Unordered addition from biproducts

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A fundamental fact about commutative monoids is that given a finite number of elements, their sum can be computed by adding the elements in any order. The naturals under addition form a typical example of a commutative monoid. A natural number can be understood as an isomorphism class of finite sets, and indeed the type FinSet of finite sets under coproduct forms a higher kind of commutative monoid – to be precise, a symmetric monoidal category. While $\mathbb{N}$ is a 0-type and FinSet is a 1-type, there is also a good generalisation of commutative monoids to general homotopy types, namely E_∞-spaces. The invertible counterpart to E_∞-spaces spaces are connective spectra. While (connective) spectra can easily be defined in homotopy type theory – in short, one considers a sequence of pointed types where each one is the loop space of the next – a definition of general E_∞-spaces in homotopy type theory remains elusive.

The problem is the same as with other higher algebraic notions: there is an infinite tower of coherences involved and we do not know how to state them all at once. What we can do is state some of them. In particular, the fundamental fact about commutative monoids mentioned above is expected to generalise to E_∞-spaces in the following way. We say a type I is finite if I merely is equivalent to a standard finite type $[n]$ for some natural n, called the cardinality of I. An unordered (n-)tuple of elements of X consists of a finite type I (of cardinality n) together with a map $I \to X$. Given an unordered tuple of elements of an E_∞-space, we expect to be able to define their sum. Moreover, we expect the usual rules for indexed sums to hold, such as $\sum_i \sum_j a_{ij} = \sum_{(i,j)} a_{ij}$. In short, we say that E_∞-spaces have unordered addition. For example, FinSet has unordered addition since given a tuple $J : I \to \mathsf{FinSet}$, the sigma-type $(i : I) \times J(i)$ is again a finite set.

While the notion of unordered addition is easy to express synthetically, it encapsulates a lot of information. For example, writing $\{a, b\}$ for the unordered pair given by the map $(a, b) : [2] \to A$, we have $\{a, b\} = \{b, a\}$ since there is an equivalence $[2] \simeq [2]$ swapping the two elements. Hence any addition operation defined on unordered pairs must be commutative in the naive sense that $a + b = b + a$. Moreover, the composite path $\{a, b\} = \{b, a\} = \{a, b\}$ is always reflexivity (essentially because the composite equivalence $[2] \simeq [2] \simeq [2]$ is the identity), meaning that the composite path $a + b = b + a = a + b$ is reflexivity. These are only two cases of an infinite tower of coherences.

Now if connective spectra are examples of E_∞-spaces, and E_∞-spaces have unordered addition, it should follow that (connective) spectra have unordered addition. The goal of this work is to prove this in homotopy type theory.

To do this, we first take a step back and characterise addition by a defining property.[1] Given a spectrum X and a type I, we can form a new spectrum X^I which is the categorical product of I-many copies of X [3, Definition 5.4.1]. For a finite type I, unordered addition will be a map $X^I \to X$ of spectra, characterised in the following way. For each $i : I$, we get a map $\delta_i : X \to X^I$ whose jth component for $j : I$ is id_X if $i = j$ and 0 otherwise. Here we used that finite types have decidable equality, and that there is a 'zero morphism' from any spectrum to any other. Addition will be the unique morphism $\Sigma : X^I \to X$ with the structure that for each $i : I$ the composite $\Sigma \circ \delta_i : X \to X^I \to X$ is the identity on X. An upshot

[1] More precisely, this will generally be structure and not property.

of this characterisation is that 'there is a unique morphism with the following structure' is a *proposition*, so to prove it we may assume I is purely equivalent to a standard finite type $[n]$. The statement is immediate for $n = 0$, and the following theorem deals with the case $n = 2$.

Theorem 1. *Given spectra A, B, C and morphisms $f : A \to C$, $g : B \to C$, the type of triples (h, p, q) where $h : A \times B \to C$*[2]*, $p : f = h \circ (\mathsf{id}_A, 0)$ and $q : g = h \circ (0, \mathsf{id}_B)$ is contractible. In other words, there is a certain equivalence between $A \times B \to C$ and $(A \to C) \times (B \to C)$.*

In short, we say that the (wild) category of spectra has *biproducts*. Inducting on n and following standard 1-categorical reasoning adapted to work for wild categories, we arrive at the following consequence.

Theorem 2. *Given a spectrum X and a finite type I, the type of pairs (Σ, p), where $\Sigma : X^I \to X$ and $p : (i : I) \to \mathsf{id}_X = \Sigma \circ \delta_i$, is contractible.*

Thus spectra do have unordered addition. We define addition as a map of spectra, but it in particular gives an operation $\Sigma : X_0^I \to_{\mathrm{pt}} X_0$ on the underlying type X_0 of X. From its characterisation, we can also directly verify expected properties.

We now discuss a couple of applications. First we consider Eckmann-Hilton, which says that for any type A, the second loop space $\Omega^2 A$ has commutative addition. In general, $\Omega^2 A$ will not be a spectrum, and indeed it might not have unordered addition. However by the stabilisation theorem [1], the 0-truncation $\|\Omega^2 A\|_0$ is always a spectrum in a canonical way. Hence $\|\Omega^2 A\|_0$ has unordered addition. It can be seen that this addition is compatible with the ordered addition on $\Omega^2 A$. Thus for $a, b : \Omega^2 A$, we have $|a+b|_0 = |b+a|_0$, or equivalently, $\|a + b = b + a\|_{-1}$. This can be understood as a weak form of Eckmann–Hilton. Applying this result to the universal case of $a, b : \Omega^2(S^2 \vee S^2)$ and appealing to the existence property of homotopy type theory[3], one can recover the full strength of Eckmann–Hilton. More interestingly, we can apply the same reasoning to $\|\Omega^3(S^3 \vee S^3)\|_1$ to get a proof of syllepsis on general third loop spaces.

Our second application is to the definition of the sign homomorphism. We present a version of Cartier's argument (see also [4]). Synthetically, the sign homomorphism consists of a pointed map $BS_n \to_{\mathrm{pt}} BS_2$ from the type BS_n of n-element types to the type BS_2 of 2-element types. We describe what this function does to a general n-element type A. Note that the type $\binom{A}{2}$ of 2-element subtypes of A is a finite type of cardinality $\binom{n}{2}$. Importantly, because the symmetric group S_2 on two elements is abelian, BS_2 is a spectrum, and so has unordered addition. Hence the sum of all 2-element subtypes of A is a well-defined element of BS_2. We may think of this 2-element type as the type of *orientations* of A. The map sending A to its type of orientations is the (delooping of the) sign homomorphism.

Our third application is to the Barratt–Priddy–Quillen theorem. This concerns the sphere spectrum $\mathbb{S}$, which is freely generated by an element $e : \mathbb{S}_0$ of its underlying type. The Barratt–Priddy–Quillen theorem describes a relationship between $\mathbb{S}$ and the type of finite sets, mediated by a map $\mathsf{FinSet} \to \mathbb{S}_0$ from finite sets to the underlying type of $\mathbb{S}$. We can describe this map in terms of unordered addition: it sends $I : \mathsf{FinSet}$ to $\sum_{i:I} e$. Many properties of this map are readily verified, such as the fact that it respects addition. One expects that the action on π_1 corresponds to the sign homomorphism $S_n \to S_2$, under the equivalence $\pi_1(\mathbb{S}) \simeq S_2$ [2].

[2] $A \times B$ denotes the product in the category of spectra, which as a sequence of pointed types is given component-wise by products of pointed types.

[3] This is an unpublished result of Kapulkin and Sattler: given a closed term of type $\|A\|_{-1}$, there is also a closed term of A.

References

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