Answer to Introduction to Smooth Manifolds - John M. Lee: Exercise Solutions

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1 Smooth Manifolds

1.1 Exercises

Exercise 1.1
Show that equivalent definitions of locally Euclidean spaces are obtained if, instead of requiring U to he homeomorphic to an open subset of $\mathbb{R}^n$, we require it to be homeomorphic to an open ball in $\mathbb{R}^n$, or to $\mathbb{R}^n$ itself.

Proof. Let X be a topological space. Assume that X is locally Euclidean in the open set sense, that is, there exist a homeomorphism $\varphi\colon U \to \widetilde{U}$ where U is a neighborhood of p and $\widetilde{U}$ is an open subset of $\mathbb{R}^n$. Because the set of open balls generates the Euclidean topology of $\mathbb{R}^n$, we can find an $\varepsilon > 0$ such that $B(p, \varepsilon) \subset \widetilde{U}$. So $\varphi^{-1}(B(p, \varepsilon))$ is a neighborhood of p that is isomorphic to an open ball.

Conversely, if X is locally Euclidean in the open ball sense, then since any open ball is an open set, we get X to be locally Euclidean in the open set sense. Thus the open subset and open ball definitions are equivalent.

Obviously the open ball is equivalent to the $\mathbb{R}^n$ itself because any open ball is isomorphic to $\mathbb{R}^n$, we get the conclusion. □

Exercise 1.2
Show that any topological subspace of a Hausdorff space is Hausdorff, and any finite product of Hausdorff spaces is Hausdorff.

Proof. Let $Y \subset X$ be a subspace of a Hausdorff topological space. For any $a, b \in Y, a \neq b$, because X is Hausdorff, there exist open subsets A, B of X that separate a and b. Hence $A \cap Y$ and $B \cap Y$ are open subsets of Y that separate a and b.

Let $X_1 \times X_2$ be the product of two Hausdorff spaces. Let (a_1, a_2) and (b_1, b_2) be two different elements of $X_1 \times X_2$. So at least there is one component is different. Without lost of generality, assume that $a_1 \neq b_1$, then they are separated by A_1 and B_1. Thus $A_1 \times X_2$ and $B_1 \times X_2$ are two disjoint open subsets of $X_1 \times X_2$ that separate (a_1, a_2) and (b_1, b_2). □

Exercise 1.3
Show that any topological subspace of a second countable space is second countable, and any finite product of second countable spaces is second countable.

Proof. Let $\mathscr{B}$ be a countable basis for a topological space X, and $Y \subset X$. Then

$$\mathscr{B}' = \{X \cap Y : X \in \mathscr{B}\}$$

is a countable basis for Y. The second part is by the definition of the product topology and the finite product of countable sets is countable. □

Exercise 1.4
Show that $\mathbb{P}^n$ is Hausdorff and second countable, and is therefore a topological n-manifold.

Proof. We know that $\mathbb{P}^n$ is a topological manifold from [1]. So this space is Hausdorff and second countable. $\square$

Exercise 1.5
Prove Lemma 1.4(b). That is two smooth atlases for M determine the same maximal smooth atlas if and only if their union is a smooth atlas.

Proof. Let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two smooth atlases for M. Assume that $\mathscr{A}_1$ and $\mathscr{A}_2$ determine the same maximal smooth atlas $\mathscr{A}$ for M. Then any two charts in $\mathscr{A}_1 \cup \mathscr{A}_2$ are also charts in $\mathscr{A}$, thus smoothly compatible with each other. So $\mathscr{A}_1 \cup \mathscr{A}_2$ is a smooth atlas for M.

Conversely, assume that $\mathscr{A}_1 \cup \mathscr{A}_2$ is a smooth atlas. By Lemma 1.4(a), $\mathscr{A}_1$ and $\mathscr{A}_1 \cup \mathscr{A}_2$ generates the same smooth structure on M. And the same thing holds for $\mathscr{A}_2$ and $\mathscr{A}_1 \cup \mathscr{A}_2$. Thus $\mathscr{A}_1$ and $\mathscr{A}_2$ generate the same smooth structure. $\square$

Exercise 1.6
If k is an integer between 0 and $\min(m, n)$, show that the set of $m \times n$ matrices whose rank is at least k is open submanifold of $M(m \times n, \mathbb{R})$.

Proof. Let $M_k(m \times n, \mathbb{R})$ be the set of matrices of rank k or above. Since $M(m \times n, \mathbb{R})$ is a smooth manifold, it is sufficient to show that $M_k(m \times n, \mathbb{R})$ is an open subset of $M(m \times n, \mathbb{R})$. For any $A \in M_k(m \times n, \mathbb{R})$, we get $\operatorname{rank}(A) = k$. Thus there is a nonsigular $k \times k$ minor φ of A. Because det is a continuous function, so is this minor φ. We have $\varphi^{-1}(\mathbb{R} \setminus \{0\})$ is open in $M(m \times n, \mathbb{R})$. Notice that any $M \in \varphi^{-1}(\mathbb{R} \setminus \{0\})$ has rank greater or equal to k, thus this coimage is an open subset of $M_k(m \times n, \mathbb{R})$, and thus is a neighborhood of A. So $M_k(m \times n, \mathbb{R})$ is a subspace of $M(m \times n, \mathbb{R})$ thus is a smooth submanifold of $M(m \times n, \mathbb{R})$. $\square$

Exercise 1.7
By identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual way, we can think of the unit circle $\mathbb{S}^1$ as a subset of the complex plane. An angle function on a subset $U \subset \mathbb{S}^1$ is a continuous function $\theta\colon U \to \mathbb{R}$ such that $e^{i\theta(p)} = p$ for all $p \in U$. Show that there exists an angle function θ on an open subset $U \subset \mathbb{S}^1$ if and only if $U \neq \mathbb{S}^1$. For any such angle function, show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

Proof. Recall that the function $\varepsilon\colon \mathbb{R} \to \mathbb{S}^1$ that maps $x \mapsto e^{ix}$ is a covering map. We want to find a function $\theta\colon U \to \mathbb{R}$ such that $\varepsilon \circ \theta(p) = e^{i\theta(p)} = p$, or the following diagram commutes.

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "\varepsilon"] \\ U \arrow[ur, dashed, "\theta"] \arrow[r, "\mathrm{Id}"'] & \mathbb{S}^1 \end{tikzcd}$$

In another words, we need to find the condition of U such that Id has a lift. But $\pi_1(\mathbb{R}) = 0$, thus $\varepsilon_* \pi_1(\mathbb{R}) = 0$. So θ exists if and only if $\mathrm{Id}_* \pi_1(U, u_0) = 0$ for some $u_0 \in U$. But this is synonymous with $U \neq \mathbb{S}^1$ (because $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ and $\pi_1(\mathbb{S}^1 \setminus \{*\}) = \pi_1(I) = 0$). So θ exists if and only if $U \neq \mathbb{S}^1$.

For the second part, we first show that (U, θ) is a chart, that is, θ is a homeomorphism onto its image. Clearly θ is continuous and injective (because $\varepsilon \circ \theta = \mathrm{Id}$). The inverse function θ^{-1} makes the following diagram commutes.

$$\begin{tikzcd} & \theta(U) \arrow[d, "\varepsilon"] \arrow[dl, dashed, "\theta^{-1}"'] \\ U & \mathbb{S}^1 \arrow[l, "\mathrm{Id}"] \end{tikzcd}$$

So $\theta^{-1}(x) = \varepsilon(x) = e^{ix}$ for $x \in \theta(U) \subset \mathbb{R}$. This map is also continuous. So we conclude that (U, θ) is a chart.

Now we show that (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with the standard smooth structure. Recall that the standard smooth structure is generated by 4 charts (U_i, φ_i) as in Example 1.11. By Lemma 1.4, it is sufficient to show that (U, θ) is smoothly compatible with (U_i, φ_i) for any i. Indeed, if $U \cap U_1 \neq \varnothing$, then

$$\theta \circ (\varphi_1^+)^{-1}(x) = \theta(\sqrt{1 - |x|^2}, x).$$

Notice that the Euler formula implies that

$$\begin{aligned} e^{i \sin^{-1}(x)} &= \cos(\sin^{-1}(x)) + i \sin(\sin^{-1}(x)) \\ &= \sqrt{1 - \sin(\sin^{-1}(x))^2} + ix \\ &= \sqrt{1 - |x|^2} + ix. \end{aligned}$$

Thus

$$\theta \circ (\varphi_1^+)^{-1}(x) = \theta(\sqrt{1 - |x|^2}, x) = \sin^{-1}(x),$$

which is smooth on its domain. Moreover,

$$(\varphi_1^+) \circ \theta(x) = \varphi_1^+(e^{ix}) = \varphi_1^+(\cos(x), \sin(x)) = \sin(x).$$

This function is also smooth on its domain. Similarly for other pairs, we conclude that (U_i, φ_i) is smoothly compatible with (U, θ) for all i. So (U, θ) is a smooth coordinate chart for $\mathbb{S}^1$ with its standard smooth structure.

$\square$

Exercise 1.8
Let $0 < k < n$ be integers, and let $P, Q \subset \mathbb{R}^n$ be the subspaces spanned by $(e_1, \cdots, e_k)$ and $(e_{k+1}, \cdots, e_n)$, respectively, where e_i is the ith standard basis vector. For any k-dimensional subspace $S \subset \mathbb{R}^n$ that has trivial intersection with Q, show that the coordinate representation $\varphi(S)$ constructed in the preceding example is the unique $(n-k) \times k$ matrix B such that S is spanned by the columns of the matrix $\begin{pmatrix} I_k \\ B \end{pmatrix}$, where I_k denotes the $k \times k$ identity matrix.

Proof. Any such space S has the form $\{x + Ax \colon x \in P\}$ where $A \colon P \to Q$ is a linear map. Let $B \in M_{(n-k)\times k}(\mathbb{R})$ such that $A(e_i) = B_{.,i}$ the i-th column of B. Then $\begin{pmatrix} e_i \\ B_{.,i} \end{pmatrix} = e_i + Ae_i \in S$. Actually this matrix B is unique for the same reason, that is, because the i-th column is a vector in S, it has to have the form $e_i + Ae_i$. It is left to prove that these columns span S. Because S has dimension k, A has rank k. Thus B has rank k and so is $\begin{pmatrix} I_k \\ B \end{pmatrix}$. Since there are k column vectors in this matrix, which has rank k, we deduce that these vectors are linearly independent. So S is spanned by the columns of the matrix $\begin{pmatrix} I_k \\ B \end{pmatrix}$. $\square$

1.2 Problems

Problem 1-1
Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = \pm 1$, and let M be the quotient of X by the equivalence relation generated by $(x, -1) \sim (x, 1)$ for all $x \neq 0$. Show that M is locally Euclidean and second countable, but not Hausdorff.

Proof. First, we will show that $X/\sim$ is locally Euclidean. Let $q : X \to X/\sim$ be a quotient map, and define $f : (X/\sim) \to \mathbb{R}$ maps $q(x, -1)$ and $q(x, 1)$ to x. Any element of $X/\sim$ has the form $q(x, -1)$ or $q(x, 1)$. If $x \neq 0 \in \mathbb{R}$, then there exists an open neighborhood of x that doesn't contain 0, say (a, b). Now we will show that $Q = \{q(x, -1) : x \in (a, b)\}$ is open in $(X/\sim)$. Indeed, $q^{-1}(Q) = \{q^{-1}(q(x)) : x \in (a, b)\} = (a, b) \times \{-1\} \cup (a, b) \times \{1\}$ which is open in $(\mathbb{R} \times \{-1\}) \cup (\mathbb{R} \times \{1\})$, so Q is open. Since $q(x, 1) = q(x, -1) \in Q$, this is a neighborhood of $q(x, 1) = q(x, -1)$. Next, we will show that $f|_Q : Q \to f(Q) = (a, b) \subset \mathbb{R}$ is a homeomorphism. For any $q(u, -1), q(v, -1) \in Q$, $f|_Q(u) = f|_Q(v)$, implies $u = v$. So $f|_Q$ is one to one. For any open set $U \subset (a, b)$, $f|_Q^{-1}(U) = \{q(x, \varepsilon) : x \in U, \varepsilon \in \{-1, 1\}$, which is open because

$q^{-1}(f|_Q^{-1}(U)) = (U \times \{-1\}) \cup (U \times \{1\})$ is open in $(\mathbb{R} \times \{-1\}) \cup (\mathbb{R} \times \{1\})$. So $f|_Q$ is continuous. Moreover, because Q is open, a set $V \subset Q$ is open in Q if and only if V is open in $X/\sim$, that is $q^{-1}(V) = (V' \times \{-1\}) \cup (V' \times \{1\})$ is open in X. But this implies $f|_Q(V) = V'$ is open in $\mathbb{R}$. So Q is homeomorphic to an open set in $\mathbb{R}$.

Now consider $q(0,-1)$, let $P = \{q(x,-1) : x \in (-1,1)\} \subset X/\sim$. Because $q^{-1}(P) = ((-1,1) \times \{-1\}) \cup ((-1,0) \times \{1\}) \cup ((0,1) \times \{1\})$, which is a union of 3 open sets thus open, we get P is open in $X/\sim$. Let $g : P \to (-1,1) \subset \mathbb{R}$ maps $q(x,-1) \mapsto x$, we will show that g is a homeomorphism. We can see g is one to one and surjective by the definition of g. For any open set $E \subset (-1,1)$, we have $g^{-1}(E) = q(E \times \{-1\})$. But this set is open because

$$q^{-1}(q(E) \times \{-1\}) = [(E \times \{-1\}) \cup (E \times \{1\})] \setminus \{(0,1)\}$$

is open in X. So g is continuous. Moreover, any open set of $P = q((-1,1) \times \{-1\})$ has the form $q(O \times \{-1\})$ where O is open in $\mathbb{R}$. Thus $g(q(O \times \{-1\})) = O$ is open in $\mathbb{R}$, which means P is a coordinate neighborhood of $q(0,-1)$. Similarly for the case $q(0,1)$, we conclude that $X/\sim$ is locally Euclidean.

Since $\mathbb{R}^2$ is second countable, we get $X \subset \mathbb{R}^2$ is second countable. Moreover, any neighborhood of $q(0,-1)$ has the form $q(V_0 \times \{-1\})$ where V_0 is a neighborhood of 0 in $\mathbb{R}$, and any neighborhood of $q(0,1)$ has the form $q(V_1 \times \{-1\})$ where V_1 is a neighborhood of 0 in $\mathbb{R}$. But then $V_0 \cap V_1$ is a nonempty neighborhood of 0, thus contain an $\varepsilon \neq 0$. So $q(\varepsilon,-1) \in q(V_0 \times \{-1\}) \cap q(V_1 \times \{-1\}) \neq \varnothing$. So $X/\sim$ is not Hausdorff. $\square$

Problem 1-3

Let $N = (0, \cdots, 0, 1)$ be the "north pole" and $S = -N$ the "south pole". Define stereographic projection $\sigma\colon \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma(x^1, \cdots, x^{n+1}) = \frac{(x^1, \cdots, x^n)}{1 - x^{n+1}}.$$

Let $\widetilde{\sigma}(x) = \sigma(-x)$ for $x \in \mathbb{S}^n \setminus \{S\}$.

(i) Show that σ is bijective, and

$$\sigma^{-1}(u^1, \cdots, u^n) = \frac{(2u^1, \cdots, 2u^n, |u|^2 - 1)}{|u|^2 + 1}.$$

(ii) Compute the transition map $\widetilde{\sigma} \circ \sigma^{-1}$ and verify that the atlas consisting of the two charts $(\mathbb{S}^n \setminus \{N\}, \sigma)$ and $(\mathbb{S}^n \setminus \{S\}, \widetilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$.

(iii) Show that this smooth structure is the same as the one defined in Example 1.11.

Proof. (i) Assume that $\sigma(x_1, \cdots, x_{n+1}) = \sigma(y_1, \cdots, y_{n+1})$, we will prove that

$$(x_1, \cdots, x_{n+1}) = (y_1, \cdots, y_{n+1}).$$

Let $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ in $\mathbb{R}^n$. Because $(x_1, \cdots, x_{n+1}) \in \mathbb{S}^n$, we get

$$x_{n+1} = \pm\sqrt{1 - |x|^2}.$$

Similarly,

$$y_{n+1} = \pm\sqrt{1 - |y|^2}.$$

So our assumption implies that

$$\frac{x}{1 \pm \sqrt{1 - |x|^2}} = \frac{y}{1 \pm \sqrt{1 - |y|^2}}. \tag{1}$$

Therefore

$$\frac{|x|}{1 \pm \sqrt{1 - |x|^2}} = \frac{|y|}{1 \pm \sqrt{1 - |y|^2}}. \tag{2}$$

Notice that

$$|x|^2 = \sum_{i=1}^{n} x_i^2 \leq \sum_{i=1}^{n+1} = 1,$$

thus

$$\frac{|x|}{1 + \sqrt{1 - |x|^2}} < |x| \leq 1. \tag{3}$$

And

$$\frac{|x|}{1 - \sqrt{1 - |x|^2}} \geq 1, \tag{4}$$

which can be proven by force as follow.

$$\begin{aligned}(|x| + \sqrt{1-|x|^2})^2 &= x^2 + 1 - x^2 + 2|x|\sqrt{1-|x|^2} \\ &= 1 + 2|x|\sqrt{1-|x|^2} \\ &\geq 1.\end{aligned}$$

So

$$|x| + \sqrt{1-|x|^2} \geq 1$$

or

$$\frac{|x|}{1-\sqrt{1-|x|^2}} \geq 1.$$

From (2), (3), and (4), there are just two cases that can occur, which are

$$\frac{|x|}{1+\sqrt{1-|x|^2}} = \frac{|y|}{1+\sqrt{1-|y|^2}},$$

and

$$\frac{|x|}{1-\sqrt{1-|x|^2}} = \frac{|y|}{1-\sqrt{1-|y|^2}}.$$

Notice that both $\frac{t}{1-\sqrt{1-t^2}}$ and $\frac{t}{1+\sqrt{1-t^2}}$ are strict monotone functions, we claim that $|x| = |y|$. Apply this to (1), we get $x = y$. Also (2), (3), and (4) imply x_{n+1} and y_{n+1} to have the same sign. Thus $x_{n+1} = y_{n+1}$. So σ is injective.

For surjective part, let $v \in \mathbb{R}^n$. Let

$$\alpha = \frac{2}{1+|v|^2},$$

we will prove that $(\alpha v, \sqrt{1-\alpha^2 v^2}) \in \mathbb{S}^n$ and $\sigma(\alpha v, \sqrt{1-\alpha^2 v^2}) = v$. The first part obvious by the distribution. Again, by force, we can check that

$$\frac{\alpha}{1-\sqrt{1-\alpha^2 v^2}} = 1.$$

So the second part is checked, that is σ is surjective. So it is bijective.

(ii) It is sufficient to prove that σ and $\widetilde{\sigma}$ are smoothly compatible. Let $u \in \mathbb{R}^n$, then

$$\begin{aligned}\widetilde{\sigma} \circ \sigma^{-1}(u) &= \widetilde{\sigma}\left(\frac{2}{|u|^2+1}u, \frac{|u|^2-1}{|u|^2+1}\right) \\ &= \sigma\left(\frac{-2}{|u|^2+1}u, \frac{1-|u|^2}{|u|^2+1}\right) \\ &= \frac{\frac{-2}{|u|^2+1}u}{1-\frac{1-|u|^2}{|u|^2+1}} \\ &= \frac{-2}{|u|^2+1-1+|u|^2}u \\ &= \frac{-1}{|u|^2} \cdot u.\end{aligned}$$

But this function is smooth with respect to each entry. Similarly for $\sigma\widetilde{\sigma}^{-1}$, we conclude that the two charts $(\mathbb{S}^n\backslash\{N\},\sigma)$ and $(\mathbb{S}^n\backslash\{S\},\widetilde{\sigma})$ define a smooth structure on $\mathbb{S}^n$.

(iii) Assume that (U_i,φ_i) and $(\mathbb{S}^n\backslash\{N\},\sigma)$ be two charts of $\mathbb{S}^n$ such that $U_i\cap\mathbb{S}^n\backslash\{N\}\neq\varnothing$. We have $\varphi_i\circ\sigma^{-1}$ to be the function that drop the i-th coordinate of $\sigma^{-1}(u)$. But $\sigma^{-1}(u)$ is smooth to each coordinate. So $\varphi\circ\sigma^{-1}$ is smooth to each coordinate. Now consider $\sigma\circ\varphi^{-1}(u)$. What σ does is forgeting the last entry and multiplying each other entry to a constant (which is $\frac{1}{1-x_{n+1}}$). But $\varphi^{-1}(u)$ is smooth to each coordinate, thus so is $\sigma\circ\varphi_i^{-1}$.

Similarly for other pairs of (U_i,φ_i), with $(\mathbb{S}^n\setminus\{N\},\sigma)$ and $(\mathbb{S}^n\setminus\{S\},\widetilde{\sigma})$, we claim that these smooth structures are the same.

□

Problem 1-4
Let M be a smooth n-manifold with boundary. Show that $\operatorname{Int}M$ is a smooth n-manifold and ∂M is a smooth $(n-1)$-manifold (both without boundary).

Proof. We first prove that if $f\colon U\to\mathbb{R}$ is smooth where $U\subset\mathbb{R}^n$ (not necessarily open), then $f\colon \operatorname{Int}U\to\mathbb{R}$ is smooth. By the definition, there is an extension $\widetilde{f}\colon\widetilde{U}\to\mathbb{R}$ of f such that $\widetilde{U}$ is open in $\mathbb{R}^n$ and that $\widetilde{f}$ is smooth. Notice that $\operatorname{Int}U$ is an open subset of $\widetilde{U}$. Hence $f|_{\operatorname{Int}U}=\widetilde{f}|_{\operatorname{Int}U}$, which is obviously smooth. So for any open subset V or $\mathbb{R}^n$ such that $V\subset U$, we have $f|_V$ is smooth.

Assume that M is a smooth n-manifold, and that $\{(U_i,\varphi_i):i\in I\}$ defines a smooth structure on M, we will show that $\{(U_i',\varphi_i):i\in I,U_i'=U_i\cap\operatorname{Int}M=\operatorname{Int}U_i\}$ defines an atlas on $\operatorname{Int}M$. (Notice that φ_i here is the restriction of φ_i onto U_i', but no relabelling is required.) Because

$$\bigcup_{i\in I}U_i'=\bigcup_{i\in I}(U_i\cap\operatorname{Int}M)=\operatorname{Int}M\cap\bigcup_{i\in I}U_i=\operatorname{Int}M\cap M=\operatorname{Int}M,$$

the U_i' cover $\operatorname{Int}M$. For any (U_i',φ_i) and (U_j',φ_j) such that $U_i'\cap U_j'\neq\varnothing$, we have $\varphi_i\circ\varphi_j^{-1}$ is smooth on $\varphi_j(U_i\cap U_j)\subset\mathbb{H}^n$. But φ_j is a homeomorphism, thus $\varphi_j(U_j'\cap U_i')$ is an open subset of $\mathbb{R}^n$ such that $\varphi_j(U_j'\cap U_i')\subset\varphi(U_i\cap U_j)$. By our remark in the first paragraph, $\varphi_i\circ\varphi_j^{-1}$ is smooth (in the open set sense). Similarly, we have $\varphi_j\circ\varphi_i^{-1}$ smooth. So $\{(U_i',\varphi_i):i\in I,U_i'=U_i\cap\operatorname{Int}M=\operatorname{Int}U_i\}$ defines an atlas on $\operatorname{Int}M$ or $\operatorname{Int}M$ is a smooth n-manifold.

If $m\in\partial M$, then there is a chart (U_m,φ_m) where U_m is a neighborhood of m, and $\varphi_m(m)\in\partial\mathbb{H}^n=\mathbb{R}^{n-1}$. And actually, by the Invariance of the Boundary, we have $\varphi_m\colon U_m\cap\partial M\to\partial\mathbb{H}^n=\mathbb{R}^{(n-1)}$. Define $\varphi_m'=\varphi_m|_{\partial M}$ and $U_m'=U_m\cap\partial M$ for each $m\in\partial M$, we will show that $\{(U_m',\varphi_m'):m\in\partial M\}$ is an atlas on ∂M. Because m runs all over ∂M and U_m' contains m, it is clear that this set covers ∂M. Moreover, for any two charts (U_m',φ_m') and (U_n',φ_n') such that $U_m'\cap U_n'\neq\varnothing$, we consider the map $\varphi_m'\circ(\varphi_n')^{-1}$. Notice that

$$\varphi_m'\circ(\varphi_n')^{-1}=\varphi_m\circ(\varphi_n)^{-1}|_{\partial\mathbb{H}^n},$$

which is just the same as $\varphi_m \circ (\varphi_n)^{-1}$ on every coordinate save the last one (which is eliminated). So this function restricted to $\partial\mathbb{H}^n$ is smooth, which is synonymous with saying $\varphi'_m \circ (\varphi'_n)^{-1}$ is smooth. Hence (U'_m, φ'_m) and (U'_n, φ'_n) are smoothly compatible for every $m, n \in \partial M$. So ∂M is a smooth $(n-1)$-manifold. □

2 Smooth Maps

2.1 Exercises

Exercise 2.1
Let $F\colon M \to N$ be a map between smooth manifolds, and suppose each point $p \in M$ has a neighborhood U such that $F|_U$ is smooth. Show that F is smooth.

Proof. Assume that each point $p \in M$ has a neighborhood U such that $F|_U$ is smooth. So there is a chart (U_p, φ_p) of M such that $F\colon U_p \to N$ is smooth, and $U_p \subset U$. (If not, then just take $U_p \cap U$.) Notice that $\{U_p : p \in M\}$ covers M, thus $\{(U_p, \varphi_p) : p \in M\}$ is an atlas on M. For any chart (V_q, ψ_q) of N, obviously $\varphi_p \circ F \circ \psi_q^{-1}$ is smooth by our assumption. Thus by Lemma 2.2, we get F to be smooth. □

Exercise 2.2
Prove the following claim. Let M, N be smooth manifolds and let $F\colon M \to N$ be any map. If $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ are smooth atlases for M and N, respectively, and if for each α and β, $\psi_\beta \circ F \circ \varphi_\alpha^{-1}$ is smooth on its domain of defininition, then F is smooth.

Proof. For any two charts (U, φ) and (V, ψ) of M and N respectively, we can find $(U_\alpha, \varphi_\alpha)$ and (V_β, ψ_β) in the atlases for M and N such that φ and φ_α are smoothly compatible, and so are ψ and ψ_β. Thus we have

$$\begin{aligned}\psi \circ F \circ \varphi &= \psi \circ (\psi_\beta^{-1} \circ \psi_\beta) \circ F \circ (\varphi_\alpha^{-1} \circ \varphi_\alpha) \circ \varphi^{-1} \\ &= (\psi \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ F \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi^{-1}).\end{aligned}$$

But the last term is a composition of three smooth maps, thus smooth. So $\psi \circ F \circ \varphi$ is smooth on its domain of definition. Thus F is smooth. □

Exercise 2.3
Let $M_1, \cdots, M_k$ and N be smooth manifolds. Show that a map $F\colon N \to M_1 \times \cdots \times M_k$ is smooth if and only if each of the "component maps" $F_i = \pi_i \circ F\colon N \to M_i$ is smooth.

Proof. For the sake of clean notation, we will prove for the case $k = 2$. The general case can be done similarly. Let $\{(A_i, \alpha_i)\}$ and $\{(B_j, \beta_j)\}$ be atlases for M_1 and M_2. From Example 1.13, we know that $\{(A_i \times B_j, \alpha_i \times \beta_j)\}$ defines an atlas on $M_1 \times M_2$.

Assume that $F\colon N \to M_1 \times M_2$ is smooth, we will show that $F_1 = \pi_1 \circ F$ is smooth. Indeed, for any (A_i, α_i) in the atlas for M_1 and (U, φ) a chart of N, we have

$$\alpha_i \circ \pi_1 \circ F \circ \varphi^{-1} = (\alpha_i \circ \pi_1 \circ (\alpha_i \times \beta_j)^{-1}) \circ ((\alpha_i \times \beta_j) \circ F \circ \varphi^{-1})$$

for some (B_j, β_j) in the atlas for M_2. Since F is smooth, we get $(\alpha_i \times \beta_j) \circ F \circ \varphi^{-1}$ to be smooth. So it is sufficient to check that $\alpha_i \circ \pi_1 \circ (\alpha_i \times \beta_j)^{-1}$ is smooth. But this is just the identity function on $\mathbb{R}^N$ by the following diagram, thus smooth.

$$\begin{tikzcd}
 & & M_1 \\
\mathbb{R}^N \arrow[urr, bend left, "\alpha_i^{-1}"] \arrow[r, "(\alpha_i \times \beta_j)^{-1}"] \arrow[drr, bend right, "\beta_j^{-1}"'] & M_1 \times M_2 \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] & \\
 & & M_2
\end{tikzcd}$$

Conversely, assume that F_1 and F_2 are smooth, we show that F is also smooth. By Exercise 2.2, it is sufficient to check that $(\alpha_i \times \beta_j) \circ F \circ \varphi^{-1}$ is smooth. But

$$(\alpha_i \times \beta_j) \circ F \circ \varphi^{-1} = (\alpha_i \circ \pi_1 \circ F \circ \varphi^{-1}) \times (\beta_j \circ \pi_2 \circ F \circ \varphi^{-1}),$$

which is a product of two smooth functions thus smooth. So F is smooth if and only if F_1 and F_2 are smooth. $\square$

Exercise 2.4
Show that "diffeomorphic" is an equivalence relation.

Proof. Assume that $M \cong N$, then there exists $F\colon M \to N$ a diffeomorphism. Thus $F^{-1}\colon N \to M$ is a diffeomorphism, which implies $N \cong M$. Clearly $Id_M\colon M \to M$ is a diffeomorphism, thus $M \cong M$. Lastly, if $M \cong N$ and $N \cong P$, then there exist $F\colon M \to N$ and $G\colon N \to P$ to be diffeomorphisms. It is not hard to see that $G \circ F\colon M \to P$ is a diffeomorphism. Thus $M \cong P$. So diffeomorphic is an equivalence relation. $\square$

Exercise 2.5
Show that a map $F\colon M \to N$ is a diffeomorphism if and only if it is a bijective local diffeomorphism.

Proof. Assume that $F\colon M \to N$ is a diffeomorphism, then obviously F is bijective. For any $m \in M$, M itself is a neighborhood m and $M \cong N = F(M)$. So F is locally diffeomorphism.

Conversely, assume that $F\colon M \to N$ is a bijective local diffeomorphism. For any $m \in M$, our hypothesis implies the existence of a neighborhood U_m that is diffeomorphic to $F(U_m)$. Assume that (M_i, φ_i) is an atlas for M, then there is some M_i that contain m. Let $\bar{U}_m = M_i \cap U_m$, then $\bar{U}_m$ is diffeomorphic to $F(\bar{U}_m)$ and $\{(\bar{U}_m, \varphi_i)\}$ defines an atlas on M. For any chart (V, ψ) of N that has m, because F is a diffeomorphism restricting on $\bar{U}_m$, we get $\psi \circ F \circ \varphi_i^{-1}$ to be smooth. So F is smooth. Similarly, we can check that F^{-1} is smooth. So F is a diffeomorphism. $\square$

Exercise 2.6
Prove that

(i) Any smooth covering map is local diffeomorphism and an open map.

(ii) An injective smooth covering map is a diffeomorphism.

(iii) A topological covering map is a smooth covering map if any only if it is a local diffeomorphism.

Proof. (i) Assume that $\pi\colon \widetilde{M} \to M$ is a smooth covering map. For any $m \in \widetilde{M}$, consider $\pi(m) \in M$. There is a neighborhood U of $\pi(m)$ that is diffeomorphic to a neighborhood V of m through π. So π is a local diffeomorphism.

For any $m \in \widetilde{M}$, construct such V_m as above, we get an open covering $\{V_m\}$ of $\widetilde{M}$ such that V_m is diffeomorpic to $\pi(V_m)$. So for open subset $U \subset \widetilde{M}$, we have

$$\pi(U) = \pi\left(\bigcup_m (U \cap V_m)\right) = \bigcup_m \pi(U \cap V_m).$$

But the right hand side is a union of open sets, thus open. So π is an open map.

(ii) Assume that $\pi\colon \widetilde{M} \to M$ is an injective smooth covering map, then (i) implies π to be a local diffeomorphism. Since π is surjective, it is also bijective. Exercise 2.5 yields π to be a diffeomorphism.

(iii) Assume that $\pi\colon \widetilde{M} \to M$ is a topological covering map. If π is a smooth covering map, then part (i) implies that π is a local diffeomorphism. Conversely, if π is a local diffeomorphism, then we show that π is a smooth covering. Indeed, for any $m \in M$, there is a neighborhood U_m of M such that each component of $\pi^{-1}(U_m)$ is homeomorphic to U_m. But homeomorphisms are bijective; combining with the fact that π is locally diffeomorphic, we get each component of $\pi^{-1}(U_m)$ to be diffeomorphic to U_m. So π is a smooth covering.

□

Exercise 2.7
Prove that the smooth structure constructed above on $\widetilde{M}$ is the unique one such that π is a smooth covering map.

Proof. We will recall the construction of the smooth structure of $\widetilde{M}$. Let $\{(U, \varphi)\}$ be an atlas of M, let $\widetilde{U}$ be as sheet of U and $\widetilde{\varphi} = \varphi \circ \pi$. Then $\{(\widetilde{U}, \widetilde{\varphi})\}$ defines an atlas for $\widetilde{M}$.

Assume that (V_m, ψ) is a chart of $\widetilde{M}$ that makes π smooth, we get $\varphi \circ \pi \circ \psi^{-1}$ to be smooth for any chart (U, φ) of M. But this is the same as $\widetilde{\varphi} \circ \psi^{-1}$ to be smooth. Conversely, for any chart (U_m, φ) of M such that $\pi(V_m) \cap U_m \neq \varnothing$, there is a smooth local section $\sigma\colon U_m \to \widetilde{M}$. That means $\psi \circ \sigma \circ \varphi^{-1}$ is smooth. But since $\sigma = \pi^{-1}$ on its domain, we get $\psi \circ \widetilde{\varphi}^{-1}$ to be smooth. So (V_m, ψ) is smoothly compatible with any $(\widetilde{U}, \widetilde{\varphi})$. Thus the smooth structure on $\widetilde{M}$ is unique. □

Exercise 2.8
Prove the following claim. Suppose G is a smooth manifold with agroup structure such that the map $G \times G \to G$ given by $(g, h) \mapsto gh^{-1}$ is smooth. Then G is a Lie group.

Proof. Assume that the map $\varphi\colon (g, h) \mapsto gh^{-1}$ is smooth for any $g, h \in G$. Let $\psi\colon G \to G\times G$ that maps $h \mapsto (e, h)$. Because ψ is smooth on each component, that is, $\psi = 1\times \mathrm{Id}_G$, it is also smooth. So $\varphi \circ \psi\colon G \to G$ that maps $h \to h^{-1}$ is smooth.

For the same reason, we get the map $(g, h) \mapsto (g, h^{-1})$ to be smooth (that is, because the map is smooth on each component). Because the composition of smooth maps are smooth, we get the map

$$(g, h) \mapsto (g, h^{-1}) \mapsto g(h^{-1})^{-1} = gh$$

is smooth. So G is a Lie group. □

Exercise 2.9
Show that a cover $\{U_\alpha\}$ of X by precompact open sets is locally finite if and only if each U_α intersects U_β for only finitely many β. Give a counterexample if the sets of the cover are not assumed to be open.

Proof. Assume that $\{U_\alpha\}$ is an open cover of X such that each set intersects only finitely many others. For any point $p \in X$, there is some α such that $p \in U_\alpha$. By the hypothesis, U_α is a neighborhood of p that intersects finitely many sets of $\{U_\alpha\}$. So $\{U_\alpha\}$ is locally finite.

Conversely, assume that $\mathscr{U} = \{U_\alpha\}$ is an open cover of X by precompact sets, and $\{U_\alpha\}$ is also locally finite. For any $U \in \mathscr{U}$, we show that it intersects only finitely many elements of $\mathscr{U}$. For any $x \in U$, because $\mathscr{U}$ is locally finite, there is some neighborhood V_x that intersects with finitely many elements of $\mathscr{U}$. Because U is paracompact, it is covered by finitely many such V_x's. So $\bigcup V_x$ intersects with finitely many elements of $\mathscr{U}$. But $U \subset \bigcup V_x$, thus it intersects with finitely many $\mathscr{U}$.

This result doesn't hold if we remove the open requirement. Let $X = \mathbb{R}$, and $\mathscr{U} = \{\{r\} \subset \mathbb{R}\}$. Clearly $\mathscr{U}$ is a cover of $\mathbb{R}$ and any element of $\mathscr{U}$ intersect with no other element, thus finite. However, any open neighborhood of 0 must have infinitely many elements, thus $\mathscr{U}$ is not locally finite. □

Exercise 2.10
Show that the assumption that A is closed is necessary in the extension lemma, by giving an example of a smooth function on a nonclosed subset of a manifold that admits no smooth extension to the whole manifold.

Proof. Let $f\colon (0, \infty) \to \mathbb{R}$ define by $x \mapsto \frac{1}{x}$. Clearly f is smooth on the open set $A := (0, \infty)$. Since $\mathbb{R}$ is an open set that contains A, if the conclusion of Lemma 2.20 is correct, then f can be extended to a function $\widetilde{f}$ where $\operatorname{supp} \widetilde{f} \subset \mathbb{R}$. Notice that

$$[0, \infty) = \overline{\operatorname{supp} f} \subset \overline{\operatorname{supp} \widetilde{f}} = \operatorname{supp} \widetilde{f}$$

so $\widetilde{f}$ is defined at 0. But $\widetilde{f}$ is smooth, we get $\lim_{x\to 0} \frac{1}{x} = \widetilde{f}(0)$, which is impossible. So the closed property of A in Lemma 2.20 is necessary. □

2.2 Problems

Problem 2-1
Compute the coordinate representation for each of the following maps, using stereographic coordinates for shperes; use this to conclude that the map $A\colon \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map $A(x) = -x$ is smooth.

Proof. From Problem 1-3, there are two charts in the atlas of $\mathbb{S}^n$, which are $(\mathbb{S}^n \setminus \{N\}, \sigma)$ and $(\mathbb{S} \setminus \{S\}, \widetilde{\sigma})$. So it is sufficient to show that $\psi \circ A \circ \varphi^{-1}$ is smooth for $\psi, \varphi \in \{\sigma, \widetilde{\sigma}\}$. We have

$$\begin{aligned}
\sigma \circ A \circ \sigma^{-1}(x) &= \sigma \circ A \left(\frac{(2u^1, \cdots, 2u^n, |u|^2 - 1)}{|u|^2 + 1} \right) \\
&= \sigma \left(-\frac{(2u^1, \cdots, 2u^n, |u|^2 - 1)}{|u|^2 + 1} \right) \\
&= -\frac{(2u^1, \cdots, 2u^n, |u|^2 - 1)}{|u|^2(|u|^2 + 1)}.
\end{aligned}$$

Clearly this function is smooth by each component, thus smooth. Similarly for other three cases, we can conclude that A is smooth. □

Problem 2-4
For any topological space M, let $C(M)$ denote the vector space of continuous functions $f\colon M \to \mathbb{R}$. If $F\colon M \to N$ is continuous map, define $F^*\colon C(N) \to C(M)$ by $F^*(f) = f \circ F$.

(a) Show that F^* is linear.

(b) If M and N are smooth manifolds, show that F is smooth if and only if $F^*(C^\infty(N)) \subset C^\infty(M)$.

(c) If $F\colon M \to N$ is a homeomorphism between smooth manifolds, show that it is a diffeomorphism if and only if $F^*\colon C^\infty(N) \to C^\infty(M)$ is an isomorphism.

Thus in a certain sense the entire smooth structure of M is encoded in the space $C^\infty(M)$.

Proof. (a) For $f, g \in C(N)$ and $\alpha \in \mathbb{R}$, we have

$$F^*(f+g) = (f+g) \circ F = f \circ F + g \circ F = F^*(f) + F^*(g),$$

and

$$F^*(\alpha f) = (\alpha f) \circ F = \alpha(f \circ F) = \alpha F^*(f).$$

So F^* is linear.

(b) Assume that $F\colon M \to N$ is smooth, then for any charts (U, φ) and (V, ψ) of M and N respectively, we have $\psi \circ F \circ \varphi^{-1}$ to be smooth. If $f \in C^\infty(N)$, then $f \circ \psi^{-1}$ is smooth. Because the composition of smooth function is smooth, we get

$$(f \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) = f \circ F \circ \varphi^{-1} = F^*(f) \circ \varphi$$

to be smooth. So $F^*(f) \in C^\infty(M)$ or $F^*(C^\infty(N)) \subset C^\infty(M)$.

Conversely, assume that $F^*(C^\infty(N)) \subset C^\infty(M)$. For any charts (U, φ) and (V, ψ) of M and N respectively, clearly ψ is in $C^\infty(N)$. Therefore, we have $F^*(\psi) \in C^\infty(M)$ or

$$F^*(\psi) \circ \varphi^{-1} = \psi \circ F \circ \varphi$$

is smooth. So F is smooth.

(c) Assume that $F\colon M \to N$ is a homeomorphism that is also a diffeomorphism. By part (b), we get $F^*(C^\infty(N)) \subset C^\infty(M)$, so $F^*\colon C^\infty(N) \to C^\infty(M)$ is well defined. It is sufficient to check that F^* is a bijection. Since $F^{-1}\colon N \to M$ is also a diffeomorphism, we get $(F^{-1})^*\colon C^\infty(M) \to C^\infty(N)$ to be well defined. Notice that

$$(F^*) \circ (F^{-1})^*(f) = F^*(f \circ F^{-1}) = f \circ F^{-1} \circ F = f,$$

and

$$(F^{-1})^* \circ F^*(f) = (F^{-1})^*(f \circ F) = f \circ F \circ F^{-1} = f.$$

So $(F^{-1})^*$ is the two sided inverse of F^*. Thus F is an isometry.

Conversely, if $F^*\colon C^\infty(N) \to C^\infty(M)$ is an isometry, then we have $F^*(C^\infty(N)) \subset C^\infty(M)$ and $(F^{-1})^*(C^\infty(M)) \subset C^\infty(N)$. From part (b), we claim that both F and F^{-1} are smooth. So F is a diffeomorphism.

□

3 The Tangent Bundle

3.1 Exercises

Exercise 3.1
Prove Lemma 3.4. Suppose M is a smooth manifold, $p \in M$, and $X \in T_p(M)$.

(a) If f is a constant function, then $Xf = 0$.

(b) If $f(p) = g(p) = 0$, then $X(fg) = 0$.

Proof. (a) Assume that f is a constant function. For any $g \in C^\infty(M)$, because X is a linear function, we get

$$X(fg) = f(p)Xg.$$

But because X is a derivation, we have

$$X(fg) = f(p)Xg + g(p)Xf.$$

So $g(p)Xf = 0$ for any $g \in C^\infty(M)$. We can let g varies among the constant functions to deduce that $Xf = 0$.

(b) If $f(p) = g(p) = 0$, then by the derivation formula, we have

$$X(fg) = f(p)Xg + g(p)Xf = 0 + 0 = 0.$$

□

Exercise 3.2
Prove Lemma 3.5. Let $F\colon M \to N$ and $G\colon N \to P$ be smooth maps and let $p \in M$.

(a) $F_*\colon T_pM \to T_{F(p)}N$ is linear.

(b) $(G \circ F)_* = G_* \circ F_*\colon T_pM \to T_{G\circ F(p)}P$.

(c) $(\mathrm{Id}_M)_* = \mathrm{Id}_{T_pM}\colon T_pM \to T_pM$.

(d) If F is a diffeomorphism, then $F_*\colon T_pM \to T_{F(p)}N$ is an isomorphism.

Proof. (a) Let $X, Y \in T_p(M)$ be derivations at p and $c \in R$. Then for any $f \in C^\infty(N)$, we have

$$\begin{aligned} F_*(X+Y)(f) &= (X+Y)(f\circ F) \\ &= X(f\circ F) + Y(f\circ F) \\ &= F_*(X)(f) + F_*(Y)(f) \\ &= (F_*(X) + F_*(Y))(f). \end{aligned}$$

Moreover,

$$\begin{aligned} F_*(cX)(f) &= (cX)(f\circ F) \\ &= c\cdot X(f\circ F) \\ &= c\cdot F_*(X)(f). \end{aligned}$$

So F_* is linear.

(b) For any $X \in T_pM$, it is sufficient to show that

$$(G\circ F)_*(X) = G_* \circ F_*(X).$$

For any $f \in C^\infty(P)$, we have

$$\begin{aligned} (G\circ F)_*(X)(f) &= X(f\circ G\circ F) \\ &= F_*(X)(f\circ G) \\ &= G_*\circ F_*(X)(f). \end{aligned}$$

So $(G\circ F)_* = G_* \circ F_*$.

(c) It is sufficient to show that $(\mathrm{Id}_M)_*(X) = X$ for all $X \in T_pM$. Indeed, for any $f \in C^\infty(M)$, we have

$$(\mathrm{Id}_M)_*(X)(f) = X(f\circ \mathrm{Id}_M) = X(f).$$

So $(\mathrm{Id}_M)_*(X) = X$ or $(\mathrm{Id}_M)_* = \mathrm{Id}_{T_pM}$.

(d) Let $X \in T_pM$, if $F_*X = 0$, we show that $X = 0$. Indeed, for any $f \in C^\infty(M)$, we have $f\circ F^{-1} \in C^\infty(N)$. Since $F_*X = 0$, we get

$$\begin{aligned} 0 &= F_*X(f\circ F^{-1}) \\ &= X(f\circ F^{-1}\circ F) \\ &= X(f). \end{aligned}$$

So $X(f) = 0$ for all $f \in C^\infty(M)$, that is, $X = 0$. Therefore, F_* is injective.

For any $Y \in T_{F(p)}N$, we define $\widetilde{Y} \in T_pM$ as follow. For any $f \in C^\infty(M)$, we let $\widetilde{Y}(f) = Y(f \circ F^{-1})$. This is well defined because $f \circ F^{-1} \colon N \to \mathbb{R}$ is smooth, and $Y \colon C^\infty(N) \to \mathbb{R}$. Moreover, we have

$$F_*\widetilde{Y}(f) = F_*Y(f \circ F^{-1}) = Y(f).$$

So it is sufficient to prove that $\widetilde{Y} \in T_pM$. Clearly the base point is at p. For any $f, g \in C^\infty(M)$, we have

$$\begin{aligned}
\widetilde{Y}(fg) &= Y(fg \circ F^{-1}) \\
&= Y((f \circ F^{-1})(g \circ F^{-1})) \\
&= f \circ F^{-1}(F(p)) \cdot Y(g \circ F^{-1}) + g \circ F^{-1}(F(p)) \cdot Y(f \circ F^{-1}) \\
&= f(p)\widetilde{Y}(g) + g(p)\widetilde{Y}(f).
\end{aligned}$$

So our proof is done.

□

Exercise 3.3
If $F \colon M \to N$ is a local diffeomomrphism, show that $F_* \colon T_pM \to T_{F(p)}N$ is an isomorphism for every $p \in M$.

Proof. Assume that $F \colon M \to N$ is a local diffeomorphism. For any $p \in M$, there is some neighborhood U_p such that $F(U_p)$ is open and U_p is diffeomorphic to $F(U_p)$. By Exercise 3.2 (d), we have T_pU_p isomorphic to $T_{F(p)}F(U_p)$. But by Proposition 3.7, we have T_pM and $T_{F(p)}N$ are isomorphic to T_pU_p and $T_{F(p)}F(U_p)$ respectively. Because isomorphism is an equivalent relation, we get T_pM to be isomorphic to $T_{F(p)}N$. □

Exercise 3.4
Suppose $F \colon M \to N$ is a smooth map. By examining the local expression (3.5) for F_* in coordinates, show that $F_* \colon TM \to TN$ is a smooth map.

Exercise 3.5
Show that the zero section of any smooth vector bundle is smooth.

Exercise 3.6
Show that $T\mathbb{R}^n$ is isomorphic to the trivial bundle $\mathbb{R}^n \times \mathbb{R}^n$.

Exercise 3.7
If $f \in C^\infty(M)$ and $Y \in \mathcal{T}(M)$, show that fY is a smooth vector field.

Exercise 3.8
Show that $\mathcal{T}(M)$ is a module over the ring $C^\infty(M)$.

3.2 Problems

Problem 3-1
Suppose M and N are smooth manifolds with M connected, and $F\colon M \to N$ is a smooth map such that $F_*\colon T_*M \to T_{F(p)}N$ is the zero map for each $p \in M$. Show that F is a constant map.

Problem 3-2
Let $M_1, \cdots, M_k$ be smooth manifolds, and let $\pi_j\colon M_1 \times \cdots \times M_k \to M_j$ be the projection onto the j-th factor. For any choices of points $p_i \in M_i$, $i = 1, \cdots, k$, show that the map

$$\alpha\colon T_{(p_1,\cdots,p_k)}(M_1 \times \cdots \times M_k) \to T_{p_1}(M_1) \times \cdots \times T_{p_k}M_k$$

defined by

$$\alpha(X) = (\pi_{1*}X, \cdots, \pi_{k*}X)$$

is an isomorphism, with inverse

$$\alpha^{-1}(X_1, \cdots, X_k) = (j_{1*}X_1, \cdots, j_{k*}X_k),$$

where $j_i\colon M_i \to M_1 \times \cdots \times M_k$ is given by $j_i(q) = (p_1, \cdots, p_{i-1}, q, p_{i+1}, \cdots, p_k)$. [Using this isomorphism, we will routinely identify T_pM, for example, as a subspace of $T_{(p,q)}(M \times N)$.]

Problem 3-3
If a nonempty n-manifold is diffeomorphic to an m-manifold, prove that $n = m$.

Problem 3-4
Show that there is a smooth vector field on $\mathbb{S}^2$ that vanishes at exactly one point.

Problem 3-5
Let E be a smooth vector bundle over M. Show that E admits a local frame over an open subset $U \subset M$ if and only if it admits a local trivialization over U, and E admits a global frame if and only if it is trivial.

Problem 3-6
Show that $\mathbb{S}^1, \mathbb{S}^3$, and $\mathcal{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ are all parallelizable.

References

[1] J. M. Lee. *Introduction to Topological Manifolds.* Springer, 2010.