Models of HoTT

(simplicial & cubical)

 

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0. Introduction & Overview
1. What is a model of type theory?
2. Presheaf models

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4. Cubical models and open questions

Exercises

0. Introduction & overview

Model of type theory:

- Interpretation of language that turns judgmental equality into actual equality. Slogan

There are many frameworks for models, sometimes equivalent, other times differing in some aspects:

- collection of types over context Γ seen as set or groupoid or category.
- split vs. non-split
- contextual / democratic
- algebraic or categorical notion

  algebraic — category of models
  categorical — bicategory of models
- terms or context projections (display maps) as primitive

We will work with categories with families.

Model of HoTT: Just a model of type theory with elements witnessing function extensionality + univalence.

History of models of HoTT:

Set model
- univalent universe of propositions

Groupoid model
- univalence for (strict) sets

Hofmann-Streicher

Simplicial model
- full univalence
- non-constructive
- models spaces/ ∞-groupoids

Voevodsky

⋮

Inverse diagram models

Shulman

⋮

Model for injective fibrations in simplicial presheaves

Coquand + coworkers

Cubical models
- full univalence
- often does not model spaces/∞-groupoids
- computational

- BCH model
- connection-based models
  - unknown if can model spaces
- Cartesian-based models
  - can model spaces (equivariant model)

HoTT and ∞-toposes

We want to "interpret" HoTT in ∞-toposes.

But models of HoTT are not (directly) ∞-categories!

↳ They are 1-categories with homotopical structure, <u>presenting</u> an ∞-category.

Like model categories!

$$\begin{tikzcd} \text{models of HoTT} \arrow[r, "\text{present}^*"] & \infty\text{-toposes} \\ \text{HoTT} \arrow[u, "\text{interprets into}"'] & \end{tikzcd}$$

So to interpret HoTT into a given ∞-topos $\mathcal{X}$, first need to find suitable <u>presentation</u> of $\mathcal{X}$.

Shulman found such a presentation for every ∞-topos!

So far, this is a classical (non-constructive) story.

1. What is a model of type theory?

Don't want to bother with modelling "named variables".

↳ Abstract over contexts as lists of hypotheses $x_1 : A_1, \dots, x_n : A_n$ by modelling contexts + substitutions as a category $\mathcal{C}$.

$$\Delta \xrightarrow{\sigma} \Gamma$$

$$[y_1 : B_1, \dots, y_m : B_m] \qquad [x_1 : A_1, \dots, x_n : A_n]$$

$$\begin{aligned} x_1 &= t_1[y_1, \dots, y_m] \\ &\vdots \\ x_n &= t_n[y_1, \dots, y_m] \end{aligned}$$

Every context Γ has a set of types $Ty(\Gamma)$.
We can substitute types: $A \in Ty(\Gamma) \rightsquigarrow A[\sigma] \in Ty(\Delta)$.
↳ Ty is a presheaf over C.

Same for terms Tm,
but they additionally depend on a type.

Excursion: Presheaves and discrete fibrations

Def Presheaves over category $\mathcal{C}$ are functors $\mathcal{C}^{op} \to Set$.

Notation: $\widehat{\mathcal{C}} = \mathrm{Presheaf}(\mathcal{C})$

Traditional definition

Grothendieck construction*

*restricted version

$$\int : [\mathcal{C}^{op}, Set] \xrightarrow[\text{ff}]{\text{fully faithful}} \{\text{category over } \mathcal{C}\}$$

$$F \longmapsto \begin{array}{c} \int F \\ \downarrow \\ \mathcal{C} \end{array} \quad \text{"category of elements"}$$

$(\int F)_0(A) = F(A)$

$(\int F)_1(x, y, f) = [F(f)(y) = x]$

$\begin{array}{c} \mathcal{E} \\ \downarrow \\ \mathcal{C} \end{array}$ We regard $\mathcal{E}$ as "displayed" over $\mathcal{C}$:

- $\mathcal{E}_0(A)$ set for $A \in \mathcal{C}_0$
- $\mathcal{E}_1(X, Y, f)$ set for
  $X \in \mathcal{E}_0(A)$
  $Y \in \mathcal{E}_0(B)$
  $f \in \mathcal{C}_1(A, B)$

That way, we can strictly reindex categories over a base.

The essential image of $\int$ are the discrete fibrations:

$$\begin{array}{cc} ? \overset{u}{\dashrightarrow} Y & \mathcal{E} \\ \downarrow & \\ A \xrightarrow{f} B & \mathcal{C} \end{array}$$

$\forall f \in \mathcal{C}_1(A, B),\ Y \in \mathcal{E}_0(B)$ have unique lift $X \in \mathcal{E}_0(A),\ u \in \mathcal{E}_1(f, X, Y)$.

Often, discrete fibrations are a better model for presheaves $\widehat{\mathcal{C}}$, than $\mathcal{C}^{op} \to Set$!

We use both interchangeably.

Categories with families

Definition

A <u>category with families</u> consists of: cwf

- Category $\mathcal{C}$
- Terminal object $1 \in \mathcal{C}$
- $\mathrm{Ty} \in \widehat{\mathcal{C}}$
- $\mathrm{Tm} \in \widehat{\int \mathrm{Ty}}$
- <u>Context extension:</u>
  for $\Gamma \in \mathcal{C}$ and $A \in \mathrm{Ty}(\Gamma)$, a representation of:

$$(\mathcal{C} \downarrow \Gamma)^{op} \longrightarrow \mathrm{Set}$$
$$\Delta \xrightarrow{\sigma} \Gamma \longmapsto \mathrm{Tm}(\Delta, A[\sigma])$$

$\Leftrightarrow$ a terminal object $(\Gamma.A, p_A, q_A)$ in the category of tuples (Δ, σ, t) where:

- $\Delta \in \mathcal{C}$
- $\Delta \xrightarrow{\sigma} \Gamma$
- $t \in \mathrm{Tm}(\Delta, A[\sigma])$

$\square$

$$\begin{array}{l} \Gamma.A \quad \text{context extension} \\ \downarrow \quad \text{context projection / display map for } A \\ \Gamma \end{array}$$

$q_A \in \mathrm{Tm}(\Gamma.A, A[p_A])$ generic term, 'last variable'

$$\begin{tikzcd} \Delta \arrow[rr, dashed, "{\langle \sigma, t \rangle}"] \arrow[dr, "\sigma"'] & & \Gamma.A \arrow[dl, "p_A"] \\ & \Gamma & \end{tikzcd}$$

For algebraic notion of model:

- context extension is structure, strictly preserved by morphisms of models
- obtain <u>category</u> of models
- "Syntax" $=_{\mathrm{def}}$ initial object

Context extension is really a property of the presheaf Tm!

Alternative definition (cwf)

$$\begin{tikzcd}
\mathrm{Tm} \arrow[d, "\dashv"'] \\
\mathrm{Ty} \arrow[u, bend right=60, "\text{context extension}"'] \arrow[d] \\
\mathcal{C} \arrow[d, "\dashv"'] \\
1 \arrow[u, bend right=60, "\text{terminal object}"']
\end{tikzcd}$$

discrete fibrations

Have right adjoints to indicated projections

$\square$

Notation for adjunctions:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend right, "\text{right adjoint}"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend right, "\text{left adjoint}"']
\end{tikzcd}$$

Exercise: show that this means the same as the first definition.

Type formers

Dependent sums

For $\Gamma \in \mathcal{C}$, $A \in \mathrm{Ty}(\Gamma)$, $B \in \mathrm{Ty}(\Gamma . A)$,
have $\Sigma(A, B) \in \mathrm{Ty}(\Gamma)$ with a bijection

$$\underset{\eta}{\mathrm{Tm}(\Gamma, \Sigma(A,B))} \underset{\pi_1, \pi_2}{\overset{\text{pair}}{\rightleftarrows}} \underset{\beta}{\left\{ \left(\begin{array}{l} a \in \mathrm{Tm}(\Gamma, A), \\ b \in \mathrm{Tm}(\Gamma, B[\langle \mathrm{id}, a \rangle]) \end{array} \right) \right\}},$$

natural in Γ.

Dependent products

For $\Gamma \in \mathcal{C}$, $A \in \mathrm{Ty}(\Gamma)$, $B \in \mathrm{Ty}(\Gamma . A)$,
have $\Pi(A, B) \in \mathrm{Ty}(\Gamma)$ with a bijection

$$\underset{\eta}{\mathrm{Tm}(\Gamma, \Pi(A,B))} \underset{\text{app}}{\overset{\lambda}{\rightleftarrows}} \underset{\beta}{\mathrm{Tm}(\Gamma . A, B)},$$

natural in Γ.

Exercise: identity types

Adjoint perspective on Σ/Π

$$\begin{tikzcd}
\Gamma.\Sigma(A,B) \simeq \Gamma.A.B \arrow[d] \arrow[dd, bend right] \\
\Gamma.A \arrow[d] \\
\Gamma
\end{tikzcd}
\qquad
\begin{tikzcd}
\Gamma.\Sigma(A,B) \arrow[rr, dashed] \arrow[dr] & & \Delta \arrow[dl, "\sigma"] \\
& \Gamma &
\end{tikzcd}
\underset{\text{natural}}{\simeq}
\begin{tikzcd}
\Gamma.A.B \arrow[rr, dashed] \arrow[dr] & & \Delta.A[\sigma] \arrow[dl] \\
& \Gamma.A &
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta \arrow[rr, dashed] \arrow[dr] & & \Gamma.\Pi(A,B) \arrow[dl] \\
& \Gamma &
\end{tikzcd}
\underset{\text{natural}}{\simeq}
\begin{tikzcd}
\Delta.A[\sigma] \arrow[rr, dashed] \arrow[dr] & & \Gamma.A.B \arrow[dl] \\
& \Gamma.A &
\end{tikzcd}$$

$$\Sigma_A \dashv p_A^* \dashv \Pi_A$$

after switching to "display map" presentation.

Caveat: Σ_A / Π_A only defined on types / display maps!

$$\mathrm{Tm}(\Gamma, A) \simeq \left\{ \begin{tikzcd} \Gamma.A \arrow[d, shift left] \\ \Gamma \arrow[u, dashed, shift left] \end{tikzcd} \text{ section} \right\}$$

Lifting perspective on Id

$$\begin{tikzcd}
\Gamma.A \arrow[r, "d"] \arrow[d, "\mathrm{refl}_A"'] & \Gamma.A.A.\mathrm{Id}_A.C \arrow[d, "p_C"] \\
\Gamma.A.A.\mathrm{Id}_A \arrow[r, equal] \arrow[ur, dashed, "J_{C,d}"'] & \Gamma.A.A.\mathrm{Id}_A
\end{tikzcd}$$

$C \in \mathrm{Ty}(\Gamma.A.A.\mathrm{Id}_A)$ motive

$d \in \mathrm{Tm}(\Gamma.A, \cancel{C[\langle p_A, q, q\rangle]}\ C[\mathrm{refl}_A])$ witness

Universe

$U \in Ty(\Gamma)$, $El \in Ty(\Gamma.U)$
natural in Γ

Equivalently:
$U \in Ty(1)$, $El \in Ty(1.U)$

$A \in Tm(\Gamma, U) \longmapsto El[\langle id_\Gamma, A\rangle]$
decodes elements of U into types.

$Ty_U \in \widehat{\mathcal{C}}$ $\qquad$ $Tm_U \in \widehat{\int Ty_U}$

$Ty_U(\Gamma) = Tm(\Gamma, U)$ $\qquad$ $Tm_U(A) = Tm(\Gamma, El[\langle id_\Gamma, A\rangle])$

defines <u>cwf structure on $\mathcal{C}$ induced by U.</u>

U closed under type formers means:
(1) induced cwf structure has type formers, $\quad Ty_U$
(2) map of cwf structures preserves type formers. $\quad Ty_U \to Ty$

Can abstractly define cumulative hierachy using morphisms of cwf structures.

Axioms (FunExt + univalence)

Witnessed by a term of the type for the axiom (naturally in the context).

Examples

- Set as cwf:

  $Ty(\Gamma) = \Gamma \to Set$

  $Tm(\Gamma, A) = (x : \Gamma) \to A(x)$

- Groupoids as cwf:

  $Ty(\Gamma) = \Gamma \to Gpd$

  $Tm(\Gamma, A) = \left\{ \begin{tikzcd} \int A \arrow[d] \\ \Gamma \arrow[u, dashed, bend left] \end{tikzcd} \right\}$

Split fibration model.
There is also the (cloven) fibration model:
pseudofunctors $\Gamma \to Gpd$.

- Cube category $\square$:

  $Ty(X) = 1$

  $Tm(X, *) = \square(X, I)$

  for interval I generating $\square$

- For a cwf $\mathcal{C}$, the <u>core</u> $\mathcal{C}_{fib}$ has
  - objects $Ty(1)$
  - maps $1.A \dashrightarrow 1.B$
  - $Ty_{fib}(A) = Ty(1.A)$

- For a cwf $\mathcal{C}$, and $\Gamma \in \mathcal{C}$, the slice $\mathcal{C} \downarrow \Gamma$ inherits a cwf structure.

...

2. Presheaf models

Goal: For category $\mathcal{C}$, make presheaves $\widehat{\mathcal{C}}$ into a model of "extensional" type theory

equality reflection

Extraordinarily useful:

- Can use ETT as internal language for presheaves.
- Can use presheaves to express naturality of type formers.
- Bootstrapping basis for <u>all known semantic models</u> of HoTT.

natural models
HOAS
LF

Let $\Gamma \in \widehat{\mathcal{C}}$.

In displayed form

$Ty(\Gamma) = \{\text{discrete fibration over } \int \Gamma\}$

$A \in Ty(\Gamma)$ written $\begin{array}{c} A \\ \downarrow \\ \Gamma \end{array}$ or $\begin{array}{c} \Gamma.A \\ \downarrow \\ \Gamma \end{array}$

$Tm(\Gamma, A) = \{\text{section of } A\}$

Context extension given by taking the "total space"/ dependent sum of presheaves.

<u>Simplification</u>

Build a cwf on <u>categories</u> with discrete fibrations as types. The presheaf model over $\mathcal{C}$ is the <u>slice model</u> over $\mathcal{C}$.

$\underline{\Sigma}$ Given by composition of discrete fibrations.

$\underline{\mathrm{Id} + \text{equality reflection}}$

Given by levelwise equality.

For Π and $\mathcal{U}$, we revisit the Grothendieck construction:

$$\begin{tikzcd}[column sep=huge]
 & \{\text{category over } \mathcal{C}\} \arrow[dl, bend right] \arrow[dl, bend left] \arrow[dd, "\text{forget}"] \\
{[\mathcal{C}^{op}, Set]} \arrow[ur, "\mathrm{ff}"] \arrow[dr, "\mathrm{ff}"'] & \\
 & \{\text{graph over } \mathcal{C}\} \arrow[ul, bend left] \arrow[ul, bend right] \arrow[uu, bend left, "\dashv"]
\end{tikzcd}$$

$\mathcal{E}_0, \mathcal{E}_1$ with no operations

Π-types are easy in graphs.

presheaves over $[0] \rightrightarrows [1]$

Compute Π-types in $\widehat{\mathcal{C}}$ by moving to graphs over $\mathcal{C}$, doing the Π-type there, and then applying the right adjoint to go back to presheaves.

Universe in categories over $\mathcal{C}$: just Set!

$$\begin{array}{c} \mathcal{C} \times Set \\ \downarrow \\ \mathcal{C} \end{array}$$

Transport it to $\widehat{\mathcal{C}}$ using the right adjoint

Exercises: work out the details.

HOAS

Higher-order abstract syntax

For cwf $\mathcal{C}$, can describe type formers using the internal language of $\hat{\mathcal{C}}$.

$$\varepsilon \vdash \mathrm{Ty} \text{ type}$$

$$A : \mathrm{Ty} \vdash \mathrm{Tm}(A) \text{ type}$$

Π-types

Given $A : \mathrm{Ty}$, $B : \mathrm{Tm}(A) \to \mathrm{Ty}$, have $\Pi(A, B) : \mathrm{Ty}$ and

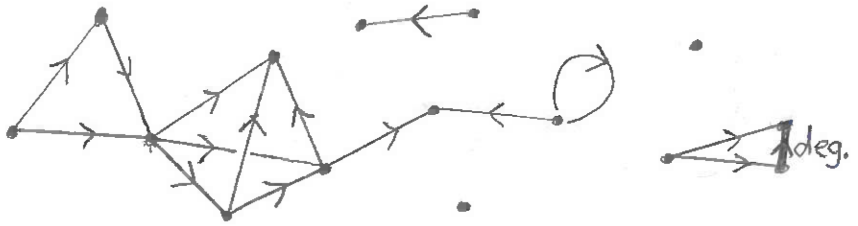
$$\mathrm{Tm}(\Pi(A, B)) \simeq \prod_{a : \mathrm{Tm}(A)} \mathrm{Tm}(B(a)).$$

Don't have to mention context Γ anymore!

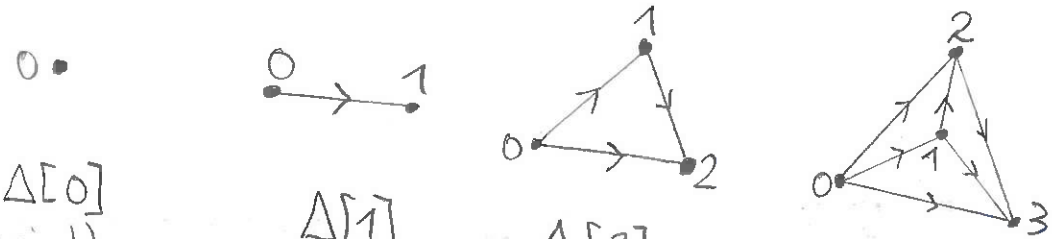
Simplicial sets

Combinatorial model for spaces/∞-groupoids*

| *among other things



Built by gluing basic geometric shapes called <u>simplices</u>, $\Delta[n]$ in dimension n:



$\Delta[0]$ (point) $\quad$ $\Delta[1]$ (line) $\quad$ $\Delta[2]$ (triangle) $\quad$ $\Delta[3]$ (tetrahedron)

Category of simplices

$\Delta =$ "inhabited finite total orders"
$\simeq \{[n] \in \mathrm{Poset} \mid n \geq 0\}$
$\quad\;\; \| $
$\{0 < 1 < \cdots < n-1 < n\}$

- injections give <u>faces</u>
- surjections give <u>degeneracies</u>

<u>Simplicial sets</u> are presheaves over Δ ← Definition

$sSet = \widehat{\Delta} = \mathrm{Presheaf}(\Delta)$ ← Notations

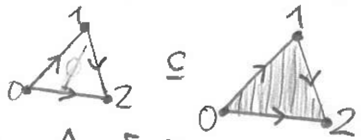
$X \in \widehat{\Delta}$:

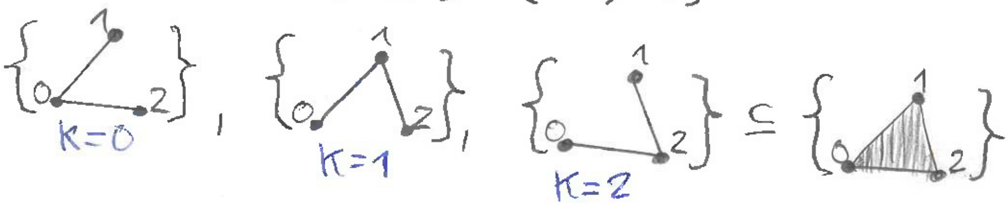
$$X_0 \leftleftarrows X_1 \; \cdots \; X_2 \; \cdots\cdots$$

points of X $\quad$ lines of X $\quad$ triangles of X

Homotopical structure on sSet

Important simplicial sets:

- The boundary $\partial\Delta[n] \subseteq \Delta[n]$ misses the "inside".
  - $n=0$: $\emptyset \subseteq \{0\}$
  - $n=1$: $\{0,1\} \subseteq \{0 \to 1\}$
  - $n=2$: 

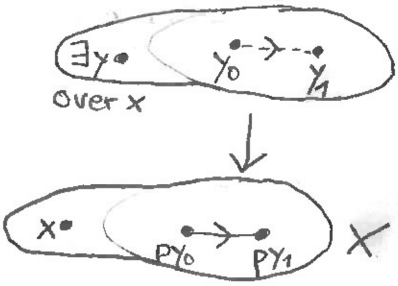
- The horn $\Lambda_k[n] \subseteq \Delta[n]$ misses the "inside" and k-th face. $|\, n \geq 1$
  - $n=1$: $\{0\} \subseteq \{0 \to 1\}$, $\{1\} \subseteq \{0 \to 1\}$
  - $n=2$: 

Definition

A map $Y \xrightarrow{p} X$ in sSet is:

- A trivial fibration if it lifts against boundary inclusions

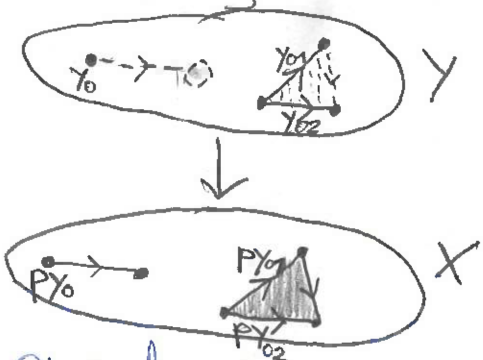
$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed, "\exists \text{ lift}"] & X \end{tikzcd}$$



These will interpret contractible types!

- A (Kan) fibration if it lifts against horn inclusions

$$\begin{tikzcd} \Lambda_k[n] \arrow[r] \arrow[d] & Y \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed, "\exists \text{ lift}"] & X \end{tikzcd}$$



These will interpret types!

encodes transport

□

Excursion: Weak factorization systems

Definition

A *weak factorization system* (L,R) on a category $\mathcal{C}$ consists of classes of maps L and R such that: **wfs**

(1) every map factors using L and R;

$$\begin{tikzcd} \bullet \arrow[rr] \arrow[dr, "\in L"'] & & \bullet \\ & \bullet \arrow[ur, "\in R"'] & \end{tikzcd}$$

(2) $R = L^{\pitchfork}$: R is the class of maps right lifting against L,

(3) $L = {}^{\pitchfork}R$: L is the class of maps left lifting against R.

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\in L"'] & \bullet \arrow[d, "\in R"] \\ \bullet \arrow[r] \arrow[ur, dashed, "\exists \text{ lift}"] & \bullet \end{tikzcd} \qquad L \pitchfork R$$

$\square$

L and R are closed under many operations:

- retracts
- composition
- pushout (for L), pullback (for R)
- coproducts (for L), products (for R)

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\in L"'] & \bullet \arrow[d, "\Rightarrow \in L"] \\ \bullet \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \bullet \end{tikzcd}$$

Note If $\exists$ is replaced by $\exists!$ (unique lift),
one speaks of L and R being orthogonal $(L \perp R)$
and a factorization system.

E.g. (n-connected, n-truncated) in HoTT

Generalizations:

- Algebraic wfs (Grandis–Tholen, Garner)
- Fibred awfs (Swan)

There are several general theorems constructing a wfs (L,R) generated by some set/category I of maps. Small object arguments

$$R \overset{!}{=} I^{\pitchfork}$$

Excursion: Secret sauce of homotopy theory

Pushout/pullback constructions!

Your homotopy theorist does not want you to know about this!

Given a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$, the pushout or pullback construction of F is a functor $\hat{F} : \mathcal{C}^{\to} \times \mathcal{D}^{\to} \to \mathcal{E}^{\to}$.

By example:

- The pushout product $f \hat{\times} g$ of $\begin{array}{c} A \\ \downarrow f \\ B \end{array}$ and $\begin{array}{c} C \\ \downarrow g \\ D \end{array}$:

$$\begin{tikzcd}
A \times C \arrow[rr, "A \times g"] \arrow[dd, "f \times C"'] & & A \times D \arrow[dl] \arrow[dd, "f \times D"] \\
& \bullet \arrow[dr, "f \hat{\times} g"] & \\
B \times C \arrow[ur] \arrow[rr, "B \times g"'] & & B \times D
\end{tikzcd}$$

- The pullback exponential $\widehat{\exp}(f,u)$ of $\begin{array}{c} Y \\ \downarrow u \\ X \end{array}$ with $\begin{array}{c} A \\ \downarrow f \\ B \end{array}$:

$$\begin{tikzcd}
Y^B \arrow[rr, "Y^f"] \arrow[dr, "\widehat{\exp}(f,u)"] \arrow[dd, "u^B"'] & & Y^A \arrow[dd, "u^A"] \\
& \bullet \arrow[ur] \arrow[dl] & \\
X^B \arrow[rr, "X^f"'] & & X^A
\end{tikzcd}$$

The magic: properties of F lift to $\hat{F}$!

- Monoidal (closed) structure on $\mathcal{C}$ lifts to $\mathcal{C}^{\to}$ ← Special case of Day convolution
- (co)continuity in each argument
- $\hat{F}$ preserves composition (up to pushout/pullback) in each argument.
- $\hat{F}$ preserves pushout/pullback in each argument.

If $F(X,-) \dashv G(X,-)\ \forall X$, then $\underset{\text{pushout construction}}{\hat{F}(f,g)} \pitchfork h \Leftrightarrow g \pitchfork \underset{\text{pullback construction}}{\hat{G}(f,h)}$ interaction with lifting!

Amazing!

Homotopical structure on sSet (cont.)

Boundary inclusions generate wfs (C, TF).

cofibrations $\rightarrowtail$ trivial fibrations $\overset{\sim}{\twoheadrightarrow}$ triv

Horn inclusions generate wfs (TC, F).

trivial cofibrations $\overset{\sim}{\rightarrowtail}$ triv fibrations $\twoheadrightarrow$

These form the Kan model structure on simplicial sets.

Lemma

{Horn inclusion} and $\left\{ \begin{array}{c} \{0\} \\ \downarrow \\ \Delta[1] \end{array}, \begin{array}{c} \{1\} \\ \downarrow \\ \Delta[1] \end{array} \right\} \hat{\times}$ {boundary inclusion} generate the same wfs.

Can also use C

endpoint inclusions for interval $\Delta[1]$

So can describe fibrations without using horns! Instead, use interval $\Delta[1]$ and reduce to trivial fibrations:

$\begin{array}{c} Y \\ \downarrow p \\ X \end{array}$ fibration

$\Leftrightarrow \{\{i\} \hookrightarrow \Delta[1]\} \hat{\times} \{\text{bound. incl.}\} \pitchfork p$

$\Leftrightarrow \{\text{bound. incl.}\} \pitchfork \widehat{\exp}(\{i\} \hookrightarrow \Delta[1], p)$ for $i = 0, 1$

$\Leftrightarrow \widehat{\exp}\left(\begin{array}{c} \{i\} \\ \downarrow \\ \Delta[1] \end{array}, \begin{array}{c} Y \\ \downarrow p \\ X \end{array} \right)$ trivial fibration

$$\begin{array}{c} \| \\ Y^{\Delta[1]} \\ \downarrow \\ Y \times_X X^{\Delta[1]} \end{array}$$

Because we only rely on an interval, this approach to fibrations makes sense also in other settings:

- groupoids
- cubical sets*

* the cartesian cubical model uses $\widehat{\exp}_I\left(\begin{array}{c} I \\ \downarrow \\ I \times I \end{array}, I^* p \right)$

sSet as a model of HoTT

The presheaf category $sSet = \widehat{\Delta}$ supports a model of "extensional" MLTT as covered in a previous part.

We refine it by adding a component to the types (contexts, substitutions, terms, context extensions do not change):

$$Ty(\Gamma) = \left\{ \left(\begin{array}{c} \Gamma.A \\ \downarrow p \\ \Gamma \end{array}, \ p \ \underline{\text{(Kan) fibration}} \right) \right\}$$

In displayed form, i.e. $A \in \widehat{\int \Gamma}$, to interpret substitution strictly functorially

Total space (non-displayed)

Two interpretations:
- Proof irrelevant: just a property of p, i.e. a proposition.
  - Voevodsky's original model
- Proof relevant: a lifting operation.
  - Better when working constructively

Dependent sums

Given by closure of fibrations under binary composition:

$$\begin{tikzcd}
\Gamma.\Sigma(A,B) \arrow[r, phantom, "\simeq"] \arrow[ddr, bend right] & \Gamma.A.B \arrow[d, two heads] \\
 & \Gamma.A \arrow[d, two heads] \\
 & \Gamma
\end{tikzcd}$$

Unit types

Same, but using nullary composition.

Path types*

*Like identity types, but:
- β-rule holds only up to path
- strictly functorial ap

Given $A \in Ty(\Gamma)$, have $Path_A(-,-) \in Ty(\Gamma.(x,y:A))$ given by:

Pullback exponent over Γ of $\Gamma.A \to \Gamma$ with $\partial\Delta[1] \to \Delta[1]$

$$\begin{tikzcd}
\Gamma.A.A.Path_A \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & (\Gamma.A)^{\Delta[1]} \arrow[d, "\widehat{\exp}\left(\begin{smallmatrix}\partial\Delta[1]\\ \downarrow \\ \Delta[1]\end{smallmatrix}, \begin{smallmatrix}\Gamma.A\\ \downarrow \\ \Gamma\end{smallmatrix}\right)"] & \\
\Gamma.A.A \simeq \Gamma \times_{\Gamma^2} (\Gamma.A)^2 \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & \Gamma^{\Delta[1]} \times_{\Gamma^2} (\Gamma.A)^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\Gamma.A)^2 \arrow[d] \\
\Gamma \arrow[rr] & & \Gamma^{\Delta[1]} \arrow[r] & \Gamma^2
\end{tikzcd}$$

r_A, J, β: exercise, using that $\Gamma.A \xrightarrow{r_A} \Gamma.A.A.Path_A$ is a strong deformation retract. SDR

Identity types

Using classical logic every mono is a cofibration.

$f \in C$, f SDR $\Rightarrow f \in TC$

Since r mono: $r_A \in TC$.

Given $T \in Ty(\Gamma.A.A.Path_A)$ with $d \in Tm(\Gamma.A, T[r])$:

$$\begin{tikzcd}
\Gamma.A \arrow[rr, "d"] \arrow[d, "triv"'] \arrow[drr, phantom, "\beta"] & & \Gamma.A.A.Path_A.T \arrow[d] \\
\Gamma.A.A.Path_A \arrow[rr, equal] \arrow[urr, dashed, "J_{c,d}"] & & \Gamma.A.A.Path_A
\end{tikzcd}$$

This is avoided for Path-types when working proof-relevantly.

Caveat: This is not substitution-stable!

Solution: Construct J once in "universal context Γ" and define J in general by restriction.

Needs closure of TC under pullback along F

$F^*(TC) \subseteq TC$

Then $Path_A$ functions as Id_A.

Another approach: Swan

Define Id_A from Path_A using (C, TF)-factorization:

$$\begin{tikzcd}
\Gamma.A \arrow[rr, "r_A", "\mathrm{SDR}"'] \arrow[dr, "\mathrm{refl}_A"'] & & \Gamma.A.A.\mathrm{Path}_A \\
& \Gamma.A.A.\mathrm{Id}_A \arrow[ur, two heads, "\mathrm{triv}"'] &
\end{tikzcd}$$

$\mathrm{refl}_A \in TC$ (constructively)

But to interpret everything in substitution-stable way, more is needed:

- switch to "uniform fibrations" as types (as in cubical approaches),
- restricting to cofibrant fragment could help.

Dependent products

Given $A \in \mathrm{Ty}(\Gamma)$, $B \in \mathrm{Ty}(\Gamma.A)$,

interpret $\Pi(A,B) \in \mathrm{Ty}(\Gamma)$ as in "extensional" model.

To show: $\Gamma.\Pi(A,B) \to \Gamma$ is fibration.

$$\begin{tikzcd}
\Gamma.A.B \arrow[d, two heads] & \Gamma.\Pi(A,B) \arrow[d, "\leftarrow ?"] \\
\Gamma.A \arrow[r, two heads] & \Gamma
\end{tikzcd}$$

By adjointness, equivalent to:

Pullback along $\Gamma.A \twoheadrightarrow \Gamma$ preserves trivial cofibrations.

This holds for $\underline{\Gamma.A \text{ cofibrant}}$. One approach:

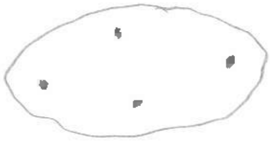
$$\begin{tikzcd}
0 \arrow[d, "\text{cofibration}"] \\
\Gamma.A
\end{tikzcd}$$

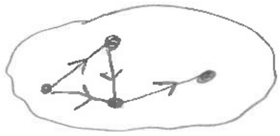
$$TC = \underbrace{C \cap \{\text{"strong" homotopy equivalences}\}}_{\text{separately stable under pullback along } \Gamma.A \twoheadrightarrow \Gamma}$$

Digression: Classicality of "all objects are cofibrant"

$X \in sSet$ is cofibrant iff we can "build up" X from the empty object by successively filling sets of boundaries with "insides".

Representative strategy:

1. Fill in all the points X_0: 

2. Fill in all the lines X_1: 

3. Fill in all the triangles X_2: 

...

Try to spot the problem!

The problem

When we add the points X_0, we automatically add degeneracies of points in higher dimensions: "constant" lines, triangles, etc.

So when filling in the lines, we must restrict ourselves to the non-degenerate lines $(X_1)_{nd}$.

Similarly for triangles and higher.

This requires us to decide degeneracy in all dimensions:

$$(X_n)_{deg} + (X_n)_{nd} \xrightarrow{\simeq} X_n.$$

This is the meaning of "X cofibrant".

Function extensionality

Simplest version to verify:

(*) Given $A \in Ty(\Gamma)$, $B \in Ty(\Gamma.A)$ contractible, have $\Pi(A,B)$ contractible.

Exercise: $\Leftrightarrow$ FunExt

Lemma $X \in Ty(\Gamma)$ contractible $\Leftrightarrow$ $\Gamma.X \downarrow \Gamma$ trivial fibration

Proof: Exercise!

By adjointness, (*) $\Leftrightarrow$ Pullback along $\Gamma.A \to \Gamma$ preserves cofibrations

Holds for $\Gamma.A$ cofibrant.

Universe

Our notion of type is <u>local</u>:

$$\mathrm{Ty}(\Gamma) \xrightarrow{\simeq} \{\text{coherent family } A_x \in \mathrm{Ty}(\Delta[n]) \text{ for } \Delta[n] \xrightarrow{x} \Gamma\}$$

Reason: fibrations defined via lifting against maps with representable target (horn inclusions).

Thus, any universe V in the "extensional" model induces a corresponding universe U classifying fibrations.

Lemma (Alignment):

Given a cofibration $\Delta \overset{\sigma}{\rightarrowtail} \Gamma$
and $A_1 \in \mathrm{Tm}(\Gamma, U)$, $B_0 \in \mathrm{Tm}(\Delta, U)$, $h \in \mathrm{Tm}(B_0 \simeq A_1[\sigma])$,
there is $A_0 \in \mathrm{Tm}(\Gamma, U)$, $g \in \mathrm{Tm}(\Gamma, A_0 \simeq A_1)$
restricting to B_0, h on σ.

$$\begin{tikzcd}
\Delta.B_0 \arrow[rr] \arrow[dd] \arrow[dr] & & U.El \arrow[dd, two heads] \\
& \Gamma.A_1 \arrow[ur, dashed] \arrow[dd] & \\
\Delta \arrow[rr] \arrow[dr, tail] & & U \\
& \Gamma \arrow[ur, dashed] &
\end{tikzcd}$$

Remaining:

(1) $U \in \mathrm{Ty}(1)$, i.e. U fibrant
(2) univalence

We show a version of (2) and use it to deduce (1)!

Univalence

In HoTT (exercise): $\mathcal{U}$ univalent $\Leftrightarrow \prod_{B:\mathcal{U}}$ is $\mathrm{Contr}\left(\sum_{A:\mathcal{U}} A \simeq B\right)$

Assuming (1), this follows from:

(2')
$$\begin{tikzcd} {[B:\mathcal{U}, A:\mathcal{U}, e:A\simeq B]} \arrow[d, "\text{trivial fibration}"] \\ {[B:\mathcal{U}]} \end{tikzcd}$$

For (2'), need lift

$$\begin{tikzcd} \partial\Delta[n] \arrow[r] \arrow[d, tail] & {[B:\mathcal{U}, A:\mathcal{U}, e:A\simeq B]} \arrow[d] \\ \Delta[n] \arrow[r] \arrow[ur, dashed] & {[B:\mathcal{U}]}. \end{tikzcd}$$

Using alignment*, this reduces to:

*and a version of it for the h-prop isEquiv

Lemma (Equivalence extension)

$$\begin{tikzcd} B_0 \arrow[rr, dashed] \arrow[dr, "\text{h-equiv}"] \arrow[dd, two heads] & & A_0 \arrow[dr, dashed, "\text{h-equiv}"] \arrow[dd, dashed, two heads] & \\ & B_1 \arrow[rr] \arrow[dl, two heads] & & A_1 \arrow[dl, two heads] \\ \Delta \arrow[rr, tail] & & \Gamma & \end{tikzcd}$$

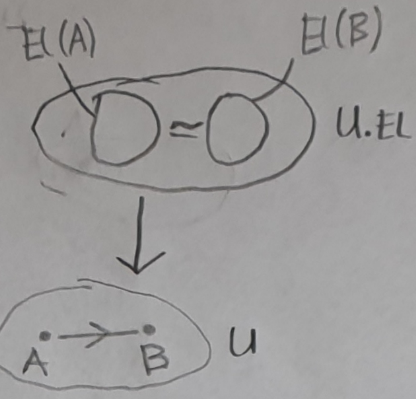
Can find A_0 as indicated.
$A_0 \to \Gamma$ classified by V if $A_1 \to \Gamma$ and $B_0 \to \Gamma$ are.

Proof: Take $A_0 \to A_1$ to be the dependent product of $B_0 \to B_1$ along $B_1 \to A_1$.
All assertions have elementary (but not so short) proofs. $\square$

Fibrancy of $\mathcal{U}$

Assuming that every object is cofibrant (classical), we have a map

$$\begin{tikzcd} \mathcal{U}^{\Delta[1]} \arrow[rr] \arrow[dr, two heads] & & {[A,B:\mathcal{U},\, A\simeq B]} \arrow[dl, two heads] \\ & \mathcal{U}^2. & \end{tikzcd}$$



To show $\mathcal{U}$ fibrant, we have to show:

- $\mathcal{U}^{\Delta[1]} \longrightarrow \mathcal{U}^{\{0\}} \in TF$
- $\mathcal{U}^{\Delta[1]} \longrightarrow \mathcal{U}^{\{1\}} \in TF.$

We only show the second claim:

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d, tail] & \mathcal{U}^{\Delta[1]} \arrow[d, "\text{can.}"] \\ \Gamma \arrow[r] \arrow[ur, dashed, "\exists"] & \mathcal{U}^{\{1\}} \end{tikzcd}$$

Using a trick ("filling from composition"), this follows (after quantifying over all such problems) from just a certain partial lift:

$$\begin{tikzcd} \Delta \arrow[r] \arrow[d, tail] & \mathcal{U}^{\Delta[1]} \arrow[r] \arrow[d] & \mathcal{U}^{\{0\}} \arrow[d] \\ \Gamma \arrow[r] \arrow[urr, dashed] & \mathcal{U}^{\{1\}} \arrow[r] & 1. \end{tikzcd}$$

This holds since $\mathcal{U}^{\Delta[1]} \to \mathcal{U}^{\{0,1\}}$ factors via $[A,B:\mathcal{U},\, A\simeq B]$ and the resulting map $[A,B:\mathcal{U},\, A\simeq B] \to \mathcal{U}^{\{0\}}$ is (2'), a trivial fibration.