TTK4115

Lecture 1

Introduction, LTI systems - representations and solutions

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This lecture

1. Motivation, Course goals
   - What will you learn?
   - 7 main topics
2. LTI systems
   - State space models
   - Transfer functions
   - Solutions
     - Convolution
   - More connections
3. Summary
4. Next time

Topic

Learning objectives

- Basic *theory* of linear multi-input multi-output (MIMO) systems, important concepts like controllability, observability, stability
- Understanding state-space theory as an alternative to frequency domain theory
- Basic introduction to the theory of random systems and signals
- Knowledge about design of linear controllers and state estimators, including the Linear-Quadratic optimal controller and the Kalman Filter
- Skills in practical use of the theory through projects and labs (experimental and simulations using Matlab and Simulink)
- Get a solid basis for further studies in control engineering; Optimization and Control; Nonlinear Systems; Adaptive Control; Navigation and Vehicle Control; Advanced Process Control, Robotics; .....

Motivation

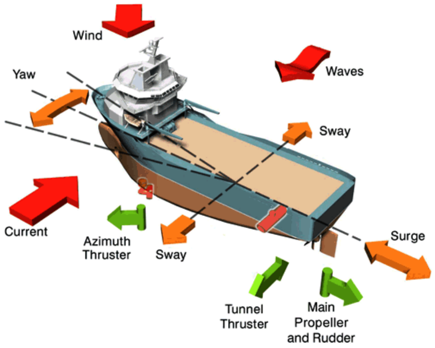
In the first course on control engineering, most of you have focused on frequency domain methods and SISO-systems.

Why state space theory?

- Mathematical description which easily handles many physical models
- Easier way to handle multiple-input multiple-output (MIMO) systems (in most cases)
- Generalizes to nonlinear systems
- Many advanced control design and estimation methods have been developed for systems on state space form

Example: Dynamic positioning system

Most of what you learn in this course is applied here!



Curriculum

1. Chi-Tsong Chen: Linear system theory and design. 4th International. New York: Oxford University Press, 2014. ISBN: 9780199964543
2. Robert G. Brown & Patrick Y. C. Hwang: Introduction to random signals and applied Kalman filtering : with MATLAB exercises. 4th. Hoboken, NJ: John Wiley & Sons, Inc., 2012. ISBN: 9780470609699

Style of lectures

- Due to the corona virus we can't have ordinary lectures in the auditorium
- Plan for this year (may be revised!):
  - One or more video lectures will be published for each week's topics
  - Full set of slides will be made available on Blackboard for each week
  - Weekly Zoom/Blackboard session for answering questions

Topic

What will you learn? 7 main topics

1. Solution of state equations

Consider the *state equation*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

and the *output equation*

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t).$$

Given the initial condition $\mathbf{x}(t_0)$ and the input $\mathbf{u}(t)$, $t_0 \leq t \leq t_1$. What is $\mathbf{x}(t_1)$ and $\mathbf{y}(t_1)$?

Why?

Basic knowledge. Understand the structure of the solution.

What will you learn? 7 main topics

2. Stability

What happens if we change the initial conditions $\mathbf{x}(0)$ of $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ slightly?
Given the model $\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$: if our input signals are bounded, will the outputs be bounded?

Why?

- Fundamental property of a dynamic system.
- Simpler to analyse stability than to solve the differential equations.
- Different formulations and techniques.
- Extensions to nonlinear systems.

What will you learn? 7 main topics

3. Canonical forms of state equations

By a change of coordinates, transform from

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

to

$$\begin{aligned}\dot{\bar{\mathbf{x}}}(t) &= \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}\mathbf{u}(t)\\ \mathbf{y}(t) &= \bar{\mathbf{C}}\bar{\mathbf{x}}(t) + \bar{\mathbf{D}}\mathbf{u}(t)\end{aligned}$$

where the new matrices $\bar{\mathbf{A}}, \bar{\mathbf{B}}, \bar{\mathbf{C}}, \bar{\mathbf{D}}$ have some desired properties, e.g. $\bar{\mathbf{A}}$ being diagonal, or makes the control design easier.

Why?

Basic knowledge. Understand your alternatives.

What will you learn? 7 main topics

4. Realizations

Given $\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$. Find a state space model

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned}$$

Why?

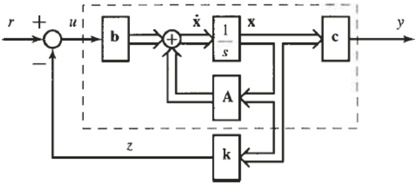
- Enables state space analysis.
- Basis for implementing filters, controllers and estimators in software and analog hardware.

What will you learn? 7 main topics

5. Controllability, State Feedback, Linear Quadratic Regulator (LQR)

We want to use the feedback $\mathbf{u}(t) = -\mathbf{k}\mathbf{x}(t)$. How do we choose **k** to get a desirable response? When can we control a system to a desired state?

Controllability: Do we have the right actuators to control the system to the desired set-point?



Why?

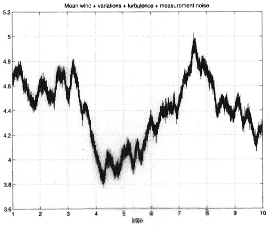
Control design approach for MIMO systems.

What will you learn? 7 main topics

6. Representing random signals and systems

How to describe a stochastic/random signal $u(t)$ in term of its statistical properties?

How to describe the response $\hat{y}(s) = \hat{g}(s)\hat{u}(s)$ when $u(t)$ is a random input signal?



Why?

Analyse and minimize the effects of unknown measurement noise, disturbances and other uncertainty.

What will you learn? 7 main topics

7. Observability, state estimation, Kalman Filter

Assume the model

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t).\end{aligned}$$

Given $\mathbf{u}(t)$, $\mathbf{y}(t)$ for any $t_0 \leq t \leq t_1$. What is $\mathbf{x}(t_0)$? What is $\mathbf{x}(t_1)$?
Observability: Do we have the right measurements? What if our measurements are correlated with noise? What if estimated states are used for control feedback?

Why?

In real world applications we often cannot measure all states
Estimation allows us to compute what we cannot measure - combining sensors and model

Topic

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Representations of linear systems

We will encounter three different representations of linear systems in this course. Our models will appear as:

State-space models

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)\end{aligned}$$

Transfer functions

$$\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s)$$

Input-output descriptions

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{G}(t,\tau)\mathbf{u}(\tau)d\tau$$

First week's lectures

In the first week's lectures we will see how these forms are connected.
Only lumped LTI systems are considered.

LTI: **L**inear **T**ime **I**nvariant

Topic

LTI systems overview

$$
\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}}
$$

State-space model

Most of our models will be put on the state space form.

System states

Definition of state

The state $\mathbf{x}(t_0)$ of a system at a time t_0 is the information at t_0 that, together with the input $\mathbf{u}(t)$, for $t \geq t_0$, determines uniquely the output $\mathbf{y}(t)$ for all $t \geq t_0$.

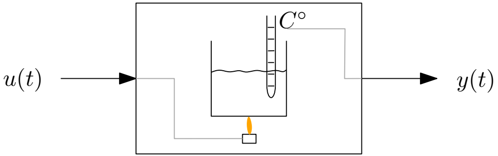
Explanation

- We do not need $\mathbf{u}(t)$ for $t < t_0$ to predict what will happen after t_0.
- We do not need $\mathbf{x}(t)$ for $t < t_0$ to predict what will happen after t_0.
- The state acts as a memory of these!

Concise statement

$$\left.\begin{array}{ll} \mathbf{x}(t_0) & \\ \mathbf{u}(t), & t > t_0 \end{array}\right\} \rightarrow \quad \mathbf{y}(t), \quad t \geq t_0$$

Water heater



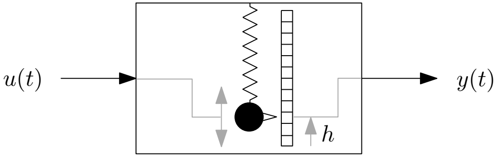
Question

What could be the state of the system?

Hints

If we know how hot the water is now at t_0....
... do we need to know the past temperatures to predict the future ones?

Pendulum



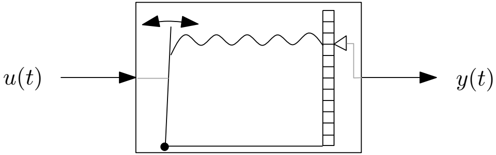
Question

What could be the states of the system? How many are there?

Hints

We know the position of the mass at t_0.
Since the mass may be moving vertically, we also need to know its speed at t_0.

Wave tank



Question

What could be the states of the system? How many are there?

Answer

The height of the water surface varies with time and along the horizontal axis.
This a **distributed** system, it has *infinitely*[1] many states!

[1] Systems with a *finite* number of states are called **lumped**

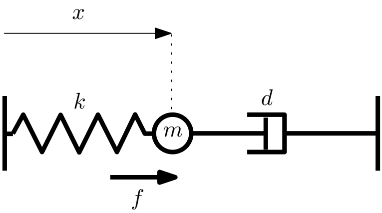
About states

- The choice of state variables is not unique
- The state does not have to be measurable
- The state does not have to be a physical quantity
- The state summarizes the history of the system

Example: Mass spring damper

Second order dynamics:

$$m\ddot{x}(t) + d\dot{x}(t) + kx(t) = f(t)$$



Input/Output: force and displacement

$$y(t) \triangleq x(t), \quad u(t) \triangleq f(t)$$

States: displacement and velocity

$$x_1(t) \triangleq x(t), \quad x_2(t) \triangleq \dot{x}(t)$$

Example: Mass spring damper

Second order dynamics:

$$m\ddot{x}(t) + d\dot{x}(t) + kx(t) = f(t)$$

Input/Output and states

$$y(t) \triangleq x(t), \quad u(t) \triangleq f(t)$$
$$x_1(t) \triangleq x(t), \quad x_2(t) \triangleq \dot{x}(t)$$

Array form

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}^{\mathbf{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}} u$$

Topic

LTI systems overview

$$\boxed{\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}} \xrightarrow{\mathcal{L}(\ldots)} \boxed{\begin{aligned}\hat{\mathbf{x}}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)]\\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)] + \mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}}$$

Laplace Transform

We can convert state space models (time domain) to the frequency domain with the **Laplace Transform**.

Transfer functions

Transfer functions model the system in the *frequency domain*.

State-space model

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\
\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)
\end{aligned}
$$

Transform rule

$$
\dot{\mathbf{x}}(t) \quad \Leftrightarrow \quad s\hat{\mathbf{x}}(s) - \mathbf{x}_0
$$

Applying the rule

$$
\begin{aligned}
s\hat{\mathbf{x}}(s) - \mathbf{x}_0 = \mathbf{A}\hat{\mathbf{x}}(s) + \mathbf{B}\hat{\mathbf{u}}(s) \\
(s\mathbb{I} - \mathbf{A})\hat{\mathbf{x}}(s) = \mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)
\end{aligned}
$$

Laplace Transform of the state space model

$$
\begin{aligned}
\hat{\mathbf{x}}(s) &= (s\mathbb{I} - \mathbf{A})^{-1}\left[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)\right] \\
\hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\left[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)\right] + \mathbf{D}\hat{\mathbf{u}}(s)
\end{aligned}
$$

Zero-state response, Zero-input response

Response

$$\hat{\mathbf{y}}(s) = \overbrace{\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{x}_0}^{\text{Zero-input}} + \overbrace{\left[\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right]\hat{\mathbf{u}}(s)}^{\text{Zero-state}}$$

Transfer matrix

In many cases the zero-input response is uninteresting because:

1. The zero-input response dies away exponentially fast if the system is stable.
2. The plant is initialized at $\mathbf{x}_0 = \mathbf{0}$.

Then the we may focus on the input/output response:

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s), \quad \hat{\mathbf{G}}(s) = \underbrace{\mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}}_{\text{transfer matrix}}$$

which is determined by the *transfer matrix* $\hat{\mathbf{G}}(s)$.

Example: Mass spring damper in the Laplace domain

Model

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}^{\mathbf{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}} u$$

Transform

Using:

$$\hat{\mathbf{y}}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A})^{-1} \left[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)\right] + \mathbf{D}\hat{\mathbf{u}}(s)$$

leads to:

$$\hat{y}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \hat{u}(s) \right)$$

Example: Mass spring damper in the Laplace domain

Laplace model:

$$\hat{y}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} \hat{u}(s) \right)$$

Inversion:

$$\left(s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix} \right)^{-1} = \frac{1}{ms^2 + sd + k} \begin{bmatrix} d + ms & m \\ -k & ms \end{bmatrix}$$

Result

$$\hat{y}(s) = \underbrace{\frac{1}{ms^2 + sd + k} \begin{bmatrix} d + ms & m \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{\text{Zero input response}} + \underbrace{\frac{1}{ms^2 + sd + k}}_{\hat{g}(s)} \hat{u}(s)$$

Topic

LTI systems overview

$\mathcal{L}(\dots)$

$$\hat{\mathbf{x}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)]$$
$$\hat{\mathbf{y}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)] + \mathbf{D}\hat{\mathbf{u}}(s)$$

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

LTI sol.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$
$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

Solutions in time

We may need to know the behavior of the system in time. This requires a solution of the differential state space model.

Solutions of LTI state equations

We want to solve:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)\end{aligned}$$

Initial value problem

We wish to find $\mathbf{y}(t)$ given $\mathbf{x}_0 = \mathbf{x}(t_0)$ and $\mathbf{u}(t)$.

First order homogenous case

Let's try something simpler first:

$$\dot{x}(t) = ax(t), \quad y(t) = cx(t)$$

No input, *one* state.

Step 1: Premultiply and reformulate

$$e^{-at}\dot{x}(t) = e^{-at}ax(t) \quad \Rightarrow \frac{d}{dt}[e^{-at}x(t)] = 0$$

Step 2: Integrate

$$\int_0^t \frac{d}{d\tau}[e^{-a\tau}x(\tau)]d\tau = e^{-at}x(t) - \underbrace{e^{-a0}x(0)}_{x(0)} = 0$$

Step 3: Recover solution

$$e^{-at}x(t) = x_0 \quad \Rightarrow x(t) = x_0 e^{at}$$

First order homogenous case

Insertion of solution:

$$\overbrace{\frac{d}{dt}[e^{at}x_0]}^{\dot{x}} = \overbrace{ae^{at}x_0}^{ax}$$

Check of the initial condition

$$\overbrace{e^{at}x_0\Big|_{t=0}}^{x(0)} = x_0$$

First order nonhomogenous case

Let's try something harder:

$$\dot{x}(t) = ax(t) + bu(t), \quad y(t) = cx(t) + du(t)$$

One input, *one* state.

Step 1: Premultiply and reformulate

$$e^{-at}\dot{x}(t) = e^{-at}ax(t) + e^{-at}bu(t) \quad \Rightarrow \frac{d}{dt}[e^{-at}x(t)] = e^{-at}bu(t)$$

Step 2: Integrate

$$\int_0^t \frac{d}{d\tau}[e^{-a\tau}x(\tau)]d\tau = \int_0^t e^{-a\tau}bu(\tau)d\tau \quad \Rightarrow e^{-at}x(t) - \underbrace{e^{-a0}x(0)}_{x(0)} = \int_0^t e^{-a\tau}bu(\tau)d\tau$$

Step 3: Recover solution

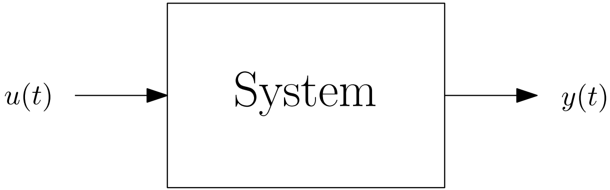
$$e^{-at}x(t) = x_0 + \int_0^t e^{-a\tau}bu(\tau)d\tau \quad \Rightarrow x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau$$

Output response

$$y(t) = cx(t) + du(t) = \overbrace{ce^{at}x_0}^{\text{Free decay}} + c\overbrace{\int_0^t e^{a(t-\tau)}bu(\tau)d\tau}^{\text{Convolution integral}} + du(t)$$

Convolution

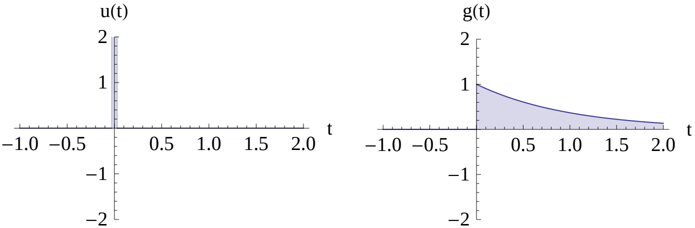
Consider now the SISO system:



Convolution

Experiment 1

Jolt at t=0: $u(t) = 1 \times \delta(t)$



Impulse response

$y(t) = g(t)$ is the system's response to an impulse.

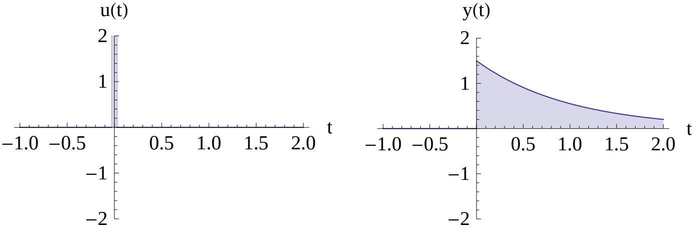
Causality

There is no response *before* the impulse.

Convolution

Experiment 2

Larger jolt: $u(t) = 1.5 \times \delta(t)$



Result

The response grows proportionally with the jolt magnitude.

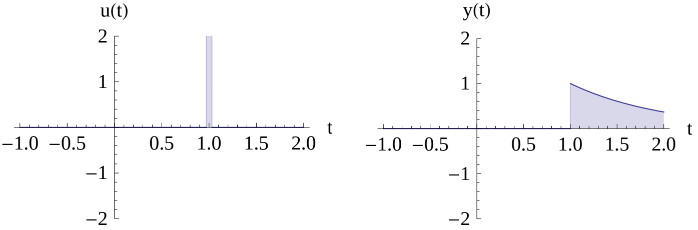
Homogeneity

$$\alpha u(t) \rightarrow \alpha y(t)$$

Convolution

Experiment 3

Jolt is shifted to t=1: $u(t) = 1 \times \delta(t-1)$



Result

The response is the same, just shifted $y(t) = g(t-1)$

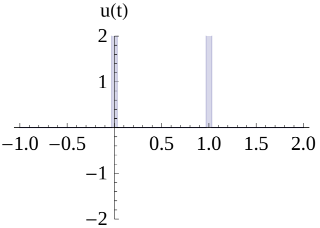
Time invariance

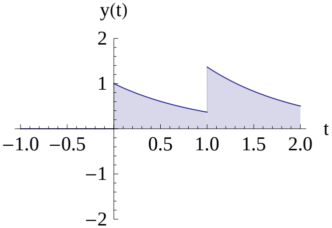
$$u(t-T) \rightarrow y(t-T)$$

Convolution

Experiment 4

Two jolts are superposed: $u(t) = 1 \times \delta(t) + 1 \times \delta(t-1)$





Result

The responses are superposed.

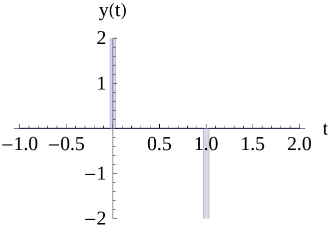
Additivity

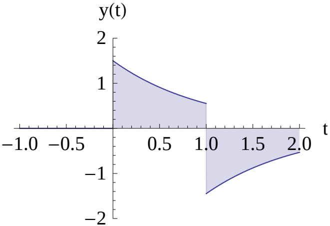
$$u_1(t) + u_2(t) \to y_1(t) + y_2(t)$$

Convolution

Experiment 5

Two jolts with different amplitudes are added: $u(t) = 1.5 \times \delta(t) - 2 \times \delta(t-1)$





Result

We see that the responses have been added proportionally.

Superposition

$$\alpha_1 u_1(t) + \alpha_2 u_2(t) \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Convolution

The idea

Convolution is simply:

- adding up all the impulse responses
- shifted to the time when the input happened: τ
- scaled proportionally with the input

In mathematics

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

Output response

$$y(t) = cx(t) + du(t) = \overbrace{ce^{at}x_0}^{\text{Free decay}} + \overbrace{c\int_0^t e^{a(t-\tau)}bu(\tau)d\tau}^{\text{Convolution integral}} + du(t)$$

The output response of an LTI system is described in terms of *convolution*.

Solution of an LTI system

Let's go for the whole thing:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

p inputs, n states, q outputs.

Step 1: Premultiply and reformulate

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) = e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Wait a minute!

What is $e^{-\mathbf{A}t}$?

The matrix exponential

A matrix analogue to the scalar exponential function
$e^{\mathbf{A}t}$ is the most important function of the matrix $\mathbf{A}$.

Analogy to e^{at}

The ordinary exponential function is defined by series expansion:

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}$$

Series expansion

The matrix exponential is defined analogously:

$$e^{\mathbf{A}t} = \mathbb{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{A}^k$$

Computing $e^{\mathbf{A}t}$

We will learn how to compute the matrix exponential *exactly* next lesson.

Important properties of $e^{\mathbf{A}t}$

Properties

$$\begin{aligned} e^{\mathbf{0}} &= \mathbb{I} \\ e^{\mathbf{A}(t_1+t_2)} &= e^{\mathbf{A}t_1} e^{\mathbf{A}t_2} \\ \left[e^{\mathbf{A}t}\right]^{-1} &= e^{-\mathbf{A}t} \end{aligned}$$

Derivative of $e^{\mathbf{A}t}$

$$\begin{aligned} \frac{d}{dt}[e^{\mathbf{A}t}] = \frac{d}{dt}\left[\sum_{k=0}^{\infty}\frac{t^k}{k!}\mathbf{A}^k\right] = \sum_{k=1}^{\infty}\frac{t^{(k-1)}}{(k-1)!}\mathbf{A}^k \\ = \quad \sum_{k=0}^{\infty}\frac{t^k}{k!}\mathbf{A}^{k+1} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A} \end{aligned}$$

Step 1: Premultiply and reformulate

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) = e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) + e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t) \quad \Rightarrow \frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Step 2: Integrate

$$\int_0^t \frac{d}{d\tau}\left[e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\right] d\tau = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \quad \Rightarrow e^{-\mathbf{A}t}\mathbf{x}(t) - \underbrace{e^{-\mathbf{A}0}\mathbf{x}(0)}_{\mathbf{x}(0)} = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Step 3: Recover solution

$$e^{-\mathbf{A}t}\mathbf{x}(t) = \mathbf{x}_0 + \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau \quad \Rightarrow \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Output response

$$\mathbf{y}(t) = \overbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0}^{\text{Zero-input}} + \overbrace{\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)}^{\text{Zero-state}}$$

Impulse response matrix

$$\mathbf{G}(t-\tau) = \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B} + \mathbf{D}\delta(t-\tau)$$

Example: Mass spring damper solution

Definitions

We redefine the constants for simplicity:

$$k/m = \omega_0^2, \quad d = 0$$

Model

$$\overbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -\omega_0^2 & 0 \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ 1/m \end{bmatrix}}^{\mathbf{B}} u$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{\mathbf{D}} u$$

Example: Mass spring damper solution

Response

$$y(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau + \mathbf{D}u(t)$$

F.y.i.

$$e^{\mathbf{A}t} = \begin{bmatrix} \cos(t\omega_0) & \frac{\sin(t\omega_0)}{\omega_0} \\ -\omega_0 \sin(t\omega_0) & \cos(t\omega_0) \end{bmatrix}$$

Response

$$y(t) = \underbrace{\begin{bmatrix} \cos(t\omega_0) & \frac{\sin(t\omega_0)}{\omega_0} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{\text{Zero input response}} + \int_0^t \underbrace{\frac{\sin[(t-\tau)\omega_0]}{m\omega_0}}_{g(t-\tau)} u(\tau)d\tau$$

Topic

LTI systems overview

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$\mathcal{L}(\dots)$

$$\hat{\mathbf{x}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)]$$
$$\hat{\mathbf{y}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)] + \mathbf{D}\hat{\mathbf{u}}(s)$$

$\mathcal{L}^{-1}(\dots)$

LTI sol.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$
$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

Inverse Laplace transform

The inverse Laplace transform gives us another route to the solution.

Frequency domain model

$$\mathbf{y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\left[\mathbf{x}_0 + \mathbf{B}\mathbf{u}(s)\right] + \mathbf{D}\mathbf{u}(s)$$

Important relations

$$\begin{aligned} \mathcal{L}^{-1}[(s\mathbb{I} - \mathbf{A})^{-1}] &= e^{\mathbf{A}t} \\ \mathcal{L}^{-1}[f(s)g(s)] &= \int_0^t f(t-\tau)g(\tau)d\tau \end{aligned}$$

Inverse Laplace Transform

$$\begin{aligned} \mathbf{y}(t) = \mathcal{L}^{-1}[\mathbf{y}(s)] &= \mathcal{L}^{-1}[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\left[\mathbf{x}_0 + \mathbf{B}\mathbf{u}(s)\right] + \mathbf{D}\mathbf{u}(s)] \\ &= \underbrace{\mathcal{L}^{-1}[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0]}_{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0} + \underbrace{\mathcal{L}^{-1}[\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{u}(s)]}_{\int_0^t \mathbf{C}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau} + \underbrace{\mathcal{L}^{-1}[\mathbf{D}\mathbf{u}(s)]}_{\mathbf{D}\mathbf{u}(t)} \end{aligned}$$

LTI systems overview

$\mathcal{L}(\dots)$

$$
\begin{aligned}
\mathbf{x}(s) &= (s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}_0+\mathbf{B}\mathbf{u}(s)] \\
\mathbf{y}(s) &= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}[\mathbf{x}_0+\mathbf{B}\mathbf{u}(s)]+\mathbf{D}\mathbf{u}(s)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t) \\
\mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t)
\end{aligned}
$$

$\mathcal{L}^{-1}(\dots)$ $\mathcal{L}(\dots)$

LTI sol.

$$
\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}_0+\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \\
\mathbf{y}(t) &= \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0+\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau+\mathbf{D}\mathbf{u}(t)
\end{aligned}
$$

Laplace transform

It is also possible to reverse the procedure.

Topic

LTI systems overview

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$\mathcal{L}(\ldots)$

$$\hat{\mathbf{x}}(s) = (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)]$$
$$\hat{\mathbf{y}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}_0 + \mathbf{B}\hat{\mathbf{u}}(s)] + \mathbf{D}\hat{\mathbf{u}}(s)$$

$\mathcal{L}^{-1}(\ldots)$

$\mathcal{L}(\ldots)$

LTI sol.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$
$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0 + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)$$

Lumped LTI systems

We have seen the different flavors of Lumped LTI systems and how to convert one representation to another.

Topic

Next time

Coming up next lesson

The matrix exponential

How do we calculate $e^{\mathbf{A}t}$?

Equivalent representations

Is it possible to find more convenient state equations that do the same job?

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{Ax} + \mathbf{Bu} \\ \mathbf{y}(t) &= \mathbf{Cx} + \mathbf{Du} \end{aligned}$$

Discretization

Useful for simulating systems