Syntax for two-level type theory

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1 Introduction

In homotopy type theory [13] (HoTT), properties that are not invariant under homotopy cannot be expressed internally. An important case is the concept of semisimplicial types, whose definition is so far elusive in HoTT. Voevodsky defined a special Homotopy Type System [14] (HTS) as a formal theory which allows constructions that require access to non-homotopy-invariant notions. Two-Level Type Theory [2] (2LTT) is envisioned to be a variant of HTS, and is composed of two separate levels of types: the outer level is Martin-Löf type theory plus the uniqueness of identity proofs [12] (UIP); the inner level is HoTT. These levels are related by a conversion function from the inner to the outer level that preserves context extensions.

The paper [2] proposes a semantics for 2LTT based on categories with families [7], which justifies reasoning *inside* the inner system with the full power of HoTT, and reasoning *about* the inner system within the outer system to circumvent a number of expressive limits of the former. With this approach it is possible to study properties of HoTT syntactically in the two-level system, and, by conservativity [4], to reflect them back in the HoTT world. Among the applications of this approach are results on Reedy fibrant diagrams [2], the Univalence Principle [1], and internal ∞-categories with families [8], which have been suggested as a way to overcome known difficulties one encounters when formalising type theory in type theory. In summary, despite the intrinsic expressive and proving power of HoTT, a wide range of results rely on meta-reasoning and meta-principles, which cannot entirely be formalised within the theory. The two-level approach formalises these meta-principles in a theory which is compatible both technically and philosophically with HoTT, allowing for their mechanisation. However, the syntax of 2LTT is just sketched in [2].

2 Syntax

In this contribution, we propose a system of inference rules for 2LTT with an infinite hierarchy of Tarski-style universes as uniform constructions [10]; the rules allow us to define the syntax in detail, clearly illustrating the behaviour of the two levels, and how they interact. In contrast to [2], we pay particular attention to the definition of Tarski-style universes, following the guidelines of [10]: other than the function El_i, which maps the codes $A : \mathcal{U}_i$ into types $\mathsf{El}_i(A)\,\mathsf{type}$ and is present in [2], we introduce a function lift_i mapping terms of one universe $A : \mathcal{U}_i$ into terms of the next one, $\mathsf{lift}_i(A) : \mathcal{U}_{i+1}$. In [2], the lift operation is not present, and the universes are *cumulative*. In our system those two functions commute: if $\Gamma \vdash A : \mathcal{U}_i$, then $\Gamma \vdash \mathsf{El}_{i+1}(\mathsf{lift}_i(A)) \equiv \mathsf{El}_i(A)\,\mathsf{type}$. The same happens for inner types; indeed, $A\,\mathsf{type}$ means that A is an outer type, while $A\,\mathsf{type}^o$ means that A is an inner type. This emphasises another difference between our approach and the 2LTT paper: we do not have a *size* for types; on the contrary, in [2] it is specified as $A\,\mathsf{type}_i$ or $A\,\mathsf{type}^o_i$: if $A : \mathcal{U}_i$, then $\mathsf{El}_i(A)\,\mathsf{type}_i$. Moreover, besides the conversion function c from inner to outer types introduced in [2], we define a conversion function c' from inner to outer codes, i.e., terms of the universes: if $A : \mathcal{U}_i^o$, then $c'(A) : \mathcal{U}_i$. It is required that $\mathsf{El}, \mathsf{lift}, c$ and c' commute. We formalise the fact that the conversion function preserves context extension by introducing a notion of equivalence between contexts together with the rule

$$\frac{\Gamma \vdash A\,\mathsf{type}^o}{\Gamma, x : A\,\mathsf{ctx} \equiv \Gamma, y : c(A)\,\mathsf{ctx}}\ \equiv\!-\mathsf{ctx}-\mathsf{EXT}$$

Then, we define a generalisation of the notion of category with families which allows us to interpret our formalisation of the two levels and the Tarski-style universes, called *two-level model*, together with a notion of morphism between models. We plan to show the compatibility of our system with the (almost) standard semantics for 2LTT by proving an initiality result; this will essentially extend recent work for Martin-Löf type theory by Brunerie, de Boer, Lumsdaine, and Mörtberg [3, 9, 6]. We define the syntactical two-level model by quotienting the syntax, similar to [11, 5], and prove that it is the initial object in the category of models.

Our long term goal is to develop the basis for a proof assistant that implements 2LTT and allows one to use additional inner and outer axioms, some of which have been already suggested [2], to formalise in parallel the inner and outer levels, and their relations.

3 Open questions

There are some open issues, which we hope to understand better in the future:

1. Can some of the conversion, lift and El maps be made "silent", to make the use of a potential proof assistant more convenient?
2. The rule $\equiv\!-\mathsf{ctx}-\mathsf{EXT}$ aims to avoid cumulativity (which can create some difficulties with typing), but remain as close as possible to it. What models can we hope for?
3. In the current version, we use judgmental equality of contexts; is this too strict for the purpose of construction of models? What are the proof-theoretic consequences?

References

[1] Benedikt Ahrens, Paige Randall North, Michael Shulman, and Dimitris Tsementzis. The Univalence Principle, 2021. https://arxiv.org/abs/2102.06275.

[2] Danil Annenkov, Paolo Capriotti, Nicolai Kraus, and Christian Sattler. Two-level type theory and applications. *arXiv:1705.03307v3*, 2019.

[3] Guillaume Brunerie, Menno de Boer, Peter LeFanu Lumsdaine, and Anders Mörtberg. A formalization of the initiality conjecture in Agda. Talk given by Brunerie at the HoTT 2019 conference, slides available at https://guillaumebrunerie.github.io/pdf/initiality.pdf, 2019.

[4] Paolo Capriotti. *Models of type theory with strict equality.* PhD thesis, University of Nottingham, 2016. http://eprints.nottingham.ac.uk/39382/1/thesis.pdf.

[5] Thierry Coquand and Simon Castellan. Dependent type theory as the initial category with families. 2014.

[6] Menno de Boer. *A Proof and Formalization of the Initiality Conjecture of Dependent Type Theory.* Licentiate dissertation, Department of Mathematics, Stockholm University, 2020.

[7] Peter Dybjer. Internal type theory. In Stefano Berardi and Mario Coppo, editors, *Types for Proofs and Programs*, pages 120–134, Berlin, Heidelberg, 1996. Springer Berlin Heidelberg.

[8] Nicolai Kraus. Internal ∞-categorical models of dependent type theory: Towards 2LTT eating HoTT. *Logic in Computer Science (LICS'21)*, 2021. Available as arXiv:2009.01883.

[9] Peter LeFanu Lumsdaine and Guillaume Brunerie. Initiality for Martin-Löf type theory. Talk at the Homotopy Type Theory Electronic Seminar Talks (HOTTEST), https://www.youtube.com/watch?v=1ogUFFUfU_M, 2020.

[10] Erik Palmgren. On universes in type theory. In G. Sambin and J. Smith, editors, *Twenty-Five Years of Constructive Type Theory*. Oxford University Press, 1998.

[11] Thomas Streicher. *Semantics of Type Theory*. Progress in Theoretical Computer Science. Springer, 1991.

[12] Thomas Streicher. *Investigations into intensional type theory*. Habilitationsschrift, Ludwig-Maximilians-Universität München, 1993.

[13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: https://homotopytypetheory.org/book, 2013.

[14] Vladimir Voevodsky. A simple type system with two identity types. Unpublished note, 2013.