

Computational Physics

Lecture 4

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git clone <https://github.com/ukzncompphys/lecture4.git>
wget www.cita.utoronto.ca/~sievers/compphys/lecture4.tar.gz
(followed by tar -xzf lecture4.tar.gz)

Tutorial Problems

- How can you list all files in a directory, with the most recent one displayed last? (5)
- How can you display the first 25 lines of a file? How can you display just lines 16 through 25? (5)
- How can you search for all commands relating to python? Hint - it will be a use of “man”. (5)
- Make a new text file with your name, email address, what you’re working on for your honours project, plus a few things you’d like to learn in the course. Initialize a git repository and commit this file. (5)
- Put your answers to the other tutorial questions into another text file. Add this to the repository also. (5)
- Make a github account and push the repository onto github. Email me your github name so I can have a look at your file/answers. (10)

Fourier Transforms

- Functions can be represented in many different ways
- We normally use “real” space - $f(x)$
- Generally, arbitrarily many transforms exist to represent functions in different spaces - $F(y)=Af(x)$ for some matrix A and some new variable y .
Iff A is invertible, $f(x)=A^{-1}F(y)$
- One important basis nature has picked out is complex exponentials/sines and cosines. Fundamental across physics, particularly quantum mechanics.

Fundamental Definition

- $F(k) = \int f(x) \exp(2\pi i k x) dx$ (where $k = 1/\omega$)
- Integral gets rid of x , replaces with k . New function has amplitude and phase as a function of k .
- Quantum mechanics - de Broglie says $p = \hbar k$. So, Fourier transform position to get momentum.
- Fourier transform electric field $E(t)$ to get frequency spectrum.
- Fourier transform to get fast correlations, convolutions, many other things.

DFT (Discrete FT)

- Computers don't do continuous. Not enough RAM for starters...
- Function exists over finite range in x at finite number of points.
- If input function has n points, output can only have n k 's.
- Gives rise to discrete Fourier Transform (DFT)
- $F(k) = \sum f(x) \exp(2\pi i k x / N)$ for N points and $0 \leq k < N$
- What would DFT of $f(0)=1$, otherwise $f(x)=0$ look like?
- What would DFT of $f(x)=1$ look like?
- DFTs have subtle behaviours not seen in continuous, infinite FTs.

Inverse

- One way to think about DFT is as a matrix multiply.
- $F(k) = Af$, $A_{mn} = \exp(2\pi i mn/N)$
- But look - $A_{mn} = A_{nm}$, so matrix is symmetric.
- Also, columns are orthogonal under conjugation:
 $\sum \exp(-2\pi i kx) \exp(2\pi i k'x) = \sum \exp(2\pi i (k' - k)x)$. N if $k' = k$, otherwise 0 .
- So, $A^{-1} = 1/N \cdot \text{conj}(A)$. $\text{IFT} = 1/N \sum F(k) \exp(-2\pi i kx)$.
- Get back to where we started by just doing another DFT with a sign flip, then divide by # of data points.
- Alternative: divide by \sqrt{N} in both DFT and IFT, (not standard computationally)

Numpy Complex

```
import numpy
def exp_prod(m,n,N):
    #define imaginary unity
    J=numpy.complex(0,1)
    #now rest of code is just like for real numbers
    x=numpy.arange(0.0,N)*2*J*numpy.pi/N
    return numpy.sum(numpy.exp(-1*x*m)*numpy.exp(x*n))
if __name__=="__main__":
    print exp_prod(0,0,8)
    print exp_prod(2,4,8)
    print exp_prod(3,3,8)
    print exp_prod(0,7,8)
```

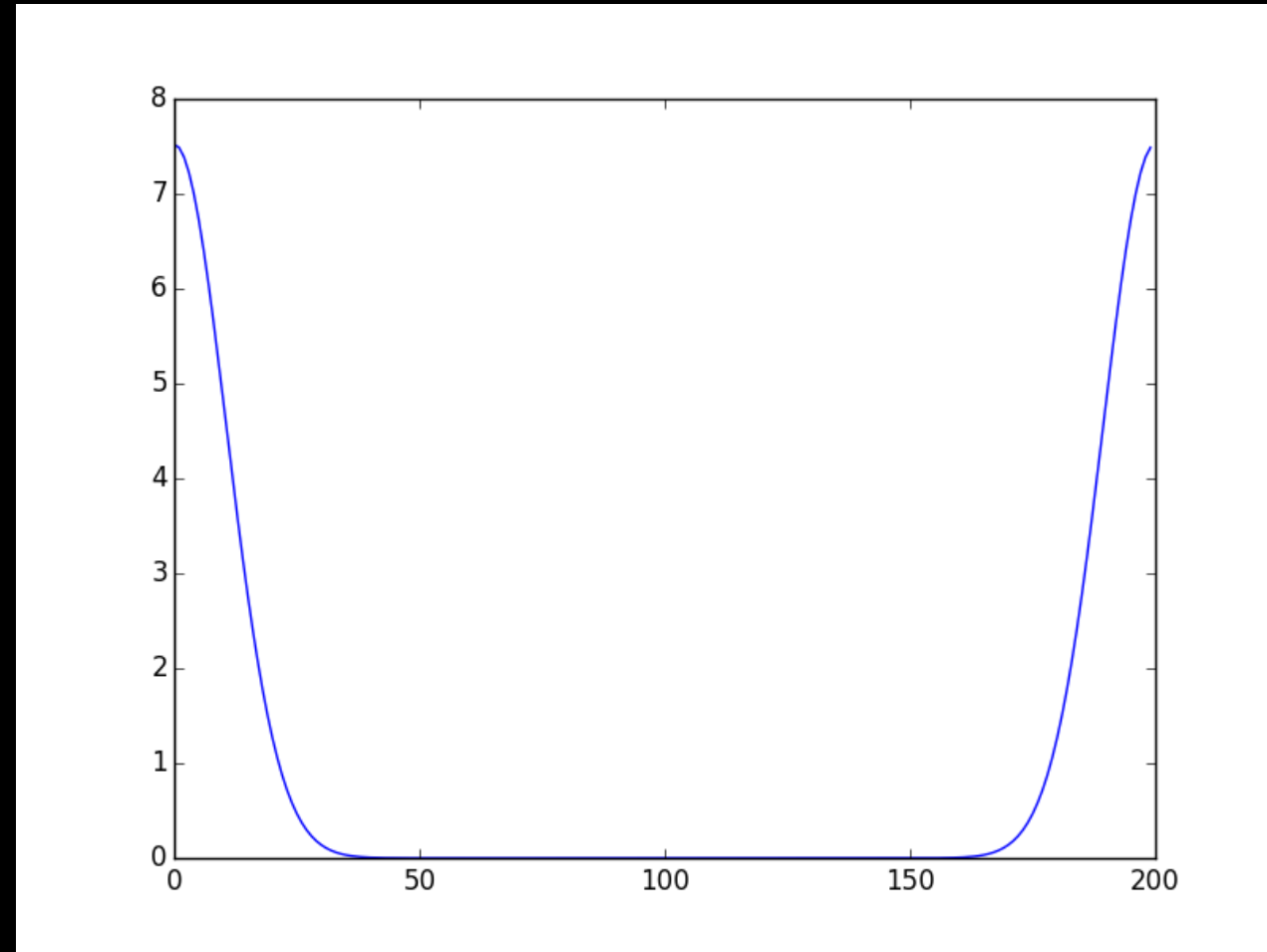
```
Jonathans-MacBook-Pro:lecture4 sievers$ python dft_columns.py
(8+0j)
(-4.28626379702e-16+4.4408920985e-16j)
(8+0j)
(3.44169137634e-15-1.11022302463e-15j)
Jonathans-MacBook-Pro:lecture4 sievers$
```

- Let's check orthogonality, need complex #'s.
- `numpy.complex(re,im)` will make a complex #
- numpy functions usually defined for complex #'s.

DFTs with Numpy

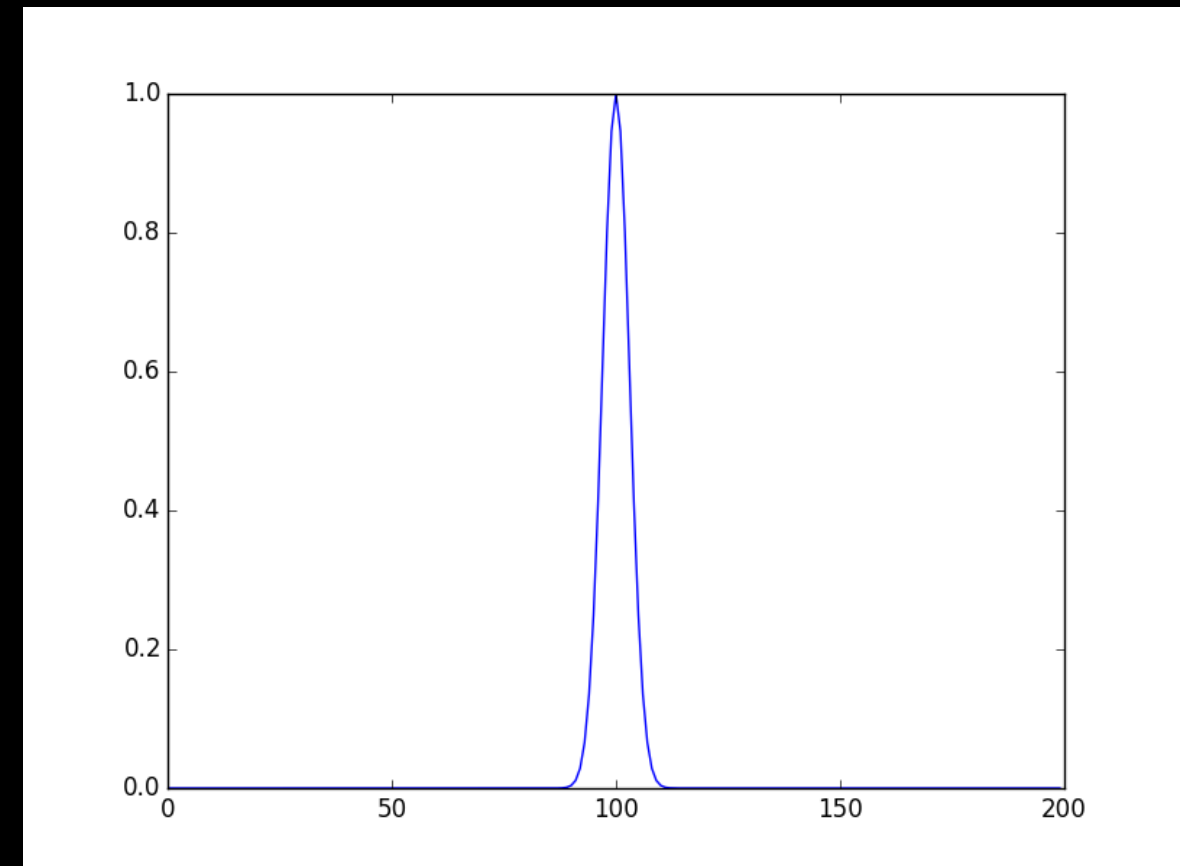
- Numpy has many Fourier Transform operations
- (for reasons to be seen) they are called *Fast* Fourier Transforms - FFT is one way of implementing DFTs.
- FFT's live in a submodule of numpy called FFT
- `xft=numpy.fft.fft(x)` takes DFT
- `x=numpy.fft.ifft(x)` takes inverse DFT
- Numpy normalizes such that $f == \text{fft}(\text{ifft}(f)) == \text{ifft}(\text{fft}(f))$

DFT in Action



```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-10,10,0.1)
y=numpy.exp(-0.5*x**2/(0.3**2))
yft=numpy.fft.fft(y)
plt.plot(numpy.abs(yft))
plt.savefig('gauss_dft')
plt.show()
```



- Left: input Gaussian
- Top: DFT of the Gaussian

Periodicity

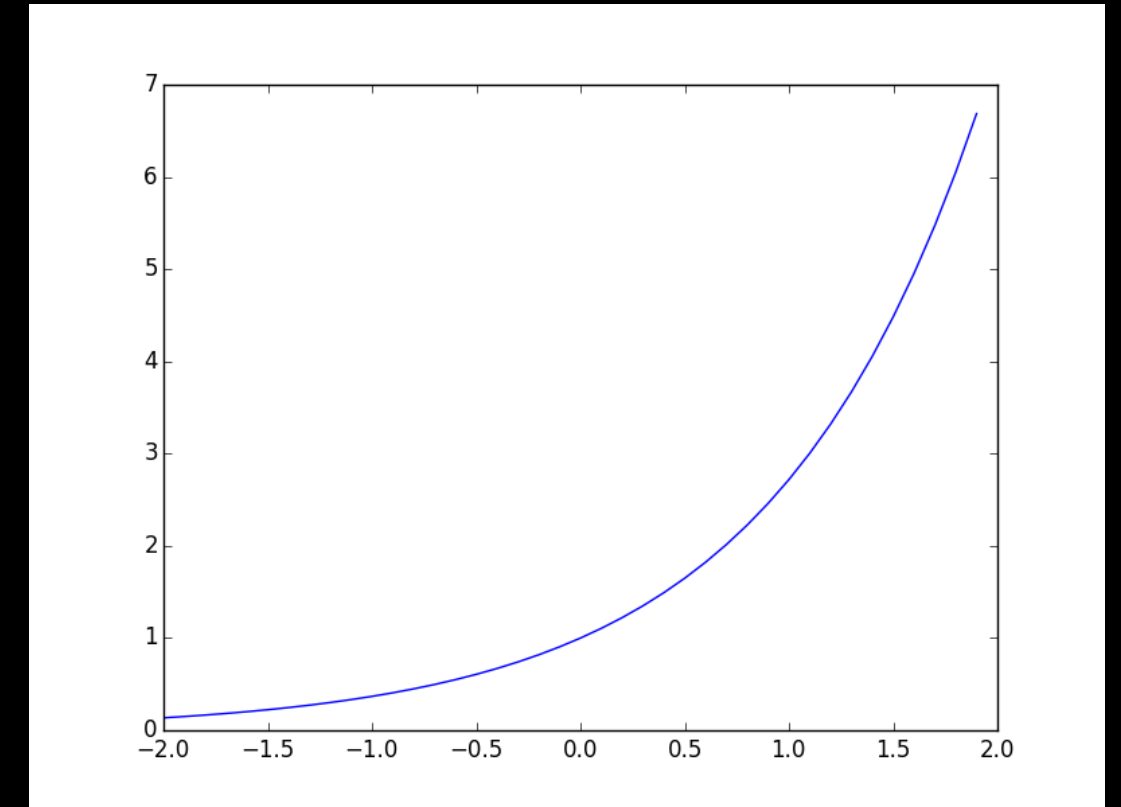
- $f(x) = \sum F(k) \exp(2\pi i k x / N)$
- What is $f(x+N)$? $\sum F(k) \exp(2\pi i k (x+N) / N)$
- $= \sum F(k) \exp(2\pi i k) \exp(2\pi i k x / N)$.
- $\exp(2\pi i k) = 1$ for integer k , so $f(x+N) = f(x)$
- DFT's are periodic - they just repeat themselves ad infinitum.

Periodicity

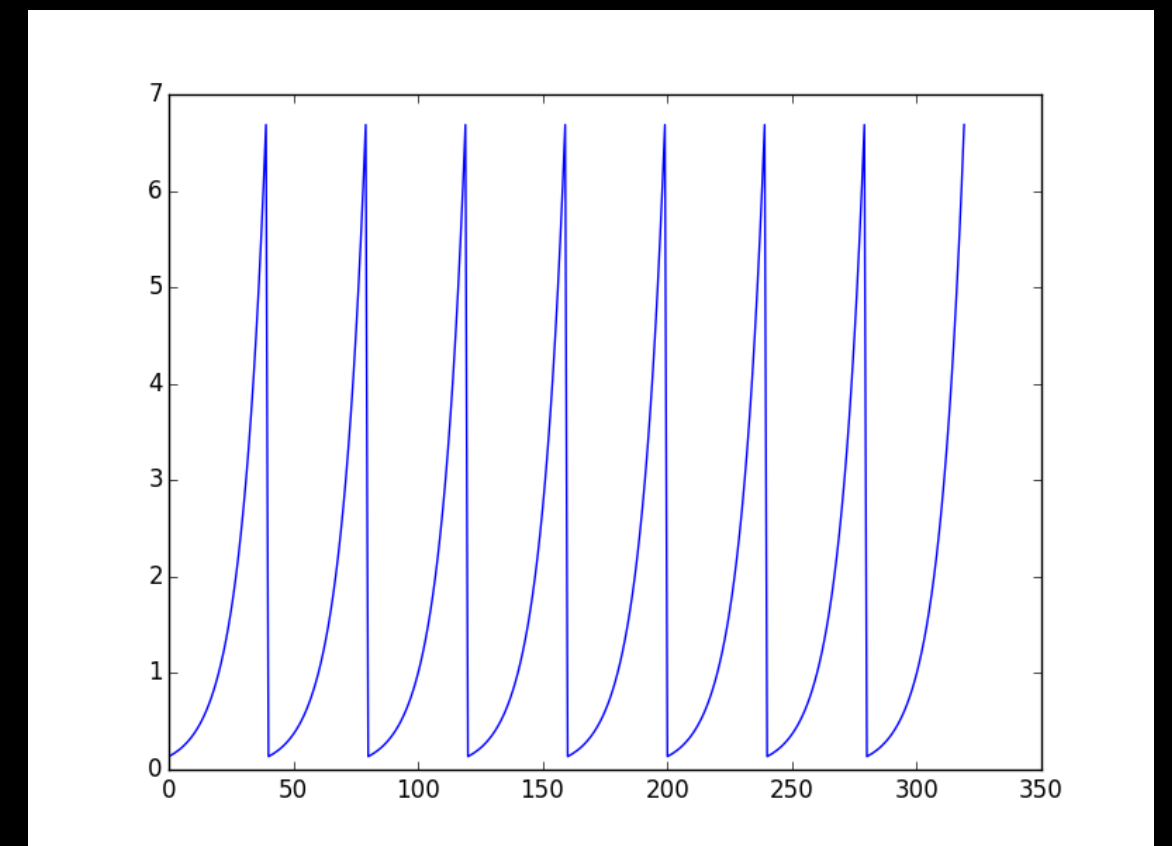
```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-2,2,0.1)
y=numpy.exp(x)
plt.plot(x,y)
plt.savefig('fft_exp')
plt.show()

yy=numpy.concatenate((y,y))
yy=numpy.concatenate((yy,yy))
yy=numpy.concatenate((yy,yy))
plt.plot(yy)
plt.savefig('fft_exp_repeating')
plt.show()
```



- You may think you're taking top transform. You're not - you're taking the bottom one.
- In particular, jumps from right edge to left will strongly affect DFT



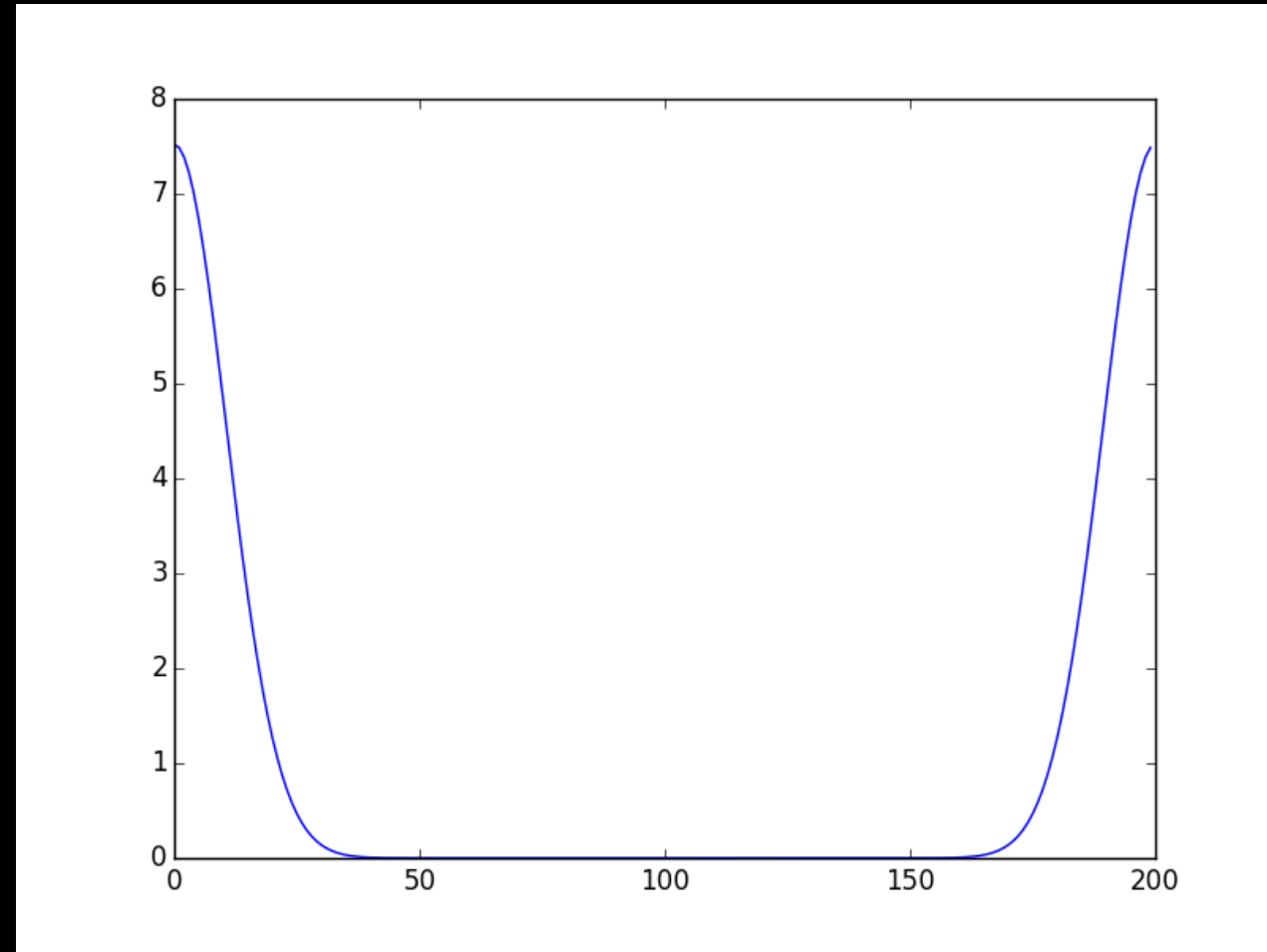
Aliasing

- $f(x) = \sum F(k) \exp(-2\pi i k x / N)$
- What if I had higher frequency, $k > N$? let $k^* = k - N$ (i.e. k^* low freq.)
- $f(x) = \sum F(k) \exp(-2\pi i (k^* + N)x / N) = \sum F(k) \exp(-2\pi i x) \exp(-2\pi i k^* x / N)$
- for x integer, middle term goes away: $\sum F(k^* + N) \exp(-2\pi i k^* x / N)$
- High frequencies behave exactly like low frequencies - power has been *aliased* into main frequencies of DFT.
- Always keep this in mind! Make sure samples are fine enough to prevent aliasing.

Negative Frequencies

- All frequencies that are N apart behave identically
- DFT has frequencies up to $(N-1)$.
- Frequency $(N-1)$ equivalent to frequency (-1) . You will do better to think of DFT as giving frequencies $(-N/2, N/2)$ than frequencies $(0, N-1)$
- *Sampling theorem*: if function is band-limited - highest frequency is ν - then I get full information if I sample *twice* per frequency, $\Delta t = 1/(2\nu)$. Factor of 2 comes from aliasing.

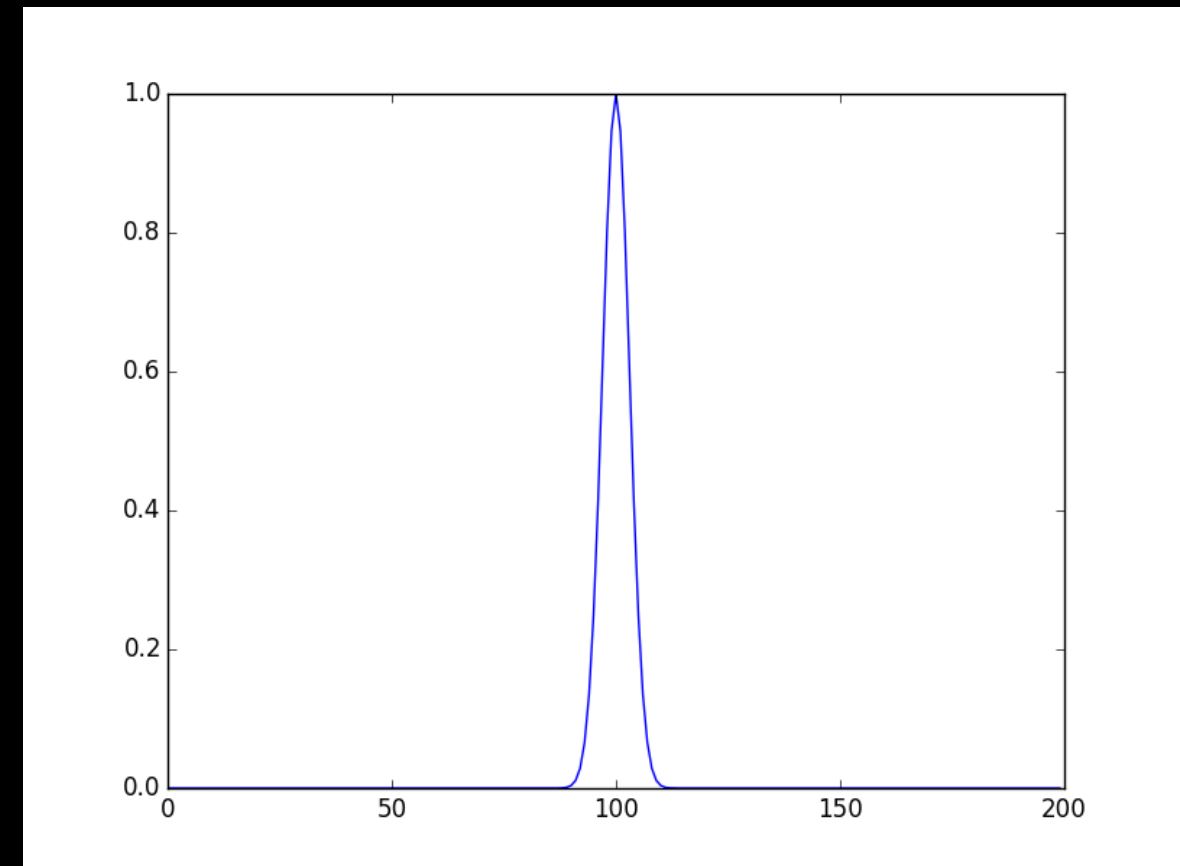
DFT in Action, Redux



```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-10,10,0.1)
y=numpy.exp(-0.5*x**2/(0.3**2))
yft=numpy.fft.fft(y)
plt.plot(numpy.abs(yft))
plt.savefig('gauss_dft')
plt.show()
```

- FFT makes more sense now - negative frequencies have been aliased to high frequency.



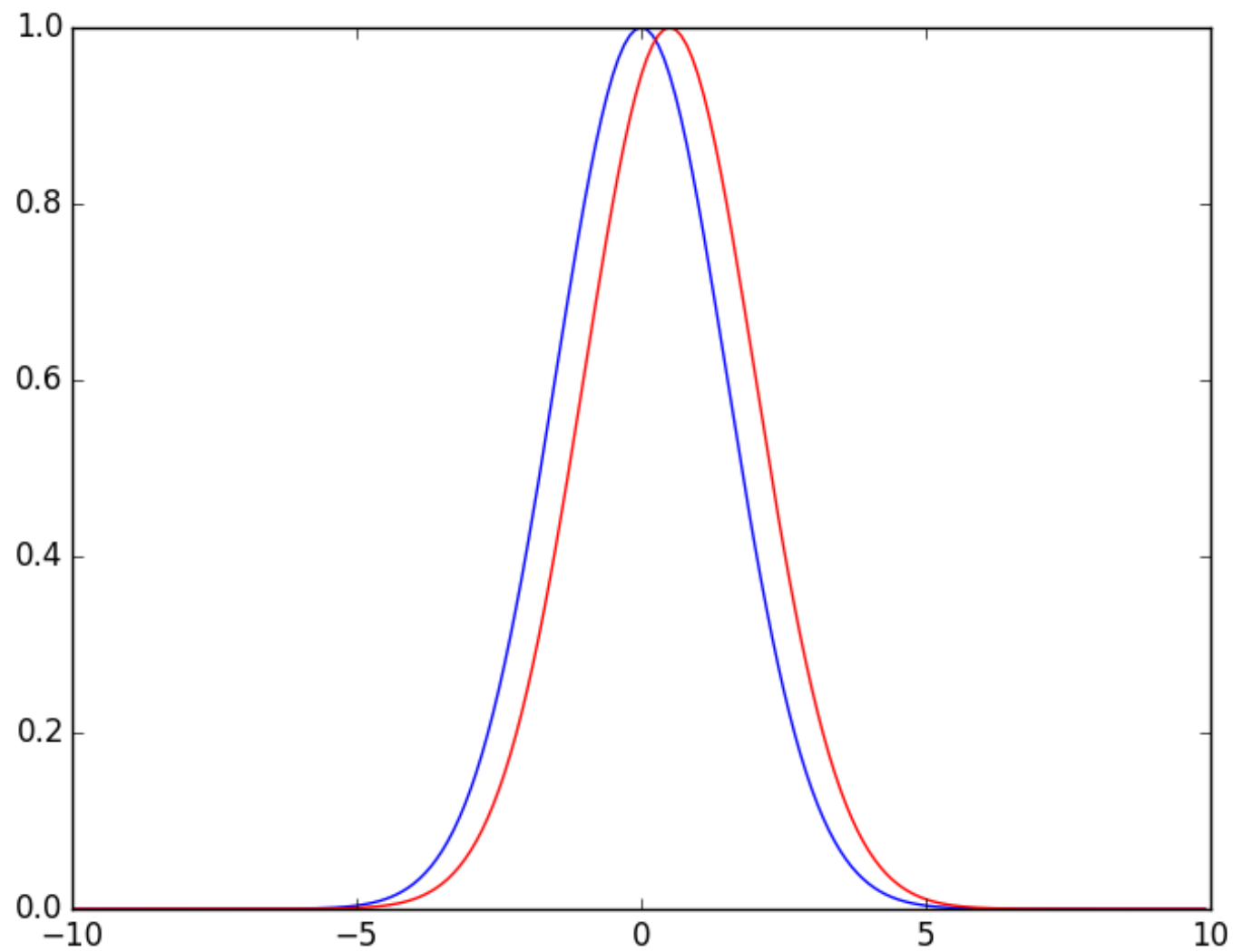
Flipping

- What is DFT of $f(-x)$?
- $\sum f(-x) \exp(2\pi i k x / N)$, $x^* = -x$, $\sum f(x^*) \exp(2\pi i k (-x) / N)$
- $\text{DFT}(f(-x)) = \sum f(x) \exp(-2\pi i k x / N) = \text{conj}(F(k))$

Shifting

- What is $\text{FFT}(x+dx)$? $\sum f(x+dx)\exp(2\pi i k x/N)$.
- $x^*=x+dx$: $F(k)=\sum f(x^*)\exp(2\pi i k (x^*-dx)/N)$
- $F(k)=\exp(-2\pi i k dx/N)\sum f(x^*)\exp(2\pi i k x^*/N)$
- So, just apply a phase gradient to the DFT to shift in x

Shifting Example



```
import numpy
from matplotlib import pyplot as plt

x=numpy.arange(-10,10,0.1)
y=numpy.exp(-0.5*x**2/(1.5**2))
N=x.size
kvec=numpy.arange(N)
yft=numpy.fft.fft(y)
J=numpy.complex(0,1)
dx=5.0;
yft_new=yft*numpy.exp(-2*numpy.pi*J*kvec*dx/N)
y_new=numpy.real(numpy.fft.ifft(yft_new))
plt.plot(x,y)
plt.plot(x,y_new,'r')
plt.savefig('shifted_gaussian')
plt.show()
```

Real Data Symmetry

- If I know $F(k)$, what is $F(N-k)$ if $f(x)$ is real?
- $F(N-k)=F(-k)$ (from alias theorem)
- $F(-k)=\sum f(x)\exp(2\pi i(-k)x/N)$. let $x^*=-x$
- $F(-k)=\sum f(-x^*)\exp(2\pi i k x^*/N) = \text{conj}(F(k))$ by flipping
- So, if $f(x)$ is real, $F(k)=\text{conj}(F(N-k))$
- If N even, $F(N/2)=\text{conj}(F(N/2))$, so $F(N/2)$ must be real.

```
>>> x=numpy.random.randn(8)
>>> xft=numpy.fft.fft(x)
>>> for xx in xft:
...     print xx
...
(-4.53568815727+0j)
(-0.174046761579+2.08827239558j)
(2.15348308858+2.32162497273j)
(-0.423040513854-3.72126858798j)
(2.75685372591+0j)
(-0.423040513853+3.72126858798j)
(2.15348308858-2.32162497273j)
(-0.174046761579-2.08827239558j)
>>>
```

Convolution Theorem

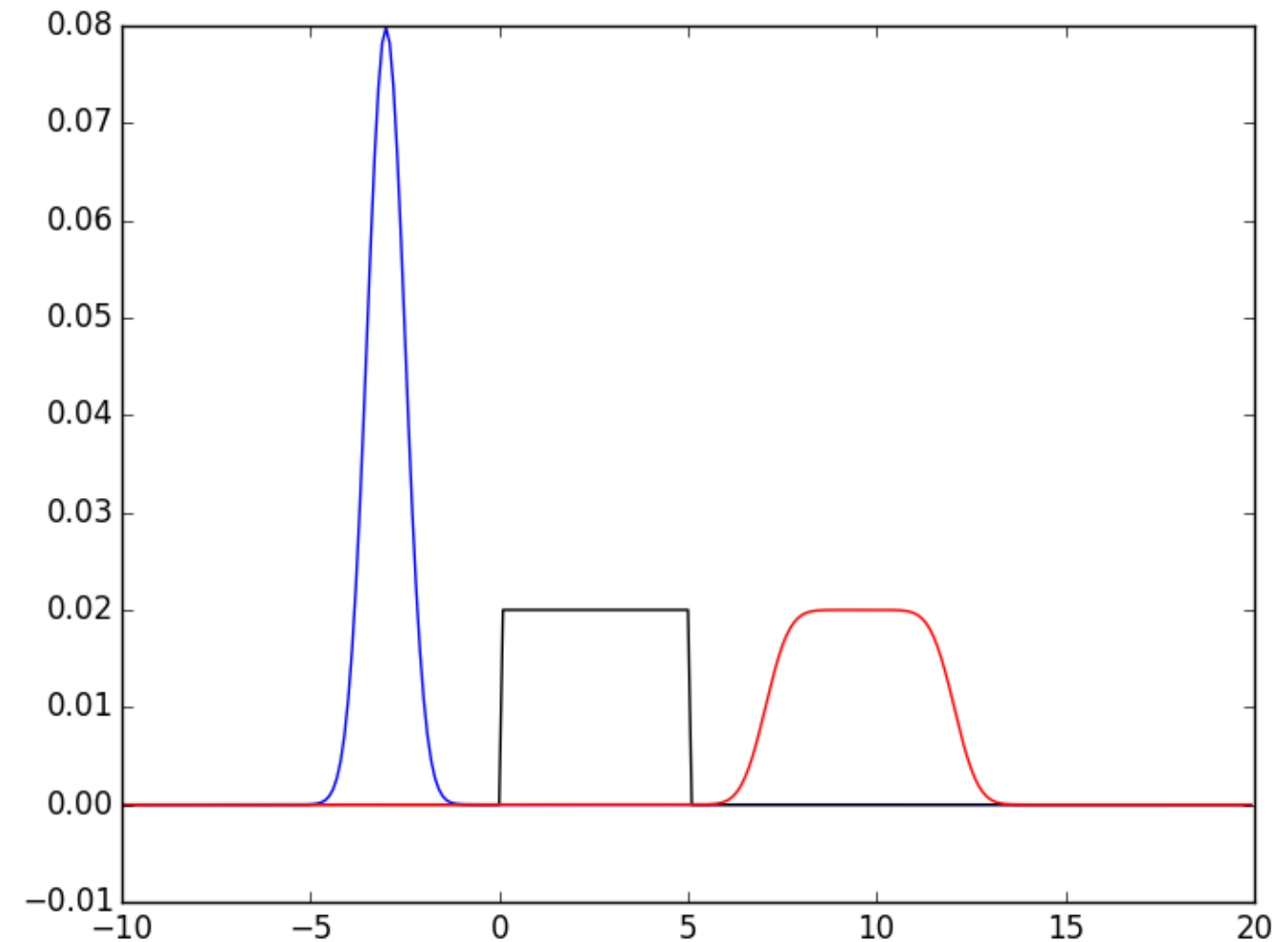
- Convolution defined to be $\text{conv}(y) = f \otimes g = \int f(x)g(y-x)dx$
- $\sum_x \sum F(k) \exp(-2\pi i k x) \sum \text{conj}(G(k')) \exp(-2\pi i k' x) \exp(2\pi i k' y/N)$
- Reorder sum: $\sum \sum F(k) \text{conj}(G(k')) \exp(2\pi i k' y/N) \sum_x \exp(-2\pi i (k+k')x)$
- equals zero unless $k' = -k$. Cancels one sum, conjugates G
- $f \otimes g = \sum F(k)G(k) \exp(-2\pi i k y/N) = \text{ift}(\text{dft}(f) * \text{dft}(g))$
- So, to convolve two functions, multiply their DFTs and take the IFT

Convolution Example

```
from numpy import arange,exp,real
from numpy.fft import fft,ifft
from matplotlib import pyplot as plt
def conv(f,g):
    ft1=fft(f)
    ft2=fft(g)
    return real(ifft(ft1*ft2))

x=arange(-10,20,0.1)
f=exp(-0.5*(x+3)**2/0.5**2)
g=0*x;g[(x>0)&(x<5)]=1
g=g/g.sum()
f=f/f.sum()
h=conv(f,g)

plt.plot(x,f,'b')
plt.plot(x,g,'k')
plt.plot(x,h,'r')
plt.savefig('convolved')
plt.show()
```



Fast Fourier Transform

- How many operations does a DFT take?
- Have an N by N matrix operating on a vector of length N - clearly N^2 operations, right?
- Nope! Otherwise we'd never use them. What's actually going on?
- Note $\text{DFT} = \sum f(x) \exp(2\pi i k x / N) = \sum f_{\text{even}}(x) \exp(2\pi i k (2x) / N) + \sum f_{\text{odd}}(x) \exp(2\pi i k (2x + 1) / N)$
- $= F_{\text{even}} + \exp(2\pi i k / N) F_{\text{odd}}$. Let $W_k = \exp(2\pi i k / N)$
- if $k > N/2$, then $k^* = k - N/2$ and $\text{DFT} = F_{\text{even}} + \exp(2\pi i k^* / N + i\pi) F_{\text{odd}} = F_{\text{even}} - W_k F_{\text{odd}}$.

FFT cont'd

- So $F(k) = F_{\text{even}}(k) + W_k F_{\text{odd}}(k)$ ($k < N/2$) or $F_{\text{even}}(k) - W_k F_{\text{odd}}(k)$ ($k \geq N/2$)
- So, can get *all* the frequencies if I have 2 half-length FFTs.
- Well, just do the same thing again. FFT of a single element is itself.
- This algorithm works for arrays whose length is a power of 2
- Popularized by Cooley/Tukey in early computer days. Later found to go back to Gauss in 1805. Changes computational work from n^2 to $n \log n$.

Sample FFT

- Routine uses *recursion* - function calls itself. Recursion can be very powerful, but also easy to goof.
- `numpy.concatenate` will combine arrays - note that they have to be passed in as a tuple, hence the extra set of parenthesis
- Modern FFT routines deal with arbitrary length arrays. Fastest Fourier Transform in the West (FFTW) standard packaged - usually used by numpy.

```
from numpy import concatenate,exp,pi,arange,complex
def myfft(vec):
    n=vec.size
    #FFT of length 1 is itself, so quit
    if n==1:
        return vec
    #pull out even and odd parts of the data
    myeven=vec[0::2]
    myodd=vec[1::2]

    nn=n/2;
    j=complex(0,1)
    #get the phase factors
    twid=exp(-2*pi*j*arange(0,nn)/n)

    #get the dfts of the even and odd parts
    eft=myfft(myeven)
    oft=myfft(myodd)

    #Now that we have the partial dfts, combine them with
    #the phase factors to get the full DFT
    myans=concatenate((eft+twid*oft,eft-twid*oft))
    return myans
```

```
>>> import myft
>>> x=numpy.random.randn(32)
>>> xft1=numpy.fft.fft(x)
>>> xft2=myft.myfft(x)
>>> print numpy.sum(numpy.abs(xft1-xft2))
2.33937690259e-13
>>>
```

Tutorial Problems

- Write a function that will shift an array by an arbitrary amount using a convolution (yes, I know there are easier ways to do this). The function should take 2 arguments - an array, and an amount by which to shift the array. Plot a gaussian that started in the centre of the array shifted by half the array length. (10)
- The correlation function $f \star g$ is $\int f(x)g(x+y)$. Through a similar proof, one can show $f \star g = \text{ift}(\text{dft}(f) * \text{conj}(\text{dft}(g)))$. Write a routine to take the correlation function of two arrays. Plot the correlation function of a Gaussian with itself. (10)
- Using the results of part 1 and part 2, write a routine to take the correlation function of a Gaussian (shifted by an arbitrary amount) with itself. How does the correlation function depend on the shift? Does this surprise you? (10)
- The circulant (wrap-around) nature of the dft can sometimes be problematic. Write a routine to take the convolution of two arrays *without* any danger of wrapping around. You may wish to add zeros to the end of the input arrays. (10)

Tutorial Bonus Problem

- You have a sample code that calculates an FFT of an array whose length is a power of 2. Using that routine as a guideline, write an FFT routine that works on an array whose length is a power of 3 (e.g. 9, 27, 81). Verify that it gives the same answer as `numpy.fft.fft` (10)