The FOLDS theory associated to a contextual category

Chaitanya Leena Subramaniam[1*] and Michael Shulman[2]

[1] Department of Mathematics, University of San Diego
cleenasubramaniam@sandiego.edu
[2] Department of Mathematics, University of San Diego
shulman@sandiego.edu

Makkai's FOLDS (first order logic with dependent sorts) [Mak95] and Cartmell's GATs (generalised algebraic theories) [Car86] are both languages that extend ordinary logic with dependent sorts.

The primary advantage of GATs over FOLDS is that GATs allow dependently sorted functions to be represented by terms of the syntax. For instance, the GAT of categories has two dependently sorted function symbols c (for composition of arrows) and i (for identity arrows).

$$\text{Sorts:} \quad \begin{cases} \vdash O \quad \text{type} \\ x{:}O, y{:}O \vdash A(x,y) \quad \text{type} \end{cases}$$

$$\text{Function symbols:} \quad \begin{cases} x{:}O \vdash i(x) : A(x,x) \\ x,y,z{:}O, f{:}A(x,y), g{:}A(y,z) \vdash c(g,f) : A(x,z) \end{cases}$$

$$\text{Axioms:} \quad \begin{cases} x,y{:}O, f{:}A(x,y) \vdash c(i(y),f) = f : A(x,y) \\ x,y{:}O, f{:}A(x,y) \vdash c(f,i(x)) = f : A(x,y) \\ x,y,z,w{:}O, f{:}A(x,y), g{:}A(y,z), h{:}A(z,w) \vdash c(h,c(g,f)) = c(c(h,g),f) : A(x,w) \end{cases}$$

Figure 1: The GAT of categories.

On the other hand, FOLDS is purely relational—in FOLDS, a function must be presented by a relation symbol (a dependent sort) along with axioms requiring the relation to be functional. The FOLDS theory of categories has two sorts T and I that are propositions and that, along with axioms, encode the operations of composition and of identity arrows as functional relations.

$$\text{Sorts:} \quad \begin{cases} \vdash O \quad \text{type} \\ x{:}O, y{:}O \vdash A(x,y) \quad \text{type} \\ x,y,z{:}O, f{:}A(x,y), g{:}A(y,z), h{:}A(x,z) \vdash T(f,g,h) \quad \text{prop.} \\ x{:}O, f{:}A(x,x) \vdash I(f) \ \text{prop.} \end{cases}$$

$$\text{Axioms:} \quad \begin{cases} x{:}O \vdash \exists! i{:}A(x,x).I(i) \\ x,y,z{:}O, f{:}A(x,y), g{:}A(y,z) \vdash \exists! h{:}A(x,z).T(f,g,h) \\ x,y{:}O, i{:}A(x,x), f{:}A(x,y), I(i) \vdash T(i,f,f) \\ x,y{:}O, i{:}A(y,y), f{:}A(x,y), I(i) \vdash T(f,i,f) \\ x,y,z,w{:}O, f{:}A(x,y), g{:}A(y,z), h{:}A(z,w), j{:}A(x,z), k{:}A(y,w), l{:}A(x,w), \\ \qquad\qquad T(f,g,j), T(g,h,k), T(j,h,l) \vdash T(f,k,l) \end{cases}$$

Figure 2: The FOLDS theory of categories.

[*] Speaker.

A drawback of GATs is that two models in sets of any GAT are indistinguishable (under validity of sentences) only if they are isomorphic. Indistinguishability of models in sets of a FOLDS theory is a weaker, but often more useful, notion of equivalence (called "FOLDS-equivalence") than isomorphy. For example, FOLDS-equivalences of models of the FOLDS theory of categories coincide with equivalences of categories. Furthermore, FOLDS-equivalence induces a relation of indistinguishability between the elements of a model of a FOLDS theory, and this relation is weaker than strict equality of elements of a set—for instance, two objects of a category (a model of the FOLDS theory of categories) are indistinguishable just when they are isomorphic. Consequently, GATs allow for algebraic structure to be defined in addition to properties, whereas FOLDS theories only express properties, but provide a better notion of equivalence of models.

In the present work, we provide a way to achieve the best of both worlds through a process that converts a suitable[1] GAT into a FOLDS theory. We describe an inductive construction that, given a GAT suitably presented as a cell complex of contextual categories, produces a FOLDS theory with the same category of models. This construction is "canonical" in the following sense: it proceeds by transforming the generating function symbols of the input GAT into dependent sorts of the output FOLDS theory, along with axioms requiring that these sorts encode functional relations. For example, the construction transforms the GAT of categories into the FOLDS theory of categories, and more generally, produces the FOLDS theories that one would "write down by hand".

Both GATs and FOLDS can be interpreted in homotopy type theory (HoTT), by interpreting dependent sorts as types; however, models in HoTT of a FOLDS theory satisfy a particularly strong structure identity principle. In [ANST21], the type of *indiscernibilities* between two elements (of the same type) of a model in HoTT of a suitable FOLDS theory is defined, and a model is said to be *univalent* if its types of indiscernibilities are equivalent to the corresponding identity types. Univalent models of a FOLDS theory satisfy a *univalence principle*: for any two univalent models A and B, the identity type $A = B$ is equivalent to the type of FOLDS-equivalences between A and B. Since FOLDS theories are purely relational, an important problem (left open in [ANST21]) is to describe a univalence principle for structures in HoTT that have algebraic operations in addition to properties. Since our construction produces a FOLDS theory from a GAT, the techniques of [ANST21] can be applied directly to provide a univalence principle for structures defined by a suitable GAT.

References

[ANST21] Benedikt Ahrens, Paige Randall North, Michael Shulman, and Dimitris Tsementzis. The Univalence Principle. *arXiv e-prints*, pages arXiv–2102, 2021.

[Car86] John Cartmell. Generalised algebraic theories and contextual categories. *Annals of pure and applied logic*, 32:209–243, 1986.

[Mak95] Michael Makkai. First order logic with dependent sorts, with applications to category theory. *Preprint:* http://www.math.mcgill.ca/makkai, 1995.

[1] Our construction applies only to GATs without sort equations (i.e., axioms of the form $\Gamma \vdash A = B$ type) among their axioms.