Carothers - Chapter 4
Open Sets and Closed Sets

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Exercise 3. *Some authors say that two metrics d and p on a set M are equivalent if they generate the same open sets. Prove this.*

Proof. If d and p generate the same open set in M, then assume that $x_n \to x$ respect to d, we will prove that $x_n \to x$ respect to p. Indeed, for any $\delta > 0$, we have $B^p_\delta(x)$ is an open set in M, thus it is also an open set respect to d. And since x is in that open set, there exists $\epsilon > 0$ such that $B^d_\epsilon(x) \subset B^p_\delta(x)$. But because $x_n \to x$ respect to d, x_n is eventually in $B^d_\epsilon(x) \subset B^p_\delta(x)$. Therefore, $x_n \to x$ respect to p, which means d and p are equivalent. $\square$

Exercise 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\{x : f(x) > 0\}$ is an open subset of $\mathbb{R}$ and that $\{x : f(x) = 0\}$ is a closed subset of $\mathbb{R}$.*

Proof. Assume that $f(x) > 0$ for some x, then because f is continuous, there exists $\delta > 0$ such that for any $y \in B_\delta(x)$, we have $f(y) > 0$. Thus $B_\delta(x) \in \{x : f(x) > 0\}$, which implies $\{x : f(x) > 0\}$ to be an open set. Similarly, we have $\{x : f(x) < 0\}$ is also an open set, which means

$$\{x : f(x) = 0\} = \mathbb{R} \setminus (\{x : f(x) > 0\} \cup \{x : f(x) < 0\})$$

is a close set. $\square$

Exercise 7. *Show that every open set in $\mathbb{R}$ is the union of (countably many) open intervals with rational endpoints. Use this to show that the collection U of all open subsets of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$ itself.*

Proof. First, we will prove that for any open interval (a, b), $a, b \in \mathbb{R}$, there is countably many rational endpoint interval whose union is (a, b). Indeed, there exists an increasing sequence of rational numbers $b_n \to b$ and a decreasing sequence of rational numbers $a_n \to a$. Clearly, we have $\cup_{n=1}^{\infty}(a_n, b_n) = (a, b)$.

Therefore, by theorem 4.6, if M is an open set on $\mathbb{R}$, then M can be broken into countably many disjoint interval. We continue to break each interval into countably many unions of rational endpoint intervals. Thus any open set on $\mathbb{R}$ can be written as a union of countably many rational endpoint intervals.

Notice that the cardinality of (a, b) where $a, b \in \mathbb{Q}$ is $card(\mathbb{Q} \times \mathbb{Q}) = card(\mathbb{N}) = \aleph_0$. Therefore, the collection U of all open subsets of $\mathbb{R}$ has the cardinality equals $card(\mathcal{P}(\mathbb{N})) = card(\mathbb{R})$. Thus the two sets have the same cardinality. $\square$

Exercise 8. *Show that every open interval (and hence every open set) in $\mathbb{R}$ is a countable union of closed intervals and that every closed interval in $\mathbb{R}$ is a countable intersection of open intervals.*

Proof. Let (a, b) be an open interval in $\mathbb{R}$, there exists an increasing sequence (b_n) and a decreasing function (a_n) such that $b_n \to b$ and $a_n \to a$. And since $a < b$, there exists n_0 such that $a_n < b_n$ for any $n > n_0$. Therefore, without loss of generality, we can assume that $a_n < b_n$ for all n. We will claim that $\cup_{n\in\mathbb{N}}[a_n, b_n] = (a, b)$. Indeed, since $[a_n, b_n] \subset (a, b)$ for all n, we have $\cup_{n\in\mathbb{N}}[a_n, b_n] \subset (a, b)$. Now for any $x \in (a, b)$, there exists m such that $a_m < x < b_m$. Thus $x \in [a_m, b_m] \in \cup_{n\in\mathbb{N}}[a_n, b_n]$, which means $(a, b) \subset \cup_{n\in\mathbb{N}}[a_n, b_n]$. For that reason, $\cup_{n\in\mathbb{N}}[a_n, b_n] = (a, b)$.

Now, for any closed interval $[a, b]$, let a_n, b_n be increasing and decreasing sequences respectively, such that $a_n \to a$ and $b_n \to b$. We claim that $\cap_{n\in\mathbb{N}}(a_n, b_n) = [a, b]$. Well, it's kinda obvious, the proof is similar to the previous case. □

Before doing exercise 10, we first prove a little lemma.

Lemma 1. *For any $x, z \in H^\infty$, if $d(x, z) < 2^{-N}t$, then $|x_k - z_k| < t$ for all $k = 1, \cdots, N$.*

Proof. notice that for any $z \in H^\infty$, we have

$$d(x, z) = \sum_{n=1}^{\infty} 2^{-n}|x_n - z_n| = \sum_{n=1}^{N} 2^{-n}|x_n - z_n| + \sum_{n=N+1}^{\infty} 2^{-n}|x_n - z_n|.$$

Because $|x_n - z_n| \geq 0$, we have

$$\sum_{n=1}^{N} 2^{-n}|x_n - z_n| \leq \sum_{n=1}^{N} 2^{-n}|x_n - z_n| + \sum_{n=N+1}^{\infty} 2^{-n}|x_n - z_n| = d(x, z) \leq 2^{-N}t.$$

Therefore, $2^{-k}|x_k - y_k| < 2^{-N}t$ for any $k = 1, \cdots, N$. That is $|x_k - y_k| < 2^{k-N}t$. But $k \leq N$, hence $2^{k-N} \leq 1$, which implies $|x_k - y_k| < t$ for all $k = 1, \cdots, N$. □

Exercise 10. *Given $y = (y_n) \in H^\infty, N \in \mathbb{N}$, and $\epsilon > 0$, show that $\{x = (x_n) \in H^\infty : |x_k - y_k| < \epsilon, k = 1, \cdots, N\}$ is open in H^∞.*

Proof. For any $x \in H^\infty$, we will prove that there exists δ such that $B_\delta(x) \in S = \{x = (x_n) \in H^\infty : |x_k - y_k| < \epsilon, k = 1, \cdots, N\}$, so we can conclude that S is open. Indeed, by the assumption, we have $x \in S$, therefore $|x_k - y_k| < \epsilon$ for $k = 1, \cdots N$, which implies $M = \max\{|x_i - y_i| : i = 1, \cdots, N\} < \epsilon$. Using the density of real number, there exists $t > 0$ such that $M + t < \epsilon$. Now let $\delta = 2^{-N}t$, then for any $z \in H^\infty \cap B_\delta(x)$, we have $d(x, z) < 2^{-N}t$. By Lemma 1, we conclude that $|x_k - z_k| \leq t$ for all $k = 1, \cdots, N$. Notice that for such k, using the triangular inequality, we have

$$|z_k - y_k| \leq |z_k - x_k| + |x_k - y_k| < t + M < \epsilon.$$

Thus, $z \in S$, which implies $B_\delta(x) \in S$. That is S indeed open. □

Exercise 11. *Let $e^{(k)} = (0, \cdots, 0, 1, 0, \cdots)$, where the kth entry is 1 and the rest are 0s. Show that $\{e^{(k)} : k \geq 1\}$ is closed as a subset of ℓ_1.*

Proof. One thing to notice is that for any $m, n \in \mathbb{N}$, we have

$$\|e^{(m)} - e^{(n)}\|_1 = \sum_{i=1}^{\infty} |e_i^{(m)} - e^{(n)}| = 2$$

whenever $m \neq n$. Back to the problem, assume that there exists $(x_n) \to a$ for some $x_n \in \{e^{(k)} : k \geq 1\}$. It is sufficient to prove that $a \in \{e^{(k)} : k \geq 1\}$. Indeed, by the definition of convergence, there exists $N \in \mathbb{N}$ such that $x_n \in B_{\frac{1}{2}}(a)$ for all $n \geq N$. But then, for any $m, n > N$, we have

$$\|x_m - x_n\|_1 \leq \|x_m - a\|_1 + \|a - x_n\|_1 \leq \frac{1}{2} + \frac{1}{2} = 1,$$

which implies $e^{(m)} = e^{(n)}$. Therefore, $e^{(n)}$ is a constant when $n \geq N$. That is $a = e^{(N)} \in \{e^{(k)} : k \geq 1\}$. □

Exercise 12. *Let F be the set of all $x \in \ell_\infty$ such that $x_n = 0$ for all but finitely many n. Is F closed? open? neither? Explain.*

Proof. First, notice that $0 \in F$, but for any $\epsilon > 0$, we have $t = (\epsilon, \epsilon, \cdots)$ where $\|t - 0\|_\infty = \epsilon$, that is $t \in B_\epsilon(0)$. However, clearly $t \notin F$. So F is not open.

Second, let $x^{(i)} = (1 - \frac{1}{i}, \frac{1}{2} - \frac{1}{i}, \cdots, \frac{1}{i} - \frac{1}{i}, 0, 0, \cdots)$ and $a = (1, \frac{1}{2}, \cdots)$. For any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then, for $n > N$, we have

$$\|a - x^{(n)}\|_\infty = \left\| \left(\frac{1}{n}, \cdots, \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \cdots \right) \right\|_\infty = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Thus $x^{(i)} \to a$. But by the definition of $x^{(i)}$ and a, we have $x^{(i)} \in F$ but $a \notin F$. Therefore F is not closed.

So F is neither closed or open. □

Exercise 13. *Show that c_0 is a closed subset of ℓ_∞*

Proof. We will prove that $\ell_\infty \setminus c_0$ is an open set. For any $x \in \ell_\infty \setminus c_0$, we get $x \notin c_0$. Remind that $x \in c_0$ means for any $\delta > 0$, there exists $N > 0$ such that for all $n > N$, we have $|x_n| < \delta$. Therefore, $x \notin c_0$ means exists $\delta > 0$ such that for any $N > 0$, there exists $n > N$ so that $|x_n| > \delta$.

We will claim that $B_{\delta/2}(x) \cap c_0 = \varnothing$, thus $B_{\delta/2}(x) \in \ell_\infty \setminus c_0$, which leads to $\ell_\infty \setminus c_0$ be an open set.

Indeed, if $y \in B_{\delta/2}(x) \cap c_0$, then because $y \in c_0$, there exists N' such that $|y_n| < \frac{\delta}{2}$ for any $n > N'$. And because $y \in B_{\delta/2}(x)$, we get $\max\{|y_n - x_n| : n \in \mathbb{N}\} < \frac{\delta}{2}$. Thus,

$$|x_n| \leq |y_n - x_n| + |y_n| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

for any $n > N'$, which contradicts to the fact that there exists $n > N'$ such that $|x_n| > \delta$. So there is no such y. □

Exercise 14. *Show that the set $A = \{x \in \ell_2 : |x_n| \le 1/n,\ n = 1, 2, \cdots\}$ is a closed set in ℓ_2 but that $B = \{x \in \ell_2 : |x_n| < 1/n,\ n = 1, 2, \cdots\}$ is not an open set.*

Proof. Assume that $x^{(k)} \in A$ and $\|x^{(k)}\|_2 \to \|x\|_2$. By exercise 3.33, $\|\bullet\|_1$ and $\|\bullet\|_2$ on $\mathbb{R}^\infty$ are equivalent, thus $\|x^{(k)}\|_1 \to \|x\|_1$, which implies $|x_n^{(k)}| \to |x_n|$ for any $n \in \mathbb{N}$. Since $x^{(k)} \in A$, we have $|x_n^{(k)}| \le \frac{1}{n}$ for all k, hence $|x_n| \le \frac{1}{n}$ too. Thus $x \in A$, which implies A is a closed set.

Notice that $0 \in B$. For any $\epsilon > 0$, there exists $0 < \delta < \epsilon$ and $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Let $a = (0, \cdots, \delta, 0, \cdots)$, that is $a_n = \delta$ and 0 everywhere else. Since $\|a\|_2 = \delta < \epsilon$, we have $a \in B_\epsilon(0)$. However, because $a_n = \delta > \frac{1}{n}$, we have $a_n \notin B$. Thus for any $\epsilon > 0$, we have $B_\epsilon(0) \not\subset B$. That is, B is not an open set. $\square$

Exercise 15. *The set $A = \{y \in M : d(x, y) \le r\}$ is sometimes called the closed*