# Relating ordinals in set theory to ordinals in type theory

Tom de Jong<sup>1</sup> Nicolai Kraus<sup>1</sup> Fredrik Nordvall Forsberg<sup>2</sup> Chuangjie Xu<sup>3</sup>

<sup>1</sup>University of Nottingham, UK

<sup>2</sup>University of Strathclyde, UK

<sup>3</sup>SonarSource GmbH, Germany

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## Ordinals in set theory

- ▶ <u>Def.</u> A set x is transitive if for every  $y \in x$  and  $z \in y$ , we have  $z \in x$ .
- ▶ <u>Def.</u> A <u>set-theoretic ordinal</u> is a transitive set whose elements are all transitive.
- <u>Lemma</u> The elements of a set-theoretic ordinal are again set-theoretic ordinals.
  - Thus, a set is a set-theoretic ordinal if and only if it is hereditarily transitive.
- ▶  $\underline{\mathsf{Ex}}$ . The sets  $\emptyset$ ,  $\{\emptyset\}$  and  $\{\emptyset, \{\emptyset\}\}$  are all set-theoretic ordinals, but  $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$  isn't, as  $\{\{\emptyset\}\}$  is a non-transitive member.

#### Ordinals in homotopy type theory

- ► In HoTT, a (type-theoretic) ordinal is defined as a type X with a prop-valued binary relation < that is transitive, extensional and wellfounded.</p>
- Extensionality means that we have

$$x = y \iff \forall (u : X).(u < x \iff u < y)$$

It follows that X is an hset.

▶ Wellfoundedness is defined in terms of accessibility, but is equivalent to the assertion that for every  $P: X \to \mathcal{U}$ , we have  $\Pi(x:X).P(x)$  as soon as  $\Pi(x:X).(\Pi(y:X).(y< x \to P(y))) \to P(x)$ .

#### Types of ordinals in HoTT

- We write Ord for the type of (small) type-theoretic ordinals.
- ▶ HoTT hosts a model  $(\mathbb{V}, \in)$  of a constructive set theory. The type  $\mathbb{V}$  is a HIT with point-constructor

$$\mathbb{V}$$
-set $(A, f)$ :  $\mathbb{V}$  for  $A : \mathcal{U}$  and  $f : A \to \mathbb{V}$ 

quotiented by bisimilarity:  $\mathbb{V}$ -set(A, f) and  $\mathbb{V}$ -set(B, g) are identified exactly when f and g have the same image.

▶ We define set-membership  $\in$  :  $\mathbb{V} \to \mathbb{V} \to \mathsf{Prop}$  by

$$x \in \mathbb{V}$$
-set $(A, f) :\equiv \exists (a : A).f(a) = x$ 

Thus, we can define the subtype V<sub>ord</sub> of V of set-theoretic ordinals in HoTT.

## Set-theoretic and type-theoretic ordinals are equivalent

- ▶ Thm. The types  $V_{ord}$  and Ord are equivalent.
- **Proof sketch** Define  $\Phi : \mathsf{Ord} \to \mathbb{V}_{\mathsf{ord}}$  by transfinite recursion:

$$\Phi(\alpha) := \mathbb{V}\text{-set}(\alpha, \lambda(a : \alpha).\Phi(\alpha \downarrow a)),$$

where

$$\alpha \downarrow a \cong \Sigma(b:\alpha).b < a.$$

Its inverse  $\Psi : \mathbb{V}_{ord} \to Ord$  is the rank function:

$$\Psi(\mathbb{V}\operatorname{-set}(A,f)) \coloneqq \bigvee_{a:A} (\Psi(f(a)) + \mathbf{1}).$$

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It is possible to give nonrecursive descriptions of the rank:

$$\Psi(x) \simeq \Sigma(y: \mathbb{V}_{\mathsf{ord}}).y \in x \quad \mathsf{and} \quad \Psi(\mathbb{V}\mathsf{-set}(A, f)) = A/\sim,$$

where  $a \sim b \iff f(a) = f(b)$ . (But be careful about size.)

## The big picture

▶  $\underline{\mathsf{Thm}}$ . The types  $\mathbb{V}_{\mathsf{ord}}$  and  $\mathsf{Ord}$  are equivalent.

But more is true...

▶ The type Ord is actually a large type-theoretic ordinal itself:

$$\alpha \prec \beta \iff \alpha \text{ is an initial segment of } \beta \iff \Sigma(y:\beta).(\alpha=\beta\downarrow y)$$

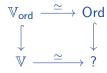
- $\blacktriangleright \ \ \mathsf{Membership} \in \mathsf{makes} \ \mathbb{V}_{\mathsf{ord}} \ \mathsf{into} \ \mathsf{a} \ \mathsf{large} \ \mathsf{type\text{-}theoretic} \ \mathsf{ordinal}.$
- ▶ Thm. The type-theoretic ordinals ( $\mathbb{V}_{ord}$ ,  $\in$ ) and (Ord,  $\prec$ ) are isomorphic.

Thus, in HoTT, set-theoretic and type-theoretic ordinals coincide.

#### The *bigger* picture

Can we realize the full cumulative hierarchy V as a type of ordered structures?

That is, can we find a type making the square



#### commute?

► An initial naive attempt may be to simply drop transitivity, i.e., to take

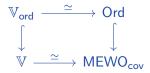
? = type of extensional wellfounded orders.

#### Generalizing from ordinals to sets

- We consider extensional wellfounded orders (X, <) with a marking: a predicate on X that picks out the top-level elements of a set.
- E.g., the sets {∅, {∅}} and {{∅}} are both represented by the two-element type ordered as 0 < 1; we mark both 0 and 1 for the first set, but only 1 in the representation of the second set.</p>
- ➤ A marking is covering if any element can be reached from a marked top-level element, i.e., if the order contains no "junk".
- ► The idea of encoding sets as wellfounded structures isn't new. The above approach worked well for our purposes of generalizing the theory of ordinals.

#### Filling the bigger picture

- We write MEWO<sub>cov</sub> for the type of covered marked extensional wellfounded orders.
- Every ordinal can be equipped with the trivial covering by marking everything. Thus, the type Ord of ordinals is a subtype of MEWO<sub>cov</sub>.
- ▶ We get the bottom isomorphism by generalizing the constructions used to establish  $\mathbb{V}_{ord} \simeq Ord$ :



#### Conclusion

- In HoTT, the set-theoretic ordinals in V coincide with the type-theoretic ordinals.
- By generalizing from type-theoretic ordinals to covered mewos, we capture all sets in V.
- Question: Do the type-theoretic ordinals in the cubical sets model of HoTT coincide with the set-theoretic ordinals?
- Question: Can we use covered mewos to pin down the exact constructive set theory that V models? E.g., can we show strong collection is independent?
- Set-Theoretic and Type-Theoretic Ordinals Coincide. TdJ, Nicolai Kraus, Fredrik Nordvall Forsberg and Chuangjie Xu. arXiv:2301.10696. Accepted for presentation at LICS'23. Fully formalized in AGDA.