

A refinement of Gabriel-Ulmer duality

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Essentially algebraic theories and *generalized algebraic theories* provide an extension of the notion of algebraic theory that is fundamental to the semantics of type theory since the doctrines in which we interpret type theories – such as the (2-)category of CwFs – are themselves categories of models of essentially algebraic theories. This insight is sometimes summarized by the slogan ‘type theory is algebraic’.

Cartmell’s generalized algebraic theories – which extend algebra by type dependency – and Freyd’s essentially algebraic theories – which permit a controlled form of partiality – are commonly recognized as being equally expressive, and are both subsumed by the less syntactic and more abstract *finite-limit theories*, which are simply small finite-limit categories.

More specifically, for each generalized/essentially algebraic theory \mathcal{T} there exists a small finite-limit category \mathbb{C} such that

$$\mathrm{Mod}(\mathcal{T}) \simeq \mathbf{Lex}(\mathbb{C}, \mathbf{Set}),$$

i.e. the category of models of \mathcal{T} is equivalent to the category of finite-limit-preserving (‘lex’) functors from \mathbb{C} to the category of sets.

Classical Gabriel-Ulmer duality states that the categories of models of such theories are precisely the *locally finitely presentable categories*, giving rise to a biequivalence

$$\mathbf{Lex} \simeq \mathbf{LFP}$$

between the 2-categories of finite-limit categories and locally finitely presentable categories.

We propose a refinement of this duality based on Taylor/Joyal’s notion of clan.

0.1 Definition A *clan* is a small category \mathcal{T} with terminal object 1 equipped with a class \mathcal{D} of *display maps* such that pullbacks of display maps along arbitrary maps exist and are again display maps, and display maps contain isomorphisms and terminal projections and are closed under composition.

A *model* of a clan \mathcal{T} is a functor $A : \mathcal{T} \rightarrow \mathbf{Set}$ which preserves 1 and pullbacks of display maps. \diamond

Clans refine finite-limit theories in that the ‘same’ finite-limit theory can be represented by different clans, thus we cannot expect to reconstruct a clan from its category of models alone.

To recover a duality we equip the category $\text{Mod}(\mathcal{T})$ of models of \mathcal{T} with additional information in form of a weak factorization system $(\mathcal{E}, \mathcal{F})$ which is cofibrantly generated by the morphisms of models induced by display maps¹.

Our main result is a way to recover \mathcal{T} from $\text{Mod}(\mathcal{T})$ and $(\mathcal{E}, \mathcal{F})$ up to Cauchy-completion, and a conjectured characterization of those locally finitely presentable categories equipped with weak factorization systems that arise as models of clans.

Moreover we showcase a major advantage of clans over finite-limit theories, which is that different clanic representations of the same finite-limit theory when interpreted in higher types give rise to different ‘homotopy-coherent generalizations’ of the set-level algebraic structure. This way we can e.g. obtain a ‘higher algebraic’ description of a $(2, 1)$ -category of CwFs, which is relevant to non-strict initiality results.

¹The same w.f.s. construction was independently proposed by Simon Henry in a recent HoTTTEST seminar.