# Syntax for two-level type theory

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## 1 Introduction

In homotopy type theory [13] (HoTT), properties that are not invariant under homotopy cannot be expressed internally. An important case is the concept of semisimplicial types, whose definition is so far elusive in HoTT. Voevodsky defined a special Homotopy Type System [14] (HTS) as a formal theory which allows constructions that require access to non-homotopy-invariant notions. Two-Level Type Theory [2] (2LTT) is envisioned to be a variant of HTS, and is composed of two separate levels of types: the outer level is Martin-Löf type theory plus the uniqueness of identity proofs [12] (UIP); the inner level is HoTT. These levels are related by a conversion function from the inner to the outer level that preserves context extensions.

The paper [2] proposes a semantics for 2LTT based on categories with families [7], which justifies reasoning *inside* the inner system with the full power of HoTT, and reasoning *about* the inner system within the outer system to circumvent a number of expressive limits of the former. With this approach it is possible to study properties of HoTT syntactically in the two-level system, and, by conservativity [4], to reflect them back in the HoTT world. Among the applications of this approach are results on Reedy fibrant diagrams [2], the Univalence Principle [1], and internal $\infty$-categories with families [8], which have been suggested as a way to overcome known difficulties one encounters when formalising type theory in type theory. In summary, despite the intrinsic expressive and proving power of HoTT, a wide range of results rely on meta-reasoning and meta-principles, which cannot entirely be formalised within the theory. The two-level approach formalises these meta-principles in a theory which is compatible both technically and philosophically with HoTT, allowing for their mechanisation. However, the syntax of 2LTT is just sketched in [2] and its proof theory is still largely unexplored.

## 2 Syntax

In this contribution, we propose a system of inference rules for 2LTT with an infinite hierarchy of Tarski-style universes as uniform constructions [10]; the rules allow us to define the syntax in detail, clearly illustrating the behaviour of the two levels, and how they interact. In contrast to [2], we pay particular attention to the definition of Tarski-style universes, following the guidelines of [10]: other than the function $\mathsf{El}_i$, which maps the codes $A : \mathcal{U}_i$ into types $\mathsf{El}_i(A)\,\mathsf{type}$ and is present in [2], we introduce a function $\mathsf{lift}_i$ mapping terms of one universe $A : \mathcal{U}_i$ into terms of the next one, $\mathsf{lift}_i(A) : \mathcal{U}_{i+1}$. In [2], the $\mathsf{lift}$ operation is not present, and the universes are *cumulative*. In our system those two functions commute: if $\Gamma \vdash A : \mathcal{U}_i$, then $\Gamma \vdash \mathsf{El}_{i+1}(\mathsf{lift}_i(A)) \equiv \mathsf{El}_i(A)\,\mathsf{type}$. The same happens for inner types; indeed, $A\,\mathsf{type}$ means that $A$ is an outer type, while $A\,\mathsf{type}^o$ means that $A$ is an inner type. This emphasises another difference between our approach and the 2LTT paper: we do not have a *size* for types; on the contrary, in [2] it is specified as $A\,\mathsf{type}_i$ or $A\,\mathsf{type}^o_i$: if $A : \mathcal{U}_i$, then $\mathsf{El}_i(A)\,\mathsf{type}_i$. Moreover, besides the conversion function $c$ from inner to outer types introduced in [2], we define a conversion function $c'$ from inner to outer codes, i.e., terms of the universes: if $A : \mathcal{U}_i^o$, then $c'(A) : \mathcal{U}_i$. It is required that $\mathsf{El}, \mathsf{lift}, c$ and $c'$ commute. We formalise the fact that the conversion function preserves context extension by introducing a notion of equivalence between contexts together with the rule

$$\frac{\Gamma \vdash A\ \mathsf{type}^o}{\Gamma, x : A\ \mathsf{ctx} \equiv \Gamma, y : c(A)\ \mathsf{ctx}}\ \equiv\mathsf{-ctx-EXT}$$

Then, we define a generalisation of the notion of category with families which allows us to interpret our formalisation of the two levels and the Tarski-style universes, called *two-level model*, together with a notion of morphism between models. We plan to show the compatibility of our system with the (almost) standard semantics for 2LTT by proving an initiality result; this will essentially extend recent work for Martin-Löf type theory by Brunerie, de Boer, Lumsdaine, and Mörtberg [3, 9, 6]. We define the syntactical two-level model by quotienting the syntax, similar to [11, 5], and prove that it is the initial object in the category of models.

Our long term goal is to develop the basis for a proof assistant that implements 2LTT and allows one to use additional inner and outer axioms, some of which have been already suggested [2], to formalise in parallel the inner and outer levels, and their relations.

## References

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