■ first order linear differential equation.

- **separable eq.**

$\frac{dy}{dx} = \frac{g(x)}{h(y)}$ 의 형태를 가지는 경우 적용

$$\int h(y)dy = \int g(x)dx$$

$$\frac{dy}{dx} = \frac{x^2}{y^2} \rightarrow \int y^2 dy = \int x^2 dx \rightarrow \frac{1}{3}y^3 = \frac{1}{3}x^3 + C$$

■ non-homogeneous second order differential equation.

$$ay'' + by' + cy = G(x)$$

- **complementary eq.**

$$ay'' + by' + cy = 0$$

- **particular solution.**

$$y(x) = y_p(x) + y_c(x)$$

- **undetermined coefficients.**

$G(x)$ => polynomial, exponential, sin, cos.

$G(x) = x^2 \rightarrow \; y_p(x) = Ax^2 + Bx + C$

- **variation of parameters.**

$G(x)$ => tan(kx)

- **series solution: ex)** $y'' - 2xy' + y = 0$

■ **Taylor series.**

$$f(x+h) = C_0 + C_1h + C_2h^2 + C_3h^3 + \cdots + C_nh^n \quad (n \to \infty)$$

$$f'(x+h) = C_1 + 2C_2h + 3C_3h^2 + \cdots + nC_nh^{n-1}$$

$$f''(x+h) = 2C_2 + 3 \cdot 2C_3h + \cdots + n(n-1)C_nh^{n-2}$$

$$C_0 = f(x)$$

$$C_1 = f'(x)$$

$$C_2 = \frac{f''(x)}{2}$$

$$\therefore\ f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2}h^2 + \cdots + \frac{f^n(x)}{n!}h^n \qquad ①$$

$$h = -h \to f(x-h) = f(x) - f'(x)h + \frac{f''(x)}{2}h^2 - \cdots + \frac{f^n(x)}{n!}(-h)^n \qquad ②$$

① - ②

$$f(x+h) - f(x-h) = 2f'(x)h + R \to\ f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$$

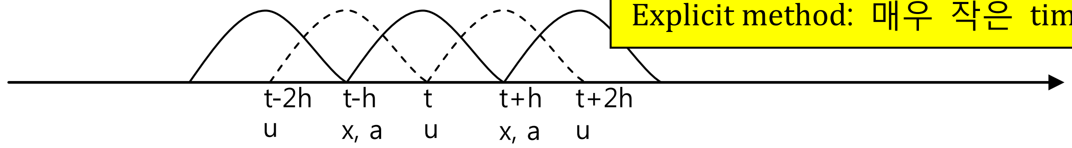
① + ②

$$f(x+h) + f(x-h) = 2f(x) + f''(x)h^2 + R \to\ f''(x) \approx \frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

ex) $F = m\ddot{x} + c\dot{x} + kx = ma + cu + kx \to\ x(t), u(t), a(t)$

$$u(t) \approx \frac{x(t+h)-x(t-h)}{2h}\ ,\quad a(t-h) \approx \frac{u(t)-u(t-2h)}{2h}$$

Explicit method: 매우 작은 time step



■ **Maclaurin series.**

$$f(x) = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_n x^n \quad (n \to \infty)$$

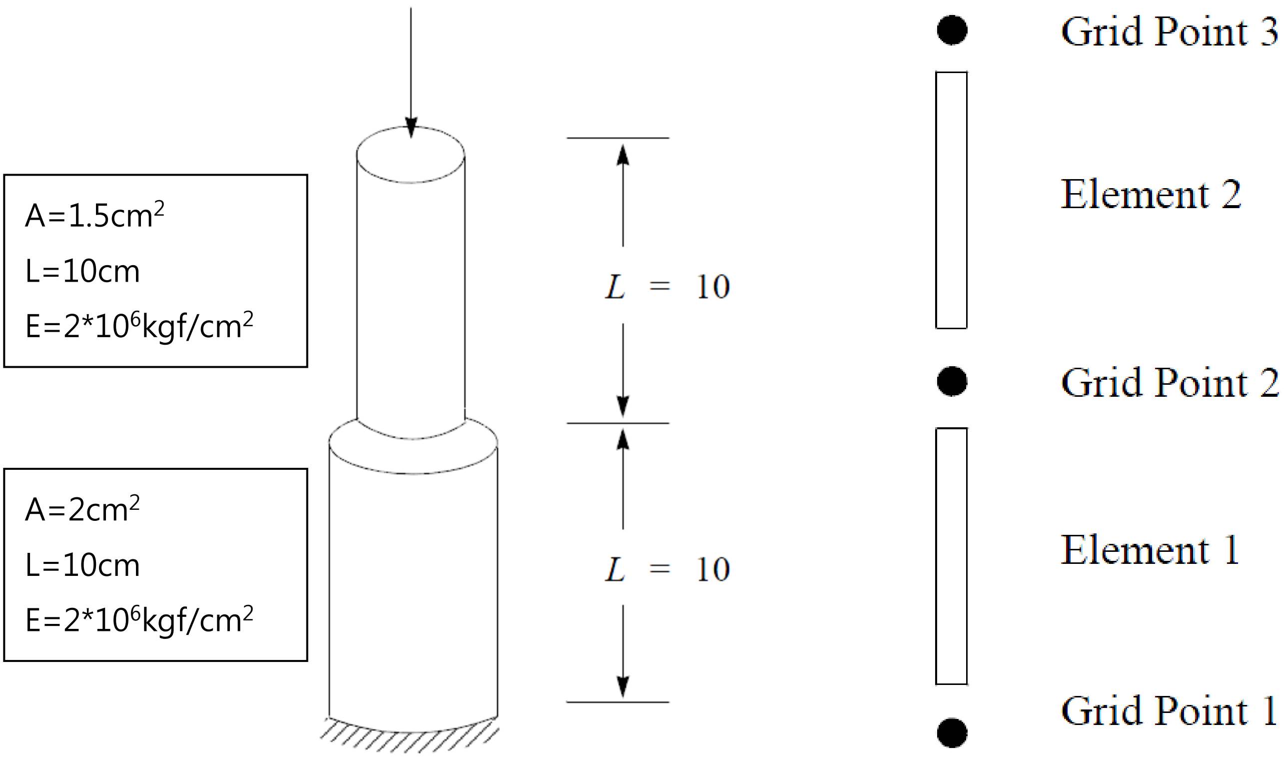
$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^n(0)}{n!}x^n$$

$$e^x = 1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n$$

$$\sin x = x - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{(2n+1)}$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \cdots + \frac{(-1)^n}{(2n)!}x^{(2n)}$$

■ **선형 정적(linear static)해석 문제의 이해.**



- **stiffness matrix(참고: Laplacian matrix).**

$$K = \begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix}$$

$$K_1 = \begin{array}{cc} \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} \frac{4*10^6}{10} & -\frac{4*10^6}{10} \\ -\frac{4*10^6}{10} & \frac{4*10^6}{10} \end{bmatrix} \\ & \begin{matrix} 1 & \qquad\qquad 2 \end{matrix} \end{array} \qquad K_2 = \begin{array}{cc} \begin{matrix} 2 \\ 3 \end{matrix} & \begin{bmatrix} \frac{3*10^6}{10} & -\frac{3*10^6}{10} \\ -\frac{3*10^6}{10} & \frac{3*10^6}{10} \end{bmatrix} \\ & \begin{matrix} 2 & \qquad\qquad 3 \end{matrix} \end{array}$$

grid 1=>고정구속

$$K_{global} = \begin{array}{cc} \begin{bmatrix} \frac{4*10^6}{10} & -\frac{4*10^6}{10} & 0 \\ -\frac{4*10^6}{10} & \frac{4*10^6}{10}+\frac{3*10^6}{10} & -\frac{3*10^6}{10} \\ 0 & -\frac{3*10^6}{10} & \frac{3*10^6}{10} \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \\ \begin{matrix} 1 & \qquad\qquad 2 & \qquad\qquad 3 \end{matrix} & \end{array}$$

$$K_{g\_구속} = \begin{array}{cc} \begin{bmatrix} \frac{4*10^6}{10}+\frac{3*10^6}{10} & -\frac{3*10^6}{10} \\ -\frac{3*10^6}{10} & \frac{3*10^6}{10} \end{bmatrix} & \begin{matrix} 2 \\ 3 \end{matrix} \\ \begin{matrix} 2 & \qquad\qquad 3 \end{matrix} & \end{array}$$

$$[K]\{u\} = \{f\}$$

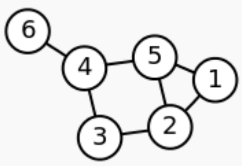
$$\begin{bmatrix} 7*10^5 & -3*10^5 \\ -3*10^5 & 3*10^5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_2 \\ f_3 \end{Bmatrix}$$ if) $f_2 = 0$, $f_3 = -1000$ 이면

$$\begin{bmatrix} 7*10^5 & -3*10^5 \\ -3*10^5 & 3*10^5 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \end{Bmatrix}$$

$$7u_2 - 3u_3 = 0 \ , \quad -3u_2 + 3u_3 = -0.01$$

$$u_2 = -0.0025 \quad u_3 = -0.00583333\cdots$$

■ Laplacian matrix.

Labeled graph	Degree matrix	Adjacency matrix	Laplacian matrix
	$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$

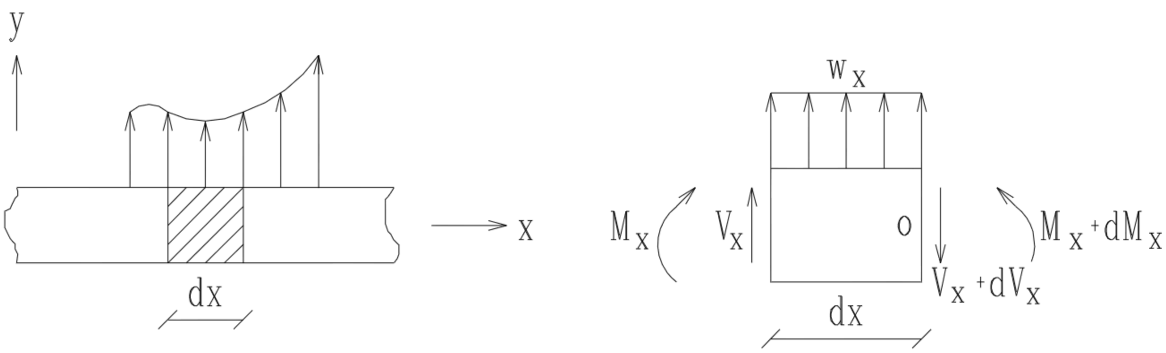
$$\sigma = E\varepsilon$$

$$\frac{F}{A} = E\frac{u}{L}$$

$$F = \frac{AE}{L}u$$

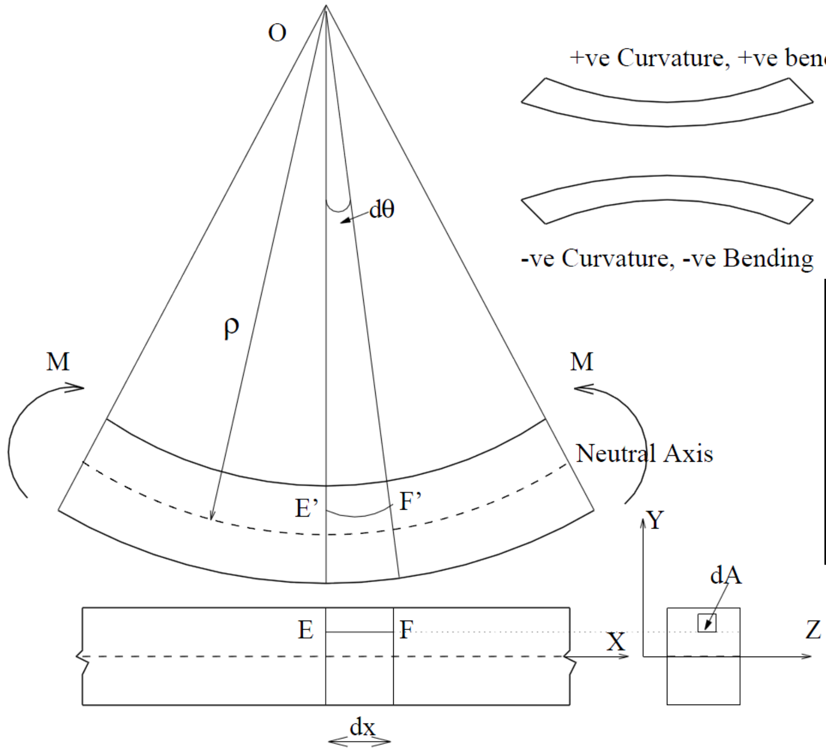
$$\frac{AE}{L}\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + (-1)\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\} = \begin{bmatrix}\frac{AE}{L} & -\frac{AE}{L}\\ -\frac{AE}{L} & \frac{AE}{L}\end{bmatrix}$$

■ 보(beam theory)

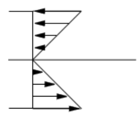


$$\sum F_y = V_x + w_x dx - (V_x + dV_x) = 0 \rightarrow w(x) = \frac{dV}{dx}$$

$$\sum M_z = M_x + V_x dx \quad w_x dx \frac{dx}{2} - (M_x + dM_x) = 0 \rightarrow V(x) = \frac{dM}{dx}$$



***** Beam 변형 모델:**
Galileo(1564-1642)
Jacob Bernoulli(1654-1705)
Leonhard Euler(1707-1783)
Navier(1785-1836)



$$\varepsilon_x = -\frac{EF - E'F'}{EF} = -\frac{dx - (\rho - y)d\theta}{dx} = -\frac{y}{\rho}\ , \ dx = \rho d\theta$$

$$\kappa = \frac{1}{\rho}\ , \quad \varepsilon_x = -\kappa y\ , \quad \sigma_x = E\varepsilon_x = -E\kappa y$$

기하학적 구속조건

$$\sum F_x = \int -E\kappa y dA = 0\ , \ \int y dA = 0 \rightarrow \text{neutral axis}$$

$$\sum M_z = 0 \rightarrow M = -\int \sigma_x y dA = \int E\kappa y^2 dA\ , \ \int y^2 dA = I$$

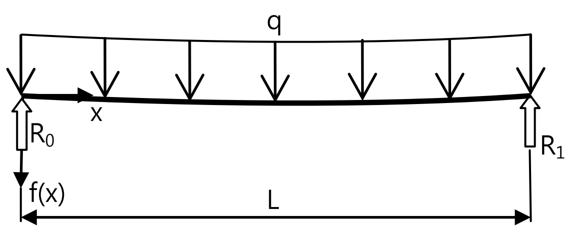
$$\frac{M}{EI} = \kappa = \frac{1}{\rho}\ , \ \sigma_x = -E\kappa y = -\frac{My}{I}$$

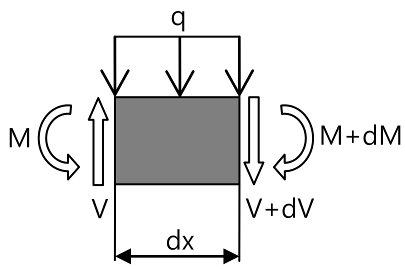
$$\kappa(x) = \frac{|f''(x)|}{[1+\{f'(x)\}^2]^{3/2}}$$

beam theory=> 전단변형 고려 안 함.
단면형상 유지 가정.
작은 변형 가정.

$$f''(x) \approx \frac{M}{EI}$$

■ 등분포하중을 받는 양단 지지보





$\sum F_y = 0 \rightarrow R_0 + R_1 = qL$,

internal equilibrium $-q = \frac{dV}{dx}$

$\sum M_z = 0 \rightarrow 0 = \frac{qL^2}{2} - R_1 L$,

$V(x) = -\frac{dM}{dx} = -qx + \frac{qL}{2}$

$$M(x) = \frac{qx^2}{2} - \frac{qLx}{2} + C1 \rightarrow C1 = M_0 = 0$$

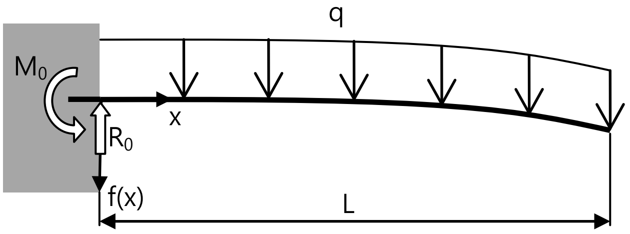
$$\frac{M}{EI} = \kappa = f''(x) \rightarrow f''(x) = \frac{1}{EI}\left(\frac{qx^2}{2} - \frac{qLx}{2}\right)$$

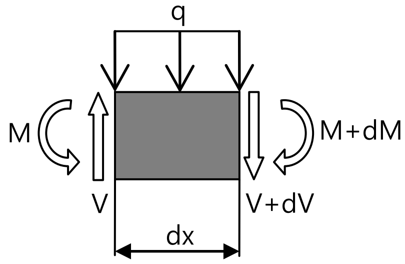
$$f'(x) = \frac{1}{EI}\left(\frac{qx^3}{6} - \frac{qLx^2}{4} + C2\right) \rightarrow f'(\frac{L}{2}) = 0 \rightarrow C2 = \frac{qL^3}{24}$$

$$f(x) = \frac{1}{EI}\left(\frac{qx^4}{24} - \frac{qLx^3}{12} + \frac{qL^3x}{24} + C3\right) \rightarrow f(0) = 0 \rightarrow C3 = 0$$

$$f(x) = \frac{1}{EI}\left(\frac{qx^4}{24} - \frac{qLx^3}{12} + \frac{qL^3x}{24}\right)$$

■ 등분포하중을 받는 일단 고정보





$\sum F_y = 0 \rightarrow R_0 = qL$, internal equilibrium $-q = \frac{dV}{dx}$

$\sum M_z = 0 \rightarrow M_0 = \frac{qL^2}{2}$, internal equilibrium $V(x) = -\frac{dM}{dx} = -qx + qL$

$$M(x) = \frac{q(x-L)^2}{2} + C1 \rightarrow M(0) = \frac{qL^2}{2} \rightarrow C1 = 0$$

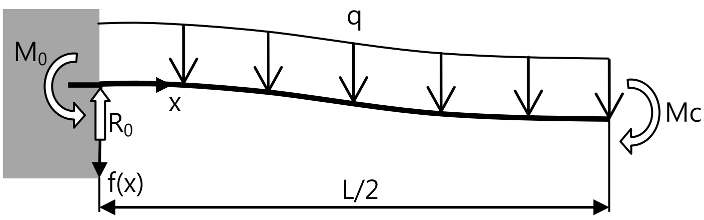
$$\frac{M}{EI} = \kappa = f''(x) \rightarrow f''(x) = \frac{1}{EI}\frac{q(x-L)^2}{2}$$

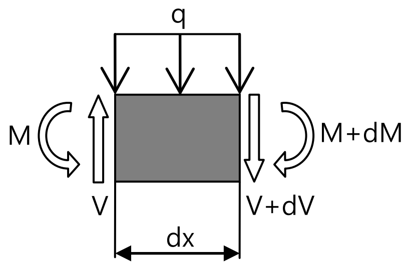
$$f'(x) = \frac{1}{EI}\left(\frac{q(x-L)^3}{6} + C2\right) \rightarrow f'(0) = 0 \rightarrow C2 = \frac{qL^3}{6}$$

$$f(x) = \frac{1}{EI}\left(\frac{q(x-L)^4}{24} + \frac{qL^3x}{6} + C3\right) \rightarrow f(0) = 0 \rightarrow C3 = -\frac{qL^4}{24}$$

$$f(x) = \frac{1}{EI}\left(\frac{q(x-L)^4}{24} + \frac{qL^3x}{6} - \frac{qL^4}{24}\right)$$

■ 등분포하중을 받는 양단 고정보





$\sum F_y = 0 \rightarrow R_0 = \frac{qL}{2},$

internal equilibrium $-q = \frac{dV}{dx}$

$\sum M_z = 0 \rightarrow M_0 = \frac{qL^2}{8} + M_c,$

$V(x) = -\frac{dM}{dx} = -qx + \frac{qL}{2}$

$$M(x) = \frac{qx^2 - qLx}{2} + C1 \rightarrow C1 = M_0$$

$$\frac{M}{EI} = \kappa = f''(x) \rightarrow f''(x) = \frac{1}{EI}\left(\frac{qx^2 - qLx}{2} + M_0\right)$$

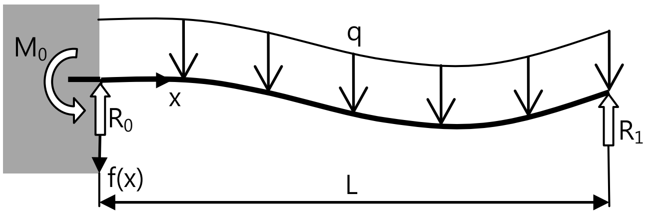
$$f'(x) = \frac{1}{EI}\left(\frac{qx^3}{6} - \frac{qLx^2}{4} + M_0 x + C2\right) \rightarrow f'(0) = 0 \rightarrow C2 = 0$$

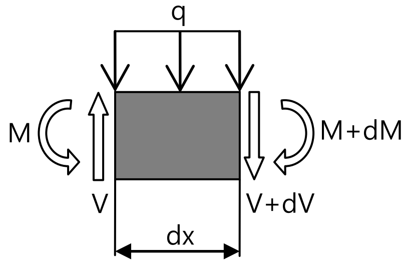
$$f'(L) = 0 \rightarrow \frac{qL^3}{6} - \frac{qL^3}{4} + M_0 L = 0 \rightarrow \ M_0 = \frac{qL^2}{12}\ ,\ M_c = -\frac{qL^2}{24}$$

$$f(x) = \frac{1}{EI}\left(\frac{qx^4}{24} - \frac{qLx^3}{12} + \frac{qL^2x^2}{24} + C3\right) \rightarrow f(0) = 0 \rightarrow C3 = 0$$

$$f(x) = \frac{1}{EI}\left(\frac{qx^4}{24} - \frac{qLx^3}{12} + \frac{qL^2x^2}{24}\right)$$

■ 등분포하중을 받는 일단 고정 일단 단순 지지보





$\sum F_y = 0 \rightarrow R_0 + R_1 = qL$, internal equilibrium $-q = \frac{dV}{dx}$

$\sum M_z = 0 \rightarrow M_0 = \frac{qL^2}{2} - R_1 L$, $V(x) = -\frac{dM}{dx} = -qx + R_0$

$$M(x) = \frac{qx^2}{2} - R_0 x + C1 \rightarrow C1 = M_0 = \frac{qL^2}{2} - R_1 L = -\frac{qL^2}{2} + R_0 L$$

$$\frac{M}{EI} = \kappa = f''(x) \rightarrow f''(x) = \frac{1}{EI}\left(\frac{qx^2}{2} - R_0 x + R_0 L - \frac{qL^2}{2}\right)$$

$$f'(x) = \frac{1}{EI}\left(\frac{qx^3}{6} - \frac{R_0 x^2}{2} + R_0 Lx - \frac{qL^2 x}{2} + C2\right) \rightarrow f'(0) = 0 \rightarrow C2 = 0$$

$$f(x) = \frac{1}{EI}\left(\frac{qx^4}{24} - \frac{R_0 x^3}{6} + \frac{R_0 L x^2}{2} - \frac{qL^2 x^2}{4} + C3\right) \rightarrow f(0) = 0 \rightarrow C3 = 0$$

$$f(L) = 0 \rightarrow \frac{qL^4}{24} - \frac{R_0 L^3}{6} + \frac{R_0 L^3}{2} - \frac{qL^4}{4} = 0 \rightarrow R_0 = \frac{5qL}{8}, R_1 = \frac{3qL}{8}$$

$$M_0 = \frac{qL^2}{8}$$

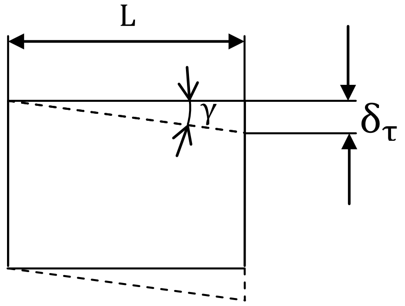
$$f(x) = \frac{1}{EI}\left(\frac{qx^4}{24} - \frac{5qLx^3}{48} + \frac{qL^2 x^2}{16}\right)$$

Beam formula

■ 전단변형

$$\tau = G\gamma\,,\quad G = \frac{E}{2(1+\nu)}$$

$$\delta_\tau = \gamma L$$



■ 등분포하중을 받는 일단 고정보의 전단변형

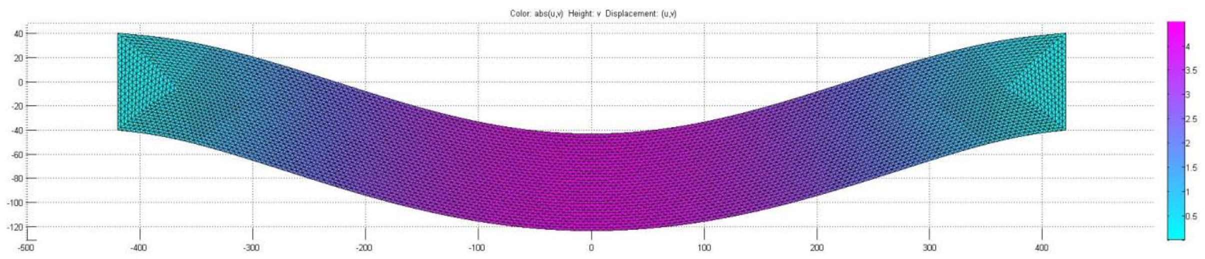
$$\tau = G\gamma = \frac{V(x)}{A} \rightarrow \ \gamma = \frac{V(x)}{GA} \rightarrow \delta_\tau = \int_0^L \gamma dx = \int_0^L \frac{V(x)}{GA} dx$$

$$V(x) = -qx + qL$$

$$\delta_\tau = \int_0^L \frac{-qx+qL}{GA} dx = \frac{1}{GA}\left(\frac{-qx^2}{2} + qLx\right)\Big]_0^L = \frac{qL^2}{2GA}$$

$$\tau_{avg} = \frac{qL}{2A} \rightarrow \ \gamma_{avg} = \frac{qL}{2GA} \rightarrow \ \delta_\tau = \gamma_{avg}L = \frac{qL^2}{2GA}$$

■ MATLAB PDETOOL



■ 유한요소법 개념 이해.

- **Stress - strain relation (constitutive equation).**

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

- **Polynomial equation.**

$$\mathrm{w} = \mathbf{Nu}$$

- **Kinematic equation.**

$$\boldsymbol{\varepsilon} = \mathbf{uB}$$

- **Stiffness matrix (equivalent nodal forces).**

$$\mathbf{F} = \mathbf{Ku}$$

$$\mathbf{K} = \int_{\mathrm{V}} \mathbf{B}^{\mathrm{T}}\,\mathrm{D}\mathbf{B}\mathrm{dv}$$

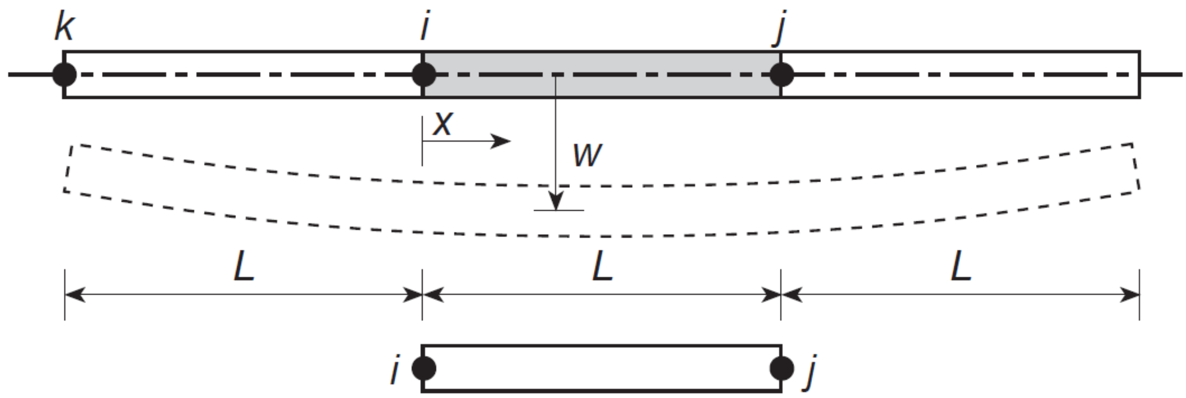
- **Principle of virtual work.**

$$\boldsymbol{\delta}\mathbf{uF} = \int_{\boldsymbol{V}} \boldsymbol{\delta\varepsilon}\,\boldsymbol{\sigma}\mathrm{dv} = \boldsymbol{\delta}\mathbf{u}\int_{\boldsymbol{V}} \mathbf{B}\,\boldsymbol{\sigma}\mathrm{dv}$$

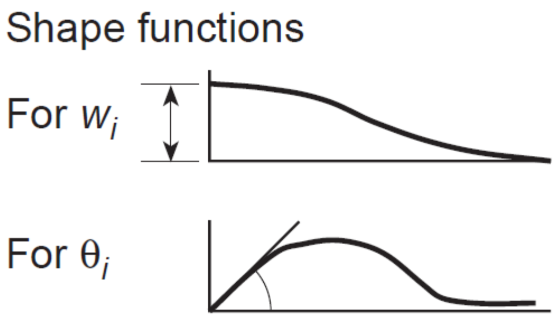
$$\mathbf{F} = \int_{\boldsymbol{V}} \mathbf{B}\,\boldsymbol{\sigma}\mathrm{dv} = \int_{\boldsymbol{V}} \mathbf{B}\,\mathrm{D}\boldsymbol{\varepsilon}\mathrm{dv} = \left[\int_{\mathrm{V}} \mathbf{B}^{\mathrm{T}}\,\mathrm{D}\mathbf{B}\mathrm{dv}\right]\mathbf{u}$$

$$\therefore \mathbf{K} = \int_{\mathrm{V}} \mathbf{B}^{\mathrm{T}}\,\mathrm{D}\mathbf{B}\mathrm{dv}$$

■ **Beam theory의 간단한 유한요소법 적용.**



$$\mathbf{d}_i = \begin{Bmatrix} w_i \\ w_{xi} \end{Bmatrix} \equiv \begin{pmatrix} w_i \\ \theta_i \end{pmatrix} \qquad \mathbf{d}_i = \begin{Bmatrix} w_j \\ w_{xj} \end{Bmatrix} \equiv \begin{pmatrix} w_j \\ \theta_j \end{pmatrix}$$



$$\boxed{\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}}$$

1차원 2-자유도(w, θ) 문제

$$\frac{M}{EI} = \kappa = -\frac{d^2w}{dx^2}\ , \quad \sigma \equiv M\,,\ \varepsilon \equiv \kappa\,,\ D \equiv EI\ , \quad \theta = \frac{dw}{dx}$$

$$w(x) = \alpha_1 + \alpha_2\left(\frac{x}{L}\right) + \alpha_3\left(\frac{x}{L}\right)^2 + \alpha_4\left(\frac{x}{L}\right)^3 = \alpha_1 + \alpha_2 s + \alpha_3 s^2 + \alpha_4 s^3$$

상기 수식의 계수를 절점에서의 변위 (w_i, θ_i, w_j, θ_j)로 표시하면,

$$w = w_i\left(1 - 3s^2 + 2s^3\right) + \theta_i\{L\left(s - 2s^2 + s^3\right)\} + w_j\left(3s^2 - 2s^3\right) + \theta_j\{L\left(-s^2 + s^3\right)\}$$

절점에 대한 형상함수(절점에서의 변위가 변형함수에 미치는 영향을 표시)

$\mathbf{N}^e_{ij} = [(1 - 3s^2 + 2s^3),\ L(s - 2s^2 + s^3),\ (3s^2 - 2s^3),\ L(-s^2 + s^3)]$ 로 정의하고

변위 $\mathbf{u}^e_{ij} = [w_i,\ \theta_i,\ w_j,\ \theta_j]$ 으로 정의하면

$w = \mathbf{N}^e_{ij}\mathbf{u}^e_{ij}$ (**Polynomial equation**) 으로 표시할 수 있다.

$$\mathrm{w(x)} = \alpha_1 + \alpha_2\left(\frac{\mathrm{x}}{\mathrm{L}}\right) + \alpha_3\left(\frac{\mathrm{x}}{\mathrm{L}}\right)^2 + \alpha_4\left(\frac{\mathrm{x}}{\mathrm{L}}\right)^3 = \alpha_1 + \alpha_2 \mathrm{s} + \alpha_3 \mathrm{s}^2 + \alpha_4 \mathrm{s}^3$$

$$\mathrm{w}'(x) = \frac{\alpha_2}{L} + 2\frac{\alpha_3}{L}\left(\frac{\mathrm{x}}{\mathrm{L}}\right) + 3\frac{\alpha_4}{L}\left(\frac{\mathrm{x}}{\mathrm{L}}\right)^2 = \frac{\alpha_2}{L} + 2\frac{\alpha_3}{L}s + 3\frac{\alpha_4}{L}s^2$$

$$\mathrm{w}(0) = \alpha_1$$

$$\mathrm{w}'(0) = \frac{\alpha_2}{L}$$

$$\mathrm{w}(1) = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \mathrm{w}(0) + L\mathrm{w}'(0) + \alpha_3 + \alpha_4$$

$$\mathrm{w}'(1) = \frac{\alpha_2}{L} + 2\frac{\alpha_3}{L} + 3\frac{\alpha_4}{L} = \mathrm{w}'(0) + 2\frac{\alpha_3}{L} + 3\frac{\alpha_4}{L}$$

$$\mathrm{w}(1) - \mathrm{w}(0) - L\mathrm{w}'(0) = \alpha_3 + \alpha_4$$

$$L\{\mathrm{w}'(1) - \mathrm{w}'(0)\} = 2\alpha_3 + 3\alpha_4$$

$$\alpha_4 = L\{\mathrm{w}'(1) - \mathrm{w}'(0)\} - 2\{\mathrm{w}(1) - \mathrm{w}(0) - L\mathrm{w}'(0)\} = 2\mathrm{w}(0) - 2\mathrm{w}(1) + L\mathrm{w}'(1) + L\mathrm{w}'(0)$$

$$\mathrm{w}(1) - \mathrm{w}(0) - L\mathrm{w}'(0) = \alpha_3 + 2\mathrm{w}(0) - 2\mathrm{w}(1) + L\mathrm{w}'(1) + L\mathrm{w}'(0)$$

$$\alpha_3 = 3\mathrm{w}(1) - 3\mathrm{w}(0) - 2L\mathrm{w}'(0) - L\mathrm{w}'(1)$$

$$\mathrm{w(x)} = \mathrm{w}(0) + \mathrm{L}\mathrm{w}'(0)\mathrm{s} + \{3\mathrm{w}(1) - 3\mathrm{w}(0) - 2L\mathrm{w}'(0) - L\mathrm{w}'(1)\}\mathrm{s}^2$$
$$+\{2\mathrm{w}(0) - 2\mathrm{w}(1) + L\mathrm{w}'(1) + L\mathrm{w}'(0)\}\mathrm{s}^3$$

$$\mathrm{w(x)} = \mathrm{w}(0)(1 - 3\mathrm{s}^2 + 2\mathrm{s}^3) + \mathrm{w}'(0)\{\mathrm{L}(\mathrm{s} - 2\mathrm{s}^2 + \mathrm{s}^3)\} + \mathrm{w}(1)(3\mathrm{s}^2 - 2\mathrm{s}^3) + \mathrm{w}'(1)\{\mathrm{L}(-\mathrm{s}^2 + \mathrm{s}^3)\}$$

$$\mathrm{w} = \mathrm{w_i}(1 - 3\mathrm{s}^2 + 2\mathrm{s}^3) + \theta_\mathrm{i}\{\mathrm{L}(\mathrm{s} - 2\mathrm{s}^2 + \mathrm{s}^3)\} + \mathrm{w_j}(3\mathrm{s}^2 - 2\mathrm{s}^3) + \theta_\mathrm{j}\{\mathrm{L}(-\mathrm{s}^2 + \mathrm{s}^3)\}$$

$$\boxed{\boldsymbol{\varepsilon} = \mathbf{u}\mathbf{B}}$$

정의에 의해 $\{\varepsilon\} = \left\{-\frac{d^2w}{dx^2}\right\} = -\frac{d^2\mathbf{N}_{ij}^e}{dx^2}\mathbf{u}_{ij}^e$ 이고, $\mathbf{B} = -\frac{d^2\mathbf{N}_{ij}^e}{dx^2}$ 이라 하면,

$\mathbf{B} = \frac{1}{L^2}[6-12s, L(4-6s), -6+12s, L(2-6s)]$ 이다.

$$\text{stiffness matrix}\quad \mathbf{K}_{ij}^e = \int_0^L \mathbf{B}^T D\mathbf{B}dx = \frac{EI}{L^3}\begin{array}{c} \begin{array}{cccc} \boxed{w_i} & \boxed{\theta_i} & \boxed{w_j} & \boxed{\theta_j} \end{array} \\ \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \end{array} \begin{array}{c} \boxed{w_i} \\ \boxed{\theta_i} \\ \boxed{w_j} \\ \boxed{\theta_j} \end{array}$$

예제) 절점 i를 고정구속($w_i = 0$, $\theta_i = 0$)하고 절점 j에 P의 하중을 가했을 때 절점 j에서 변형량을 구하라.

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ w_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} \\ \\ P \\ 0 \end{Bmatrix} \quad \text{from} \quad [K]\{u\} = \{f\}$$

$$\frac{EI}{L^3}\{12w_j + (-6L)\theta_j\} = P,\ (-6L)w_j + 4L^2\theta_j = 0 \rightarrow \theta_j = \frac{6Lw_j}{4L^2} = \frac{3w_j}{2L}$$

$$\frac{EI}{L^3}\left\{12w_j + (-6L)\frac{3w_j}{2L}\right\} = P \rightarrow \ w_j = \frac{PL^3}{3EI}$$

$$\mathbf{B}^T D\mathbf{B} = \frac{EI}{L^4}\begin{bmatrix} 6-12s \\ L(4-6s) \\ -6+12s \\ L(2-6s) \end{bmatrix}[6-12s, L(4-6s), -6+12s, L(2-6s)]$$

$$= \frac{EI}{L^4}\begin{bmatrix} (6-12s)^2 & (6-12s)L(4-6s) & (6-12s)(-6+12s) & (6-12s)\{L(2-6s)\} \\ & \{L(4-6s)\}^2 & \{L(4-6s)\}(-6+12s) & \{L(4-6s)\}\{L(2-6s)\} \\ \boxed{\text{대칭 행렬}} & & (-6+12s)^2 & (-6+12s)\{L(2-6s)\} \\ & & & \{L(2-6s)\}^2 \end{bmatrix}$$

예제) 절점 i를 고정구속($w_i = 0$, $\theta_i = 0$)하고 절점 j에 P의 하중 및 $-\frac{PL}{2}$의 moment 하중을 가했을 때 절점 j에서 변형량을 구하라.

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ w_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} \\ \\ P \\ -\frac{PL}{2} \end{Bmatrix} \text{ from } [K]\{u\} = \{f\}$$

$$\frac{EI}{L^3}\left\{12w_j + (-6L)\theta_j\right\} = P, \qquad \frac{EI}{L^3}\left\{(-6L)w_j + 4L^2\theta_j\right\} = -\frac{PL}{2}$$

$$\rightarrow \theta_j = \frac{1}{4L^2}\left\{-\frac{PL^4}{2EI} + 6Lw_j\right\} = -\frac{PL^2}{8EI} + \frac{3w_j}{2L}$$

$$\frac{EI}{L^3}\left\{12w_j + (-6L)\left(\frac{3w_j}{2L} - \frac{PL^2}{8EI}\right)\right\} = P \rightarrow \quad w_j = \frac{PL^3}{12EI} \rightarrow \theta_j = 0$$

$$w = w_i(1 - 3s^2 + 2s^3) + \theta_i\{L(s - 2s^2 + s^3)\} + w_j(3s^2 - 2s^3) + \theta_j\{L(-s^2 + s^3)\}$$

$$= \frac{PL^3}{12EI}(3s^2 - 2s^3) = \frac{P}{12EI}(3Lx^2 - 2x^3) \quad (0 \leq x \leq L)$$