Exercise 4.1. *Label the following statements as true or false.*

(a) *The function* $\det : M_{2\times 2}(F) \to F$ *is a linear transformation.*

Proof. False. The det is not a linear transformation. □

(b) *The determinant of a* 2×2 *matrix is a linear function of each row of the matrix when the other row is held fixed.*

Proof. True. This is theorem 4.1. □

(c) *If* $A \in M_{2\times 2}(F)$ *and* $\det(A) = 0$*, then* A *is invertible.*

Proof. False. It should be the opposite. □

(d) *If* u *and* v *are vectors in* R^2 *emanating from the origin, then the area of the parallelogram having* u *and* v *as adjacent sides is*

$$\det \begin{pmatrix} u \\ v \end{pmatrix}.$$

Proof. False. The determinant can be negative when the area cannot. □

(e) *A coordinate system is right-handed if and only if its orientation equals* 1.

Proof. True, that is the definition of orientation. □

Exercise 4.2.1. *Label the following statements as true or false.*

(a) *The function* $\det : M_{n\times n}(F) \to F$ *is a linear transformation.*

Proof. False, clearly. □

(b) *The determinant of a square matrix can be evaluated by cofactor expansion along any row.*

Proof. True. This is theorem 4.4. □

(c) *If two rows of a square matrix* A *are identical, then* $det(A) = 0$.

Proof. True, this is the corollary for theorem 4.4. □

(d) *If* B *is a matrix obtained from a square matrix* A *by interchanging any two rows, then* $\det(B) = -\det(A)$

Proof. True. Theorem 4.5 □

(e) *If* B *is a matrix obtained from a square* A *by multiplying a row of* A *by a scalar, then* $\det(B) = \det(A)$.

Proof. False, $\det(B) = k\det(A)$. □

(f) *If* B *is a matrix obtained from a square matrix* A *by adding* k *times row* i *to row* j*, then* $\det(B) = k\det(A)$.

Proof. False, $\det(A) = \det(B)$. □

(g) *If* $A \in M_{n\times n}(F)$ *has rank* n*, then* $\det(A) = 0$.

Proof. False, look at the $n \times n$ identical matrix. Its rank is n and its det is 1. □

(h) *The determinant of an upper triangular matrix equals the product of its diagonal entries.*

Proof. True. □

Exercise 4.2.23. *Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.*

Proof. The proof is by mathematical induction. Assume that this result holds for $(n-1) \times (n-1)$ matrices, consider a $n \times n$ triangular matrix

$$\begin{pmatrix} a_11 & B \\ O & C \end{pmatrix}$$

where B is a $1 \times (n-1)$ matrix, O is a $(n-1) \times 1$ zero matrix and C is an $(n-1) \times (n-1)$ triangular matrix. Now applying the determinant formula for the first column, we get

$$\det(A) = a_{11} \det(C).$$

By the induction assumption, $\det(C)$ is the product of $(n-1)$ diagonal entries. Thus $\det(A)$ is the product of its diagonal entries. □

Exercise 4.2.24. *Prove that if $A \in M_{n\times n}(F)$ has a row consisting entirely of zeros, then $\det(A) = 0$.*

Proof. Assume that the rth row of A contains only zeros. Multiply row r by a scalar k, the matrix doesn't change. However, the determination of A increase k time. Therefor

$$\det(A) = k \det(A)$$

for all k. Thus $\det(A) = 0$. □

Exercise 4.2.25. *Prove that $\det(kA) = k^n \det(A)$ for any $A \in M_{n\times n}(F)$.*

Proof. What to prove? Multiply one row by k, the determinant increase k times. So Multiply n rows by k, the determinant increase by k^n times. □

Exercise 4.2.26. *Let $A \in M_{n\times n}(F)$. Under what conditions is $\det(-A) = \det(A)$.*

Proof. If n is even, by exercise 25, we have

$$\det(-A) = (-1)^n \det(A) = \det(A).$$

If n is odd, similar to the case above, we get $\det(A) = -\det(A)$, therefor $\det(A) = 0$. □

Exercise 4.2.27. *Prove that if $A \in M_{n\times n}(F)$ has two identical columns, then $\det(A) = 0$.*

Proof. Clearly, $rank(A) < n$, thus by the corollary of theorem 4.6, we have $\det(A) = 0$. □

Exercise 4.3.1. *Label the following statements as true or false.*

(i) *If E is an elementary matrix, then* $\det(E) = \pm 1$.

Proof. False. In the case of multiplying one row to k, $\det(E) = k$. □

(ii) *For any $A, B \in M_{n\times n}(F)$,* $\det(AB) = \det(A) \cdot \det(B)$.

Proof. True. Theorem 4.7. □

(iii) *A matrix $M \in M_{n\times n}(F)$ is invertible if and only if* $\det(M) = 0$.

Proof. False. If $\det(A) = 0$, then A is not invertible. □

(iv) *A matrix $M \in M_{n\times n}(F)$ has rank n if and only if* $\det(M) \neq 0$.

Proof. True. The matrix M has rank n, M is invertible and $\det(M) = 0$ are the same if M is a square matrix. □

(v) *For any $A \in M_{n\times n}(F)$,* $\det(A^t) = -\det(A)$.

Proof. False, because $\det(A^t) = \det(A)$. □

(vi) *The determinant of a square matrix can be evaluated by cofactor expansion along any column.*

Proof. True. □

(vii) *Every system of n linear equations in n unknowns can be solved by Cramer's rule.*

Proof. False. We can use Cramer's rule only if its determinant is nonzero. □

(viii) *Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \cdots, x_n)^t$. If* $\det(A) \neq 0$ *and if M_k is the $n \times n$ matrix obtained from A by replacing row k of A by b^t, then the unique solution of $Ax = b$ is*

$$x_k = \frac{\det(M_k)}{\det(A)} \quad \textit{for } k = 1, 2, \cdots, n.$$

Proof. False. By Cramer's rule, if M_k is the $n \times n$ matrix obtained from A by replacing **column** k of A by b, then you get a solution. If we define M_k this way, in most cases, we will get an identical solution. But since $\det(A) \neq 0$, the solution must be unique. Thus this statement is false. $\square$

Exercise 9. *Prove that an upper triangular $n \times n$ matrix is invertible if and only if all its diagonal entries are nonzero.*

Proof. Let M be that upper triangular $n \times n$ matrix. If M is invertible, then $\det(M) \neq 0$. Let's remind that $\det(M)$ is the product of the diagonal entries. Since their product is nonzero, each entry must be nonzero itself. Conversely, if all the diagonal entries are nonzero, then $\det(M) \neq 0$. Hence, M is invertible. $\square$

Exercise 10. *A matrix $M \in M_{n\times n}(C)$ is called nilpotent if, for some positive integer k, $M^k = O$, where O is the $n \times n$ zero matrix. Prove that if M is nilpotent, then* $\det(M) = 0$.

Proof. Since $M^k = O$, we have $\det(M)^k = \det(M^k) = 0$. Thus $\det(M) = 0$. $\square$

Exercise 11. *A matrix $M \in M_{n\times n}(C)$ is called skew-symmetric if $M^t = -M$. Prove that if M is skew-symmetric and n is odd, then M is not inveri-ble. What happens if n is even?*

Proof. We have $\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M)$. If n is odd, then $\det(M) = -\det(M)$, which easily leads to $\det(M) = 0$. Therefore, M is invertible. Otherwise, if n is even, M isn't necessarily invertible. One example is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. $\square$

Exercise 12. *A matrix $Q \in M_{n\times n}(R)$ is called orthogonal if $QQ^t = I$. Prove that if Q is orthogonal, then* $\det(Q) = \pm 1$.

Proof. We have $1 = \det(I) = \det(QQ^t) = \det(Q)\det(Q^t) = \det(Q)^2$. Thus $\det(Q) = \pm 1$. $\square$

Exercise 13. *For $M \in M_{n\times n}(C)$, let $\overline{M}$ be the matrix such that $(\overline{M})_{ij} = \overline{M_{ij}}$ for all i, j, where $\overline{M_{ij}}$ is the complex conjugate of M_{ij}.*

(a) Prove that $\det(\overline{M}) = \overline{\det(M)}$.

Proof. First, we have a few properties about complex conjugate as follow:

$$\overline{ab} = \overline{a}\overline{b}$$
$$\overline{a+b} = \overline{a} + \overline{b}$$

for any $a, b \in \mathbb{C}$. Indeed, let $a = x + yi$ and $b = z + ti$, then

$$\begin{aligned}
\overline{ab} &= \overline{(x+yi)(z+ti)} \\
&= \overline{xz - yt + (xt + yz)i} \\
&= xz - yt - (xt + yz)i \\
&= xz - yzi - yt - xti \\
&= z(x - yi) - ti(x - yi) \\
&= (x - yi)(z - ti) \\
&= \overline{a} \cdot \overline{b}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\overline{a+b} &= \overline{x + z + (y+t)i} \\
&= x + z - (y + t)i \\
&= x - yi + z - ti \\
&= \overline{a} + \overline{b}.
\end{aligned}$$

Now the proof of (a) is by mathematical induction on n. For $n = 1$, the result is trivial. Assume that this result holds for $n - 1$ and let $A \in M_{n\times n}(C)$. Let $\tilde{A}_{ij}$ denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j, then we have

$$\begin{aligned}
\overline{\det(A)} &= \overline{\sum_{i=1}^{n} A_{1i} \det(\tilde{A}_{1i})} \\
&= \sum_{i=1}^{n} \overline{A_{1i} \det(\tilde{A}_{1i})} \\
&= \sum_{i=1}^{n} \overline{A_{1i}} \cdot \overline{\det(\tilde{A}_{1i})} \\
&= \sum_{i=1}^{n} \overline{A_{1,i}} \cdot \det\left(\tilde{\overline{A}}\right) \\
&= \det(\overline{A}).
\end{aligned}$$

$\square$

(b) A matrix $Q \in M_{n\times n}(C)$ is called unitary if $QQ^* = I$, where $Q^* = \overline{Q^t}$. Prove that if Q is a unitary matrix, then $|\det(Q)| = 1$.

Proof. Since Q is a unitary matrix, we have $\det(QQ^*) = \det(I) = 1$. Thus $\det(Q)\det(Q^*) = 1$. Notice that

$$\begin{aligned}\det(Q^*) &= \det(\overline{Q^t}) \\ &= \overline{\det(Q^t)} \\ &= \overline{\det(Q^t)} \\ &= \overline{\det(Q)}.\end{aligned}$$

Remind that for a complex number c, we have $c \cdot \overline{c} = |c|$, using the calculation above, we have

$$1 = \det(Q)\det(Q^*) = \det(Q)\overline{\det(Q)} = |\det(Q)|.$$

□

Exercise 15. *Prove that if $A, B \in M_{n\times n}(F)$ are similar, then* $\det(A) = \det(B)$.

Proof. If A and B are similar, then there exists a matrix Q such that

$$A = Q^{-1}BQ.$$

Thus

$$\begin{aligned}\det(A) &= \det(Q^{-1}BQ) \\ &= \det(Q^{-1})\det(B)\det(Q) \\ &= \det(Q^{-1}Q)\det(B) \\ &= \det(I)\det(B) \\ &= \det(B).\end{aligned}$$

□

Exercise 16. *Use determinants to prove that if $A, B \in M_{n\times n}(F)$ are such that $AB = I$, them A is invertible (and hence $B = A^{-1}$).*

Proof. Since $\det(A)\det(B) = \det(AB) = \det(I) = 1$, we have $\det(A) \neq 0$. Thus A is invertible. Notice that the matrix B such that $AB = I$ is unique. Because $AA^{-1} = I$ too, we have $B = A^{-1}$.

Indeed, if there exists B and C such that $AB = AC$, then $A^{-1}AB = A^{-1}AC$. Thus $B = C$, which means such B is unique. □

Exercise 18. *Complete the proof of Theorem 4.7 by showing that if A is an elementary matrix of type 2 or type 3, then* $\det(AB) = \det(A) \cdot \det(B)$.

Proof. If A is an elementary matrix obtained by multiplying row jth to k, then $\det(A) = k$. But AB is also obtained from B by multiplying k to jth row. Thus $\det(AB) = k \det(B) = \det(A) \det(B)$.

If A is an elementary matrix obtained by adding a multiply of some row of I to another row, then $\det(A) = 1$. We can easily see that $\det(AB) = \det(B)$ because type 3 elementary row operation doesn't change the determinant. Thus $\det(AB) = \det(B) = \det(A) \det(B)$. $\square$

Exercise 4.4.1. *Label the following statements as true or false.*

(a) *The determinant of a square matrix may be computed by expanding the matrix along any row or column.*

Proof. True. □

(b) *In evaluating the determinant of a matrix, it is wise to expand along a row or column containing the largest number of zero entries.*

Proof. True. If there are k zeros, then we will calculate k less determinant. And since calculating determinant is a nightmare, wise people will try to avoid that. □

(c) *If two rows or columns of A are identical, then* $\det(A) = 0$.

Proof. True. □

(d) *If B is a matrix obtained by interchanging two rows or two columns of A, then* $\det(B) = \det(A)$.

Proof. False, $\det(A) = -\det(B)$. □

(e) *If B is a matrix obtained by multiplying each entry of some row or column of A by a scalar, then* $\det(B) = \det(A)$.

Proof. False. If that scalar is k, then $\det(B) = k\det(A)$. □

(f) *If B is a matrix obtained from A by adding a multiple of some row to a different row, then* $\det(B) = \det(A)$.

Proof. True. □

(g) *The determinant of an upper triangular $n \times n$ matrix is the product of its diagonal entries.*

Proof. True. □

(h) *For every $A \in M_{n\times n}(F)$,* $\det(A^t) = -\det(A)$.

Proof. False, $\det(A) = \det(A^t)$. $\square$

(i) *If $A, B \in M_{n\times n}(F)$, then* $\det(AB) = \det(A)\det(B)$.

Proof. True. $\square$

(j) *If Q is an invertible matrix, then* $\det(Q^{-1}) = [\det(Q)]^{-1}$.

Proof. True. Another way to write this is $\det(Q^{-1}) = \frac{1}{\det(Q)}$. $\square$

(k) *A matrix Q is invertible if and only if* $\det(Q) \neq 0$.

Proof. True. $\square$

Exercise 4.5.1. *Label the following statements as true or false.*

(a) *Any n-linear function $\delta : M_{n\times n}(F) \to F$ is a linear transformation.*

Proof. False. By the definition, it is linear of each row, when the other $(n-1)$ rows are fixed. $\square$

(b) *Any n-linear function $\delta : M_{n\times n}(F) \to F$ is a linear function of each row of an $n \times n$ matrix when the other $n-1$ rows are held fixed.*

Proof. True. $\square$

(c) *If $\delta : M_{n\times n}(F) \to F$ is an alternating n-linear function and the matrix $A \in M_{n\times n}(F)$ has two identical rows, then $\delta(A) = 0$.*

Proof. True. $\square$

(d) *If $\delta : M_{n\times n}(F) \to F$ is an alternating n-linear function and B is obtained from $A \in M_{n\times n}(F)$ by interchanging two rows of A, then $\delta(B) = \delta(A)$.*

Proof. False because $\delta(B) = -\delta(A)$. $\square$

(e) *There is a unique alternating n-linear function $\delta : M_{n\times n}(F) \to F$.*

Proof. False because $\delta(x) = k\det(x)$ is a unique alternating n-linear function for each scalar k. Thus δ is not unique. $\square$

(f) The function $\delta : M_{n\times n}(F) \to F$ defined by $\delta(A) = 0$ for every $A \in M_{n\times n}(F)$ is an alternating n-linear function.

Proof. True. $\square$

Exercise 4.5.2. *Determine all the* 1*-linear function $\delta : M_{1\times 1}(F) \to F$.*

Proof. Since f is a 1-linear function, we have $f(ka) = kf(a)$ for a vector a and a scalar k. Now let $a = 1$, then we have $f(k) = kf(1)$ for all k. Thus all the 1-linear functions $\delta : M_{1\times 1}(F) \to F$ has the form $f(x) = ax$ for a scalar a. $\square$

Exercise 4.5.11. *Prove Corollaries 2 and 3 of Theorem 4.10. That is, let $\delta : M_{n\times n}(F) \to F$ be an alternating n-linear function. If $M \in M_{n\times n}(F)$ has rank less than n, then $\delta(M) = 0$. Moreover, let E_1, E_2, and E_3 in $M_{n\times n}(F)$ be elementary matrices of type 1,2, and 3, respectively. Suppose that E_2 is obtained by multiplying some row of I by the nonzero scalar k. Then $\delta(E_1) = -\delta(I)$, $\delta(E_2) = k \cdot \delta(I)$, and $\delta(E_3) = \delta(I)$.*

Proof. By Corollary 1, we have $\det(B) = \det(A)$ if B is obtained from A by adding a multiple of some row of A to another row of A. Also by theorem 4.10. if B is obtained from A by interchanging two any two rows of A, then $\delta(B) = -\delta(A)$. And since δ is a n-linear function, if B is obtained from A by multiply a row of A by k, then $\delta(B) = k\delta(A)$.

Back to the problem, because rank M is less than M, after a finite number of elementary operations on M, we can obtain a matrix M' where M' has two identical rows. Thus, by Theorem 4.10, $\delta(M') = 0$, which leads to $\delta(M) = 0$.

Moreover, in the first paragraph, let $A = I$, then we have $\delta(E_1) = -\delta(I), \delta(E_2) = k\delta(I)$ and $\delta(E_3) = \delta(I)$. $\square$

Exercise 4.5.12. *Prove Theorem 4.11.*

Proof. If $rank(B) = 0$, then $rank(AB) = 0$. Thus by Corollary 2, we have $\delta(AB) = 0 = \delta(A) \cdot \delta(B)$. If $rank(B) > 0$, then B can be written as a product of elementary matrices. Thus we only need to check $\delta(AB) = \delta(A) \cdot \delta(B)$ in case B is an elementary matrix. Indeed, if B is an elementary matrix type 1, then $\delta(AE_1) = -\delta(A)$ by theorem 4.10. Moreover, we have $\delta(E_1) = -\delta(I) = -1$. Thus $\delta(AE_1) = \delta(A) \cdot \delta(E_1)$. If B is an elementary matrix type 2, then $\delta(AE_2) = k\delta(A)$. Moreover, $\delta(E_2) = k\delta(I) = k$. Thus $\delta(AE_2) = \delta(A) \cdot \delta(E_2)$. Similarly, if B is a type 3 elementary matrix, then $\delta(AE_3) = \delta(A)$ and $\delta(E_3) = \delta(I) = 1$. Thus $\delta(AE_3) = \delta(A) \cdot \delta(E_3)$. To sum up, $\delta(AB) = \delta(A) \cdot \delta(B)$ for any $A, B \in M_{n \times n}(F) \to F$. $\square$

Exercise 4.5.19. *Let $\delta : M_{n\times n}(F) \to F$ be an n-linear function and F a field that does not have characteristic two. Prove that if $\delta(B) = -\delta(A)$ whenever B is obtained from $A \in M_{n\times n}(F)$ by interchanging any two rows of A, then $\delta(M) = 0$ whenever $M \in M_{n\times n}(F)$ has two identical rows.*

Proof. First, we will prove that if F is even characteristic, then F is characteristic 2. Indeed, for any $a \in F$, we have $1 = a \cdot a^{-1} = a(0 + a^{-1}) = a.0 + 1$. Thus $0 = a \cdot 0$. Assume that $char(F) = 2\beta$, then let γ equals to $1 + 1 + \cdots + 1$ β times. Then $\gamma + \gamma = 0$, hence $\gamma^{-1}(\gamma + \gamma) = \gamma^{-1} \cdot 0 = 0$. Thus $1 + 1 = 0$. So if F has even characteristic, $char(F) = 2$.

If M has two identical rows, let M' is a matrix obtained from M by interchanging two identical rows of M. Thus $\delta(M) = -\delta(M') = -\delta(M)$. Thus $\delta(M) + \delta(M) = 0$. Because F have characteristic 2, we have $\delta(M) = 0$. $\square$