TTK4115

Lecture 5

Canonical forms, realizations, observability

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This lecture

1. Canonical Forms

2. Realizations

3. Observability

    Duality

Topic

Canonical Forms

Canonical Forms

- Using the change of basis $\mathbf{x} = \mathbf{T}\bar{\mathbf{x}}$ we can change a system into infinitely many similar forms.
- Some of these forms are more useful than others.
- Some of these are called *canonical*.

Equivalence/Similarity transform

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{aligned}$$

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \mathbf{T}^{-1}\mathbf{A}\mathbf{T}\bar{\mathbf{x}} + \mathbf{T}^{-1}\mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{T}\bar{\mathbf{x}} + \mathbf{D}\mathbf{u} \end{aligned}$$

Canonical Forms

Canonical Forms[1]

- Jordan Form
- Modal Form
- Companion form
- Controllable form
- Observable form

[1] This list is not exhaustive.

Canonical Forms

Jordan Form

The Jordan form is the most convenient to use when solving the system. We have seen that this form is very practical for finding solutions to LTI systems:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

Diagonal matrix

$$\mathbf{A} = \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \Rightarrow e^{\mathbf{A}t} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{bmatrix}$$

Jordan Block

$$\mathbf{A} = \mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \Rightarrow e^{\mathbf{A}t} = \begin{bmatrix} e^{t\lambda} & e^{t\lambda}t & \frac{1}{2!}e^{t\lambda}t^2 & \frac{1}{3!}e^{t\lambda}t^3 \\ 0 & e^{t\lambda} & e^{t\lambda}t & \frac{1}{2!}e^{t\lambda}t^2 \\ 0 & 0 & e^{t\lambda} & e^{t\lambda}t \\ 0 & 0 & 0 & e^{t\lambda} \end{bmatrix}$$

Topic

1. Canonical Forms

2. Realizations

3. Observability

Duality

LTI systems overview

$\mathcal{L}(\dots)$

$$\mathbf{x}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\mathbf{u}(s)]$$
$$\mathbf{y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\mathbf{u}(s)] + \mathbf{D}\mathbf{u}(s)$$

Realization

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$\mathcal{L}^{-1}(\dots)$

$\mathcal{L}(\dots)$

LTI sol.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$
$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

Realizations

The final piece in the diagram

Realizations

Key points

Realization

- We have seen that a transformation $\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}$ can change the state equation..
- but the transfer function remains the same.
- When we realize, we start with a transfer function $\mathbf{H}(s)$..
- and generate a state-space $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$
- that yields $\mathbf{H}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

Note

There are infinitely many state-spaces we could realize to!

Note

We usually go for a *canonical* form.

Realizations
Conditions

Proper

A transfer function must be proper to have a realization:

$$h(s) = \frac{n(s)}{d(s)} \quad \Rightarrow \deg d(s) \geq n(s)$$

$$|h_p(j\infty)| < \infty, \quad |h_{sp}(j\infty)| = 0$$

Rational

A transfer function must be rational to have a realization.

- The degrees of the numerator and denominator must be finite.
- All lumped LTI systems are rational.

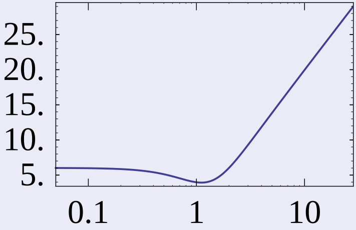
Proper transfer functions

We must have a proper transfer function for realization.

Example

$$h(s) = \frac{2 + 2s + s^2}{1 + s}$$

$|h(i\omega)|$



Question:

Is this a proper transfer function?

Signals are amplified

at infinite frequencies.. no device can do this

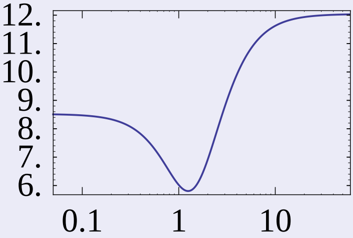
Proper transfer functions

Proper transfer functions behave nicely at high frequencies

Example

$$h(s) = \frac{4\left(2 + 2s + s^2\right)}{3 + 4s + s^2}$$

$|h(i\omega)|$



Question:

Is this a proper transfer function?

Answer

Yes, *but not strictly proper*. The transfer function is finite at infinite frequencies:

$$\lim_{\omega \to \infty} h(j\omega) = h_\infty \neq 0$$

Quiz

Are these transfer functions realizable?

- $g_1(s) = \frac{1}{s}$ (yes, strictly proper)
- $g_2(s) = s$ (no, not proper)
- $g_3(s) = \frac{1}{s+1}$ (yes, strictly proper)
- $g_4(s) = \frac{1}{s-1}$ (yes, strictly proper)
- $g_5(s) = \frac{s}{s+1}$ (yes, proper)
- $g_6(s) = e^{-\tau s}, \quad \tau > 0$ (no, not rational)
- $g_7(s) = \frac{1-\frac{\tau}{2}s}{1+\frac{\tau}{2}s}, \quad \tau > 0$ (yes, proper)

Improper/Proper/Strictly proper

Improper

$$H_{i.p.}(s) = k_P + k_D s + \frac{k_I}{s} = \frac{k_D s^2 + k_P s + k_I}{s} \quad \text{PID regulator}$$

Proper

$$H_p(s) = \frac{s}{Ts + 1} \quad \text{Band-limited differentiator}$$

Strictly proper

$$H_{s.p.}(s) = \frac{1}{s^2 m + ds + k} \quad \text{Mass-spring-damper}$$

Decomposition

We decompose the proper transfer function as:

$$\mathbf{G}(s) = \overbrace{\mathbf{G}_{sp}(s)}^{\text{strictly proper}} + \overbrace{\mathbf{G}_{\infty}}^{\text{constant}}$$

Relation between system matrices and decomposed transfer function

$$\begin{aligned}
\hat{\mathbf{y}}(s) &= \mathbf{G}_{sp}(s)\hat{\mathbf{u}}(s) + \mathbf{G}_{\infty}\hat{\mathbf{u}}(s) \\
\hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}\hat{\mathbf{u}}(s) + \mathbf{D}\hat{\mathbf{u}}(s)
\end{aligned}$$

Matching

The crucial next step is to select a state-space model with unknown coefficients:

$$\Sigma_r : \ \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$$

that can represent our transfer-function.

Matching

We shall use the **Controllable Canonical Form** today. This is one of many choices.

Realizations

Controllable form

Let's pick a nice **A** for the realization

Four states $\rightarrow$ up to s^4 in the denominator of $g(s)$.

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$(s\mathbb{I} - \mathbf{A})^{-1}$

$$(s\mathbb{I}-\mathbf{A})^{-1} = \frac{1}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4} \begin{bmatrix} s^3 & -s^2\alpha_2 - s\alpha_3 - \alpha_4 & -s^2\alpha_3 - s\alpha_4 & -s^2\alpha_4 \\ s^2 & s^3 + s^2\alpha_1 & -s\alpha_3 - \alpha_4 & -s\alpha_4 \\ s & s^2 + s\alpha_1 & s^3 + s^2\alpha_1 + s\alpha_2 & -\alpha_4 \\ 1 & s + \alpha_1 & s^2 + s\alpha_1 + \alpha_2 & s^3 + s^2\alpha_1 + s\alpha \end{bmatrix}$$

Let's pick a nice **B** for the realization too

Four states $\rightarrow$ up to s^4 in the denominator of $g(s)$.

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} \frac{1}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

What about $\mathbf{C}$?

Four states $\rightarrow$ up to s^4 in the denominator of $g(s)$.

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \end{bmatrix} \begin{bmatrix} s^3 \\ s^2 \\ s \\ 1 \end{bmatrix} \frac{1}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{G}_{sp}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{s^3 n_1 + s^2 n_2 + s n_3 + n_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

Example: Mass spring damper transfer-function realization

$$\frac{x(s)}{f(s)} = \frac{y(s)}{u(s)} = \frac{1}{ms^2 + sd + k} = \frac{1/m}{s^2 + s(d/m) + (k/m)}$$

Controllable canonical form, $n = 2$

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 & -\alpha_2 \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} n_1 & n_2 \end{bmatrix}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{G}_{sp}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{s\overbrace{n_1}^{0} + \overbrace{n_2}^{1/m}}{s^2 + s\underbrace{\alpha_1}_{d/m} + \underbrace{\alpha_2}_{k/m}}$$

Realization

The mass spring damper back on state-space form:

$$\mathbf{A} = \begin{bmatrix} -d/m & -k/m \\ 1 & 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1/m \end{bmatrix}$$

Controllable canonical form: p inputs, q outputs

$$\mathbf{A} = \begin{bmatrix} -\alpha_1 \mathbb{I}_p & -\alpha_2 \mathbb{I}_p & -\alpha_3 \mathbb{I}_p & -\alpha_4 \mathbb{I}_p \\ \mathbb{I}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{I}_p & \mathbf{0} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbb{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 & \mathbf{N}_4 \end{bmatrix}$$

$\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}$

$$\mathbf{G}_{sp}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{s^3\mathbf{N}_1 + s^2\mathbf{N}_2 + s\mathbf{N}_3 + \mathbf{N}_4}{s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4}$$

$d(s)$

We have to find the common denominator of $\mathbf{G}_{sp}(s)$: $d(s) = s^4 + s^3\alpha_1 + s^2\alpha_2 + s\alpha_3 + \alpha_4$

Realizations

Example 1

Realize $\mathbf{G}(s)$ to controllable canonical form:

$$\mathbf{G}(s) = \begin{bmatrix} \frac{-10+4s}{1+2s} & \frac{3}{2+s} \\ \frac{1}{(2+s)(1+2s)} & \frac{1+s}{(2+s)^2} \end{bmatrix}$$

Find $\mathbf{G}_\infty$

$$\mathbf{D} = \mathbf{G}_\infty = \lim_{s \to \infty} \begin{bmatrix} \frac{-10+4s}{1+2s} & \frac{3}{2+s} \\ \frac{1}{(2+s)(1+2s)} & \frac{1+s}{(2+s)^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Find $\mathbf{G}_{sp} = \mathbf{G}(s) - \mathbf{G}_\infty$

$$\mathbf{G}_{sp} = \begin{bmatrix} \frac{-10+4s}{1+2s} & \frac{3}{2+s} \\ \frac{1}{(2+s)(1+2s)} & \frac{1+s}{(2+s)^2} \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{12}{1+2s} & \frac{3}{2+s} \\ \frac{1}{2+5s+2s^2} & \frac{1+s}{(2+s)^2} \end{bmatrix}$$

Find common denominator $d(s)$

$$\mathbf{G}_{sp} = \frac{1}{s^3 + (9/2)s^2 + 6s + 2} \begin{bmatrix} -6(2+s)^2 & 3(1+s/2)(1+2s) \\ 1+s/2 & (1/2+s)(1+s) \end{bmatrix}$$

Realizations

Example 1

Realize $\mathbf{G}(s)$ to controllable canonical form:

$$\mathbf{G}_{sp} = \frac{1}{d(s)}\left(\begin{bmatrix} -24-24s & 3+\frac{15s}{2} \\ 1+\frac{s}{2} & \frac{1}{2}+\frac{3s}{2} \end{bmatrix} + s^2\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}\right)$$

$$d(s) = s^3 + (9/2)s^2 + 6s + 2$$

Find numerator matrices $\mathbf{N}_i$

$$\mathbf{G}_{sp} = \frac{1}{d(s)}\left(\overbrace{\begin{bmatrix} -24 & 3 \\ 1 & \frac{1}{2} \end{bmatrix}}^{\mathbf{N}_3} + s\overbrace{\begin{bmatrix} -24 & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}}^{\mathbf{N}_2} + s^2\overbrace{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}}^{\mathbf{N}_1}\right)$$

$$d(s) = s^3 + (9/2)s^2 + 6s + 2$$

Realizations

Example 1

Realize $\mathbf{G}(s)$ to controllable canonical form:

$$\mathbf{G}_{sp} = \frac{1}{d(s)}\left[\mathbf{N}_3 + s\mathbf{N}_2 + s^2\mathbf{N}_1\right] \quad \mathbf{G}_\infty = \mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{N}_1 = \begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix} \quad \mathbf{N}_2 = \begin{bmatrix} -24 & \frac{15}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \quad \mathbf{N}_3 = \begin{bmatrix} -24 & 3 \\ 1 & \frac{1}{2} \end{bmatrix}$$

$$d(s) = s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3 = s^3 + (9/2)s^2 + 6s + 2$$

Realize:

$$\mathbf{A} = \begin{bmatrix} -\alpha_1\mathbb{I}_p & -\alpha_2\mathbb{I}_p & -\alpha_3\mathbb{I}_p \\ \mathbb{I}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbb{I}_p & \mathbf{0} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbb{I}_p \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \mathbf{N}_3 \end{bmatrix}$$

Topic

Observability

Observability: definition

The system:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}
\end{aligned}
$$

is said to be **observable** if for any unknown initial state $\mathbf{x}(0)$ there exists a finite $t_1 > 0$ such that the knowledge of the input $\mathbf{u}$ and the output $\mathbf{y}$ over $[0, t_1]$ suffices to determine uniquely the initial state $\mathbf{x}(0)$.
Otherwise, the system is **unobservable**.

Observability

Observability Gramian

If the **Observability Gramian**:

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau$$

is nonsingular for any $t > 0$, then the state equation is observable.

Explanation

The solution to the state equation is:

$$\mathbf{y}(t) = \overbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)}^{\text{We don't know the in. conds.}} \overbrace{+\,\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)}^{\text{We know the input.}}$$

Definition

The **unknown** part of the solution is:

$$\mathbf{y}_{\text{nat.}}(t) \triangleq \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

AKA the *natural response*.

Observability

1: Premultiply

$$e^{\mathbf{A}^\mathsf{T} t}\mathbf{C}^\mathsf{T}\mathbf{y}_{\text{nat.}}(t) = e^{\mathbf{A}^\mathsf{T} t}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

2: Integrate

$$\int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{y}_{\text{nat.}}(\tau)d\tau = \int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}d\tau\,\mathbf{x}(0)$$

3: Recover the Gramian

$$\int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{y}_{\text{nat.}}(\tau)d\tau = \underbrace{\int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}d\tau}_{\mathbf{W}_o(t_1)}\,\mathbf{x}(0)$$

4: Invert[2] and premultiply

$$\mathbf{W}_o^{-1}(t_1)\int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{y}_{\text{nat.}}(\tau)d\tau = \mathbf{W}_o^{-1}(t_1)\mathbf{W}_o(t_1)\mathbf{x}(0) = \mathbb{I}\mathbf{x}(0)$$

[2] Only possible if the system is observable.

Observability

Only *one possible* $\mathbf{x}(0)$ for a given $\mathbf{y}_{\text{nat.}}(t)$

$$\mathbf{W}_o^{-1}(t_1)\int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{y}_{\text{nat.}}(\tau)d\tau = \mathbf{x}(0)$$

Observability Gramian

If the **Observability Gramian**:

$$\mathbf{W}_o(t) = \int_0^{t} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}d\tau$$

is nonsingular for any $t > 0$, then the state equation is observable.
-**The initial conditions can be recovered from the output**

Observability

Observability Gramian

Otherwise the Gramian:

$$\exists\, \mathbf{v} : \quad \mathbf{v}^\mathsf{T}\mathbf{W}_o(t)\mathbf{v} = \int_0^t \mathbf{v}^\mathsf{T} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\mathbf{v}\, d\tau = \int_0^t \left\| \mathbf{C}e^{\mathbf{A}\tau}\mathbf{v} \right\|^2 d\tau = 0$$

is **singular**.
-Then the state equation is **unobservable**.

$\mathbf{x}(0)$ cannot be identified uniquely given $\mathbf{y}_{\text{nat.}}(t)$

$$\int_0^{t_1} e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{y}_{\text{nat.}}(\tau) d\tau = \mathbf{W}_o(t_1)[\mathbf{x}(0) + \mathbf{v}]$$

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Duality

Observability Gramian

If the **Observability Gramian**:

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau$$

is nonsingular for any $t > 0$, then the state equation is **observable**.

Controllability Gramian

If the **Controllability Gramian**:

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\mathsf{T}e^{\mathbf{A}^\mathsf{T}\tau}\,d\tau$$

is nonsingular for any $t > 0$, then the state equation is **controllable**.

Duality

System Gramians

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau \quad \mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\mathsf{T}e^{\mathbf{A}^\mathsf{T}\tau}\,d\tau$$

Imagine a system:

$$\dot{\mathbf{z}} = \mathbf{A}^\mathsf{T}\mathbf{z} + \mathbf{C}^\mathsf{T}\mathbf{u}$$

What is the controllability gramian?

$$\mathbf{W}_c^z(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau$$

Duality

System Gramians

$$\underline{\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau} \quad \mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^\mathsf{T}e^{\mathbf{A}^\mathsf{T}\tau}\,d\tau$$

Dual system:

$$\dot{\mathbf{z}} = \mathbf{A}^\mathsf{T}\mathbf{z} + \mathbf{C}^\mathsf{T}\mathbf{u}$$

Implication?

$$\underline{\mathbf{W}_c^z(t) = \int_0^t e^{\mathbf{A}^\mathsf{T}\tau}\mathbf{C}^\mathsf{T}\mathbf{C}e^{\mathbf{A}\tau}\,d\tau}$$

Theorem of duality

The pair $\{\mathbf{A}, \mathbf{C}\}$ is observable if and only if
the pair $\{\mathbf{A}^\mathsf{T}, \mathbf{C}^\mathsf{T}\}$ is controllable.

Duality

The pair $\{\mathbf{A}, \mathbf{B}\}$ is controllable iff:

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^{\mathsf{T}} e^{\mathbf{A}^{\mathsf{T}}\tau} d\tau$$

is nonsingular for all $t > 0$.

The **controllability matrix**:

$$\mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$$

has full *row rank*.

The pair $\{\mathbf{A}, \mathbf{C}\}$ is observable iff:

$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^{\mathsf{T}}\tau}\mathbf{C}^{\mathsf{T}}\mathbf{C} e^{\mathbf{A}\tau} d\tau$$

is nonsingular for all $t > 0$.

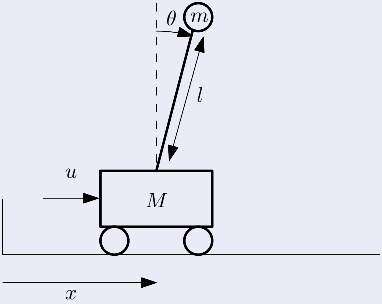
The **observability matrix**:

$$\mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}$$

has full *column rank*.

Observability

Example



Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Observability

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Case I

The *angle* is measured:

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 15 & 0 & 0 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

Observability

Linearized EOM: $M = 2kg$, $m = 1kg$, $l = 1m$, $g = 10\frac{m}{s^2}$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -5 & 0 & 0 \\ 0 & 15 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} u$$

Case II

The *position* is measured:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}$$