Choice axioms and Postnikov completeness [1]

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The talk is going to be in the language of category theory, not HoTT.

By a category I mean an $(\infty, 1)$-category.

By a n-topos I mean an n-topos in the sense of Lurie.

By a surjection in an n-topos, I mean an effective epimorphisms in the sense of Lurie
($\Leftrightarrow$ a map left orthogonal to monomorphisms
$\Leftrightarrow$ a map which is the quotient of its associated equivalence relation).

An object X in an n-topos is inhabited if the map $X \to 1$ is a surjection.

Choice axioms

In a 1-topos, the axiom of countable choice (CC) can be stated as

a countable product of inhabited objects is inhabited.

$$
\begin{tikzcd}
A \arrow[d, two heads, "Surj"] & & \prod_n A_n \arrow[d, two heads, "Surj"] \\
\mathbb{N} & \Longrightarrow & 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
& A \arrow[d, two heads, "Surj"] \\
\mathbb{N} \times U \arrow[ur, dashed, "\exists"] \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{N} \arrow[d] \\
U \arrow[r, dashed, two heads, "\exists", "Surj"'] & 1
\end{tikzcd}
$$

Choice axioms

Examples

1. Set.
2. $Set_{/I} = Set^I$ for a set I.
3. Set^G for a group G.
4. $Sh(\text{Cantor})$.
5. $Sh(\text{Alexandrov})$ (opens are closed under arbitrary intersections).
6. any 1-topos where $\mathbb{N}$ is projective
   (the axiom is in fact equivalent to $\mathbb{N}$ being projective).

Counterexamples : $Sh([0,1])$

Choose a cover by open sets $U_{k,n}$ of diameter less than $1/n$,
then each $A_n = \coprod_k U_{k,n} \to [0,1]$ is inhabited,
but $\prod A_n = \varnothing$ since it has no local sections.

n-topoi

The same axiom makes sense in an n-topos for $1 \leq n \leq \infty$.

But in this context, it is natural to introduce some homotopical variations of CC.

Need recollections/notations on m-connected maps.

Truncation modalities

The diagonal of a map $f : A \to B$ is $\Delta(f) : A \to A \times_B A$ (family of identity types of B indexed by A).

We put $\Delta^0(f) := f$ and $\Delta^{k+1}(f) := \Delta(\Delta^k(f))$.

A map $f : A \to B$ is m-truncated (T_m) if $\Delta^{m+2}(f)$ is an isomorphism.

(-1)-truncated = monomorphism.

The subcategory of m-truncated objects is $E^{\leq m} \subset E$.

It is reflective.

It is an $(m+1)$-topos.

Truncation modalities

A map $f : A \to B$ is *m-connected* (C_m) if $\Delta^k(f)$ is a surjection for every $0 \leq k \leq m+1$.

(-1)-connected = surjection.

The pair $(\mathsf{C}_m, \mathsf{T}_m)$ form a modality (a factorization system stable under base change)

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathsf{C}_m"'] & & B \\ & C \arrow[ur, "\mathsf{T}_m"'] & \end{tikzcd}$$

∞-Truncation modality

A map is ∞-connected (C_∞) if it is m-connected for every m.

A map is ∞-truncated (T_∞) if it is right orthogonal to ∞-connected maps. (Lurie: hypercomplete)

In an ∞-topos E, the $(\mathrm{C}_\infty, \mathrm{T}_\infty)$ form a modality.

The subcategory of ∞-truncated objects is $E^{\leq\infty} \subset E$.

It is reflective, and the reflection is the hypercompletion of $E \to E^{\leq\infty}$.

It is an ∞-topos.

Homotopical choice axioms

Definition

For $-1 \leq d \leq \infty$, the axiom of countable choice of dimension $\leq d$ ($\mathrm{CC_d}$) holds in an n-topos E if

a countable product of $(d-1)$-connected objects is inhabited.

Lemma

TFAE

1. $\mathrm{CC_d}$.
2. *For every $-1 \leq n \leq \infty$, a countable product of $(n+d)$-connected objects is an n-connected.*
3. *Countable products of d-connected maps are surjections.*
4. *For every $-1 \leq m \leq \infty$, countable products of $(m+d)$-connected maps are m-connected.*

$$\mathrm{CC}_{-1} \Rightarrow \mathrm{CC}_0 \Rightarrow \mathrm{CC}_1 \Rightarrow \ldots \Rightarrow \mathrm{CC}_\infty$$

Logical interpretation

Surjection = semantics for existential conditions $\exists$.

Isomorphism = semantics for unique-existential conditions $\exists!$.

n-connected maps = semantics for intermediate existential conditions $\exists_n$ (= unicity up to n-truncation = iterated existential condition on all identity types of level $\leq n$).

∞-connected maps = semantics for iterated existential conditions on all identity types $\exists_\infty$ (notice that $\exists_\infty \neq \exists!$).

$$\begin{array}{ccccccccccc} \mathit{Iso} & \subset & C_\infty & \subset & \dots & \subset & C_1 & \subset & C_0 & \subset & C_{-1} = \mathit{Surj} \\ \exists! & \Rightarrow & \exists_\omega & \Rightarrow & \dots & \Rightarrow & \exists_1 & \Rightarrow & \exists_0 & \Rightarrow & \exists_{-1} = \exists\,. \end{array}$$

Operations sending $(n+d)$-connected maps to n-connected maps = operations weakening the 'unicity level' of existential conditions.

Homotopical choice axioms

Examples

1. Only the trivial n-topos $E = 1$ has CC_{-1}.
2. $S^{\leq n}$ has CC_0 (in fact, *any discrete product* of inh. obj. is inh.).
3. $[C, S^{\leq n}]$ for any category (or space) C has CC_0.
4. In a 1-topos CC_1 is always true (0-connected maps are iso).
5. In an n-topos CC_n is always true (n-connected maps are iso).
6. Not every ∞-topos has CC_∞ (see counterexample below).
   But every hypercomplete ∞-topos has (trivially) CC_∞.
7. If CC_d holds in E it holds in $E^{\leq n}$ for $0 \leq n \leq \infty$ (in particular in the hypercompletion). The converse is false.

An ∞-topos without CC_∞

The notion of ∞-connected objects is geometric.

$$X : \mathrm{C}_\infty \quad \Longleftrightarrow \quad \forall n,\ \Delta^n X : Surj\,.$$

The ∞-topos classifying ∞-connected objects is

$$S[X^{(\infty)}] \quad = \quad [Fin, S]^{\text{atomic top.}}\,.$$

The universal ∞-connected object $X^{(\infty)}$ is the sheafification of the canonical inclusion $Fin \to S$.

Evaluation at $X^{(\infty)}$ induces an equivalence of categories

$$\big[S[X^{(\infty)}], E\big]^{lex}_{cc} \quad = \quad \{X \in E \mid X \text{ is } \infty\text{-connected}\} \ \subset \ E\,.$$

Fact:
There exists ∞-connected objects with no global sections (next slide).
Thus $X^{(\infty)} : Fin \to S$ must verify

$$X^{(\infty)}(\varnothing) = \varnothing\,.$$

An ∞-topos without CC_∞

How to get an ∞-connected objects with no global sections.

Take an ∞-connected X in E which is not contractible.

There exists Z such that $\mathrm{Map}\,(Z, X) \neq 1$ in S.

One the π_ns of $\mathrm{Map}\,(Z, X)$ must be have two different elements.

There exists n, such that $\mathrm{Map}\,(Z \times S^n, X)$ has 2-connected components.

In $E' := E_{/Z \times S^n}$, the ∞-connected object $X' := \ (X \times Z \to Z)$ has two different global sections a and b.

The path object $\Omega_{a,b}X$ is ∞-connected in E' but with no global sections.

An ∞-topos without CC_∞

Consider the ∞-topos classifiying a countable number of ∞-connected objects

$$S[X_1^{(\infty)}, X_2^{(\infty)}, \dots] = \left[Fin^{(\mathbb{N})}, S \right]^{\text{at. top. in each var.}}$$

where $Fin^{(\mathbb{N})}$ is the free cocompletion of the set $\mathbb{N}$ ($= \mathbb{N} \to Fin$ whose values are almost all $\varnothing$).
Then

$$\prod_n X_n^{(\infty)} = \varnothing.$$

Proof:

$$X_n^{(\infty)} = Fin^{(\mathbb{N})} \xrightarrow{p_n} Fin \xrightarrow{X^{(\infty)}} S.$$

For $A \in Fin^{(\mathbb{N})}$, we have

$$\prod_n X_n^{(\infty)}(A) = X^{(\infty)}(A_1) \times X^{(\infty)}(A_2) \times \cdots \times X^{(\infty)}(A_k = \varnothing) \times \cdots = \varnothing\,.$$

Homotopical dimension

An object $X \in E$ is of homotopy dimension $\leq d$ if $\Gamma : E_{/X} \to S$ sends $(n+d)$-connected objects to n-connected objects.

An n-topos has enough objects of homotopy dimension $\leq d$ ($\mathrm{EOHD_d}$) if every object can be covered by objects of homotopy dimension $\leq d$.

Examples

1. objects of covering dimension ≤ 0 = externally projective objects
2. Any space of covering dimension $\leq d$ has $\mathrm{EOHD_d}$
   2.1 $[0,1]^d$ has $\mathrm{EOHD_d}$
   2.2 any d-manifold has $\mathrm{EOHD_d}$
3. the ∞-topos envelope of a 1-topos with $\mathrm{EOHD_0}$ has $\mathrm{EOHD_0}$ (don't know for $d > 0$)

Proposition (AB)

Any n-topos with enough objects of homotopy dimension $\leq d$ has $\mathrm{CC_d}$.

Homotopical choice axioms

More examples:

4. $Sh_\infty([0,1])$ has CC_1 but not CC_0.
5. $Sh_\infty\left([0,1]^d\right)$ has CC_d but not $\mathsf{CC}_{\mathsf{d-1}}$.
6. $Sh_\infty\left(\coprod_d [0,1]^d\right)$ has CC_∞ but not CC_d for $d < \infty$.

Proof that $Sh([0,1])$ has CC_1

$A_k \to [0,1]$ family of 0-connected sheaves, need to show that:
for every $x \in [0,1]$, there exists a neighborhood $x \in U$ such that every A_k has a section on U.

Gonna prove stronger result that every A_k has a global section.

A_k has local sections on a cover U_i of $[0,1]$.

Can refine U_i such that on U_{ij} the two local sections are connected by a homotopy.

Use these homotopies to build a section of A_k on $[0,1]$.

No need of coherence because can chose U_i without triple intersections ($= [0,1]$ is of covering dimension ≤ 1).

This shows each A_k has a global section.

This shows $\prod_k A_k$ has a global section and is therefore inhabited.

Similar for $[0,1]^d$ using that is of covering dimension $\leq d$.

Application

Theorem (AB)

If CC_{d} holds for $-1 \leq d < \infty$, then every formal Postnikov tower of E is the Postnikov tower of some object in E.

Will make things more precise.

Postnikov towers

$P_n : E \to E^{\leq n}$ reflection onto n-truncated objects

We have a tower of categories

$$E \longrightarrow E^{\leq \infty} \longrightarrow \ldots \xrightarrow{P_1} E^{\leq 2} \xrightarrow{P_0} E^{\leq 0} \xrightarrow{P_{-1}} E^{\leq -1}\,.$$

The category $\mathrm{Post}\,(E) = \lim_n E^{\leq n}$ is an ∞-topos.

The objects of $\mathrm{Post}\,(E)$ are formal Postnikov towers

$$\ldots \longrightarrow X_2 \xrightarrow{C_1} X_1 \xrightarrow{C_0} X_0 \xrightarrow{Surj} X_{-1}\,.$$

$P_\bullet : E \to \mathrm{Post}\,(E)$ sends an object X to its Postnikov tower

$$\ldots \longrightarrow P_2X \xrightarrow{C_1} P_1X \xrightarrow{C_0} P_0X \xrightarrow{Surj} P_{-1}X\,.$$

Postnikov towers

The functor $P_\bullet$ preserves colimits and finite limits. Its right adjoint is the limit of towers.

$$E \underset{\lim}{\overset{P_\bullet}{\rightleftarrows}} \operatorname{Post}(E)$$

1. E is Postnikov complete if $\lim : \operatorname{Post}(E) \to E$ is an equivalence.

2. E is Postnikov effective if $E \to \operatorname{Post}(E)$ is a localization
   $\Leftrightarrow$ $\lim : \operatorname{Post}(E) \to E$ is fully faithful
   $\Leftrightarrow$ every FPT is the PT of its limit

   $$P_k\left(\lim X_n\right) = X_k\,.$$

3. E is Postnikov convergent if $E \to \operatorname{Post}(E)$ is fully faithful.
   $\Leftrightarrow$ limit of PT of X is X

   $$X = \lim_n P_n X\,.$$

Application

Theorem (AB)

If $\mathrm{CC_d}$ *holds in* E *for* $-1 \leq d < \infty$*, then* E *is Postnikov effective: every FPT is the PT of its limit*

$$P_k\big(\lim X_n\big) = X_k\,.$$

Maps inverted by $E \to \mathrm{Post}\,(E)$ are exactly the ∞-connected maps.
Get a localization/conservative factorization

$$E \quad\longrightarrow\quad E^{\leq\infty} \quad\longrightarrow\quad \mathrm{Post}\,(E)\,.$$

Corollary (AB)

If $\mathrm{CC_d}$ *holds in* E *for* $-1 \leq d < \infty$*, then* $E^{\leq\infty}$ *is Postnikov complete.*

The theorem is false for $d = \infty$ because there exists hypercomplete topoi ($\Rightarrow \mathrm{CC}_\infty$) which are not Postnikov complete.

The converse of the theorem is false because the CC_∞ counter-example has S for his hypercompletion.

Proof

Let X_n be a FPT. We put $X = \lim X_n$

We want to prove that the projection $p_k : X \to X_k$ is k-connected. We have

$$\begin{tikzcd} X \arrow[d, "p_k"'] \\ X_k \end{tikzcd} = \lim_{\overleftarrow{\mathbb{N}}} \begin{tikzcd} \dots \arrow[r] & X_{k+2} \arrow[r] \arrow[d, "c_k"] & X_{k+1} \arrow[r] \arrow[d, "c_k"] & X_k \arrow[d, "c_k"] \\ \dots \arrow[r, equal] & X_k \arrow[r, equal] & X_k \arrow[r, equal] & X_k\,. \end{tikzcd}$$

$$= \lim_{\rightrightarrows} \begin{tikzcd} \prod_n X_{k+n} \arrow[r, shift left] \arrow[r, shift right, "="'] \arrow[d] & \prod_n X_{k+n} \arrow[d] \\ \prod_n X_k \arrow[r, equal] & \prod_n X_k\,. \end{tikzcd}$$

Proof

We have a countable product of k-connected maps

$$\begin{array}{c}\prod_n X_{k+n} \\ \Big\downarrow{\scriptstyle \prod_n \mathsf{C}_k \subset \mathsf{C}_{k-d}} \\ \prod_n X_k\end{array}$$

which is an $(k-d)$-connected map by $\mathrm{CC_d}$.

Thus the equalizer of

$$\begin{tikzcd}
\prod_n X_{k+n} \arrow[r, shift left] \arrow[r, shift right, "="'] \arrow[d, "\mathsf{C}_{k-d}"] & \prod_n X_{k+n} \arrow[d, "\mathsf{C}_{k-d}"] \\
\prod_n X_k \arrow[r, equal] & \prod_n X_k\,.
\end{tikzcd}$$

is an $(k-d-1)$-connected map.

Proof

The map $p_k : X = \lim X_n \to X_k$ is $(k - d - 1)$-connected.

The map $X_k \to X_{k-d-1}$ is $(k - d - 1)$-connected.

The map $X \to X_{k-d-1}$ is the $(k - d - 1)$-truncation of X.

Using that d is finite and chosing $k = d + 1 + n$,

we get that $X \to X_n$ is the n-truncation of X for every n.

$\square$

Thank you!

Bonus

In Blass' Cohomology detects failures of the axiom of choice, he considers the following statement:

If, for every set X and every group G, $\prod_X BG$ is connected, then, for every family of inhabited sets Y_x, $\prod_{x \in X} Y_x$ is inhabited

This suggest to consider the axiom $\mathsf{CC}_{\mathsf{d}}^{\geq \mathsf{b}}$

Countable products of $(d+b)$-connected objects are b-connected.

For $b = -1$, we have $\mathsf{CC}_{\mathsf{d}}^{\geq -1} = \mathsf{CC}_{\mathsf{d}}$.

For $b = 0$, Blass' theorem essentially says that

$$\mathsf{CC}_{0}^{\geq 0} \quad \Rightarrow \quad \mathsf{CC}_{0}$$

but the assumptions of his proof are not clear to me (in which 1-topoi is this true?).

This raises the question to find conditions implying

$$\mathsf{CC}_{\mathsf{d}}^{\mathsf{b}} \quad \Rightarrow \quad \mathsf{CC}_{\mathsf{d}}^{\mathsf{b}-1}.$$