

Easy Parametricity

Jem Lord

Department of Computer Science and Technology
University of Cambridge

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Overview

1. Parametricity?
2. Models of the Axiom
3. The Main Theorem
4. How To Get Parametricity (Proof of Main Theorem)
5. Scope of the Technique
6. Extras

Parametricity?

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if $\beta : \prod_{X:\mathcal{U}} (X \rightarrow X) \rightarrow (X \rightarrow X)$ is suitably uniformly defined then we would hope that $\beta = -^{\circ n}$ for some $n : \mathbb{N}$.

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Today, we’re going to look at conditions on a universe \mathcal{U} which ensure *all* families are appropriately parametric.

The Parametricity Axiom

For A, B types, write $A \perp B$ for the statement that $a \mapsto \lambda_.a : A \rightarrow (B \rightarrow A)$ is an equivalence.

Axiom ($\text{PA}_{\mathcal{U}}$)

\mathcal{U} is a universe; for any type $A : \mathcal{U}$, $\mathcal{U} \perp A$ i.e. the map

$$a \mapsto \lambda_.a : A \rightarrow \prod_{X:\mathcal{U}} A$$

is an equivalence.

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We can think of this as a simpler version of the instance of parametricity which states that

$$a \mapsto \lambda_.\lambda f.f(a) : A \rightarrow \prod_{X:\mathcal{U}} (A \rightarrow X) \rightarrow X$$

is an equivalence.

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This is also equivalent to asking that there are unique diagonal fillers in squares like

$$\begin{array}{ccc} \mathcal{U} & \dashrightarrow & \sum_{A:\mathcal{U}} A \\ \downarrow ! & \nearrow & \downarrow \pi_0 \\ 1 & \dashrightarrow & \mathcal{U} \end{array}$$

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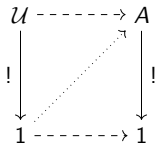
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Or that for any $A : \mathcal{U}$ there are unique diagonal fillers in squares like



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Or as some arcane relative of logical relations.

It expresses validity of the following:

Let T be a \mathcal{U} -small type.

Say that $t_0 R t_1$ iff there exist $\langle \tau_A \rangle_{A:\mathcal{U}} : T$ with $\tau_0 = t_0$ and $\tau_1 = t_1$.

Then $t_0 R t_1$ implies $t_0 = t_1$.

(Here a “relation” between $T_0, T_1 : \mathcal{U}$ would be a type family $\langle T'_A \rangle_{A:\mathcal{U}} : \mathcal{U}$ with $T'_0 = T_0$ and $T'_1 = T_1$.)

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Theorem (Main)

Assume $\text{PA}_{\mathcal{U}}$.

Let \mathbf{C} be a \mathcal{U} -complete univalent category and \mathbf{D} be a locally \mathcal{U} -small category.

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Let \mathbf{C} be a \mathcal{U} -complete univalent category and \mathbf{D} be a locally \mathcal{U} -small category.

- (a) Let $F, G : \mathbf{C} \rightarrow \mathbf{D}$ be functors and let $\alpha : \prod_{X:\text{Ob}(\mathbf{C})} \mathbf{D}(F(X), G(X))$. Then α is natural.
- (b) Let $F, G : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{D}$ be bifunctors and let $\beta : \prod_{X:\text{Ob}(\mathbf{C})} \mathbf{D}(F(X, X), G(X, X))$. Then β is dinatural.
- (c) Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a function on objects and morphisms which respects sources, targets and identity morphisms. Then F respects composition, so is a functor.

Models from Modalities

- Let \mathcal{V} be a univalent universe and $\diamond : \mathcal{V} \rightarrow \mathcal{V}$ be an (idempotent monadic) modality on \mathcal{V} .
- Write \mathcal{V}_\diamond for the reflective subuniverse of \diamond -modal types. \mathcal{V}_\diamond has 1 , \times , \rightarrow , Σ , Π and $=$ but may fail to have HITs.

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- Suppose there is a type \mathbb{I} with $0_{\mathbb{I}}, 1_{\mathbb{I}} : \mathbb{I}$, $0_{\mathbb{I}} \neq 1_{\mathbb{I}}$ and $\diamond \mathbb{I} \cong 1$. (This is equivalent to the ‘axiom of sufficient cohesion’.)
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Then $\text{PA}_{\mathcal{V}_\diamond}$.
- If \diamond is the shape modality left adjoint to the flat \flat modality of modal HoTT ($\diamond = \int \dashv \flat$), then \mathcal{V}_\diamond has all discrete types including $0, 2, \mathbb{N}$ and (I think?) has HITs.

Some Models

- In simplicial type theory, PA holds for the type of groupoids (those C with $[1] \perp C$).
- More generally, the subuniverse of discrete types in cohesive HoTT satisfies PA as soon as the axiom of sufficient cohesion (axiom C2) holds.
- The (internally defined) subuniverse of discrete types in the 1-toposes of cubical sets or simplicial sets satisfy PA.
- Similarly, in any stably locally connected (1-)topos (e.g. simplicial sets, cubical sets), PA is implied by the axiom of sufficient cohesion (axiom C2).
- The univalent universe \mathcal{U} of modest types in the cubical assemblies model satisfies $PA_{\mathcal{U}}$.
- The universe \mathcal{U} of modest sets of a category of assemblies satisfies $PA_{\mathcal{U}}$.

Theorem (Main)

Assume $\text{PA}_{\mathcal{U}}$.

Let \mathbf{C} be a \mathcal{U} -complete univalent category and \mathbf{D} be a locally \mathcal{U} -small category.

- (a) Let $F, G : \mathbf{C} \rightarrow \mathbf{D}$ be functors and let $\alpha : \prod_{X:\text{Ob}(\mathbf{C})} \mathbf{D}(F(X), G(X))$. Then α is natural.
- (b) Let $F, G : \mathbf{C}^{\text{op}} \times \mathbf{C} \rightarrow \mathbf{D}$ be bifunctors and let $\beta : \prod_{X:\text{Ob}(\mathbf{C})} \mathbf{D}(F(X, X), G(X, X))$. Then β is dinatural.
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Aside: factorisations

Let \mathbf{C} be a category and $A \xrightarrow{f} B$ be a morphism of \mathbf{C} . We denote $\text{Fact}(f)$ the type of factorisations of f :

$$\begin{array}{ccc} & X & \\ f_l \nearrow & & \searrow f_r \\ A & \xrightarrow{f} & B \end{array}$$

Note that this is equivalently the fibre of f under $\circ : \mathbf{C}^{\bullet \rightarrow \bullet \rightarrow \bullet} \rightarrow \mathbf{C}^{\bullet \rightarrow \bullet}$.

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Amongst $\text{Fact}(f)$ we in particular have the factorisations (f, id) and (id, f) :

$$\begin{array}{ccc} & B & \\ f \nearrow & & \searrow \text{id} \\ A & \xrightarrow{f} & B \end{array} \qquad \begin{array}{ccc} & A & \\ \text{id} \nearrow & & \searrow f \\ A & \xrightarrow{f} & B \end{array}$$

Main Theorem (but better)

A category \mathbf{C} is $\perp \in \mathcal{U}$ -tame iff for any $A : \mathcal{U}$, $\text{Fact}(f) \perp A$ i.e. all functions from $\text{Fact}(f)$ to a \mathcal{U} -small type are constant.

Theorem (A)

*Assume $\text{PA}_{\mathcal{U}}$. Then any univalent category with limits of shape $p * 1$ for all propositions p is $\perp \in \mathcal{U}$ -tame.*

Theorem (B)

Let \mathbf{C} be a $\perp \in \mathcal{U}$ -tame category and \mathbf{D} be a locally \mathcal{U} -small category. Then Theorem Main(a, b, c) hold.

Approach to proof of Theorem B

Theorem (B)

*Let \mathbf{C} be a $\perp \in \mathcal{U}$ -tame category and \mathbf{D} be a locally \mathcal{U} -small category.
Then Theorem 1(a,b,c) hold.*

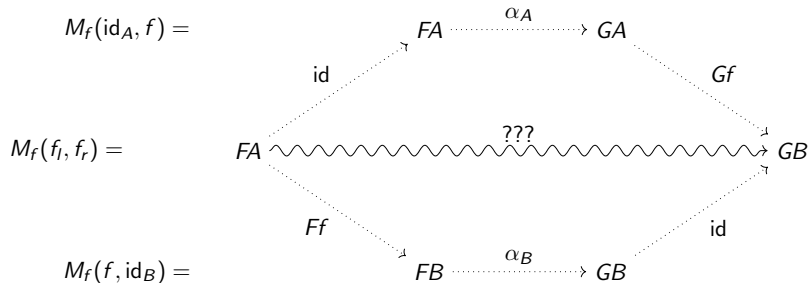
- B.1 Massage the goal identity into the form $M(f, \text{id}) = M(\text{id}, f)$ for some function $M : \text{Fact}(f) \rightarrow \text{hom}(c, d)$.
- B.2 By $\perp \in \mathcal{U}$ -tameness of \mathbf{C} and \mathcal{U} -smallness of $\text{hom}(c, d)$, $M(f, \text{id}) = M(\text{id}, f)$ as desired.

Proof of B.1(a)

Let F, G, α be as in hypotheses of Theorem 1(a). For any morphism f in \mathbf{C} , we wish to show that $\alpha_B \circ Ff = Gf \circ \alpha_A$.

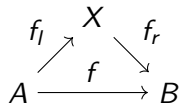
Goal: Outer hexagon commutes.

Construct $M_f : \text{Fact}(f) \rightarrow \text{hom}_{\mathbf{D}}(FA, GB)$ to interpolate.

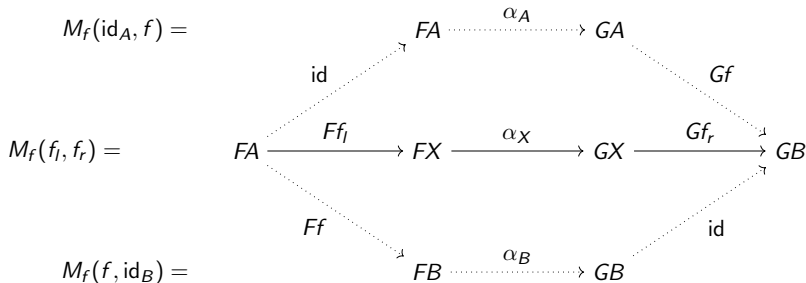


Proof of B.1(a)

Construct $M_f : \text{Fact}(f) \rightarrow \text{hom}_{\mathbf{D}}(FA, GB)$ to interpolate. So let



be a factorisation of f .



Hence it suffices to show that for all f , $M_f(f, \text{id}) = M_f(\text{id}, f)$.

Approach to proof of Theorem A

Theorem (A)

*Assume $\text{PA}_{\mathcal{U}}$. Then any univalent category with limits of shape $p * 1$ for all propositions p is $\perp \in \mathcal{U}$ -tame.*

- A.1 Show (using univalence and completeness of \mathbf{C}) that for any f there is a function $F_f : \mathcal{U} \rightarrow \text{Fact}(f)$ with $F_f(\mathbf{0}) = (f, \text{id})$ and $F_f(\mathbf{1}) = (\text{id}, f)$.
- A.2 For any g , any $M : \text{Fact}(g) \rightarrow A$ and any factorisation (f, h) of g , let $M' : \text{Fact}(f) \rightarrow A$ be $M'(f_l, f_r) = M(f_l, h \circ f_r)$.
Then by $\text{PA}_{\mathcal{U}}$, $M'F_f$ is constant, so $M(f, h \circ \text{id}) = M'F_f\mathbf{0} = M'F_f\mathbf{1} = M(\text{id}, h \circ f)$ as desired.

Proof of A.1

Let $f : X \rightarrow Y$ be a morphism of \mathbf{C} . We now want to show (using univalence and completeness of \mathbf{C}) that there is a function $F : \mathcal{U} \rightarrow \text{Fact}(f)$ with $F(\mathbf{0}) = (f, \text{id})$ and $F(\mathbf{1}) = (\text{id}, f)$.

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Hence

we want to construct an interpolating factorisation $F(S) = X \xrightarrow{l_S} M_S \xrightarrow{r_S} Y$ dependent on a type $S : \mathcal{U}$.

By univalence

of \mathbf{C} , we only need isomorphisms $F(\mathbf{0}) \cong (f, \text{id})$ and $F(\mathbf{1}) \cong (\text{id}, f)$.

$$\begin{array}{lcl} F(\mathbf{0}) = & & \begin{array}{ccccc} & & Y & & \\ & \nearrow f & \downarrow \cong & \searrow \text{id} & \\ X & \xrightarrow{l_0} & M_0 & \xrightarrow{r_0} & Y \end{array} \\ F(S) = & & \begin{array}{ccccc} X & \xrightarrow{l_S} & M_S & \xrightarrow{r_S} & Y \end{array} \\ F(\mathbf{1}) = & & \begin{array}{ccccc} X & \xrightarrow{l_1} & M_1 & \xrightarrow{r_1} & Y \\ & \searrow \text{id} & \downarrow \cong & \nearrow f & \\ & & X & & \end{array} \end{array}$$

Proof of A.1, continued

For S a category (pictured, a set), let $S * 1$ be the result of freely adjoining a terminal object to S .

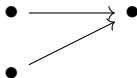
$0 * 1$



$1 * 1$



$2 * 1$



Proof of A.1, continued

For S a category (pictured, a set), let $S * 1$ be the result of freely adjoining a terminal object to S .

Let $J_S : (S * 1) \rightarrow \mathbf{C}$ be the diagram for a wide pullback of S copies of $f : X \rightarrow Y$.

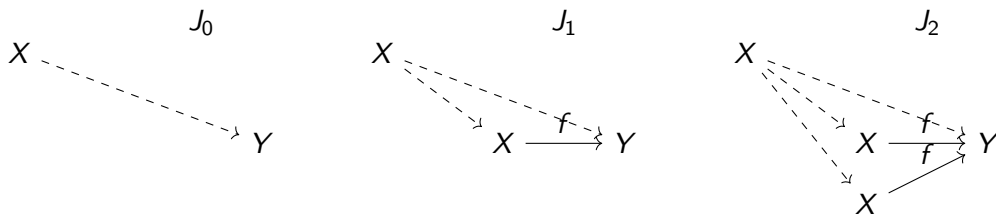
 J_0 Y J_1 $X \xrightarrow{f} Y$ J_2
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The J_S admit cones from X .



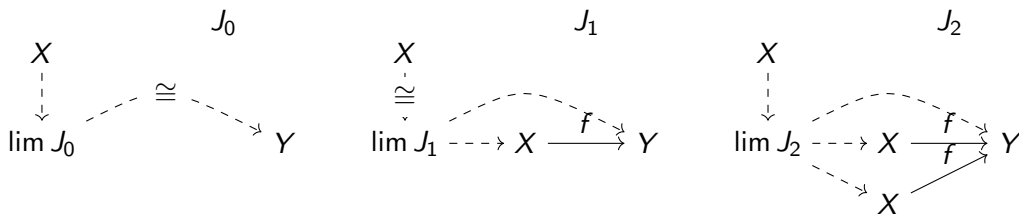
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We shall take the limit of $J_S : (S * 1) \rightarrow \mathbf{C}$.



Hence, let $F(T) = X \xrightarrow{(\langle \text{id}_X \rangle_{s: \|T\|_{-1}}, f)} \lim J_{\|T\|_{-1}} \xrightarrow{\pi} Y$.

“That’s nice, but I’m a *homotopy* type theorist.”

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Fortunately, we didn’t really use much of that the category \mathbf{C} was a 1-category – the same techniques work for higher categories we write down.

Take a *wild category* to be defined like a category, but without restricting the homotopy types of objects or morphisms, removing associativity, and adding that $\text{idl}_{\text{id}} = \text{idr}_{\text{id}} : (\text{id} \circ \text{id}) = \text{id}$.

Theorem

If \mathbf{C} is a locally \mathcal{U} -small $\perp \in \mathcal{U}$ -tame wild category, then \mathbf{C} has an associator α , and moreover α satisfies the pentagon equation.

Theorem

If \mathbf{C} is a locally \mathcal{U} -small $\perp \in \mathcal{U}$ -tame wild category, then so is \mathbf{C}^{\rightarrow} .

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What about versions of the axiom for non-univalent universes?

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Call a category \mathbf{C} *glueable* if for any proposition p , object A and p -indexed family $(B_u, i_u : A \cong B_u)_{u:p}$ of objects isomorphic to A , there’s a specified object and isomorphism $(B', i' : A \cong B')$ which if $u : p$ equals $(B_u, i_u : A \cong B_u)$.

Theorem (A')

*Assume $\text{PA}_{\mathcal{U}}$. Then any glueable category with limits of shape $p * 1$ for all propositions p is $\perp \in \mathcal{U}$ -tame.*

When **Set** is glueable, so are plenty of other categories, such as presheaf categories and their replete full subcategories.

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All models on the Models slide have their category of sets glueable.

Caveat: \mathcal{U} doesn't contain $\text{Prop}_{\mathcal{U}}$

Lemma

$\text{PA}_{\mathcal{U}}$ implies \mathcal{U} doesn't contain its subobject classifier. (Hence $\text{LEM}_{\mathcal{U}}$ also fails.)

Explicitly, we take $\text{SC}_{\mathcal{U}}$ to be the statement that there is $\Omega : \mathcal{U}$ and some $i : \text{hProp}_{\mathcal{U}} / \leftrightarrow \cong \Omega$. (The quotient is only necessary in absence of univalence.)

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Proof.

Suppose (by $\text{SC}_{\mathcal{U}}$) there is $\Omega : \mathcal{U}$ such that $i : \text{hProp}_{\mathcal{U}} / \leftrightarrow \cong \Omega$.

Define $f : \prod_{X:\mathcal{U}} \Omega$ as $f(X) = i(\|X\|_{-1})$. Then $f(\mathbf{0}) = i([0]) \neq i([1]) = f(\mathbf{1})$.

f is nonconstant, contradicting $\text{PA}_{\mathcal{U}}$. □

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Despite this failure, it's still possible to have a subuniverse $\mathcal{S} \subseteq \mathcal{U}$ for which $\text{LEM}_{\mathcal{S}}$ (hence $\text{SC}_{\mathcal{S}}$) holds and a superuniverse $\mathcal{V} \supseteq \mathcal{U}$ for which $\text{SC}_{\mathcal{V}}$ holds: PA remains useful for reasoning about situations with SC.

The End

Questions?

Proof of B.1(b)

Let F, G, β be as in hypotheses of Theorem Main(b).

Goal: Outer diamond commutes.

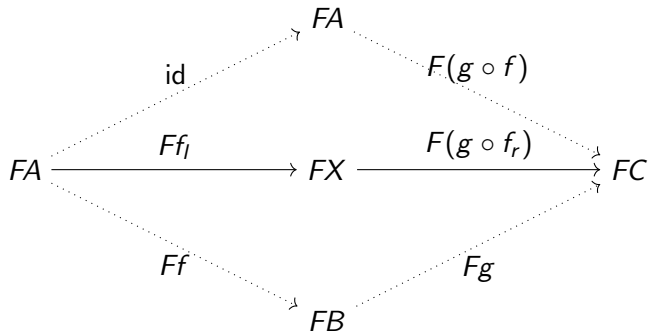
Construct $M_f : \text{Fact}(f) \rightarrow \text{hom}_{\mathbf{D}}(F(A, B), G(B, A))$ to interpolate.

$$\begin{array}{lcl}
 M_f(\text{id}_A, f) = & & \begin{array}{c} F(A, A) \xrightarrow{\beta_A} G(A, A) \\ \nearrow F(\text{id}_A, f) \quad \searrow G(f, \text{id}_B) \end{array} \\
 M_f(f_l, f_r) = & & \begin{array}{c} F(A, B) \xrightarrow{F(f_l, f_r)} F(X, X) \xrightarrow{\beta_X} G(X, X) \xrightarrow{G(f_r, f_l)} G(B, A) \\ \nearrow F(f, \text{id}_B) \quad \searrow G(\text{id}_B, f) \end{array} \\
 M_f(f, \text{id}_B) = & & \begin{array}{c} F(B, B) \xrightarrow{\beta_B} G(B, B) \end{array}
 \end{array}$$

Proof of B.1(c)

Goal: Outer diamond commutes.

Construct $M_f : \text{Fact}(f) \rightarrow \text{hom}_{\mathbf{D}}(FA, FC)$ to interpolate.



An Impredicative Univalent Universe

Let \mathcal{U} be an univalent universe which has all (possibly \mathcal{U} -large) products.

Proposition

If $A : \mathcal{U}$ satisfies $\forall a, b. \neg\neg(a = b) \rightarrow (a = b)$, then $a \mapsto \lambda_.a : A \rightarrow (\mathcal{U} \rightarrow A)$ is an equivalence.

Proof sketch.

Let $f : \mathcal{U} \rightarrow A$ and assume $f(0) \neq f(1)$. Denote $\text{Prop}_{\neg\neg} := \{a : \text{Prop} \mid a = \neg\neg a\}$.

Then $f : \text{Prop}_{\neg\neg} \hookrightarrow A$ is a split embedding, so $\text{Prop}_{\neg\neg}$ is essentially \mathcal{U} -small.

By large completeness, $\text{Prop}_{\neg\neg}^- : \mathbf{Set} \rightarrow \mathbf{Set}$ has an initial algebra $A \cong \text{Prop}_{\neg\neg}^A$, contradicting Cantor's diagonalisation argument.

Hence $f(0) = f(1)$. Similarly, $f(1) = f(X^0) = f(X^1) = f(X)$. □

Conjecture

$\text{PA}_{\mathcal{U}}$