TTK4115

Lecture 2

Equivalent representations, useful forms, functions of square matrices

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This lecture

1. Equivalent Representations

2. Diagonalization

3. Recovering the Diagonal and Jordan Forms

4. Complex eigenvalues: Modal Form

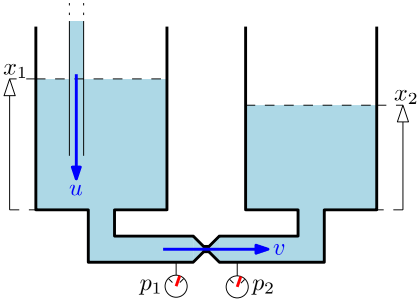
5. Physical significance of Eigenvalues/vectors

6. Functions of a Square Matrix

   Matrix Exponentials - Special Properties

Topic

Example: Tank system



Hydraulic model

The flow between the tanks is assumed proportional to the pressure differential over the constriction. The hydraulic head is $p = \rho g x$. Then:

$$v(t) = k[p_2(t) - p_1(t)] = k\rho g[x_1(t) - x_2(t)]$$

Dynamics

Tank 1 balance:

$$S\dot{x}_1 = u - v = u - k\rho g[x_1 - x_2]$$

Tank 2 balance:

$$S\dot{x}_2 = v = k\rho g[x_1 - x_2]$$

Output: Averaged tank level

$$y = \frac{1}{2}[x_1 + x_2]$$

ρ : Density of fluid
S : Tank cross-section
k : Constriction constant
g : Gravitational constant

Tank system state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t), \quad y(t) = \mathbf{c}\mathbf{x}(t)$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} -\frac{k\rho g}{S} & \frac{k\rho g}{S} \\ \frac{k\rho g}{S} & -\frac{k\rho g}{S} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} \frac{1}{S} \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Change of basis

Now define two *alternative* states:

$$\bar{x}_1(t) \triangleq \underbrace{\frac{1}{2}[x_1(t) + x_2(t)]}_{\text{Average}}, \quad \bar{x}_2(t) \triangleq \underbrace{[x_1(t) - x_2(t)]}_{\text{Difference}}$$

Transformation matrix

A transformation matrix **T** relates the two state vectors:

$$\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \bar{\mathbf{x}} = \mathbf{T}\mathbf{x}$$

System transformation

The system dynamics may be expressed in terms of the new states, if the transformation matrix is *invertible*:

$$\bar{\mathbf{x}} = \mathbf{T}\mathbf{x}, \quad \mathbf{x} = \mathbf{T}^{-1}\bar{\mathbf{x}}$$

Equivalence transformation

$$\begin{aligned}
\dot{\bar{\mathbf{x}}} &= \overbrace{\mathbf{T}\mathbf{A}\mathbf{T}^{-1}}^{\bar{\mathbf{A}}}\bar{\mathbf{x}} + \overbrace{\mathbf{T}\mathbf{B}}^{\bar{\mathbf{B}}}\mathbf{u} \\
\mathbf{y} &= \underbrace{\mathbf{C}\mathbf{T}^{-1}}_{\bar{\mathbf{C}}}\bar{\mathbf{x}} + \underbrace{\mathbf{D}}_{\bar{\mathbf{D}}}\mathbf{u}
\end{aligned}$$

Algebraic equivalence

If we can find an invertible matrix **T** that relate the two systems:

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{B}}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \quad \mathbf{y} = \bar{\mathbf{C}}\bar{\mathbf{x}} + \bar{\mathbf{D}}\mathbf{u}
\end{aligned}$$

they are **algebraically equivalent**.

Tank example, cont.

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1 & -1 \end{bmatrix}}^{\mathbf{T}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}}^{\mathbf{T}^{-1}} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Transform of **A**:

$$\bar{\mathbf{A}} \triangleq \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -\frac{k\rho g}{S} & \frac{k\rho g}{S} \\ \frac{k\rho g}{S} & -\frac{k\rho g}{S} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2gk\rho}{S} \end{bmatrix}$$

Transform of **B**:

$$\bar{\mathbf{B}} \triangleq \mathbf{T}\mathbf{B} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{S} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2S} \\ \frac{1}{S} \end{bmatrix}$$

Transform of **C**:

$$\bar{\mathbf{C}} \triangleq \mathbf{C}\mathbf{T}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Algebraic equivalence

Below are two **equivalent** tank models, that represent the **same dynamics**.

Representation 1 - **Original**

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} -\frac{k\rho g}{S} & \frac{k\rho g}{S} \\ \frac{k\rho g}{S} & -\frac{k\rho g}{S} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{S} \\ 0 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\end{aligned}
$$

Representation 2 - **Transformed**

$$
\begin{aligned}
\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2gk\rho}{S} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{2S} \\ \frac{1}{S} \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}
\end{aligned}
$$

Change of state variables

Key points

- We converted the original state variables to a *linear combination* of an alternative set of state variables: $\mathbf{x} = \mathbf{T}^{-1}\bar{\mathbf{x}}$.
- A new *basis* $\mathbf{T} = \begin{bmatrix} \mathbf{t}_1 & \mathbf{t}_2 \end{bmatrix}$ is used to represent the system.
- The transformation $\bar{\mathbf{A}} = \mathbf{TAT}^{-1}$ is called an **equivalence/similarity** transformation.
- The choice of $\mathbf{T}$ is *not unique*. Some choices may be better than others, depending on the application.

Transfer function invariance

State equation

$$\dot{\bar{\mathbf{x}}} = \overbrace{\mathbf{TAT}^{-1}}^{\bar{\mathbf{A}}}\bar{\mathbf{x}} + \overbrace{\mathbf{TB}}^{\bar{\mathbf{B}}}\mathbf{u}, \quad \mathbf{y} = \overbrace{\mathbf{CT}^{-1}}^{\bar{\mathbf{C}}}\bar{\mathbf{x}} + \mathbf{Du}$$

Laplace transform of equivalent system

The transfer matrix is invariant under a similarity transformation:

$$\hat{\mathbf{G}}(s) = \bar{\mathbf{C}}(s\mathbb{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}} + \mathbf{D} = \mathbf{CT}^{-1}(s\mathbb{I} - \mathbf{TAT}^{-1})^{-1}\mathbf{TB} + \mathbf{D} = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Since poles and zeros are encoded in $\hat{\mathbf{G}}(s)$, these are invariant also.

Zero state equivalence

Tank example: representation 2

$$\begin{bmatrix} \dot{\bar{x}}_1(t) \\ \dot{\bar{x}}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2gk\rho}{S} \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{2S} \\ \frac{1}{S} \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix}$$

Note that the two states are <u>decoupled</u> and cannot affect each other.

Tank example: representation 3

Remove the unmeasured state $x_2(t)$ to obtain a reduced order model:

$$\dot{\bar{x}}_1(t) = \frac{1}{2S} u(t), \quad y(t) = \bar{x}_1(t)$$

Zero-state equivalence

<u>Claim</u>: Representations 1-3 all have the same transfer function $g(s)$. They are **zero-state equivalent**.

Zero state equivalence

Transfer function for Representation 1/2[1]

$$g(s) = \mathbf{c}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s + \frac{2gk\rho}{S} \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{1}{2S} \\ \frac{1}{S} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s} & 0 \\ 0 & \frac{1}{s+\frac{2gk\rho}{S}} \end{bmatrix} \begin{bmatrix} \frac{1}{2S} \\ \frac{1}{S} \end{bmatrix} = \frac{1}{2Ss}$$

Transfer function for Representation 3

$$g(s) = \frac{1}{2Ss}$$

[1] Recall that the transfer function is invariant to a similarity transformation.

Zero state equivalence

Zero-state equivalence

If the system:

$$\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$$

has the same transfer function as the system:

$$\{\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}}\}$$

they are *zero-state equivalent*.

Caution

- Algebraic equivalence $\Rightarrow$ Zero-state equivalence
- Zero-state equivalence $\not\Rightarrow$ Algebraic equivalence

Topic

Recall the definition of **eigenvalues** and **eigenvectors**:

$$\mathbf{Aq} = \lambda \mathbf{q}$$

These are very important in dynamics:

Consider the case where the state coincides with an eigenvector at some time:

$$\mathbf{x}(t) = \mathbf{q}\alpha(t), \quad t = 0$$

where $\alpha(t)$ scales the *constant*[a] eigenvector. Assume $\mathbf{u}(t) = \mathbf{0},\ t > 0$, so that:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{Ax}(t) \\ \Rightarrow \mathbf{q}\dot{\alpha}(t) &= \mathbf{Aq}\alpha(t) \\ \Rightarrow \mathbf{q}\dot{\alpha}(t) &= \lambda\mathbf{q}\alpha(t) \\ \Rightarrow \dot{\alpha}(t) &= \lambda\alpha(t) \end{aligned}$$

Solutions along an eigenvector stays along the eigenvector. Solving this problem is simple:

$$\alpha(t) = e^{\lambda t}\alpha(0) \quad \Rightarrow \mathbf{x}(t) = \mathbf{q}e^{\lambda t}\alpha(0)$$

..only the scalar factor changes in time.

[a] Eigenvectors are often normalized so that: $\mathbf{q}^T\mathbf{q} = 1$.

Diagonalization

Generalization

Consider next the case where $\mathbf{x}(t) \in \mathbb{R}^n$ coincides with a *linear combination* of n eigenvectors[a]:

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{q}_1\alpha_1(t) + \mathbf{q}_2\alpha_2(t) + \ldots + \mathbf{q}_n\alpha_n(t) \\
&= \overbrace{\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \ldots & \mathbf{q}_n \end{bmatrix}}^{\mathbf{Q}} \underbrace{\begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix}}_{\boldsymbol{\alpha}(t)} = \mathbf{Q}\boldsymbol{\alpha}(t)
\end{aligned}$$

[a] We have n linearly independent eigenvectors and n eigenvalues in cases where $\mathbf{A}$ is *semisimple*. This is most often the case.

Diagonalization: $\mathbf{x}(t) = \mathbf{Q}\alpha(t)$

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) \\
\mathbf{Q}\dot{\alpha}(t) &= \begin{bmatrix} \mathbf{A}\mathbf{q}_1 & \mathbf{A}\mathbf{q}_2 & \dots & \mathbf{A}\mathbf{q}_n \end{bmatrix} \boldsymbol{\alpha}(t) \\
\mathbf{Q}\dot{\alpha}(t) &= \begin{bmatrix} \lambda_1\mathbf{q}_1 & \lambda_2\mathbf{q}_2 & \dots & \lambda_n\mathbf{q}_n \end{bmatrix} \boldsymbol{\alpha}(t) \\
\mathbf{Q}\dot{\alpha}(t) &= \overbrace{\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_n \end{bmatrix}}^{\mathbf{Q}} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\boldsymbol{\Lambda}} \boldsymbol{\alpha}(t)
\end{aligned}
$$

Finally: $\mathbf{Q}\dot{\alpha}(t) = \mathbf{Q}\boldsymbol{\Lambda}\alpha(t)$

$$
\Rightarrow \begin{bmatrix} \dot{\alpha}_1(t) \\ \dot{\alpha}_2(t) \\ \vdots \\ \dot{\alpha}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1\alpha_1(t) \\ \lambda_2\alpha_2(t) \\ \vdots \\ \lambda_n\alpha_n(t) \end{bmatrix}
$$

..this is far simpler than solving the system as is!

With: $\mathbf{Q}\dot{\alpha}(t) = \mathbf{Q}\mathbf{\Lambda}\alpha(t)$

$$\begin{bmatrix} \dot{\alpha}_1(t) \\ \dot{\alpha}_2(t) \\ \vdots \\ \dot{\alpha}_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} = \begin{bmatrix} \lambda_1\alpha_1(t) \\ \lambda_2\alpha_2(t) \\ \vdots \\ \lambda_n\alpha_n(t) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \\ \vdots \\ \alpha_n(t) \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t}\alpha_1(0) \\ e^{\lambda_2 t}\alpha_2(0) \\ \vdots \\ e^{\lambda_n t}\alpha_n(0) \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}}_{e^{\mathbf{\Lambda} t}} \begin{bmatrix} \alpha_1(0) \\ \alpha_2(0) \\ \vdots \\ \alpha_n(0) \end{bmatrix}$$

..we can solve large systems easily[a]:

$$\boldsymbol{\alpha}(t) = e^{\mathbf{\Lambda} t}\boldsymbol{\alpha}(0)$$

[a] This trick only works for *diagonal* matrices.

Finally with $\alpha(t) = e^{\Lambda t}\alpha(0)$:

Initial conditions are obtained as:

$$\mathbf{x}(0) = \mathbf{Q}\alpha(0) \quad \Rightarrow \mathbf{Q}^{-1}\mathbf{x}(0) = \alpha(0)$$

Hence:

$$\mathbf{x}(t) = \mathbf{Q}\alpha(t) = \mathbf{Q}e^{\Lambda t}\mathbf{Q}^{-1}\mathbf{x}(0)$$

Recall:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$$

which implies:

$$\mathbf{Q}e^{\Lambda t}\mathbf{Q}^{-1} \equiv e^{\mathbf{A}t}$$

Equivalence transform: $\mathbf{x} = \mathbf{Q}\bar{\mathbf{x}}$

The eigenvector matrix defines an equivalence transformation with $\mathbf{T} = \mathbf{Q}^{-1}$:

$$
\begin{aligned}
\dot{\bar{\mathbf{x}}} &= \overbrace{\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}}^{\mathbf{\Lambda}}\,\bar{\mathbf{x}} + \overbrace{\mathbf{B}\mathbf{Q}^{-1}}^{\bar{\mathbf{B}}}\,\mathbf{u} \\
\mathbf{y} &= \underbrace{\mathbf{C}\mathbf{Q}}_{\bar{\mathbf{C}}}\,\bar{\mathbf{x}} + \underbrace{\mathbf{D}}_{\bar{\mathbf{D}}}\,\mathbf{u}
\end{aligned}
$$

Diagonalization

If the transform above is possible, the system has been **diagonalized**. ($\mathbf{\Lambda}$ has elements only on the main diagonal.)

Solutions

Compute:

$$\mathbf{y}(t) = \bar{\mathbf{C}} e^{\mathbf{\Lambda} t} \bar{\mathbf{x}}_0 + \bar{\mathbf{C}} \int_0^t e^{\mathbf{\Lambda}(t-\tau)} \bar{\mathbf{B}} \mathbf{u}(\tau) d\tau + \bar{\mathbf{D}}(t)$$

with:

$$e^{\mathbf{\Lambda} t} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

Transfer functions

Compute

$$\mathbf{G}(s) = \bar{\mathbf{C}}(s\mathbb{I} - \mathbf{\Lambda})^{-1} \bar{\mathbf{B}} + \bar{\mathbf{D}}$$

with:

$$(s\mathbb{I} - \mathbf{\Lambda})^{-1} = \begin{bmatrix} \frac{1}{s-\lambda_1} & & & \\ & \frac{1}{s-\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{s-\lambda_n} \end{bmatrix}$$

The Jordan Form

The Jordan Form

- If there are repeated eigenvalues, some eigenvectors may also be repeated.
- Then **Q** will not have full rank, and no inverse exists.
- The Jordan form captures these cases, using *generalized eigenvectors*, to give an invertible **Q**

Topic

Diagonal & Jordan forms

Eigenvalues & Eigenvectors

Definition:

$$\mathbf{Aq} = \lambda\mathbf{q} \quad \Rightarrow (\lambda\mathbb{I} - \mathbf{A})\mathbf{q} = \mathbf{0}$$

Idea

1. If $(\lambda\mathbb{I} - \mathbf{A})$ has full rank, only $\mathbf{q} = \mathbf{0}$ is possible[a]
2. The determinant of a matrix with full rank is never zero: $\det(\mathbf{M}) \neq 0$.
3. ..so we search for eigenvalues that make $|\lambda\mathbb{I} - \mathbf{A}| = 0$.
4. This is done by solving the characteristic polynomial $\Delta(\lambda) = |\lambda\mathbb{I} - \mathbf{A}| = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n = 0$.
5. There are n solutions to the characteristic polynomial $\Delta(\lambda) = 0$, not necessarily distinct.
6. For each eigenvalue λ_i we identify the corresponding eigenvector $\mathbf{Aq}_i = \mathbf{q}_i\lambda_i$
7. ..by finding the *null space* of $(\lambda_i\mathbb{I} - \mathbf{A})$.

[a] This is known as the trivial solution.

Diagonal & Jordan forms

Eigenvalues & Eigenvectors

For λ_i, $i = 1 \ldots n$, we have:

$$\mathbf{A}\mathbf{q}_i = \lambda_i \mathbf{q}_i$$

with the associated eigenvectors $\mathbf{q}_i$.

We may also write: $\mathbf{AQ} = \mathbf{Q}\boldsymbol{\Lambda}$

$$\begin{bmatrix} \mathbf{A}\mathbf{q}_1 & \mathbf{A}\mathbf{q}_2 & \cdots & \mathbf{A}\mathbf{q}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{q}_1 & \lambda_2\mathbf{q}_2 & \cdots & \lambda_n\mathbf{q}_n \end{bmatrix}$$

$$\Rightarrow \mathbf{A}\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{q}_1 & \lambda_2\mathbf{q}_2 & \cdots & \lambda_n\mathbf{q}_n \end{bmatrix}$$

$$\Rightarrow \mathbf{A}\underbrace{\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{bmatrix}}_{\mathbf{Q}} = \underbrace{\begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{bmatrix}}_{\mathbf{Q}} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{\boldsymbol{\Lambda}}$$

Repeated eigenvalues

With:

$$\mathbf{AQ} = \mathbf{Q}\boldsymbol{\Lambda}$$

Our aim is to transform the original system to the form:

$$\begin{aligned} \overbrace{\mathbf{Q}^{-1}\mathbf{Q}}^{\mathbb{I}}\,\dot{\bar{\mathbf{x}}} &= \overbrace{\mathbf{Q}^{-1}\mathbf{Q}}^{\mathbb{I}}\,\boldsymbol{\Lambda}\bar{\mathbf{x}} + \mathbf{Q}^{-1}\mathbf{Bu} \\ \mathbf{y} &= \mathbf{CQ}\bar{\mathbf{x}} + \mathbf{Du} \end{aligned}$$

Q must be invertible to do this

This requires that $\mathbf{q}_1 \ldots \mathbf{q}_n$ are linearly independent. This is not always the case with repeated eigenvalues..

Repeated eigenvalues

Option 1

For a repeated eigenvalue λ_r: If $(\lambda_r \mathbb{I} - \mathbf{A})$ has *nullity*[a] larger than 1, we can find several linearly independent solutions to:

$$(\lambda_r \mathbb{I} - \mathbf{A})\mathbf{n} = \mathbf{0}$$

[a] $\text{null}(M) + \text{rank}(M) = n$

Distinct eigenvectors

If the nullity of $(\lambda_r \mathbb{I} - \mathbf{A})$ equals the number of repetitions of λ_r we can find distinct eigenvectors to form a full rank Q matrix:

$$\mathbf{A} \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{bmatrix}$$

Repeated eigenvalues

Option 2

If the nullity of $(\lambda_r \mathbb{I} - \mathbf{A})$ is *less* than the repetitions of the eigenvalue, we don't have sufficient distinct eigenvectors.
In this case we can use a *Jordan block*:

$$\mathbf{J}_1 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$

Jordan Block, application

$$\mathbf{A} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & & \\ & & \lambda_2 & \\ & & & \lambda_3 \end{bmatrix}$$

Modified equations

$$\Rightarrow \mathbf{A} \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{q}_2 & \mathbf{q}_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 \mathbf{v}_1 & \lambda_1 \mathbf{v}_2 + 1\mathbf{v}_1 & \lambda_2 \mathbf{q}_2 & \lambda_3 \mathbf{q}_3 \end{bmatrix}$$

with linear combinations of eigenvectors:

$$\Rightarrow \mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \quad \mathbf{A}\mathbf{v}_2 = \lambda_1 \mathbf{v}_2 + 1\mathbf{v}_1$$

Example: 4 Repeated eigenvalues

$$\mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

Nullity $(\lambda\mathbb{I} - \mathbf{A}) = 1$

Find the generalized eigenvectors

$$\begin{aligned}
\mathbf{A}\mathbf{v}_1 &= \lambda\mathbf{v}_1 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})\mathbf{v}_1 &= 0 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})^4\mathbf{v} &= 0 \\
\mathbf{A}\mathbf{v}_2 &= \lambda\mathbf{v}_2 + \mathbf{v}_1 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})\mathbf{v}_2 &= \mathbf{v}_1 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})^3\mathbf{v} &= \mathbf{v}_1 \\
\mathbf{A}\mathbf{v}_3 &= \lambda\mathbf{v}_3 + \mathbf{v}_2 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})\mathbf{v}_3 &= \mathbf{v}_2 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})^2\mathbf{v} &= \mathbf{v}_2 \\
\mathbf{A}\mathbf{v}_4 &= \lambda\mathbf{v}_4 + \mathbf{v}_3 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})\mathbf{v} &= \mathbf{v}_3 & \rightarrow (\mathbf{A} - \lambda\mathbb{I})^1\mathbf{v} &= \mathbf{v}_3
\end{aligned}$$

All the eigenvectors are seen to issue from a chain generated by $\mathbf{v}$.

Procedure

1. Find the multiplicity of the repeated eigenvalue $\lambda_i : n_i$.
2. Find the nullity N of $(\lambda_i \mathbb{I} - \mathbf{A})$.
3. Generate N linearly independent eigenvectors, $\mathbf{v}_k$, $k = 1 \ldots N$, from the null-space of $(\lambda_i \mathbb{I} - \mathbf{A})$.
4. We are left with $n_i - N$ eigenvectors to find.
5. Use the generalized eigenvector scheme to generate the remaining vectors: $(\mathbf{A} - \lambda_i \mathbb{I})\mathbf{v}_{k,2} = \mathbf{v}_k$.
6. ..$(\mathbf{A} - \lambda_i \mathbb{I})\mathbf{v}_{k,3} = \mathbf{v}_{k,2}$
7. You can choose which of $\mathbf{v}_k$ to use.
8. Associate chains of these generated vectors with Jordan blocks.

Jordan form

With

- $\mathbf{Q}$ consisting of eigenvectors and generalized eigenvectors
- $\mathbf{J}$ the matrix with Jordan blocks for repeated eigenvalues and distinct eigenvalues along the diagonal

The system can be transformed like this:

$$
\begin{aligned}
\dot{\bar{\mathbf{x}}} &= \mathbf{J}\bar{\mathbf{x}} + \mathbf{Q}^{-1}\mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}\mathbf{Q}\bar{\mathbf{x}} + \mathbf{D}\mathbf{u}
\end{aligned}
$$

Example

$$\mathbf{J} = \begin{bmatrix} \lambda_1 & & & & & & & & & & & \\ & \lambda_2 & 1 & 0 & 0 & & & & & & & \\ & & \lambda_2 & 1 & 0 & & & & & & & \\ & & & \lambda_2 & 1 & & & & & & & \\ & & & & \lambda_2 & & & & & & & \\ & & & & & \lambda_3 & 1 & 0 & & & & \\ & & & & & & \lambda_3 & 1 & & & & \\ & & & & & & & \lambda_3 & & & & \\ & & & & & & & & \lambda_3 & 1 & & \\ & & & & & & & & & \lambda_3 & & \\ & & & & & & & & & & \lambda_4 & 0 \\ & & & & & & & & & & & \lambda_4 \end{bmatrix}$$

Topic

Modal Form

Modal Form

The modal form is useful when we have pairs of complex conjugated eigenvalues. It allows us to deal with only real numbers, as opposed to the Jordan form.

Modal form

Example: Mass spring damper

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}^{\mathbf{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}} u$$

Characteristic equation

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix} \right|$$

$$= \left(\lambda + \frac{d + \sqrt{d^2 - 4km}}{2m} \right) \left(\lambda + \frac{d - \sqrt{d^2 - 4km}}{2m} \right)$$

Modal form

Example: Mass spring damper; $d = k = m = 1$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

Characteristic equation

$$\begin{aligned} & \left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \right| \\ = \quad & \left(\lambda + \frac{1 + \sqrt{1-4}}{2} \right) \left(\lambda + \frac{1 - \sqrt{1-4}}{2} \right) \\ = \quad & \left(\lambda + \frac{1}{2} + \frac{i\sqrt{3}}{2} \right) \left(\lambda + \frac{1}{2} - \frac{i\sqrt{3}}{2} \right) \end{aligned}$$

Eigenvalues & Eigenvectors

$$\lambda_{1,2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \quad \mathbf{Q} = \begin{bmatrix} \frac{1}{2}\left(-1 - i\sqrt{3}\right) & \frac{1}{2}\left(-1 + i\sqrt{3}\right) \\ 1 & 1 \end{bmatrix}$$

Similarity transform

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \mathbf{Q}^{-1} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \mathbf{Q} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \mathbf{Q}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{Q} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Result (with complex coefficients)

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{i\sqrt{3}}{2} & 0 \\ 0 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} + \frac{i}{2\sqrt{3}} \\ \frac{1}{2} - \frac{i}{2\sqrt{3}} \end{bmatrix} u$$

$$y = \begin{bmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

Modal form

Do yet another similarity transform with

$$\mathbf{M} = \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}$$

The modal form avoids imaginary numbers in the state equation.

Similarity transform to Modal Form

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \mathbf{M}^{-1} \begin{bmatrix} -\frac{1}{2} + \frac{i\sqrt{3}}{2} & 0 \\ 0 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{bmatrix} \mathbf{M} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \mathbf{M}^{-1} \begin{bmatrix} \frac{1}{2} + \frac{i}{2\sqrt{3}} \\ \frac{1}{2} - \frac{i}{2\sqrt{3}} \end{bmatrix} u$$

$$y = \begin{bmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{bmatrix} \mathbf{M} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

The system on **modal** form

$$\begin{aligned} \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -\frac{1}{\sqrt{3}} \end{bmatrix} u \\ y &= \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} \end{aligned}$$

Modal form

General case

A diagonalized state equation with complex eigenvalues:

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 + i\beta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 - i\beta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 + i\beta_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha_2 - i\beta_2 \end{bmatrix}$$

Modal transform: $\boldsymbol{\Lambda}_m = \mathbf{M}^{-1}\boldsymbol{\Lambda}\mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{i}{2} \end{bmatrix} \quad \mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & i & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & i & -i \end{bmatrix}$$

Modal form

General case

$$\mathbf{\Lambda}_m = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}$$

Modal transform: $\mathbf{\Lambda}_m = \mathbf{M}^{-1}\mathbf{\Lambda}\mathbf{M}$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{i}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{i}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{i}{2} \end{bmatrix} \quad \mathbf{M}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & i & -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & i & -i \end{bmatrix}$$

Canonical forms

Modal form

Modal form: $\mathbf{\Lambda}_m = \mathbf{M}^{-1}\mathbf{\Lambda}\mathbf{M}$

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix}$$

Matrix exponential:

$$e^{\mathbf{\Lambda}_m t} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{t\alpha_1}\cos(t\beta_1) & e^{t\alpha_1}\sin(t\beta_1) & 0 & 0 & 0 \\ 0 & -e^{t\alpha_1}\sin(t\beta_1) & e^{t\alpha_1}\cos(t\beta_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{t\lambda_2} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{t\alpha_2}\cos(t\beta_2) & e^{t\alpha_2}\sin(t\beta_2) \\ 0 & 0 & 0 & 0 & -e^{t\alpha_2}\sin(t\beta_2) & e^{t\alpha_2}\cos(t\beta_2) \end{bmatrix}$$

Summary

Usage

- Distinct real eigenvalues: diagonal blocks: $\begin{bmatrix} \lambda_i & 0 \\ 0 & \lambda_{i+1} \end{bmatrix}$
- Repeated real eigenvalues: Jordan blocks: $\begin{bmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{bmatrix}$
- Complex eigenvalues: modal blocks: $\begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix}$

Topic

Modal analysis

Definition: Modal analysis

The study of the dynamic properties of structures under vibrational excitation.

Applications

- Earthquake engineering
- Acoustics
- Aeroelasticity
- Fatigue analysis
- Architecture

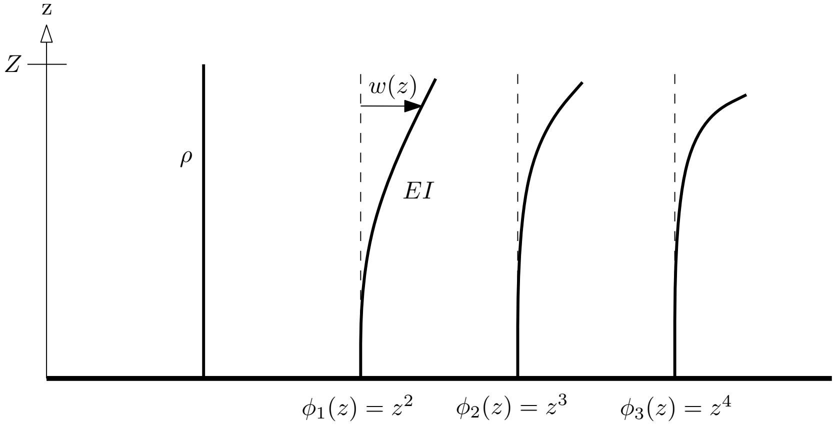
Methodology

Construct a dynamic model of the structure. Then find:

Eigenvalues: Complex part of eigenvalue corresponds to the *resonance frequency*.

Eigenvectors: These encode the *shape* of the resonant motion.

Example: Elastic rod



Kinematics of elastic rod

The rod's deflection $w(z)$ is approximated by the superposition of n test modes $\phi_i(z) = z^{i+1}$ scaled by states x_i:

$$w(z) = \sum_{i=1}^{n} \phi_i(z) x_i$$

Kinetic energy

Kinematics of elastic rod

The rod's deflection $w(z)$ is approximated by the superposition of n test modes $\phi_i(z) = z^{i+1}$ scaled by states x_i:

$$w(z) = \sum_{i=1}^{n} \phi_i(z) x_i$$

Kinetic energy

Let the mass-distribution be uniform with density ρ. Then the kinetic energy is a *quadratic form*:

$$\mathcal{K} = \frac{1}{2}\rho \int_0^Z \dot{w}^2(z) dz = \frac{1}{2}\rho \int_0^Z \left[\sum_{i=1}^{n} \phi_i(z)\dot{x}_i\right] \left[\sum_{j=1}^{n} \phi_j(z)\dot{x}_j\right] dz$$

$$= \frac{1}{2}\rho \sum_{i=1}^{n}\sum_{j=1}^{n} \left(\left[\int_0^Z \phi_i(z)\phi_j(z) dz\right] \dot{x}_i\dot{x}_j\right) = \frac{1}{2}\dot{\mathbf{x}}^\top \mathbf{M}\dot{\mathbf{x}}, \quad M_{ij} = \rho \sum_{i=1}^{n}\sum_{j=1}^{n} \left[\int_0^Z \phi_i(z)\phi_j(z) dz\right]$$

Potential energy

Kinematics of elastic rod

The rod's deflection $w(z)$ is approximated by the superposition of n test modes $\phi_i(z) = z^{i+1}$ scaled by states:

$$w(z) = \sum_{i=1}^{n} \phi_i(z) x_i$$

Potential energy

The potential energy is proportional to the specific elastic modulus EI and quadratic in beam *curvature*:

$$\mathcal{U} = \frac{1}{2} EI \int_0^Z w''^2(z) dz = \frac{1}{2} EI \int_0^Z \left[\sum_{i=1}^{n} \phi_i''(z) x_i \right] \left[\sum_{j=1}^{n} \phi_j''(z) x_j \right] dz$$

$$= \frac{1}{2} EI \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\left[\int_0^Z \phi_i''(z) \phi_j''(z) dz \right] x_i x_j \right) = \frac{1}{2} \mathbf{x}^\mathsf{T} \mathbf{K} \mathbf{x}, \quad K_{ij} = EI \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\int_0^Z \phi_i''(z) \phi_j''(z) dz \right]$$

Equations of motion

Kinetic energy

$$\mathcal{K} = \frac{1}{2}\dot{\mathbf{x}}^{\mathsf{T}}\mathbf{M}\dot{\mathbf{x}}, \quad M_{ij} = \rho \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\int_0^Z \phi_i(z)\phi_j(z)dz\right]$$

Potential energy

$$\mathcal{U} = \frac{1}{2}\mathbf{x}^{\mathsf{T}}\mathbf{K}\mathbf{x}, \quad K_{ij} = EI \sum_{i=1}^{n}\sum_{j=1}^{n}\left[\int_0^Z \phi_i''(z)\phi_j''(z)dz\right]$$

Lagrangian equations of motion, $\mathcal{L} = \mathcal{K} - \mathcal{U}$

$$\frac{d}{dt}\left[\frac{\partial\mathcal{K}}{\partial\dot{\mathbf{x}}}\right] + \frac{\partial\mathcal{U}}{\partial\mathbf{x}} = \ddot{\mathbf{x}}^{\mathsf{T}}\mathbf{M} + \mathbf{x}^{\mathsf{T}}\mathbf{K} = \mathbf{0}^{\mathsf{T}}$$

State-space model

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}, \quad \mathbf{z} \triangleq \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbb{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix}$$

Eigenvalues/vectors

State-space model

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z}, \quad \mathbf{z} \triangleq \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbb{I} \\ -\mathbf{M}^{-1}\mathbf{K} & \mathbf{0} \end{bmatrix}$$

Eigenvalues

$$|\lambda\mathbb{I} - \mathbf{A}| = \begin{vmatrix} \lambda\mathbb{I} & -\mathbb{I} \\ \mathbf{M}^{-1}\mathbf{K} & \lambda\mathbb{I} \end{vmatrix} = |\lambda^2\mathbb{I} + \mathbf{M}^{-1}\mathbf{K}| = |\lambda^2\mathbf{M} + \mathbf{K}| = 0$$

Imaginary eigenvalues result:

$$\lambda = 0 \pm j\omega \quad \Rightarrow |\mathbf{K} - \omega^2\mathbf{M}| = 0$$

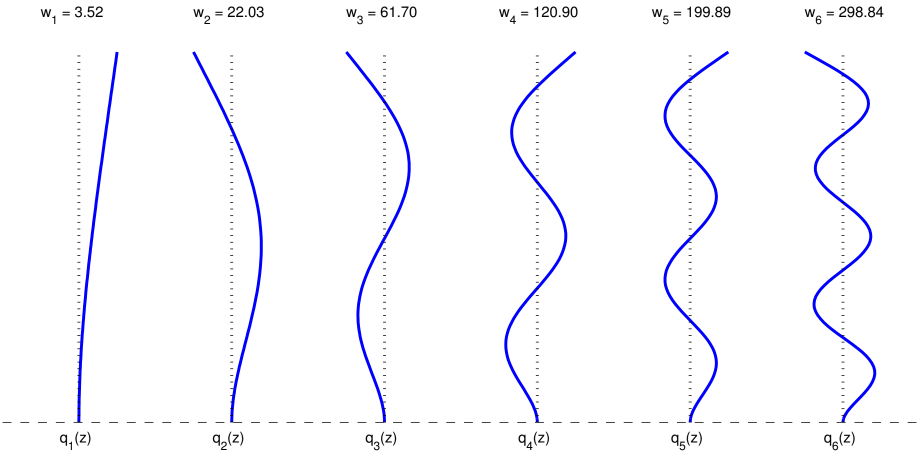
Eigenvectors

$$[\lambda\mathbb{I} - \mathbf{A}]\mathbf{q} = \begin{bmatrix} \lambda\mathbb{I} & -\mathbb{I} \\ \mathbf{M}^{-1}\mathbf{K} & \lambda\mathbb{I} \end{bmatrix} \begin{bmatrix} \mathbf{q}_x \\ \mathbf{q}_{\dot{x}} \end{bmatrix} = \begin{bmatrix} \lambda\mathbf{q}_x - \mathbf{q}_{\dot{x}} \\ \mathbf{M}^{-1}\mathbf{K}\mathbf{q}_x + \lambda\mathbf{q}_{\dot{x}} \end{bmatrix}$$

Thus:

$$\mathbf{q}_{\dot{x}} = \lambda\mathbf{q}_x \quad \Rightarrow (\lambda^2\mathbb{I} + \mathbf{M}^{-1}\mathbf{K})\mathbf{q}_x = (\mathbf{K} - \omega^2\mathbf{M})\mathbf{q}_x = \mathbf{0}$$

Eigenvalues/vectors



Results: $\rho = 1,\ EI = 1,\ Z = 1$

The first six modeshapes[2] $m_i(z) = \sum_j^n (\phi_j(z) q_j^i)$ and frequencies ω_i are shown.

[2] q_j^i: i'th component of j'th eigenvector

Topic

Last lecture

State-space model

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

Solution

$$\mathbf{y}(t) = \mathbf{C}e^{At}\mathbf{x_0} + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

This lecture

Computation and properties of the matrix exponential $e^{\mathbf{A}t}$

Last lecture

State-space model

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\end{aligned}$$

Laplace transform

$$\mathbf{y}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s) + \mathbf{D}\mathbf{u}(s)$$

This lecture

Computation and properties of the matrix: $(s\mathbb{I} - \mathbf{A})^{-1}$

Properties of $e^{\mathbf{A}t}$

- $\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t}$
- $e^{\mathbf{A}t}e^{\mathbf{A}\tau} = e^{\mathbf{A}(t+\tau)}$
- $(e^{\mathbf{A}t})^{-1} = e^{-\mathbf{A}t}$
- $\mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$
- $e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k$

Warning

$$e^{\mathbf{A}t}e^{\mathbf{B}t} \neq e^{(\mathbf{A}+\mathbf{B})t}$$

Note

$$e^{\lambda t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\lambda^k$$

Functions of a square matrix

Computation of $e^{\mathbf{A}t}$

1. It is inconvenient to use an infinite series to compute $e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k$
2. There is a shortcut that allows a finite summation $e^{\mathbf{A}t} = \sum_{k=0}^{n-1} a_k(t)\mathbf{A}^k$
3. The *Cayley Hamilton Theorem* provides the recipe.

Cayley-Hamilton: A matrix satisfies its own characteristic polynomial

$$\Delta(\lambda) = \det(\lambda\mathbb{I} - \mathbf{A}) = \lambda^n + \alpha_1\lambda^{n-1} + \cdots + \alpha_{n-1}\lambda + \alpha_n$$

so:

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_1\mathbf{A}^{n-1} + \cdots + \alpha_{n-1}\mathbf{A} + \alpha_n\mathbb{I} = \mathbf{0}$$

Why is $\Delta(\mathbf{A}) = \mathbf{0}$ important?

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_1\mathbf{A}^{n-1} + \cdots + \alpha_{n-1}\mathbf{A} + \alpha_n\mathbb{I} = \mathbf{0}$$

$\mathbf{A}^n$

$$\mathbf{A}^n = -\alpha_1\mathbf{A}^{n-1} - \cdots - \alpha_{n-1}\mathbf{A} - \alpha_n\mathbb{I}$$

$\mathbf{A}^n$

Can be written as a linear combination of $\{\mathbf{A}^{n-1}, \mathbf{A}^{n-2}, \ldots \mathbb{I}\}$

$\mathbf{A}^{n+1}$

$$\overbrace{\mathbf{A}\mathbf{A}^n}^{\mathbf{A}^{n+1}} = -\alpha_1 \overbrace{\mathbf{A}\mathbf{A}^{n-1}}^{\mathbf{A}^n} - \cdots - \alpha_{n-1} \overbrace{\mathbf{A}\mathbf{A}}^{\mathbf{A}^2} - \alpha_n \overbrace{\mathbf{A}}^{\mathbf{A}} \mathbb{I}$$

$\mathbf{A}^{n+1}$

Can be written as a linear combination of $\{\mathbf{A}^n, \mathbf{A}^{n-1}, \ldots \mathbf{A}\}$
..which is a linear combination of $\{\mathbf{A}^{n-1}, \mathbf{A}^{n-2}, \ldots \mathbb{I}\}$

Functions of a square matrix

Why is $\Delta(\mathbf{A}) = \mathbf{0}$ important?

$$\Delta(\mathbf{A}) = \mathbf{A}^n + \alpha_1 \mathbf{A}^{n-1} + \cdots + \alpha_{n-1}\mathbf{A} + \alpha_n \mathbb{I} = \mathbf{0}$$

Because:

It tells us that any polynomial function can be written as a linear combination of $\{\mathbf{A}^{n-1}, \mathbf{A}^{n-2}, \ldots \mathbb{I}\}$!

Linear combination:

$$f(\mathbf{A}) = \beta_0 \mathbb{I} + \beta_1 \mathbf{A} + \cdots + \beta_{n-1}\mathbf{A}^{n-1}$$

Functions of a square matrix

Linear combination:

$$h(\mathbf{A}) = \beta_0 \mathbb{I} + \beta_1 \mathbf{A} + \cdots + \beta_{n-1} \mathbf{A}^{n-1}$$

Linear combination in terms of λ:

$$h(\lambda) = \beta_0 + \beta_1 \lambda + \cdots + \beta_{n-1} \lambda^{n-1}$$

Functions of a square matrix

Procedure to compute $f(\mathbf{A})$

1. Given a function we wish to find: $f(\mathbf{A})$
2. Define the function of unknown coefficients $h(\lambda) = \beta_0 + \beta_1\lambda + \cdots + \beta_{n-1}\lambda^{n-1}$
3. For each eigenvalue, make an equation: $f(\lambda_i) = h(\lambda_i)$.
4. If the eigenvalue is repeated n_i times:
5. Use the derivatives $\left.\frac{d^l f(\lambda)}{d\lambda^l}\right|_{\lambda=\lambda_i} = \left.\frac{d^l h(\lambda)}{d\lambda^l}\right|_{\lambda=\lambda_i}$ for $l = 1 \ldots n_i - 1$ to generate additional equations.
6. Solve the n equations for $\beta_0 \ldots \beta_{n-1}$
7. Insert: $f(\mathbf{A}) = h(\mathbf{A}) = \beta_0\mathbb{I} + \beta_1\mathbf{A} + \cdots + \beta_{n-1}\mathbf{A}^{n-1}$

Example: $\mathbf{A}^{10}$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \quad f(\mathbf{A}) = \mathbf{A}^{10}$$

Computation

Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$

$$h(\lambda) = \beta_0 + \beta_1 \lambda, \quad f(\lambda) = \lambda^{10}$$

Equations:

$$\begin{aligned} h(2) &= f(2) \quad \Rightarrow \beta_0 + 2\beta_1 = 2^{10} \\ h(1) &= f(1) \quad \Rightarrow \beta_0 + 1\beta_1 = 1^{10} \end{aligned}$$

Solution:

$$\beta_0 = -1022, \quad \beta_1 = 1023$$

Result

$$f(\mathbf{A}) = \beta_0 \mathbb{I} + \beta_1 \mathbf{A} = -1022 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 1023 \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1024 & 4092 \\ 0 & 1 \end{bmatrix}$$

Example: $e^{\mathbf{A}t}$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \quad f(\mathbf{A}) = e^{\mathbf{A}t}$$

Computation

Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 1$

$$h(\lambda) = \beta_0 + \beta_1 \lambda, \quad f(\lambda) = e^{\lambda t}$$

Equations:

$$\begin{aligned} h(2) &= f(2) \quad \Rightarrow \beta_0 + 2\beta_1 = e^{2t} \\ h(1) &= f(1) \quad \Rightarrow \beta_0 + 1\beta_1 = e^{t} \end{aligned}$$

Solution:

$$\beta_0 = -e^t\left(-2 + e^t\right), \quad \beta_1 = e^t\left(-1 + e^t\right)$$

Result

$$f(\mathbf{A}) = \beta_0 \mathbb{I} + \beta_1 \mathbf{A} = -e^t\left(-2 + e^t\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + e^t\left(-1 + e^t\right)\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

Example: $(s\mathbb{I} - A)^{-1}$

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}, \quad f(\mathbf{A}) = (s\mathbb{I} - A)^{-1}$$

Computation

Equations:

$$\begin{aligned} h(2) &= f(2) \quad \Rightarrow \beta_0 + 2\beta_1 = \frac{1}{s-2} \\ h(1) &= f(1) \quad \Rightarrow \beta_0 + 1\beta_1 = \frac{1}{s-1} \end{aligned}$$

Solution:

$$\beta_0 = \frac{-3+s}{2-3s+s^2}, \quad \beta_1 = \frac{1}{2-3s+s^2}$$

Result

$$f(\mathbf{A}) = \beta_0\mathbb{I} + \beta_1\mathbf{A} = \frac{-3+s}{2-3s+s^2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{2-3s+s^2}\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$$

Topic

Property 1:

$$f(\mathbf{PAP}^{-1}) = \mathbf{P}f(\mathbf{A})\mathbf{P}^{-1}$$

Recall that all matrix functions are linear combinations:

$$f(\mathbf{A}) = \beta_0\mathbb{I} + \beta_1\mathbf{A} + \beta_2\mathbf{AA}\ldots + \beta_{n-1}\mathbf{A}^{n-1}$$

Insert similar matrix:

$$f(\mathbf{PAP}^{-1}) = \beta_0\overbrace{\mathbb{I}}^{\mathbf{PP}^{-1}} + \beta_1\mathbf{PAP}^{-1} + \beta_2\mathbf{PA}\overbrace{\mathbf{P}^{-1}\mathbf{P}}^{\mathbb{I}}\mathbf{AP}^{-1} + \ldots$$

Clean up and rearrange:

$$f(\mathbf{PAP}^{-1}) = \beta_0\mathbf{PP}^{-1} + \beta_1\mathbf{PAP}^{-1} + \beta_2\mathbf{PA}^2\mathbf{P}^{-1} + \ldots$$

Q.E.D.:

$$f(\mathbf{PAP}^{-1}) = \mathbf{P}[\beta_0 + \beta_1\mathbf{A} + \beta_2\mathbf{A}^2 + \ldots]\mathbf{P}^{-1} = \mathbf{P}f(\mathbf{A})\mathbf{P}^{-1}$$

Property 2:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & a_3 & \\ & & & a_4 \end{bmatrix} \quad \Rightarrow f(\mathbf{A}) = \begin{bmatrix} f(\mathbf{A}_1) & & & \\ & f(\mathbf{A}_2) & & \\ & & f(a_3) & \\ & & & f(a_4) \end{bmatrix}$$

Functions of a square matrix

Special cases

Diagonal matrix

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \Rightarrow e^{\boldsymbol{\Lambda} t} = \begin{bmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{bmatrix}$$

Jordan Block

$$\mathbf{J} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \quad \Rightarrow e^{\mathbf{J}t} = \begin{bmatrix} e^{t\lambda} & e^{t\lambda}t & \frac{1}{2!}e^{t\lambda}t^2 & \frac{1}{3!}e^{t\lambda}t^3 \\ 0 & e^{t\lambda} & e^{t\lambda}t & \frac{1}{2!}e^{t\lambda}t^2 \\ 0 & 0 & e^{t\lambda} & e^{t\lambda}t \\ 0 & 0 & 0 & e^{t\lambda} \end{bmatrix}$$