

# (Truncated) Simplicial Models of Type Theory

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- 1 Introduction
- 2 Reflexive graphs
- 3 2-truncated simplicial spaces
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# Outline

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# Introduction I

In order to develop *synthetic higher category theory*, Riehl and Shulman introduced a *Type Theory with Shapes* (RSTT) in [RS17]: MLTT with types of simplices, allowing for defining *synthetic*  $(\infty, 1)$ -categories as *complete Segal/Rezk types*.

As a main feature, RSTT postulates *extension types*, i.e. for shape inclusions  $\Phi \hookrightarrow \Psi$ , families  $A : \Phi \rightarrow \mathcal{U}$ , and terms  $a : \prod_{t:\Phi} A(t)$  there exists the type of liftings

$$\left\langle \prod_{t:\Psi} A(t) \middle| \begin{array}{c} \Phi \\ \downarrow \\ \Psi \end{array} \right\rangle_a \triangleq \left\{ \begin{array}{ccc} \Phi & \xrightarrow{a} & A \\ \downarrow & \nearrow \bar{a} & \\ \Psi & & \end{array} \right\}$$

**Example & Definition:** For a type  $A$  and terms  $x, y : A$ , define the *hom-types*

$$\mathrm{hom}_A(x, y) := \left\langle \prod_{t:\Delta^1} A(t) \middle| \begin{array}{c} \partial\Delta^1 \\ [x, y] \end{array} \right\rangle.$$

# Introduction II

**Goal:** Understanding simplicial type theory and its models with the aim of doing synthetic  $(\infty, 1)$ -category theory with appropriate universes.

There is a model of RSTT in *simplicial spaces*, i.e. the model category  $[\mathbb{A}^{\text{op}}, \mathbf{sSet}_{\text{Quillen}}]_{\text{Reedy}}$ . This model structure presents the  $(\infty, 1)$ -category  $\text{PSh}_{\infty}(\mathbb{A})$ .

We consider variations such as  $\text{PSh}_{\infty}(\mathbb{A}_{\leq k})_{\text{Reedy}}$  for  $k = 1, 2$ .

We get a model of RSTT whenever we have  $\Delta^1$  together with  $\leq: \Delta^1 \times \Delta^1 \rightarrow \text{Prop}$  as an interval type.

# Intended model: simplicial spaces I

Definitions from [RS17]:

- A type  $A$  is a *Segal type* if  $(\Delta^2 \rightarrow A) \xrightarrow{\simeq} (\Lambda_1^2 \rightarrow A)$ .
- A Segal type  $A$  is a *Rezk type* if  $\text{idtoiso}_A : \prod_{x,y:A} \text{Id}_A(x,y) \xrightarrow{\simeq} \text{iso}_A(x,y)$ .
- A type  $A$  is a *discrete type* if  $\text{idtorarr}_A : \prod_{x,y:A} \text{Id}_A(x,y) \xrightarrow{\simeq} \text{hom}_A(x,y)$ .

RSTT with a univalent universe can be modeled on  $[\Delta^{\text{op}}, \mathbf{sSet}_{\text{Quillen}}]_{\text{Reedy}}$ , cf. [Shu15].

The notions just introduced semantically coincide with their classical analogues, *at the level of objects*.

## Intended model: simplicial spaces II

- Types are interpreted as Reedy fibrant objects. Families of types are interpreted as Reedy fibrations. A map  $f$  is a fibration if  $m \perp f$  for all  $m$  which are componentwise trivial cofibrations in  $\mathbf{sSet}$ .
- Segal types are interpreted as Segal spaces, i.e. Reedy fibrant objects  $X$  with  $m \otimes I(i) \perp X$  for all monomorphisms  $m$ , and  $i : \Lambda_1^2 \rightarrow \Delta^2$ ,  $I : \mathbf{sSet} \hookrightarrow [\Delta^{\mathrm{op}}, \mathbf{sSet}]$ . *Segal types are  $\infty$ -precategories (i.e. non-univalent).*
- Rezk types are interpreted as complete Segal spaces, aka Rezk spaces, i.e. Segal spaces  $X$  where  $X_0 \simeq X_{\mathrm{hoeq}}$ . *Rezk types are univalent  $\infty$ -categories.*
- Discrete types are Rezk types  $X$  such that all  $X_n$  are discrete simplicial sets. *Discrete types are (univalent)  $\infty$ -groupoids.*

# Subuniverses of simplicial spaces I

In RSTT, define

$$\text{isSegal}(A) := \text{isEquiv}((\Delta^2 \rightarrow A) \rightarrow (\Lambda_1^2 \rightarrow A)),$$

$$\text{isRezk}(A) := \text{isSegal}(A) \times \text{isEquiv}(\text{idtoiso}_A),$$

$$\text{isDisc}(A) := \text{isEquiv}(\text{idtorarr}_A) \simeq \text{isRezk}(A) \times \prod_{x,y:A} \prod_{f:\text{hom}_A(x,y)} \text{isIso}(f),$$

giving rise to contexts

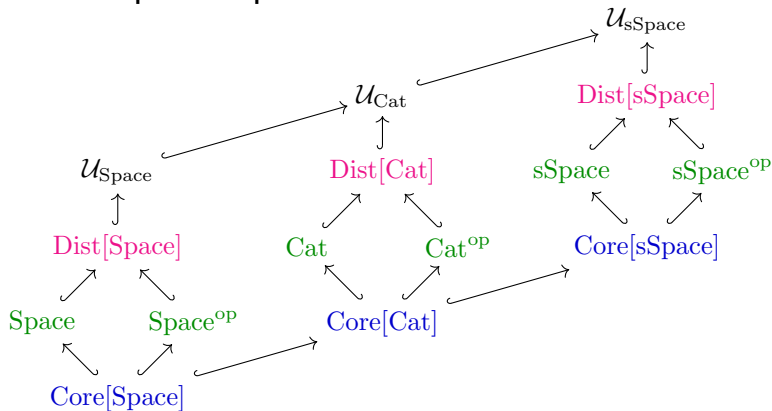
$$\text{Segal} := \llbracket A : \mathcal{U}, p : \text{isSegal}(A) \rrbracket, \quad \text{Rezk} := \llbracket A : \mathcal{U}, p : \text{isRezk}(A) \rrbracket,$$

$$\text{Disc} := \llbracket A : \mathcal{U}, p : \text{isDisc}(A) \rrbracket,$$

in either the “full” simplicial space model  $\text{PSh}_\infty(\Delta)$ , or the truncated ones  $\text{PSh}_\infty(\Delta_{\leq k})$ .



# Subuniverses of simplicial spaces II



$\text{sSpace} := \text{CSS of simpl. spaces}$

$\text{Cat}_n := (\mathcal{U}_{\text{Cat}})^*(\text{sSpace}_n) \simeq \Gamma(\text{coCart}(\Delta^n))$

$\text{Space}_n := \Gamma(\text{LFib}(\Delta^n)) \simeq (\mathcal{U}_{\text{Space}})^*(\text{Cat}_n)$

Where are Disc and Rezk?

$\text{Dist}[\text{sSpace}]_n := \Gamma(\text{ExpFib}(\Delta^n))$

$\text{Dist}[\text{Cat}]_n := (\mathcal{U}_{\text{Cat}})^*(\text{Dist}[\text{sSpace}]_n)$

$\text{Dist}[\text{Space}]_n := (\mathcal{U}_{\text{Space}})^*(\text{Dist}[\text{Cat}]_n)$

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# Reflexive graphs I

Taking  $\text{PSh}_\infty(\Delta_{\leq 1})$  gives rise to the reflexive graph model, already investigated by Rijke-Spitters [Rij17]:

**Contexts**  $\Gamma$

$$\Gamma_0 : \mathcal{U}$$

$$\Gamma_1 : \Gamma_0 \rightarrow \Gamma_0 \rightarrow \mathcal{U}$$

$$\rho_\Gamma : \prod_{x:\Gamma_0} \Gamma_1(x, x)$$

**Families**  $\Gamma \vdash A$

$$A_0 : \Gamma_0 \rightarrow \mathcal{U}$$

$$A_1 : \prod_{\{x,y:\Gamma_0\}} \Gamma_1(x, y) \rightarrow A_0(x) \rightarrow A_0(y) \rightarrow \mathcal{U}$$

$$\rho_A : \prod_{\{x:\Gamma_0\}} \prod_{u:A_0(x)} A_1(\rho_\Gamma(x), u, u)$$

**Context substitution**  $\varphi : \Gamma \rightarrow \Delta$

$$\varphi_0 : \Gamma_0 \rightarrow \Delta_0$$

$$\varphi_1 : \prod_{\{x,y:\Gamma_0\}} \Gamma_1(x, y) \rightarrow \Delta_1(\varphi_0 x, \varphi_0 y)$$

$$\rho_\varphi : \prod_{x:\Gamma_0} (\varphi_1(\rho_\Gamma x) = \rho_\Delta(\varphi_0 x))$$

**Terms**  $a : A$

$$a_0 : \prod_{x:\Gamma_0} A_0(x)$$

$$a_1 : \prod_{\{x,y:\Gamma_0\}} \prod_{e:\Gamma_1(x,y)} A_1(e, a_0(x), a_0(y))$$

$$\rho_a : \prod_{x:\Gamma_0} (a_1(\rho_\Gamma x) = \rho_A(a_0 x))$$

We obtain a model of (cohesive) *simplicial type theory* on top of  $\text{MLTT}_{\Pi, \Sigma, \text{Id}, \mathcal{U}}$ , with  $\Delta^n$  as types.

# Disc as a reflexive graph

## Theorem

$\llbracket \text{Disc} \rrbracket_1$  is equivalent to the reflexive graph  $\text{Disc}_1$  with:

$$\begin{aligned}(\text{Disc}_1)_0 &::= \text{Set} \\ (\text{Disc}_1)_1 \ X \ Y &::= X \rightarrow Y \rightarrow \text{Prop} \\ \rho_{\text{Disc}_1} \ X &::= \lambda x y. (x = y)\end{aligned}$$

*Proof idea:* For  $A : \mathcal{U}$  we have  $\text{isDisc}(A) \simeq \prod_{x:\Gamma_0} \text{isContr}(\text{arrfrom}_A(x))$ . This leads to

$$\begin{aligned}\llbracket \text{Disc} \rrbracket_1 \ X \ Y &\simeq \sum_{Z:X \rightarrow Y \rightarrow \mathcal{U}} \prod_{\substack{x:X, \\ y:Y}} \prod_{z:Z \ x \ y} \text{isContr}(Z \ x \ y) \\ &\simeq \sum_{Z:X \rightarrow Y \rightarrow \mathcal{U}} \prod_{\substack{x:X, \\ y:Y}} \text{isProp}(Z \ x \ y).\end{aligned}$$

# Rezk as a reflexive graph

## Theorem

$\llbracket \text{Rezk} \rrbracket_1$  is equivalent to the reflexive graph  $\text{Rezk}_1$  with:

$$(\text{Rezk}_1)_0 \equiv \text{Poset}$$

$$(\text{Rezk}_1)_1 X Y \equiv \text{“full relations } X \rightarrow Y \rightarrow \text{Prop”}$$

$$\rho_{\text{Rezk}_1} X \equiv X.\text{PO}$$

Hence,  $\text{Disc}_1$  is *not* Rezk.

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# 2-truncated simplicial spaces I

## Contexts $\Gamma$

$$\Gamma_0 : \mathcal{U}$$

$$\Gamma_1 : \Gamma_0 \rightarrow \Gamma_0 \rightarrow \mathcal{U}$$

$$\rho_\Gamma : \prod_{x:\Gamma_0} \Gamma_1(x, x)$$

$$\Gamma_{\text{comp}} : \prod_{x,y,z:\Gamma_0} \Gamma_1(x, y) \rightarrow \Gamma_1(y, z) \rightarrow \Gamma_1(x, z) \rightarrow \mathcal{U}$$

$$\Gamma_{\text{left}} : \prod_{x,y:\Gamma_0} \prod_{f:\Gamma_1(x,y)} \Gamma_{\text{comp}}(\rho_\Gamma x, f, f)$$

$$\Gamma_{\text{right}} : \prod_{x,y:\Gamma_0} \prod_{f:\Gamma_1(x,y)} \Gamma_{\text{comp}}(f, \rho_\Gamma y, f)$$

$$\Gamma_{\text{coh}} : \prod_{x:\Gamma_0} (\Gamma_{\text{left}}(\rho_\Gamma x) = \Gamma_{\text{right}}(\rho_\Gamma x))$$

## Context substitution $\varphi : \Gamma \rightarrow \Delta$

$$\varphi_0 : \Gamma_0 \rightarrow \Delta_0$$

$$\varphi_1 : \prod_{\{x,y:\Gamma_0\}} \Gamma_1(x, y) \rightarrow \Delta_1(\varphi_0 x, \varphi_0 y)$$

$$\rho_\varphi : \prod_{x:\Gamma_0} (\varphi_1(\rho_\Gamma x) = \rho_\Delta(\varphi_0 x))$$

$$\varphi_{\text{comp}} : \prod_{\{\dots\}} \Gamma_{\text{comp}}(f, g, h) \rightarrow \Delta_{\text{comp}}(\varphi_1 f, \varphi_1 g, \varphi_1 h)$$

$$\varphi_{\text{left}} : \prod_{\{x,y:\Gamma_0\}} \prod_{f:\Gamma_1(x,y)} (\varphi_{\text{comp}}(\Gamma_{\text{left}}(f)) =_{\rho_\varphi(x)} \Delta_{\text{left}}(\varphi_1 f))$$

$$\varphi_{\text{right}} : \prod_{\{x,y:\Gamma_0\}} \prod_{f:\Gamma_1(x,y)} (\varphi_{\text{comp}}(\Gamma_{\text{right}}(f)) =_{\rho_\varphi(y)} \Delta_{\text{right}}(\varphi_1 f))$$

$$\varphi_{\text{coh}} : \prod_{x:\Gamma_0} (\varphi_{\text{left}}(\rho_\Gamma x) \stackrel{\dots}{=} \varphi_{\text{right}}(\rho_\Gamma x))$$

# 2-truncated simplicial spaces II

**Families**  $\Gamma \vdash A$

$$A_0 : \Gamma_0 \rightarrow \mathcal{U}$$

$$A_1 : \prod_{\{x,y:\Gamma_0\}} \Gamma_1(x,y) \rightarrow A_0(x) \rightarrow A_0(y) \rightarrow \mathcal{U}$$

$$\rho_A : \prod_{\{x:\Gamma_0\}} \prod_{u:A_0(x)} A_1(\rho_\Gamma(x), u, u)$$

$$\begin{aligned} A_{\text{comp}} : \prod_{\{\cdots\}} \Gamma_{\text{comp}}(f,g,h) &\rightarrow A_1(a,b) \rightarrow A_1(b,c) \\ &\rightarrow A_1(a,c) \rightarrow \mathcal{U} \end{aligned}$$

$$A_{\text{left}} : \prod_{\{\cdots\}} \cdots \rightarrow A_{\text{comp}}(\Gamma_{\text{left}}f, \rho_A x, g, g)$$

$$A_{\text{right}} : \prod_{\{\cdots\}} \cdots \rightarrow A_{\text{comp}}(\Gamma_{\text{right}}f, g, \rho_A y, g)$$

$$A_{\text{coh}} : \cdots$$

**Terms**  $a : A$

$$a_0 : \prod_{x:\Gamma_0} A_0(x)$$

$$a_1 : \prod_{\{x,y:\Gamma_0\}} \prod_{e:\Gamma_1(x,y)} A_1(e, a_0(x), a_0(y))$$

$$\rho_a : \prod_{x:\Gamma_0} (a_1(\rho_\Gamma x) = \rho_A(a_0 x))$$

$$\begin{aligned} a_{\text{comp}} : \prod_{\cdots, \sigma:\Gamma_{\text{comp}}(f,g,h), \cdots} A_{\text{comp}}(\sigma, a_1(f), a_1(g), a_1(h)) \end{aligned}$$

$$a_{\text{left}} : \prod_{\{\cdots\}} \cdots \rightarrow A_{\text{left}}(g) =_{\rho_a x} a_{\text{comp}}(\Gamma_{\text{left}}f)$$

$$a_{\text{right}} : \prod_{\{\cdots\}} \cdots \rightarrow A_{\text{right}}(g) =_{\rho_a y} a_{\text{comp}}(\Gamma_{\text{right}}f)$$

$$a_{\text{coh}} : \cdots$$



# Disc and Rezk as 2-truncated simplicial spaces

$\llbracket \text{Disc} \rrbracket_2$  has as vertices 1-types, as edges Set-valued relations, and as triangles 2-spans of propositions.

We conjecture  $\llbracket \text{Rezk} \rrbracket_2$  to have as vertices 1-categories. The edges are some form of “generalized distributors”.

Extrapolating to dimension  $n$ , one might expect  $\llbracket \text{Disc} \rrbracket_n$  to have as vertices the  $(n - 1)$ -types, as edges  $(n - 2)$ -type valued spans, as triangles  $(n - 3)$ -type valued 2-spans etc.

We suspect  $\llbracket \text{Rezk} \rrbracket_n$  to have as vertices  $(n - 1, 1)$ -categories, as edges “generalized  $(n - 2)$ -type valued distributors”, as triangles “ $(n - 3)$ -type-valued 2-distributors (?)” etc.

# Direct replacement

Direct replacement of  $\Delta_{\leq 1}$ , construction due to Szumiło, cf. [Szu14], [KS17]:

$$\begin{array}{ccc} (1) & \rightrightarrows & (1, 1) \\ \downarrow \wr & \swarrow & \\ (2) & & \end{array}$$

A context  $\Gamma$  is given by the following data:

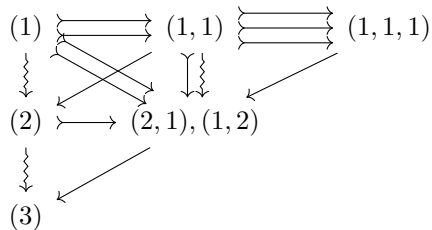
$$\Gamma_0 : \mathcal{U}$$

$$\Gamma_1 : \Gamma_0 \rightarrow \Gamma_0 \rightarrow \mathcal{U}$$

$$\Gamma_{\text{refl}} : \prod_{x:\Gamma_0} \Gamma_1(x, x) \rightarrow \mathcal{U}$$

$$\begin{aligned} \sigma_\Gamma : \text{isEquiv} & \left( \left( \sum_{x:\Gamma_0} \sum_{f:\Gamma_1(x,x)} \Gamma_{\text{refl}}(x, f) \right) \rightarrow \Gamma_0 \right) \\ & \simeq \prod_{x:\Gamma_0} \text{isContr} \left( \sum_{f:\Gamma_1(x,x)} \Gamma_{\text{refl}}(x, f) \right) \end{aligned}$$

Direct replacement of  $\Delta_{\leq 2}$ :



$$\Gamma_0 : \mathcal{U}$$

$$\sigma_\Gamma : \prod_{x:\Gamma_0} \text{isContr} \left( \sum_{f:\Gamma_1(x,x)} \Gamma_2(x,f) \right)$$

$$\Gamma_1 : \Gamma_0 \rightarrow \Gamma_0 \rightarrow \mathcal{U}$$

$$\Gamma_2 : \prod_{x,y,z:\Gamma_0} \Gamma_1(x,y) \rightarrow \Gamma_1(y,z) \rightarrow \Gamma_1(x,z) \rightarrow \mathcal{U}$$

+ witnesses for degeneration

$$\Gamma_{\text{refl}} : \prod_{x:\Gamma_0} \Gamma_1(x,x) \rightarrow \mathcal{U}$$

$$\Gamma_{\text{left}} : \prod_{\{x,y:\Gamma_0\}} \prod_{f:\Gamma_1(x,y)} \prod_{g:\Gamma_1(y,y)} \Gamma_{\text{comp}}(f,g,f) \rightarrow \mathcal{U}$$

$$\Gamma_{\text{right}} : \prod_{\{x,y:\Gamma_0\}} \prod_{f:\Gamma_1(x,x)} \prod_{g:\Gamma_1(x,y)} \Gamma_{\text{comp}}(f,g,g) \rightarrow \mathcal{U}$$

$$\Gamma_{\text{triangle}} : \dots$$

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# Perspectives

Given a predicate  $P : X \rightarrow \mathbf{Prop}$  internally in RSTT, is there a way to define the corresponding full predicate? (Would yield definitions of  $\mathcal{U}_{\mathbf{Space}}$ ,  $\mathcal{U}_{\mathbf{Cat}}$ .)

Can we define  $\mathbf{Cat}$  as the intersection of  $\mathbf{Rezk}$  and  $\mathbf{sSpace}$ ?

If we define  $\mathbf{sSpace} := (\Delta^{\mathbf{op}} \rightarrow \mathbf{Space})$ , what would be the element fibration?

Externally, universes and notions of fibrations are interdefinable. What about internally?

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