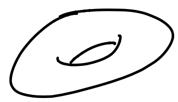
- Motivation

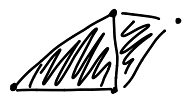
  - HoTT provides the theory for spaces.

    Benefits:

    - All encompassing (one theory applies to many models)
    - Machine checkable
    - A different perspective (from classical homotopy theory)

    Spaces are:





T

Actual topological spaces

Collections of points, paths, higher paths, ... (Kan complex)

Types in HoTT

The paths are:
- reflexive
- reversible
- composable

But many interesting structures are made of paths which are
- reflexive
- composable

<u>Examples</u>.

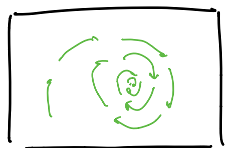
1. (Higher) categories
   - 0-cells (objects)
   - Between 0-cells A, B: 1-cells $\hom(A,B)$
   - Between 1-cells $f, g \in \hom(A,B)$: 2-cells $\hom(f,g)$

     $\vdots$

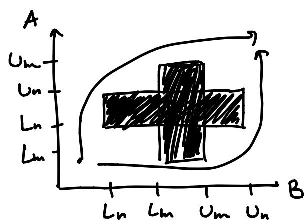
2. Directed spaces
   - Topological space T + subset $D \subseteq \mathrm{Paths}(T)$ of 'directed paths' closed under reflexivity paths, concatenation, etc...
   - Certain bicubical sets

Examples from applications:

A. Dynamical systems



B. Concurrent prosseces

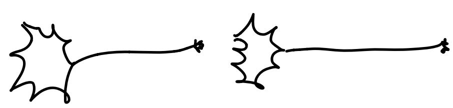


Processes A, B

Resources n, m

You can lock (L_n), unlock (U_n) the resources.

C. Networks of neurons?



I. A (directed) type theory for $(\infty,1)$-categories (Riehl, Shulman)

Def. An $(\infty,1)$-category is a higher category with 0-cells, 1-cells, 2-cells, ... where the 2-cells, 3-cells, 4-cells, ... are all reversible.

Note that (at every level) there are:

'undirected/reversible' paths (equivalences) $\leq$ 'directed/nonreversible' paths (cells)

Id-type

internal like in CuTT

Syntax: There are three layers:

- Cubes:
  - 2 cube
  - $0:$ cube
  - $1:$ cube

- Topes:

$$x,y:2 \vdash (x \leq y) \text{ tope}$$

$$x,y:2 \vdash (x \equiv y) \text{ tope}$$

$$x:2 \mid \cdot \vdash (x \leq x)$$

$$x,y,z:2 \mid (x \leq y),(y \leq z) \vdash (x \leq z)$$

$$x,y:2 \mid (x \leq y),(y \leq x) \vdash (x \equiv z)$$

$$x,y:2 \mid \cdot \vdash (x \leq y) \vee (y \leq x)$$

$$x:2 \vdash (0 \leq x),\ (x \leq 1)$$

$$\cdot \mid (0 \equiv 1) \vdash \bot$$

- ML Type Theory:

  - $$\frac{\Xi \mid \Phi \mid \Gamma \vdash a,b:A}{\Xi \mid \Phi \mid \Gamma \vdash \mathrm{Id}_A(a,b) \text{ type}}$$

    $\vdots$

  - $$\frac{\begin{array}{c} t:I \mid \phi \vdash \psi \\ \Xi, t:I \mid \Phi, \psi \mid \Gamma \vdash A \text{ type} \\ \Xi, t:I \mid \Phi, \phi \mid \Gamma \vdash a:A \end{array}}{\Xi \mid \Phi \mid \Gamma \vdash \left\langle \prod_{t:I\mid\psi} A \middle| {}^{\phi}_{a} \right\rangle}$$

    Only depends on $\Xi \mid \Phi$

    Dependent functions $\{t:I \mid \psi\} \to A$

    $$\begin{array}{ccc} \{t:I\mid\psi\} & \longrightarrow & A \\ \uparrow & \circlearrowleft & \nearrow_{a} \\ \{t:I\mid\phi\} & & \end{array}$$

    st $\{t:I\mid\phi\}$

    If A doesn't depend on $\{t:I\mid\psi\}$, write $\langle \{t:I\mid\psi\} \to A \mid {}^{\phi}_{a} \rangle$

  - $$\frac{\begin{array}{c} + \\ \Xi, t:I \mid \Phi, \psi \mid \Gamma \vdash b:A \\ \Xi, t:I \mid \Phi, \phi \mid \Gamma \vdash a \equiv b:A \end{array}}{\Xi \mid \Phi \mid \Gamma \vdash \lambda t.b : \left\langle \prod_{t:I\mid\psi} A \middle| {}^{\phi}_{a} \right\rangle}$$

Shapes:

$$\begin{array}{l} t:I \vdash \phi \text{ tope} \\ \quad \Updownarrow \\ \{t:I \mid \phi\} \text{ shape} \end{array}$$

Ex.

$$\Delta^0 := \{x:\mathbb{1} \mid \top\}$$

$$\Delta^1 := \{x:2 \mid \top\}$$

$$\Delta^2 := \{x,y:2 \mid y \leq x\}$$

$$\Delta^3 := \{x,y,z:2 \mid z \leq y \leq x\}$$

$\vdots$

$$\partial\Delta^1 := \{x:2 \mid (x \equiv 0) \vee (x \equiv 1)\}$$

$$\partial\Delta^2 = \{x,y:2 \mid (y \equiv 0) \vee (x \equiv 1) \vee x \equiv y\}$$

Def. Given $\Xi \mid \Phi \mid \Gamma \vdash x, y : A$, we get

$$\Xi, \underline{x : 2} \mid \Phi, \underline{(x \equiv 0) \vee (x \equiv 1)} \mid \Gamma \vdash [x,y] : A,$$

$\partial\Delta^1$

so define

$$\hom_A(x,y) := \left\langle \Delta^1 \to A \,\middle|^{\partial\Delta^1}_{[x,y]} \right\rangle.$$

functions

$$\begin{tikzcd} & \Delta^1 \arrow[r] & A \\ \text{st} & \partial\Delta^1 \arrow[u] \arrow[ur, "{[x,y]}"'] & \end{tikzcd}$$

Def. Given $\Xi \mid \Phi \mid \Gamma \vdash f : \hom_A(x,y),\ g : \hom_A(y,z),\ h : \hom_A(x,z)$, we get

$$\Xi, \underline{x, y : 2} \mid \Phi, \underline{(y \equiv 0) \vee (x \equiv 1) \vee (x \equiv y)} \mid \Gamma \vdash [f,g,h] : A,$$

$\partial\Delta^2$

so define

$$\hom^2_A\left(\begin{tikzcd}[column sep=small] & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd} \right) := \left\langle \Delta^2 \to A \,\middle|^{\partial\Delta^2}_{[f,g,h]} \right\rangle.$$

A type is <u>Segal</u> if for all $x \xrightarrow{f} y \xrightarrow{g} z$ in A,

$$\sum_{h : \hom_A(x,z)} \hom^2_A\left(\begin{tikzcd}[column sep=small] & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd} \right)$$

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d] & A \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

is contractible. In this case, the unique h is the <u>composite</u> of f and g.

We get

$$\mathrm{Id}_A(x,y) \subseteq \hom_A(x,y)$$

equivalences $\qquad$ cells

& a lot of higher category theory...

Not all dependent types $x : A \vdash D(x)$ type are functorial in A.

We always have

$$x,y : A,\ p : \mathrm{Id}_A(x,y) \vdash p_* : D(x) \to D(y),$$

but we don't always have

$$x,y : A,\ f : \hom_A(x,y) \vdash f_* : D(x) \to D(y).$$

Riehl – Shulman introduce 'covariant' dependent types where this <u>is</u> true, and Cavallo – Riehl – Sattler have proven <u>directed</u> univalence for it.

- Models: Segal spaces (bisimplicial sets which model $(\infty,1)$-categories) and similar

<u>II. A directed homotopy type theory</u> (North)

- Meant to represent directed spaces.

directed/irreversible paths >< undirected/reversible paths

↑ chosen 'flow' in the space — hom-type based on ML-type

↑ paths in ambient space — ML Id type

- Goal: reproduce ML Id-type with everything but reversibility

  In particular, any dependent type $x : A \vdash D(x)$ is functorial (in specific homomorphisms.)

- <u>Models</u>: Categories with directed wfs's (WIP, van den Berg – McCloskey – N)
  - categories, simplicial sets, directed spaces...

<u>Foundation</u>:

$$\frac{T\ \mathrm{TYPE}}{T^{\mathrm{core}}\ \mathrm{TYPE}} \qquad \frac{T\ \mathrm{TYPE}}{T^{\mathrm{op}}\ \mathrm{TYPE}} \qquad \frac{t : T^{\mathrm{core}}}{it : T} \qquad \frac{t : T^{\mathrm{core}}}{i^{\mathrm{op}}t : T^{\mathrm{op}}}$$

– Can think of OP, CORE as modalities (cf. modal type theory).

Rules for hom

Formation:

$$\frac{A \ \mathrm{TYPE} \quad a : A \quad b : A}{\mathrm{Id}_A(a,b) \ \mathrm{TYPE}} \qquad\qquad \frac{A \ \mathrm{TYPE} \quad a : A^{op} \quad b : A}{\hom_A(a,b) \ \mathrm{TYPE}}$$

Introduction:

$$\frac{A \ \mathrm{TYPE} \quad a : A}{r_a : \mathrm{Id}_A(a,a)} \qquad\qquad \frac{A \ \mathrm{TYPE} \quad a : A^{core}}{1_a : \hom_A(i^{op}a, ia)}$$

Elimination:

$$\frac{\begin{array}{c} A \ \mathrm{TYPE} \\ a : A, b : A, p : \mathrm{Id}_A(a,b) \vdash D(p) \ \mathrm{TYPE} \\ a : A \vdash d(a) : D(r_a) \end{array}}{a : A, b : A, p : \mathrm{Id}_A(a,b) \vdash \bar{d}(p) : D(p)}$$

Right

$$\frac{\begin{array}{c} A \ \mathrm{TYPE} \\ a : A^{core}, b : A, f : \hom_A(i^{op}a, b) \vdash D(p) \ \mathrm{TYPE} \\ a : A^{core} \vdash d(a) : D(1_a) \end{array}}{a : A^{core}, b : A, f : \hom_A(i^{op}a, b) \vdash \bar{d}(f) : D(f)}$$

Left

$$\frac{\begin{array}{c} A \ \mathrm{TYPE} \\ a : A, b : A^{core}, f : \hom_A(a, ib) \vdash D(p) \ \mathrm{TYPE} \\ a : A^{core} \vdash d(a) : D(1_a) \end{array}}{a : A, b : A^{core}, f : \hom_A(a, ib) \vdash \bar{d}(f) : D(f)}$$

Computation:

$$\frac{\cdots}{a : A \vdash d(a) \equiv \bar{d}(r_a) : D(r_a)}$$

Right/left

$$\frac{\cdots}{a : A^{core} \vdash d(a) \equiv \bar{d}(1_a) : D(1_a)}$$

Functoriality / transport: Given $x : A \vdash D(x)$ TYPE, $f : \hom_A(i^{op}a, b)$, $d : (i^{op}a)$,
get $f_* d : D(b)$.

$\rightsquigarrow$ Composition: $f : \hom_A(a, ib)$, $g : \hom_A(i^{op}b, c) \vdash g \circ f : \hom_A(a, c)$.

Many questions: • Σ, Π, other type formers
• Univalence ...

Many other interesting ideas:

Directed type theory:

1. Finster, Mimram (2017): A type-theoretical definition of weak ω-categories
2. Licata, Harper (2011): 2-Dimensional directed type theory
3. Nuyts (2015): Towards a directed homotopy type theory using 4 kinds of variance
4. Warren (2013): Directed type theory.

Modal type theory

1. Licata, Shulman (2016): Adjoint logic with a 2-category of modes
2. Rijke, Shulman, Spitters (2020): Modalities in homotopy type theory

Directed topology

1. Fajstrup, Goubault, Haucourt, Mimram, Raussen (2016): Directed algebraic topology and concurrency
2. Krishnan (2013): Cubical approximation for directed topology
3. Dłotko et al (2016): Topological analysis of the connectome of digital reconstructions of neural microcircuits