## Karatsuba's algorithm for polynomials over rings without decidable equality

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Workshop on Homotopy Type Theory/ Univalent Foundations Vienna, 23.04.2023

## A brief history of fast multiplication

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- 1962: Kolmogorov publishes the result under Karatsuba's name: Karatsuba and Ofman [1962]

## Karatsuba for polynomials I

Following [Abdeljaoued and Lombardi, 2004, Sec. 6.1]:

$$p(x) = a_0 + a_1 x + \dots + a_n x^n$$

$$= \underbrace{p_e}_{a_0 + a_2 x + \dots} (x^2) + x \cdot \underbrace{p_o}_{a_1 + a_3 x + \dots} (x^2)$$

 $\Rightarrow$  we can write  $p(x) \cdot q(x)$  as:

$$p_e q_e(x^2) + x \cdot [(p_e + p_o)(q_e + q_o) - p_e q_e - p_o q_o](x^2) + x^2 \cdot p_o q_o(x^2)$$

## Karatsuba for polynomials II

```
Algorithm: karatsuba
Input: polynomials p, q
if p or q constant then
    scalar multiplication with constant
    -- remark: p = a_0 \Rightarrow p = p_e
else
    r \leftarrow \text{karatsuba } p_e \ q_e
    s \leftarrow \text{karatsuba } p_o \ q_o
    t \leftarrow \text{karatsuba} (p_e + p_o) (q_e + q_o)
    return r(x^2) + x \cdot [t - r - s](x^2) + x^2 \cdot s(x^2)
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- $\Rightarrow$  3 recursive calls with input roughly half the size
- $\Rightarrow$  runs in  $\mathcal{O}(n^{\log_2 3}) \approx \mathcal{O}(n^{1.585})$  instead of  $\mathcal{O}(n^{\log_2 4}) = \mathcal{O}(n^2)$  time

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- karatsuba $\operatorname{Rec}(n+1) p q := \operatorname{above algorithm} w / \operatorname{recursive calls}$ 
  - karatsubaRec n p<sub>e</sub> q<sub>e</sub>
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Today: stick with this solutions

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Today: Use HITs to get best of both worlds! (see this PR)

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## A HIT for Polynomials

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One can prove (without assuming that R has decidable equality):

- R[X] has a commutative ring structure (See bachelor's thesis of Åkerman Rydbeck [2022])
- It has the universal property of the polynomial ring

## Karatsuba in Cubical Agda

Because of drop0, need **mutually recursive** definition:

```
karatsubaRec : \mathbb{N} \to R[X] \to R[X] \to R[X]
karatsubaRec\equiv : \forall (n : \mathbb{N}) (p \ q : R[X]) \to \text{karatsubaRec} \ n \ p \ q \equiv p \cdot q
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#### Because of drop0, need **mutually recursive** definition:

karatsubaRec zero  $p q = p \cdot q$  -- base case

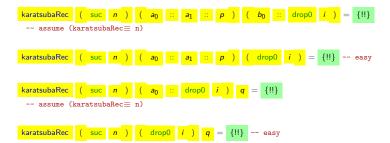
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```

#### Base algorithm as presented:

```
karatsubaRec (suc n) [] q = \{!!\} -- mult. w/ 0 karatsubaRec (suc n) [ a_0 ] q = \{!!\} -- mult. w/ scalar a_0 karatsubaRec (suc n) (a_0 :: a_1 :: p) [] = \{!!\} -- mult. w/ 0 karatsubaRec (suc n) (a_0 :: a_1 :: p) [ b_0 ] = \{!!\} -- mult. w/ scalar b_0 karatsubaRec (suc n) (a_0 :: a_1 :: p) (b_0 :: b_1 :: q) = \{!!\} -- rec. call
```

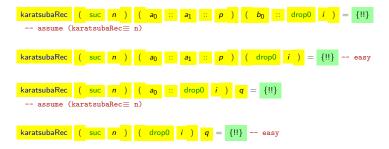
## Filling in the paths

#### For the higher constructor:



## Filling in the paths

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Then, since  $\forall n \ p \ q \rightarrow \text{isProp (karatsubaRec} \equiv n \ p \ q)$   $\Rightarrow$  only cumbersome but straightforward algebra remains

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#### But we can define something like length:

```
truncLength : R[X] \rightarrow || N ||
truncLength [] = | 0 |
truncLength (a :: p) = map suc (truncLength p)
truncLength (drop0 i) = squash | 1 | | 0 | i
```

## Factor through $|\underline{\phantom{a}}|: \mathbb{N} \to ||\mathbb{N}||$

## Lemma (Kraus [2015])

Let A be a type and B a set. Given a function  $f: A \to B$  with a proof of

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Now, we have

- isSet  $(R[X] \rightarrow R[X] \rightarrow R[X])$
- $\forall n \ m \rightarrow \text{karatsubaRec} \ n \equiv \underline{\phantom{a}} \cdot \underline{\phantom{a}} \equiv \text{karatsubaRec} \ m$

And thus get

 $\mathsf{karatsubaTruncRec}: \parallel \mathbb{N} \parallel \to \mathsf{R}[\mathsf{X}] \to \mathsf{R}[\mathsf{X}] \to \mathsf{R}[\mathsf{X}]$ 

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**Exercise**: Define a different karatsubaTruncRec' using the contractibility of the singleton type of  $\_\cdot\_$ .

How does their computational behavior compare?

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## Putting it all together

```
karatsuba : R[X] \rightarrow R[X] \rightarrow R[X]
karatsuba p q = let fuel = map2 max (truncLength p) (truncLength q) in karatsubaTruncRec fuel p q
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in karatsubaTruncRec fuel \ p \ q
```

#### Exercises/sanity checks:

- Convince yourself that this computes as it should (on list polynomials without drop0)
- ② Prove karatsuba ≡ \_-\_

- Implement a provably correct version of some algorithm
  - ▶ In our setting: *Fuel* for the termination checker
  - Bound for fuel depends on implementation of input data

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#### Future Work:

- make a runable program using --erased-cubical
- 2 more algorithm examples

## Thank You

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