Easy Parametricity

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16th April 2025

Overview

1. Parametricity?
2. Models of the Axiom
3. The Main Theorem
4. How To Get Parametricity (Proof of Main Theorem)
5. Scope of the Technique
6. Extras

Parametricity?

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e.g. if $\alpha : \prod_{X:\mathcal{U}} (X \times X) \to X$ is suitably uniformly defined then we would hope that either $\alpha = \pi_0$ or $\alpha = \pi_1$.
if $\beta : \prod_{X:\mathcal{U}} (X \to X) \to (X \to X)$ is suitably uniformly defined then we would hope that $\beta = -^{\circ n}$ for some $n : \mathbb{N}$.

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Today, we're going to look at conditions on a universe $\mathcal{U}$ which ensure *all* families are appropriately parametric.

The Parametricity Axiom

For A, B types, write $A \perp B$ for the statement that $a \mapsto \lambda\_.a : A \to (B \to A)$ is an equivalence.

Axiom ($\mathsf{PA}_{\mathcal{U}}$)

$\mathcal{U}$ is a universe; for any type $A : \mathcal{U}$, $\mathcal{U} \perp A$ i.e. the map

$$a \mapsto \lambda\_.a : A \to \prod_{X:\mathcal{U}} A$$

is an equivalence.

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We can think of this as a simpler version of the instance of parametricity which states that

$$a \mapsto \lambda\_.\lambda f.f(a) : A \to \prod_{X:\mathcal{U}} (A \to X) \to X$$

is an equivalence.

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This is also equivalent to asking that there are unique diagonal fillers in squares like

$$\begin{tikzcd}
\mathcal{U} \arrow[r, dashed] \arrow[d, "!"'] & \sum_{A:\mathcal{U}} A \arrow[d, "\pi_0"] \\
1 \arrow[r, dashed] \arrow[ur, dotted] & \mathcal{U}
\end{tikzcd}$$

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Or that for any $A : \mathcal{U}$ there are unique diagonal fillers in squares like

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Or as some arcane relative of logical relations.

It expresses validity of the following:
Let T be a $\mathcal{U}$-small type.
Say that $t_0 \; R \; t_1$ iff there exist $\langle \tau_A \rangle_{A:\mathcal{U}} : T$ with $\tau_{\mathbf{0}} = t_0$ and $\tau_{\mathbf{1}} = t_1$.
Then $t_0 \; R \; t_1$ implies $t_0 = t_1$.
(Here a "relation" between $T_0, T_1 : \mathcal{U}$ would be a type family $\langle T'_A \rangle_{A:\mathcal{U}} : \mathcal{U}$ with $T'_{\mathbf{0}} = T_0$ and $T'_{\mathbf{1}} = T_1$.)

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Main Theorem

Theorem (Main)

Assume $\mathrm{PA}_{\mathcal{U}}$.
Let $\mathbf{C}$ *be a* $\mathcal{U}$*-complete univalent category and* $\mathbf{D}$ *be a locally* $\mathcal{U}$*-small category.*

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Let $\mathbf{C}$ *be a* $\mathcal{U}$*-complete univalent category and* $\mathbf{D}$ *be a locally* $\mathcal{U}$*-small category.*

(a) *Let* $F, G : \mathbf{C} \to \mathbf{D}$ *be functors and let* $\alpha : \prod_{X:\mathsf{Ob}(\mathbf{C})} \mathbf{D}(F(X), G(X))$. *Then* α *is natural.*

(b) *Let* $F, G : \mathbf{C}^{\mathsf{op}} \times \mathbf{C} \to \mathbf{D}$ *be bifunctors and let* $\beta : \prod_{X:\mathsf{Ob}(\mathbf{C})} \mathbf{D}(F(X,X), G(X,X))$. *Then* β *is dinatural.*

(c) *Let* $F : \mathbf{C} \to \mathbf{D}$ *be a function on objects and morphisms which respects sources, targets and identity morphisms. Then* F *respects composition, so is a functor.*

Models from Modalities

- Let $\mathcal{V}$ be a univalent universe and $\lozenge : \mathcal{V} \to \mathcal{V}$ be an (idempotent monadic) modality on $\mathcal{V}$.
- Write $\mathcal{V}_\lozenge$ for the reflective subuniverse of $\lozenge$-modal types. $\mathcal{V}_\lozenge$ has 1, $\times$, $\to$, Σ, Π and $=$ but may fail to have HITs.

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- Suppose there is a type $\mathbb{I}$ with $0_\mathbb{I}, 1_\mathbb{I} : \mathbb{I}$, $0_\mathbb{I} \neq 1_\mathbb{I}$ and $\lozenge\mathbb{I} \cong 1$. (This is equivalent to the 'axiom of sufficient cohesion'.)
  Then $\mathrm{PA}_{\mathcal{V}_\lozenge}$.

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  Then $\mathsf{PA}_{\mathcal{V}_\lozenge}$.
- If $\lozenge$ is the shape modality left adjoint to the flat $\flat$ modality of modal HoTT ($\lozenge = \int \dashv \flat$), then $\mathcal{V}_\lozenge$ has all discrete types including $0, 2, \mathbb{N}$ and (I think?) has HITs.

Some Models

- In simplicial type theory, PA holds for the type of groupoids (those C with $[1] \perp C$).
- More generally, the subuniverse of discrete types in cohesive HoTT satisfies PA as soon as the axiom of sufficient cohesion (axiom C2) holds.

- The (internally defined) subuniverse of discrete types in the 1-toposes of cubical sets or simplicial sets satisfy PA.
- Similarly, in any stably locally connected (1-)topos (e.g. simplicial sets, cubical sets), PA is implied by the axiom of sufficient cohesion (axiom C2).

- The univalent universe $\mathcal{U}$ of modest types in the cubical assemblies model satisfies $\mathsf{PA}_{\mathcal{U}}$.
- The universe $\mathcal{U}$ of modest sets of a category of assemblies satisfies $\mathsf{PA}_{\mathcal{U}}$.

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Let $\mathbf{C}$ *be a* $\mathcal{U}$*-complete univalent category and* $\mathbf{D}$ *be a locally* $\mathcal{U}$*-small category.*

(a) *Let* $F, G : \mathbf{C} \to \mathbf{D}$ *be functors and let* $\alpha : \prod_{X:\mathrm{Ob}(\mathbf{C})} \mathbf{D}(F(X), G(X))$*. Then* α *is natural.*

(b) *Let* $F, G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$ *be bifunctors and let* $\beta : \prod_{X:\mathrm{Ob}(\mathbf{C})} \mathbf{D}(F(X, X), G(X, X))$*. Then* β *is dinatural.*

(c) *Let* $F : \mathbf{C} \to \mathbf{D}$ *be a function on objects and morphisms which respects sources, targets and identity morphisms. Then* F *respects composition, so is a functor.*

Aside: factorisations

Let $\mathbf{C}$ be a category and $A \xrightarrow{f} B$ be a morphism of $\mathbf{C}$. We denote $\mathrm{Fact}(f)$ the type of factorisations of f:

$$\begin{tikzcd} & X \arrow[dr, "f_r"] & \\ A \arrow[ur, "f_l"] \arrow[rr, "f"] & & B \end{tikzcd}$$

Note that this is equivalently the fibre of f under $\circ : \mathbf{C}^{\bullet \to \bullet \to \bullet} \to \mathbf{C}^{\bullet \to \bullet}$.

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Amongst $\mathrm{Fact}(f)$ we in particular have the factorisations (f, id) and (id, f):

$$\begin{tikzcd} & B \arrow[dr, "\mathrm{id}"] & \\ A \arrow[ur, "f"] \arrow[rr, "f"] & & B \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dr, "f"] & \\ A \arrow[ur, "\mathrm{id}"] \arrow[rr, "f"] & & B \end{tikzcd}$$

Main Theorem (but better)

A category **C** is $\perp\in\mathcal{U}$-*tame* iff for any $A : \mathcal{U}$, $\mathrm{Fact}(f) \perp A$ i.e. all functions from $\mathrm{Fact}(f)$ to a $\mathcal{U}$-small type are constant.

Theorem (A)

Assume $\mathrm{PA}_{\mathcal{U}}$. *Then any univalent category with limits of shape* $p * 1$ *for all propositions* p *is* $\perp\in\mathcal{U}$*-tame.*

Theorem (B)

Let **C** *be a* $\perp\in\mathcal{U}$*-tame category and* **D** *be a locally* $\mathcal{U}$*-small category.*
Then Theorem Main(a,b,c) hold.

Approach to proof of Theorem B

Theorem (B)

Let **C** *be a* $\perp\in\mathcal{U}$*-tame category and* **D** *be a locally* $\mathcal{U}$*-small category. Then Theorem 1(a,b,c) hold.*

B.1 Massage the goal identity into the form $M(f, \mathrm{id}) = M(\mathrm{id}, f)$ for some function $M : \mathrm{Fact}(f) \to \hom(c, d)$.

B.2 By $\perp\in\mathcal{U}$-tameness of **C** and $\mathcal{U}$-smallness of $\hom(c, d)$, $M(f, \mathrm{id}) = M(\mathrm{id}, f)$ as desired.

Proof of B.1(a)

Let F,G,α be as in hypotheses of Theorem 1(a). For any morphism f in $\mathbf{C}$, we wish to show that $\alpha_B \circ Ff = Gf \circ \alpha_A$.
Goal: Outer hexagon commutes.
Construct $M_f : \mathrm{Fact}(f) \to \hom_{\mathbf{D}}(FA, GB)$ to interpolate.

$$\begin{tikzcd}
M_f(\mathrm{id}_A, f) = & & FA \arrow[r, dotted, "\alpha_A"] & GA \arrow[dr, dotted, "Gf"] & \\
M_f(f_l, f_r) = & FA \arrow[ur, dotted, "\mathrm{id}"] \arrow[rrr, squiggly, "???"] \arrow[dr, dotted, "Ff"'] & & & GB \\
M_f(f, \mathrm{id}_B) = & & FB \arrow[r, dotted, "\alpha_B"] & GB \arrow[ur, dotted, "\mathrm{id}"'] &
\end{tikzcd}$$

Proof of B.1(a)

Construct $M_f : \mathrm{Fact}(f) \to \hom_{\mathbf{D}}(FA, GB)$ to interpolate. So let

$$\begin{tikzcd}
& X \arrow[dr, "f_r"] & \\
A \arrow[ur, "f_l"] \arrow[rr, "f"] & & B
\end{tikzcd}$$

be a factorisation of f.

$$\begin{tikzcd}
M_f(\mathrm{id}_A, f) = & & FA \arrow[r, dotted, "\alpha_A"] & GA \arrow[dr, dotted, "Gf"] & \\
M_f(f_l, f_r) = & FA \arrow[ur, dotted, "\mathrm{id}"] \arrow[r, "Ff_l"] \arrow[dr, dotted, "Ff"'] & FX \arrow[r, "\alpha_X"] & GX \arrow[r, "Gf_r"] & GB \\
M_f(f, \mathrm{id}_B) = & & FB \arrow[r, dotted, "\alpha_B"] & GB \arrow[ur, dotted, "\mathrm{id}"'] &
\end{tikzcd}$$

Hence it suffices to show that for all f, $M_f(f, \mathrm{id}) = M_f(\mathrm{id}, f)$.

Approach to proof of Theorem A

Theorem (A)

Assume $\mathsf{PA}_{\mathcal{U}}$. *Then any univalent category with limits of shape* $p * 1$ *for all propositions* p *is* $\bot \in \mathcal{U}$*-tame.*

A.1 Show (using univalence and completeness of $\mathbf{C}$) that for any f there is a function $F_f : \mathcal{U} \to \mathrm{Fact}(f)$ with $F_f(\mathbf{0}) = (f, \mathrm{id})$ and $F_f(\mathbf{1}) = (\mathrm{id}, f)$.

A.2 For any g, any $M : \mathrm{Fact}(g) \to A$ and any factorisation (f, h) of g, let $M' : \mathrm{Fact}(f) \to A$ be $M'(f_l, f_r) = M(f_l, h \circ f_r)$.
Then by $\mathsf{PA}_{\mathcal{U}}$, $M' F_f$ is constant, so $M(f, h \circ \mathrm{id}) = M' F_f \mathbf{0} = M' F_f \mathbf{1} = M(\mathrm{id}, h \circ f)$ as desired.

Proof of A.1

Let $f : X \to Y$ be a morphism of $\mathbf{C}$. We now want to show (using univalence and completeness of $\mathbf{C}$) that there is a function $F : \mathcal{U} \to \mathrm{Fact}(f)$ with $F(\mathbf{0}) = (f, \mathrm{id})$ and $F(\mathbf{1}) = (\mathrm{id}, f)$.

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Hence
we want to construct an interpolating factorisation $F(S) = X \xrightarrow{l_S} M_S \xrightarrow{r_S} Y$ dependent on a type $S : \mathcal{U}$.

By univalence
of $\mathbf{C}$, we only need isomorphisms $F(\mathbf{0}) \cong (f, \mathrm{id})$ and $F(\mathbf{1}) \cong (\mathrm{id}, f)$.

$$F(\mathbf{0}) = \begin{tikzcd} & Y \arrow[dr, "\mathrm{id}"] \arrow[d, dash, "\cong"] & \\ X \arrow[ur, "f"] \arrow[r, "l_0"] & M_0 \arrow[r, "r_0"] & Y \end{tikzcd}$$

$$F(S) = \begin{tikzcd} X \arrow[r, "l_S"] & M_S \arrow[r, "r_S"] & Y \end{tikzcd}$$

$$F(\mathbf{1}) = \begin{tikzcd} X \arrow[r, "l_1"] \arrow[dr, "\mathrm{id}"'] & M_1 \arrow[r, "r_1"] \arrow[d, dash, "\cong"] & Y \\ & X \arrow[ur, "f"'] & \end{tikzcd}$$

Proof of A.1, continued

For S a category (pictured, a set), let $S * 1$ be the result of freely adjoining a terminal object to S.

$0 * 1$

$$\bullet$$

$1 * 1$

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \end{tikzcd}$$

$2 * 1$

$$\begin{tikzcd} \bullet \arrow[r] & \bullet \\ \bullet \arrow[ur] & \end{tikzcd}$$

Proof of A.1, continued

For S a category (pictured, a set), let $S * 1$ be the result of freely adjoining a terminal object to S.
Let $J_S : (S * 1) \to \mathbf{C}$ be the diagram for a wide pullback of S copies of $f : X \to Y$.

J_0

$$Y$$

J_1

$$X \xrightarrow{f} Y$$

J_2

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X \arrow[ur, "f"] & \end{tikzcd}$$

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The J_S admit cones from X.

J_0

$$\begin{tikzcd} X \arrow[dr, dashed] & \\ & Y \end{tikzcd}$$

J_1

$$\begin{tikzcd} X \arrow[dr, dashed] \arrow[drr, dashed] & & \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

J_2

$$\begin{tikzcd} X \arrow[dr, dashed] \arrow[drr, dashed] \arrow[ddr, dashed] & & \\ & X \arrow[r, "f"] & Y \\ & X \arrow[ur, "f"] & \end{tikzcd}$$

Proof of A.1, continued

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Let $J_S : (S * 1) \to \mathbf{C}$ be the diagram for a wide pullback of S copies of $f : X \to Y$.
The J_S admit cones from X.
We shall take the limit of $J_S : (S * 1) \to \mathbf{C}$.

$$\begin{tikzcd}
X \arrow[d, dashed] & & J_0 \\
\lim J_0 \arrow[rr, dashed, bend left, "\cong"] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, dashed, "\cong"] & & J_1 \\
\lim J_1 \arrow[r, dashed] \arrow[rr, dashed, bend left] & X \arrow[r, "f"] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[d, dashed] & & J_2 \\
\lim J_2 \arrow[r, dashed] \arrow[dr, dashed] \arrow[rr, dashed, bend left] & X \arrow[r, "f"] & Y \\
& X \arrow[ur, "f"] &
\end{tikzcd}$$

Hence, let $F(T) = X \xrightarrow{(\langle \mathrm{id}_X \rangle_{s:\|T\|_{-1}}, f)} \lim J_{\|T\|_{-1}} \xrightarrow{\pi} Y$.

"That's nice, but I'm a *homotopy* type theorist."

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Fortunately, we didn’t really use much of that the category $\mathbf{C}$ was a 1-category – the same techniques work for higher categories we write down.

Take a *wild category* to be defined like a category, but without restricting the homotopy types of objects or morphisms, removing associativity, and adding that $\mathsf{idl}_{\mathsf{id}} = \mathsf{idr}_{\mathsf{id}} : (\mathsf{id} \circ \mathsf{id}) = \mathsf{id}$.

Theorem

If $\mathbf{C}$ is a locally $\mathcal{U}$-small $\perp\in\mathcal{U}$-tame wild category, then $\mathbf{C}$ has an associator α, and moreover α satisfies the pentagon equation.

Theorem

If $\mathbf{C}$ is a locally $\mathcal{U}$-small $\perp\in\mathcal{U}$-tame wild category, then so is $\mathbf{C}^{\to}$.

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Well, *we never assumed univalence*.

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What about versions of the axiom for non-univalent universes?
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Call a category $\mathbf{C}$ *glueable* if for any proposition p, object A and p-indexed family $(B_u, i_u : A \cong B_u)_{u:p}$ of objects isomorphic to A, there's a specified object and isomorphism $(B', i' : A \cong B')$ which if $u : p$ equals $(B_u, i_u : A \cong B_u)$.

Theorem (A')

Assume $\mathrm{PA}_{\mathcal{U}}$*. Then any glueable category with limits of shape* $p * 1$ *for all propositions* p *is* $\perp\in\mathcal{U}$*-tame.*

When **Set** is glueable, so are plenty of other categories, such as presheaf categories and their replete full subcategories.

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All models on the Models slide have their category of sets glueable.

Caveat: $\mathcal{U}$ doesn't contain $\mathsf{Prop}_{\mathcal{U}}$

Lemma

$\mathsf{PA}_{\mathcal{U}}$ *implies* $\mathcal{U}$ *doesn't contain its subobject classifier. (Hence* $\mathsf{LEM}_{\mathcal{U}}$ *also fails.)*

Explicitly, we take $\mathsf{SC}_{\mathcal{U}}$ to be the statement that there is $\Omega : \mathcal{U}$ and some $i : \mathsf{hProp}_{\mathcal{U}}/\leftrightarrow \cong \Omega$. (The quotient is only necessary in absence of univalence.)

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Proof.

Suppose (by $\mathsf{SC}_{\mathcal{U}}$) there is $\Omega : \mathcal{U}$ such that $i : \mathsf{hProp}_{\mathcal{U}}/\leftrightarrow \cong \Omega$.
Define $f : \prod_{X:\mathcal{U}} \Omega$ as $f(X) = i(\|X\|_{-1})$. Then $f(\mathbf{0}) = i([\mathbf{0}]) \neq i([\mathbf{1}]) = f(\mathbf{1})$.
f is nonconstant, contradicting $\mathsf{PA}_{\mathcal{U}}$. $\square$

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f is nonconstant, contradicting $\mathsf{PA}_{\mathcal{U}}$. $\square$

Despite this failure, it's still possible to have a subuniverse $\mathcal{S} \subseteq \mathcal{U}$ for which $\mathsf{LEM}_{\mathcal{S}}$ (hence $\mathsf{SC}_{\mathcal{S}}$) holds and a superuniverse $\mathcal{V} \supseteq \mathcal{U}$ for which $\mathsf{SC}_{\mathcal{V}}$ holds: PA remains useful for reasoning about situations with SC.

The End

Questions?

Proof of B.1(b)

Let F,G,β be as in hypotheses of Theorem Main(b).
Goal: Outer diamond commutes.
Construct $M_f : \mathrm{Fact}(f) \to \hom_{\mathbf{D}}(F(A,B), G(B,A))$ to interpolate.

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M_f(f, \mathrm{id}_B) = & & F(B,B) \arrow[r, dotted, "\beta_B"] & G(B,B) \arrow[ur, dotted, "G(\mathrm{id}_B, f)"'] &
\end{tikzcd}
$$

Proof of B.1(c)

Goal: Outer diamond commutes.
Construct $M_f : \mathrm{Fact}(f) \to \hom_{\mathbf{D}}(FA, FC)$ to interpolate.

$$\begin{tikzcd}
 & FA \arrow[dr, dotted, "F(g \circ f)"] & \\
FA \arrow[ur, dotted, "\mathrm{id}"] \arrow[r, "Ff_l"] \arrow[dr, dotted, "Ff"'] & FX \arrow[r, "F(g \circ f_r)"] & FC \\
 & FB \arrow[ur, dotted, "Fg"'] &
\end{tikzcd}$$

An Impredicative Univalent Universe

Let $\mathcal{U}$ be an univalent universe which has all (possibly $\mathcal{U}$-large) products.

Proposition

If $A : \mathcal{U}$ satisfies $\forall a, b. \neg\neg(a = b) \to (a = b)$, then $a \mapsto \lambda\_.a : A \to (\mathcal{U} \to A)$ is an equivalence.

Proof sketch.

Let $f : \mathcal{U} \to A$ and assume $f(0) \neq f(1)$. Denote $\mathrm{Prop}_{\neg\neg} := \{a : \mathrm{Prop} \mid a = \neg\neg a\}$.
Then $f : \mathrm{Prop}_{\neg\neg} \hookrightarrow A$ is a split embedding, so $\mathrm{Prop}_{\neg\neg}$ is essentially $\mathcal{U}$-small.
By large completeness, ${\mathrm{Prop}_{\neg\neg}}^{-} : \mathbf{Set} \to \mathbf{Set}$ has an initial algebra $A \cong {\mathrm{Prop}_{\neg\neg}}^{A}$, contradicting Cantor's diagonalisation argument.
Hence $f(0) = f(1)$. Similarly, $f(1) = f(X^0) = f(X^1) = f(X)$. □

Conjecture

$\mathrm{PA}_{\mathcal{U}}$