TTK4115

Lecture 10

Random state space systems

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Topic

Deterministic state-space model

Assume that $\mathbf{x}_0$ and $\mathbf{u}(t)$ are *known*. Then, the state space model given below is a deterministic process

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x}(0) = \mathbf{x}_0$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

It is in fact straightforward to compute the deterministic solution which is given by

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\, d\tau, \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

Problem

What happens if

- $\mathbf{u}(t)$ is unknown and random? (Denoted $\mathbb{u}(t)$)
- $\mathbf{x}_0$ is unknown and random? (Denoted $\mathbb{x}_0$)

Then it follows that $\mathbf{y}(t)$ and $\mathbf{x}(t)$ must also be random and unknown! Here denoted by the symbols $\mathbb{y}(t)$ and $\mathbb{x}(t)$.

Uncertain state-space model

The state space model given below describes a *random* process

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbb{u}, \quad \mathbb{x}(0) = \mathbb{x}_0$$
$$\mathbb{y} = \mathbf{C}\mathbb{x}$$

The *uncertain* solution follows from

$$\mathbb{x}(t) = e^{\mathbf{A}t}\mathbb{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbb{u}(\tau)\, d\tau, \quad \mathbb{y}(t) = \mathbf{C}\mathbb{x}(t)$$

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*Rather than attempting to find out what **will** happen, it is possible to find out what is **likely** to happen.*

What can one *expect* from $\mathbb{x}(t)$, $\mathbb{x}_0$, $\mathbb{y}(t)$, $\mathbb{u}(t)$?

The expectation operator[1] E can be used to identify a series of important quantities at each time t.

Mean : $m_x(t) = \mathrm{E}[\mathbb{x}(t)]$

Variance : $\mathrm{var}[\mathbb{x}(t)] = \mathrm{E}[(\mathbb{x}(t) - m_x(t))^2]$

Covariance : $\mathrm{cov}[\mathbb{x}_1(t), \mathbb{x}_2(t)] = \mathrm{E}[(\mathbb{x}_1(t) - m_{x_1}(t))(\mathbb{x}_2(t) - m_{x_2}(t))^\mathsf{T}]$

[1] A *linear* operator satisfying $\mathrm{E}[\mathbb{x} + c] = \mathrm{E}[\mathbb{x}] + c$, $\mathrm{E}[\mathbb{x}_1 + \mathbb{x}_2] = \mathrm{E}[\mathbb{x}_1] + \mathrm{E}[\mathbb{x}_2]$, $\mathrm{E}[a\mathbb{x}] = a\mathrm{E}[\mathbb{x}]$.

Uncertain state-space model

$$\mathbb{x}(t) = e^{\mathbf{A}t}\mathbb{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbb{u}(\tau)\, d\tau, \quad \mathbb{y}(t) = \mathbf{C}\mathbb{x}(t)$$

Mean of random process

Where to *expect* $\mathbb{x}(t)$ is found in the following manner

$$\begin{aligned}\mathbf{m}_{\mathbf{x}}(t) \triangleq \mathrm{E}[\mathbb{x}(t)] = \mathrm{E}\left[e^{\mathbf{A}t}\mathbb{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbb{u}(\tau)\, d\tau, \quad \mathbb{y}(t) = \mathbf{C}\mathbb{x}(t)\right] \\ = e^{\mathbf{A}t}\mathrm{E}[\mathbb{x}_0] + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathrm{E}[\mathbb{u}(\tau)]\, d\tau, \quad \mathrm{E}[\mathbb{y}(t)] = \mathbf{C}\mathrm{E}[\mathbb{x}(t)]\end{aligned}$$

Model for the mean

Differentiating on both sides produces a simple model for the mean

$$\begin{aligned}\dot{\mathbf{m}}_{\mathbf{x}}(t) = \mathbf{A}\left[e^{\mathbf{A}t}\mathrm{E}[\mathbb{x}_0] + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{m}_{\mathbf{u}}(\tau)\, d\tau\right] + \mathbf{B}\mathbf{m}_{\mathbf{u}}(t) \\ = \mathbf{A}\mathbf{m}_{\mathbf{x}}(t) + \mathbf{B}\mathbf{m}_{\mathbf{u}}(t), \quad \mathbf{m}_{\mathbf{y}}(t) = \mathbf{C}\mathbf{m}_{\mathbf{x}}(t)\end{aligned}$$

Here, $\mathbf{m}_{\mathbf{u}}(t) \triangleq \mathrm{E}[\mathbb{u}(t)]$ and $\mathbf{m}_{\mathbf{x}_0} \triangleq \mathrm{E}[\mathbb{x}_0]$, whilst $\mathbf{m}_{\mathbf{y}}(t) \triangleq \mathrm{E}[\mathbb{y}(t)]$.

Uncertain state-space model

The state space model given below describes a *random* process

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbb{u}, \quad \mathbb{x}(0) = \mathbb{x}_0$$
$$\mathbb{y} = \mathbf{C}\mathbb{x}$$

Mean of random process

Its **mean** (expected value) follows from using the expectancy operator on the preceding equation

$$\dot{\mathbf{m}}_{\mathbf{x}} = \mathbf{A}\mathbf{m}_{\mathbf{x}} + \mathbf{B}\mathbf{m}_{\mathbf{u}}, \quad \mathbf{m}_{\mathbf{x}}(0) = \mathbf{m}_{\mathbf{x}_0}$$
$$\mathbf{m}_{\mathbf{y}} = \mathbf{C}\mathbf{m}_{\mathbf{x}}$$

This result implies that deterministic models are found in the limit $\text{var}[\mathbb{x}] \to \mathbf{0}$.

Covariance & variance of random process

The *variance* provides a measure of the spread of the variable in question whereas the *covariance* measures the relation between two random variables. It is customary to collect this information in a *covariance matrix*.

A vector $\mathbb{x}(t) \in \mathbb{R}^n$ is thus equipped with the covariance matrix

$$\mathcal{C}_{\mathbf{x}}(t) \triangleq \mathrm{E}\begin{bmatrix} (\mathbb{x}_1 - m_{x_1})(\mathbb{x}_1 - m_{x_1}) & (\mathbb{x}_1 - m_{x_1})(\mathbb{x}_2 - m_{x_2}) & \cdots & (\mathbb{x}_1 - m_{x_1})(\mathbb{x}_n - m_{x_n}) \\ (\mathbb{x}_2 - m_{x_2})(\mathbb{x}_1 - m_{x_1}) & (\mathbb{x}_2 - m_{x_2})(\mathbb{x}_2 - m_{x_2}) & \cdots & (\mathbb{x}_2 - m_{x_2})(\mathbb{x}_n - m_{x_n}) \\ \dots & \dots & \ddots & \vdots \\ (\mathbb{x}_n - m_{x_n})(\mathbb{x}_1 - m_{x_1}) & (\mathbb{x}_n - m_{x_n})(\mathbb{x}_2 - m_{x_2}) & \cdots & (\mathbb{x}_n - m_{x_n})(\mathbb{x}_n - m_{x_n}) \end{bmatrix}$$

A compact vectorial representation is given by

$$\mathcal{C}_{\mathbf{x}}(t) = \mathrm{E}[(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))^{\mathsf{T}}]$$

Note

- Variances are located on the *diagonal*.
- The covariance matrix is symmetric.

Uncertain state-space model about the *mean*

$$\begin{aligned}\dot{\mathbb{x}} - \dot{\mathbf{m}}_{\mathbf{x}} &= \mathbf{A}(\mathbb{x} - \mathbf{m}_{\mathbf{x}}) + \mathbf{B}(\mathbb{u} - \mathbf{m}_{\mathbf{u}}) \\ \mathbb{y} - \mathbf{m}_{\mathbf{y}} &= \mathbf{C}(\mathbb{x} - \mathbf{m}_{\mathbf{x}})\end{aligned}$$

Covariance & variance of random process

The *variance* provides a measure of the spread of the variable in question whereas the *covariance* measures the relation between two random variables. It is customary to collect this information in a *covariance matrix*. A compact vectorial representation is given by

$$\mathcal{C}_{\mathbf{x}}(t) = \mathrm{E}[(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))^{\mathsf{T}}]$$

Computation

Direct differentiation yields a *covariance update equation*, viz.

$$\begin{aligned}\dot{\mathcal{C}}_{\mathbf{x}} &= \mathrm{E}[(\dot{\mathbb{x}} - \dot{\mathbf{m}}_{\mathbf{x}})(\mathbb{x} - \mathbf{m}_{\mathbf{x}})^{\mathsf{T}}] + \mathrm{E}[(\mathbb{x} - \mathbf{m}_{\mathbf{x}})(\dot{\mathbb{x}} - \dot{\mathbf{m}}_{\mathbf{x}})^{\mathsf{T}}] \\ &= \mathrm{E}[(\mathbf{A}(\mathbb{x} - \mathbf{m}_{\mathbf{x}}) + \mathbf{B}(\mathbb{u} - \mathbf{m}_{\mathbf{u}}))(\mathbb{x} - \mathbf{m}_{\mathbf{x}})^{\mathsf{T}}] + \mathrm{E}[(\mathbb{x} - \mathbf{m}_{\mathbf{x}})(\mathbf{A}(\mathbb{x} - \mathbf{m}_{\mathbf{x}}) + \mathbf{B}(\mathbb{u} - \mathbf{m}_{\mathbf{u}}))^{\mathsf{T}}] \\ &= \mathbf{A}\mathcal{C}_{\mathbf{x}} + \mathcal{C}_{\mathbf{x}}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathrm{E}[(\mathbb{u} - \mathbf{m}_{\mathbf{u}})(\mathbb{x} - \mathbf{m}_{\mathbf{x}})^{\mathsf{T}}] + \mathrm{E}[(\mathbb{x} - \mathbf{m}_{\mathbf{x}})(\mathbb{u} - \mathbf{m}_{\mathbf{u}})^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}}\end{aligned}$$

But, what is the covariance between $\mathbb{x}$ and $\mathbb{u}$?

Uncertain state-space model about the *mean*

$$\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t) = e^{\mathbf{A}t}(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0}) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}(\mathbb{u}(\tau) - \mathbf{m}_{\mathbf{u}}(\tau))\, d\tau$$

Covariance computation

$$\begin{aligned}
&\mathrm{E}[(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}} \\
&= \mathrm{E}[e^{\mathbf{A}t}(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}] + \mathrm{E}\left[\left(\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}(\mathbb{u}(\tau) - \mathbf{m}_{\mathbf{u}}(\tau))\, d\tau\right)(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}\mathbf{B}^{\mathsf{T}}\right] \\
&= e^{\mathbf{A}t}\mathrm{E}[(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}} + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathrm{E}[(\mathbb{u}(\tau) - \mathbf{m}_{\mathbf{u}}(\tau))(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}}\, d\tau
\end{aligned}$$

Covariance computation

$$\begin{aligned}&\mathrm{E}[(\mathbb{x}(t) - \mathbf{m_x}(t))(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}]\mathbf{B}^\mathsf{T} \\ &= e^{\mathbf{A}t}\mathrm{E}[(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}]\mathbf{B}^\mathsf{T} + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathrm{E}[(\mathbb{u}(\tau) - \mathbf{m_u}(\tau))(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}]\mathbf{B}^\mathsf{T}\, d\tau\end{aligned}$$

Causality

The input given in the interval $[0, t)$ cannot affect the initial conditions at $t = 0$ by having impacts *backwards* in time. Arguing from causality, one can assume

$$\mathrm{E}[(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}] = \mathbf{0}$$

Autocovariance

The factor $\mathrm{E}[(\mathbb{u}(\tau) - \mathbf{m_u}(\tau))(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}]$ from the expression above is by definition the autocovariance of the input signal $\mathbb{u}(t)$:

$$\mathcal{A}_\mathbf{u}(t, \tau) \triangleq \mathrm{E}[(\mathbb{u}(\tau) - \mathbf{m_u}(\tau))(\mathbb{u}(t) - \mathbf{m_u}(t))^\mathsf{T}]$$

White noise

White noise is a theoretical signal that is *completely uncorrelated* to itself over time. The autocovariance of white noise is:

$$\mathcal{A}_n(t,\tau) = \delta(t-\tau)q_n(\tau)$$

where $\delta(t)$ represents Dirac's function and $q_n(t) > 0$.
Knowing the white noise $\mathbb{n}(t)$ at the instant t_1 does not inform us in any way whatsoever about its value at time t_2:

$$\mathcal{A}_n(t,\tau) = \mathrm{E}[(\mathbb{n}(t) - m_n(t))(\mathbb{n}(\tau) - m_n(\tau))] = 0, \quad t \neq \tau$$

At $\tau = t$, the autocovariance reduces to a simple *variance*. This variance is given by

$$\mathcal{A}_n(t,t) = \mathrm{E}[(\mathbb{n}(t) - m_n(t))^2] = \delta(0)q_n(t), \quad t = \tau$$

Remember

White noise is a theoretical construct aimed at simplifying analysis and modeling
- No physical signal has infinite variance.

Autocovariance with $\mathbb{u}$ modeled as **white noise**.

For the random input used in the present process we have:

$$\mathcal{A}_{\mathbf{u}}(t,\tau) \triangleq \mathrm{E}[(\mathbb{u}(\tau) - \mathbf{m}_{\mathbf{u}}(\tau))(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]$$

Assuming that $\mathbb{u}(t)$ represents white noise permits the simplification

$$\mathcal{A}_{\mathbf{u}}(t,\tau) = \delta(t-\tau)\mathbf{Q}_{\mathbf{u}}(\tau), \quad \mathbf{Q}_{\mathbf{u}} \succ \mathbf{0}$$

Covariance computation

The particular properties of white noise permit significant simplifications to the analysis:

$$\begin{aligned}\mathrm{E}[(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}} &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathcal{A}_{\mathbf{u}}(t,\tau)\mathbf{B}^{\mathsf{T}}\, d\tau \\ &= \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(t-\tau)\mathbf{Q}_{\mathbf{u}}(\tau)\mathbf{B}^{\mathsf{T}}\, d\tau\end{aligned}$$

We use the half-maximum convention on Heaviside's function $\Theta(0) = 1/2$ to arrive at:

$$\mathrm{E}[(\mathbb{x}(t) - \mathbf{m}_{\mathbf{x}}(t))(\mathbb{u}(t) - \mathbf{m}_{\mathbf{u}}(t))^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}} = \int_0^\infty \Theta(t-\tau)e^{\mathbf{A}(t-\tau)}\mathbf{B}\delta(t-\tau)\mathbf{Q}_{\mathbf{u}}(\tau)\mathbf{B}^{\mathsf{T}}\, d\tau = \frac{1}{2}\mathbf{B}\mathbf{Q}_{\mathbf{u}}(t)\mathbf{B}^{\mathsf{T}}$$

Covariance computation

We previously found the following expression for the covariance update equation:

$$\dot{\mathcal{C}}_{\mathbf{x}} = \mathbf{A}\mathcal{C}_{\mathbf{x}} + \mathcal{C}_{\mathbf{x}}\mathbf{A}^{\mathsf{T}} + \mathbf{B}\mathrm{E}[(\mathbb{u} - \mathbf{m_u})(\mathbb{x} - \mathbf{m_x})^{\mathsf{T}}] + \mathrm{E}[(\mathbb{x} - \mathbf{m_x})(\mathbb{u} - \mathbf{m_u})^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}}$$

We then found that:

$$\mathrm{E}[(\mathbb{x} - \mathbf{m_x})(\mathbb{u} - \mathbf{m_u})^{\mathsf{T}}]\mathbf{B}^{\mathsf{T}} = \frac{1}{2}\mathbf{B}\mathbf{Q_u}\mathbf{B}^{\mathsf{T}}$$

Since $\mathbf{Q_u}$ is symmetric, it follows that:

$$\mathbf{B}\mathrm{E}[(\mathbb{u} - \mathbf{m_u})(\mathbb{x} - \mathbf{m_x})^{\mathsf{T}}] = \left(\frac{1}{2}\mathbf{B}\mathbf{Q_u}\mathbf{B}^{\mathsf{T}}\right)^{T} = \frac{1}{2}\mathbf{B}\mathbf{Q_u}\mathbf{B}^{\mathsf{T}}$$

Covariance update equation

$$\dot{\mathcal{C}}_{\mathbf{x}} - \mathbf{A}\mathcal{C}_{\mathbf{x}} - \mathcal{C}_{\mathbf{x}}\mathbf{A}^{\mathsf{T}} = \mathbf{B}\mathbf{Q_u}(t)\mathbf{B}^{\mathsf{T}}$$

It will be assumed in the following that $\mathbf{Q_u}$ is a constant matrix, although this need not be the case.

Uncertain state-space model

It is in fact possible to say quite a lot about what to *expect* from the random process given below, even though both $\mathbb{u}$ and $\mathbb{x}_0$ are *random*.

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbb{u}, \quad \mathbb{y} = \mathbf{C}\mathbb{x}$$

Results

The following quantities are assumed *known*.

Means : $\mathrm{E}[\mathbb{u}(t)] = \mathbf{m_u}(t)$ and $\mathrm{E}[\mathbb{x}_0] = \mathbf{m}_{\mathbf{x}_0}$.

Covariances : $\mathrm{E}[(\mathbb{u} - \mathbf{m_u})(\mathbb{u} - \mathbf{m_u})^\mathsf{T}] = \delta(0)\mathbf{Q_u}$ and $\mathrm{E}[(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})(\mathbb{x}_0 - \mathbf{m}_{\mathbf{x}_0})^\mathsf{T}] = \mathcal{C}_\mathbf{x}(0)$.

Adopting the assumption that $\mathbb{u}(t)$ is well represented by white noise informs us what to *expect* from the uncertain model. Verify, and note **linearity**, of the following.

The **means** are given by

$$\begin{aligned} \dot{\mathbf{m}}_\mathbf{x} &= \mathbf{A}\mathbf{m_x} + \mathbf{B}\mathbf{m_u}, \quad \mathbf{m_x}(0) = \mathbf{m}_{\mathbf{x}_0} \\ \mathbf{m_y} &= \mathbf{C}\mathbf{m_x} \end{aligned}$$

The **covariance matrices** follow from

$$\begin{aligned} \dot{\mathcal{C}}_\mathbf{x} &= \mathbf{A}\mathcal{C}_\mathbf{x} + \mathcal{C}_\mathbf{x}\mathbf{A}^\mathsf{T} + \mathbf{B}\mathbf{Q_u}\mathbf{B}^\mathsf{T}, \quad \mathcal{C}_\mathbf{x}(0) = \mathcal{C}_{\mathbf{x}_0} \\ \mathcal{C}_\mathbf{y} &= \mathbf{C}\mathcal{C}_\mathbf{x}\mathbf{C}^\mathsf{T} \end{aligned}$$

Here $\mathcal{C}_\mathbf{y} \triangleq \mathrm{E}[(\mathbb{y} - \mathbf{m_y})(\mathbb{y} - \mathbf{m_y})^\mathsf{T}] = \mathbf{C}\mathrm{E}[(\mathbb{x} - \mathbf{m_x})(\mathbb{x} - \mathbf{m_x})^\mathsf{T}]\mathbf{C}^\mathsf{T} = \mathbf{C}\mathcal{C}_\mathbf{x}\mathbf{C}^\mathsf{T}$.

Topic

1. Random state space systems

2. Optimal estimation preview

Physical model

Let a general plant model be given by a random process

$$\dot{\mathbb{x}} = \mathbf{A}\mathbb{x} + \mathbf{B}\mathbf{u} + \mathbf{G}\mathbb{w}, \quad \mathbb{y} = \mathbf{C}\mathbb{x} + \mathbb{v}$$

The random signals giving rise to the uncertainty are

Noise $\mathbb{v}$: represented by a zero mean white Gaussian signal with autocovariance/autocorrelation $\mathcal{A}_{\mathbf{v}}(t,\tau) = \mathrm{E}[\mathbb{v}(t)\mathbb{v}(\tau)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{R}_{\mathbf{v}}$.

Disturbance $\mathbb{w}$: represented by a zero mean white Gaussian signal with autocovariance/autocorrelation $\mathcal{A}_{\mathbf{w}}(t,\tau) = \mathrm{E}[\mathbb{w}(\tau)\mathbb{w}(t)^{\mathsf{T}}] = \delta(t-\tau)\mathbf{Q}_{\mathbf{w}}$

The noise and disturbance are assumed to be *uncorrelated* implying that $\mathcal{A}_{\mathbf{vw}}(t,\tau) = \mathrm{E}[\mathbb{v}(t)\mathbb{w}(\tau)^{\mathsf{T}}] \equiv \mathbf{0}$.

Luenberger observer

It will be of interest to perform estimation on the random process representing the plant. Let a *Luenberger observer* be given by

$$\dot{\hat{\mathbb{x}}} = \mathbf{A}\hat{\mathbb{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(t)(\mathbb{y} - \mathbf{C}\hat{\mathbb{x}})$$

Note that the estimate is not deterministic since it is perturbed by the random process $\mathbb{y}$. We let $\mathbf{L}(t)$ be undetermined for now.

Dynamics of the estimation error

The random estimation error is defined by $\mathbb{e} = \mathbb{x} - \hat{\mathbb{x}}$. Verify that

$$\dot{\mathbb{e}} = (\mathbf{A} - \mathbf{L}(t)\mathbf{C})\mathbb{e} + \mathbf{G}\mathbb{w} - \mathbf{L}\mathbb{v}$$

Unbiased estimation

At $t = 0$ the observer is initialized at the *mean* of the true state vector so that $\hat{\mathbf{x}}_0 = \mathrm{E}[\mathbb{x}_0]$. Taking expectations, noting the unbiased noise and disturbance, shows that no mean error is committed

$$\dot{\mathbf{m}}_{\mathbf{e}} = (\mathbf{A} - \mathbf{L}(t)\mathbf{C})\mathbf{m}_{\mathbf{e}}, \quad \mathbf{m}_{\mathbf{e}}(0) = \mathrm{E}[\mathbb{x}_0] - \hat{\mathbf{x}}_0 = \mathbf{0} \quad \Rightarrow \mathbf{m}_{\mathbf{e}}(t) = \mathbf{0}$$

This result implies that the estimate is *unbiased*.

Covariance dynamics

The covariance matrix for the estimation error is equipped with the special notation

$$\mathbf{P}(t) \triangleq \mathrm{E}[\mathbb{e}(t)\mathbb{e}(t)^{\mathsf{T}}]$$

The matrix **P** quantifies the uncertainty in the estimate; low variances (found along the diagonal) imply good estimates!

The Kalman filter gives the gain matrix **K** that reduces the uncertainty in the estimate, tr(**P**), at the fastest rate.