Differential Cohesive Type Theory

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Overview

Want a syntax for

$$
\begin{array}{ccccccccc}
\Re & \dashv & \Im & \dashv & \& & & \\
 & & \cup & & \cup & & \\
 & & \int & \dashv & \flat & \dashv & \sharp
\end{array}
$$

Overview

- Modal Type Theory
- Simple Differential Cohesive Type Theory
- Towards Differential Cohesive Homotopy Type Theory

Modal Type Theory

- Why do we need Modal Type Theory?

No-Go Theorem for Comonads

(Shulman '15) Any internal comonadic modality is of the form

$$\Box A \cong A \times U$$

for some proposition U

No-Go Theorem for Comonads

(Shulman '15) Any **internal** comonadic modality is of the form

$$\Box A \cong A \times U$$

for some proposition U

The solution: comonads can't be applied in every context, so add a syntax for restricted contexts.

Change the Judgmental structure!

Modal Logic

Traditionally: encode different **modes** of “truth”

$$A : A \text{ is true in the current world}$$
$$\Box A : A \text{ is true in all possible worlds}$$
$$\diamond A : A \text{ is true in some possible world}$$

$$\frac{\varnothing \vdash A}{\Gamma \vdash \Box A} \qquad \frac{A \vdash \Diamond C}{\Diamond A \vdash \Diamond C}$$

Modal Type Theory

- Encode different **modes** of “proof” or construction:
  - Smooth variation
  - Continuous variation
  - Discontinuity

Real-Cohesive Homotopy Type Theory
Shulman ‘15

$$\int \dashv \flat \dashv \sharp$$

- Extends Homotopy Type Theory with an extra context of “discontinuous dependency”.
- flat, sharp defined using a modal type theory
- Shape is **defined** as localization at the reals.

Simple Differential Cohesive Type Theory

- A non-dependent type theory that includes all of the modalities of differential cohesion.
- Dependent type theory rules will be generalizations of these.

Simple DCTT

$$
\begin{array}{llll}
\text{Types} & A, B, C & ::= & \int A \mid \flat A \mid \sharp A \mid \Re A \mid \Im A \mid \& A \\
 & & & \mid A \to B \mid 1 \mid A \times B \mid X \\
\text{Modes} & m, n & ::= & \flat \mid \Re \mid 1 \mid \Im \mid \int
\end{array}
$$

compare real-cohesion which only has flat, 1.

Modes

$$\frac{\Gamma\ \mathsf{ctx} \qquad A\ \mathsf{type} \qquad m \in \{\flat, \Re, 1, \Im, \textstyle\int\}}{\Gamma, x :_m A\ \mathsf{ctx}}$$

Modes

Modes are the **left adjoints**, monadic or comonadic

$$\begin{array}{ccccccc} \Re & \dashv & \Im & \dashv & \& & & \\ & & \cup & & \cup & & \\ & & \int & \dashv & \flat & \dashv & \sharp \end{array}$$

Modes

Monads

$$\flat \leq \Re \leq 1 \leq \Im \leq \int$$

Comonads

Easier to Use $\longrightarrow$ Harder to Use

Modes

$$\flat \leq \Re \leq 1 \leq \Im \leq \int$$

$$\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash t : A$$

Alternative notation for contexts

Modes

$$\flat \leq \Re \leq 1 \leq \Im \leq \int$$

$$\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash t : A$$

Alternative notation for contexts

Meaning of the Modes

$$\Gamma, x :_m A \vdash t : B$$

denotes

$$[\![t]\!] : [\![\Gamma]\!] \times [\![m]\!][\![A]\!] \to [\![B]\!]$$

Meaning of the Modes

$$\Gamma, x :_m A \vdash t : B$$

$\flat$: is possibly discontinuous
$\Re$: possibly ignores infinitesimal extensions
1 : is smooth
$\Im$: has 0 derivatives
$\int$: is constant on each connected component

Variables

Variable rule is a counit

$$\frac{m \leq 1}{\Gamma, x :_m A \vdash x : A}$$

Variables

Can't directly use m > 1 vars

$$\frac{m \leq 1}{\Gamma, x :_m A \vdash x : A}$$

Variables

Modal Promotion

$$\frac{m \leq n \qquad \Gamma, x :_n A \vdash t : B}{\Gamma, x :_m A \vdash t : B}$$

Examples:

1. Locally constant functions have 0 derivative
2. Functions with 0 derivative are smooth
3. Smooth functions are “at most discontinuous”

Cut

Can only cut against restricted substitutions

$$\frac{\Gamma \vdash^{m} u : A \qquad \Gamma, x :_{m} A \vdash t : C}{\Gamma \vdash t[u/x] : C}$$

$$\frac{\Gamma \vdash^{m} u : A \qquad \Gamma, x :_{m} A \vdash t : C}{\Gamma \vdash t[u/x] : C}$$

Rule out substitutions like

$$\frac{y :_{1} B \vdash u : A \qquad x :_{\flat} A, y :_{1} B \vdash t : C}{y :_{1} B \vdash t[u/x] : C}$$

$$\frac{\Gamma \vdash^m u : A \qquad \Gamma, x :_m A \vdash t : C}{\Gamma \vdash t[u/x] : C}$$

Rule out substitutions like

$$\frac{y :_1 B \vdash u : A \qquad x :_\flat A, y :_1 B \vdash t : C}{y :_1 B \vdash t[u/x] : C}$$

DISALLOWED

u is smooth in y, but t is discontinuous in x, so the composition is not necessarily smooth in y

$$\frac{\Gamma \vdash^{m} u : A \qquad \Gamma, x :_{m} A \vdash t : C}{\Gamma \vdash t[u/x] : C}$$

$$\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash^{\flat} A :\equiv \Gamma \mid \cdot \mid \cdot \mid \cdot \mid \Xi \vdash A$$

$$\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash^{\Re} A :\equiv \Gamma \mid \Delta \mid \cdot \mid \cdot \mid \Xi \vdash A$$

$$\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash^{1} A :\equiv \Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash A$$

$$\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash^{\Im} A :\equiv \Gamma \mid \Lambda \mid \cdot \mid \cdot \mid \Xi \vdash A$$

$$\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash^{\int} A :\equiv \Gamma \mid \Xi \mid \cdot \mid \cdot \mid \cdot \vdash A$$

would not keep Xi in real-cohesion

Example: Reduction

$$\frac{\Gamma \mid \Delta \mid \cdot \mid \Theta, \Lambda \mid \Xi \vdash e : A}{\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash e^{\Re} : \Re A}\ \Re\mathrm{I}$$

Example: Reduction

$$\frac{\Gamma \mid \Delta \mid \cdot \mid \Theta, \Lambda \mid \Xi \vdash e : A}{\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash e^{\Re} : \Re A} \; \Re\mathrm{I}$$

Comonadic introduction: variables become *harder* to use

Example: Reduction

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \mid \Delta \mid \cdot \mid \Theta, \Lambda \mid \Xi \vdash t : A}{t : \flat\Gamma \times \Re\Delta \times \Im\Theta \times \Im\Lambda \times \textstyle\int\Xi \to A \qquad m \neq 1 \implies \Re m = m}}{\Re t : \flat\Gamma \times \Re\Delta \times \Im\Theta \times \Im\Lambda \times \textstyle\int\Xi \to \Re A \qquad\qquad 1 \leq \Im}}{\Re t \circ (\cdots) : \flat\Gamma \times \Re\Delta \times \Theta \times \Im\Lambda \times \textstyle\int\Xi \to \Re A}}{\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash t^{\Re} : \Re A}$$

Justified by functoriality

Example: Reduction

$$\frac{\Gamma \vdash^{m} u : \Re A \qquad \Gamma, x :_{m\Re} A \vdash t : C}{\Gamma \vdash \mathsf{let}\ x^{\Re} :_{m}= u\ \mathsf{in}\ t : C}\ \Re\mathrm{E}$$

- same semantics as substitution
- producing RA under different restrictions means you can use A at different modes

Example: Shape

how do we use monadically modal variables?
Monadic introduction rules **promote** variables

$$\frac{\Gamma \mid \Delta, \Theta, \Lambda, \Xi \mid \cdot \mid \cdot \mid \cdot \vdash e : A}{\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash e^{\int} : \int A} \int \mathrm{I}$$

Increased expressivity over cohesion
In cohesion, they would be promoted to 1
but in differential cohesion promoted to R

Example: Shape

$$
\begin{array}{c}
\Gamma \mid \Delta, \Theta, \Lambda, \Xi \mid \cdot \mid \cdot \mid \cdot \vdash t : A \\
\hline
t : \flat\Gamma \times \Re\Delta \times \Re\Theta \times \Re\Lambda \times \Re\Xi \to A \qquad \int\flat = \flat \qquad \int\Re = \Re \\
\hline
\int t : \flat\Gamma \times \int\Delta \times \int\Theta \times \int\Lambda \times \int\Xi \to \int A \qquad \Re \leq 1 \leq \Im \leq \int \\
\hline
\int t \circ (\cdots) : \flat\Gamma \times \Re\Delta \times \Theta \times \Im\Lambda \times \int\Xi \to \int A \\
\hline
\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash t^{\int} : \int A
\end{array}
$$

Example: Shape

$$\frac{\Gamma \mid \Delta, \Theta, \Lambda, \Xi \mid \cdot \mid \cdot \mid \cdot \vdash e : \int\!A \qquad \Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi, x : A \vdash e' : C}{\Gamma \mid \Delta \mid \Theta \mid \Lambda \mid \Xi \vdash \mathsf{let}\ x^{\int} := e\ \mathsf{in}\ e' : C}\ \int\!\mathrm{E}$$

Unlike Reduction, one mode of cut is optimal,
so we only need one rule.

Simple DCTT

- Linear order on modes makes it easy to pick a “best” mode for a rule.
- Of 12 intro/elim rules for modalities, only 2: reduction/co-reduction elimination need mode annotations

Complicated…

- 6 modalities, 5 modes, too complex to do “manually”

Fibrational Framework

A Fibrational Framework for Substructural and Modal Logics (extended version)

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Mode Theory
=>
Modal Type Theory

- Provide a Mode Theory:
  - 0-cells ~ Categories
  - 1-cells ~ Adjunctions
  - 2-cells ~ Morphisms of Adjunctions
- Get an “Assembly code” modal logic
  - simplify manually after

Fibrational Framework

Uniform rules for left adjoints

$$\frac{\Gamma, \Gamma', \Delta \vdash_{\beta[\alpha/x]} C}{\Gamma, x : \mathsf{F}_{\alpha}(\Delta), \Gamma' \vdash_{\beta} C}\ \mathsf{FL} \qquad \frac{\beta \Rightarrow \alpha[\gamma] \quad \Gamma \vdash_{\gamma} \Delta}{\Gamma \vdash_{\beta} \mathsf{F}_{\alpha}(\Delta)}\ \mathsf{FR}$$

Fibrational Framework

Uniform rules for right adjoints

$$\frac{x : \mathsf{U}_{x.\alpha}(\Delta \mid A) \in \Gamma \quad \beta \Rightarrow \beta'[\alpha[\gamma]/z] \quad \Gamma \vdash_{\gamma} \Delta \quad \Gamma, z : A \vdash_{\beta'} C}{\Gamma \vdash_{\beta} C}\ \mathsf{UL} \qquad \frac{\Gamma, \Delta \vdash_{\alpha[\beta/x]} A}{\Gamma \vdash_{\beta} \mathsf{U}_{x.\alpha}(\Delta \mid A)}\ \mathsf{UR}$$

Mode Theory

- Multiple possibilities for a mode theory
  - Split monads/comonads into adjunctions, relationship encoded by assembling these adjunctions in various ways, add in equalities for (co)-reflections.
    - 3 categories (Discrete, Smooth, Formal Smooth)
    - At least 9 functors.
  - Treat the monads/comonads *directly* as the adjoints of the theory, add in 2-cells to make them monads/comonads
    - 1 category (Formal Smooth)
    - 4 adjunctions

Mode Theory

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  - Split monads/comonads into adjunctions, relationship encoded by assembling these adjunctions in various ways, add in equalities for (co)-reflections.
    - 3 categories (Discrete, Smooth, Formal Smooth)
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  - Treat the monads/comonads *directly* as the adjoints of the theory, add in 2-cells to make them monads/comonads
    - **1 category (Formal Smooth)**
    - 4 adjunctions

Mode Theory

0	p
1	$\Re, \Im, \int, \flat : p \to p$ and a product that the above respect $\times : (p, p) \to p \quad \top : () \to p$
2	$\flat \Rightarrow \Re \Rightarrow 1 \Rightarrow \Im \Rightarrow \int$
eq	$m\flat = \flat$ $m\int = \int$ $m\Im = m$ $m\Re = m$

Mode Theory

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Every 1 cell generates left and right adjoint (named for left)

Mode Theory

0	p
1	$\Re, \Im, \int, \flat : p \to p$ and a product that the above respect $\times : (p, p) \to p \qquad \top : () \to p$
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Note: product means we function types too

Mode Theory

0	p
1	$\Re, \Im, \int, \flat : p \to p$ and a product that the above respect $\times : (p,p) \to p \qquad \top : () \to p$
2	$\flat \Rightarrow \Re \Rightarrow 1 \Rightarrow \Im \Rightarrow \int$
eq	$m\flat = \flat$ $m\int = \int$ $m\Im = m$ $m\Re = m$

Note: The "Infinitesimal" modalities don't survive interaction with a cohesive modality because the infinitesimal structure is destroyed

Fibrational Framework

- Gives us meta-theoretic properties: substitution, variables working properly together
- F -| U, so all of our adjunctions hold
- Manually we have verified that the F,U presentation of middle adjoints are equivalent.

Simple DCTT

- Nice simple type theory, but not much use for Differential Geometry
- Can prove that the modalities have the right relationship and preserve products, but not much else.

Towards Differential Cohesive **Homotopy** Type Theory

- How to make a *dependent* modal type theory with differential cohesive \infty-toposes as models?

Comonadic Modalities

- We have a nice description of **monadic** modalities (see previous talk), but less worked out on the comonadic side.

Who can you depend on?

- Summary:
  - flat can depend on flat (assumptions)
  - & can depend on flat, &
  - Reduction can depend on flat, but not itself.

Differential Cohesive HoTT

- Try adding dependent types to our simple type theory.
- Problems even with just cohesion: $\int \dashv \flat \dashv \sharp$
- Monadic assumptions don't work out so nicely.

Real-Differential Cohesion

- In the models, the shape modality is usually a **localization**, can define them internally and add axioms (Shulman '15)
  - Working on doing the same thing for inf. shape
  - May avoid weird dependency issues by avoiding monadic modes.

Progress on Real-DCoHoTT

Previous work: Shulman '15, "Real Cohesion"

$$\int \quad \dashv \quad \flat \quad \dashv \quad \sharp$$

defined

Progress on Real-DCoHoTT

Works

$$\begin{array}{ccccc} & & \& & & \\ & & \cup & & \\ \int & \dashv & \flat & \dashv & \sharp \end{array}$$

defined

Progress on Real-DCoHoTT

Needs semantics work

defined

$$
\begin{array}{ccccc}
\Im & \dashv & \& & & \\
\cup & & \cup & & \\
\int & \dashv & \flat & \dashv & \sharp
\end{array}
$$

defined

Progress on Real-DCoHoTT

???

defined

$$\begin{array}{ccccccc} \Re & \dashv & \Im & \dashv & \& & & \\ & & \cup & & \cup & & \\ & & \int & \dashv & \flat & \dashv & \sharp \end{array}$$

defined

Conclusion

- Simple Type Theory for full Diff. Cohesion.
- “Abstract” DC-HoTT requires a more sophisticated understanding of modal dependency.
- Extending Shulman’s Real-Cohesion may be easier, less general.
- https://github.com/jpaykin/DifferentialCohesiveHoTT
  - Full rules, WIP Agda formalization

Shapely Dependence

$$\frac{\Gamma\ \mathsf{ctx} \qquad \Gamma \vdash mA\ \mathsf{type} \qquad m \in \{\textstyle\int, \flat\}}{\Gamma, x :_m A\ \mathsf{ctx}}$$

$$\frac{\Gamma \vdash A\ \mathsf{type}}{\Gamma \vdash \textstyle\int A\ \mathsf{type}} \qquad \frac{\Gamma \vdash A\ \mathsf{type} \qquad \Gamma\ \mathsf{discrete}}{\Gamma, \Delta \vdash \flat A\ \mathsf{type}}$$

Shapely Dependence

- Can't order the context since dependency structure and natural transformations don't align.

- Shape and flat **both** define the discrete types, but since $\int A$ can depend on non-discrete types, need to look at transitive dependence

Shapely Dependence

No best choice sometimes

$$\frac{? \vdash t : C}{x :_{\int} A, y :_{\flat} B[x] \vdash t^{\int} : \int C}$$

Shapely Dependence

No best choice sometimes

$$\frac{x :_1 A \vdash t : C}{x :_\int A, y :_\flat B[x] \vdash t^\int : \int C}$$

$$\frac{x :_\int A, y :_\flat B[x] \vdash t : C}{x :_\int A, y :_\flat B[x] \vdash t^\int : \int C}$$

Shapely Dependence

Mike's proposal: keep track of *how* types depend on terms

$$\frac{x : A, y :_\flat B[x_\int] \vdash t : C \qquad \flat\int \sim 1}{x :_\int A, y :_\flat B[x_\int] \vdash t^\int : \int C}$$

unclear how to justify in general