Answer to Algebra Chapter 0 by Paolo Aluffi

Hoang Vo Ke

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Chapter I. Preliminaries: Set theory and categories

1.3. Categories

Exercise 1
Let C be a category. Consider a structure C^{op} with

- $\mathrm{Obj}(C^{op}) := \mathrm{Obj}(C)$;
- for A, B objects of C^{op} (hence objects of C), $\mathrm{Hom}_{C^{op}}(A, B) := \mathrm{Hom}_C(B, A)$.

Show how to make this into a category (that is, define composition of morphisms in C^{op} and verify the properties listed in 3.1.

Proof. For any $f \in \mathrm{Hom}_{C^{op}}(A, B)$ and $g \in \mathrm{Hom}_{C^{op}}(B, C)$, we define the composition $g \circ f$ of C^{op} to be the composition fg of C. (We will denote the composition in C^{op} with "$\circ$" and nothing for the composition in C). With this definition, we have

$$h \circ (g \circ f) = h \circ fg = (fg)h = f(gh) = f(h \circ g) = (h \circ g) \circ f,$$

which says this composition law is associative.

For any object A of C, let the identity of $\mathrm{Hom}_{C^{op}}(A, A)$ equals the identity of $\mathrm{Hom}_C(A, A)$. So for any $f \in \mathrm{Hom}_{C^{op}}(A, B) = \mathrm{Hom}_C(B, A)$, we have $f \circ 1_A = 1_A f = f$. Similarly, we get $1_B \circ f = f 1_B = f$. So C^{op} is a category. $\square$

Exercise 3
Formulate precisely what it means to say that 1_a is an identity with respect to composition in Example 3.3, and prove this assertion.

Proof. To show that 1_A is an identity, we must show that for $f \in \mathrm{Hom}(a, b)$, we have $1_b f = f = f 1_a$. Indeed, we have $1_b f = (b, b)(a, b) = (a, b) = f$ and $f 1_a = (a, b)(a, a) = (a, b) = f$. So 1_a is an identity with respect to the composition in Example 3.3. $\square$

Exercise 4
Can we define a category in the style of Example 3.3 using the relation $<$ on the set $\mathbb{Z}$?

Proof. No we cannot define a category in style of Example 3.3 using the relation $<$ because it is not reflexive. Therefore, there is no identity morphism. $\square$

Exercise 5
Explain in what sense Example 3.4 is an instance of the categories considered in Example 3.3

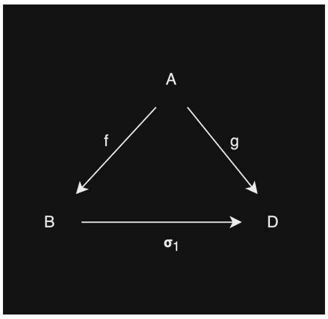
Proof. Because the $\subseteq$ relation is transitive and reflexive, we can define a category out of $P(S)$ similar to Example 3.3. $\square$

Exercise 7
Define carefully objects and morphisms in Example 3.7, and draw the diagram corresponding to composition.

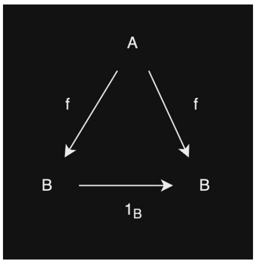
Proof. Let C be a category, we will define C_A as follow

$$\mathrm{Obj}(C_A) = \{f : f \in \mathrm{Hom}(A, B) \text{ for some } B \in \mathrm{Obj}(C)\}.$$

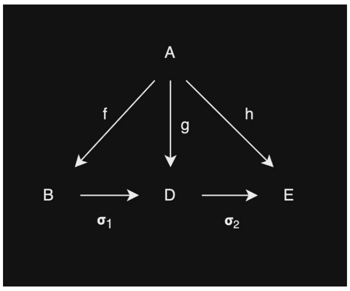
Let $f \in \mathrm{Hom}_C(A, B)$, $g \in \mathrm{Hom}_C(A, D)$, and $h \in \mathrm{Hom}_C(A, E)$, then we will define the morphism $f \to g$ be the commutative diagram.



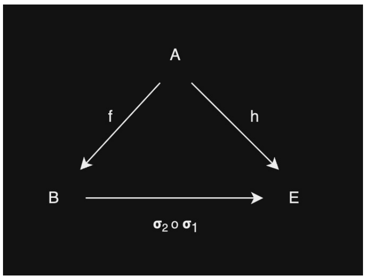
The identity of f would be this diagram.



Similar to Example 3.7, we can define the composition of two diagrams $f \to g$ and $g \to h$ as follow.



Because C is a Category, the previous diagram is the same as this.



And it is not hard to check that is definition satisfies all the properties of a Category. (Trust me, I have done it on paper.) $\square$

4. Morphisms

Exercise 4.3
Let A, B be objects of a category C, and let $f \in \mathrm{Hom}_C(A, B)$ be a morphism.

- Prove that if f has a right-inverse, then f is an epimorphism.
- Show that the converse does not hold, by giving an explicit example of a category and an epimorphism without a right-inverse.

Proof.

- Assume that $f \in \mathrm{Hom}_C(A, B)$ has a right inverse, say f', then $f \circ f' = 1_A$. For any β and β' in C such that $\beta \circ f = \beta' \circ f$, then we would have

$$\beta = \beta \circ (f \circ f') = (\beta \circ f) \circ f' = (\beta' \circ f) \circ f' = \beta'.$$

  So f is an epimorphism.

- The converse is not true however. Take the category $\mathbb{Z}$ with the relation $\leq$ as an example. Any morphism is an epimorphism but $(3, 5)$ doesn't have an inverse.

$\square$

5. Universal properties

Exercise 5.1
Prove that a final object in a category C is initial in the opposite category C^{op}.

Proof. Let A be a final object of C, so for any $B \in C$, we have $\mathrm{Hom}_{C^{op}}(A, B) = \mathrm{Hom}_C(B, A)$ is a singleton. So A is an initial object in C^{op}. □

Exercise 5.2
Prove that $\varnothing$ is the unique initial object in Set.

Proof. Let $A \neq \varnothing$ be an initial object in Set. Let $\{x, y\} \in Set$ be an object of Set that has two elements. We can define two distinct functions in $\mathrm{Hom}(A, \{x, y\})$, namely $f(a) = x$ and $f(a) = y$ for all $a \in A$. But this is impossible since A is an initial object, thus $\varnothing$ is the unique initial object of Set. □

Exercise 5.3
Prove that final objects are unique up to isomorphism.

Proof. Let A and B be two final objects of a category C. Notice that the unique element of $\mathrm{Hom}(A, A)$ is 1_A and the same for B. Let $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, A)$. Then $f \circ g \in \mathrm{Hom}(B, B)$, which implies $f \circ g = 1_B$. Similarly we get $g \circ f = 1_A$. So A is isomorphic to B. □

Exercise 5.6
Consider the category corresponding to endowing (as in Example 3.3) the set $\mathbb{Z}^+$ of positive integers with the divisibility relation. Thus there is exactly one morphism $d \to m$ in this category if and only if d divides m without remainder; there is no morphism between d and m otherwise. Show that this category has products and coproducts. What are their "conventional" names?

Exercise 5.8
Show that in every category C the products $A \times B$ and $B \times A$ are isomorphic if they exist.

Exercise 5.10
Push the envelope a little further still, and define products and coproducts for families (i.e., indexed sets) of objects of a category.
Do these exists in Set?
It is common to denote the product $A \times A \times \cdots \times A$ by A^n.