

The rules of type theory

Overview

- following¹ consider MLTT with
 - Σ -types (dependent sums)
 - Π -types (dependent products)
 - **Id**-types (identity types), and
 - a base type **N** of natural numbers (no universes and (higher) inductive types)
- use a very verbose syntax with lots of *type annotations*
 - good for theoretical reasoning, but not practical
 - omitting type decorations can be justified a posteriori since in most cases they can be uniquely reconstructed
- basic syntactic entities are **types** A, B, C, \dots , **terms** s, t, u, \dots , **contexts**

$$\Gamma, \Delta, \dots = (x_1:A_1, x_2:A_2, \dots, x_n:A_n)$$

and **judgments**

¹M. Hofmann. “Syntax and semantics of dependent types”. In: *Extensional Constructs in Intensional Type Theory*. Springer, 1997, pp. 13–54.

6 forms of judgments

$\Gamma \vdash \text{cxt}$	' Γ is a context'
$\Gamma \equiv \Delta \vdash \text{cxt}$	' Γ and Δ are equal contexts'
$\Gamma \vdash A \text{ type}$	A is a type in context Γ
$\Gamma \vdash A \equiv B \text{ type}$	A and B are equal types in context Γ
$\Gamma \vdash t : A$	t is a term of type A in context Γ
$\Gamma \vdash t \equiv u : A$	t and u are equal terms of type A in context Γ

Types of rules

For each type former – i.e. Σ , Π , Id , \mathbf{N} – there are

- formation
- introduction
- elimination
- computation
- congruence

rules.

Furthermore, there are the following basic rules.

- empty context and context extension
- variable, weakening, substitution
- reflexivity, transitivity, symmetry for definitional equality
- conversion

Context rules

$$\frac{}{\diamond \vdash \text{cxt}}$$

empty context

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x:A \vdash \text{cxt}}$$

context extension

$$\frac{\Gamma \equiv \Delta \vdash \text{cxt} \quad \Gamma \vdash A \equiv B \text{ type}}{(\Gamma, x:A) \equiv (\Delta, x:B) \vdash \text{cxt}}$$

congruence for context extension

Variable, weakening and substitution

$$\frac{\Gamma, x:A, \Delta \vdash \text{cxt}}{\Gamma, x:A, \Delta \vdash x : A}$$

variable

$$\frac{\Gamma, \Delta \vdash \mathcal{J} \quad \Gamma \vdash A \text{ type}}{\Gamma, x:A, \Delta \vdash \mathcal{J}}$$

weakening

$$\frac{\Gamma, x : A, \Delta \vdash \mathcal{J} \quad \Gamma \vdash t : A}{\Gamma, \Delta[t/x] \vdash \mathcal{J}[t/x]}$$

substitution

Equivalence rules for definitional equality

$$\frac{\Gamma \vdash \text{cxt}}{\Gamma \equiv \Gamma \vdash \text{cxt}}$$

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A \equiv A \text{ type}}$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash t \equiv t : A}$$

$$\frac{\Gamma \equiv \Delta \vdash \text{cxt}}{\Delta \equiv \Gamma \vdash \text{cxt}}$$

$$\frac{\Gamma \vdash A \equiv B \text{ type}}{\Gamma \vdash B \equiv A \text{ type}}$$

$$\frac{\Gamma \vdash t \equiv u : A}{\Gamma \vdash u \equiv t : A}$$

$$\frac{\Gamma \equiv \Delta \vdash \text{cxt} \quad \Delta \equiv \Theta \vdash \text{cxt}}{\Gamma \equiv \Theta \vdash \text{cxt}}$$

$$\frac{\Gamma \vdash A \equiv B \text{ type} \quad \Gamma \vdash B \equiv C \text{ type}}{\Gamma \vdash A \equiv C \text{ type}}$$

$$\frac{\Gamma \vdash t \equiv u : A \quad \Gamma \vdash u \equiv v : A}{\Gamma \vdash t \equiv v : A}$$

Conversion rules

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \equiv \Delta \vdash \text{cxt}}{\Delta \vdash A \text{ type}}$$

type conversion

$$\frac{\Gamma \vdash t : A \quad \Gamma \equiv \Delta \vdash \text{cxt} \quad \Gamma \vdash A \equiv B \text{ type}}{\Delta \vdash t : B}$$

term conversion

Π types

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x:A \vdash B \text{ type}}{\Gamma \vdash \Pi x:A. B \text{ type}}$$

formation

$$\frac{\Gamma, x:A \vdash t : B}{\Gamma \vdash (\lambda x:A. t^B) : \Pi x:A. B}$$

introduction

$$\frac{\Gamma \vdash t : \Pi x:A. B \quad \Gamma \vdash u : A}{\Gamma \vdash \text{app}_{[x:A]B}(t, u) : B[u/x]}$$

elimination

$$\Gamma \vdash \text{app}_{[x:A]B}(\lambda x. t^B, u) \equiv t[u/x] : B[u/x]$$

computation (β -rule)

$$\Gamma \vdash t \equiv (\lambda x:A. \text{app}_{[x:A]B}(t, x)^B) : \Pi x:A. B$$

computation (η -rule)

$$\frac{\Gamma \vdash A \equiv A' \text{ type} \quad \Gamma, x:A \vdash B \equiv B' \text{ type}}{\Gamma \vdash \Pi x:A. B \equiv \Pi x:A'. B'}$$

formation congruence

$$\frac{\Gamma, x:A \vdash t \equiv u : B}{\Gamma \vdash (\lambda x:A. t^B) \equiv (\lambda x:A. u^B) : \Pi x:A. B}$$

introduction congruence

$$\frac{\Gamma \vdash t \equiv t' : \Pi x:A. B \quad \Gamma \vdash u \equiv u' : A}{\Gamma \vdash \text{app}_{[x:A]B}(t, u) \equiv \text{app}_{[x:A]B}(t', u') : B[u/x]}$$

elimination congruence

Σ types

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x:A \vdash B \text{ type}}{\Gamma \vdash \Sigma x:A. B \text{ type}}$$

formation

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : B[t/x]}{\text{pair}_{[x:A]B}(t, u) : \Sigma x : A. B}$$

introduction

$$\frac{\begin{array}{l} \Gamma, z : \Sigma x : A. B \vdash C \text{ type} \\ \Gamma, x:A, y:B \vdash t : C[\text{pair}_{[x:A]B}(x, y)/z] \\ \Gamma \vdash u : \Sigma x : A. B \end{array}}{\Gamma \vdash \text{rec}_{[z]C}^{\Sigma x:A. B}([x, y]t, u) : C[u/z]}$$

elimination

$$\begin{array}{l} \Gamma \vdash \text{rec}_{[z]C}^{\Sigma x:A. B}([x, y]u, \text{pair}_{[x:A]B}(s, t)) \\ \equiv u[s/x, t/y] \quad : \quad C[\text{pair}_{[x:A]B}(s, t)/z] \end{array}$$

computation

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congruence rules

The type \mathbb{N} of natural numbers

$$\frac{\Gamma \vdash \text{cxt}}{\Gamma \vdash \mathbf{N} \text{ type}}$$

formation

$$\frac{\Gamma \vdash \text{cxt}}{\Gamma \vdash 0 : \mathbf{N}} \quad \frac{\Gamma \vdash t : \mathbf{N}}{\Gamma \vdash \text{succ}(t) : \mathbf{N}}$$

introduction

$$\begin{array}{l} \Gamma, x : \mathbf{N} \vdash A \text{ type} \\ \Gamma \vdash s : A[0/x] \\ \Gamma, x : \mathbf{N}, y : A \vdash t : A[\text{succ}(x)/x] \\ \Gamma \vdash u \in \mathbf{N} \end{array} \quad \frac{}{\Gamma \vdash \text{rec}_{[x]A}^{\mathbf{N}}(s, [x, y]t, u) : A[u/x]}$$

elimination

$$\Gamma \vdash \text{rec}_{[x]A}^{\mathbf{N}}(s, [x, y]t, 0) \equiv s : A[u/x]$$

computation

$$\begin{array}{l} \Gamma \vdash \text{rec}_{[x]A}^{\mathbf{N}}(s, [x, y]t, \text{succ}(u)) \\ \equiv t[u/x, \text{rec}_{[x]A}^{\mathbf{N}}(s, [x, y]t, u)/y] : A[u/x] \end{array}$$

computation

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2 congruence rules

Identity types

$$\frac{\Gamma \vdash t : A \quad \Gamma \vdash u : A}{\Gamma \vdash \text{Id}_A(t, u) \text{ type}}$$

formation

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \text{refl}_A(t) : \text{Id}_A(t, t)}$$

introduction

$$\begin{array}{l} \Gamma, x:A, y:A, p:\text{Id}_A(x, y) \vdash B \text{ type} \\ \Gamma, z:A \vdash s : B[z/x, z/y, \text{refl}_A(z)/p] \\ \Gamma \vdash t : A \quad \Gamma \vdash u : A \quad \Gamma \vdash v : \text{Id}_A(t, u) \\ \hline \Gamma \vdash \text{rec}_{[x,y,p]B}^{\text{Id}_A}([z]s, t, u, v) : B[t/x, u/y, v/p] \end{array}$$

elimination

$$\begin{array}{l} \Gamma \vdash \text{rec}_{[x,y,p]B}^{\text{Id}_A}([z]s, t, t, \text{refl}_A(t)) \\ \equiv s[t/z] : B[t/x, t/y, \text{refl}_A(t)/p] \end{array}$$

computation

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3 congruence rules