$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

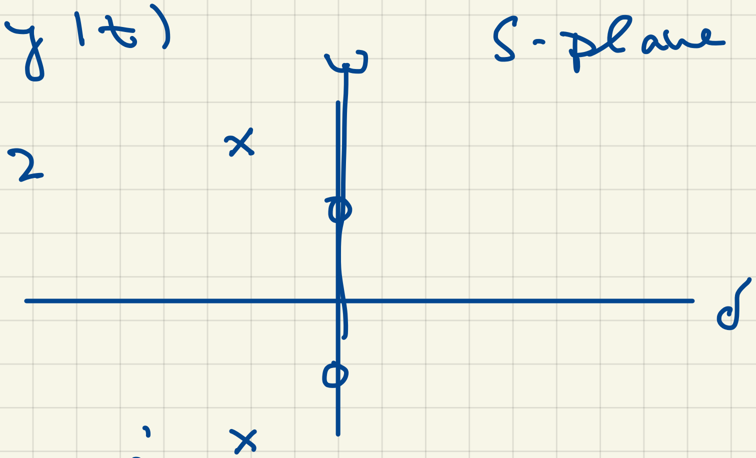
$s = \sigma + i\omega$

$F(s) \circ\!\!-\!\!\circ f(t)$

$x(t) \rightarrow \boxed{} \rightarrow y(t)$

$$H(s) = \frac{b_n s^n + \dots + b_0}{a_n s^n + \dots + a_0}$$

$k = 2$



$\downarrow$ A/D – conversion

$\underline{x[n]} = x_n \rightarrow \boxed{} \rightarrow y[n] = y_n$

Z-Transform (discrete version of Laplace Transform)

$\omega = 2\pi f$

Sample frequency f_s $\quad t_k = \frac{1}{f_s} \cdot k = \frac{2\pi}{\omega_s} \cdot k$

$f(t) \longrightarrow f(t_k) = f[k]$

$$F(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt \xrightarrow[\text{Version}]{\text{Discrete}} \sum_{n=0}^{\infty} f[n]\, e^{-s\frac{1}{f_s}n}$$

$$= \sum_{n=0}^{\infty} f[n]\, e^{-(\delta + i\omega)\frac{2\pi}{\omega_s}\cdot n}$$

$$= \sum_{n=0}^{\infty} f[n] \Big(\underbrace{e^{\delta\frac{2\pi}{\omega_s}}\, e^{i2\pi\frac{\omega}{\omega_s}}}_{z}\Big)^{-n}$$

$$\boxed{\sum_{n=0}^{\infty} f[n]\, z^{-n} = F(z)}$$

z-Transform

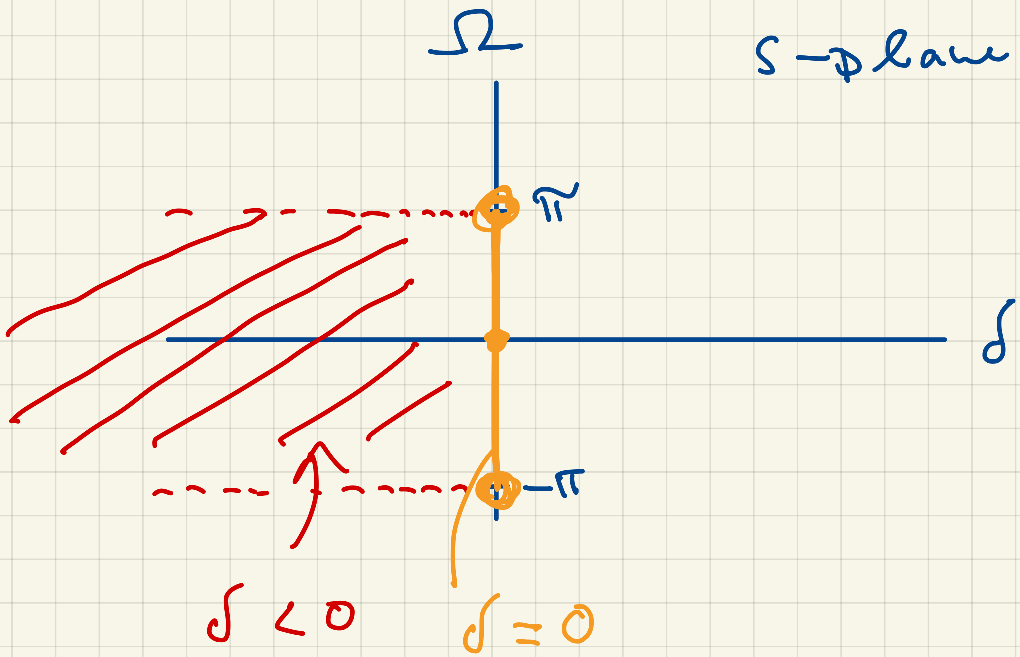
$$z = e^{\sigma \frac{2\pi}{\omega_s}} \; e^{i 2\pi \frac{\omega}{\omega_s}} \qquad e^{i\alpha} = e^{i(\alpha + 2\pi \cdot n)}$$

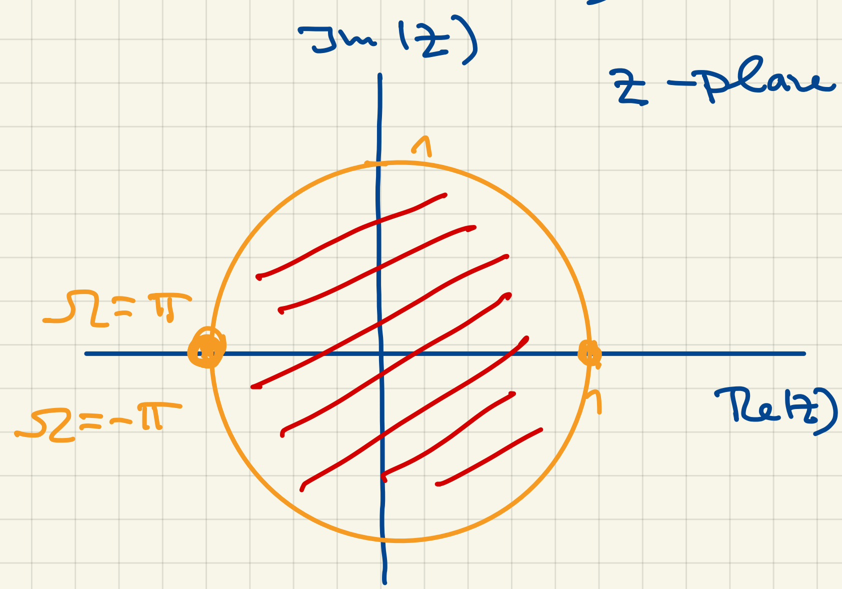
$$e^{i 2\pi \frac{\omega + \omega_s}{\omega_s}} = e^{i 2\pi \left(1 + \frac{\omega}{\omega_s}\right)} = e^{i 2\pi \frac{\omega}{\omega_s}}$$

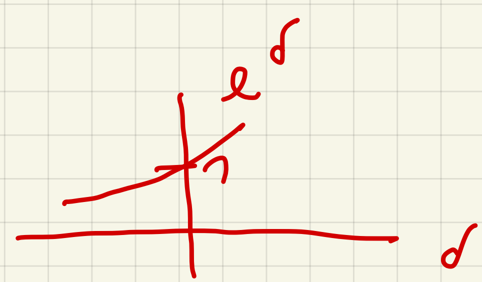
$$z = e^{\sigma \frac{2\pi}{\omega_s}} \; e^{i\Omega} \qquad \Omega = 2\pi \frac{\omega}{\omega_s}$$

$$\omega = \left[-\frac{\omega_s}{2}, \frac{\omega_s}{2}\right] \longleftrightarrow \Omega = [-\pi, \pi]$$

$$z = e^{\sigma \frac{2\pi}{\omega_s}} \; e^{i\Omega}$$







$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \longrightarrow F(z) = \sum_{n=0}^{\infty} f[n]\, z^{-n}$$

Exercise 1: $f[n] = a^n$ $\quad a \in \mathbb{C}, \; |a| < 1$

$a = \frac{1}{2}, -\frac{1}{2}$ $\quad a = \frac{1}{2}: \; f[n] = 1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}$

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \qquad |q| < 1$$

$$F(z) = \sum_{n=0}^{\infty} a^n \underbrace{z^{-n}}_{\frac{1}{z^n}} = \sum_{n=0}^{\infty} \underbrace{\left(\frac{a}{z}\right)}_{q}^n = \frac{1}{1-\frac{a}{z}} \cdot \frac{z}{z}$$

$$\left(\frac{|a|}{|z|} < 1 \quad |z| > |a| \right) \qquad = \frac{z}{z-a}$$

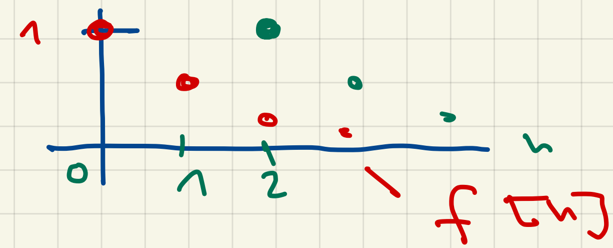
11:15

$$\sum_{n=0}^{\infty} q^n = 1 + q + q^2 + q^3 + \dots = \frac{1}{1-q}$$

$|q| < 1$

$$q = \frac{1}{2}: \quad 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \frac{1}{1-\frac{1}{2}} = 2$$

$$q = -\frac{1}{2}: \quad 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \pm \dots = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$



$$f[n] \circ\!\!-\!\!\bullet\ F(z)$$

$$f[n-n_0] \circ\!\!-\!\!\bullet\ z^{-n_0} F(z)$$

2

$$F(z) = \sum_{n=n_0}^{\infty} f[n-n_0]\, z^{-n} = \sum_{n=0}^{\infty} f[n]\, \underbrace{z^{-(n+n_0)}}_{z^{-n} z^{-n_0}} \qquad f[n-2]$$

$$= z^{-n_0} \quad F(z)$$

$$f[n] \;\circ\!\!-\!\!\bullet\; F(z)$$

$$f[n-n_0] \;\circ\!\!-\!\!\bullet\; z^{-n_0} F(z)$$

Discrete time LTI - Systems:

$$x[n] \longrightarrow \boxed{\text{System}} \longrightarrow y[n]$$

Difference equation

$$a_n y[n] + a_{n-1} y[n-1] + \dots = b_n x[n] + b_{n-1} x[n-1] + \dots$$

Examples: 3 - point moving average

$$y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$$

Transfer function:

$$\boxed{H(z) = \frac{b_n + b_{n-1} z^{-1} + \dots}{a_n + a_{n-1} z^{-1} + \dots}}$$

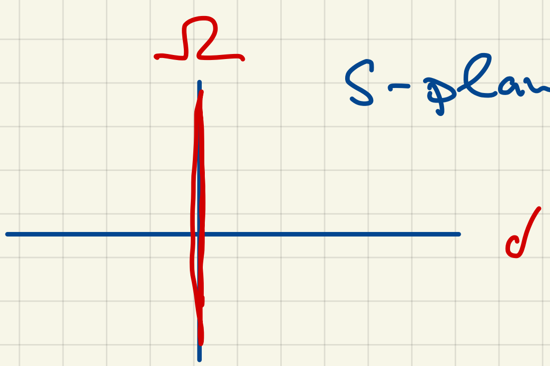
$$H(z) = \frac{b_n + b_{n-1} z^{-1} + \dots}{a_n + a_{n-1} z^{-1} + \dots}$$

examples: 1. $y[n] - \frac{1}{4} y[n-2] = 3x[n] - 3x[n-1]$

$$H(z) = \frac{3 - 3z^{-1}}{1 - \frac{1}{4} z^{-2}}$$

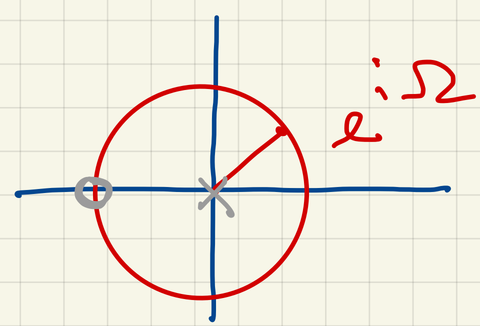
$$y[n] = \frac{1}{2} x[n] + \frac{1}{2} x[n-1]$$

$$H(z) = \frac{\frac{1}{2} + \frac{1}{2}\,\frac{1}{z}}{1} \cdot \frac{z}{z} = \frac{1}{2}\,\frac{z+1}{z}$$



S-plane

$z = \cancel{\sigma}\, e^{i\Omega}$



$H(e^{i\Omega})$

$\downarrow$

frequency response

$$H(z) = \frac{1}{2}\,\frac{z+1}{z} \qquad H(e^{i\Omega}) = \frac{1}{2}\,\frac{e^{i\Omega}+1}{e^{i\Omega}} \cdot \frac{e^{-i\frac{\Omega}{2}}}{e^{-i\frac{\Omega}{2}}}$$

$$= e^{-i\frac{\Omega}{2}}\;\frac{1}{2}\left(e^{i\frac{\Omega}{2}} + e^{-i\frac{\Omega}{2}}\right) = \underbrace{e^{-i\frac{\Omega}{2}}}_{e^{i\varphi}} \cdot \underbrace{\cos\left(\frac{\Omega}{2}\right)}_{|H|}$$