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A Transformation for a Multiple Depot, Multiple Traveling Salesman Problem

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***Abstract*— In this paper, a Multiple Depot, Multiple Traveling Salesman Problem is transformed into a Single, Asymmetric Traveling Salesman Problem if the cost of the edges satisfy the triangle inequality. This improves on the previously known transformation for a 2-Depot, Multiple Traveling Salesman Problem in the literature. To test the effectiveness of the transformation, some computational results are presented by applying the well known LKH heuristic on the transformed problem for instances involving Dubins vehicles. Results show that the transformation is effective and high quality solutions can be found for large instances in a relatively short time.**

***Index Terms*— Traveling salesman, Unmanned Aerial Vehicle, Multiple Depot Routing.**

I. Introduction

A Multiple Depot, Multiple Traveling Salesman Problem is an important problem that arises in robotic applications involving ground and aerial vehicles having fuel constraints. This problem can be stated as follows: Let there be n targets and m salesmen located at distinct depots. Let V be the set of vertices that correspond to the initial locations of the salesmen and the targets, with the first m vertices $V_1, \ldots, V_m$ representing the salesmen (i.e., the vertex V_i corresponds to the i^{th} salesman) and $V_{m+1}, \ldots, V_{m+n}$ representing the targets. Let $E = V \times V$ denote the set of all edges (pairs of vertices) and let $C : E \to \Re_+$ denote the cost function with $C(a, b)$ representing the cost of traveling from vertex a to vertex b. We consider costs that are asymmetric, and satisfy the triangle inequality, namely, $C(a, b) + C(b, c) \geq C(a, c)$ for all $a, b, c \in V$. A tour of salesman V_i, $TOUR_i$, is empty and its corresponding tour cost, $C(TOUR_i)$, is zero if the i^{th} salesman does not visit any target. If the i^{th} salesman visits at least one target, then his tour is an ordered set, $TOUR_i$, of at least $r_i + 2,\ r_i \geq 1$ elements of the form $\{V_i, V_{i_1}, \ldots, V_{i_{r_i}}, V_i\}$, where $V_{i_l},\ l = 1, \ldots, r_i$ corresponds to r_i distinct targets being visited in that sequence by the i^{th} salesman. In this case, there is a cost, $C(TOUR_i)$, associated with a tour for the i^{th} salesman and is defined as $C(TOUR_i) = C(V_i, V_{i_1}) + \sum_{k=1}^{r_i-1} C(V_{i_k}, V_{i_{k+1}}) + C(V_{i_{r_i}}, V_i)$. This paper addresses the following Multiple Depot, Multiple Traveling Salesmen Problem (**MDMTSP**): Given the graph, $G = (V, E)$, find tours for the salesmen so that

- each target is visited by at least one salesman, and,
- the overall cost defined by $\sum_{i \in V} C(TOUR_i)$ is minimized.

We are motivated to address the MDMTSP, as it occurs as a subproblem in some of the basic routing problems that arises in applications involving Unmanned Aerial Vehicles (UAVs) with motion and fuel constraints. A fixed wing UAV is typically modeled as a vehicle traveling at a constant speed subject to constraints on its minimum turning radius. Given two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ with their visiting angles as θ_1 and θ_2 respectively, the minimum distance required for an UAV to travel from A to B may not be equal to the minimum distance to travel from B to A (please refer to Fig. 1 for an illustration). As a result, the cost of traveling between any two points for a fixed wing UAV can be asymmetric but satisfy the triangle inequality.

A fundamental routing problem that arises in some surveillance applications require a team of multiple UAVs to visit a group of targets such that the total travel distance is minimized. This routing problem is a challenging optimization problem because it involves two difficult subproblems, namely 1) the combinatorial problem that requires assigning a sequence of targets for each vehicle such that each target is visited by some vehicle, and 2) the motion planning problem where the path of the each vehicle must be determined subject to the turning radius constraints of the vehicles. Currently, there are no approaches available in the literature that can find an optimal solution for this routing problem. However, there are few approximation algorithms, heuristics [3],[4] available to find feasible solutions. In any of these approaches, the combinatorial problem is solved as a first step by assuming an initial angle to visit at each of the targets. For the second step, the path planning problem is solved for each of the vehicles using the target sequences obtained from the first step. This two-step approach can also be iterated multiple times if it can improve the quality of the feasible solutions. In essence, the combinatorial subproblem in these routing problems involving UAVs is essentially the MDMTSP. This is our primary motivation for studying MDMTSPs. *The focus of this paper is on developing fast algorithms for the MDMTSP. As we are not dealing with the path planning problem in this paper, we assume that the angle at which each target must be visited is given apriori.*

MDMTSP is a generalization of the single, Traveling Salesman Problem (TSP) and is NP-Hard [1]. The TSP and its variants have received significant attention in the area of combinatorial optimization [1]. There are several approaches including exact methods [1],[2], approximation algorithms [5] and heuristics [1],[2] that can be used to address TSPs.

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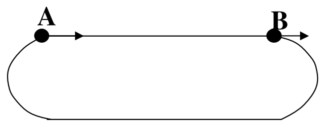

Fig. 1. An example illustrating that the minimum distance to travel from A to B and B to A for an UAV can be asymmetric.

Variants of the multiple TSPs have also been transformed to single TSPs in the literature. By doing so, one can make use of the algorithms available for the single TSPs to solve the multiple TSPs.

In this paper, we present a transformation that can convert the MDMTSP to a single, Asymmetric Traveling Salesman Problem (ATSP). This improves on the result in Rao [8] where, a transformation is given for a 2-Depot, Multiple TSP. To show the effectiveness of the transformation, we also present computational results by applying the well known LKH heuristic available for the single TSP to the transformed problem. The results show that high quality, feasible solutions are found very quickly (less than 20 seconds) for instances involving 50 UAVs and 500 targets. Also, the cost of the feasible solution is, on an average, 3% away from its optimum.

A. Literature Review

Multiple Traveling Salesman Problems can be classified based on whether all the salesmen start from a single depot or from multiple depots. There are several transformations currently available for the variants of a Single Depot, Multiple Traveling Salesman Problem (SDMTSP) in the literature. In [6], Bellmore and Hong consider a SDMTSP where each salesman is available for service at a specific cost and the edge costs need not satisfy triangle inequality. Since the objective is to reduce the total cost travelled by the salesmen, there could be situations when the optimal solution will not necessitate using all the salesmen. Bellmore and Hong provide a way of transforming this single depot MTSP to a standard TSP for the asymmetric case. Hong and Padberg present a more elegant transformation for the same problem in [10]. Rao discusses the symmetric version of the SDMTSP in [8]. Jonker and Volgenant [9] give an improved transformation for a variant of the symmetric, SDMTSP where each salesman has to visit at least one target.

For the multiple depot case, Rao [8] gives a transformation where salesmen start from 2 depots. For the variant of the multiple depot TSP where each salesman need not return to his initial depot and must visit at least one target, GuoXing [7] provides a transformation to a single ATSP. Currently, there is no transformation available for the MDMTSP when each salesman must return to his initial depot for more than 2 depots. In the next section, we present a transformation of the MDMTSP to a single ATSP.

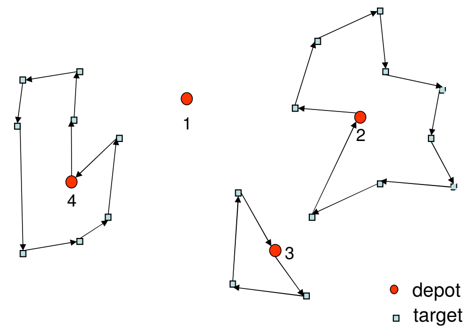

Fig. 2. Given feasible solution for a MDMTSP: 4 depots.

II. Transformation to a Single ATSP

The MDMTSP is transformed by adding a set of copy nodes, where each copy corresponds to a depot vertex. Let the set of copy nodes be denoted by $V_1', \ldots, V_m'$. For each $i \in \{1, \ldots, m\}$, V_i' is the copy corresponding to the depot V_i. The new transformed graph is represented by (V_T, E_T) where $V_T = V \bigcup \{V_1', \ldots, V_m'\}$ and E_T is the set of all the directed edges joining any two vertices in V_T. Let $I_v = \{1, \ldots, m\}$ and $I_t = \{m+1, \ldots, m+n\}$. The cost of the edges in E_T is defined as follows:

$$\begin{aligned}
C_T(V_i, V_j') &= C(V_i, V_j), \text{ for all } i \in I_t, j \in I_v, \\
C_T(V_i, V_j) &= C(V_i, V_j), \text{ for all } i, j \in I_t, \\
C_T(V_i, V_j) &= C(V_i, V_j), \text{ for all } i \in I_v, j \in I_t, \\
C_T(V_i, V_i') &= 0, \quad \text{for all } i \in I_v, \\
C_T(V_i', V_{i+1}) &= 0, \quad \text{for all } i \in \{1, \ldots, m-1\}, \\
C_T(V_m', V_1) &= 0, \\
C_T(u, v) &= M, \quad \text{for all other edges } (u, v) \in E_T.
\end{aligned} \tag{1}$$

In the above equations, M is a large positive constant and is chosen to be equal to $(m+n)\max_{i,j=1,\ldots,m+n} C(V_i, V_j)$. The main result of this paper is the following theorem:

Theorem 2.1: Let the optimal cost of the single ATSP on the transformed graph (V_T, E_T) be C_{atsp}^{opt}. Also, let the optimal cost of the MDMTSP on the original graph (V, E) be C_{mdmtsp}^{opt}. Then, $C_{atsp}^{opt} = C_{mdmtsp}^{opt}$. Moreover, given an optimal solution for the ATSP on the transformed graph, a corresponding optimal solution can be obtained for the MDMTSP in $n + 3m$ steps.

To prove theorem 2.1, we first prove the following lemma:

Lemma 2.1: If there is a feasible solution for the MDMTSP on the original graph (V, E), then there exists a corresponding feasible solution for the single ATSP on

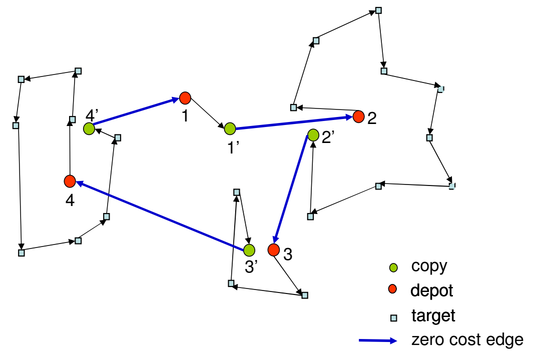

Fig. 3. Corresponding feasible solution for the ATSP on the transformed graph.

the transformed graph (V_T, E_T) with the same cost. Hence, $C^{opt}_{atsp} \leq C^{opt}_{mdmtsp}$.

Proof: Without loss of generality, we assume that the given feasible solution for the MDMTSP consists of p ($p <= m$) nonempty tours, *i.e.*, $Tour_1, ..., Tour_p$. The remaining tours, $Tour_{p+1}, ..., Tour_m$, corresponding to vehicles at $\{V_{p+1}, \dots, V_m\}$ are all empty. If $Tour_i$ is not empty, recall that it is an ordered set represented as $\{V_i, V_{i_1}, \dots, V_{i_{r_i}}, V_i\}$. On the transformed graph, construct the following sequence of vertices for vehicle V_i to visit based on $TOUR_i$:

$$PATH_i = \{V_i, V_{i_1}, \dots, V_{i_{r_i}}, V'_i\} \quad \forall i = 1, \dots, p.$$

For the remaining $(m-p)$ vehicles that do not visit any targets, construct the following sequence of nodes:

$$PATH_i = \{V_i, V'_i\} \quad \forall i = p+1, \dots, m.$$

Now, consider the following ordered sequence of nodes on the transformed graph:

$$TOUR_T = \{PATH_1, PATH_2, \dots, PATH_m, V_1\}.$$

$TOUR_T$ visits all the vertices in V_T exactly once and returns to V_1. Therefore, $TOUR_T$ is a feasible solution for the single ATSP on the transformed graph. Refer to Fig.2 and Fig.3 illustrating the construction of a tour for a 4 depot example. To prove that the costs are the same, first consider the case when $p < m - 1$.

$$\begin{aligned} C_T(TOUR_T) \;=\; & \sum_{i=1..p} [C_T(V_i, V_{i_1}) + \sum_{k=1}^{r_i - 1} C_T(V_{i_k}, V_{i_{k+1}}) \\ & + C_T(V_{i_{r_i}}, V'_i) + C_T(V'_i, V_{i+1})] \\ & + \sum_{i=p+1}^{m} C_T(V_i, V'_i) + \\ & \sum_{i=p+1}^{m-1} C_T(V'_i, V_{i+1}) + C_T(V'_m, V_1) \end{aligned} \tag{2}$$

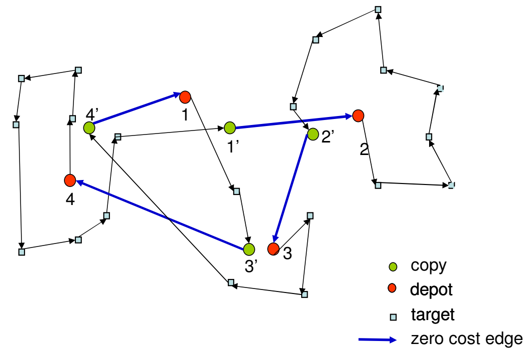

Fig. 4. A given optimal solution for the ATSP on the transformed graph.

Substituting for costs as given in equation (1), we get,

$$\begin{aligned} C_T(TOUR_T) \;=\; & \sum_{i=1..p} [C(V_i, V_{i_1}) + \sum_{k=1}^{r_i - 1} C(V_{i_k}, V_{i_{k+1}}) + \\ & C(V_{i_{r_i}}, V_i)] \\ =\; & \sum_{i=1..p} C(TOUR_i). \end{aligned} \tag{3}$$

The above argument can also be shown for $p = m-1$ and $p = m$. Therefore, the cost of the constructed ATSP tour on the transformed graph is the same as the cost of the feasible solution for the MDMTSP.

■

Proposition 2.1: The following are the properties of any optimal solution for the ATSP on the transformed graph:

1) An optimal solution will not have any edge whose cost is M.
2) For any $i = 1, \dots, m$, if the directed edge from depot V_i to V'_i is not present in an optimal solution, then there is an outgoing edge from V_i to a target and an incoming edge from a target to V'_i.
3) All the zero cost, directed edges in the set $\{(V'_1, V_2), (V'_2, V_3), \dots, (V'_{m-1}, V_m), (V'_m, V_1)\}$ must be present in any optimal solution.

Proof: The proof for properties (1) and (2) are trivial. To prove property (3), note that for any depot vertex, the only incoming edge whose cost is not equal to M is the zero cost edge coming from an adjacent copy vertex. Therefore, all the zero cost, directed edges in the set $\{(V'_1, V_2), (V'_2, V_3), \dots, (V'_{m-1}, V_m), (V'_m, V_1)\}$ must be present in an optimal solution.

■

Lemma 2.2: Given an optimal solution for the single ATSP on the transformed graph, a corresponding feasible solution can be found for the MDMTSP whose cost is at most C^{opt}_{atsp} in $n + 3m$ steps. Hence, $C^{opt}_{mdmtsp} \leq C^{opt}_{atsp}$.

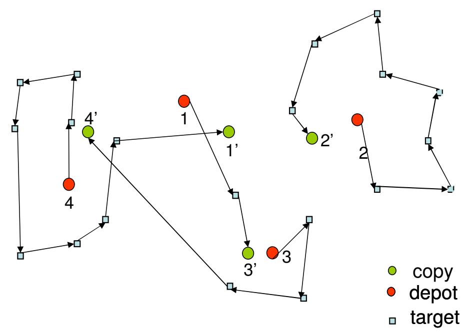

Fig. 5. Remove the zero cost edges.

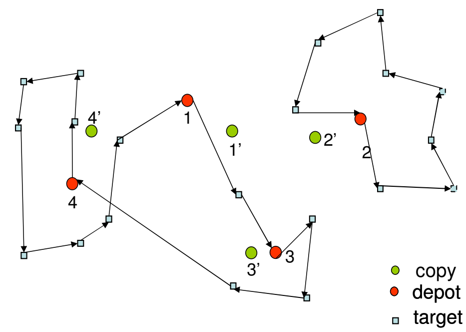

Fig. 6. Transfer any incident edges on the copies to their respective depots.

Proof: Let the set of edges in the optimal solution be denoted by E_{opt}. Also, let the zero cost edges (refer to Fig. 4) in the optimal solution be denoted by E_{zero}. Now, apply the following algorithm to the given optimal solution for the single ATSP on the transformed graph:

1) Remove all the zero cost edges in the optimal solution (refer to Fig. 5). After removing the zero cost edges, the incoming (outgoing) degree of all the depots (copies) in the graph $(V_T, E_{opt} \backslash E_{zero})$ is equal to 0. This follows from property (3) in $Proposition$ 2.1. Also, if the incoming degree of a copy vertex is equal to 1 in $(V_T, E_{opt} \backslash E_{zero})$, then the outgoing degree of its corresponding depot will also be equal to 1. This follows from property (2) in $Proposition$ 2.1. Also, the incoming and the outgoing degree of each target vertex in the graph $(V_T, E_{opt} \backslash E_{zero})$ is equal to 1.
2) Let $E_0 := E_{opt} \backslash E_{zero}$. For $k = 1, \ldots, m$ do the following:
If any directed edge (V_{t_i}, V_i') is incident on a copy V_i', then let $E_i = E_{i-1} \backslash (V_{t_i}, V_i') \bigcup (V_{t_i}, V_i)$. This step essentially transfers any edges incident on the copy vertices to its corresponding depots (refer to Fig.6). For each transferred edge, $C^T(V_{t_i}, V_i') = C(V_{t_i}, V_i)$.

Note that the above algorithm requires at most $2m$ steps. Since only zero cost edges have been pruned from E_{opt} in step 1, the total cost of the edges in E_0, $C^T(E_0)$, is equal to C_{atsp}^{opt}. After step 2, $C(E_m) = C^T(E_0)$ as the cost of the transferred edges does not change. Now, consider the graph (V, E_m). Every target vertex in (V, E_m) has an incoming and an outgoing degree equal to 1. If a depot is connected to any target, then it will also have an incoming degree and an outgoing degree equal to 1. Therefore, the graph (V, E_m) consists of a collection of directed cycles and a set of unused depots. Each cycle in (V, E_m) consists of at least one depot. If any directed cycle consists of more than one depot, then the depots can be shortcut such that each directed cycle consists of exactly one depot. Since the costs satisfy the triangle inequality, short cutting some of the depots will not increase the total cost of the directed cycles, i.e., $C(E_m)$. After short cutting, we have a collection of directed cycles such that each target is visited exactly once and there is exactly one depot in each directed cycle (refer to Fig.7). We have now constructed a feasible solution for the MDMTSP whose cost is at most the cost of the optimal solution for the single ATSP on the transformed graph. Therefore, $C_{mdmtsp}^{opt} \leq C(E_m) = C^T(E_0) = C_{atsp}^{opt}$. Note that short cutting of depots to restrict exactly one depot in each directed cycle could take $n+m$ steps. Hence, given an optimal solution for the ATSP, the total number of steps to find a feasible solution for the MDMTSP is $n+3m$. ■

$Theorem$ 2.1 follows from $Lemma$ 2.1 and $Lemma$ 2.2.

III. Computational Results

The objective of this section is to show that standard algorithms that are known to successfully address the single ATSP can also be used to solve the MDMTSP using the transformation given in this paper. Specifically, we apply the modified LKH heuristic [11] which is one the best heuristics [1] available to solve the single ATSP on the transformed graph.

In all the simulations, the number of UAVs were fixed to be equal to 20. The minimum turning radius (r) of all the UAVs in the simulations was chosen to be 100 meters. Dubins' [12] result was used to calculate the minimum distance to travel for an UAV between any two targets. Dubins' [12] result states that the path joining the two targets (x_1, y_1, θ_1) and (x_2, y_2, θ_2) that has minimal length subject to the minimum turning radius constraints, is one of RSR, RSL, LSR, LSL, RLR and LRL. Here, any curved segment of radius r along which the vehicle executes a clockwise (counterclockwise) rotational motion is denoted by $R(L)$, and the segment along which the vehicle travels straight is denoted by S.

Three different scenarios were considered where targets are uniformly generated in a square of area 1×1 km^2, 3×3 km^2 and 5×5 km^2. For each generated target, an approach angle was selected uniformly in the interval $[0, 2\pi]$. The number of targets were allowed to vary from 20 to 400. For a given number of vehicles (m) and targets (n), 100 instances were randomly generated. The upper bound on the solution quality of an instance I is defined as

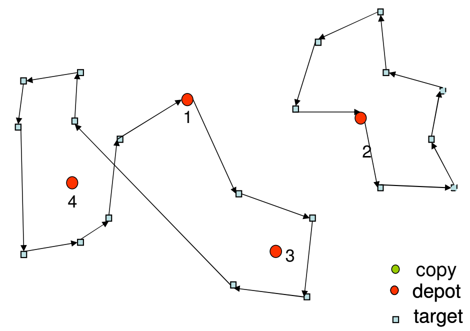

Fig. 7. Remove all the copies; shortcut the depots such that there is exactly one depot in each cycle.

$$100\left(\frac{C^I_{LKH} - C^I_{LB}}{C^I_{LB}}\right),$$

where, C^I_{LKH} is the cost of the solution obtained by applying the LKH heuristic on the transformed graph and C^I_{LB} is the lower bound for the single ATSP on the transformed graph. The LKH program by Helsgaun available at http://www.akira.ruc.dk/ keld/research/LKH/ was used to solve the ATSP. The program was run on a Pentium 4 CPU with 3GHz processing power and 1.24 GB RAM.

The results regarding the mean solution quality and their computation times are shown in Fig.8 and Fig.9 respectively. The results show that the mean solution quality decreases as the number of targets increases. In general, the solution quality gets better as the size of the area decreases. As expected, the mean computation times increased with the number of targets. *The results indicate that a multiple UAV tour with a solution quality of approximately 0.2% can be obtained for a 20 UAV-400 target instance by applying the LKH heuristic on the transformed graph in approximately 20 seconds.* These computational results validate that the transformation is effective and can be successfully used for the MDMTSP involving UAVs.

Results about the mean solution quality may not exactly reflect the solution quality that could be obtained for a given instance. Therefore, in Fig. 10, we also plot the maximum solution quality computed among the 100 randomly generated instances for each (n, m) as a function of the number of targets. These results show that the LKH heuristic performs well on the transformed graph in the worst case also. Fig. 11 shows the comparison between the mean and maximum solution quality when the targets are sampled in a 1×1 km^2.

IV. CONCLUSIONS

In this paper, a transformation was presented for a Multiple Depot, Multiple Traveling Salesman Problem. Computational results were also presented by applying the well known LKH heuristic to the transformed graph for instances involving UAVs. The results show that the transformation is effective

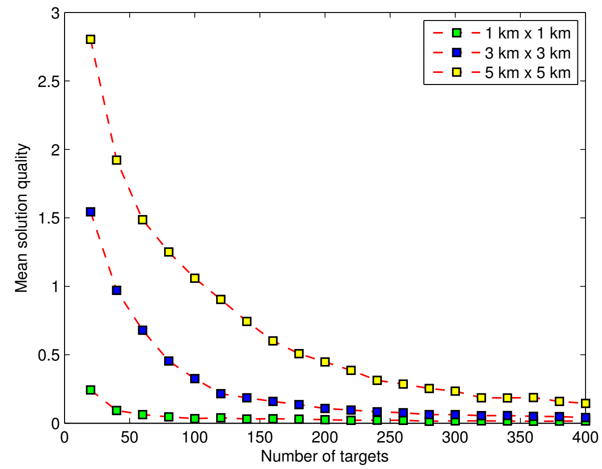

Fig. 8. Mean solution quality as a function of the number of targets.

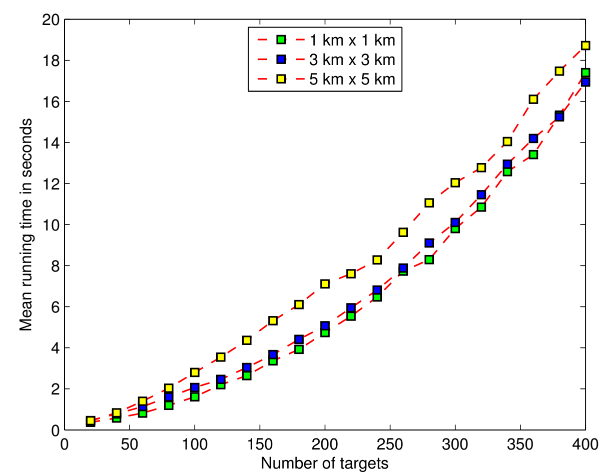

Fig. 9. Mean computation time in seconds as a function of the number of targets.

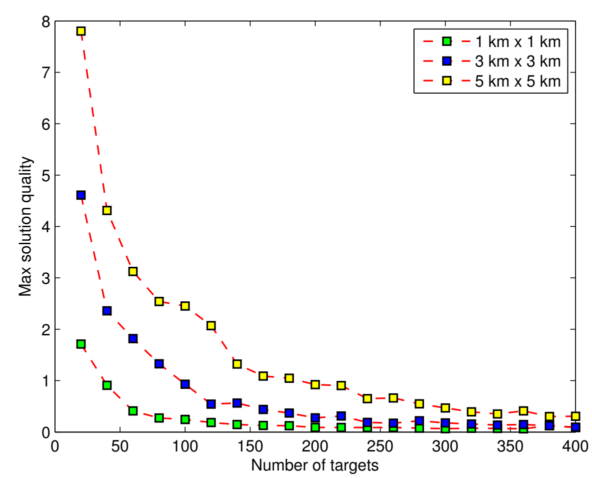

Fig. 10. Max solution quality as a function of the number of targets.

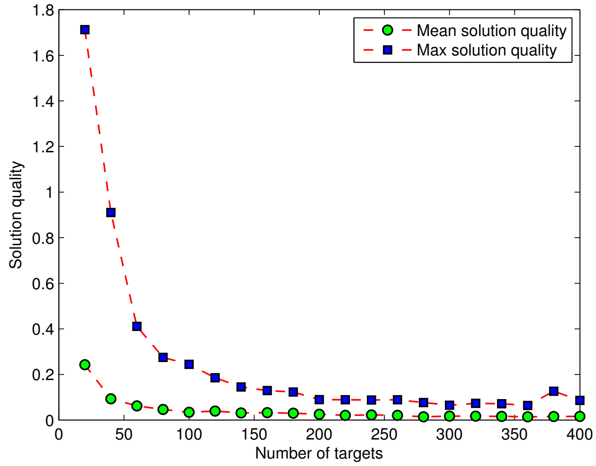

Fig. 11. Comparison of the min and max solution quality for points sampled in 1 x 1 km^2.

and standard algorithms can be used to produced high quality solutions in a relatively short time.

V. ACKNOWLEDGEMENTS

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