WIP: Coherence via big categories with families of locally cartesian closed categories

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The coherence problem

Locally cartesian closed (lcc) categories are natural categorical models of dependent type theory.

Substitution	Pullback
strictly functorial	functorial up to iso
$\tau[s_2][s_1] = \tau[s_2[s_1], s_1]$	$s_1^*(s_2^*(\tau)) \cong (s_2 \circ s_1)^*(\tau)$
commutes with type formers	preserves structure up to iso
$(\tau_1 \to \tau_2)[s] = \tau_1[s] \to \tau_2[s]$	$s^*(\tau_2^{\tau_1}) \cong s^*(\tau_2)^{s^*(\tau_1)}$

$\Longrightarrow$ Cannot interpret syntax directly

Coherence constructions

Prior art:

- Curien — Substitution up to isomorphism
- Hofmann — Giraud-Bénabou construction
- Lumsdaine, Warren, Voevodsky — (local) universes

These constructions interpret type theory in a given single lcc category.

This talk: Interpret type theory in the ("gros") category of all lcc categories.

- Interpretation of extensional type theory in a single lcc 1-category can be recovered by slicing
- Expected to interpret a (as of now, hypothetical) weak variant of intensional dependent type theory in arbitrary lcc quasi-categories

The big cwfs of lcc categories

Definition

Let $r \in \{1, \infty\}$. The cwf $\mathbf{Lcc}_r$ is given as follows:

- A context is a cofibrant lcc r-category Γ.
- $\mathrm{Ty}(\Gamma) = \mathsf{Ob}\,\Gamma$
- $\mathrm{Tm}(\Gamma, \sigma) = \mathrm{Hom}_\Gamma(1_\Gamma, \sigma)$
- $\mathrm{Hom}_{\mathrm{Ctx}}(\Gamma, \Delta) = \mathrm{Hom}_{\mathrm{sLcc}}(\Delta, \Gamma)$
- $\Gamma.\sigma$ is obtained from Γ by adjoining freely a morphism $v : 1 \to \sigma$:

$$
\begin{tikzcd}
\Gamma.\sigma \arrow[rr, dashed, "\exists! \langle F, w\rangle"] & & \mathcal{C} \\
\Gamma \arrow[u] \arrow[urr, "F"'] & &
\end{tikzcd}
$$

with F strict, $w : 1 \to F(\sigma)$ in $\mathcal{C}$ and $\langle F, w\rangle(v) = w$.

Main results

Theorem

Let $r \in \{1, \infty\}$*.*

- *The functors* $\mathbf{Lcc}_r^{\mathrm{op}} \to \mathrm{Lcc}_r$ *are equivalences of* $(2,1)$ *resp.* $(\infty, 1)$ *categories.*
- *Context extension in* $\mathbf{Lcc}_r$ *is well-defined.*
- *Denote by* $(\mathbf{Lcc}_r)_*$ *the category of pairs* (Γ, σ) *of contexts equipped with a base type* $\sigma \in \mathrm{Ty}(\Gamma)$*. Then the two functors* $(\mathbf{Lcc}_r)_*^{\mathrm{op}} \to \mathrm{Lcc}_r$

  $$(\Gamma, \sigma) \mapsto \Gamma.\sigma \qquad\qquad (\Gamma, \sigma) \mapsto \Gamma_{/\sigma}$$

  are strictly naturally equivalent.
- $\mathbf{Lcc}_1$ *supports* $\mathbf{\Pi}$, $\mathbf{\Sigma}$, *(extensional)* **Eq** *and* **Unit** *types.*

Recovering an interpretation in a single lcc category

Corollary

Every lcc 1-category $\mathcal{C}$ is equivalent to a cwf supporting $\mathbf{\Pi}$, $\mathbf{\Sigma}$, *(extensional)* **Eq** *and* **Unit** *types.*

Proof.

Let $\Gamma_{\mathcal{C}} \in \mathbf{Lcc}$ such that $\Gamma_{\mathcal{C}} \simeq \mathcal{C}$ as lcc categories. Let $\mathbf{C}$ be the least full on types and terms sub-cwf of $\mathbf{Lcc}_{/\Gamma_{\mathcal{C}}}$ supporting the type constructors above. Then $\mathbf{C} \simeq \Gamma_{\mathcal{C}} \simeq \mathcal{C}$ as lcc categories. $\square$

J-algebras

Definition

Let J be a set of morphisms in a category $\mathcal{C}$. A *J-algebra* is an object X of $\mathcal{C}$ *equipped with* lifts

$$\begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "j"'] & X \\ \cdot \arrow[ur, dashed, "\ell_j(p)"'] & \end{tikzcd}$$

for all $j \in J$ and arbitrary p. A J-algebra morphism is a morphism in $\mathcal{C}$ compatible with the $\ell_j(p)$. The category of J-algebras is denoted by $\mathcal{A}(J)$.

Duality of structure and property

Theorem (Nikolaus 2011)

Let $\mathcal{M}$ be a cofibrantly generated locally presentable model category whose cofibrations are the monomorphisms. Let J be a set of trivial cofibrations such that an object (!) X is fibrant iff it has the rlp. wrt. J. Denote by $\mathcal{A} = \mathcal{A}(J)$ the category of J-algebras. Then the evident forgetful functor $R : \mathcal{A} \rightleftarrows \mathcal{M} : L$ is a right adjoint (even monadic). The model category structure of $\mathcal{M}$ can be transferred to $\mathcal{A}$, and (R, L) is a Quillen equivalence.

in $\mathcal{M}$	in $\mathcal{A}$
all objects are cofibrant	all objects are fibrant
codomains might not have enough properties	domains might be too structured
$X \to R(L(X))$ is fib. replacement	$L(R(Y)) \to Y$ is cof. replacement

Model categories of lcc categories

Assumption

Let $r \in \{1, \infty\}$. There are cofibrantly generated locally presentable model categories Lcc_r such that

- *the cofibrations are the monomorphisms,*
- *the fibrant objects are lcc 1-categories resp. lcc quasi-categories, and*
- *the weak equivalences of fibrant objects are equivalences of (quasi-)categories.*

Definition

Fix sets $J_r \subseteq \mathrm{Lcc}_r$ as in Nikolaus's theorem. The category of strict lcc r-categories is given by $\mathrm{sLcc}_r = \mathcal{A}(J_r)$.

Thus $(\mathbf{Lcc}_r)^{\mathrm{op}} \subseteq \mathrm{sLcc}_r$ is the full subcategory of cofibrant objects.

Marking universal objects

Idea to construct Lcc_r: A category of (separated) presheaves over some base category S containing objects corresponding to universal objects.

Example

$(S_{\mathrm{Pb}})^{\mathrm{op}}$ is generated by $\Delta \to S_{\mathrm{Pb}}$ and a commuting square

$$\begin{tikzcd}
\mathrm{Pb} \arrow[r, "\mathrm{tr}"] \arrow[d, "\mathrm{bl}"] & {[2]} \arrow[d, "\delta^1"] \\
{[2]} \arrow[r, "\delta^1"] & {[1]}\,.
\end{tikzcd}$$

Then $\mathcal{M} = \{X \in \widehat{S_{\mathrm{Pb}}} \mid \mathrm{tr}, \mathrm{bl} : X_{\mathrm{Pb}} \rightrightarrows X_2 \text{ is jointly mono}\}$ and J_{Pb} is chosen such that $(\Lambda^n_k \subset \Delta^n) \in J_{\mathrm{Pb}}$ and

- marked squares have the universal property of pullback squares;
- there is a marked square completing any given cospan;
- marked squares are closed under isomorphism.

Proofs of the main results

Proposition

The functors $(\mathbf{Lcc}_r)^{\mathrm{op}} \to \mathrm{Lcc}_r$ *are equivalences of* $(2,1)$ *resp.* $(\infty, 1)$ *categories.*

Proof.

$\mathrm{sLcc}_r \to \mathrm{Lcc}_r$ is an equivalence by Nikolaus's theorem and $(\mathbf{Lcc}_r)^{\mathrm{op}} = \mathrm{sLcc}_{\mathrm{cof}}$. $\square$

Proposition

Context extension in $\mathbf{Lcc}_r$ is well-defined.

Proof.

$$\begin{tikzcd} & L(\langle\sigma\rangle) \arrow[r, tail, "L(i)"] \arrow[d] & L(\langle v : 1 \to \sigma\rangle) \arrow[d] \\ 0 \arrow[r, tail] & \Gamma \arrow[r, tail] & \Gamma.\sigma. \end{tikzcd} \tag{1}$$

□

Proposition

Denote by $(\mathbf{Lcc}_r)_*$ the category of pairs (Γ, σ) of contexts Γ equipped with a base type $\sigma \in \mathrm{Ty}(\Gamma)$. Then the two functors $(\mathbf{Lcc}_r)_*^{\mathrm{op}} \to \mathrm{Lcc}_r$

$$(\Gamma, \sigma) \mapsto \Gamma.\sigma \qquad\qquad (\Gamma, \sigma) \mapsto \Gamma_{/\sigma}$$

are strictly naturally equivalent.

Proof.

$\Gamma_{/\sigma}$ is a homotopy pushout of (1) in Lcc_r. □

Proposition

$\mathbf{Lcc}_1$ *supports* $\mathbf{\Pi}$, $\mathbf{\Sigma}$, *(extensional)* **Eq** *and* **Unit** *types.*

Proof ($\mathbf{\Pi}$).

Suppose $\Gamma.\sigma \vdash \tau$. Define $\Gamma \vdash \mathbf{\Pi}_\sigma(\tau)$ as image of τ under

$$\Gamma.\sigma \xrightarrow{D} \Gamma_{/\sigma} \xrightarrow{\Pi_\sigma} \Gamma$$

Now suppose $\Gamma \vdash u : \mathbf{\Pi}_\sigma\tau$. Then $\tilde{u} : \sigma^*(1) \to D(\tau)$ in $\Gamma_{/\sigma}$ by transposing along $\sigma^* \dashv \Pi_\sigma$. Map $\tilde{u}$ via $E : \Gamma_{/\sigma} \xrightarrow{\sim} \Gamma.\sigma$ and compose with component of natural equivalence $E(D(\tau)) \simeq \tau$ to obtain $\Gamma.\sigma \vdash \mathrm{App}(u) : \tau$. □

For $r = \infty$ all non-trivial equalities hold only up to path equality, e.g. the β law

$$\mathrm{App}(\lambda u)) = u$$

holds only up to a contractible choice of path.

Future work

Can some kind of weak type theory be interpreted in $\mathbf{Lcc}_\infty$?

The unit $\mathcal{C} \to \mathcal{C}^s$ arising from freely turning a monoidal category $\mathcal{C}$ into a *strict* monoidal category $\mathcal{C}^s$ is not generally an equivalence, but MacLane's theorem shows that it is if $\mathcal{C}$ is cofibrant.
Does this also work with lcc quasi-categories and a notion of strictness where e.g. the canonical simplex corresponding to β is required to be a degenerate?

Conclusion

- Yet another solution to the coherence problem for extensional dependent type theory
- Interpret in category of all lcc categories instead of a single one, recover interpretation in a single lcc by slicing
- Restrict to cofibrant objects in algebraic presentation of model categories of lcc categories
- Model context extension as 1-categorical pushout, not slice category
- Solves at least pullback coherence for quasi-categories

Stephen Lack, *Homotopy-theoretic aspects of 2-monads.*, Journal of Homotopy & Related Structures **2** (2007), no. 1.

Thomas Nikolaus, *Algebraic models for higher categories*, Indagationes Mathematicae **21** (2011), no. 1-2, 52–75.