Colimits in the category of pointed types

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Homotopy type theory is a useful system for developing synthetic homotopy theory. We treat types as spaces and function types as hom-groupoids, thereby making the universe of types an internal version of the ∞-category of spaces. In certain cases, we want to prove things about the category of *pointed* types and basepoint-preserving functions $A \to_* B :\equiv \sum_{f:A\to B} f(a_0) = b_0$. In our ongoing work on a type-theoretic version of the Brown representability theorem, we need a Yoneda-like lemma whose standard proof requires pointed function types to take colimits to limits. Since we must carry proofs of the identity $f(a_0) = b_0$, this property is tricky to prove compared to the unpointed version. To prove it, we must place coherence conditions on the colimits we take.

We exhibit a large class of colimits preserved by pointed function types. These colimits are taken over graphs equipped with *strong-tree* structure, called *strong trees* for short. Such colimits include, for example, pushouts, sequential colimits, and wedge sums. Specifically, consider a strong tree Γ with basepoint j_0, a diagram F over Γ, and a cocone (K, ι_K, κ_K) under F. For each vertex i, suppose that $F(i)$ has a basepoint b_i. Further, suppose that all maps in F are pointed. For every pointed type P, we construct a function

$$((K, \iota_{j_0}(b_{j_0})) \to_* P) \xrightarrow{e^{j_0}_{K,P}} \mathsf{lim}_{i:\Gamma^{\mathrm{op}}}(F(i) \to_* P)$$

and prove that e^{j_0} is an equivalence for the canonical cocone $(\mathsf{colim}_\Gamma(F), \iota, \kappa)$ under F. Moreover, we prove that if $e^{j_0}_{K,P}$ is an equivalence for all P, then K and $\mathsf{colim}_\Gamma(F)$ are equivalent as pointed types. In this sense, we prove that colimits over strong trees are preserved by the forgetful functor from pointed types to types. Since all strong trees are connected, this property of colimits is reminiscent of the classical theorem that colimits over connected categories in the 1-category of pointed spaces can be computed in the unpointed category.

All strong trees are trees in the traditional sense, i.e., have contractible quotients. It is unclear, however, whether the converse holds. Our development relies on the combinatorial flavor of strong-tree structure. Extending our main theorem to all graphs with contractible quotients is left as an open problem.

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