Answer to Steps in Commutative Algebra - Sharp: Exercise Solutions

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1 Chapter 1

Exercise [1.19 - Sharp]
Let K be an infinite field, let Λ be a finite subset of K, and let $f \in K[X_1, \cdots, X_n]$, the ring of polynomials over K in the indeterminates $X_1, \cdots, X_n$. Suppose that $f \neq 0$. Show that there exist infinitely many choices of

$$(\alpha_1, \cdots, \alpha_n) \in (K \setminus \Lambda)^n$$

for which $f(\alpha_1, \cdots, \alpha_n) \neq 0$.

Proof. The proof is by mathematical induction on n. Assume that $n = 1$. For any $f \in K[X_1]$, f can have at most $\deg(f)$ roots, which is finite. Since K is an infinite field, there are infinitely many $\alpha_1 \in K \setminus \Lambda$ such that $f(\alpha_1) \neq 0$.

Assume that this statement is true when $n \leq k$, we show that it still holds for $n = k+1$. For any $f \in K[X_1, \cdots, X_{k+1}]$, let $t = \deg(f)$, we write

$$f = X_{k+1}^t g_t + X_{k+1}^{t-1} g_{t-1} + \cdots + g_0.$$

Here $g_i \in K[X_1, \cdots, X_k]$. By our induction hypothesis, there exists $(\alpha_1, \cdots, \alpha_k) \in (K \setminus \Lambda)^k$ such that $g_t(\alpha_1, \cdots, \alpha_k) \neq 0$. So

$$f^*(X_{k+1}) = f(\alpha_1, \cdots, \alpha_k, X_{k+1})$$

is nonzero. Notice that $f^* \in K[X_{k+1}]$, we again applying our induction hypothesis to find infinitely many $\alpha_{k+1}^{(i)} \in (K \setminus \Lambda)$ such that

$$f(\alpha_1, \cdots, \alpha_k, a_{k+1}^{(i)}) = f^*(\alpha_{k+1}^{(i)}) \neq 0.$$

Since each $(\alpha_1, \cdots, \alpha_k, a_{k+1}^{(i)}) \in (K \setminus \Lambda)^{k+1}$, by mathematical induction, we complete our proof. □

2 Chapter 4

Exercise 4.4
Complete the proof of 4.3(i).

Proof. Assume taht R/I is not trivial, then clearly $I \in R$, $I \neq R$. Moreover, if $a, b \in R$, $ab \in I$, and $a \notin I$, then $\bar{a}\bar{b} = 0$ and $\bar{a} \neq 0$. Then $\bar{b}$ is a zero divisior, and by our assumption, be nilpotent. So $b^n \in I$ for some $n \in \mathbb{N}$. □

Exercise 4.7
Let $f\colon R \to S$ be a surjective homomorphism of commutative rings. Let $I \in C_R$. Show that

1. I is a primary ideal of R iff I^e is a primary ideal of S;

2. when this is the case, $\sqrt{I} = (\sqrt{I^e})^c$ and $\sqrt{I^e} = (\sqrt{I})^e$.

Proof. 1. First we show that $f^{-1}(f(I)) = I$. For each $a \in f^{-1}(f(I))$, we have $f(a) \in f(I)$, which means there exists $i \in I$ such that $a - i = k \in \ker f$. So $a = k + i \in I$ (because $\ker f \subset I$ and I is a group under addition). So $f^{-1}(f(I)) = I$, or $f(a) \in f(I)$ if and only if $a \in I$.

Notice that

$$I^e = f(I)S = f(I)f(R) = f(IR) = f(I),$$

and $f(I) \neq f(R)$, thus $ab \in I$ and $a \notin I$ impling $b^n \in I$ is synonymous with $\bar{a}\bar{b} \in f(I)$ and $\bar{a} \notin I$ impling $\bar{b}^n \in f(I)$.

2. We prove that $\sqrt{I^e} = \sqrt{I}^e$ by showing $\sqrt{f(I)} = f(\sqrt{I})$. Indeed, $f(a) \in \sqrt{f(I)}$ is the same as $f(a)^n \in f(I)$ or $f(a^n) \in f(I)$. But this is synonymous with $a^n \in I$ or $a \in \sqrt{I}$. This is the same as $f(a) \in f(\sqrt{(I)})$. Notice that $\sqrt{I} \subset \sqrt{I}^{ec} = (\sqrt{I^e})^c$. Because $\ker f \subset I \subset \sqrt{I}$, thus we have

$$\sqrt{I} = f^{-1}(f(\sqrt{I})) = f^{-1}(\sqrt{I^e}) = (\sqrt{I^e})^c.$$

□

Exercise 4.8
Let I be a proper ideal of the commutative ring R, and let P, Q be ideals of R which contain I. Prove that Q is a P-primary ideal of R if and only if Q/I is a P/I-primary ideal of R/I.

Proof. If Q is a P-primary ideal, then $\sqrt{Q} = P$. Therefore

$$\sqrt{Q/I} = \sqrt{Q^e} = \sqrt{Q}^e = P^e = P/I.$$

Conversely, if Q/I is a P/I-primary ideal, then $\sqrt{Q/I} = P/I$, or $\sqrt{Q^e} = P^e$. Thus

$$\sqrt{Q} = (\sqrt{Q^e})^c = P^{ec} = f^{-1}(f(P)) = P.$$

□

Exercise 4.21
Let $f\colon R \to S$ be a homomorphism of commutative rings, and use the contraction notation of 2.41 in conjunction with f. Let I be a decomposable ideal of S.

(i) Let

$$I = Q_1 \cap \cdots \cap Q_n \quad \text{with } \sqrt{Q_i} = P_i$$

be a primary decomposition of I. Show that

$$I^c = Q_1^c \cap \cdots \cap Q_n^c \text{ with } \sqrt{Q_i^c} = P_i^c$$

is a primary decomposition of I^c. Deduce that I^c is a decomposition ideal of R and that

$$\mathrm{ass}_R(I^c) \subset \{P^c : P \in \mathrm{ass}_S I\}.$$

(ii) Now suppose that f is surjective. Show that, if the first primary decomposition in (i) is minimal, then so too is the second, and deduce that, in this circumstances,

$$\operatorname{ass}_R(I^c) = \{P^c : P \in \operatorname{ass}_S I\}.$$

Proof.

(i) From 2.43(iii) we have

$$I^c = (Q_1 \cap \cdots \cap Q_n)^c = Q_1^c \cap \cdots \cap Q_n^c.$$

We will show that Q_i^c is a P_i^c-primary ideal for $i = 1 \cdots n$. Indeed, for each $i \in \overline{1,n}$, note that from 3.27(ii) we know $P_i^c \in \operatorname{Spec}(R)$ as $P_i \in \operatorname{Spec}(S)$ and from 2.43(iv) $\sqrt{Q_i^c} = P_i^c$. Let a, b be elements of R such that $ab \in Q_i^c$. We obtain that

$$f(a)f(b) = f(ab) \in Q_i.$$

Since Q_i is a primary ideal of S, we have $f(a) \in Q_i$ or $f(b) \in \sqrt{Q_i}$. Hence $a \in Q_i^c$ or $b \in (\sqrt{Q_i})^c = \sqrt{Q_i^c}$. Now if

$$I = Q_1 \cap \cdots \cap Q_n \quad \text{with } \sqrt{Q_i} = P_i$$

is a minimal primary decomposition of I then

$$I^c = Q_1^c \cap \cdots \cap Q_n^c \text{ with } \sqrt{Q_i^c} = P_i^c$$

is a primary decomposition of I^c. Therefore, from the construction of mininal primary decompostions, we deduce that the associated prime ideals of I^c must be P_i^c for some $i \in \overline{1,n}$. In other words

$$\operatorname{ass}_R(I^c) = \{P^c : P \in \operatorname{ass}_S I\}.$$

(ii) Suppose that

$$I = Q_1 \cap \cdots \cap Q_n \quad \text{with } \sqrt{Q_i} = P_i$$

is a minimal primary decomposition of I. Firstly, for $i, j \in \overline{1,n}$ with $i \neq j$, without loss of generality, we assume that $P_i \nsubseteq P_j$. Let $a \in P_i \backslash P_j$. Since f is surjective, $f^{-1}(a) \neq \varnothing$. We obtain that

$$f^{-1}(a) \subset P_i^c / P_j^c.$$

It yields that $P_i^c \neq P_j^c$.
Now, for each $j \in \overline{1,n}$, let $b \in (\bigcap_{i \neq j} Q_i) \backslash Q_j$. Similarly, we have $f^{-1}(b) \neq \varnothing$ and

$$f^{-1}(b) \subset (\bigcap_{i \neq j} Q_i^c) \backslash Q_j^c.$$

Thus $(\bigcap_{i \neq j} Q_i) \nsubseteq Q_j$. In conclusion, we have

$$I^c = Q_1^c \cap \cdots \cap Q_n^c \text{ with } \sqrt{Q_i^c} = P_i^c$$

is a primary decomposition of I^c. It also yields that

$$\operatorname{ass}_R(I^c) = \{P^c : P \in \operatorname{ass}_S I\}.$$

$\square$

Exercise 4.22
Let $f\colon R \to S$ be a surjective homomorphism of commutative rings; use the extension notation of 2.41 in conjunction with f. Let $I, Q_1, \cdots, Q_n, P_1, \cdots, P_n$ be ideals of R all of which contain $\ker f$. Show that

$$I = Q_1 \cap \cdots \cap Q_n \quad \text{with } \sqrt{Q_i} = P_i$$

is a primary decomposition of I if and only if

$$I^e = Q_1^e \cap \cdots \cap Q_n^e \quad \text{with } \sqrt{Q_i^e} = P_i^e$$

is a primary decomposition of I^e, and that, when this is the case, the first of these is minimal if and only if the second is.

Deduce that I is a decomposable ideal of R if and only if I^e is a decomposable ideal of S, and, when this is the case,

$$\operatorname{ass}_S(I^e) = \{P^e : P \in \operatorname{ass}_R I\}.$$

Proof. Notice that as f is surjective, we have

$$J^e = f(J)S = f(J)f(R) = f(JR) = f(J)$$

for any ideal J of R. Moreover, as $I, Q_1, \cdots, Q_n, P_1, \cdots, P_n$ are ideals of R all of which contain $\operatorname{Ker} f$, we can again use the proof in 4.7 to show that $I^{ec} = I$, similarly, $Q_i^{ec} = Q_i$, $P_i^{ec} = P_i$ for $i = 1 \cdots n$.
Now, suppose that

$$I = Q_1 \cap \cdots \cap Q_n \quad \text{with } \sqrt{Q_i} = P_i$$

is a primary decomposition of I. It follows from 4.7 that Q_i^e is P_i^e-primary. Now we only need to show that

$$I^e = (Q_1 \cap \cdots \cap Q_n)^e = Q_1^e \cap \cdots \cap Q_n^e.$$

It is obvious that

$$(Q_1 \cap \cdots \cap Q_n)^e \subset Q_1^e \cap \cdots \cap Q_n^e.$$

We will prove that

$$Q_1^e \cap \cdots \cap Q_n^e \subset (Q_1 \cap \cdots \cap Q_n)^e$$

by induction with respect to n. It is obvious in the case of $n = 1$. For $n > 1$, from induction hypothesis we obtain that

$$Q_1^e \cap \cdots \cap Q_n^e \subset (Q_1 \cap \cdots \cap Q_{n-1})^e \cap Q_n^e.$$

Let $x \in (Q_1 \cap \cdots \cap Q_{n-1})^e \cap Q_n^e$. There exists $x_1 \in Q_1 \cap \cdots \cap Q_{n-1}$, $x_2 \in Q_n$ such that $x = f(x_1) = f(x_2)$. Then

$$x_1 - x_2 \in \operatorname{Ker} f.$$

Since $\operatorname{Ker} f \subset Q_i$ for $i = 1 \cdots n$, we have

$$x_1 \in Q_1 \cap \cdots \cap Q_n.$$

Thus $x \in (Q_1 \cap \cdots \cap Q_n)^2 = e$. Henceforth,

$$Q_1^e \cap \cdots \cap Q_n^e \subset (Q_1 \cap \cdots \cap Q_n)^e.$$

Conversely, suppose that

$$I^e = Q_1^e \cap \cdots \cap Q_n^e \quad \text{with } \sqrt{Q_i^e} = P_i^e$$

is a primary decomposition of I^e. From 4.7 we know that Q_i is a primary ideal of R for $i = 1 \cdots n$, moreover,

$$\sqrt{Q_i} = \sqrt{Q_i^{ec}} = (\sqrt{Q_i^e})^c = P_i^{ec} = P_i.$$

Besides, from 2.43 we have

$$I = I^{ec} = (Q_1^e \cap \cdots \cap Q_n^e)^c = Q_1^{ec} \cap \cdots \cap Q_n^{ec} = Q_1 \cap \cdots \cap Q_n.$$

The proof of the first part is hence obtained.
We move to the next part. From 4.21 and the beginning of this proof, if

$$I^e = Q_1^e \cap \cdots \cap Q_n^e \quad \text{with } \sqrt{Q_i^e} = P_i^e$$

is a minimal primary decomposition of I^e, then

$$I = Q_1 \cap \cdots \cap Q_n \quad \text{with } \sqrt{Q_i} = P_i$$

is a minimal primary decomposition of I.
Conversely, for $i \neq j \in \overline{1,n}$, as $P_i \neq P_j$, without lost of generality, we assume that $P_i \subsetneq P_j$. Let $x \in P_i \backslash P_j$, then $f(x) \in P_i^e$. We will show that $f(x) \notin P_j^e$ and therefore $P_i^e \neq P_j^e$. Indeed, assume to the contrary that $f(x) \in P_j^e$, there exists $y \in P_j^{ec} = P_j$ such that $f(x) = f(y)$. Then $x - y \in \operatorname{Ker} f \subset P_j$, hence,

$$x = y + x - y \in P_j,$$

it is the contradiction. The proof for the statement

$$Q_j \not\supseteq \bigcap_{i \neq j} Q_i \quad \forall j \in \{1 \cdots n\}$$

is similar. Thus

$$I = Q_1 \cap \cdots \cap Q_n$$

is a minimal primary decomposition of I. In conclusion, I is a decomposable ideal of R if and only if I^e is a decomposable ideal of S, and,

$$\operatorname{ass}_S(I^e) = \{P^e : P \in \operatorname{ass}_R I\}.$$

□

Exercise 4.26
Suppose that the decomposable ideal I of the commutative ring R satisfies $\sqrt{I} = I$. Show that I has no embedded prime.

Proof. Assume $I = Q_1 \cap \cdots \cap Q_n$ and $\sqrt{Q_i} = P_i$ be the minimal decomposition of I and $\sqrt{I} = I$. Then we have

$$\bigcap P_i = \bigcap \sqrt{Q_i} = \sqrt{\bigcap Q_i} = \sqrt{I} = I.$$

If P_j is an embedded prime, then there eixsts P'_j such that $P_j \supsetneq P'_j \supset I$. Then

$$P'_j \supset \bigcap P_i.$$

So $P_j \supset P'_j \supset P_i$ for some $i \neq j$. Therefore we can rewrite

$$I = \cap_{i \neq j} P_i,$$

which is another decomposition with one less element, contradicting to our minimal hypothesis. So I has no embedded prime. □

Exercise 4.28
Let K be a field and let $R = K[X, Y]$ be the ring of polynomials over K in indeterminates X, Y. In R, let $I = (X^3, XY)$.

(i) Show that, for every $n \in \mathbb{N}$, the ideal (X^3, XY, Y^n) of R is primary.

(ii) Show that $I = (X) \cap (X^3, Y)$ is a minimal primary decomposition of I.

(iii) Construct infinitely many different minimal primary decompositions of I.

Proof. (i) We will prove that all zero divisor of $A_n = K[X, Y]/(X^3, XY, Y^n)$ is nilpotent, thus Lemma 4.3 implies that (X^3, XY, Y^n) is primary. The quotient ring above can be viewed as $\langle X, Y \mid X^3 = XY = Y^n = 0\rangle$, thus $\{1, X, X^2, Y, Y^2, \cdots, Y^{n-1}\}$ form a basis for this ring. For any polynomial $p \in A_n$, if p has nonzero constant, that is nonzero coefficient associating with 1 in the basis, then p is not a zero divisor. If p has only terms of X and Y, then it is nilpotent. Thus (X^3, XY, Y^n) is a primary for any $n \in \mathbb{N}$.

(ii) Notice that we have $(X) \cap (X^3) = (X^3)$ and $(X) \cap (Y) = (XY)$. Thus

$$\begin{aligned} I &= (X^3, XY) \\ &= (X^3) + (XY) \\ &= (X) \cap (X^3) + (X) \cap (Y) \\ &= (X) \cap [(X^3) + (Y)] \\ &= (X) \cap (X^3, Y). \end{aligned}$$

Moreover, $\sqrt{(X)} = X$, which doesn't contain Y, so is different from $\sqrt{(X^3, Y)}$. Since these primary ideals have different radicals, this is a minimal primary decomposition.

(iii) We will show that for any $n \in \mathbb{N}$, we have $I = (X^3, XY, Y^n) \cap (X)$ is a minimal primary decomposition. With a similar calculation as in part (ii), we get $I = (X^3, XY, Y^n) \cap (X)$. Moreover, they have different radicals, thus this decomposition is minimal.

$\square$

Exercise 4.30
Show that the zero ideal in the ring $C[0,1]$ of all continuous real-valued functions defined on the closed interval $[0,1]$ is not decomposable, that is, it does not have a primary decomposition.

Proof. Assume to the contrary that the zero ideal is decomposable. Let $P \in \operatorname{ass}_{C[0,1]} 0$, from 4.17 there exists $f \in C[0,1]$ such that $\sqrt{(0 : f)} = P$. For $g \in P$, there is $n \in \mathbb{N}$ such that $g^n f = 0$. For each $x \in [a,b]$, we have $(g(x))^n f(x) = 0$ yielding that $g(x) = 0$ or $f(x) = 0$, thus $g(x)f(x) = 0$. Hence $gf = 0$ or $g \in (0 : f)$. We conclude that $(0 : f) = P$. Since $f \neq 0$ and f is continuous, there exist $[c,d] \subset [a,b]$ such that $f \neq 0$ on $[c,d]$.Let $t \in (c,d)$. Define that

$$h(x) = \begin{cases} 0 & a \leq x \leq t \\ \dfrac{f(d)}{d-t}(x-t) & t \leq x \leq d \\ f(x) & d \leq x \leq b \end{cases}$$

and

$$k(x) = \begin{cases} f(x) & a \leq x \leq c \\ \dfrac{f(c)}{c-t}(x-t) & c \leq x \leq t \\ 0 & t \leq x \leq b \end{cases}$$

We can check that h,k is continuous and $hk = 0$ on $[a,b]$, and therefore, $hk \in (0 : f)$. However, $hf \neq 0$ on $(t,b]$ and $kf \neq 0$ on $[a,c)$, thus $h,k \notin (0 : f)$, it is the contracdiction. The proof is hence obtained. $\square$

Exercise 4.32
Let $f\colon R \to S$ be a surjective homomorphism of commutative rings, and use the extension notation of 2.41 in conjunction with f. Let I be an ideal of R which contains $\ker f$. Show that I is an irreducible ideal of R if and only if I^e is an irreducible ideal of S.

Proof. By Exercise 4.7, we have $I_i^e = f(I_i)$ for all $I_i \supset I$. If I is irreducible, and $I^e = I_1^e \cap I_2^e$, then $I = I_1 \cap I_2$. Thus without lost of generality, we have $I = I_1$ or $I^e = I_1^e$. Similarly for the converse. $\square$

Exercise 4.36
Let R be a commutative ring and let X be an indeterminate; use the extension and contraction notation of 2.41 in conjunction with the natural ring homomorphism $f\colon R \to R[X]$. Let Q and I be ideals of R.

1. Show that Q is a primary ideal of R if and only if Q^e is a primary ideal of $R[X]$.

2. Show that, if I is a decomposable ideal of R and

$$I = Q_1 \cap \cdots \cap Q_n \quad \text{with } \sqrt{Q_i} = P_i$$

is a primary decomposition of I, then

$$I^e = Q_1^e \cap \cdots \cap Q_n^e \quad \text{with } \sqrt{Q_i^e} = P_i^e$$

is a primary decomposition of the ideal I^e of $R[X]$.

3. Show that, if I is a decomposable ideal of R, then

$$\mathrm{ass}_{R[X]} I^e = \{P^e : P \in \mathrm{ass}_R I\}.$$

Proof.

1. Suppose that Q is a primary ideal of R. Let

$$f(X) = a_n X^n + \cdots + a_0,$$
$$g(X) = b_m X^m + \cdots + b_0,$$

where $n, m > 0$, $a_n, b_m \neq 0$, be polynomials of $R[X]$ such that

$$f(X)g(X) \in Q^e = QR[x].$$

We have

$$f(X)g(X) = c_{mn}X^{mn} + \cdots + c_0$$

where

$$c_k = a_k b_0 + a_{k-1}b_1 + \cdots a_1 b_{k-1} + a_0 b_k$$

for $k = 1, \cdots, mn$. Then $c_k \in Q$ for all k. Suppose that $f(X) \notin Q^e$, that is, $a_k \notin Q$ for $k = 1, \cdots, n$. We will prove that $g(X) \in \sqrt{Q^e}$,i.e. $b_k \in \sqrt{Q}$ for $k = 1, \cdots, m$ by induction with respect to k. In the case of $k = 0$, we have $a_0 b_0 = c_0 \in Q$, since Q is primary and $a_0 \notin Q$, $b_0 \in \sqrt{Q}$. For $k > 0$, we have

$$a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0 = c_k \in Q,$$

by the induction hypothesis, $b_{k-1}, b_{k-2}, \cdots, b_0 \in \sqrt{Q}$, yielding that $a_0 b_k \in \sqrt{Q}$, and as $a_0 \notin Q$, we deduce that $b_k \in \sqrt{Q}$. Notice that we can use again this proof to show that the image of a prime ideal of R is also a prime ideal of $R[X]$.
Conversely, suppose that Q_i^e is a primary ideal of $R[X]$. Let $a, b \in R$ such that $ab \in Q$ and $a \notin Q$. Since Q can be consider as a subset of Q^e, we obtain that $b \in \sqrt{Q^e} = (\sqrt{Q})^e$, that is, there is $n \in N$ such that $b^n \in Q^e$. However, b^n is only a free coefficient in R, thus $b^n \in Q$, henceforth, $b \in \sqrt{Q}$.
In conclusion, Q is a primary ideal of R if and only if Q^e is a primary ideal of $R[X]$. Moreover, we can show that, in this case,

$$\sqrt{Q^e} = (\sqrt{Q})^e.$$

From 2.47(ii) $Q^{ec} = Q$, therefore, from 2.43(iv) and 2.44(ii) we obtain that

$$(\sqrt{Q})^e = (\sqrt{Q^{ec}})^e = (\sqrt{Q^e})^{ce} \subset \sqrt{Q^e}.$$

On the other hand, we have $Q^e \subset (\sqrt{Q})^e$, thus

$$\sqrt{Q^e} \subset \sqrt{(\sqrt{Q})^e} = (\sqrt{Q})^e.$$

We completed the proof for this problem.

2. From 1. we know that Q_i^e is P_i^e-primary ideal of $R[X]$ for $i = 1, \cdots, n$. Moreover, from 2.47(iv) we obtain that

   $$I^e = (Q_1 \cap \cdots \cap Q_n)^e = (Q_1 \cap \cdots \cap Q_n)R[X] = Q_1 R[x] \cap \cdots \cap Q_n R[x] = Q_1^e \cap \cdots \cap Q_n^e.$$

   We conclude that this expression is a primary decomposition of I^e.

3. Suppose that

   $$I = Q_1 \cap \cdots \cap Q_n \quad \text{with } \sqrt{Q_i} = P_i$$

   is a minimal primary decomposition of I. We know that

   $$I^e = Q_1^e \cap \cdots \cap Q_n^e \quad \text{with } \sqrt{Q_i^e} = P_i^e$$

   is a primary decomposition of I^e. We will show that it is minimal, and therefore,

   $$\operatorname{ass}_{R[X]} I^e = \{P^e : P \in \operatorname{ass}_R I\}.$$

   Indeed, for $i \neq j \in \{1, \cdots, n\}$, without lost of generality we assume that $P_i \subsetneq P_j$. Let $a \in P_i \backslash P_j$. Note that P_k can be consider as subset of P_k^e for all k, moreover, $a \notin P_j^e$ as in this case, $a \in P_j$. Thus we have $a \in P_i^e \backslash P_j^e$, hence, $P_i^e \neq P_j^e$. We can prove that

   $$Q_j \not\supseteq \bigcap_{i \neq j} Q_i$$

   for $i = 1, \cdots, n$ similarly.
   This arises our proof.

□

Exercise 4.37
Let R be a commutative Noetherian ring, and let Q be a P-primary ideal of R. By 4.33, Q can be expressed as an intersection of finitely many irreducible ideal of R. One can refine such an expression to obtain

$$Q = \bigcap_{i=1}^{n} J_i,$$

where each J_i (for $1 \leq i \leq n$) is irreducible and irredundant in the intersection, so that, for all $i = 1, \cdots, n$,

$$\bigcap_{j=1, j \neq i}^{n} J_j \not\subset J_i.$$

By 4.34, the ideals $J_1, \cdots, J_n$ are all primary.
Prove that J_i is P-primary for all $i = 1, \cdots, n$.

Proof. Because Q is a primary ideal, the expression

$$Q = Q$$

is a minimal primary decomposition of Q. Our aim is to show that $\sqrt{J_i} = \sqrt{J_k}$ for any $i, k \in \overline{1, n}$. Suppose that this is not the case, that is, there exists $i, k \in \overline{1, n}$ such that $\sqrt{J_i} \neq \sqrt{J_k}$. Now we define a relation on $\{1 \cdots n\}$ by

$$a \sim b \iff \sqrt{J_a} = \sqrt{J_b}.$$

We can check that $\sim$ is a equivalent realtion, so it construct a partition $\{I_1 \cdots I_m\}$ on $\{1 \cdots n\}$. For each $j \in \overline{1, m}$, let $P_j = \sqrt{J_i}$ with $i \in I_j$ and let $Q_j = \bigcap_{i \in I_j} J_i$, then we have Q_j is a P_j-primary ideal. Since

$$Q = \bigcap_{i=1}^{n} J_i$$

is a primary decomposition of Q and for all $i = 1, \cdots, n$,

$$\bigcap_{j=1, j \neq i}^{n} J_j \not\subset J_i,$$

we can rewrite the above expression by

$$Q = \bigcap_{j=1}^{m} (Q_j).$$

It is another minimal primary decomposition of Q. However, notice that because there exists $i, k \in \overline{1, n}$ such that $\sqrt{J_i} \neq \sqrt{J_k}$, the number of terms appearing in this decomposition must be larger than 1, from 4.18 it is a contradiction. Thus for all i, $\sqrt{J_i}$ is the same, and therefore they equals to P. □

Exercise 4.38
Let R be the polynomial ring $K[X_1, \cdots, X_n]$ over the field K in the indeterminates $X_1, \cdots, X_n$, and let $\alpha_1, \cdots, \alpha_n \in K$. Let $r \in \mathbb{N}$ with $1 \leq r \leq n$. Show that, for all choices of $t_1, \cdots, t_r \in \mathbb{N}$, the ideal

$$((X_1 - \alpha_1)^{t_1}, \cdots, (X_r - \alpha_r)^{t_r})$$

of R is primary.

Proof. Let $R = K[X_1, \cdots, X_n]$ and $Q = ((X_1 - \alpha_1)^{t_1}, \cdots, (X_r - \alpha_r)^{t_r})$. We will show that all zero divisor of R/Q is nilpotent. Let $M = ((X_1 - \alpha_1), \cdots, (X_r - \alpha_r))$, then we know that M is maximal and for $k = \max\{t_1, \cdots, t_r\}$, we have $M^{nk} \subset Q \subset M$. Thus $\mathrm{rad}(Q) = M$. Assume that $ab \in Q$ and $b \notin M = \sqrt{(Q)}$, it is sufficient to prove that $a \in Q$. Indeed, because

$$M^{nk} \subset Q \subset M,$$

we get M is the only maximal ideal that contain Q. Since $b \notin M$, $(Q)+(b)$ is not contained in any maximal ideal. So $(Q)+(b)=R$. Thus

$$a \in (a) = a(Q)+(b)) = a(Q)+(ab) \subset Q.$$

So Q is indeed a primary. $\square$