linear systems

- we will introduce the concept of a linear system of equations
- informally discuss the types of solutions such systems can have.
- focus for first 1/4 of class
- linear systems are simple but powerful

Linear Equation

- A *linear equation* in two variables is an expression of the following form:

$$a_1x_1 + a_2x_2 = b$$

  where the a_i and b are specific chosen numbers. The a_i are called the *coefficients*, and b is called the *right hand side* (RHS)
- coefficients and RHS will always be *real numbers* (elements of $\mathbb{R}$).
- symbols, x_1 and x_2, are called the *variables*.
- In a linear equation we cannot do anything else funky to our variables, like have terms with forms such as x_1x_2, x_2^3, $\frac{1}{x_1}$, $\sqrt{x_1}$, $\sin x_1$, ...
- Here is a specific example

$$7x_1 - x_2 = -2$$

solutions

- The *solutions* to this equation are the set of real numbers that can be substituted for the variables that will make the equation true.
- one solution to our example is setting x_1 to 1 and x_2 to 9.
- Another solution is to set x_1 to 2 and x_2 to 16.
- We can plot one solution $(1, 9)$ as a point in the plane.
- One can plot another solution $(2, 16)$ as a second point in the plane.
- all of the solutions to this equation, form a line in our plane.

scalar multiples

- If we multiply our linear equation by some non-zero, real number k, we obtain a new linear equation

$$k7x_1 - kx_2 = -k2$$

- must have same solution set.
- we say that our two equations are *scalar multiples* of each other.

exceptional behaviour

- One linear equation in two variables will typically have a line of solutions,
- but $0x_1 - 0x_2 = -1$ will have no solutions.
- and $0x_1 - 0x_2 = 0$ will be solved by any x_1 and x_2 will be solutions
  - We might call such an equation *trivial*.

Linear systems

- A *linear system* is a collection of a finite number of linear equations using a common set of variables. such as:

$$\begin{aligned} 7x_1 - x_2 &= -2 \\ 2x_1 + x_2 &= 11 \end{aligned}$$

- solutions make both equations simultaneously true.
  - $(1, 9)$ is solution.

geometry

- let L_1 be the line of solutions for first equation alone,
- and likewise for L_2
- then the solutions of the linear system must be at the intersection $L_1 \cap L_2$.
- For our system, these two lines intersect at a single point, making the solution $(1, 9)$ the *unique* solution.
- "most" line pairs intersect at a single point, so this is the "expected" solution set.

exceptions 1

- exceptionally, the solution set can be larger than expected, or completely empty.

$$\begin{aligned} 7x_1 - x_2 &= -2 \\ 14x_1 - 2x_2 &= -4 \end{aligned}$$

- the two equations are independently solved with the same line of points
  - the second equation is a scalar multiple of the first
- Q: what is the geometry here
  - In this case, $L_1 = L_2$ and so $L_1 \cap L_2$ will be the entire line of points.
  - This solution set is larger than expected.

exceptions 2

- consider

$$\begin{aligned} 7x_1 - x_2 &= -2 \\ 14x_1 - 2x_2 &= -20 \end{aligned}$$

- Q: what is the geometry here
- the two equations are independently solved by a set of parallel lines.
  - lhs is a scalar muliple but not full equation
- $L_1 \cap L_2$ is empty, and so there are no solutions

exceptions 3

- if both equations are trivial, then we will have the entire plane as solutions.

three equations, two variables

- consider

$$\begin{aligned} 7x_1 - x_2 &= -2 \\ 2x_1 + x_2 &= 11 \\ 3x_1 + 1x_2 &= 4 \end{aligned}$$

- common intersection of three lines.
- these three lines do no intersect, so we have no solutions.
- most triplets of lines in the plane do not have a common point of intersection, so this is the expected behaviour.
- but as above, for specially chosen equations, we may still have a single solution, a line of solutions, or an entire plane of solutions.

shape of solution space

- With two variables, our solution sets were always in form of empty set, single point, single line, entire plane.
- these are "flat" looking sets
  - We will use the fancy word *affine* to refer to this flat property.
- To talk about how big these solution sets are, we might start talking about the *dimension* of the solution set.
  - captures how many ways we can move in the set, starting at some point in the set.
  - plane has dimension 2, a line has dimension 1, and a single point has dimension 0.

summary

- Let us use the symbol n to denote the number of variables, which currently is 2.
- Let us use the symbol m to denote the number of equations in our system.
- Then we expect that the solution to be an affine set of dimension

$$n - m$$

  - with a negative dimension meaning the empty set.
- But we know that we can exceptionally get larger solution sets, or an empty solution set.
- In future chapters, we will explore these issues in significant depth.

Three variables

- consider

$$7x_1 - 2x_2 + 4x_3 = 8$$

- solutions are all the points that lie on a single plane P_1 in three dimensions.
- BQ: What do you expect the solution set to look like with three variables and 2, 3, or 4 equations?

two equations three variables

- consider

$$\begin{aligned} 7x_1 - 2x_2 + 4x_3 &= 8 \\ 0x_1 - 1x_2 - 2x_3 &= 1 \end{aligned}$$

- let P_2 be the solution plane for the second equation
- the solutions for the system is the intersection $P_1 \cap P_2$
  - which in this case is a line in three dimensions, we will call L_{12}.
  - Indeed, we expect that two planes in 3D will intersect in a line.

three equations three variables

- consider

$$\begin{aligned} 7x_1 - 2x_2 + 4x_3 &= 8 \\ 0x_1 - 1x_2 - 2x_3 &= 0 \\ -5x_1 + .7x_2 - \sqrt{5}x_3 &= 10 \end{aligned}$$

- and letting P_3 be the solution plane for the third equation,
- the solutions for this system is the intersection $P_1 \cap P_2 \cap P_3 = L_{12} \cap P_3$, which in this case is a single point $\mathbf{x}$.
- Indeed, we expect generally that three planes in 3D, (or one plane and one line) will intersect in a point.

four equations three variables

- Adding a fourth equation to the system

$$\begin{aligned} 7x_1 - 2x_2 + 4x_3 &= 8 \\ 0x_1 - 1x_2 - 2x_3 &= 0 \\ -5x_1 + .7x_2 - \sqrt{5}x_3 &= 10 \\ -3x_1 + 2x_2 - x_3 &= \sqrt{2} \end{aligned}$$

- letting P_4 be the solution plane for the fourth equation
- the the solutions for the system is the intersection $P_1 \cap P_2 \cap P_3 \cap P_4 = \mathbf{x} \cap P_4$, which in this case is the empty set.
- Indeed, we expect generally that four planes in 3D, (or one plane and one point) will have no common intersection.

example no solutions

$$\begin{aligned} 0x_1 - 2x_2 - 20x_3 &= 8 \\ 20x_1 - 4x_2 + 2x_3 &= 0 \\ 10x_1 - 1x_2 - 9x_3 &= 34 \end{aligned}$$

- none of the equations are inconsistant, and no two of the three solution planes P_i are parallel, but we have L_{12} parallel to P_3, thus having no intersection!
- New feature: with 3 variables, we can get exceptional behaviour without scalar multiples or trivial equations

example with bigger solution set

$$\begin{aligned} 0x_1 - 2x_2 - 20x_3 &= 0 \\ 20x_1 - 4x_2 + 2x_3 &= 0 \\ 10x_1 - 1x_2 - 9x_3 &= 0 \end{aligned}$$

- For this system, we get three solution planes P_i that intersect in a common line.
- This system is related to the previous one, as they share the same LHS

smaller solution set?

- notably, we cannot get a solution set that is smaller than expected, but still non-empty
- lets loook at two equations (planes)
- with two equations, the intersection of two planes usually gives us a line
- we can arrange two planes to have no intersection
- but we cannot arrange two planes to have a single point of intersection.

summary

- with $n = 3$ variables and m equations the solution space is expected to be $n - m$.
- there can be exceptional behaviour. We can either obtain a larger than expected (but still affine) solution set or no solution at all.
- This makes sence from a 3d Geometric point of view as we add in each plane
  - a plane and a plane should intersect in a line
  - a plane and a line should intersect in a point
  - a plane and a point should not intersect
  - exceptionally: a plane can miss the prior solution set
  - exceptionally: a plane can contain the prior solution set
- so far we have just argued using pictures and informal reasoning.
- we will want to understand how to prove these things, and esp. how to prove that similar behaviour holds in higher dimensions.

plane geometry

- for the plane satisfiying $0x_1 - 2x_2 - 20x_3 = 8$
- draw the "vector" $\mathbf{n}$ going from the point $(0, 0, 0)$ to $(0, -2, -20)$, with an arrow at the end.
- The first point is just the *origin* and the second point was obtained by looking at the coordinates on the *LHS* (left hand side) of the equation.
- $\mathbf{n}$ meets P_1 at a right angle.
  - that $\mathbf{n}$ is a *normal vector* of P_1.
- note that the normal computation does not look at rhs.

geometry of the exceptional cases

- For the three solution planes P_i, we can draw three normal vectors.
- in the exeptional cases, the three normal vectors all lie in a single plane!
- compare this the normals in the expected case.
- This surprising geometry is somehow related to the unexpected behaviour in the solution set dimension of the linear system
- We will get a better grip on this coincidence in future chapters.

n variables

- when we get to applications, it will be useful to look at problems with many, many variables.
- We will often use the symbol n to refer the number of variables in our system.
- In general a linear equation will be an expression of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

  where the a_i and b are specific chosen real numbers.

plan

- We will soon want to have "n-dimensional" versions of the various concepts we saw earlier such as plane, affine set, dimension. We will do this fairly carefully in the next chapters.
- But when all of that dust settles, we will still find that the solution set is always affine. Its expected dimension is $n - m$, but exceptionally it can be larger, or empty.
- In future chapters we also to study algorithmic techniques to solve these problems.
- We will also want to understand how to find approximate solution to problems with no exact solution.

- We will also see how to work with such problems in a computational environment.

Why linear systems

- variables are quantities that we want to know about
- equations gives us rules that must hold for these quantities
  - the rules may describe some physical law about the variables
- equality (=) relations are easier to deal with than inequalities ($\leq$)
- linearity of the expressions are the easiest possible case to study.

rules from measurement

- the equations may describe some measurement process linearly combines the unknown quantities
- CT scan
  - unknown data: 3d density field
  - measured: density combinations along varios x-ray rays.
- Debluring
  - unknown: true sharp image
  - measured: blurry image with each blurred pixels being a combitation of true sharp pixels.

Rules from laws

- the rules might come from physical laws.

$$x_1CH_4 + x_2O_2 \rightarrow x_3CO_2 + x_4H_2O$$

- the variables x_i are the number of each kind of molecules
- rule: we know that the number of each type of atom is preserved.
- for hydrogen, we have $4x_1 = 2x_4$
- for all elements we have

$$\begin{aligned}
1x_1 + 0x_2 - 1x_3 + 0x_4 &= 0 \\
4x_1 + 0x_2 + 0x_3 - 2x_4 &= 0 \\
0x_1 + 2x_2 - 2x_3 - 1x_4 &= 0
\end{aligned}$$

- extra requirements here: we want integer solutions that are positive.
  - in general these requirement can make a problem harder, but for problems with a zero RHS, this is actually not especially hard.

rules from models

- the rules might come from an assumed model-with-knobs (unknown knob settings) that we are using to fit some data points.
- ie. find knob settings so that the model then fits the data.
- example: quadratic smog vs population model $s = ap^w + bp + c$
  - unknown knobs: coefficients, a, b and c
  - data: a table of (p, s) values.

Echelon Form

- Lets look at the following linear system

$$2x_1 + 4x_2 - 3x_3 + 2x_4 + 10x_5 = -2 \quad (1)$$
$$0x_1 + 1x_2 - 5x_3 + 2x_4 - 5x_5 = 4 \quad (2)$$
$$0x_1 + 0x_2 + 0x_3 + 1x_4 - 2x_5 = 3 \quad (3)$$

- in each row, its first non-zero coefficient, called its *leading* term, is to the right of the leading term of any of the rows above it.
- we say that the system is in *echelon form*.
  - echelon form also allows for all-zero LHS
- we call the collection of leading variables, in this case x_1, x_2 and x_4, the *basic variables*.
- We call the remainder of the variables, in this case x_3 and x_5 the *free variables*.
- note: usually we are not given a system directly in echelon form, but we will disucss algorithms that take a general system and replace it with one in echelon form!

backsub

- A linear system in echelon form is very easy to solve, using a method called *backsubstitution*
- We start with the bottom row, and rewrite the equation to get its basic variable alone, on the left side, with a coefficient of 1 (by dividing if needed).

$$x_4 = 3 + 2x_5$$

  - Notice that on the RHS, we only have free variables.

backsub.

- Then we move up one row and do the same thing and substititue for x_4

$$\begin{aligned} x_2 &= 4 + 5x_3 - 2x_4 + 5x_5 \\ &= 4 + 5x_3 - 2(3 + 2x_5) + 5x_5 \\ &= -2 + 5x_3 + 1x_5 \end{aligned}$$

- Again, notice that on the RHS, we only have free variables.

backsub..

- Then we move up one row and apply the same process:

$$\begin{aligned} x_1 &= 1/2(-2 - 4x_2 + 3x_3 - 2x_4 - 10x_5) \\ &= 1/2(-2 - 4(-2 + 5x_3 + 1x_5) + 3x_3 - 2(3 + 2x_5) - 10x_5) \\ &= 1/2(-16 - 17x_3 - 18x_5) \\ &= -8 - 17/2x_3 - 9x_5 \end{aligned}$$

- Again, notice that on the RHS, we only have free variables.

conclusion

- Since there are no more rows, we are done, and we can conclude

$$\begin{aligned} x_4 &= 3 + 2x_5 \\ x_2 &= -2 + 5x_3 + 1x_5 \\ x_1 &= -8 - 17/2x_3 - 9x_5 \end{aligned}$$

- to get a solution, we can set the free variables x_3 and x_5 to be any real numbers that we want, and the calculate the, uniquely determined, values for the basic variables.

- Since we have 2 free variables, we will say that the solution set is 2 dimensional.

exceptions 1

- From our definition, the following system is also in echelon form.

$$
\begin{aligned}
2x_1 + 4x_2 - 3x_3 + 2x_4 + 10x_5 &= -2 \\
0x_1 + 1x_2 - 5x_3 + 2x_4 - 5x_5 &= 4 \\
0x_1 + 0x_2 + 0x_3 + 1x_4 - 2x_5 &= 3 \\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 0
\end{aligned}
$$

- Notice that the last equation is trivial, so we can just erase it, and apply backsubstitution to the remaining system.

exceptions 1

- From our definition, the following system is also in echelon form.

$$
\begin{aligned}
2x_1 + 4x_2 - 3x_3 + 2x_4 + 10x_5 &= -2 \\
0x_1 + 1x_2 - 5x_3 + 2x_4 - 5x_5 &= 4 \\
0x_1 + 0x_2 + 0x_3 + 1x_4 - 2x_5 &= 3 \\
0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 &= 6
\end{aligned}
$$

- Notice that the last equation is never solvable, and so we know that this system has no-solutions.
- We see that testing for trivial and unsolvable equations must be included in our back substitution algorithm.

back to our chemical reaction

- is our chemical reaction equation in echelon form?
- hw: there is a easy way, using reordering, to turn it into echelon form.
- which then makes this problem easy to solve.
- this is lucky, and why you were able to solve this problem in chemistry class.
- for some chemical systems, you will might not get so lucky!

Triangular

- When $n = m$ we say that our linear system is *square*.
- here is a square system is in echelon form, and it has no zero rows

$$
\begin{aligned}
2x_1 - 4x_2 + 3x_3 &= 1 \\
0x_1 + 1x_2 - 4x_3 &= -5 \\
0x_1 + 0x_2 + 1x_3 &= 4
\end{aligned}
$$

- so the leading variables, shift once to the right as we drop down one row.
- when you run backsubstitution on such a system you will get a unique real numbered solution.
  - there will be no free variables.
- The pattern of this system has all of the non-zero coefficients in an upper triangle.
- this is related to the notion of an upper triangular matrix which we will see later.

Gaussian elimination

- There are three simple operations that we can apply to a linear system that will not change the solution set.
  - We can permute the rows.

  - We can replace a row by a (non-zero) scalar multiple of itself.
  - We can replace a row by its sum with a (scalar multiple of a) different row

- By applying these operations in a systematic way, we can turn any linear system in to one in echelon form.
- This process is called *Gaussian elimination*
- Once Gaussian elimination has been performed, you can apply backsubstitution

computation

- with more variables, you will probably want to use a computer.
- But once you have a computer, there are methods that are typically preferred, such as one called LU-decomposition.
- We will cover these later,
- also on a computer, we use a floating point representation with some fixed precision.
- Using fixed precision also means that our calculations may only be approximate.

matrix form

- we will briefly introduce matrix notation and how they relate to linear systems.
- A *vector in* $\mathbb{R}^m$ is an ordered collection of m real numbers. in a vertical column, such as

$$\mathbf{b} := \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$$

- We index the entries starting at the top, with the index 1.
- To save white space, I may also write this vertical collection as $[-2; 4; 3]$, using semicolons.

variables

- If have a collection of variables we can use them to form a variable vector. Here is an example in $\mathbb{R}^5$.

$$\mathbf{x} := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

matrix

- An m-by-n matrix is a two dimensional array of real numbers where the array has m rows and n columns. Here is a 3-by-5 matrix

$$A := \begin{bmatrix} 2 & 4 & -3 & 2 & 10 \\ 0 & 1 & -5 & 2 & -5 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

- A matrix is indexed by two numbers. The upper left entry is called a_{11}. The first index increases as we move down the rows. The second index increases as we move rightward on the columns.

system

- Given this setting, we will write the complete linear system as

$$\begin{bmatrix} 2 & 4 & -3 & 2 & 10 \\ 0 & 1 & -5 & 2 & -5 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}$$

- Or using their symbol names as

$$A\mathbf{x} = \mathbf{b}$$

MATLAB

- see intro1.m and matlabtutorial.m
- The MATLAB backslash symbol is its general symbol for solving a linear system. Depending on the system, it may apply different algorithms. When the system is upper triangular, it simply does backsubstitution.

summary

- basics of linear systems and their solutions.
- expected solution set and the possibility of exceptional behaviour.
- terms like affine, and dimension.
- solution techniques.

next

- learn the topic of linear algebra.
- better understand vectors and matrices, and their properties.
- this will let us deeply understand linear sytems
- this will let us then learn about approximation and optimization.