On left adjoints preserving colimits in HoTT

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Goals

1. See whether left adjoints preserve colimits in wild categories.
2. Find a reasonably nice sufficient condition for it to hold.
3. Apply this condition to $\Sigma \dashv \Omega$.

   Use a higher version of *Cavallo's trick* to enable mechanization in Book HoTT.

Why?

- Originally, show that pointed colimits preserve acyclic types.
- Construct colimits in various wild categories of higher groups by describing them as reflective subcategories.
- Simplify the construction of stable homotopy as a homology theory.

The classical proof

Consider a diagram $F : \mathcal{J} \to \mathcal{C}$ with a colimit $T \coloneqq \operatorname{colim}_{\mathcal{J}}(F)$.

Short and sweet:

$$
\begin{aligned}
& \hom_{\mathcal{D}}(L(T), Y) \\
\cong\ & \hom_{\mathcal{C}}(T, R(Y)) \\
\cong\ & \lim_i(\hom_{\mathcal{C}}(F_i, R(Y))) \\
\cong\ & \lim_i(\hom_{\mathcal{D}}(L(F_i), Y))
\end{aligned}
$$

This is *almost* the universal property of the colimit.

Need to ensure **the composite equals the canonical function**.

Not guaranteed to hold for *wild categories*.

The wild setting

A *wild category* is a pre-category except with untruncated hom-types.

Suppose $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ are functors of wild categories.

Suppose $L \dashv R$:

- a family of hom-equivalences

$$\alpha \ : \ \prod_{X:\mathrm{Ob}(\mathcal{D})} \prod_{A:\mathrm{Ob}(\mathcal{C})} \mathrm{hom}_{\mathcal{D}}(LA, X) \xrightarrow{\simeq} \mathrm{hom}_{\mathcal{C}}(A, RX)$$

- proofs V_1 and V_2 of the naturality of α in X and A, respectively.

Let Γ be a graph and a diagram $F : \Gamma \to \mathcal{C}$.

Consider a cocone

$$\begin{tikzcd} F_i \arrow[rr, "F_{i,j,g}"] \arrow[dr, "r_i"'] & {} \arrow[d, phantom, "K_{i,j,g}"] & F_j \arrow[dl, "r_j"] \\ & T & \end{tikzcd}$$

under F.

Suppose the cocone (T, r, K) is colimiting.

Replaying the standard proof

We still have the chain of equivalences

$$\begin{aligned} & \hom_{\mathcal{D}}(L(T), Y) \\ \cong\ & \hom_{\mathcal{C}}(T, R(Y)) \\ \cong\ & \lim_i(\hom_{\mathcal{C}}(F_i, R(Y))) \\ \cong\ & \underbrace{\lim_i(\hom_{\mathcal{D}}(L(F_i), Y))}_{\substack{\textit{type of cocones} \\ \textit{on } Y \textit{ under } L(F)}} \end{aligned}$$

Problem: This composite need not be post-composition.[1]

- Legs of the cocones are still equal.
- The triangle homotopies may be different.

[1] See the abstract for a counterexample based on the H-space S^1.

A sufficient condition

Our definition of *adjunction* is fine for 1-categories but not coherent enough for wild categories.

Nothing about the interaction between

- the naturality sqaures of the adjunction
- the equational axioms of the categories and functors.

We need a condition on this interaction to make *composite* = *post-comp*.

We say that L is 2-*coherent* if the diagram

$$\begin{tikzcd}
(\alpha(h_1) \circ h_2) \circ h_3 \arrow[r, equal, "\mathsf{assoc}(\alpha(h_1),h_2,h_3)"] \arrow[d, equal, "\mathsf{ap}_{-\circ h_3}(V_2(h_2,h_1))"'] & \alpha(h_1) \circ (h_2 \circ h_3) \arrow[d, equal, "V_2(h_2 \circ h_3,h_1)"] \\
\alpha(h_1 \circ L(h_2)) \circ h_3 \arrow[d, equal, "V_2(h_3,h_1 \circ L(h_2))"'] & \alpha(h_1 \circ L(h_2 \circ h_3)) \arrow[d, equal, "\mathsf{ap}_\alpha(\mathsf{ap}_{h_1 \circ -}(L_\circ(h_2,h_3)))"] \\
\alpha((h_1 \circ L(h_2)) \circ L(h_3)) \arrow[r, equal, "\mathsf{ap}_\alpha(\mathsf{assoc}(h_1,L(h_2),L(h_3)))"'] & \alpha(h_1 \circ (L(h_2) \circ L(h_3)))
\end{tikzcd}$$

commutes for all suitable morphisms h_1, h_2, and h_3.

Theorem

If L is 2-coherent, then $(L(T), L(r), L(K))$ is colimiting in $\mathcal{D}$.

Suspension is 2-coherent

Goal: Show that $\Sigma : \mathcal{U}^* \to \mathcal{U}^*$ is a 2-coherent left adjoint to Ω.

The SIP turns 2-coherence into a *(pointed) homotopy between pointed homotopies*:

Definition

Let f_1 and f_2 be pointed maps and let $(H_1, \kappa_1), (H_2, \kappa_2) : f_1 \sim_* f_2$.

A *homotopy between* (H_1, κ_1) *and* (H_2, κ_2) consists of

- a homotopy $\mu : H_1 \sim H_2$
- a path $M_\mu : \kappa_1 =_\mu \kappa_2$ over μ.

In the case of Σ,

- μ: messy but doable
- M_μ: *real* nasty.

But we're landing in a loop space, which is *strongly homogeneous*.[2]

Lemma (*yaCt*)

Let $f_1, f_2 : X_1 \to_* X_2$ with X_2 strongly homogeneous.

Let $(H_1, \kappa_1), (H_2, \kappa_2) : f_1 \sim_* f_2$. If $H_1 \sim H_2$, then (H_1, κ_1) and (H_2, κ_2) are homotopic.

Result: We ignore M_μ and are done!

[2] A pointed type is *strongly homogeneous* if it's homogeneous such that the automorphism is the identity for the basepoint.

Future work

- A trick for showing that $\wedge : \mathcal{U}^* \to \mathcal{U}^*$ is 2-coherent?

- Show that all modalities on $\mathcal{U}$ satsisfy 2-coherence (not hard).

- Show that all reflective subuniverses of $\mathcal{U}$ satisfy 2-coherence.

  "For any reflective subuniverse, we can prove all the familiar facts about reflective subcategories from category theory, in the usual way" (*The HoTT Book*, p. 248).

  This seems non-obvious for preservation of colimits.

Conclusion

Takeaway: Left adjoints preserve colimits under a reasonable condition, which Σ satisfies.

Agda code: `https://github.com/PHart3/colimits-agda`

Thanks!