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Models for Axiomatic Type Theory

Daniël Otten and Matteo Spadetto

Contents

We explain and motivate Axiomatic Type Theory (ATT). (type theory without reductions)

We compare two semantics for a minimal version of ATT:

- comprehension categories: more traditional and well-studied closely follow the syntax and intricacies of type theory.
- path categories (Van den Berg, Moerdijk 2017): more concise take inspiration from homotopy theory.

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Our Contributions

Path categories are equivalent to certain comprehension categories. This allows us to turn path categories into actual models as well.

We introduce a more fine-grained notion: display path categories, and show a similar equivalence.

We obtain the following diagram of 2-categories:

DisplayPathCat $\stackrel{\sim}{\longrightarrow}$ ComprehensionCat_{Contextual.=}

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- Extensional Type Theory (ETT): everything is definitional,
- Axiomatic Type Theory (ATT): nothing is definitional.

Larger

If we define

$$0+n \equiv n,$$

 $(Sm)+n \equiv S(m+n),$
 $m+0 = m,$
 $m+(Sm)=S(m+n),$

then we can prove

But these proven eq are not definitional.

Agda allows you to make them definitional.

- Cubical Type Theory: only propositional β -rule for =-types.
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Complexity and Conservativity

The complexity of type checking:

■ ETT: undecidable

Axiomatic Type Theory

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- ITT: nonelementary,
- ATT: quadratic

Does ETT prove more than ATT? Yes, namely

- binder extensionality (bindext).
- uniqueness of identity proofs (uip)

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$$\begin{array}{c} \Gamma, x, x': A, p: x =_A x' \vdash C \text{ type} \\ \frac{\Gamma, x: A \vdash d: C[x/x', \operatorname{refl}_x/p]}{\Gamma, x, x': A, p: x =_A x' \vdash \operatorname{ind}_{C,d,p}^{\equiv}: C} (=\mathcal{E}), \end{array}$$

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Minimal ATT

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Minimal ATT

Lets start by considering the normal rules for =-types:

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Without Π -types, we have to strengthen the rules:

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Minimal ATT

In ATT, we change the reduction to an axiom:

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Axiomatic Type Theory

How do we model this minimal ATT

- Follow the syntax and rules. (comprehension category)
 - We require: $=_A$, refl_A, ind $^=_{C,c,v}$, and $\beta^=_{G,c,v}$.
- Use intuition from homotopy theory. (path category)
 - $\,\,\,\,\,\,\,\,\,$ We require: $\,=_A$, refl $_A$, and that refl $_A$ is an equivalence

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- Use intuition from homotopy theory. (path category)
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- lacksquare a category of contexts with terminal object ϵ
- a category of types
- for every type A a context map $p_A : \Gamma.A \to \Gamma$. (display map)
- for every type A in context Γ and context map $\sigma:\Delta\to\Gamma$, a type $A[\sigma]$ in context Δ . (substitution
- satisfying some universal properties.

The terms of A are the maps $a:\Gamma\to\Gamma.A$ such that $p_A\circ a=\mathrm{id}_{\Gamma.A}$

- =-types: for A a type $=_A$ and terms refl_A , $\operatorname{ind}_{A,C,d}^=$, $\beta_{A,C,d}^=$
- weak stability: for σ we have that $=_A[\sigma]$ is also an =-type

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To model ATT, we need choices that are split:

$$A[\mathrm{id}_{\Gamma}] = A,$$

$$A[\tau \circ \sigma] = A[\sigma][\tau]$$

And strongly stable

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A path category is a category C with two classes of maps:

- fibrations: closed under pullbacks and compositions,
- (weak) equivalences: satisfying 2-out-of-6, so, if we have

$$A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \stackrel{h}{\longrightarrow} D$$

where $g \circ f$ and $h \circ g$ are equivalences, then f, g, h, and $h \circ g \circ h$ are equivalences.

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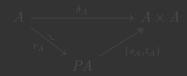
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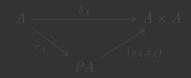
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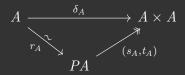
Path Objects

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Homotopy Theory

We call two maps $f,g:A\to B$ homotopic, written $f\simeq g$, if there exists a map $h:A\to PB$ such that $s_B\circ h=f$ and $t_B\circ h=g$.

We call $f:A\to B$ an homotopy equivalence, if there exists a map $g:B\to A$ such that $g\circ f\simeq \mathrm{id}_A$ and $f\circ g\simeq \mathrm{id}_B$.

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Path Category → **Comprehension Category**

We can view a path category \mathcal{C} as a comprehension category:

- the contexts are given by C
- the types are given by the full subcategory $\mathcal{C}^{\mathsf{fib}} \subseteq \mathcal{C}^{\to}$,
- the display map for $p \in \mathcal{C}^{\mathrm{fib}}$ is p itself,
- the substitution $p[\sigma]$ is the pullback σ^*p .

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- weakly stable Σ -types with β and η reductions,
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For a type A we define

$$=_A \coloneqq (s_A, t_A) : P_A \twoheadrightarrow A \times A, \qquad \qquad \text{(formation)}$$

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The elimination and β -axiom follow from our lifting theorem and the fact that r_A is an equivalence.

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We can turn a comprehension category C with weakly stable = $\sum_{\beta,n}$ and contextuality into a path category by taking:

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In a display path category we distinguish $\Gamma.A$ and $\Gamma.A_0.\ldots.A_{n-1}$.

Instead of fibrations we use display maps as a primitive notion.

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In addition, we replace path objects for objects Γ with a seemingly weaker notion: path objects for display maps $A \to \Gamma$.

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We obtain the following diagram of 2-categories

PathCat
$$\stackrel{\sim}{\longrightarrow}$$
 ComprehensionCat<sub>Contextual,=,\Sigma_{\beta\eta}}
$$U \uparrow + \downarrow C \qquad \qquad F \uparrow + \downarrow U$$</sub>

- Can we simplify other type formers as we did with =-types?
- In particular, are propositional Σ -types and Π -types homotopical left and right adjoints of pullback.
- Connect with (Maietti 2005) and (Clairembault, Dybjer 2013).

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