Modal Two-Level Type Theory with Graphical Syntax*

Julian Gould[1]

University of Pennsylvania
jjgould@sas.upenn.edu

1 Introduction

Annenkov, Capriotti, Kraus, and Sattler introduced *two-level type theory (2LTT)* [1] to simultaneously accommodate two different type theories. In this talk, we present our ongoing work developing a modal type theory inspired by ACKS's 2LTT. While ACKS use multiple universe hierarchies to accommodate inner and outer types, our *modal two-level type theory (M2LTT)* uses a single universe hierarchy and two *dependency modalities* that capture inner and outer dependency. This talk makes two primary contributions by introducing modal two-level type theory. First, it does not appear that M2LTT fits in the framework of Gratzer, Kavvos, Nuyts, and Birkedal's *multimodal dependent type theory (MTT)* [2]. Second, the syntax for M2LTT allows for a graphical calculus of contexts. This graphical calculus brings string-diagram style reasoning to type theory, allowing clear proofs of properties like exchange. This tool may be applied to other type theories including basic Martin-Löf type theory.

2 Overview of Modal Two-Level Type Theory

Modal two-level type theory has a single universe hierarchy and a pair of inner and outer dependency modalities. These modalities are not applied types themselves, but to pairs of types with a dependency. A simple two type example is illustrative.

Example 1. For each type A, there are two different identity types. The *outer* identity type $\mathrm{Id}^o_A(x, y)$ satisfies UIP. We think of the outer identity type as capture equality "on the nose" like in a set model of MLTT. The *inner* identity type $\mathrm{Id}^i_A(x, y)$ is not assumed to satisfy UIP. We think of the inner identity type as capturing equality "up to homotopy" as in Voevodsky's simplicial set model of MLTT.

Now suppose a type $B(x)$ depends on the variable $x : A$. In M2LTT, we must specify if $B(x)$ depends on $x : A$ in an *inner way* or in an *outer way* by applying one of the modalities. If this dependence is *outer*, we may only derive transport maps $\mathrm{tr}(p, B) : B(x) \to B(y)$ for outer equalities $p : \mathrm{Id}^o_A(x, y)$. Since outer equalities are strict, this transport map is just the identity. If this dependence is *inner* though, we may also derive nontrivial transport maps for $q : \mathrm{Id}^i_A(x, y)$ like in homotopy type theory.

With more than two types, much more complicated dependency structures are obtained that appear to preclude M2LTT from fitting in the framework of MTT.

We define semantics for M2LTT as a variant of *display map categories*. A *class of displays* in a category $\mathcal{M}$ with terminal object $*$ is a pullback-closed class of arrows $\mathcal{D}$ that contains all isomorphisms and maps into $*$.

Definition 2. A *double display map category (DDMC)* is a triple $\langle \mathcal{M}, \mathcal{D}^i, \mathcal{D}^o \rangle$ of a category $\mathcal{M}$ with terminal object, and two classes of displays $\mathcal{D}^i, \mathcal{D}^o$.

*Special thanks to my advisors Paige North and Robert Ghrist

Models of M2LTT are defined to be DDMCs that model certain type formers. $\mathcal{D}^i$ and $\mathcal{D}^o$ are the classes of *inner and outer displays* that model inner and outer dependency respectively. Interpretations of other type formers like Σ and Π types can be given. We prove the following theorem.

Theorem 3. *There is a model of M2LTT where* $\mathcal{M} := \underline{sSet}$*,* $\mathcal{D}^i$ *is the class of Kan fibrations, and* $\mathcal{D}^o$ *is the class of all simplicial maps.*

Inner and outer identity types both give rise to weak factorization systems on the underlying category [5]. Lifting problems in the style of [4] are used to interpret the interplay of these weak factorization systems. The machinery of Lumsdaine and Warren [3] likely provides a strictification of DDMCs to a strict model of M2LTT.

3 Graphical Syntax

The syntax of M2LTT is novel because the contexts are *diagrams* instead of lists. Specifically, a context is a edge-labeled directed graph of variables of types.

$$\begin{tikzcd}
z : C(x,y) \arrow[r, "o"] \arrow[dr, "i"'] & y : B(x) \arrow[d, "o"] \\
& x : A
\end{tikzcd}$$

This context says that C depends on the variable $x : A$ in an inner way and on $y : B$ in an outer way, while B depends on $x : A$ in an outer way. This graphical syntax is convenient because the inner/outer modalities may be though of as decorations on the arrows representing dependency. This has the benefit of allowing us to more easily recognize the structure of dependencies in contexts. For example, it is clear that the variables x and y are independent of one another in the following context.

$$x : X \xleftarrow{i} z : C(x,y) \xrightarrow{o} y : Y$$

The rules of M2LTT may be translated to inline representations of contexts as lists of triples that encode inner/outer dependencies. We define an equivalence relation on these inline contexts to capture standard judgemental equalities of contexts. This gives rise to a calculus akin to that of string diagrams for symmetric monoidal categories is given that allows the derivation of rules like exchange.

References

[1] Danil Annenkov, Paolo Capriotti, and Nicolai Kraus. Two-level type theory and applications. *CoRR*, abs/1705.03307, 2017.

[2] Daniel Gratzer, G. A. Kavvos, Andreas Nuyts, and Lars Birkedal. Multimodal Dependent Type Theory. *Logical Methods in Computer Science*, Volume 17, Issue 3, July 2021.

[3] Peter Lefanu Lumsdaine and Michael A. Warren. The local universes model. *ACM Transactions on Computational Logic*, 16(3):1–31, jul 2015.

[4] Erin McCloskey. Relative weak factorization systems. Master's thesis, Universiteit van Amsterdam, 2022.

[5] P. R. North. *Type Theoretic Weak Factorization Systems*. PhD thesis, University of Cambridge, 2017.