Assignment

problem: prove that the set of rational numbers Q, equipped with the two binary operation of addition and multiplication, forms a field.

Ans: To prove that the set of rational numbers Q, equipped with the standard operations of addition (+) and multiplication (·), froms a field, we must verify that it satisfies the 11 fields axioms.

the set of rational numbers is defined as:

$$Q = \{ \frac{a}{b} \mid a \in Z, b \in Z, b \neq 0$$

Let x, y, z be arbitrary rational numbers in Q.

Axiom of Addition:

1. Closers under addition

the sum of two rational numbers is a rational number.

let $x = \frac{a}{b}$ and $y = \frac{c}{d}$ when, $a, c, \in Z$ and $b, d \in Z/\{0\}$

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Since $a, b, c, d \in Z$, then $ad + bc \in Z$ and $bd \in Z$. Also since $b \neq 0$ and $d \neq 0$, then $bd \neq 0$. thus $x + y \in Q$.

2. Associativity of Addition:

the grouping of numbers not affect them sum.

$$(x+y)+z = x+(y+z)$$

this property is inherited from the associativity of addition in the integers (Z).

3. Commutativity of addition:

the order of numbers does not affect them sum.

$$x+y = y+x$$

this property is also inherited from the commutativity of addition in Z.

4. Additive Identity (zero Elements):

there exists a unique elements, 0, in Q such that for all $x \in Q$:

$$x+0 = x$$

the additive identity is $0 = \frac{0}{1} = 0$, since $x+0 = x$.

5. Additive inverse:

for any $a = \frac{x}{y} \in Q$ consider, $-a = \frac{-x}{y}$

Since x is an integer, $-x$ is also an integer, so, $-a \in Q$

$$a+(-a) = \frac{x}{y} + \frac{-x}{y} = \frac{x-x}{y} = \frac{0}{y} = 0$$

thus, every elements is Q has an additive inverse.

Axioms of multiplications:

6. Closure under multiplication: the product of two rational numbers is a rational number

$$\text{let } x = \frac{a}{b} \text{ and } y = \frac{c}{d}$$

$$x \cdot y = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Since $a, b, c, d \in \mathbb{Z}$ then $ac \in \mathbb{Z}$ and $bd \in \mathbb{Z}$. Since $b \neq 0$ and $d \neq 0$, then $bd \neq 0$, thus $x \cdot y \in \mathbb{Q}$

7. Associtivity of multiplication: the grouping of numbers does not affect their product.

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

this property is inherited from the associativity of multiplication in $\mathbb{Z}$.

8. Commutativity of multiplication: the order of numbers does not affect their product. $x \cdot y = y \cdot x$ this property is inherited from the commutativity of multiplication in $\mathbb{Z}$.

9. multiplicative identity (unity elements):

there exists a unique element, 1, in $\mathbb{Q}$ such that for all $x \in \mathbb{Q}$:

$$x \cdot 1 = x$$

the multiplicative identity is $1 = \frac{1}{1} = 1$,

since $x \cdot 1 = x$

10. multiplicative inverse:

For any non-zero $a = \frac{x}{y} \in Q$, we must have $x \neq 0$. consider the element $a^{-1} = \frac{y}{x}$, since $x \neq 0$.

a^{-1} is a well-defined Rational number. $a \cdot a^{-1} = \frac{x}{y} \cdot \frac{y}{x} = \frac{xy}{yx} = 1$

thus, every non-zero element in Q has a multiplicative inverse.

Axiom of Distribution:

11. Distributivity:

Multiplication is distributive over addition.

$$x \cdot (y+z) = (x \cdot y) + (x \cdot z)$$

this property is inherited from the distributivity in Z.

Since the set of Rational numbers Q Satisfies all eleven field axioms with respect to addition and multiplication. so $(Q, +, \cdot)$ is a field.