbf{0}$$

Say we want to move the system from initial value $e^{-\mathbf{A}t_1}\mathbf{v} \neq \mathbf{0}$ to $\mathbf{0}$

$$\mathbf{x}_1 = \mathbf{0} = \mathbf{v} + \int_0^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau) d\tau$$

Premultiplying by $\mathbf{v}^T$:

$$\mathbf{0} = \mathbf{v}^T\mathbf{v} + \int_0^{t_1} \overbrace{e^{\mathbf{v}^T\mathbf{A}(t_1-\tau)}\mathbf{B}}^{\mathbf{0}}\,\mathbf{u}(\tau) d\tau = ||\mathbf{v}||^2 + 0$$

Which contradicts $\mathbf{v} \neq \mathbf{0}$. So if the system is controllable, $\mathbf{W}_c(t)$ cannot be singular.

The gramian and the controllability matrix

Note:

- $e^{\mathbf{A}t}$ may be expressed as a linear combination of $\{\mathbb{I}, \mathbf{A}, \ldots, \mathbf{A}^{n-1}\}$
- $e^{\mathbf{A}t}\mathbf{B}$ may be expressed as a linear combination of $\{\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\}$

If $\mathbf{W}_c(t)$ is invertible and $\mathcal{C}$ does not have full row rank:

There exists a nonzero $\mathbf{v}$ such that

$$\mathbf{v}^T\mathcal{C} = \mathbf{v}^T[\mathbf{B} \quad \mathbf{AB} \quad \ldots \quad \mathbf{A}^{n-1}\mathbf{B}] = \mathbf{0}$$

which implies that

$$\mathbf{v}^T\mathbf{A}^k\mathbf{B} = \mathbf{0} \qquad \text{for } k = 0, 1, 2, \ldots, n-1$$

Since $e^{\mathbf{A}t}\mathbf{B}$ can be expressed as a linear combination of $\{\mathbf{B}, \mathbf{AB}, \ldots, \mathbf{A}^{n-1}\mathbf{B}\}$:

$$\mathbf{v}^T e^{\mathbf{A}t}\mathbf{B} = \mathbf{0}$$

which contradicts the nonsingularity of $\mathbf{W}_c(t)$.

Which shows:

If $\mathcal{C}$ doesn't have full rank, $\mathbf{W}_c(t)$ cannot be invertible.

The gramian and the controllability matrix

If $\mathcal{C}$ has full row rank but $\mathbf{W}_c(t)$ is not invertible:

There exists a nonzero $\mathbf{v}$ such that

$$\mathbf{v}^T e^{\mathbf{A}t}\mathbf{B} = \mathbf{0} \qquad \text{for all } t$$

which implies that for $t = 0$, $\mathbf{v}^T\mathbf{B} = \mathbf{0}$.

Differentiating k times and setting $t = 0$, we get $\mathbf{v}^T\mathbf{A}^k\mathbf{B} = \mathbf{0}$, or:

$$\mathbf{v}^T[\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}] = \mathbf{v}^T\mathcal{C} = \mathbf{0}$$

which contradicts the hypothesis that $\mathcal{C}$ has full row rank.

Which shows:

If $\mathbf{W}_c(t)$ is not invertible, $\mathcal{C}$ cannot have full row rank.

Equivalent Statements on Controllability

Controllability Gramian

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)} d\tau$$

Iff $\mathbf{W}_c(t)$ is invertible, the system is controllable.

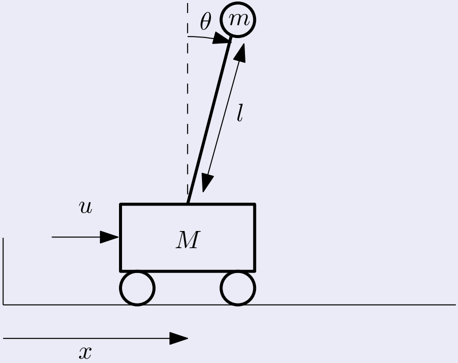
Controllability Matrix

$$\mathcal{C} = \overbrace{\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}}^{np} \}n$$

Iff the controllability matrix has full rank: $\text{rank}(\mathcal{C}) = n$, the system is controllable.

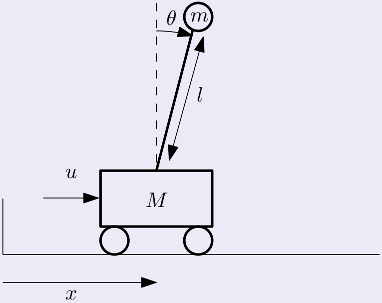
Controllability

Example



Controllability

Example



Linearized EOM

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(m+M)}{lM} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{lM} \end{bmatrix} u$$

Controllability

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Controllability Matrix: Full row rank

$$\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} & \mathbf{A}^3\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{15}{2} \\ \frac{1}{2} & 0 & \frac{5}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{15}{2} & 0 \end{bmatrix}$$

Controllability Gramian: Invertible

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t-\tau)} d\tau$$

Controllability

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Move the cart from $\mathbf{x}_0 = \mathbf{0}$ to $\mathbf{x}_1 = [1, 0, 0, 0]^\top$, $t_1 = 3s$

Let's use the Gramian:

$$\mathbf{u}(t) = -\mathbf{B}^\top e^{\mathbf{A}^\top (t_1 - t)} \mathbf{W}_c^{-1}(t_1) \left[e^{\mathbf{A}t_1} \mathbf{x}_0 - \mathbf{x}_1 \right]$$

Controllability Gramian

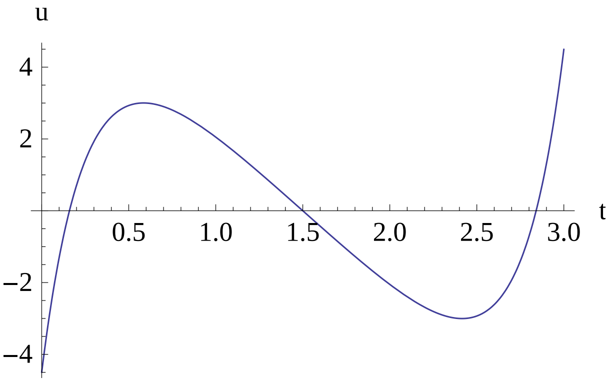
$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}(t-\tau)} \mathbf{B}\mathbf{B}^\top e^{\mathbf{A}^\top (t-\tau)} d\tau$$

Controllability

Move the cart from $\mathbf{x}_0 = \mathbf{0}$ to $\mathbf{x}_1 = [1, 0, 0, 0]^\mathsf{T}$, $t_1 = 3s$

Let's use the Gramian:

$$\mathbf{u}(t) = -\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1 - t)} \mathbf{W}_c^{-1}(t_1) \left[e^{\mathbf{A}t_1} \mathbf{x}_0 - \mathbf{x}_1 \right]$$

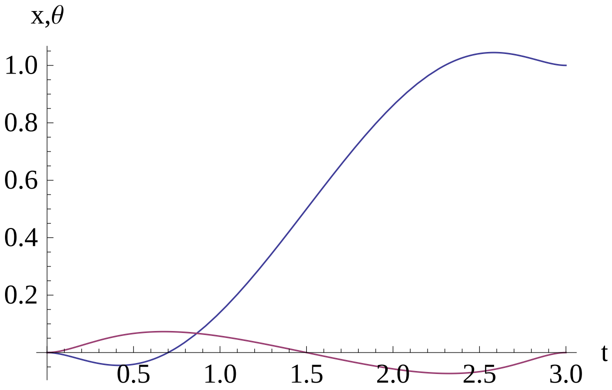


Controllability

Move the cart from $\mathbf{x}_0 = \mathbf{0}$ to $\mathbf{x}_1 = [1, 0, 0, 0]^\mathsf{T}$, $t_1 = 3s$

Let's use the Gramian:

$$\mathbf{u}(t) = -\mathbf{B}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}(t_1 - t)} \mathbf{W}_c^{-1}(t_1) \left[e^{\mathbf{A}t_1}\mathbf{x}_0 - \mathbf{x}_1 \right]$$



Topic

Eigenvector test

Basic idea

If the Controllability matrix has full rank, there is no vector $\mathbf{v}$ such that:

$$\mathcal{C}^{\mathsf{T}}\mathbf{v} = \begin{bmatrix} \mathbf{B}^{\mathsf{T}} \\ \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} \\ \dots \\ \mathbf{B}^{\mathsf{T}}(\mathbf{A}^{\mathsf{T}})^{(n-1)} \end{bmatrix} \mathbf{v} = \mathbf{0}, \quad \forall \mathbf{v} \neq \mathbf{0}$$

Let $\mathbf{q}$ be an eigenvector of $\mathbf{A}^{\mathsf{T}}$: $\mathbf{A}^{\mathsf{T}}\mathbf{q} = \lambda\mathbf{q}$

$$\begin{bmatrix} \mathbf{B}^{\mathsf{T}} \\ \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}} \\ \dots \\ \mathbf{B}^{\mathsf{T}}(\mathbf{A}^{\mathsf{T}})^{(n-1)} \end{bmatrix} \mathbf{q} = \begin{bmatrix} \mathbf{B}^{\mathsf{T}} \\ \lambda\mathbf{B}^{\mathsf{T}} \\ \vdots \\ \lambda^{n-1}\mathbf{B}^{\mathsf{T}} \end{bmatrix} \mathbf{q}$$

Controllability and eigenvectors

Controllability is only possible if every eigenvector of $\mathbf{A}^{\mathsf{T}}$ is not in the null-space of $\mathbf{B}^{\mathsf{T}}$.

Eigenvector test

Controllability and eigenvectors

Controllability is only possible if every eigenvector of $\mathbf{A}^\mathsf{T}$ is not in the null-space of $\mathbf{B}^\mathsf{T}$.

Popov-Belevitch-Hautus - test

The PBH test gives an elegant test based on this insight.
An LTI system is controllable iff:

$$\operatorname{rank}[\mathbf{A} - \lambda \mathbb{I} \quad \mathbf{B}] = n, \quad \text{for all eigenvalues } \lambda \text{ of } \mathbf{A}$$

Topic

Controllable systems

Three important points

- If the pair $\{\mathbf{A}, \mathbf{B}\}$ is controllable, so is $\{\mathbf{A} - \mathbf{BK}, \mathbf{B}\}$.
- If the system is controllable we can place the eigenvalues of the system exactly as desired.
- A controllable system can always be transformed to the controllable canonical form.

Controllability in practice

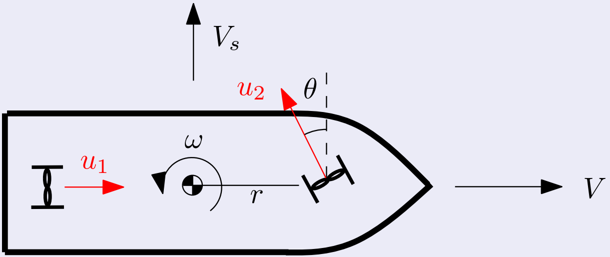
Caveats

Even if the controllability matrix has full rank, this does not mean that the system is easy to control in practice.

- The controller may require too large inputs.
- The closed loop response may be higly sensitive to modeling errors in $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$.
- The closed loop eigenvalues may have been chosen unrealistically fast.
- Fast response requires powerful actuators and an accurate model.
- The system may be ”almost uncontrollable” in practice.

Controllable?

Dynamic positioning

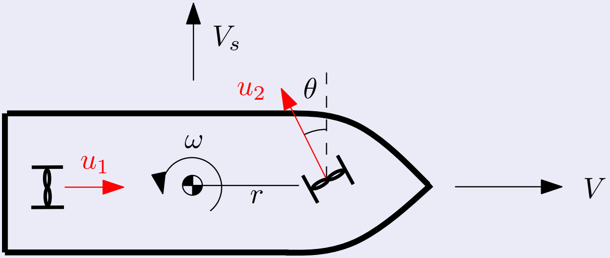


State equation

$$\begin{bmatrix} \dot{V} \\ \dot{V}_s \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} -\frac{d}{m} & 0 & 0 \\ 0 & -\frac{d_s}{m} & 0 \\ 0 & 0 & -\frac{d_\omega}{J} \end{bmatrix} \begin{bmatrix} V \\ V_s \\ \omega \end{bmatrix} + \begin{bmatrix} 1/m & -\sin(\theta)/m \\ 0 & \cos(\theta)/m \\ 0 & \cos(\theta)r/J \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Controllable?

Dynamic positioning

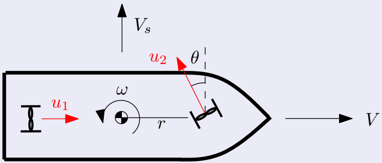


Controllability matrix

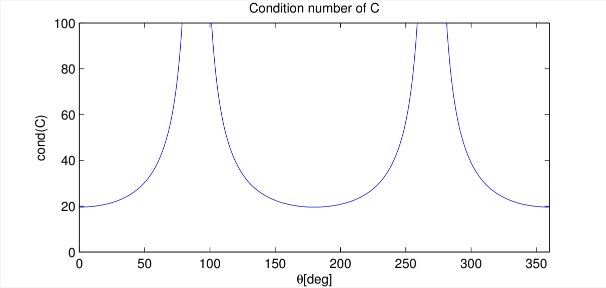
$$\mathcal{C} = \begin{bmatrix} \frac{1}{m} & -\frac{\sin(\theta)}{m} & -\frac{d}{m^2} & \frac{d\sin(\theta)}{m^2} & \frac{d^2}{m^3} & -\frac{d^2\sin(\theta)}{m^3} \\ 0 & \frac{\cos(\theta)}{m} & 0 & -\frac{\cos(\theta)d_s}{m^2} & 0 & \frac{\cos(\theta)d_s^2}{m^3} \\ 0 & \frac{r\cos(\theta)}{J} & 0 & -\frac{r\cos(\theta)d_\omega}{J^2} & 0 & \frac{r\cos(\theta)d_\omega^2}{J^3} \end{bmatrix}$$

Controllable?

Dynamic positioning



Condition number of $\mathcal{C}$

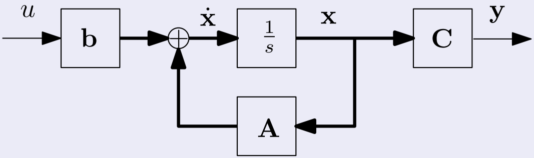


Topic

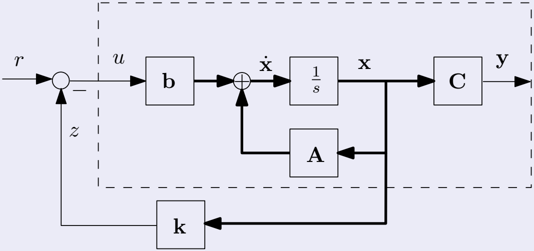
Topic

Open loop and feedback controlled systems

Open loop



State feedback: $u = r - \mathbf{kx}$



Closed loop dynamics

$$u = r - \mathbf{k}\mathbf{x}$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x} + \mathbf{b}r$$

State feedback

All states are available for feedback. Contrast this with *output feedback*.

First important point

If the pair $\{\mathbf{A}, \mathbf{b}\}$ is controllable, there is a similarity transform $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ that will transform the system to controllable canonical form:

$$
\begin{aligned}
\dot{\bar{\mathbf{x}}} &= \overbrace{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}}^{\bar{\mathbf{A}}}\,\bar{\mathbf{x}} + \overbrace{\mathbf{P}^{-1}\mathbf{b}}^{\bar{\mathbf{b}}}\,u \\
y &= \underbrace{\mathbf{c}\mathbf{P}}_{\bar{\mathbf{c}}}\,\bar{\mathbf{x}}
\end{aligned}
$$

Controllable canonical form $n = 4$

$$
\bar{\mathbf{A}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \bar{\mathbf{b}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \bar{\mathbf{c}} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix}
$$

$$
g(s) = \bar{\mathbf{c}}(s\mathbb{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{b}} = \frac{s^3 n_1 + s^2 n_2 + s n_3 + n_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}
$$

First important point

If the pair $\{\mathbf{A}, \mathbf{b}\}$ is controllable, there is a similarity transform $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$ that will transform the system to controllable canonical form:

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \overbrace{\mathbf{P}^{-1}\mathbf{A}\mathbf{P}}^{\bar{\mathbf{A}}}\,\bar{\mathbf{x}} + \overbrace{\mathbf{P}^{-1}\mathbf{b}}^{\bar{\mathbf{b}}}\,u \\ y &= \underbrace{\mathbf{c}\mathbf{P}}_{\bar{\mathbf{c}}}\,\bar{\mathbf{x}} \end{aligned}$$

Transformation

$$\mathbf{P} = \underbrace{\begin{bmatrix} \mathbf{b} & \mathbf{A}\mathbf{b} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}}_{\mathcal{C}} \underbrace{\begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\bar{\mathcal{C}}^{-1}}$$

Second important point

The pair $\{(\mathbf{A}-\mathbf{bk}),\mathbf{b}\}$ is only controllable if the pair $\{\mathbf{A},\mathbf{b}\}$ is controllable.
- Feedback cannot produce or destroy controllability.

Reason[3]

Factorize closed loop controllability matrix to obtain:

$$\begin{aligned}\mathcal{C}_k &= \begin{bmatrix} \mathbf{b} & (\mathbf{A}-\mathbf{bk})\mathbf{b} & (\mathbf{A}-\mathbf{bk})^2\mathbf{b} & (\mathbf{A}-\mathbf{bk})^3\mathbf{b} \end{bmatrix} \\ &= \mathcal{C}\underbrace{\begin{bmatrix} 1 & -\mathbf{kb} & -\mathbf{k}(\mathbf{A}-\mathbf{bk})\mathbf{b} & -\mathbf{k}(\mathbf{A}-\mathbf{bk})^2\mathbf{b} \\ 0 & 1 & -\mathbf{kb} & -\mathbf{k}(\mathbf{A}-\mathbf{bk})\mathbf{b} \\ 0 & 0 & 1 & -\mathbf{kb} \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Never singular}}\end{aligned}$$

3
$$\mathcal{C} = \begin{bmatrix} \mathbf{b} & \mathbf{Ab} & \mathbf{A}^2\mathbf{b} & \mathbf{A}^3\mathbf{b} \end{bmatrix}$$

Third important point

If the pair $\{\mathbf{A}, \mathbf{b}\}$ is controllable, we can place the eigenvalues exactly where we want!

Pole placement

Transform the plant to the controllable canonical form. Choose the **state feedback**:

$$u = \bar{\mathbf{k}}\bar{\mathbf{x}}, \quad \bar{\mathbf{k}} = \begin{bmatrix} \eta_1 - \alpha_1 & \eta_2 - \alpha_2 & \eta_3 - \alpha_3 & \eta_4 - \alpha_4 \end{bmatrix}$$

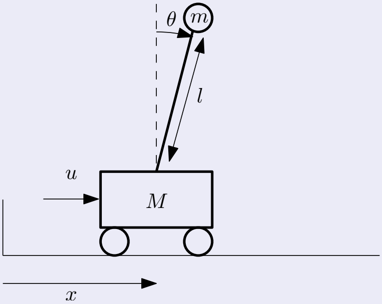
Closed loop dynamics

$$\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}} = \begin{bmatrix} -\alpha_1 - (\eta_1 - \alpha_1) & -\alpha_2 - (\eta_2 - \alpha_2) & -\alpha_3 - (\eta_3 - \alpha_3) & -\alpha_4 - (\eta_4 - \alpha_4) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Modified transfer function

$$\Rightarrow g_k(s) = \frac{s^3 n_1 + s^2 n_2 + s n_3 + n_4}{s^4 + s^3 \eta_1 + s^2 \eta_2 + s \eta_3 + \eta_4}$$

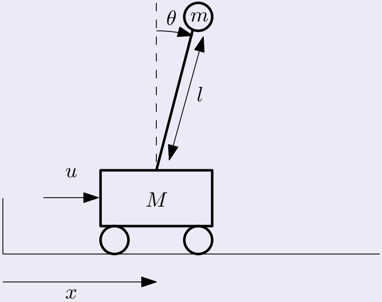
Example



Linearized EOM

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{u} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(m+M)}{lM} & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{lM} \end{bmatrix} u$$

Example



Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Controllability matrix

$$\mathcal{C} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{15}{2} \\ \frac{1}{2} & 0 & \frac{5}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{15}{2} & 0 \end{bmatrix} \quad \text{Full rank}$$

The system is controllable

It can be converted to controllable form!

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Characteristic polynomial

$$|s\mathbb{I} - \mathbf{A}| = s^4 - 15s^2 = s^2\left(-\sqrt{15} + s\right)\left(\sqrt{15} + s\right)$$

$$\alpha_2 = -15, \quad \alpha_{1,3,4} = 0$$

Transform

$$\mathbf{P} = \mathcal{C} \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \alpha_1 & \alpha_2 \\ 0 & 0 & 1 & \alpha_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{5}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{15}{2} \\ \frac{1}{2} & 0 & \frac{5}{2} & 0 \\ -\frac{1}{2} & 0 & -\frac{15}{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -15 & 0 \\ 0 & 1 & 0 & -15 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Characteristic polynomial

$$|s\mathbb{I} - \mathbf{A}| = s^4 - 15s^2 = s^2\left(-\sqrt{15} + s\right)\left(\sqrt{15} + s\right)$$

$$\alpha_2 = -15, \quad \alpha_{1,3,4} = 0$$

Better characteristic polynomial

$$|s\mathbb{I} - \mathbf{A}'| = s^2\left(\sqrt{15} + s\right)\left(\sqrt{15} + s\right) = s^4 + 2\sqrt{15}s^3 + 15s^2$$

$$\eta_1 = 2\sqrt{15}, \quad \eta_2 = 15, \quad \eta_{3,4} = 0$$

Pole placement: $u = \bar{\mathbf{k}}\bar{\mathbf{x}}$

$$\bar{\mathbf{k}} = \begin{bmatrix} \eta_1 - \alpha_1 & \eta_2 - \alpha_2 & \eta_3 - \alpha_3 & \eta_4 - \alpha_4 \end{bmatrix} = \begin{bmatrix} 2\sqrt{15} & 30 & 0 & 0 \end{bmatrix}$$

Transformed system

$$\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}} = \begin{bmatrix} -2\sqrt{15} & -15 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Reversion to original system $\bar{\mathbf{x}} = \mathbf{P}^{-1}\mathbf{x}$

$$\dot{\bar{\mathbf{x}}} = (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}})\bar{\mathbf{x}} \quad \Rightarrow \mathbf{P}^{-1}\dot{\mathbf{x}} = (\bar{\mathbf{A}} - \bar{\mathbf{b}}\bar{\mathbf{k}})\mathbf{P}^{-1}\mathbf{x} \quad \Rightarrow \dot{\mathbf{x}} = \mathbf{P}\bar{\mathbf{A}}\mathbf{P}^{-1}\mathbf{x} - \mathbf{P}\bar{\mathbf{b}}\bar{\mathbf{k}}\mathbf{P}^{-1}\mathbf{x} \quad \Rightarrow \dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x}$$

Transformed gain matrix

$$\mathbf{k} = \bar{\mathbf{k}}\mathbf{P}^{-1} = \begin{bmatrix} 0 & -60 & 0 & -4\sqrt{15} \end{bmatrix}$$

Original state equation: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{u} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ u \\ q \end{bmatrix}$$

Closed loop state equation $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{b}\mathbf{k})\mathbf{x}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{u} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 25 & 0 & 2\sqrt{15} \\ 0 & -15 & 0 & -2\sqrt{15} \end{bmatrix} \begin{bmatrix} x \\ \theta \\ u \\ q \end{bmatrix}$$

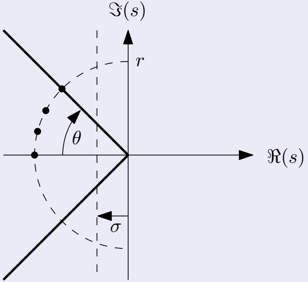
Pole placement

Procedure

1. Calculate the controllability matrix.
2. Find transform to controllable canonical form, $\mathbf{x} = \mathbf{P}\bar{\mathbf{x}}$.
3. Choose gain matrix $\bar{\mathbf{k}}$ such that $u = -\bar{\mathbf{k}}\bar{\mathbf{x}}$ leads to the desired poles. Always possible.
4. Transform $\bar{\mathbf{k}}$ back to the original coordinates, $\mathbf{k} = \bar{\mathbf{k}}\mathbf{P}^{-1}$

Pole placement

Where to place poles



Considerations

- Close spacing of eigenvalues: sluggish response, large *u*.
- Large σ: fast response, large *u*.
- Large *r*: fast response, large *u*.
- Large θ: greater overshoot

Pole placement in MIMO systems

Controllable canonical form: p inputs

$$\bar{\mathbf{A}} = \begin{bmatrix} -\alpha_1 \mathbb{I}_p & -\alpha_2 \mathbb{I}_p & -\alpha_3 \mathbb{I}_p & -\alpha_4 \mathbb{I}_p \\ \mathbb{I}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I}_p & \mathbf{0} \end{bmatrix} \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbb{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

$\mathbf{G}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

$$\mathbf{G}(s) = \frac{s^3\mathbf{N}_1 + s^2\mathbf{N}_2 + s\mathbf{N}_3 + \mathbf{N}_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4} + \mathbf{D}$$

Pole placement in MIMO systems

Controllable canonical form: p inputs

$$\bar{\mathbf{A}} = \begin{bmatrix} -\alpha_1 \mathbb{I}_p & -\alpha_2 \mathbb{I}_p & -\alpha_3 \mathbb{I}_p & -\alpha_4 \mathbb{I}_p \\ \mathbb{I}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I}_p & \mathbf{0} \end{bmatrix} \quad \bar{\mathbf{B}} = \begin{bmatrix} \mathbb{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

Pole placement

$$\bar{\mathbf{K}} = \begin{bmatrix} (\eta_1 - \alpha_1)\mathbb{I}_p & (\eta_2 - \alpha_2)\mathbb{I}_p & (\eta_3 - \alpha_3)\mathbb{I}_p & (\eta_4 - \alpha_4)\mathbb{I}_p \end{bmatrix}$$

$\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{B} + \mathbf{D}$

$$\mathbf{G}(s) = \frac{s^3\mathbf{N}_1 + s^2\mathbf{N}_2 + s\mathbf{N}_3 + \mathbf{N}_4}{s^4 + s^3\eta_1 + s^2\eta_2 + s\eta_3 + \eta_4} + \mathbf{D}$$

MIMO control

Comments

- In matlab, use: ”K = place(A,B,eig)”
- Remember to always assign complex eigenvalues in *pairs*.
- MIMO pole placement alters the common denominator of $\mathbf{G}_{sp}(s)$.
- It may be difficult to tune the individual outputs.
- Only eigenvalues are considered, not input amplitudes.

Linear **Q**uadratic **R**egulator

The LQR allows more flexibility when specifying system performance. This is the preferred method for MIMO systems.

Topic

What's wrong with this **B**?

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Answer

Linear dependency in the columns. We disregard redundant inputs.

We have p inputs. Let:

$$\mathbf{B} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix}$$

Controllability matrix

$$\mathcal{C} = \left[\begin{array}{cccc|cccc|c|cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p & \mathbf{A}\mathbf{b}_1 & \mathbf{A}\mathbf{b}_2 & \cdots & \mathbf{A}\mathbf{b}_p & \ldots & \mathbf{A}^{n-1}\mathbf{b}_1 & \mathbf{A}^{n-1}\mathbf{b}_2 & \cdots & \mathbf{A}^{n-1}\mathbf{b}_p \end{array}\right]$$

Controllability index

The controllability indices of $\mathbf{b}_i$: μ_i, are the number of linearly independent columns associated with $\mathbf{b}_i$. If $\mathcal{C}$ has rank n, these indices sum to:

$$\mu_1 + \mu_2 + \cdots + \mu_p = n$$

The largest μ_i is the controllability index.

Multi-input controllability

Using these indices, we can show that it is sufficient to check the rank of:

$$\mathcal{C} = \left[\begin{array}{c|c|c|c} \mathbf{B} & \mathbf{A}\mathbf{B} & \ldots & \mathbf{A}^{n-p}\mathbf{B} \end{array}\right]$$

Final notes

Property 1

Controllability is not affected by an equivalence transformation.

Property 2

Controllability is not affected by reordering the columns of $\mathbf{B}$.