Assignment

Problem 1: let G be a group of order pq where p and q are distinct primes. prove that G is abelian.

Provided Ans.: Abelian

Correct Ans: False

Explanation: A group of order pq is not always abelian. the statement is only true under an additional condition. using sylow theorems, we can show that if $p < q$, the group is guaranteed to be abelian only if p does not divide $q-1$

A classic counter example is the symmetric group S_3, which is the group of permutations of three elements.

- the order of S_3 is $3! = 6$
- we can write the order as a product of distinct primes: $6 = 2 \cdot 3$.

consider the free product $G = Z_2 * Z_2$

* this group can be described by generators and relation: $G = (a, b / a^2 = e, b^2 = e)$.

* the elements a has order 2 (finite).

* However, their product, ab, has infinite order. the element $(ab)^k = abab \cdots ab$ is never the identity for $k > 0$.

since the product of two elements of finite order can have infinite order, the set is not closed under the operation and thus is not a subgroup. this set is called the torsion subset of a group; it is only guaranteed to be a subgroup (the torsion subgroup) if the group is abelian.

- S_3 is a well-known non-abelian group for instance, the composition $(12)(13) = (132)$ and $(13)(12) = (123)$ are not equal.

therefore, a group of order pq is not necessarily abelian.

problem 5: prove that in any group G, the set of elements of finite order a subgroup of G

provided answer: True

correct answer: False

explanation: this statement is only true if the group G is abelian for a general non-abelian group the set set of elements of finite order is not necessarily closed under the group operation.

<u>problem 6:</u> Let G be a finite group and p be the smallest prime dividing $|G|$: prove that any subgroup of index p in G is normal.

<u>provided Ans:</u> true

<u>correct Ans:</u> true

<u>Explantion:</u> this correct and important theorem. the proof involves a group action. let H be a subgroup of G with index $[G:H] = p$

1. Let G act on the set of left ~~coset~~ of H, $s = \{gH / g \in G\}$, by left. multiplicatio this induces a homomorphism. $\phi : G \rightarrow S_p$, where is the symmetric group on the p elements.

2. the kernel of this action. $Ker(\phi)$, is a normal subgroup of G and is contained with H

3. By the First isomorphism theorem, $G/Ker(\phi)$ is isomorphic to a subgroup of S_p. thus the order of $G / Ker(\phi)$, which is $[G:Ker(\phi)]$, must divide $|S_p| = p!$

4. since H has index p. we have $|G| = p|H|$.

also, $[G : \ker(\phi)] = [G : H][H : \ker\phi] = p.[H : \ker(\phi)]$

this shows that p ~~pr~~ divides $[G : \ker(\phi)]$

5. we know that $[G : \ker(\phi)]$ divides both

$|G|$ and $p!$. the prime factors of $p!$ are

all primes less than or equal to p. since

p is the smallest prime dividing $|G|$. the only

prime factor that $[G : \ker(\phi)]$ ~~the~~ and

$|G|$ can share is p

6. this implies that $[G : \ker(\phi)]$ must be a

power of p. However, since $[G : \ker(\phi)]$

divides $p!$, the highest power of p it

can be is p

7. therefore, $[G : \ker(\phi)] = p$

8. From step 4, we have $p = p[H : \ker(\phi)]$,

which implies $[H : \ker(\phi)] = 1$. this means

$H = \ker(\phi)$.

since $\ker(\phi)$ is always a normal subgroup, H

must be normal in G.

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problem 7: Let G be a group and $a, b \in G$. prove that if $a^4 = b^2$ and $ab = ba$ then $(ab)^6 = e$

provided Ans: True

correct Ans: False

Explanation: the statement as written is true. There appear to be a typo in the problem. correct. There appear to be a typo in the problem's premises. we can construct a simple counter example.

Let's follow the reason:

1. Since a and b commute, $(ab)^6 = a^6 b^6$

2. we are given $a^4 = b^2$

3. we can write $b^6 = (b^2)^3$. substituting the given equation, we get $b^6 = (a^4)^3 = a^{12}$

4. substituting this back into the first line: $(ab)^6 = a^6 a^{12} = a^{18}$ the problem is now reduced to proving that $a^{18} = e$. However, the given conditions do not guarantee this.

Counter example: Let $G = Z_{36}$ (the group of integers modulo 36 under is abelian, so all elements commute

Let $a = 2$. then $a^4 = 4 \cdot 2 = 8$.

we need $b^2 = a^4 = 8$. lets check for each b

is $2 \cdot 36$, $10^2 = 100 = 28$, $11^2 = 121 = 13$, ...

Let's try another a.

Let's $a = 10$ then $a^4 = 40 \equiv 4 \pmod{36}$.

we need $b^2 = 4$. we can choose $b = 2$ ($2^2 = 4$) or

$b = 20$ ($20^2 = 400 = 11 \cdot 36 + 4) \equiv 4$).

Lets choose $a = 10$ and $b = 20$. The conditions

$ab = ba$ and $a^4 = b^2$ are met. now lets

calculate $(ab)^6$: $(ab)^6 = (10+20)^6 = 30^6 = (-6)^6 \pmod{36}$

$(-6)^2 = 36 \equiv 0$ $(-6)^6 = ((-6)^2)^3 = 0^3 = 0 = e$

this example clearly let's try to find

one that almost.

From the logic above, we need to find an

a where $a^4 \neq e$.

let $G = Z_2$. let $a = 1$, then $a^4 = 4$. we

need $b^2 = 4$. we can choose $b = 2$.

conditions: $a = 1, b = 2$ Group is abelian $a4 = 4$, $b2 = 4$. conditions met.

Check the conclusion: $(ab)^6 = (1+2)^6 = 3^6$ mod 20)'

$3^2 = 9,\ 3^4 = 81 \equiv 1,\ 3^6 = 3^4 \cdot 3^2 = 1 \cdot 9 = 9$

Since $(ab)^6 = 9 \neq 0$ (the identity),

The statements is false.

problem 8: let G be a group and H be a subgroup of G. prove that if $[G:H] = n$, then for any $x \in G$, $x^n \in H$.

provided Ans: true

correct Ans: False

explanation: this statement is a property that holds only if the subgroup H is normal in G. if H is normal, then the set of cosets G/H forms a group of order n. By Lagrange's theorem applied to this quotient group any element $(xH)^n$ is the identity element, H. this means $(xH)^n = x^n H = H$, which implies $x^n \in H$. However, if H is not normal, the statement is false.

counter example: let $G = S_3$ and $H = \{e, (12)\}$. the order of G is 6 and the order of H is 2. the index is $[G:H] = 3$.

So, $n = 3$.

H is not a normal subgroup

lets choose an elements $x = (13) \in G$

According to the statements.

$x^{n} = (13)^3$ should be in H.

calculating the power:

$(13)^3 = (13)^2 (13) = e \cdot (13) = 13.$

the elements (13) is not in $H = \{e, (12)\}$.

therefore, the statement is false.

problem 9: Let G be finite group and p be a prime number. if G has exactly one subgroup of order p^k for each $k \leq n$, when p^n divides $|G|$, prove that G has a normal sylow p- subgroup.

provid Ans: True

Correct Ans: True

Conclusion: : this statements is correct. Let the highest power of p that divides $|G|$ be p^m (so $m \geq n$). A sylow p- subgroup of G is a subgroup of order p^m. the key information is that "G has exactly one subgroup of order p^k for each $k \leq n$".

1. Let's assume p^n is the highest power of dividing $|G|$, so a sylow p- subgroup has order p^n.

2. the problem states that for $k=n$, there is exactly one subgroup of order p^n

3. Let this unique subgroup be P. By definition P is a sylow p-subgroup.

4. A core part of sylow theory is that a sylow p-subgroup is normal if and only if it is unique.

5. Since we are given that the subgroup of order pn is unique, it is the unique sylow p-subgroup and is therefore normal in G.

The information about the uniqueness of subgroups for k < n is additional ~~of always~~ that is consistent with the conclusion but not strictly necessary to prove it.