TTK4115

Lecture 7

Stability

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This lecture

1. Internal Stability

   Lyapunov's Method

2. Input-Output stability

Topic

1. Internal Stability

   Lyapunov's Method

2. Input-Output stability

Internal Stability

Internal dynamics

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \quad \mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$$

Stability in the sense of Lyapunov

Asymptotic stability : every finite initial state $\mathbf{x}_0$ produces a bounded response, and $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$.

Marginal stability : every finite initial state $\mathbf{x}_0$ produces a bounded response.

Eigenvalue Conditions

Marginal stability : All eigenvalues of $\mathbf{A}$ have **zero or negative** real parts. No Jordan blocks larger than 1×1 associated with zero eigenvalues[a].

Asymptotic stability : All eigenvalues of $\mathbf{A}$ have **negative real** parts.

Exponential stability : All eigenvalues of $\mathbf{A}$ have **negative real** parts. (Only for LTI systems).

Unstable : If none of the above conditions are met. One or more of the eigenvalues of $\mathbf{A}$ have **positive real** parts, or $\mathbf{A}$ has Jordan blocks larger than 1×1 associated with zero eigenvalues.

[a]

$$J = [0] : \text{ OK}, \quad J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} : \text{ NOT OK}$$

Exponential stability

Exponential stability

$$\|\mathbf{x}(t)\| = \|e^{\mathbf{A}t}\mathbf{x}_0\| \leq \|e^{\mathbf{A}t}\|\|\mathbf{x}_0\| \leq c^{-\lambda t}\|\mathbf{x}_0\|$$

Exponential stability

LTI system stability always implies *global exponential stability*. This is **not** the case for systems in general, and may be difficult to show.

Topic

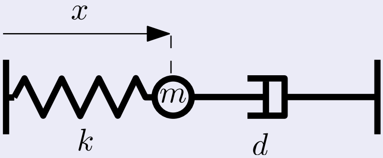
1. Internal Stability

    Lyapunov's Method

2. Input-Output stability

Lyapunov method

Example: Mass spring damper



This is the equation of motion of a mass spring damper system:

$$\overbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}^{\mathbf{x}}$$

Energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}\mathbf{x}^T \overbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}^{\mathbf{M}} \mathbf{x}$$

Lyapunov method

Example: Mass spring damper

$$\overbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}^{\mathbf{x}}$$

Energy

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}\mathbf{x}^T \overbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}^{\mathbf{M}} \mathbf{x}$$

Change in energy

$$\dot{E} = \frac{1}{2}\dot{\mathbf{x}}^T\mathbf{M}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{M}\dot{\mathbf{x}}$$

$$\dot{E} = \frac{1}{2}\overbrace{\dot{\mathbf{x}}^T}^{\mathbf{x}^T\mathbf{A}^T}\mathbf{M}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{M}\overbrace{\dot{\mathbf{x}}}^{\mathbf{A}\mathbf{x}}$$

Lyapunov method

Example: Mass spring damper

$$\overbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}^{\mathbf{x}}$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}\mathbf{x}^T \overbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}^{\mathbf{M}} \mathbf{x}$$

Change in energy

$$\dot{E} = \frac{1}{2}\mathbf{x}^T\mathbf{A}^T\mathbf{M}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{M}\mathbf{A}\mathbf{x} = \frac{1}{2}\mathbf{x}^T\left(\mathbf{A}^T\mathbf{M} + \mathbf{M}\mathbf{A}\right)\mathbf{x}$$

Lyapunov method

Example: Mass spring damper

$$\overbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}^{\mathbf{x}}$$

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}\mathbf{x}^T \overbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}^{\mathbf{M}} \mathbf{x}$$

Change in energy

$$\dot{E} = \frac{1}{2}\mathbf{x}^T \left(\mathbf{A}^T\mathbf{M} + \mathbf{M}\mathbf{A}\right) \mathbf{x} = -\frac{1}{2}\mathbf{x}^T\mathbf{N}\mathbf{x}, \quad \mathbf{N} = \begin{bmatrix} 0 & 0 \\ 0 & 2d \end{bmatrix}$$

Lyapunov method

Example: Mass spring damper

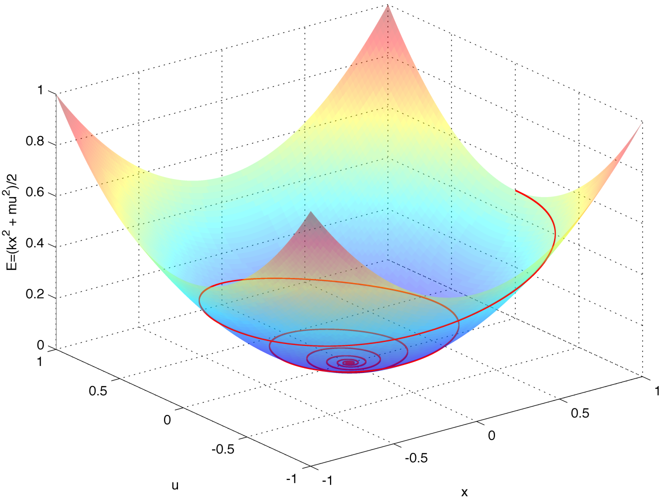
$$\overbrace{\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}}^{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -(k/m) & -(d/m) \end{bmatrix}}^{\mathbf{A}} \overbrace{\begin{bmatrix} x \\ \dot{x} \end{bmatrix}}^{\mathbf{x}}$$

$$E = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} kx^2 = \frac{1}{2}\mathbf{x}^T \overbrace{\begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}}^{\mathbf{M}} \mathbf{x}$$

Change in energy

$$\dot{E} = \frac{1}{2}\mathbf{x}^T \left(\mathbf{A}^T\mathbf{M} + \mathbf{M}\mathbf{A}\right)\mathbf{x} = \overbrace{-\frac{1}{2}\mathbf{x}^T\mathbf{N}\mathbf{x}}^{-d\dot{x}^2 \leq 0}, \quad \mathbf{N} = \begin{bmatrix} 0 & 0 \\ 0 & 2d \end{bmatrix}$$

Lyapunov method



Lyapunov method

Lyapunov function

The Lyapunov function does not have to be the energy:

$$E(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{M}\mathbf{x}$$

In fact, it can be any function, usually called V that:

$$V(\mathbf{0}) = 0, \quad V(\mathbf{x}) > 0 \quad \forall \mathbf{x} \neq 0$$

Lyapunov method

Lyapunov function: Example

Energy function:

$$E(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix} \mathbf{x}$$

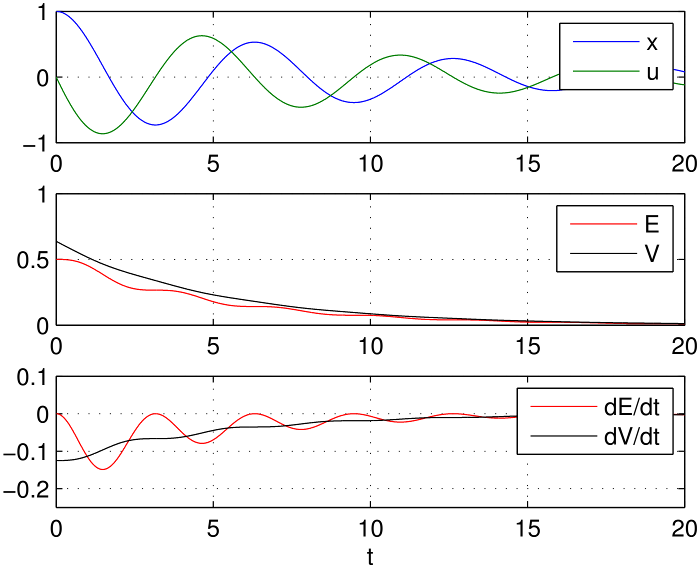
Energy "like" function

$$V(\mathbf{x}) = \mathbf{x}^T \begin{bmatrix} \frac{d^2+k(k+m)}{2dk} & \frac{m}{2k} \\ \frac{m}{2k} & \frac{m(k+m)}{2dk} \end{bmatrix} \mathbf{x}$$

Rate of change of "energy"

$$\begin{aligned} \dot{E} &= -d\dot{x}^2 = -\frac{1}{2}\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 2d \end{bmatrix} \mathbf{x} \\ \dot{V} &= -\dot{x}^2 - x^2 = -\frac{1}{2}\mathbf{x}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} \end{aligned}$$

Lyapunov method



Lyapunov method

Quadratic form

$$\frac{1}{2}\mathbf{x}^T\left(\mathbf{A}^T\mathbf{M}+\mathbf{MA}\right)\mathbf{x}=-\frac{1}{2}\mathbf{x}^T\mathbf{N}\mathbf{x}$$

Remove states:

$$\frac{1}{\cancel{2}}\cancel{\mathbf{x}^T}\left(\mathbf{A}^T\mathbf{M}+\mathbf{MA}\right)\cancel{\mathbf{x}}=-\frac{1}{\cancel{2}}\cancel{\mathbf{x}^T}\mathbf{N}\cancel{\mathbf{x}}$$

Lyapunov's equation

$$\mathbf{A}^T\mathbf{M}+\mathbf{MA}=-\mathbf{N}$$

Lyapunov method

Lyapunov's equation

$$\underline{\mathbf{A}^T\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}}$$

Stability condition:

If for any given positive definite symmetric matrix $\mathbf{N}$, the Lyapunov equation has a unique **symmetric positive definite** solution $\mathbf{M}$, the system is **asymptotically stable**.

Positive definite

A matrix $\mathbf{P}$ is **positive definite** if:

$$\mathbf{x}^T\mathbf{P}\mathbf{x} > 0 \quad \forall \mathbf{x} \neq 0$$

But **semidefinite** if:

$$\mathbf{x}^T\mathbf{P}\mathbf{x} \geq 0$$

Positive definite

Positive definiteness

A symmetric $n \times n$ matrix $\mathbf{P}$ is positive definite if and only if:

- All its eigenvalues are positive
- Its leading principal minors are all positive
- There exists a nonsingular $n \times n$ matrix $\mathbf{L}$ so that $\mathbf{P} = \mathbf{L}^\star \mathbf{L}$.

Principal minors

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{bmatrix}, \quad \underbrace{\left\{ \begin{vmatrix} 2 \end{vmatrix}, \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}, \begin{vmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 3 \end{vmatrix} \right\}}_{\text{Leading principal minors}}$$

Positive definite

Eigenvalue analysis

Symmetric matrices can be factored as:

$$\mathbf{M} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^T$$

where $\mathbf{\Lambda}$ is a diagonal matrix containing the eigenvalues of $\mathbf{M}$ and $\mathbf{Q}$ contains the eigenvectors.

Consequences

For an eigenvector $\mathbf{q}$ and corresponding eigenvalue λ:

$$\mathbf{q}^T\mathbf{M}\mathbf{q} = \lambda\mathbf{q}^T\mathbf{q} = \lambda|\mathbf{q}|^2$$

This reveals the following properties:

- $\lambda_{min}(\mathbf{M})|\mathbf{x}|^2 \leq \mathbf{x}^T\mathbf{M}\mathbf{x} \leq \lambda_{max}(\mathbf{M})|\mathbf{x}|^2$
- $\mathbf{M} > 0$ iff $\lambda_{min}(\mathbf{M}) > 0$

Summary

Equivalent conditions for the LTI system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$

1. The system is asymptotically stable
2. The system is exponentially stable
3. All eigenvalues of **A** have strictly negative real parts.
4. For every symmetric positive definite matrix **Q**, there is a unique solution **P** to the following Lyapunov equation:

   $$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$$

   and **P** is symmetric and positive definite.

Topic

Input-Output stability

LTI solution

$$\mathbf{y}(t) = \underbrace{\mathbf{C}e^{\mathbf{A}t}\mathbf{x}_0}_{\text{Zero Input Resp.}} \underbrace{+\mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)}_{\text{Zero State Resp.}}$$

Bounded Input

An input $u(t)$ is bounded if there exists a constant u_m such that $|u(t)| \leq u_m < \infty$, for all $t \geq 0$.

Bounded Input Bounded Output stability

A system is said to be BIBO stable if every bounded input excites a bounded output for $\mathbf{x}(0) = \mathbf{0}$.

Asymptotic stability implies BIBO stability

Every pole of $\mathbf{G}(s)$ is an eigenvalue of $\mathbf{A}$.

Note

$\mathbf{G}(s)$ only tells us about BIBO stability.

Input-Output stability

SISO system

$$\begin{aligned} y(t) &= \int_0^t g(\tau)u(t-\tau)d\tau \\ \Rightarrow |y(t)| &= \left|\int_0^t g(\tau)u(t-\tau)d\tau\right| \\ \Rightarrow |y(t)| &\leq \int_0^t |g(\tau)||u(t-\tau)|d\tau \\ \Rightarrow |y(t)| &\leq \int_0^t |g(\tau)|u_m d\tau \end{aligned}$$

Bounded **I**nput

An input $u(t)$ is bounded if there exists a constant u_m such that $|u(t)| \leq u_m < \infty$, for all $t \geq 0$.

Input-Output stability

SISO system

$$
\begin{aligned}
|y(t)| &\leq \int_0^\infty |g(\tau)| u_m d\tau \\
|y(t)| &\leq M u_m \leq \infty
\end{aligned}
$$

Bounded Input

An input $u(t)$ is bounded if there exists a constant u_m such that $|u(t)| \leq u_m < \infty$, for all $t \geq 0$.

Bounded Output

The output is bounded if:

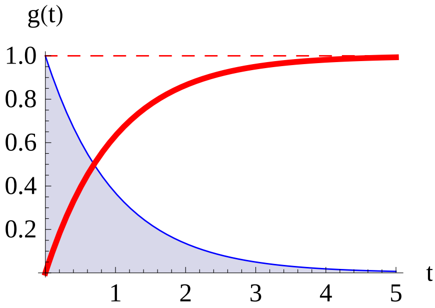
$$\int_0^\infty |g(\tau)| d\tau \leq M \leq \infty$$

SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 1

$$g(s) = \frac{1}{s+1} \quad \Rightarrow g(t) = e^{-t}$$

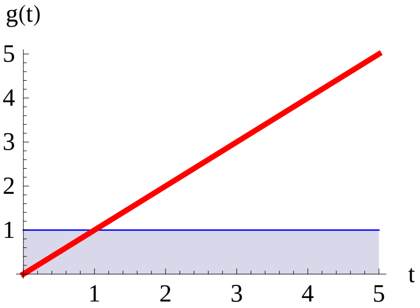


SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 2

$$g(s) = \frac{1}{s} \quad \Rightarrow g(t) = 1$$

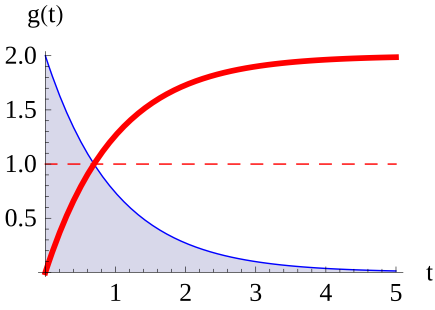


SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 3

$$g(s) = \frac{-s+1}{s+1} \quad \Rightarrow g(t) = 2e^{-t} - \text{DiracDelta}[t]$$

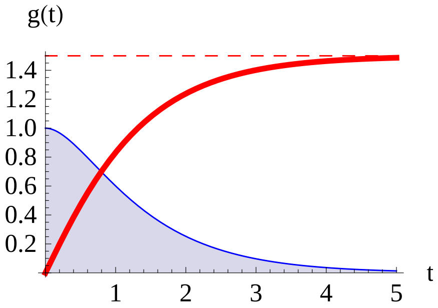


SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 4

$$g(s) = \frac{s+3}{(s+1)(s+2)} \quad \Rightarrow g(t) = e^{-2t}\left(-1+2e^{t}\right)$$

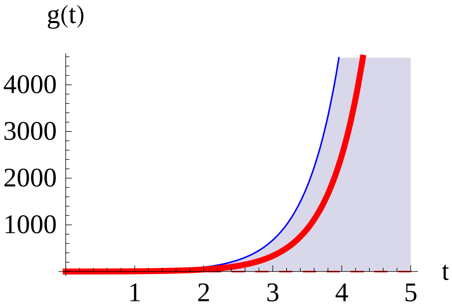


SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 5

$$g(s) = \frac{s+3}{(s+1)(s-2)} \quad \Rightarrow g(t) = \frac{1}{3}e^{-t}\left(-2+5e^{3t}\right)$$

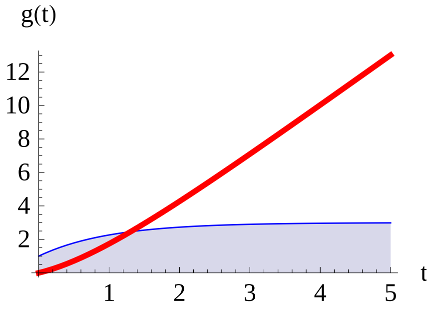


SISO system

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

Example 6

$$g(s) = \frac{s+3}{s(s+1)} \quad \Rightarrow g(t) = 3 - 2e^{-t}$$



Input-Output stability

If a system is BIBO stable:

- The output excited by $u(t) = A$ approches $\hat{g}(0)A$ as $t \to \infty$.
- The output excited by $u(t) = \sin(\omega t)$ approaches $|\hat{g}(j\omega)| \sin(\omega t + \angle\hat{g}(j\omega))$ as $t \to \infty$.

BIBO & Poles

A SISO system with proper rational transfer function $\hat{g}(s)$ is BIBO stable if and only if every pole of $\hat{g}(s)$ has a negative real part.

BIBO for MIMO

- A multivariable system is BIBO stable if every element of its impulse response matrix $\mathbf{G}(t)$: $g_{ij}(t)$ is absolutely integrable in $[0, \infty)$.
- A multivariable system is BIBO stable if and only of every pole of every element of its transfer matrix $\hat{\mathbf{G}}(s)$: $\hat{g}_{ij}(s)$ has a negative real part.