- Tensor algebra.

  - Kronecker delta.

$$\delta_{ij} = \begin{Bmatrix} = 1, & i = j \\ = 0, & i \neq j \end{Bmatrix} = \delta_{ji}$$

  - The Levi-Civita "e" tensor (permutation tensor).

$$e_{ijk} = \begin{cases} 1 & ijk = 123,231,312 \\ -1 & ijk = 321,213,132 \\ 0 & otherwise \end{cases}$$

  - e-delta identity.

$$e_{ijk}e_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

  - Operations of three-dimensional vector algebra.

Decomposition: $\mathbf{u} = u_i\mathbf{e}_i$ ($vector\ in\ x_1x_2x_3\ space\ \boldsymbol{x} = x_i\mathbf{e}_i$)

Scalar (dot) product between unit vectors: $\mathbf{e}_i \bullet \mathbf{e}_j = \delta_{ij}$(orthonormality)

Projection: $\mathbf{e}_i \bullet \mathbf{u} = \mathbf{e}_i \bullet \mathbf{e}_j u_j = \delta_{ij}u_j = u_i$

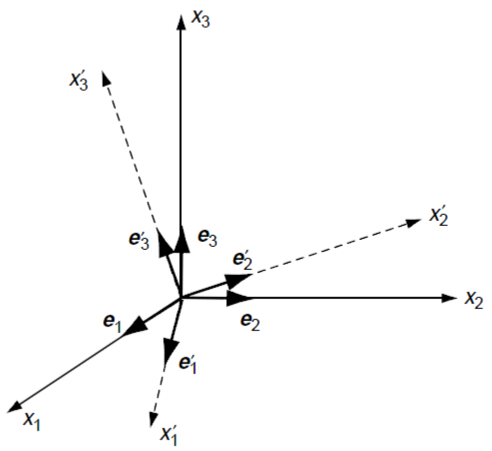
Scalar (dot) product between any two vectors: $\mathbf{u} \bullet \mathbf{v} = u_i\mathbf{e}_i \bullet \mathbf{e}_j v_j = u_i v_i$

Vector (cross) product between unit vectors: $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$

Vector (cross) product between any two vectors: $\mathbf{u} \times \mathbf{v} = \mathbf{e}_i e_{ijk} u_j v_k$

Scalar triple product: $\mathbf{u} \bullet (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \bullet w = e_{ijk}u_i v_j w_k$

  - Tensor.



$\mathbf{e}_i \bullet \mathbf{e}'_j = \beta_{ij}$ ($\beta_{ij} = cos\theta$, θ: $angle\ between$ $\mathbf{e}_i$ and $\mathbf{e}'_j$)

$$\mathbf{e}_i = \sum_j \beta_{ij}\mathbf{e}'_j \ , \quad \mathbf{e}'_j = \sum_i \beta_{ji}\mathbf{e}_i \ .$$

$$\boldsymbol{u} = \sum_i \mathrm{u}_i \mathbf{e}_i = \sum_i \mathrm{u}'_i \mathbf{e}'_i$$

$$\mathrm{u}'_i = \sum_{j=1}^{3} \beta_{ij}\mathrm{u}_j \ , \qquad \mathrm{u}_i = \sum_{j=1}^{3} \beta_{ij}\mathrm{u}'_j$$

$$\mathrm{u}'_i = \sum_{j=1}^{3} \beta_{ij}\mathrm{u}_j = \sum_{j=1}^{3} \beta_{ij} \sum_{k=1}^{3} \beta_{jk}\mathrm{u}'_k$$

$$\mathrm{u}_i = \sum_{j=1}^{3} \beta_{ij}\mathrm{u}'_j = \sum_{j=1}^{3} \beta_{ij} \sum_{k=1}^{3} \beta_{jk}\mathrm{u}_k$$

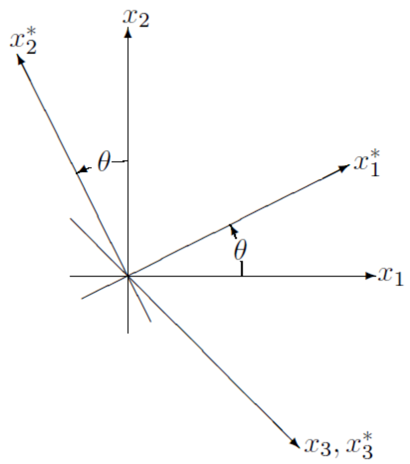
$$\mathrm{u}'_i = \delta_{ij}\mathrm{u}'_j \ , \quad \mathrm{u}_i = \delta_{ij}\mathrm{u}_j$$

$$\sum_{j=1}^{3} \beta_{ij} \sum_{k=1}^{3} \beta_{jk} = \delta_{ik} \qquad \therefore \text{matrix } \boldsymbol{\beta} = [\beta_{ij}] => \boldsymbol{\beta}\cdot\boldsymbol{\beta}^T = \boldsymbol{\beta}^T\cdot\boldsymbol{\beta} = \boldsymbol{I}(orthogonal)$$

$$|\boldsymbol{\alpha}| = |\boldsymbol{\alpha}^T| \ , \quad |\boldsymbol{\alpha\beta}| = |\boldsymbol{\alpha}|\cdot|\boldsymbol{\beta}| \qquad \therefore orthogonal: (|\boldsymbol{\beta}|)^2 = |\boldsymbol{I}| = 1 \ , \quad |\boldsymbol{\beta}| = \pm 1$$

예제)

$$\underline{\beta} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$\mathbf{e}'_j = \sum_i \beta_{ji}\mathbf{e}_i => \begin{bmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \cos(90-\theta) & 0 \\ \cos(90+\theta) & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

- Linear operators.

$$\mathbf{v} = \boldsymbol{\lambda}(\mathbf{u}) = \boldsymbol{\lambda} \bullet \mathbf{u}(\text{tensor notation}) = \boldsymbol{\lambda}\mathbf{u}(\text{matrix notation})$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \lambda_{11}u_1 + \lambda_{12}u_2 + \lambda_{13}u_3 \\ \lambda_{21}u_1 + \lambda_{22}u_2 + \lambda_{23}u_3 \\ \lambda_{31}u_1 + \lambda_{32}u_2 + \lambda_{33}u_3 \end{bmatrix}$$

$$\boldsymbol{\lambda}(a\mathbf{u} + b\mathbf{v}) = a\boldsymbol{\lambda}(\mathbf{u}) + b\boldsymbol{\lambda}(\mathbf{v}), \quad \text{a, b: scalar}$$

$$v_i = \mathbf{e}_i \bullet \boldsymbol{\lambda}(\mathbf{e}_j u_j) = \sum_j \lambda_{ij} u_j \equiv \mathbf{e}_i \bullet \boldsymbol{\lambda}(\mathbf{e}_j) u_j, \quad \sum_j \lambda_{ij} \equiv \mathbf{e}_i \bullet \boldsymbol{\lambda}(\mathbf{e}_j), \quad \boldsymbol{\lambda} = [\lambda_{ij}]$$

- $\{x_1 \quad x_2 \quad x_3\}$에서 정의된 $\boldsymbol{\lambda}$를 $\{x'_1 \quad x'_2 \quad x'_3\}$에 적용할 경우.

$\mathbf{v}' = \boldsymbol{\lambda}'(\mathbf{u}') = \boldsymbol{\lambda}' \bullet \mathbf{u}' = \boldsymbol{\lambda}'\mathbf{u}'$를 만족하는 $\boldsymbol{\lambda}'$를 구한다.

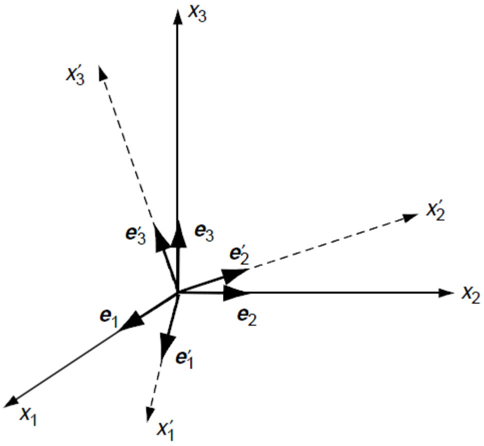
$$v'_i = \sum_j \beta_{ij} v_j, \quad v_j = \sum_k \lambda_{jk} u_k, \quad u_k = \sum_l \beta_{kl} u'_l$$

$$v'_i = \sum_j \beta_{ij} \sum_k \lambda_{jk} \sum_l \beta_{kl} u'_l,$$

$$\sum_l \lambda'_{il} = \sum_j \beta_{ij} \sum_k \lambda_{jk} \sum_l \beta_{kl}, \quad \boldsymbol{\lambda}' = \boldsymbol{\beta}\boldsymbol{\lambda}\boldsymbol{\beta}^T (\text{matrix notation})$$

$il: \{x'_1 \quad x'_2 \quad x'_3\}$*system indices*, $jk: \{x_1 \quad x_2 \quad x_3\}$*system indices*

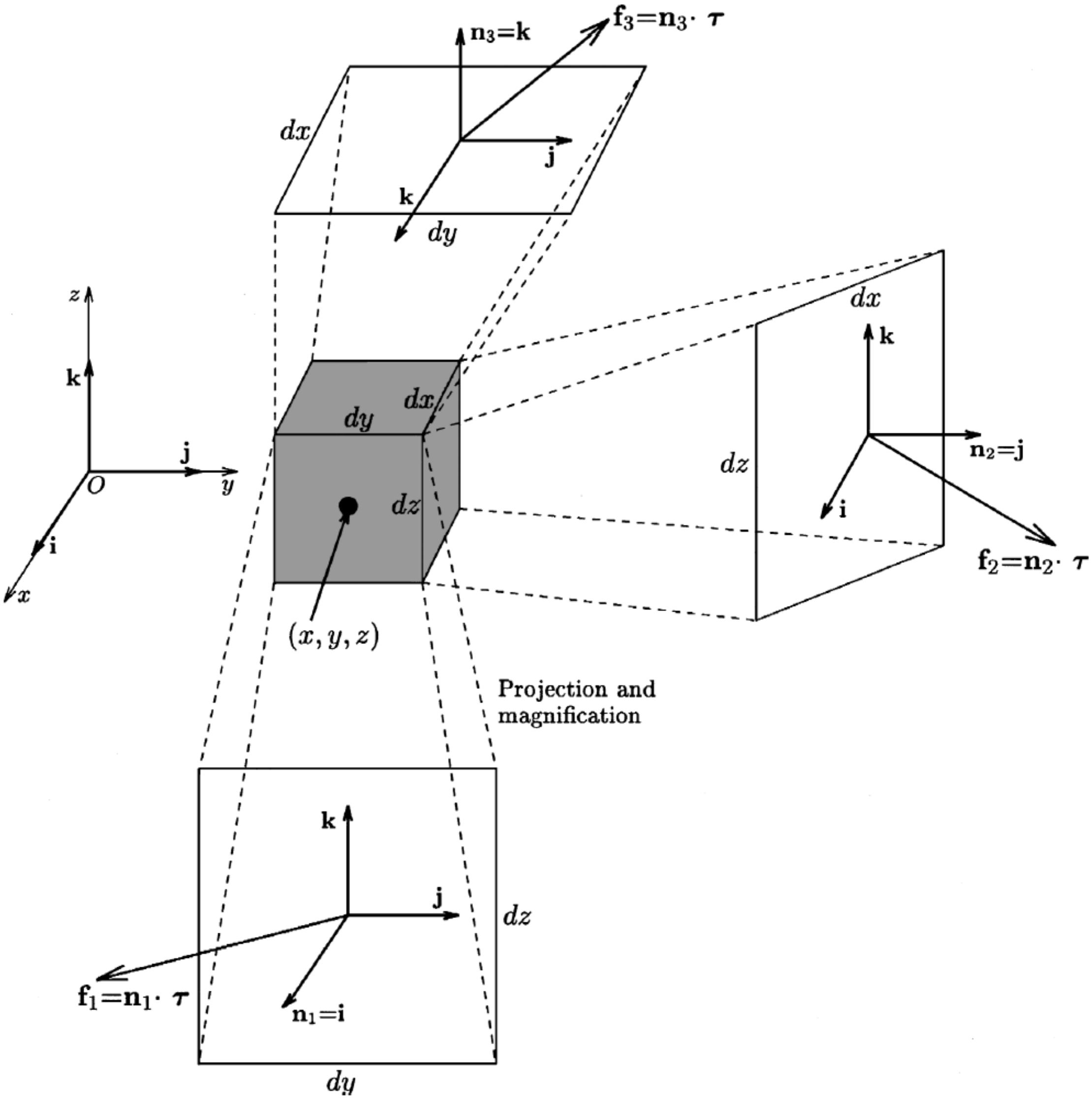
$$\begin{bmatrix} \lambda'_{11} & \lambda'_{12} & \lambda'_{13} \\ \lambda'_{21} & \lambda'_{22} & \lambda'_{23} \\ \lambda'_{31} & \lambda'_{32} & \lambda'_{33} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{32} \\ \beta_{13} & \beta_{23} & \beta_{33} \end{bmatrix}$$



- Invariant.

The property is shared by $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ (for any $\boldsymbol{\beta}$).

- Stress tensor.



$$\mathbf{a}\bullet\boldsymbol{\sigma}=\sum_{i=1}^{3}\left(\sum_{j=1}^{3}\mathrm{a}_j\sigma_{ji}\right)\mathbf{n}_i\,,\quad \boldsymbol{\sigma}\bullet\mathbf{a}=\sum_{i=1}^{3}\left(\sum_{j=1}^{3}\sigma_{ij}\mathrm{a}_j\right)\mathbf{n}_i\ (tensor\ algebra)$$

$$\mathbf{f}_1=\mathbf{n}_1\bullet\boldsymbol{\tau}\,,\qquad \mathbf{f}_2=\mathbf{n}_2\bullet\boldsymbol{\tau}\,,\qquad \mathbf{f}_3=\mathbf{n}_3\bullet\boldsymbol{\tau}$$

$$\mathbf{f}_1=\mathbf{n}_1\bullet\boldsymbol{\tau}=[\tau_{11}\quad\tau_{12}\quad\tau_{13}]=[1\quad 0\quad 0]\begin{bmatrix}\tau_{11}&\tau_{12}&\tau_{13}\\\tau_{21}&\tau_{22}&\tau_{23}\\\tau_{31}&\tau_{32}&\tau_{33}\end{bmatrix}$$

$$\mathbf{f}_2=\mathbf{n}_2\bullet\boldsymbol{\tau}=[\tau_{21}\quad\tau_{22}\quad\tau_{23}]=[0\quad 1\quad 0]\begin{bmatrix}\tau_{11}&\tau_{12}&\tau_{13}\\\tau_{21}&\tau_{22}&\tau_{23}\\\tau_{31}&\tau_{32}&\tau_{33}\end{bmatrix}$$

$$\mathbf{f}_3=\mathbf{n}_3\bullet\boldsymbol{\tau}=[\tau_{31}\quad\tau_{32}\quad\tau_{33}]=[0\quad 0\quad 1]\begin{bmatrix}\tau_{11}&\tau_{12}&\tau_{13}\\\tau_{21}&\tau_{22}&\tau_{23}\\\tau_{31}&\tau_{32}&\tau_{33}\end{bmatrix}$$

$$\boldsymbol{\tau} = \mathbf{n}_1\mathbf{f}_1 + \mathbf{n}_2\mathbf{f}_2 + \mathbf{n}_3\mathbf{f}_3 (\textit{tensor notation})$$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13}\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}\tau_{21} & \tau_{22} & \tau_{23}\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\begin{bmatrix}\tau_{31} & \tau_{32} & \tau_{33}\end{bmatrix}$$

$$= \begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13}\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\\tau_{21} & \tau_{22} & \tau_{23}\\0 & 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\\tau_{31} & \tau_{32} & \tau_{33}\end{bmatrix} = \begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13}\\\tau_{21} & \tau_{22} & \tau_{23}\\\tau_{31} & \tau_{32} & \tau_{33}\end{bmatrix}$$

$$\begin{aligned}\mathbf{f}_1 &= \tau_{11}\mathbf{n}_1 + \tau_{12}\mathbf{n}_2 + \tau_{13}\mathbf{n}_3\\ \mathbf{f}_2 &= \tau_{21}\mathbf{n}_1 + \tau_{22}\mathbf{n}_2 + \tau_{23}\mathbf{n}_3\\ \mathbf{f}_3 &= \tau_{31}\mathbf{n}_1 + \tau_{32}\mathbf{n}_2 + \tau_{33}\mathbf{n}_3\end{aligned} \quad \Rightarrow \quad \begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13}\\\tau_{21} & \tau_{22} & \tau_{23}\\\tau_{31} & \tau_{32} & \tau_{33}\end{bmatrix} = \boldsymbol{\tau}$$

$$\mathbf{f} = \begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix}^T = \mathbf{n} \bullet \boldsymbol{\tau} = \begin{bmatrix}n_1 & n_2 & n_3\end{bmatrix}\begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13}\\\tau_{21} & \tau_{22} & \tau_{23}\\\tau_{31} & \tau_{32} & \tau_{33}\end{bmatrix} = \begin{bmatrix}n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31}\\n_1\tau_{12} + n_2\tau_{22} + n_3\tau_{32}\\n_1\tau_{13} + n_2\tau_{23} + n_3\tau_{33}\end{bmatrix}^T$$

$$\left(|\mathbf{n}| = \sqrt{{n_1}^2 + {n_2}^2 + {n_3}^2} = 1\right)$$

$$\mathbf{f} = \mathbf{f}_N + \mathbf{f}_T$$

$$\mathbf{f}_N = (\mathbf{n} \bullet \mathbf{f})\,\mathbf{n} = \mathbf{n} \bullet (\mathbf{n} \bullet \boldsymbol{\tau})\,\mathbf{n} = (\mathbf{nn} : \boldsymbol{\tau})\,\mathbf{n}$$

$$\mathbf{n} \bullet \mathbf{f} = \begin{bmatrix}n_1 & n_2 & n_3\end{bmatrix}\begin{bmatrix}n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31}\\n_1\tau_{12} + n_2\tau_{22} + n_3\tau_{32}\\n_1\tau_{13} + n_2\tau_{23} + n_3\tau_{33}\end{bmatrix}$$

$$= n_1(n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31}) + n_2(n_1\tau_{12} + n_2\tau_{22} + n_3\tau_{32}) + n_3(n_1\tau_{13} + n_2\tau_{23} + n_3\tau_{33})$$

$$\mathbf{nn} : \boldsymbol{\tau} = \sum_{i=1}^{3}\{[\mathrm{nn}]\ [\tau]\}_{ii} = \sum_{i=1}^{3}\left\{\begin{bmatrix}n_1n_1 & n_1n_2 & n_1n_3\\n_2n_1 & n_2n_2 & n_2n_3\\n_3n_1 & n_3n_2 & n_3n_3\end{bmatrix}\begin{bmatrix}\tau_{11} & \tau_{12} & \tau_{13}\\\tau_{21} & \tau_{22} & \tau_{23}\\\tau_{31} & \tau_{32} & \tau_{33}\end{bmatrix}\right\}_{ii}$$

$$= (n_1n_1\tau_{11} + n_1n_2\tau_{21} + n_1n_3\tau_{31}) + (n_2n_1\tau_{12} + n_2n_2\tau_{22} + n_2n_3\tau_{32}) + (n_3n_1\tau_{13} + n_3n_2\tau_{23} + n_3n_3\tau_{33})$$

$$= n_1(n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31}) + n_2(n_1\tau_{12} + n_2\tau_{22} + n_3\tau_{32}) + n_3(n_1\tau_{13} + n_2\tau_{23} + n_3\tau_{33}) = |\mathbf{f}_N|$$

$$\mathbf{f}_T = \mathbf{f} - \mathbf{f}_N = \mathbf{n} \bullet \boldsymbol{\tau} - (\mathbf{nn} : \boldsymbol{\tau})\mathbf{n}$$

$$= \begin{bmatrix}n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31} - n_1\big(n_1(n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31}) + n_2(n_1\tau_{12} + n_2\tau_{22} + n_3\tau_{32}) + n_3(n_1\tau_{13} + n_2\tau_{23} + n_3\tau_{33})\big)\\n_1\tau_{12} + n_2\tau_{22} + n_3\tau_{32} - n_2\big(n_1(n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31}) + n_2(n_1\tau_{12} + n_2\tau_{22} + n_3\tau_{32}) + n_3(n_1\tau_{13} + n_2\tau_{23} + n_3\tau_{33})\big)\\n_1\tau_{13} + n_2\tau_{23} + n_3\tau_{33} - n_3\big(n_1(n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31}) + n_2(n_1\tau_{12} + n_2\tau_{22} + n_3\tau_{32}) + n_3(n_1\tau_{13} + n_2\tau_{23} + n_3\tau_{33})\big)\end{bmatrix}$$

$$\mathbf{f}_T = \mathbf{n} \times (\mathbf{f} \times \mathbf{n}) = \mathbf{f} \bullet (\mathbf{I} - \mathbf{nn})$$

$$\mathbf{f} \times \mathbf{n} = \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\f_1 & f_2 & f_3\\n_1 & n_2 & n_3\end{vmatrix} = \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31} & n_1\tau_{12} + n_2\tau_{22} + n_3\tau_{32} & n_1\tau_{13} + n_2\tau_{23} + n_3\tau_{33}\\n_1 & n_2 & n_3\end{vmatrix}$$

$$\mathbf{n} \times (\mathbf{f} \times \mathbf{n}) = \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\n_1 & n_2 & n_3\\f_2n_3 - f_3n_2 & f_3n_1 - f_1n_3 & f_1n_2 - f_2n_1\end{vmatrix} = \begin{bmatrix}n_2(f_1n_2 - f_2n_1) - n_3(f_3n_1 - f_1n_3)\\n_3(f_2n_3 - f_3n_2) - n_1(f_1n_2 - f_2n_1)\\n_1(f_3n_1 - f_1n_3) - n_2(f_2n_3 - f_3n_2)\end{bmatrix}$$

$$n_2(f_1n_2 - f_2n_1) - n_3(f_3n_1 - f_1n_3) = f_1(n_1n_1 + n_2n_2 + n_3n_3) - n_1(n_1f_1 + n_2f_2 + n_3f_3)$$

$$= n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31} - n_1\big(n_1(n_1\tau_{11} + n_2\tau_{21} + n_3\tau_{31}) + n_2(n_1\tau_{12} + n_2\tau_{22} + n_3\tau_{32}) + n_3(n_1\tau_{13} + n_2\tau_{23} + n_3\tau_{33})\big)$$

$$\mathbf{f} \bullet (\mathbf{I} - \mathbf{nn}) = \begin{bmatrix}f_1 & f_2 & f_3\end{bmatrix}\begin{bmatrix}1 - n_1n_1 & -n_1n_2 & -n_1n_3\\-n_2n_1 & 1 - n_2n_2 & -n_2n_3\\-n_3n_1 & -n_3n_2 & 1 - n_3n_3\end{bmatrix} = \begin{bmatrix}f_1 - n_1(n_1f_1 + n_2f_2 + n_3f_3)\\f_2 - n_2(n_1f_1 + n_2f_2 + n_3f_3)\\f_3 - n_3(n_1f_1 + n_2f_2 + n_3f_3)\end{bmatrix}$$

- Invariants.

$$\begin{vmatrix} \tau_{11}-\lambda & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22}-\lambda & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33}-\lambda \end{vmatrix}$$

$$= (\tau_{11}-\lambda)(\tau_{22}-\lambda)(\tau_{33}-\lambda) - \tau_{23}\tau_{32}(\tau_{11}-\lambda) + \tau_{12}\tau_{23}\tau_{31} - \tau_{12}\tau_{21}(\tau_{33}-\lambda) + \tau_{13}\tau_{21}\tau_{32} - \tau_{13}(\tau_{22}-\lambda)\tau_{31}$$

$$= -\{\lambda^3 - (\tau_{11}+\tau_{22}+\tau_{33})\lambda^2 + (\tau_{11}\tau_{22}+\tau_{11}\tau_{33}+\tau_{22}\tau_{33}-\tau_{12}\tau_{21}-\tau_{13}\tau_{31}-\tau_{23}\tau_{32})\lambda$$
$$-(\tau_{11}\tau_{22}\tau_{33}-\tau_{11}\tau_{23}\tau_{32}+\tau_{12}\tau_{23}\tau_{31}-\tau_{12}\tau_{21}\tau_{33}+\tau_{13}\tau_{21}\tau_{32}-\tau_{13}\tau_{22}\tau_{31})\}$$

$$= -\{\lambda^3 - J_1\lambda^2 + J_2\lambda - J_3\} = 0$$

$$\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \qquad \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$J_1 = \tau_{11}+\tau_{22}+\tau_{33} = \lambda_1+\lambda_2+\lambda_3$$

$$J_2 = \begin{vmatrix} \tau_{22} & \tau_{23} \\ \tau_{32} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{13} \\ \tau_{31} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{vmatrix} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \ (cofactor)$$

$$J_3 = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix} = \lambda_1\lambda_2\lambda_3$$

$$(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$$

$$= \lambda^3 - (\lambda_1+\lambda_2+\lambda_3)\lambda^2 + (\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3$$

- Maximum shear stress.

$$\mathbf{f} = \mathbf{f}_N + \mathbf{f}_T$$

$$\mathbf{f}_N = (\mathbf{n}\bullet\mathbf{f})\,\mathbf{n} = \mathbf{n}\bullet(\mathbf{n}\bullet\boldsymbol{\tau})\,\mathbf{n} = (\mathbf{nn}:\boldsymbol{\tau})\,\mathbf{n}$$

n$->$ principal direction

$$\mathbf{f} = \mathbf{n}\bullet\boldsymbol{\tau} = [n_1 \quad n_2 \quad n_3]\begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix} = \begin{bmatrix} n_1\sigma_{11} \\ n_2\sigma_{22} \\ n_3\sigma_{33} \end{bmatrix}^T -> |\mathbf{f}|^2 = {n_1}^2{\sigma_{11}}^2 + {n_2}^2{\sigma_{22}}^2 + {n_3}^2{\sigma_{33}}^2$$

$$|\mathbf{f}_N| = \mathbf{n}\bullet\mathbf{f} = [n_1 \quad n_2 \quad n_3]\begin{bmatrix} n_1\sigma_{11} \\ n_2\sigma_{22} \\ n_3\sigma_{33} \end{bmatrix} = {n_1}^2\sigma_{11} + {n_2}^2\sigma_{22} + {n_3}^2\sigma_{33}$$

$$|\mathbf{f}_T|^2 = |\mathbf{f}|^2 - |\mathbf{f}_N|^2 = {n_1}^2{\sigma_{11}}^2 + {n_2}^2{\sigma_{22}}^2 + {n_3}^2{\sigma_{33}}^2 - ({n_1}^2\sigma_{11} + {n_2}^2\sigma_{22} + {n_3}^2\sigma_{33})^2$$

$${n_3}^2 = 1 - {n_1}^2 - {n_2}^2$$

$$|\mathbf{f}_T|^2 = {n_1}^2{\sigma_{11}}^2 + {n_2}^2{\sigma_{22}}^2 + (1-{n_1}^2-{n_2}^2){\sigma_{33}}^2 - ({n_1}^2\sigma_{11} + {n_2}^2\sigma_{22} + (1-{n_1}^2-{n_2}^2)\sigma_{33})^2$$

$$\frac{\partial|\mathbf{f}_T|^2}{\partial n_1} = 0\,, \quad \frac{\partial|\mathbf{f}_T|^2}{\partial n_2} = 0$$

$$\frac{\partial|\mathbf{f}_T|^2}{\partial n_1} = 2n_1\sigma_{11}{}^2 - 2n_1\sigma_{33}{}^2 - 2(n_1{}^2\sigma_{11} + n_2{}^2\sigma_{22} + (1 - n_1{}^2 - n_2{}^2)\sigma_{33})(2n_1\sigma_{11} - 2n_1\sigma_{33}) = 0$$

$$n_1(\sigma_{11}{}^2 - \sigma_{33}{}^2) = 2(n_1{}^2\sigma_{11} + n_2{}^2\sigma_{22} + (1 - n_1{}^2 - n_2{}^2)\sigma_{33})n_1(\sigma_{11} - \sigma_{33})$$

$$(\sigma_{11} + \sigma_{33})n_1 = 2(n_1{}^2\sigma_{11} + n_2{}^2\sigma_{22} + (1 - n_1{}^2 - n_2{}^2)\sigma_{33})n_1$$

$$(\sigma_{22} + \sigma_{33})n_2 = 2(n_1{}^2\sigma_{11} + n_2{}^2\sigma_{22} + (1 - n_1{}^2 - n_2{}^2)\sigma_{33})n_2$$

Case1) $\sigma_{11} \neq \sigma_{22} \neq \sigma_{33}$

$$n_1{}^2\sigma_{11} + n_2{}^2\sigma_{22} + (1 - n_1{}^2 - n_2{}^2)\sigma_{33} - \frac{\sigma_{11} + \sigma_{33}}{2} = 0$$

$$n_1{}^2\sigma_{11} + n_2{}^2\sigma_{22} + (1 - n_1{}^2 - n_2{}^2)\sigma_{33} - \frac{\sigma_{22} + \sigma_{33}}{2} = 0$$

$\sigma_{11} \neq \sigma_{22} \quad \therefore n_1 = 0 \ or\ n_2 = 0$

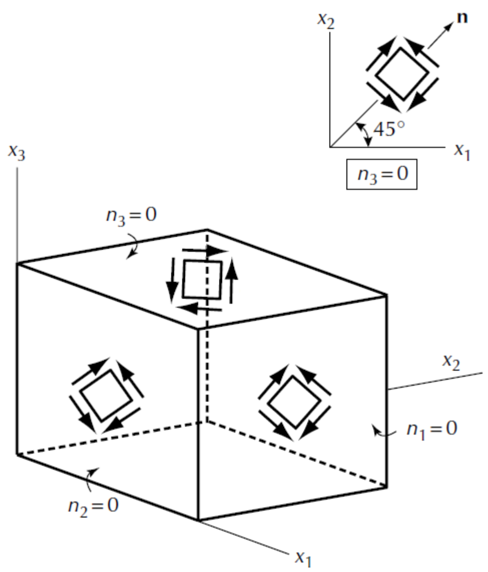
$if\ n_1 = 0$

$$n_1{}^2\sigma_{11} + n_2{}^2\sigma_{22} + (1 - n_1{}^2 - n_2{}^2)\sigma_{33} - \frac{\sigma_{22} + \sigma_{33}}{2} = \left(n_2{}^2 - \frac{1}{2}\right)(\sigma_{22} - \sigma_{33}) = 0$$

$$\therefore\ n_1 = 0,\quad n_2 = \pm\frac{1}{\sqrt{2}},\quad n_3 = \pm\frac{1}{\sqrt{2}},\ |\mathbf{f}_T|^2 = \left(\frac{\sigma_{22} - \sigma_{33}}{2}\right)^2 \ -> \ |\mathbf{f}_T| = \left|\frac{\sigma_{22} - \sigma_{33}}{2}\right|$$

$$n_1 = \pm\frac{1}{\sqrt{2}},\quad n_2 = 0,\quad n_3 = \pm\frac{1}{\sqrt{2}},\quad |\mathbf{f}_T| = \left|\frac{\sigma_{11} - \sigma_{33}}{2}\right|$$

$$n_1 = \pm\frac{1}{\sqrt{2}},\quad n_2 = \pm\frac{1}{\sqrt{2}},\quad n_3 = 0,\quad |\mathbf{f}_T| = \left|\frac{\sigma_{11} - \sigma_{22}}{2}\right|$$



Case2) $\sigma_{11} = \sigma_{22} \neq \sigma_{33}$

$$n_1{}^2\sigma_{11} + n_2{}^2\sigma_{22} + (1 - n_1{}^2 - n_2{}^2)\sigma_{33} - \frac{\sigma_{11} + \sigma_{33}}{2} = 0$$

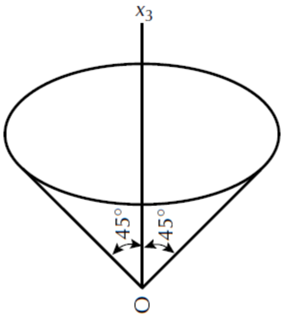
$$n_1{}^2\sigma_{11} + n_2{}^2\sigma_{22} + (1 - n_1{}^2 - n_2{}^2)\sigma_{33} - \frac{\sigma_{22} + \sigma_{33}}{2} = 0$$

$\sigma_{11} = \sigma_{22}$

$$\left(n_1{}^2 + n_2{}^2 - \frac{1}{2}\right)\sigma_{11} + \left(\frac{1}{2} - n_1{}^2 - n_2{}^2\right)\sigma_{33} = 0$$

$$n_1{}^2 + n_2{}^2 = \frac{1}{2}, \quad n_3{}^2 = \frac{1}{2}, \quad n_3 = \pm\frac{1}{\sqrt{2}}$$

$$|\mathbf{f}_T|^2 = \left(\frac{\sigma_{22} - \sigma_{33}}{2}\right)^2 \;{-}{>}\; |\mathbf{f}_T| = \left|\frac{\sigma_{22} - \sigma_{33}}{2}\right| = \left|\frac{\sigma_{11} - \sigma_{33}}{2}\right|$$



Case3) $\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma$(hydrostatic pressure)

$$|\mathbf{f}_T|^2 = n_1{}^2\sigma_{11}{}^2 + n_2{}^2\sigma_{22}{}^2 + (1 - n_1{}^2 - n_2{}^2)\sigma_{33}{}^2 - (n_1{}^2\sigma_{11} + n_2{}^2\sigma_{22} + (1 - n_1{}^2 - n_2{}^2)\sigma_{33})^2$$

$$= \sigma^2 - \sigma^2 = 0$$

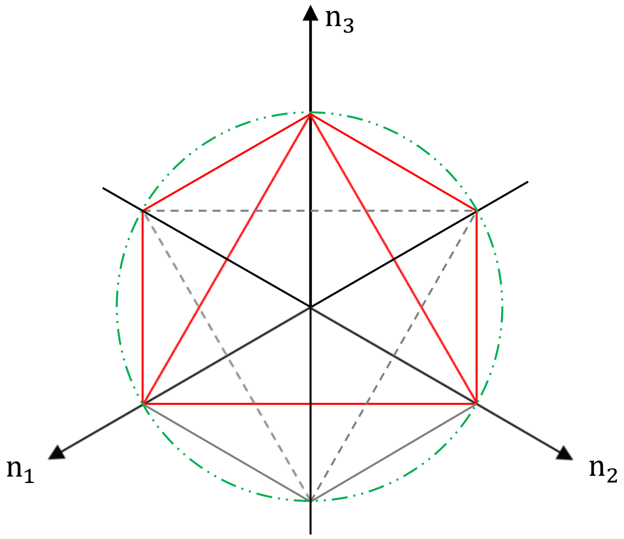
■ Hydrostatic and deviatoric stress.

$$\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} = \begin{bmatrix} \frac{\tau_{11} + \tau_{22} + \tau_{33}}{3} & 0 & 0 \\ 0 & \frac{\tau_{11} + \tau_{22} + \tau_{33}}{3} & 0 \\ 0 & 0 & \frac{\tau_{11} + \tau_{22} + \tau_{33}}{3} \end{bmatrix}$$

$$+ \begin{bmatrix} \tau_{11} - \frac{\tau_{11} + \tau_{22} + \tau_{33}}{3} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} - \frac{\tau_{11} + \tau_{22} + \tau_{33}}{3} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} - \frac{\tau_{11} + \tau_{22} + \tau_{33}}{3} \end{bmatrix}$$

deviatoric stress

- Octahedral shear stress.



n$->$ principal direction

$$n_1^{\,2} = n_2^{\,2} = n_3^{\,2} = \frac{1}{3}, \quad \text{(octahedral planes)}$$

$$|\mathbf{f}_T|^2 = |\mathbf{f}|^2 - |\mathbf{f}_N|^2 = n_1^{\,2}\sigma_{11}^{\,2} + n_2^{\,2}\sigma_{22}^{\,2} + n_3^{\,2}\sigma_{33}^{\,2} - (n_1^{\,2}\sigma_{11} + n_2^{\,2}\sigma_{22} + n_3^{\,2}\sigma_{33})^2$$

$$= \frac{1}{3}(\sigma_{11}^{\,2} + \sigma_{22}^{\,2} + \sigma_{33}^{\,2}) - \frac{1}{9}(\sigma_{11} + \sigma_{22} + \sigma_{33})^2$$

$$= \frac{1}{9}\left(2\sigma_{11}^{\,2} + 2\sigma_{22}^{\,2} + 2\sigma_{33}^{\,2} - 2\sigma_{11}\sigma_{22} - 2\sigma_{22}\sigma_{33} - 2\sigma_{33}\sigma_{11}\right)$$

$$= \frac{1}{9}\{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2\}$$

or

$$= \frac{2}{9}\{(\sigma_{11} + \sigma_{22} + \sigma_{33})^2 - 3(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11})\} = \frac{2}{9}\left(J_1^{\,2} - 3J_2\right)$$

$$= \frac{2}{9}\{(\tau_{11}+\tau_{22} + \tau_{33})^2 - 3(\tau_{11}\tau_{22} + \tau_{11}\tau_{33} + \tau_{22}\tau_{33} - \tau_{12}\tau_{21} - \tau_{13}\tau_{31} - \tau_{23}\tau_{32})\}$$

$$= \frac{1}{9}\{(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 + 6(\tau_{12}^{\,2} + \tau_{13}^{\,2} + \tau_{23}^{\,2})\}$$

$$J_1 = \tau_{11}+\tau_{22} + \tau_{33} = \lambda_1 + \lambda_2 + \lambda_3$$

$$J_2 = \begin{vmatrix}\tau_{22} & \tau_{23}\\ \tau_{32} & \tau_{33}\end{vmatrix} + \begin{vmatrix}\tau_{11} & \tau_{13}\\ \tau_{31} & \tau_{33}\end{vmatrix} + \begin{vmatrix}\tau_{11} & \tau_{12}\\ \tau_{21} & \tau_{22}\end{vmatrix} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$$

- Effective stress: tensile test$\left(\sigma_{11} = \sigma_{yield}, \ \ \sigma_{22} = \sigma_{33} = 0, \ |\tau_{oct}| < \frac{\sqrt{2}}{3}\sigma_{yield}\right)$.

$$\sigma_{eff} = \frac{3}{\sqrt{2}}|\tau_{oct}| = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2}$$

$$= \frac{1}{\sqrt{2}}\sqrt{(\tau_{11} - \tau_{22})^2 + (\tau_{22} - \tau_{33})^2 + (\tau_{33} - \tau_{11})^2 + 6(\tau_{12}^{\,2} + \tau_{13}^{\,2} + \tau_{23}^{\,2})} < \sigma_{yield}$$

■ Graph of octahedral shear stress.

$$|\tau_{oct}|^2 < \left(\frac{\sqrt{2}}{3}\sigma_{yield}\right)^2 = \frac{1}{9}\{(\sigma_{11}-\sigma_{22})^2+(\sigma_{22}-\sigma_{33})^2+(\sigma_{33}-\sigma_{11})^2\}$$

$\sigma_{33} = 0 \; -> \; {\sigma_{yield}}^2 = {\sigma_{11}}^2 + {\sigma_{22}}^2 - \sigma_{11}\sigma_{22}$

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, $B \neq 0$ 의 경우 xy 항을 제거하기 위한 좌표 계의 회전각 θ는 $\cot 2\theta = \frac{A-C}{B}$, $0 < 2\theta < 180°$

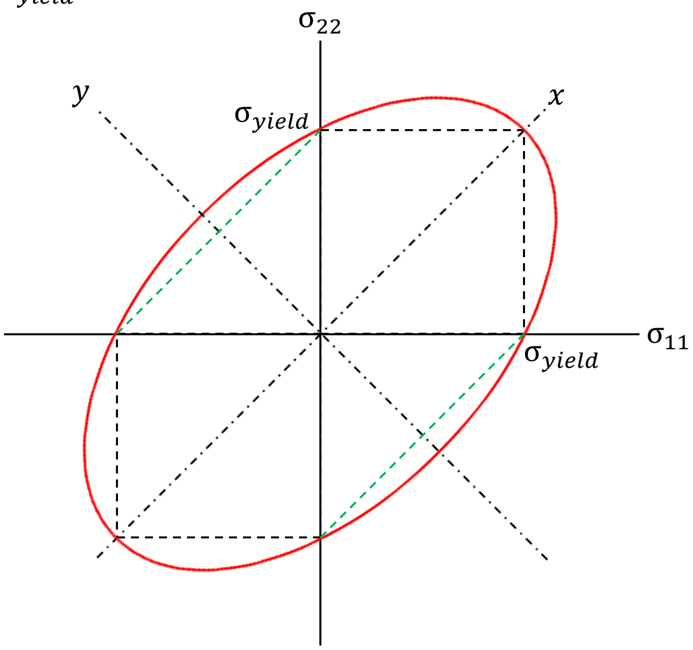
$${\sigma_{11}}^2 - \sigma_{11}\sigma_{22} + {\sigma_{22}}^2 = {\sigma_{yield}}^2$$

$$\cot 2\theta = \frac{1}{\tan 2\theta} = \frac{A-C}{B} = 0 \; -> \; \theta = 45°$$

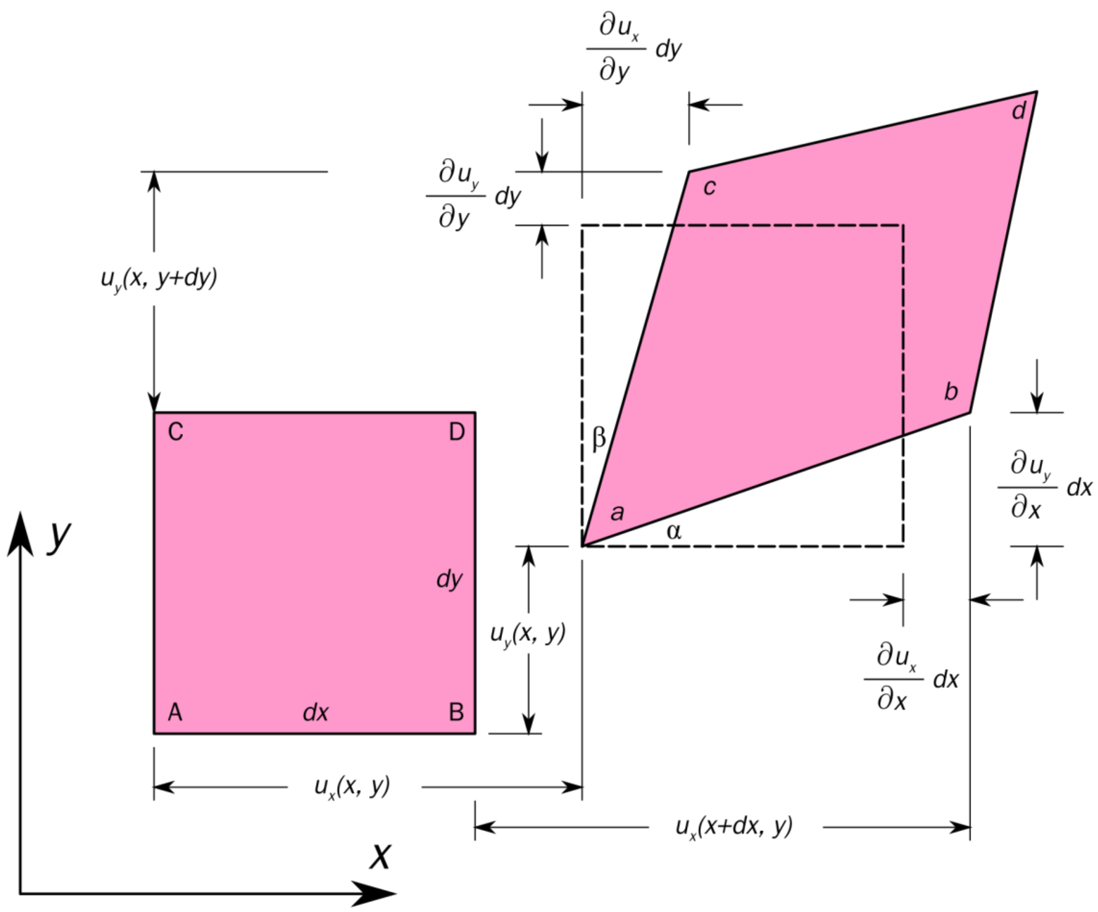
$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\end{bmatrix} = \begin{bmatrix}\cos(-\theta) & \sin(-\theta)\\ -\sin(-\theta) & \cos(-\theta)\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}\cos\theta\, x - \sin\theta\, y\\ \sin\theta\, x + \cos\theta\, y\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y\\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\end{bmatrix}$$

$$\left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y\right)^2 - \left(\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y\right)\left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right) + \left(\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y\right)^2 = {\sigma_{yield}}^2$$

$$\left(\frac{1}{\sqrt{2}}x\right)^2 + \left(\frac{\sqrt{3}}{\sqrt{2}}y\right)^2 = {\sigma_{yield}}^2$$



- Strain tensor.



- Displacement gradient tensor.

$$\nabla \mathbf{u} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} [u_x \quad u_y \quad u_z] = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} \end{bmatrix}$$

- Normal strain.

$$\varepsilon_x = \frac{\overline{ab} - \overline{AB}}{\overline{AB}} = \frac{\overline{ab} - dx}{dx}$$

$$\overline{ab} = \sqrt{\left(dx + \frac{\partial u_x}{\partial x} dx\right)^2 + \left(\frac{\partial u_y}{\partial x} dx\right)^2} \cong \left(1 + \frac{\partial u_x}{\partial x}\right) dx \quad \left[\left(\frac{\partial u_y}{\partial x} dx\right)^2 \to 0\right]$$

$$\varepsilon_x \cong \frac{\partial u_x}{\partial x} \quad (small\ strain < 0.01)$$

- Shear strain.

$$\gamma_{xy} = \alpha + \beta = \tan^{-1}\left(\frac{\frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}dx}{dx + \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}}dx}\right) + \tan^{-1}\left(\frac{\frac{\partial \mathrm{u_x}}{\partial \mathrm{y}}dy}{dy + \frac{\partial \mathrm{u_y}}{\partial \mathrm{y}}dy}\right)$$

$$\gamma_{xy} \cong \frac{\partial \mathrm{u_y}}{\partial \mathrm{x}} + \frac{\partial \mathrm{u_x}}{\partial \mathrm{y}} \quad (small\ \ normal\ and\ shear\ strain)$$

- Tensor decomposition (deformation and rigid body rotation).

$$\nabla\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}) + \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{T}}) \qquad (\nabla\mathbf{u})^{\mathrm{T}} = \begin{bmatrix} \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}} & \frac{\partial \mathrm{u_x}}{\partial \mathrm{y}} & \frac{\partial \mathrm{u_x}}{\partial \mathrm{z}} \\ \frac{\partial \mathrm{u_y}}{\partial \mathrm{x}} & \frac{\partial \mathrm{u_y}}{\partial \mathrm{y}} & \frac{\partial \mathrm{u_y}}{\partial \mathrm{z}} \\ \frac{\partial \mathrm{u_z}}{\partial \mathrm{x}} & \frac{\partial \mathrm{u_z}}{\partial \mathrm{y}} & \frac{\partial \mathrm{u_z}}{\partial \mathrm{z}} \end{bmatrix}$$

Symmetric tensor

$$e_{ij} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}) = \begin{bmatrix} \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}} & \frac{\frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}+\frac{\partial \mathrm{u_x}}{\partial \mathrm{y}}}{2} & \frac{\frac{\partial \mathrm{u_z}}{\partial \mathrm{x}}+\frac{\partial \mathrm{u_x}}{\partial \mathrm{z}}}{2} \\ \frac{\frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}+\frac{\partial \mathrm{u_x}}{\partial \mathrm{y}}}{2} & \frac{\partial \mathrm{u_y}}{\partial \mathrm{y}} & \frac{\frac{\partial \mathrm{u_z}}{\partial \mathrm{y}}+\frac{\partial \mathrm{u_y}}{\partial \mathrm{z}}}{2} \\ \frac{\frac{\partial \mathrm{u_z}}{\partial \mathrm{x}}+\frac{\partial \mathrm{u_x}}{\partial \mathrm{z}}}{2} & \frac{\frac{\partial \mathrm{u_z}}{\partial \mathrm{y}}+\frac{\partial \mathrm{u_y}}{\partial \mathrm{z}}}{2} & \frac{\partial \mathrm{u_z}}{\partial \mathrm{z}} \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{xy}}{2} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & \varepsilon_z \end{bmatrix}$$

- Large strain (ex: Green strain).

$$\mathbf{G} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}) + \frac{1}{2}\nabla\mathbf{u}(\nabla\mathbf{u})^{\mathrm{T}} = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{xy} & e_{yy} & e_{yz} \\ e_{xz} & e_{yz} & e_{zz} \end{bmatrix}$$

$$\nabla\mathbf{u}(\nabla\mathbf{u})^{\mathrm{T}} = \begin{bmatrix} \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}} & \frac{\partial \mathrm{u_y}}{\partial \mathrm{x}} & \frac{\partial \mathrm{u_z}}{\partial \mathrm{x}} \\ \frac{\partial \mathrm{u_x}}{\partial \mathrm{y}} & \frac{\partial \mathrm{u_y}}{\partial \mathrm{y}} & \frac{\partial \mathrm{u_z}}{\partial \mathrm{y}} \\ \frac{\partial \mathrm{u_x}}{\partial \mathrm{z}} & \frac{\partial \mathrm{u_y}}{\partial \mathrm{z}} & \frac{\partial \mathrm{u_z}}{\partial \mathrm{z}} \end{bmatrix} \begin{bmatrix} \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}} & \frac{\partial \mathrm{u_x}}{\partial \mathrm{y}} & \frac{\partial \mathrm{u_x}}{\partial \mathrm{z}} \\ \frac{\partial \mathrm{u_y}}{\partial \mathrm{x}} & \frac{\partial \mathrm{u_y}}{\partial \mathrm{y}} & \frac{\partial \mathrm{u_y}}{\partial \mathrm{z}} \\ \frac{\partial \mathrm{u_z}}{\partial \mathrm{x}} & \frac{\partial \mathrm{u_z}}{\partial \mathrm{y}} & \frac{\partial \mathrm{u_z}}{\partial \mathrm{z}} \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial \mathrm{u_x}}{\partial \mathrm{x}}\right)^2 + \left(\frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}\right)^2 + \left(\frac{\partial \mathrm{u_z}}{\partial \mathrm{x}}\right)^2 & \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_x}}{\partial \mathrm{y}} + \frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_y}}{\partial \mathrm{y}} + \frac{\partial \mathrm{u_z}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_z}}{\partial \mathrm{y}} & \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_x}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_y}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_z}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_z}}{\partial \mathrm{z}} \\ \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_x}}{\partial \mathrm{y}} + \frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_y}}{\partial \mathrm{y}} + \frac{\partial \mathrm{u_z}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_z}}{\partial \mathrm{y}} & \left(\frac{\partial \mathrm{u_x}}{\partial \mathrm{y}}\right)^2 + \left(\frac{\partial \mathrm{u_y}}{\partial \mathrm{y}}\right)^2 + \left(\frac{\partial \mathrm{u_z}}{\partial \mathrm{y}}\right)^2 & \frac{\partial \mathrm{u_x}}{\partial \mathrm{y}}\frac{\partial \mathrm{u_x}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_y}}{\partial \mathrm{y}}\frac{\partial \mathrm{u_y}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_z}}{\partial \mathrm{y}}\frac{\partial \mathrm{u_z}}{\partial \mathrm{z}} \\ \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_x}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_y}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_z}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_z}}{\partial \mathrm{z}} & \frac{\partial \mathrm{u_x}}{\partial \mathrm{y}}\frac{\partial \mathrm{u_x}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_y}}{\partial \mathrm{y}}\frac{\partial \mathrm{u_y}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_z}}{\partial \mathrm{y}}\frac{\partial \mathrm{u_z}}{\partial \mathrm{z}} & \left(\frac{\partial \mathrm{u_x}}{\partial \mathrm{z}}\right)^2 + \left(\frac{\partial \mathrm{u_y}}{\partial \mathrm{z}}\right)^2 + \left(\frac{\partial \mathrm{u_z}}{\partial \mathrm{z}}\right)^2 \end{bmatrix}$$

$$e_{xx} = \frac{\partial \mathrm{u_x}}{\partial \mathrm{x}} + \frac{1}{2}\left[\left(\frac{\partial \mathrm{u_x}}{\partial \mathrm{x}}\right)^2 + \left(\frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}\right)^2 + \left(\frac{\partial \mathrm{u_z}}{\partial \mathrm{x}}\right)^2\right], \quad e_{yy} = \frac{\partial \mathrm{u_y}}{\partial \mathrm{y}} + \frac{1}{2}\left[\left(\frac{\partial \mathrm{u_x}}{\partial \mathrm{y}}\right)^2 + \left(\frac{\partial \mathrm{u_y}}{\partial \mathrm{y}}\right)^2 + \left(\frac{\partial \mathrm{u_z}}{\partial \mathrm{y}}\right)^2\right], \quad e_{zz} = \frac{\partial \mathrm{u_z}}{\partial \mathrm{z}} + \frac{1}{2}\left[\left(\frac{\partial \mathrm{u_x}}{\partial \mathrm{z}}\right)^2 + \left(\frac{\partial \mathrm{u_y}}{\partial \mathrm{z}}\right)^2 + \left(\frac{\partial \mathrm{u_z}}{\partial \mathrm{z}}\right)^2\right]$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial \mathrm{u_y}}{\partial \mathrm{x}} + \frac{\partial \mathrm{u_x}}{\partial \mathrm{y}}\right) + \frac{1}{2}\left(\frac{\partial \mathrm{u_x}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_x}}{\partial \mathrm{y}} + \frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_y}}{\partial \mathrm{y}} + \frac{\partial \mathrm{u_z}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_z}}{\partial \mathrm{y}}\right), \quad e_{xz} = \frac{1}{2}\left(\frac{\partial \mathrm{u_z}}{\partial \mathrm{x}} + \frac{\partial \mathrm{u_x}}{\partial \mathrm{z}}\right) + \frac{1}{2}\left(\frac{\partial \mathrm{u_x}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_x}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_y}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_y}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_z}}{\partial \mathrm{x}}\frac{\partial \mathrm{u_z}}{\partial \mathrm{z}}\right)$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial \mathrm{u_z}}{\partial \mathrm{y}} + \frac{\partial \mathrm{u_y}}{\partial \mathrm{z}}\right) + \frac{1}{2}\left(\frac{\partial \mathrm{u_x}}{\partial \mathrm{y}}\frac{\partial \mathrm{u_x}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_y}}{\partial \mathrm{y}}\frac{\partial \mathrm{u_y}}{\partial \mathrm{z}} + \frac{\partial \mathrm{u_z}}{\partial \mathrm{y}}\frac{\partial \mathrm{u_z}}{\partial \mathrm{z}}\right)$$

Why?

Polar decomposition=> eigenvalue problem 의 연산 어려움.

- Strain transformation.

$$\sum_{l} \lambda'_{il} = \sum_{j} \beta_{ij} \sum_{k} \lambda_{jk} \sum_{l} \beta_{kl} \ , \ \boldsymbol{\lambda}' = \boldsymbol{\beta\lambda\beta}^T \text{(matrix notation)}$$

$il: \{x'_1 \quad x'_2 \quad x'_3\} system\ indices\ , \ jk: \{x_1 \quad x_2 \quad x_3\} system\ indices$

$$\begin{bmatrix} \lambda'_{11} & \lambda'_{12} & \lambda'_{13} \\ \lambda'_{21} & \lambda'_{22} & \lambda'_{23} \\ \lambda'_{31} & \lambda'_{32} & \lambda'_{33} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{bmatrix} \begin{bmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{21} & \beta_{31} \\ \beta_{12} & \beta_{22} & \beta_{32} \\ \beta_{13} & \beta_{23} & \beta_{33} \end{bmatrix}$$

예제)

$$\begin{bmatrix} e'_{11} & e'_{12} \\ e'_{21} & e'_{22} \end{bmatrix} = \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{bmatrix} \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} e_{11}cos\theta + e_{12}sin\theta & -e_{11}sin\theta + e_{12}cos\theta \\ e_{21}cos\theta + e_{22}sin\theta & -e_{21}sin\theta + e_{22}cos\theta \end{bmatrix}$$

$$e'_{11} = e_{11}cos^2\theta + e_{22}sin^2\theta + 2e_{12}sin\theta cos\theta$$

$$e'_{22} = e_{11}sin^2\theta + e_{22}cos^2\theta - 2e_{12}sin\theta cos\theta$$

$$e'_{12} = e'_{21} = e_{12}(cos^2\theta - sin^2\theta) - e_{11}sin\theta cos\theta + e_{22}sin\theta cos\theta$$

- Principal strain.

$\det(\boldsymbol{e} - \lambda\mathbf{I}) = \mathbf{0}$ => Characteristic equation.

■ Generalized Hooke's law for linear isotropic elastic solids.

$$e_{11} = \frac{1}{E}(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33}))$$

$$e_{22} = \frac{1}{E}(\sigma_{22} - \nu(\sigma_{11} + \sigma_{33}))$$

$$e_{33} = \frac{1}{E}(\sigma_{33} - \nu(\sigma_{11} + \sigma_{22}))$$

$$\left(E: Young's\ modulus, \nu: Poisson's\ ratio = \frac{e_{lateral}}{e_{longitudinal}}\right)$$

$$e_{12} = \frac{1+\nu}{E}\tau_{12} = \frac{1}{2\mu}\tau_{12} = \frac{1}{2G}\tau_{12} = \frac{\gamma_{12}}{2}$$

$$e_{13} = \frac{1+\nu}{E}\tau_{13}$$

$$e_{23} = \frac{1+\nu}{E}\tau_{23}$$

$2D\ principal\ stress\ \sigma_{11} = -\sigma_{22}\ , \sigma_{33} = 0$

$$e_{11} = \frac{1}{E}\big(\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\big)\ \rightarrow\ e_{11} = \frac{1+\nu}{E}\sigma_{11}$$

$Maximum\ shear\ stress\ \tau_{12} = \sigma_{11}\ ,\ \ Maximum\ shear\ strain\ e_{12} = e_{11}$

$$\therefore\ e_{12} = \frac{1+\nu}{E}\tau_{12} = \frac{1}{2G}\tau_{12} \rightarrow G = \mu = \frac{E}{2(1+\nu)}$$

- Volumetric strain, bulk modulus, hydrostatic stress.

$$e(\text{volumetric strain}) = (1+e_{11})(1+e_{22})(1+e_{33}) - 1$$

$$= e_{11} + e_{22} + e_{33} + e_{11}e_{22} + e_{11}e_{33} + e_{22}e_{33} + e_{11}e_{22}e_{33} \cong e_{11} + e_{22} + e_{33}$$

$$= \frac{1}{E}\big(\sigma_{11} + \sigma_{22} + \sigma_{33} - 2\nu(\sigma_{11} + \sigma_{22} + \sigma_{33})\big) = \frac{1-2\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

$$k(\text{bulk modulus}) = \frac{\sigma_{hyd}}{e} = \frac{\left(\frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3}\right)}{e} = \frac{E}{3(1-2\nu)}$$

- Lamé’s constant (λ).

$$\sigma_{11} = \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{11}$$

$$= \frac{\lambda(1-2\nu)}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \frac{2\mu}{E}\big(\sigma_{11} + \nu\sigma_{11} - \nu(\sigma_{11} + \sigma_{22} + \sigma_{33})\big)$$

$$= \frac{2\mu(1+\nu)}{E}\sigma_{11} + \frac{\lambda(1-2\nu) - 2\mu\nu}{E}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

$$\frac{2\mu(1+\nu)}{E} = 1 \rightarrow\ \lambda(1-2\nu) - 2\mu\nu = 0 \rightarrow\ \upsilon = \frac{\lambda}{2(\lambda + \mu)}$$

$$\upsilon = \frac{E}{2\mu} - 1 \rightarrow\ \mathrm{E} = \frac{\mu(3\lambda + 2\mu)}{(\lambda + \mu)}$$

■ Equilibrium equation.

운동량 보존: $\int_S \mathbf{n}\cdot\mathbf{T}dS + \int_V \rho\mathbf{g}dV = 0 \;\rightarrow\; \nabla\cdot\mathbf{T} + \mathbf{F} = 0$ (divergence theorem)

- Stress formulation.

$$\nabla\cdot\mathbf{T} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\tau_{21}}{\partial y} + \frac{\partial\tau_{31}}{\partial z} \\ \frac{\partial\tau_{12}}{\partial x} + \frac{\partial\sigma_{22}}{\partial y} + \frac{\partial\tau_{32}}{\partial z} \\ \frac{\partial\tau_{13}}{\partial x} + \frac{\partial\tau_{23}}{\partial y} + \frac{\partial\sigma_{33}}{\partial z} \end{bmatrix}^T$$

$$\frac{\partial\sigma_{11}}{\partial x} + \frac{\partial\tau_{21}}{\partial y} + \frac{\partial\tau_{31}}{\partial z} + F_x = 0$$

$$\frac{\partial\tau_{12}}{\partial x} + \frac{\partial\sigma_{22}}{\partial y} + \frac{\partial\tau_{32}}{\partial z} + F_y = 0$$

$$\frac{\partial\tau_{13}}{\partial x} + \frac{\partial\tau_{23}}{\partial y} + \frac{\partial\sigma_{33}}{\partial z} + F_z = 0$$

- Displacement formulation.

$(\lambda + \mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} + \mathbf{F} = 0$

$$e_{ij} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}}{2} & \frac{\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}}{2} \\ \frac{\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}}{2} & \frac{\partial u_y}{\partial y} & \frac{\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}}{2} \\ \frac{\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}}{2} & \frac{\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}}{2} & \frac{\partial u_z}{\partial z} \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{xy}}{2} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{xz}}{2} & \frac{\gamma_{yz}}{2} & \varepsilon_z \end{bmatrix}$$

$\mathbf{T} = \lambda(\nabla\cdot\mathbf{u})\mathbf{I} + \mu[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$

$$= \begin{bmatrix} \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_x}{\partial x} & \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \\ \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) & \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_y}{\partial y} & \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) \\ \mu\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) & \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) & \lambda\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) + 2\mu\frac{\partial u_z}{\partial z} \end{bmatrix}$$

$$\nabla\cdot\mathbf{T} = (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u}$$

$$\nabla\cdot(\nabla\cdot\mathbf{u})\mathbf{I} = \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right) & 0 & 0 \\ 0 & \left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right) & 0 \\ 0 & 0 & \left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right)\end{bmatrix}$$

$$= \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right) = \nabla(\nabla\cdot\mathbf{u})$$

$$\nabla\cdot(\nabla\mathbf{u})^{T} = \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z}\end{bmatrix} = \begin{bmatrix}\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial x\,\partial y}+\frac{\partial^2 u_z}{\partial x\,\partial z} \\ \frac{\partial^2 u_x}{\partial x\,\partial y}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_z}{\partial y\,\partial z} \\ \frac{\partial^2 u_x}{\partial x\,\partial z}+\frac{\partial^2 u_y}{\partial y\,\partial z}+\frac{\partial^2 u_z}{\partial z^2}\end{bmatrix}^{T}$$

$$= \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right) = \nabla(\nabla\cdot\mathbf{u})$$

$$\nabla\cdot\nabla\mathbf{u} = \begin{bmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\end{bmatrix}\begin{bmatrix}\frac{\partial u_x}{\partial x} & \frac{\partial u_y}{\partial x} & \frac{\partial u_z}{\partial x} \\ \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_x}{\partial z} & \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z}\end{bmatrix} = \begin{bmatrix}\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2} \\ \frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2} \\ \frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\end{bmatrix}^{T} = \nabla^2\mathbf{u}$$

$$(\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right) + \mu\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right) + F_x = 0$$

$$(\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right) + \mu\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right) + F_y = 0$$

$$(\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right) + \mu\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right) + F_z = 0$$