TTK4115

Lecture 6

State estimation, output feedback

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This lecture

1. State estimation

   Measurement equation inversion

   Open-loop estimator

   Closed-loop estimator

   Noise suppression: A simple example

   Band-limited differentiation: A simple example

2. Conveyor belt case study

3. Certainty Equivalence

4. Separation principle

Topic

1. State estimation

    Measurement equation inversion

    Open-loop estimator

    Closed-loop estimator

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3. Certainty Equivalence

4. Separation principle

State estimation

Problem

With the system:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

Given $\mathbf{y}(t)$ and $\mathbf{u}(t)$.
What is $\mathbf{x}(t)$?

Applications

- State feedback without direct measurements
- Navigation, GPS, robotics, motion control
- Radar, sonar etc..
- Fault detection & diagnosis
- Disturbance estimation

State estimation

Notation

$\mathbf{x}$: Signal to be estimated
$\hat{\mathbf{x}}$: Estimate
$\mathbf{y}$: Output (no noise)
$\mathbf{y}_m$: Measured output: $\mathbf{y}_m = \mathbf{y} + \mathbf{n}$
$\mathbf{n}$: Noise

Estimation error

The estimation error is defined as:

$$\mathbf{e} \triangleq \mathbf{x} - \hat{\mathbf{x}}$$

The error is **dynamic**:

$$\dot{\mathbf{e}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - \dot{\hat{\mathbf{x}}}$$

Observer design is concerned with choosing the **estimate update law** $\dot{\hat{\mathbf{x}}}(\mathbf{x}, \hat{\mathbf{x}}, \mathbf{u})$ in a clever way so that:

$$\mathbf{e} \to \mathbf{0}, \quad t \to \infty$$

Topic

1. State estimation

- Measurement equation inversion
- Open-loop estimator
- Closed-loop estimator
- Noise suppression: A simple example
- Band-limited differentiation: A simple example

2. Conveyor belt case study

3. Certainty Equivalence

4. Separation principle

Method I: Measurement equation inversion

If

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

and **C** is square and invertible, we can reconstruct the states:

$$\mathbf{x}(t) = \mathbf{C}^{-1}\mathbf{y}(t)$$

If **C** is *not* square and invertible, but there are *more measurements than states*, we can use:

$$\mathbf{x}(t) = \left[\mathbf{C}^T\mathbf{C}\right]^{-1}\mathbf{C}^T\mathbf{y}(t)$$

which is a *least squares* estimator, minimizing:

$$\left[\mathbf{y}(t) - \mathbf{C}\mathbf{x}(t)\right]^T\left[\mathbf{y}(t) - \mathbf{C}\mathbf{x}(t)\right]$$

Topic

Method II: Open-loop estimator

Idea

If we have a good model, we can use it to estimate the states:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u}$$

by simulating the model given the real input $\mathbf{u}(t)$.

Caveats

- Sensitive to modeling errors and disturbances
- Often slow convergence rate
- Does not work for unstable systems

Method II: Open-loop estimator

Plant:

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}$$

Open loop estimator:

$$\begin{aligned}\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} \\ \hat{\mathbf{y}} &= \mathbf{C}\hat{\mathbf{x}}\end{aligned}$$

Error dynamics

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} - (\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u}) = \mathbf{A}\mathbf{e}$$

Note

$\mathbf{A}$ must be stable: $\mathbf{e} \to \mathbf{0}, \quad t \to \infty$

Topic

Method III: Closed-loop estimator

Idea

Use an open-loop estimator in conjunction with a feedback from the measurement error:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}(\mathbf{y} - \mathbf{C}\hat{\mathbf{x}})$$

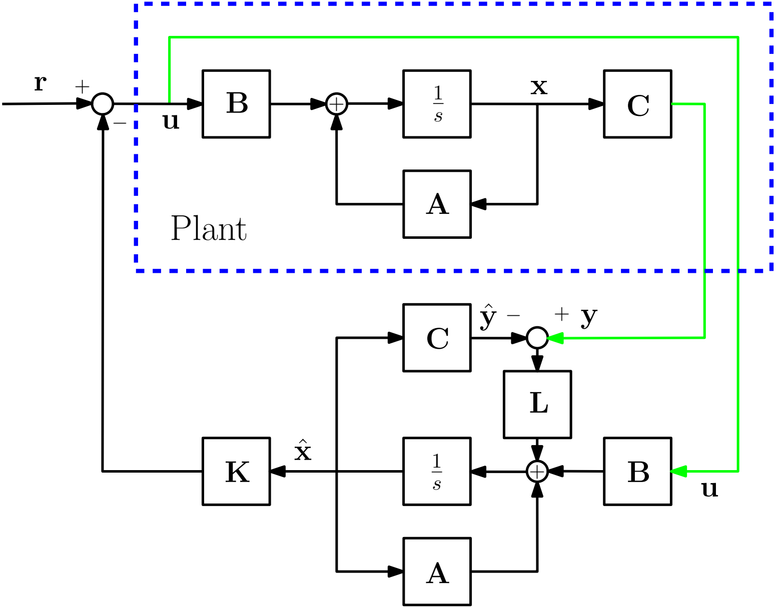
Key points

- The feedback corrects to a certain degree for uncertainties due to modeling errors and disturbances.
- Subject to certain conditions, $\mathbf{L}$ can be chosen such that $\hat{\mathbf{x}}(t) \to \mathbf{x}(t)$ with desired convergence rate.
- An asymptotically stable estimator can be achieved for an unstable system, due to stabilizing feedback.

Theorem

All eigenvalues of $\mathbf{A} - \mathbf{LC}$ can be *assigned arbitrarily* by selecting a real constant matrix $\mathbf{L}$ if and only if $\{\mathbf{A}, \mathbf{C}\}$ are **observable**.

Method III: Closed-loop estimator



Method III: Closed-loop estimator

Plant:

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\
\mathbf{y} &= \mathbf{C}\mathbf{x}
\end{aligned}
$$

Closed loop estimator:

$$
\begin{aligned}
\dot{\hat{\mathbf{x}}} &= \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \overbrace{\mathbf{L}(\mathbf{y}_m - \hat{\mathbf{y}})}^{\text{Injection term}}, \quad \mathbf{L}(\mathbf{y}_m - \hat{\mathbf{y}}) = \mathbf{L}\mathbf{C}(\mathbf{x} - \hat{\mathbf{x}}) + \mathbf{L}\mathbf{n} = \mathbf{L}\mathbf{C}\mathbf{e} + \mathbf{L}\mathbf{n} \\
\hat{\mathbf{y}} &= \mathbf{C}\hat{\mathbf{x}}
\end{aligned}
$$

Error: $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$

$$
\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e} - \mathbf{L}\mathbf{n}
$$

Note

$\mathbf{L}$ can be chosen so that $\mathbf{e} \to \mathbf{0}$ as $t \to \infty$.
We can in fact choose the eigenvalues of $\mathbf{A} - \mathbf{L}\mathbf{C}$ freely if the system is observable!

Topic

Noise suppression: A simple example

Plant

$$\dot{x} = -\frac{1}{T}x + \frac{1}{T}u, \quad y_m = x + n$$

Task: **Estimate x using an observer to suppress noise in the measurement.**

Closed loop estimator

Estimate update equation:

$$\dot{\hat{x}} = \underbrace{-\frac{1}{T}\hat{x} + \frac{1}{T}u}_{\text{Copy dynamics}} + \underbrace{l(y_m - \hat{y})}_{\text{Injection term}}$$

Error dynamics:

$$\dot{e} = \underbrace{\left(-\frac{1}{T} - l\right)}_{\mathbf{A}-\mathbf{LC}} e - ln$$

Pole placement

Apply the *observer gain*:

$$l = -\frac{1}{T} + \frac{1}{T_d}$$

to place the pole at $\lambda_1 = -\frac{1}{T_d}$. A small time constant yields fast convergence.

Noise suppression: A simple example

Error dynamics

$$\dot{e} = -\tfrac{1}{T_d} e + \left(\tfrac{1}{T} - \tfrac{1}{T_d}\right) n$$

Error due to noise: Laplace model

- The observer low-passes the noise contribution in the estimate.
- Higher observer gain yields a higher cut-off frequency.

$$\frac{\hat{e}(s)}{\hat{n}(s)} = \frac{\left(\frac{1}{T} - \frac{1}{T_d}\right)}{\left(s + \frac{1}{T_d}\right)}$$

Compare with original situation:

$$\hat{x} = y_m \quad \Rightarrow \frac{\hat{e}(s)}{\hat{n}(s)} = -1$$

Topic

Band-limited differentiation: A simple example

Plant

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y_m = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + n$$

Task: **Estimate the derivative $\dot{x}_1 = x_2$ using an observer.**

Closed loop estimator

Estimate update equation:

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}}_{\text{Copy dynamics}} + \underbrace{\overbrace{\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}}^{\mathbf{L}} \overbrace{(y_m - \hat{y})}^{\mathbf{Ce+n}}}_{\text{Injection term}}$$

Error dynamics:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} n$$

Band-limited differentiation: A simple example

Closed loop estimator

Estimate update equation:

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}}_{\text{Copy dynamics}} + \underbrace{\overbrace{\begin{bmatrix} l_1 \\ l_2 \end{bmatrix}}^{\mathbf{L}} \overbrace{(y_m - \hat{y})}^{\mathbf{Ce}+\mathbf{n}}}_{\text{Injection term}}$$

Error dynamics:

$$\begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} n$$

Pole placement

Apply the *observer gain*:

$$\begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 2\omega_n \\ \omega_n^2 \end{bmatrix}$$

to place the poles at $\lambda_1 = \lambda_2 = -\omega_n$. A high natural frequency yields fast performance.

Band-limited differentiation: A simple example

Closed loop estimator

Estimate update equation:

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -2\omega_n & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 2\omega_n \\ \omega_n^2 \end{bmatrix} y_m$$

Band-limited differentiation: Laplace model

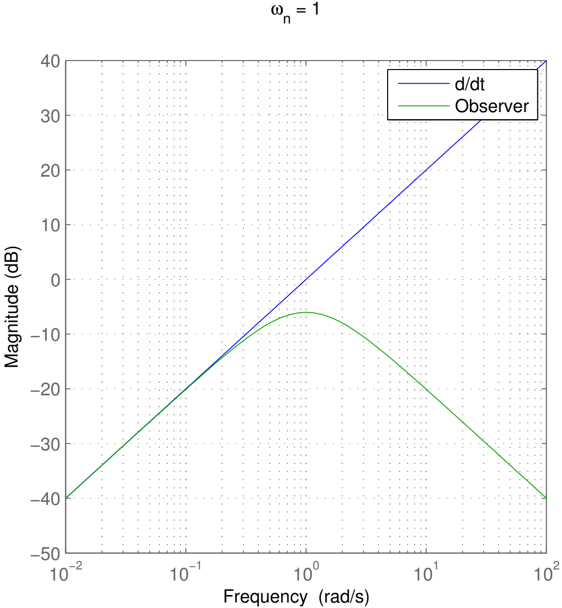
- The observer uses band-limited differentation.
- Higher observer gain yields a higher cut-off frequency.
- Warning: noise below cut-off is also differentiated.

$$\frac{\hat{\hat{x}}_2(s)}{\hat{y}_m(s)} = \frac{s\omega_n^2}{(s+\omega_n)^2}$$

Compare with original situation:

$$\frac{\hat{\hat{x}}_2(s)}{\hat{y}_m(s)} = s$$

Band-limited differentiation: A simple example



Topic

Example: Conveyor belt

Model

Transport belt model:

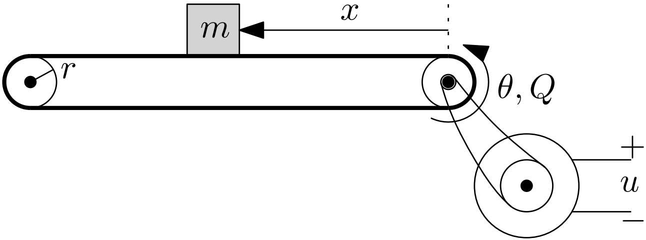
$$x = \theta r$$

Belt dynamics:

$$m\ddot{x}(t) + \frac{j}{r}\ddot{x}(t) + d\dot{x}(t) = \frac{1}{r}Q(t), \quad \text{Mass } m \text{ unknown}$$

Motor dynamics:

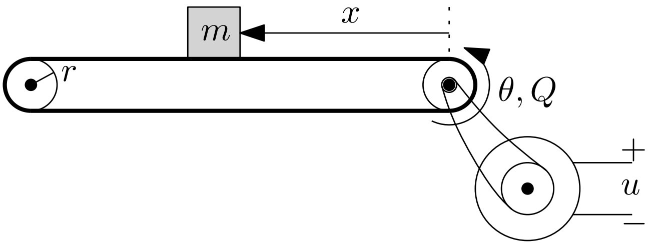
$$T\dot{Q}(t) + Q(t) = u(t)$$



Example: Conveyor belt

State-space dynamics

$$\begin{bmatrix} \dot{Q} \\ \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -\frac{1}{T} & 0 & 0 \\ \frac{1}{j+mr} & -\frac{dr}{j+mr} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} \frac{1}{T} \\ 0 \\ 0 \end{bmatrix} u$$



Example: Conveyor belt

Parameters

$$T = 1, \quad j = 10, \quad r = 1, \quad d = 1, \quad \underbrace{m = 1}_{\text{unknown}}$$

Plant State-space dynamics - numerical

$$\begin{bmatrix} \dot{Q} \\ \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{11} & -\frac{1}{11} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Eigenvalues:

$$\lambda = \left\{ 0, -1, -\frac{1}{11} \right\} \quad \text{Marginally stable}$$

Observer model: $m = 0$

$$\begin{bmatrix} \dot{Q} \\ \ddot{x} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ \frac{1}{10} & -\frac{1}{10} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

Three measurement models

Case 1: Encoder on roller wheel

$$y_m = x_m = \theta r + n_x \quad \Rightarrow y_m = \overbrace{\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}^{\mathbf{c}_1} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + n_x$$

Quantization noise on position signal: n_x.

Case 2: Add tachometer on roller shaft.

$$\mathbf{y}_m = \begin{bmatrix} \dot{x}_m \\ x_m \end{bmatrix} + \begin{bmatrix} n_{\dot{x}} \\ n_x \end{bmatrix} \quad \Rightarrow \mathbf{y}_m = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{C}_2} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} n_{\dot{x}} \\ n_x \end{bmatrix}$$

Quantization noise on speed signal: $n_{\dot{x}}$.

Case 3: Add strain transducer on motor shaft.

$$\mathbf{y}_m = \begin{bmatrix} Q \\ \dot{x}_m \\ x_m \end{bmatrix} + \begin{bmatrix} n_Q \\ n_{\dot{x}} \\ n_x \end{bmatrix} \quad \Rightarrow \mathbf{y}_m = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{\mathbf{C}_3} \begin{bmatrix} Q \\ \dot{x} \\ x \end{bmatrix} + \begin{bmatrix} n_Q \\ n_{\dot{x}} \\ n_x \end{bmatrix}$$

Vibrations/electrical noise: n_Q.

All measurement models render the plant observable. Noise is assumed to be of *high frequency*.

Method I: Measurement equation inversion

Conveyor belt: **Case 3**

Only full state measurements may be used.

$$\mathbf{y}_m = \mathbf{C}_3\mathbf{x} + \mathbf{n}, \quad \mathbf{C}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Measurement equation inversion

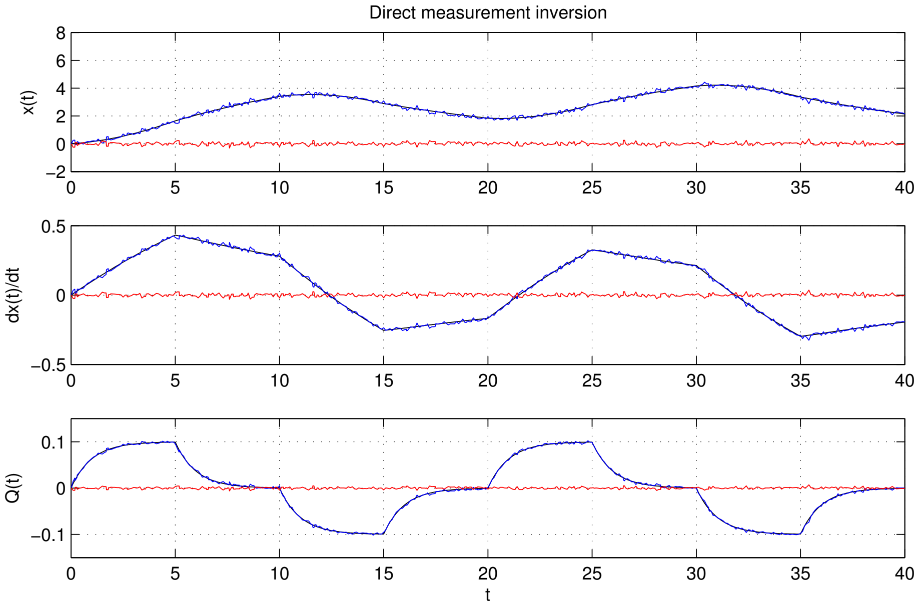
With $\mathbf{C}_3 = \mathbb{I}$:

$$\hat{\mathbf{x}}(t) = \mathbf{C}_3^{-1}\mathbf{y}_m(t) = \mathbf{x} + \mathbf{n}$$

The error reads as:

$$\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}} = -\mathbf{n}$$

Method I: Measurement equation inversion

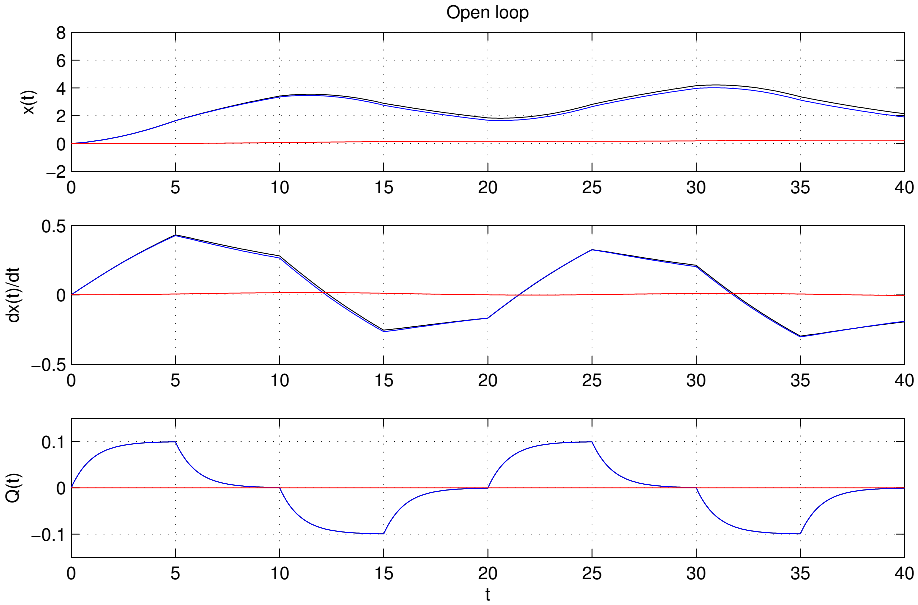


Method I: Measurement equation inversion

Comments

This method picks up the measurement noise by design: $\hat{\mathbf{x}} = \mathbf{y}_m$. With noisy sensors this may be problematic.

Method II: Open loop observer



Method II: Open loop observer

Comments

As expected the open-loop observer will diverge due to the model error in the simulator dynamics. On the other hand, the methodology is ”noise-proof”.

Method III: Closed loop observer

Pole-placement

Pole-placement for the observer gain is not a trivial exercise. Some general guidelines are:

- The error dynamics should be faster than the plant itself: $\times 2 - 20$.
- Noisy plants generally warrant a slower observer, i.e. placing poles with smaller negative real part. This to avoid amplification of noise.
- The Kalman filter is a closed loop observer. It takes the guesswork out of the tuning and is often the preferable approach. Pay attention in the last lecture weeks.

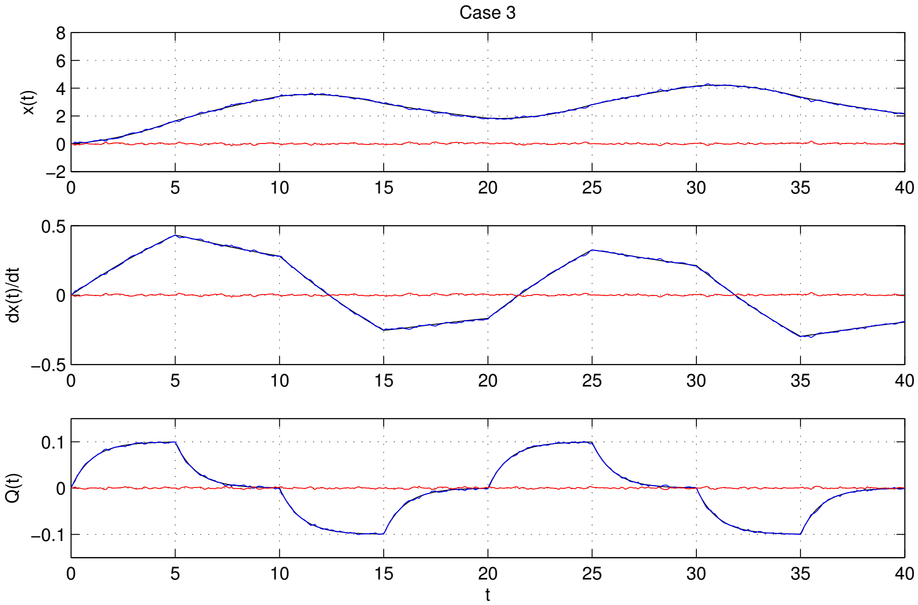
Placing poles for Cases 1,2,3

- Poles are initially placed along a circular arc of radius $R = 5$. The most negative pole in the plant is $\lambda_{\text{max}} = -1$. There is some noise in the plant so it is best to be conservative.
- The arc opens to an angle of 60°. This is to avoid bunching poles close together.

For the conveyor:

$$\lambda = -5\{e^{-30 \cdot \frac{pi}{180} i},\ e^{0 \cdot \frac{pi}{180} i},\ e^{+30 \cdot \frac{pi}{180} i}\} = \{-4.33 + 2.5i,\ -5,\ -4.33 - 2.5i\}$$

Case 3: All states measured

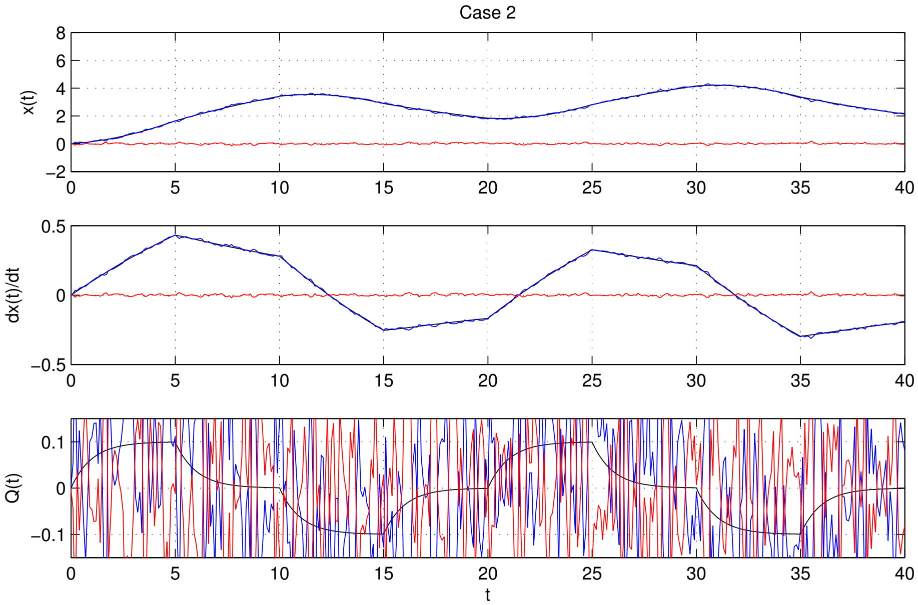


Case 3: All states measured

Comments

The observer filters out significant portions of the measurement noise. The convergence to the true states is due to the error feedback.

Case 2: Shaft torque not measured



Case 2: Shaft torque not measured

Comments

The lack of a torque measurement means *differentiation* must be employed in torque estimation, albeit not in a pure manner. This is seen in the belt dynamics:

$$\left(m + \frac{j}{r}\right)\ddot{x}(t) + d\dot{x}(t) = \frac{1}{r}Q(t)$$

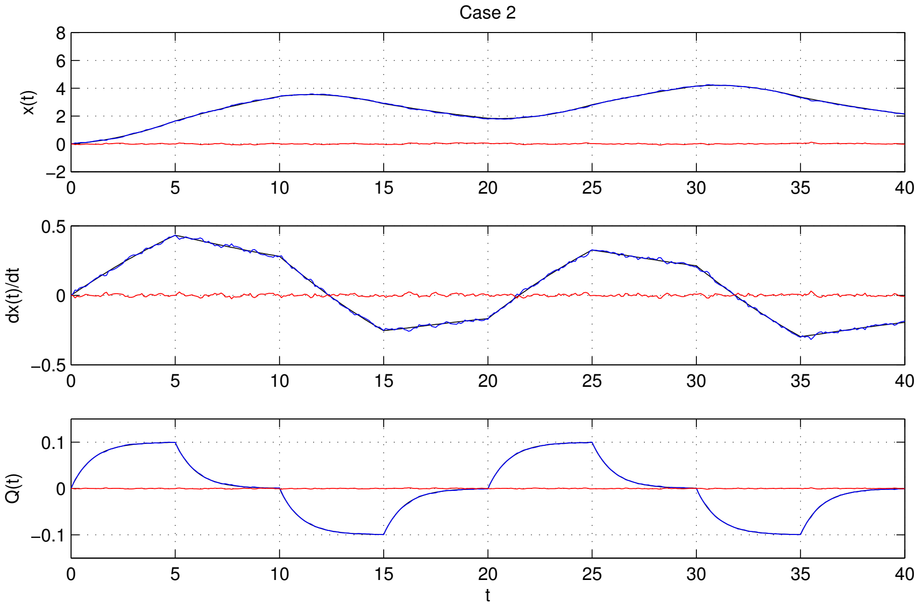
This suggests that the errors are due amplification of noise. The observer must be tuned so as to avoid this phenomena.

Better tuning

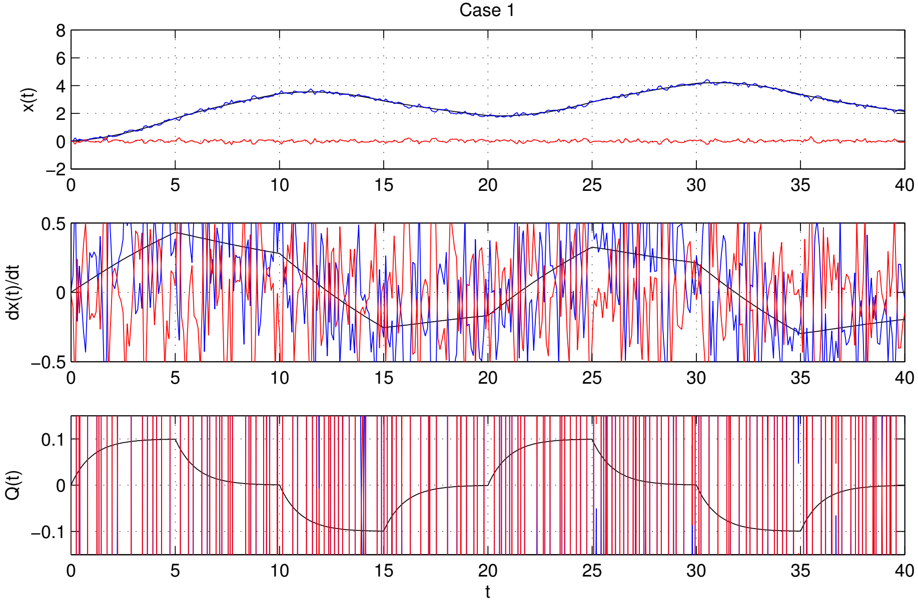
Moderating the eigenvalues and tuning yields a potentially better set of poles:

$$\lambda_{\text{tuned}} = \{-1,\ -2.3,\ -2.8\}$$

Case 2: Shaft torque not measured - Tuned



Case 1: Shaft torque and band velocity not measured



Case 1: Shaft torque and band velocity not measured

Comments

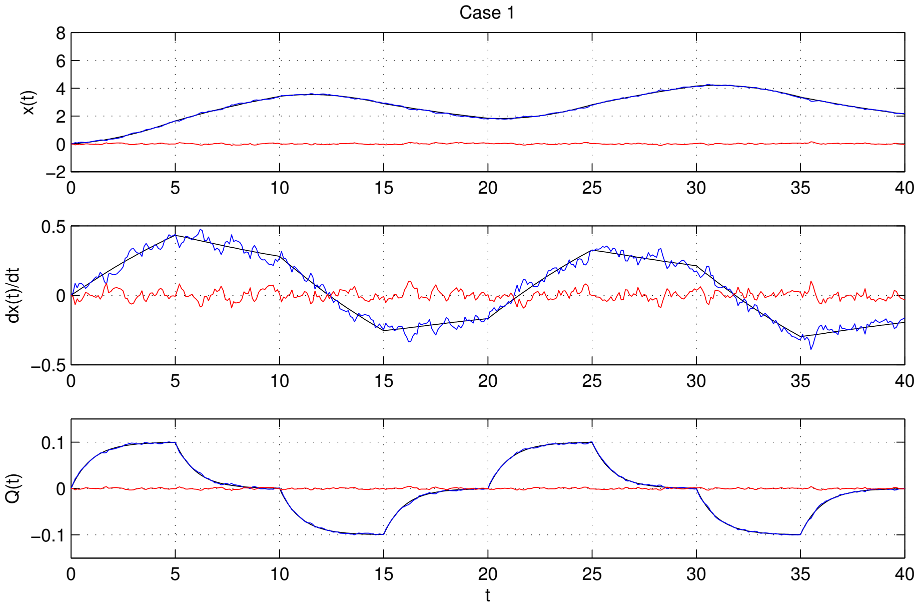
Here the noise amplification makes the observer worthless. A less aggressive tuning is necessary!

Better tuning

Moderating the eigenvalues and tuning yields a potentially better set of poles:

$$\lambda_{\text{tuned}} = \{-1,\ -1.5 + 1i,\ -1.5 - 1i\}$$

Case 1: Shaft torque and band velocity not measured - Tuned



Topic

Certainty Equivalence

If the full state is not available for feedback, one can use the estimate $\hat{\mathbf{x}}$, an approach known as *certainty equivalence*. The estimate is generated from the output $\mathbf{y}$, thus motivating the name *output feedback*.

Output feedback controller

The feedback no longer consists of a simple constant matrix $\mathbf{K}$, but is in fact a dynamic system of its own. This means that care must be exercised due to the added complexity.

In the loop

Plant :

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{y} = \mathbf{C}\mathbf{x}$$

Estimator :

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}[\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}]$$

Output feedback :

$$\mathbf{u} = -\mathbf{K}\hat{\mathbf{x}} + \mathbf{K}_r\mathbf{r}, \quad \mathbf{K}_r = [\mathbf{C}(\mathbf{B}\mathbf{K} - \mathbf{A})^{-1}\mathbf{B}]^{-1}$$

Regulator system

Estimate update equation:

$$\dot{\hat{\mathbf{x}}} = \mathbf{A}\hat{\mathbf{x}} + \mathbf{B}[-\mathbf{K}\hat{\mathbf{x}} + \mathbf{K}_r\mathbf{r}] + \mathbf{L}[\mathbf{y} - \mathbf{C}\hat{\mathbf{x}}] = (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C})\hat{\mathbf{x}} + \mathbf{B}\mathbf{K}_r\mathbf{r} + \mathbf{L}\mathbf{y}$$

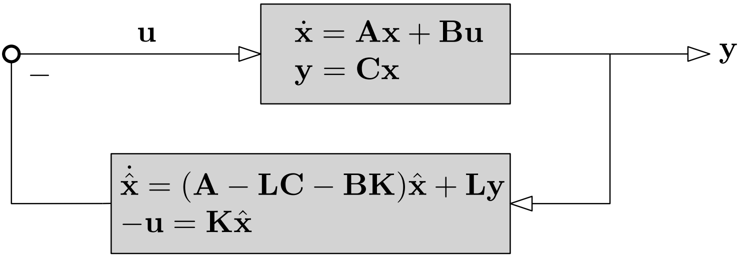
Laplace domain system[a]:

$$\hat{\hat{\mathbf{x}}}(s) = (s\mathbb{I} - \mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{L}\mathbf{C})^{-1}[\mathbf{B}\mathbf{K}_r\hat{\mathbf{r}}(s) + \mathbf{L}\hat{\mathbf{y}}(s)]$$

$$\Rightarrow \hat{\mathbf{u}}(s) = -\mathbf{K}(s\mathbb{I} - \mathbf{A} + \mathbf{B}\mathbf{K} + \mathbf{L}\mathbf{C})^{-1}[\mathbf{B}\mathbf{K}_r\hat{\mathbf{r}}(s) + \mathbf{L}\hat{\mathbf{y}}(s)] + \mathbf{K}_r\hat{\mathbf{r}}(s)$$

[a] $\hat{(\cdot)}$ denotes the Laplace Transform.

Block diagram: $\mathbf{r} \equiv \mathbf{0}$



Topic

Output feedback

Feedback from estimated states

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\hat{\mathbf{x}} + \mathbf{r}) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e} + \mathbf{B}\mathbf{r} \\ \mathbf{y} &= \mathbf{C}\mathbf{x}\end{aligned}$$

Error: $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}$$

Note

$$\hat{\mathbf{x}} = \mathbf{x} - \mathbf{e}$$

Output feedback

Feedback from estimated states

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BKe} + \mathbf{Br} \\ \mathbf{y} &= \mathbf{Cx} \end{aligned}$$

Error: $\mathbf{e} = \mathbf{x} - \hat{\mathbf{x}}$

$$\dot{\mathbf{e}} = \dot{\mathbf{x}} - \dot{\hat{\mathbf{x}}} = (\mathbf{A} - \mathbf{LC})\mathbf{e}$$

Closed loop system

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} &= \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r} \\ \mathbf{y} &= \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \end{aligned}$$

Output feedback

Closed loop system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Transfer function

$$\frac{\mathbf{y}(s)}{\mathbf{r}(s)} = \mathbf{G}(s) = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} s\mathbb{I}-\mathbf{A}+\mathbf{BK} & -\mathbf{BK} \\ \mathbf{0} & s\mathbb{I}-\mathbf{A}+\mathbf{LC} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix}$$

$$= \mathbf{C}(s\mathbb{I}-\mathbf{A}+\mathbf{BK})^{-1}\mathbf{B}$$

Output feedback

Closed loop system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A}-\mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A}-\mathbf{LC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{r}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

Separation principle

The input output response:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbb{I} - \mathbf{A} + \mathbf{BK})^{-1}\mathbf{B}$$

and error dynamics:

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{LC})\mathbf{e}$$

are separated!

Separation principle

The separation principle tells us that the overall loop is stable if the state-feedback and observer are individually stable.

Proof

The estimation error is denoted $\mathbf{e} \triangleq \mathbf{x} - \hat{\mathbf{x}}$, giving:

$$\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\hat{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e} \\
\dot{\mathbf{e}} &= \underbrace{\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}}_{\dot{\mathbf{x}}} - \underbrace{[\mathbf{A}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{L}\mathbf{C}\mathbf{e}]}_{\dot{\hat{\mathbf{x}}}} = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}
\end{aligned}$$

The full system has this equation:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ 0 & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

and the characteristic polynomial shows that the eigenvalues are independent:

$$\begin{vmatrix} \lambda\mathbb{I} - (\mathbf{A} - \mathbf{B}\mathbf{K}) & -\mathbf{B}\mathbf{K} \\ 0 & \lambda\mathbb{I} - (\mathbf{A} - \mathbf{L}\mathbf{C}) \end{vmatrix} = |\lambda\mathbb{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})||\lambda\mathbb{I} - (\mathbf{A} - \mathbf{L}\mathbf{C})|$$