Elementary fibrations and (algebraic) weak factorisation systems

Jacopo Emmenegger[1], Fabio Pasquali[2], and Giuseppe Rosolini[3]

[1]School of Computer Science, University of Birmingham, UK

[2,3]DIMA, Università di Genova, Italy

j.j.emmenegger@bham.ac.uk pasquali@dima.unige.it rosolini@unige.it

Abstract

We present a characterisation of elementary fibrations, *i.e.* fibrations with equality, that generalises the one for faithful fibrations, and employ it for a comparison with the structures used in the semantics of the identity type of Martin-Löf type theory.

Fibrations provide an algebraic framework that underlies the treatment of syntax and semantics of (fragments of) first and higher order logics, as well as of dependent type theories. The former approach dates back to Lawvere's hyperdoctrines [7, 8] where, in the spirit of functorial semantics, equality is specified requiring left adjoints to certain reindexing functors. On the other hand, models of dependent type theory that do not collapse the (whole) hierarchy of identity types do not treat equality as an adjunction. Rather, they often rely on weak factorisation systems or related structures. We provide a characterisation of elementary fibrations that contributes to shed light on the relation between the two approaches to equality. As it will become clear, the relation is based on a structure which, in type-theoretic terms, can be understood as a transport structure.

Let $K\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a (cloven) fibration, write $f^*\colon \mathcal{E}_Y \longrightarrow \mathcal{E}_X$ for the reindexing functor along $f\colon X \longrightarrow Y$ in $\mathcal{B}$. A fibration $K\colon \mathcal{E} \longrightarrow \mathcal{B}$ ***has finite products*** if the base $\mathcal{B}$ has finite products as well as each fibre $\mathcal{E}_X$, and each reindexing functor preserves products—equivalently, both $\mathcal{E}$ and $\mathcal{B}$ have finite products, and K preserves them. We denote products in fibres as $A \wedge B$, and write lists of product projections $\langle \mathrm{pr}_{i_1}, \ldots, \mathrm{pr}_{i_n}\rangle\colon X_1 \times \cdots \times X_m \longrightarrow X_{i_1} \times \cdots \times X_{i_n}$ in $\mathcal{B}$ as $\mathrm{pr}_{i_1,\ldots,i_n}$. Recall from [5] that a fibration with products $K\colon \mathcal{E} \longrightarrow \mathcal{B}$ is ***elementary*** if, for every pair of objects Y and X in $\mathcal{B}$, reindexing along the parametrised diagonal $\mathrm{pr}_{1,2,2}\colon Y \times X \longrightarrow Y \times X \times X$ has a left adjoint $\exists_{Y,X}\colon \mathcal{E}_{Y\times X} \longrightarrow \mathcal{E}_{Y\times X\times X}$, and these satisfy the Frobenius Reciprocity and the Beck-Chevalley Condition for pullbacks of the form

$$\begin{tikzcd}
Y \times X \arrow[r, "f\times X"] \arrow[d, "\mathrm{pr}_{1,2,2}"'] & Z \times X \arrow[d, "\mathrm{pr}_{1,2,2}"] \\
Y \times X \times X \arrow[r, "f\times X\times X"] & Z \times X \times X.
\end{tikzcd}$$

Example. Let $\mathcal{C}$ be a category and denote its arrow category as $\mathcal{C}^{\mathbf{2}}$ and the codomain functor as $\mathsf{cod}\colon \mathcal{C}^{\mathbf{2}} \longrightarrow \mathcal{C}$. Let $\mathcal{A}$ be a full subcategory of $\mathcal{C}^{\mathbf{2}}$ and suppose that, for every $f\colon X \longrightarrow Y$ in $\mathcal{C}$, there is a choice of a pullback square for each $g\colon B \longrightarrow Y$ in $\mathcal{A}$ and, further, that $\mathcal{A}$ is stable under pullback. Then the composite

$$\begin{tikzcd}
\mathcal{A} \arrow[r, hook] \arrow[rr, bend left, "\mathsf{cod}\restriction_{\mathcal{A}}"] & \mathcal{C}^{\mathbf{2}} \arrow[r, "\mathsf{cod}"'] & \mathcal{C}
\end{tikzcd}$$

is a fibration where reindexing is given by the chosen pullbacks. If $\mathcal{C}$ has finite products then so does $\mathsf{cod}\restriction_{\mathcal{A}}$.

1. Let $\mathcal{M}$ denote the full subcategory of the arrow category $\mathcal{C}^{\mathbf{2}}$ on the monos. If $\mathcal{C}$ has pullbacks of monos along any arrow, then $\mathsf{cod}\restriction_{\mathcal{M}}$ is a (faithful) fibration and the poset reflection of the fibre $\mathcal{M}_X$ is the poset of subobjects of X. This fibration is elementary with $\exists_{Y,X}(m) := A \xrightarrow{\mathrm{pr}_{1,2,2}m} Y \times X \times X$.

2. Let $(\mathcal{L}, \mathcal{R})$ be a weak factorisation system of a category $\mathcal{C}$. If there are pullbacks of arrows in $\mathcal{R}$ along any arrow, then $\mathsf{cod}\restriction_{\mathcal{R}} : \mathcal{R} \longrightarrow \mathcal{C}$ is a fibration.

Example 1 is the prototypical example of a faithful fibration. These have a robust theory in terms of indexed posets, see [10, 9], and can be characterised [1] as those faithful fibrations with finite products $K: \mathcal{E} \longrightarrow \mathcal{B}$ that are equipped, for every X in the base, with an element $\mathrm{I}_X \in \mathcal{E}_{X\times X}$ which is (i) reflexive, *i.e.* $\top_X \leq {\mathrm{pr}_{1,1}}^*\mathrm{I}_X$, (ii) substitutive, *i.e.* for every $A \in \mathcal{E}_X$, ${\mathrm{pr}_1}^*A \wedge \mathrm{I}_X \leq {\mathrm{pr}_2}^*A$, and (iii) product-stable, *i.e.* ${\mathrm{pr}_{1,3}}^*\mathrm{I}_X \wedge {\mathrm{pr}_{2,4}}^*\mathrm{I}_Y \leq \mathrm{I}_{X\times Y}$.

With the aim of extending this result to a general fibration $K: \mathcal{E} \longrightarrow \mathcal{B}$ with finite products, consider the following structure on an object X in $\mathcal{B}$: (I) an object I_X over $X \times X$ and an arrow $\partial_X : \top_X \longrightarrow \mathrm{I}_X$ over $\mathrm{pr}_{1,1} : X \longrightarrow X \times X$, and (II) for every $A \in \mathcal{E}_X$, an arrow $\mathrm{t}_A : ({\mathrm{pr}_1}^*A) \wedge \mathrm{I}_X \longrightarrow A$ over $\mathrm{pr}_2 : X \times X \longrightarrow X$. We refer to this structure as a ***transporter on*** X. Transporters can be found in elementary fibrations as well as in those fibrations $\mathsf{cod}\restriction_{\mathcal{R}}$ from Example 2 arising from models of Martin-Löf's identity type. In fact, transporters in these examples enjoy also other properties: a condition analogous to (iii) above, and the existence of a section $\mathrm{t}_A\delta_A = \mathrm{id}_A$ for t_A for each $A \in \mathcal{E}_X$, where $\delta_A : A \longrightarrow ({\mathrm{pr}_1}^*A) \wedge \mathrm{I}_X$ is obtained pairing $\partial_X !_A$ with the obvious cartesian arrow over $\mathrm{pr}_{1,1}$. We say that transporters satisfying these two additional conditions are ***strictly productive***. One last ingredient is needed to state the characterisation. Recall from [11] that an arrow $\varphi: A \longrightarrow B$ in $\mathcal{E}$ is ***locally epic with respect to*** K if, for every pair $\psi, \psi' : B \longrightarrow B'$ such that $K(\psi) = K(\psi')$, whenever $\psi\varphi = \psi'\varphi$ it is already $\psi = \psi'$.

Theorem. *A fibration with products $K: \mathcal{E} \longrightarrow \mathcal{B}$ is elementary if and only if*

1. *it has strictly productive transporters and,*

2. *for every X, all arrows in a certain class Ξ_X are locally epic with respect to K.*

The arrows in the class Ξ_X are obtained from ∂_X by suitable reindexing and pairing and include, in particular, all the δ_A defined above, for $A \in \mathcal{E}_X$. The proof of the Theorem builds on the observation that existence of a left adjoint to reindexing along some f is equivalent to existence of cocartesian lifts over f, and provides equivalent formulations of the Frobenius Reciprocity and the Beck-Chevalley Condition in terms of closure conditions for cocartesian arrows. As we shall illustrate, in an elementary fibration the arrows in Ξ_X are precisely the cocartesian arrows over parametrised diagonals $\mathrm{pr}_{1,2,2} : Y \times X \longrightarrow Y \times X \times X$, for Y in $\mathcal{B}$.

The complete statement of our main result lists other equivalent characterisations of an elementary fibration, which are also convenient intermediate steps in the proof of the Theorem above. We shall discuss all the concepts needed to state the main result and illustrate them with several examples. In particular, we shall see when condition 2 holds for a fibration of the form $\mathsf{cod}\restriction_{\mathcal{R}}$ from Example 2. This condition fails, for instance, in the fibration $\mathsf{cod}\restriction_{\mathcal{R}}$ associated to a (non-trivial) model of Martin-Löf's identity type.

On the other hand, the weak factorisation system $(\mathcal{L}, \mathcal{R})$, whose right class provides the comprehension category of such a model, is often the underlying w.f.s. of an algebraic weak factorisation system [2, 3]. The richer structure of algebraic weak factorisation systems produces more structured fibrations. Consider, for instance, the algebraic weak factorisation system on $\mathcal{C}at$ (and $\mathcal{G}pd$) whose underlying weak factorisation system $(\mathcal{L}, \mathcal{R})$ is the one of acyclic cofibrations and fibrations from the canonical, or "folk", model structure on $\mathcal{C}at$ (and $\mathcal{G}pd$).

In this case, the two fibrations $\mathsf{cod}\restriction_{\mathcal{R}}$ into $\mathcal{Cat}$ and $\mathcal{Gpd}$ are not elementary. This is not a surprise, as $\mathsf{cod}\restriction_{\mathcal{R}}: \mathcal{R} \longrightarrow \mathcal{Gpd}$ is the fibration underlying the Hofmann–Streicher groupoid model from [4]. However, using the Theorem we shall prove that the fibration of algebras for the monad on the right functor $\mathcal{R}$ is elementary.

References

[1] J. Emmenegger, F. Pasquali, and G. Rosolini. Elementary doctrines as coalgebras. To appear in *J. Pure Appl. Algebra*, 2020.

[2] N. Gambino and M. F. Larrea. Models of Martin-Löf type theory from algebraic weak factorisation systems. available at `arXiv:1906.01491, 2019`, 2019.

[3] N. Gambino and C. Sattler. The Frobenius condition, right properness, and uniform fibrations. *J. Pure Appl. Algebra*, 221(12):3027–3068, 2017.

[4] M. Hofmann and T. Streicher. The groupoid interpretation of type theory. In *Twenty-five years of constructive type theory* (*Venice, 1995*), volume 36 of *Oxford Logic Guides*, pages 83–111. Oxford Univ. Press, New York, 1998.

[5] B. Jacobs. *Categorical Logic and Type Theory*, volume 141 of *Studies in Logic and the foundations of mathematics*. North Holland Publishing Company, 1999.

[6] Bart Jacobs. Semantics of weakening and contraction. *Ann. Pure Appl. Logic*, 69(1):73–106, 1994.

[7] F. W. Lawvere. Adjointness in foundations. *Dialectica*, 23:281–296, 1969.

[8] F.W. Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In A. Heller, editor, *Proc. New York Symposium on Application of Categorical Algebra*, pages 1–14. Amer.Math.Soc., 1970.

[9] M.E. Maietti and G. Rosolini. Quotient completion for the foundation of constructive mathematics. *Log. Univers.*, 7(3):371–402, 2013.

[10] A.M. Pitts. Categorical logic. In S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science, Volume 5. Algebraic and Logical Structures*, chapter 2, pages 39–128. Oxford University Press, 2000.

[11] T. Streicher. Fibred Categories à la Jean Bénabou. available at `arXiv:1801.02927v7`, 2020.