Answer to Multidimensional Real Analysis - Duistermaat: Exercise Solutions

Hoang Vo Ke

Exercise 0.1. *For purposes of integration it is useful to parametrize points of the circle $\{(\cos\alpha, \sin\alpha) | \alpha \in \mathbb{R}\}$ by means of rational functions of a variable in $\mathbb{R}$.*

(i) *Verify for all $\alpha \in (-\pi, \pi) \setminus \{0\}$ we have $\frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$. Deduce that both quotients equal a number $t \in \mathbb{R}$. Next prove*

$$\cos\alpha = \frac{1-t^2}{1+t^2}, \ \sin\alpha = \frac{2t}{1+t^2}.$$

Now use the identity $\tan\alpha = \dfrac{2\tan\dfrac{\alpha}{2}}{1-\tan^2\dfrac{\alpha}{2}}$ *to conclude $t = \tan\frac{\alpha}{2}$, that is $\alpha = 2\arctan t$.*

Proof. We have $\sin^2\alpha + \cos^2\alpha = 1$, thus $\sin^2\alpha = 1 - \cos^2\alpha = (1 + \cos\alpha)(1 - \cos\alpha)$. Since $\alpha \in (-\pi, \pi)$, $1 + \cos\alpha$ and $\sin\alpha$ is nonzero. Thus

$$\frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha} = t.$$

Because $\frac{\sin\alpha}{1+\cos\alpha} = t$, hence $\sin\alpha = t + t\cos\alpha$ (1). Similarly, we have $1 - \cos\alpha = t\sin\alpha$ (2). Multiply both sides of (1) to t, we get

$$1 - \cos\alpha = t\sin\alpha = t^2 + t^2\cos\alpha.$$

Thus

$$\cos\alpha = \frac{1-t^2}{t^2+1}.$$

Replace this to (2), we easily get

$$\sin\alpha = \frac{2t}{1+t^2}.$$

Thus $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{2t}{1-t^2}$. Moreover, we can easily check that $f(x) = \frac{2x}{1-x^2}$ is strictly increase, thus f is one-to-one. We have

$$f(t) = \tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} = f(\tan\frac{\alpha}{2},$$

thus $t = \frac{\alpha}{2}$

□

(ii) *Show for* $0 \leq \alpha < \frac{\pi}{2}$ *that* $t = \tan\alpha$*, thus* $\alpha = \arctan t$*, implies*

$$\cos^2\alpha = \frac{1}{1+t^2}, \ \sin^2\alpha = \frac{t^2}{1+t^2}$$

Proof. We have

$$t^2 = \frac{\cos^2\alpha}{\sin^2\alpha} = \frac{1-\sin^\alpha}{\sin^2\alpha},$$

thus

$$t^2\sin^2\alpha = 1 - \sin^2\alpha.$$

And with some simple calculations, we get

$$\sin^2\alpha = \frac{1}{t^2+1},$$

thus, by $t = \frac{\sin\alpha}{\cos\alpha}$, we also get

$$\cos^2\alpha = \frac{t^2}{t^2+1}.$$

□

Exercise 0.2.

(i) *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is differentiable and satisfies the functional equation* $f(x+y) = f(x) + f(y)$ *for all* x *and* $y \in \mathbb{R}$*. Prove that* $f(x) = f(1)x$ *for all* $x \in \mathbb{R}$*.*

Proof. Take the derivative both sides respect to x, we get $f'(x+y) = f'(x)$. Let $x = 0$, we get $f'(y) = f'(0)$. Thus $f(x) = f'(0)x + c$. Moreover, let $x = y = 0$, we get $f(0) = 2f(0)$, thus $c = f(0) = 0$. Thus $f(x) = f'(0)x$. Moreover, let $x = y = 1$, we have $2 \cdot f'(0) = f(2) = 2f(1)$. Thus $f(x) = f(1)x$. □

(ii) *Suppose* $g : \mathbb{R} \to \mathbb{R}_+$ *is differentiable and satisfies* $g(\frac{x+y}{2}) = \frac{1}{2}(g(x) + g(y))$ *for all* x *and* $y \in \mathbb{R}$*. Show that* $g(x) = g(1)x + g(0)(1-x)$ *for all* $x \in \mathbb{R}$*.*

Proof. Let $x = a + b$ and $y = 0$, we get

$$g(\frac{a+b}{2}) = \frac{1}{2}(g(a+b) + g(0)).$$

Let $x = a$ and $y = b$, we get

$$g(\frac{a+b}{2}) = \frac{1}{2}(g(a) + g(b)).$$

Thus

$$\begin{aligned} g(a+b) + g(0) &= g(a) + g(b) \\ g(a+b) - g(0) &= g(a) - g(0) + g(b) - b(0) \end{aligned}$$

Let $h(x) = g(x) - g(0)$, we get $h(x+y) = h(x) + h(y)$. Notice that if g is differentiable then so is h. Thus by (i), $h(x) = h(1)x$. Thus $g(x) - g(0) = (g(1) - g(0))x$, or $g(x) = g(1)x + g(0)(1-x)$. (Why g has to be $\mathbb{R} \to \mathbb{R}_+$, can it just be $\mathbb{R} \to \mathbb{R}$?) $\square$

(iii) *Suppose $g : \mathbb{R} \to \mathbb{R}_+$ is differentiable and satisfies $g(x+y) = g(x)g(y)$ for all x and $y \in \mathbb{R}$. Show that $g(x) = g(1)^x$ for all $x \in \mathbb{R}$.*

Proof. Let $f(x) = \log_{g(1)}(g(x))$, then

$$\begin{aligned} f(x+y) &= \log_{g(1)}(g(x+y)) \\ &= \log_{g(1)}(g(x)g(y)) \\ &= \log_{g(1)}(g(x)) + \log_{g(1)}(g(y)) \\ &= f(x) + f(y) \end{aligned}$$

Since f is differentiable, by (i), we get $f(x) = f(1)x$. Thus $\log_{g(1)} g(x) = \log_{g(1)} g(1)x = x$. Thus $g(x) = g(1)^x$. $\square$

(iv) *Suppose $h : \mathbb{R}_+ \to \mathbb{R}_+$ is differentiable and satisfies $h(xy) = h(x)h(y)$ for all x and $y \in \mathbb{R}_+$. Verify that $h(x) = x^{\log h(e)} = x^{h'(1)}$ for all $x \in \mathbb{R}_+$.*

Proof. Let $g(y) = h(e^y)$, we get

$$\begin{aligned} g(\log(x))g(\log(y)) &= h(x)h(y) \\ &= h(xy) \\ &= h(e^{\log(x)}e^{\log(y)}) \\ &= h(e^{\log(x)+\log(y)}) \\ &= g(\log(x) + \log(y)). \end{aligned}$$

Notice that $\log(x)$ is isomorphism to $\mathbb{R}$, we have $g : \mathbb{R} \to \mathbb{R}_+$ and $g(a)g(b) = g(a+b)$. Thus, by (iii), we have $g(x) = g(1)^x$ or $h(e^x) = h(e)^x$. Now let $z = e^x$, then $h(z) = h(e)^x = e^{\log(h(e))x} = z^{\log h(e)}$. Thus $h'(1) = \log(h(e))1^{\log h(e)} = \log(h(e))$. Thus $h(x) = x^{h'(1)}$. $\square$

(v) *Suppose $k : \mathbb{R}_+ \to \mathbb{R}$ is differentiable and satisfies $k(xy) = k(x) + k(y)$ for all x and $y \in \mathbb{R}_+$. Show that $k(x) = k(e)\log x = \log_b x$ for all $x \in \mathbb{R}_+$, where $b = e^{k(e)^{-1}}$.*

Proof. Let $g(x) = e^{k(x)}$, we have

$$\begin{aligned} g(xy) &= e^{k(xy)} \\ &= e^{k(x)+k(y)} \\ &= e^{k(x)}e^{k(y)} \\ &= g(x)g(y) \end{aligned}$$

Thus, by (iv), $g(x) = x^{\log g(e)}$, thus $e^{k(x)} = x^{\log(e^{k(e)})} = x^{k(e)} = e^{k(e)\log(x)}$. Thus $k(x) = k(e)\log(x)$. Now, we can easily check that $k(x) = \log_b x$ where $b = e^{k(e)^{-1}}$. □

(vi) Let $f : \mathbb{R} \to \mathbb{R}$ be Riemann integrable over every closed interval in $\mathbb{R}$ and suppose $f(x+y) = f(x) + f(y)$ for all x and $y \in \mathbb{R}$. Verify that the conclusion of (i) still holds.

Proof. We have

$$\int_0^y f(x+t)dt = \int_0^y f(x)dt + \int_0^y f(t)dt.$$

Thus

$$\int_0^y f(x+t)dt = yf(x) + \int_0^y f(t)dt.$$

With that in mind, we have

$$\begin{aligned}
\int_0^{x+y} f(t)dt &= \int_0^x f(t)dt + \int_x^{x+y} f(t)dt \\
&= \int_0^x f(t)dt + \int_0^y f(x+t)dt \\
&= \int_0^x f(t)dt + \int_0^y f(t)dt + yf(x).
\end{aligned}$$

Thus

$$yf(x) = \int_0^{x+y} f(t)dt - \int_0^x f(t)dt - \int_0^y f(t)dt.$$

Notice that in the right side of the equation, x and y are equivalent. Thus

$$yf(x) = xf(y).$$

Thus

$$\frac{f(x)}{x} = \frac{f(y)}{y} = c$$

for all $x, y \neq 0$. Thus $f(x) = cx$ for $x \neq 0$. By $f(x+y) = f(x) + f(y)$, we get $f(0) = 0$ and $f(2) = 2f(1)$, thus $f(x) = f(1)x$. $\square$

(vii) Suppose $f : \mathbb{R} \to \mathbb{R} \cap \{\pm\infty\}$ is differentiable where real-valued and satisfies

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)} \quad (x, y \in \mathbb{R})$$

where well-defined. Check that $f(x) = \tan(f'(0)x)$ for $x \in \mathbb{R}$ with $f'(0)x \notin \frac{\pi}{2} + \pi\mathbb{Z}$.

Proof. Let $x = y = 0$, we get

$$f(0) = \frac{2f(0)}{1 - f(0)^2}.$$

Hence

$$\begin{aligned} f(0)(1 - \frac{2}{1-f(0)^2}) &= 0 \\ f(0)\frac{f(0)^2 - 1}{1 - f(0)^2} &= 0 \\ -f(0) &= 0 \\ f(0) &= 0. \end{aligned}$$

Now, take the derivative respect to x from both sides, we get

$$\begin{aligned} f'(x+y) &= \frac{f'(x)(1 - f(x)f(y)) + f'(x)f(y)(f(x) + f(y))}{(1 - f(x)f(y))^2} \\ &= \frac{f'(x) - f'(x)f(x)f(y) + f'(x)f(y)f(x) + f(y)^2 f'(x)}{(1 - f(x)f(y))^2} \\ &= \frac{f'(x) + f'(x)f(y)^2}{(1 - f(x)f(y))^2}. \end{aligned}$$

Let $x = 0$, we get

$$\begin{aligned} f'(y) &= f'(0) + f'(0)f(y)^2 \\ f'(0) &= \frac{f'(y)}{1 + f(y)^2}. \end{aligned}$$

And let $g(x) = \tan^{-1}(f(x))$, then $g(0) = \tan^{-1}(0) = 0$. Remind that $\tan^{-1}(x)' = \frac{1}{1+x^2}$, we have

$$\begin{aligned} g'(x) &= f'(x)\tan^{-1}(f(x))' \\ &= \frac{f'(x)}{1 + f^2(x)} \\ &= f'(0). \end{aligned}$$

Thus $tan^{-1}(f(x)) = g(x) = f'(0)x + c$. However, $0 = g(0) = c$. Thus $g(x) = f'(0)x$, thus $f(x) = \tan(f'(0)x)$. □

(viii) *Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable and satisfies $|f(x)| \leq 1$ for all $x \in \mathbb{R}$ and*

$$f(x+y) = f(x)\sqrt{1 - f(y)^2} + f(y)\sqrt{1 - f(x)^2} \quad (x, y \in \mathbb{R}).$$

Prove that $f(x) = \sin(f'(0)x)$ for $x \in \mathbb{R}$.

Proof. Let $x = y = 0$, we have

$$f(0) = 2f(0)\sqrt{1 - f(0)^2}.$$

Thus

$$f(0)(1 - 2\sqrt{1 - f(0)^2} = 0.$$

With some calculation, $f(0)$ can be either 0 or $\frac{\sqrt{3}}{2}$.

If $f(0) = \frac{\sqrt{3}}{2}$, then let $y = 0$, we get

$$\begin{aligned}
f(x) &= \frac{1}{2}f(x) + \frac{\sqrt{3}}{2}\sqrt{1 - f(x)^2} \\
\frac{1}{2}f(x) &= \frac{\sqrt{3}}{2}\sqrt{1 - f(x)^2} \\
f(x)^2 &= 3 - 3f(x)^2 \\
f(x) &= \frac{\sqrt{3}}{2}.
\end{aligned}$$

Well, this is weird cause $f(x) = \frac{\sqrt{3}}{2}$ seems to work, but $f'(0) = 0$ thus $\sin(f'(0)x) = 0 \neq f(x)$.

Now, if $f(0) = 0$, take the derivative respect to x from both sides, we get

$$\begin{aligned}
f'(x + y) &= f'(x)\sqrt{1 - f(y)^2} + \frac{-1}{2}f(y)\frac{2f(x)f'(x)}{\sqrt{1 - f(x)^2}} \\
&= f'(x)\sqrt{1 - f(y)^2} - f(y)\frac{f(x)f'(x)}{\sqrt{1 - f(x)^2}}.
\end{aligned}$$

Let $x = 0$, we get

$$f'(y) = f'(0)\sqrt{1 - f(y)^2}.$$

Thus

$$\begin{aligned}
f''(y) &= (f'(0)\sqrt{1 - f(y)^2})' \\
&= f'(0)\frac{-f'(y)f(y)}{\sqrt{1 - f(y)^2}} \\
&= f'(0)\frac{-f'(0)\sqrt{1 - f(y)^2}f(y)}{\sqrt{1 - f(y)^2}} \\
&= -f(0)^2 f(y).
\end{aligned}$$

This is a linear differential equation with the auxiliary polynomial

$$t^2 + f'(0)^2 = 0.$$

Thus $t = f'(0)i$ or $t = -f'(0)i$, which gives the set $\{e^{f'(0)ix}, e^{-f'(0)ix}\}$ be the basis for the zero space. So $f(x)$ has the form

$$\begin{aligned}
&ae^{f'(0)ix} + be^{-f'(0)ix} \\
=&a(\cos(f'(0)x) + i\sin(f'(0)x)) + b(\cos(-f'(0)x) + i\sin(-f'(0)x)) \\
=&(a+b)\cos(f'(0)x) + (a-b)i\sin(-f'(0)x).
\end{aligned}$$

But $f(0) = 0$, thus $a + b = 0$. So $f(x) = t\sin(f'(0)x)$ for some complex number t. Plug this to the original equation $f(x+y) = f(x)\sqrt{1-f(y)^2} + f(y)\sqrt{1-f(x)}$, we get $t^2 = t$. Thus $t = 0$ or $t = 1$. But in either case, $f(x) = sin(f'(0)x)$. □

Exercise 0.11. *Let $\alpha \in \mathbb{R}$ be fixed and define $f : (-\infty, 1)$ by $f(x) = (1-x)^{-\alpha}$.*

(i) For $k \in N_0$ show $f^{(k)}(x) = (\alpha)_k(1-x)^{-\alpha-k}$.

$$(\alpha)_0 = 1;\ (\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) \quad (k \in \mathbb{N})$$

Proof. The proof is by mathematical induction. For $k = 0$, the result becomes $f(x) = (1-x)^{-\alpha}$ which is obvious. Assume that this result holds for k, that is $f^{(k)}(x) = (\alpha)_k(1-x)^{-\alpha-k}$, we will prove that it is also holds for $k+1$. Indeed, we have

$$\begin{aligned}
f^{(k+1)}(x) &= \frac{d}{dx}f^{(k)}(x) \\
&= \frac{d}{dx}(\alpha)_k(1-x)^{-\alpha-k} \\
&= (\alpha+k)\cdot(\alpha)_k(1-x)^{-\alpha-k-1} \\
&= (\alpha)_{k+1}(1-x)^{-\alpha-(k+1)}.
\end{aligned}$$

So by mathematical induction, the result is proved. □

(ii) Using the ratio test show that the series $F(x)$ has radius of convergence equal to 1.

$$F(x) = \sum_{k \in N_0} \frac{(\alpha)_k}{k!}x^k$$

Proof. Let $a_n = \frac{(\alpha)_k x^k}{k!}$, we have

$$\frac{a_{n+1}}{a_n} = \frac{(\alpha)_{k+1} x^{k+1} k!}{(k+1)!(\alpha)_k x^k} = \frac{(\alpha+k)}{k+1} x.$$

Notice that $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(\alpha+k)}{k+1} x = x$, $F(x)$ converges if $|x| < 1$ and diverges if $|x| > 1$. Thus $F(x)$ has radius of convergence equal to 1. □

(iii) *For $|x| < 1$, prove by termwise differentiation that $(1-x)F'(x) = \alpha F(x)$ and deduce that $f(x) = F(x)$ from*

$$\frac{d}{dx}((1-x)^\alpha F(x)) = 0.$$

Proof. First, we will need to prove $(1-x)F'(x) = \alpha F(x)$. Indeed, we have

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{(\alpha)_0 x^0}{0!} + \sum_{k=1}^{\infty} \frac{(\alpha)_k}{k!} x^k\right) \\ &= \sum_{k=1}^{\infty} \frac{(\alpha)_k}{(k-1)!} x^{k-1} \quad (1) \\ &= \sum_{k\in N_0} \frac{(\alpha)_{k+1}}{k!} x^k \qquad \text{(just replace } k-1 \text{ by } k \text{ so the equation look nicer).} \end{aligned}$$

Moreover, using (1), we have

$$xF'(x) = \sum_{k=1}^{\infty} \frac{(\alpha)_k}{(k-1)!} x^k.$$

With the above calculations, subtracting termwise $F'(x)$ and $xF'(x)$, we have

$$\begin{aligned}(1-x)F'(x) &= F'(x) - xF'(x) \\ &= (\alpha)_1 + \sum_{k=1}^{\infty} \frac{(\alpha)_{k+1} - k(\alpha)_k}{k!} x^k \\ &= \alpha + \sum_{k=1}^{\infty} \frac{(\alpha)_k (a + k - k)}{k!} x^k \\ &= \alpha \frac{(\alpha)_0 x^0}{0!} + \alpha \sum_{k=1}^{\infty} \frac{(\alpha)_k}{k!} x^k \\ &= \alpha \sum_{k \in N_0} \frac{(\alpha)_k}{k!} x^k \\ &= \alpha F(x).\end{aligned}$$

Now comes the magic calculation. We have

$$\begin{aligned}\frac{d}{dx}((1-x)^{\alpha} F(x)) &= (1-x)^{\alpha} F'(x) - \alpha(1-x)^{\alpha-1} F(x) \\ &= (1-x)^{\alpha-1}[(1-x)F'(x) - \alpha F(x)] \\ &= 0,\end{aligned}$$

hence $(1-x)^{\alpha} F(x) = c$ where c is a constant in $\mathbb{R}$. Notice that $(1-x)^{\alpha} = f(x)^{-1}$, we have $\frac{F(x)}{f(x)} = c$. Because $f(0) = (1-0)^{-\alpha} = 1$ and $F(0) = \sum_{k \in N_0} \frac{(\alpha)_k}{k!} 0^k = \frac{(\alpha)_0}{0!} 0^0 = 1$, we have $c = 1$, which implies $f(x) = F(x)$ when $|x| < 1$. $\square$

P/S: What a surprise. Isn't this a clever trick that was discovered by trying many things? I mean, it is a nightmare to check the reminder of the Taylor series, and MacLaurin series shouldn't be an exception.

(iv) Conclude from (iii) that

$$(1-x)^{-(n+1)} = \sum_{k \in N_0} \binom{n+k}{n} x^k \quad (|x| < 1, n \in \mathbb{N}_0).$$

Show that this identity also follows by n-fold differentiation of the geometric series $(1-x)^{-1} = \sum_{k \in N_0} x^k$.

Proof. From (iii), let $\alpha = n + 1$, we have

$$f(x) = (1-x)^{-\alpha} = (1-x)^{-(n+1)}$$

and

$$F(x) = \sum_{k\in N_0} \frac{(n+1)_k}{k!} x^k = \sum_{k\in N_0} \frac{(n+k)!}{n!k!} x^k = \sum_{k\in N_0} \binom{n+k}{n} x^k.$$

Therefore,

$$(1-x)^{-(n+1)} = \sum_{k\in N_0} \binom{n+k}{n} x^k \quad (|x|<1, n\in N_0).$$

We can also prove this result using differentiation and mathematical induction. We know that by the geometric series, if $|x|<1$, we have

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k.$$

Thus the result holds for $n=1$. Assume that the result holds for n, that is

$$(1-x)^{-(n+1)} = \sum_{k=0}^{\infty} \binom{n+k}{n} x^k.$$

That the derivative both sides, we get

$$(n+1)(1-x)^{-(n+2)} = \sum_{k=0}^{\infty} k \cdot \binom{n+k}{n} x^{k-1}.$$

Because when $k=0$, $k\binom{n+k}{n} x^{k-1} = 0$ so we can start counting from 1. Thus the expression above equals

$$\sum_{k=0}^{\infty} (k+1) \cdot \binom{n+k+1}{n} x^k.$$

The calculation above implies

$$(1-x)^{-(n+2)} = \sum_{k=0}^{\infty} \frac{k+1}{n+1} \cdot \binom{n+k+1}{n} x^k = \sum_{k=0}^{\infty} \binom{n+k+1}{n+1} x^k.$$

So by mathematical induction, we get the result. $\square$

(v) For $|x| < 1$, prove

$$(1+x)^\alpha = \sum_{k \in N_0} \binom{\alpha}{k} x^k,$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

In particular, show for $|x| < 1$

$$(1-4x)^{-\frac{1}{2}} = \sum_{k \in N_0} \binom{2k}{k} x^k$$

For $|x| < |y|$ deduce the following identity, which generalizes Newton's Binomial Theorem:

$$(x+y)^\alpha = \sum_{k \in N_0} \binom{\alpha}{k} x^k y^{\alpha-k}.$$

Proof. First notice that

$$(-\alpha)_k = (-\alpha)(-\alpha+1)\cdots(-\alpha+k-1) = (-1)^k(\alpha)(\alpha-1)\cdots(\alpha-k+1).$$

Therefore

$$\begin{aligned}
\frac{(-\alpha)_k}{k!}(-x)^k &= \frac{(-1)^k\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} \cdot (-1)^k x^k \\
&= \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k \\
&= \binom{\alpha}{k} x^k.
\end{aligned}$$

Using (iii) with $-\alpha \in \mathbb{R}$ and $-x \in (-1,1)$, we get

$$(1+x)^\alpha = \sum_{k \in N_0} \frac{(-\alpha)_k}{k!} x^k = \sum_{k \in N_0} \binom{\alpha}{k} x^k.$$

Now for the second part, first we have a little result, that is

$$2 \cdot 6 \cdots (4k-2) = (k+1)(k+2)\cdots(2k).$$

The proof is by mathematical induction. For $k = 1$, the result becomes $2 = 2$ which is obvious. Assume that the result holds for k, that is

$$2 \cdot 6 \cdots (4k-2) = (k+1)(k+2)\cdots(2k),$$

then

$$\begin{aligned}2\cdot 6\cdots(4k-2)(4k+2) &= (k+1)(k+2)\cdots(2k)(4k+2)\\ &= 2(k+1)(k+2)\cdots(2k)(2k+1)\\ &= (k+2)\cdots(2k+2).\end{aligned}$$

So by mathematical induction, we have the result. (This result feels so random. Is is a famous result or there is an obvious way to see it?)

Now using the MacLaurin series, we get

$$\begin{aligned}(1-4x)^{-\frac{1}{2}} &= \sum_{k\in N_0}\frac{(\frac{1}{2})_k}{k!}(4x)^k\\ &= \sum_{k\in N_0}\frac{\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2k-1}{2}}{k!}4^k(x)^k\\ &= \sum_{k\in N_0}\frac{2\cdot 6\cdots(4k-2)}{k!}(x)^k\\ &= \sum_{k\in N_0}\frac{(k+1)\cdots(2k)}{k!}(x)^k\\ &= \sum_{k\in N_0}\binom{2k}{k}x^k.\end{aligned}$$

Now, $|x|<|y|$, we have $|\frac{x}{y}|<1$. Therefore, using the first result in (v), we have

$$\begin{aligned}(x+y)^\alpha &= y^\alpha\left(1+\frac{x}{y}\right)^\alpha\\ &= y^\alpha\sum_{k\in N_0}\binom{\alpha}{k}\left(\frac{x}{y}\right)^k\\ &= \sum_{k\in N_0}\binom{\alpha}{k}x^k y^{\alpha-k}.\end{aligned}$$

□

Exercise 0.12. *Define* $f : \mathbb{R} \setminus \mathbb{Z} \to \mathbb{R}$ *by*

$$f(x) = \frac{\pi^2}{\sin^2(\pi x)} - \sum_{k\in\mathbb{Z}} \frac{1}{(x-k)^2}.$$

(i) Check that f *is well-defined. Verify that the series converges uniformly on bounded and closed subsets of* $\mathbb{R}\setminus\mathbb{Z}$*, and conclude that* $f : \mathbb{R}\setminus\mathbb{Z} \to \mathbb{R}$ *is continuous. Prove, by Taylor expansion of the function* sin*, that* f *can be continued to a function, also denoted by* f*, that is continuous at* 0*. Conclude that* $f : \mathbb{R} \to \mathbb{R}$ *thus defined is a continuous periodic function, and that consequently* f *is bounded on* $\mathbb{R}$*.*

Proof. For $x \in \mathbb{R} \setminus \mathbb{Z}$, we have $\sin(\pi x) \neq 0$ and $x - k \neq 0$ for $k \in \mathbb{Z}$. Therefore, $\frac{\pi^2}{\sin^2(\pi x)}$ and $\frac{1}{(x-k)^2}$ is well defined. Now, if $x \in \mathbb{R}\setminus\mathbb{Z}$, we have

$$\sum_{k\in\mathbb{Z}} \frac{1}{(x-k)^2} = \frac{1}{x^2} + \sum_{k=1}^{\infty} \frac{1}{(x-k)^2} + \sum_{k=1}^{\infty} \frac{1}{(x+k)^2}.$$

So it's sufficient to check that two summations on the right are well defined. Let $u = x - k$ in the following expression, we get

$$\int_1^{\infty} \frac{1}{(x-k)^2}\,\mathrm{d}k = -\int_{x-1}^{-\infty} \frac{1}{u^2}\,\mathrm{d}u = \frac{1}{u}\Big|_{x-1}^{-\infty} = -\frac{1}{x-1}.$$

Because $x \in \mathbb{R} \setminus \mathbb{Z}$, the integral always converge. By the integral test, we have $\sum_{k=1}^{\infty} \frac{1}{(x-k)^2}$ converges. Similarly, we get $\sum_{k=1}^{\infty} \frac{1}{(x+k)^2}$ converge. Thus $\sum_{k\in\mathbb{Z}} \frac{1}{(x-k)^2}$ well-defined.

Let I be a bounded and closed subset of $\mathbb{R}\setminus\mathbb{Z}$, then there exists $m, n \in \mathbb{N}$ such that $x + m > 0$ and $x - n < 0$ for all $x \in I$. Hence

$$\left|\frac{1}{(x+m')^2}\right| \leq \frac{1}{\lceil x+m' \rceil^2}$$

and

$$\left|\frac{1}{(x-n')^2}\right| \leq \frac{1}{\lfloor x-n' \rfloor^2}.$$

for any $m' > m$ and $n' > n$. Since

$$\sum_{i=m}^{\infty} \frac{1}{\lceil x+i \rceil^2} \text{ and } \sum_{i=n}^{\infty} \frac{1}{\lfloor x-i \rfloor^2}$$

are convergent, by the Weierstrass M-test, we have

$$\sum_{i=m}^{\infty} \frac{1}{(x+i)^2} \text{ and } \sum_{i=n}^{\infty} \frac{1}{(x-n)^2}$$

are uniformly convergent. So

$$f(x) = \frac{\pi^2}{\sin^2(\pi x)} - \sum_{k\in\mathbb{Z}} \frac{1}{(x-k)^2}$$

is uniformly convergent on bounded and closed subsets of $\mathbb{R}\setminus\mathbb{Z}$. Now we will prove that f can be continued to a function continuous at 0 by showing $\lim_{x\to 0} f(x)$ exists. We know that $\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{(x-k)^2}$ is uniformly convergent, thus

$$\begin{aligned}\lim_{x\to 0} f(x) &= \lim_{x\to 0} \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{(x-k)^2} + \lim_{x\to 0}\left(\frac{\pi^2}{\sin^2(\pi x)} - \frac{1}{x^2}\right)\\ &= \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{(k)^2} + \lim_{x\to 0}\left(\frac{\pi^2}{\sin^2(\pi x)} - \frac{1}{x^2}\right).\end{aligned}$$

Since the summation is convergent, it's sufficient to prove that the limit exists. Indeed, applying L'hopital rule four times we get

$$\lim_{x\to 0}\left(\frac{\pi^2}{\sin^2(\pi x)} - \frac{1}{x^2}\right) = \lim_{x\to 0} \frac{\pi^2 x^2 - \sin^2(\pi x)}{x^2 \sin^2(\pi x)} = \lim_{x\to 0} \frac{g(x)}{h(x)} = \lim_{x\to 0} \frac{g^{(4)}(x)}{h^{(4)}(x)}$$

where $g(x)$ and $h(x)$ are the numerator and the denominator of the previous fraction. After some messy calculation (please tell me if you want me to include the computation,) we get $g^{(4)} = 8\pi^4\cos(2\pi x)$ and $h^{(4)}(x) = 12\pi^2 x\cos(2\pi x) + (6\pi - 4\pi^3 x^2)\sin(2\pi x)$. Thus $\lim_{x\to 0}\lim_{x\to 0}\left(\frac{\pi^2}{\sin^2(\pi x)} - \frac{1}{x^2}\right) = \frac{g^{(4)}(0)}{h^{(4)}(0)} = \frac{1}{3}\pi^2$. So $\lim_{x\to 0} f(x)$ exists, which implies f can be continued to a function that is continuous at 0. Also notice that

$$f(x+1) = \frac{\pi^2}{\sin^2(\pi x+\pi)} - \sum_{k\in\mathbb{Z}} \frac{1}{(x-k+1)^2} = \frac{\pi^2}{(-\sin(\pi x))^2} - \sum_{k\in\mathbb{Z}} \frac{1}{(x-k)^2} = f(x).$$

Thus f is periodic. Because f is a continuous periodic function, f is bounded on $\mathbb{R}$.

$\square$

(ii) Show that

$$f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) = 4f(x) \quad (x \in \mathbb{R}).$$

Use this, and the boundedness of f, to prove that $f = 0$ on $\mathbb{R}$, that is, for $x \in \mathbb{R} \setminus \mathbb{Z}$ and $x - \frac{1}{2} \in \mathbb{R} \setminus \mathbb{Z}$, respectively,

$$\frac{\pi^2}{\sin^2(\pi x)} = \sum_{k \in \mathbb{Z}} \frac{1}{(x-k)^2},$$

$$\pi^2 \tan^{(1)}(\pi x) = \frac{\pi^2}{\cos^2(\pi x)} = 2^2 \sum_{k \in \mathbb{Z}} \frac{1}{(2x-2k-1)^2}.$$

Proof. We have

$$f\left(\frac{x}{2}\right) = \frac{\pi^2}{\sin^2(\pi \cdot \frac{x}{2})} - \sum_{k \in \mathbb{Z}} \frac{1}{(\frac{x}{2}-k)^2} = \frac{\pi^2}{\sin^2(\frac{\pi}{2}x)} - \sum_{k \in \mathbb{Z}} \frac{4}{(x-2k)^2}$$

and

$$f\left(\frac{x+1}{2}\right) = \frac{\pi^2}{\sin^2(\frac{\pi}{2}(x+1))} - \sum_{k \in \mathbb{Z}} \frac{1}{(\frac{x+1}{2}-k)^2} = \frac{\pi^2}{\sin^2(\frac{\pi}{2}(x+1))} - \sum_{k \in \mathbb{Z}} \frac{4}{(x-2k+1)^2}.$$

Therefore,

$$\begin{aligned} f\left(\frac{x}{2}\right) + f\left(\frac{x+1}{2}\right) &= \frac{\pi^2}{\sin^2(\frac{\pi}{2}x)} + \frac{\pi^2}{\sin^2(\frac{\pi}{2}(x+1))} - \sum_{k \in \mathbb{Z}} \frac{4}{(x-2k)^2} - \sum_{k \in \mathbb{Z}} \frac{4}{(x-2k+1)^2} \\ &= \frac{\pi^2}{\sin^2(\frac{\pi}{2}x)} + \frac{\pi^2}{\cos^2(\frac{\pi}{2}x)} - \sum_{k \in \mathbb{N}} \frac{4}{(x-k)^2} \\ &= \frac{\pi^2(\sin^2(\frac{x}{2}) + \cos^2(\frac{x}{2}))}{\sin^2(\frac{\pi}{2}x)\cos^2(\frac{\pi}{2}x)} - \sum_{k \in \mathbb{N}} \frac{4}{(x-k)^2} \\ &= \frac{\pi^2}{(\frac{\sin^2(\pi x)}{2})^2} - \sum_{k \in \mathbb{N}} \frac{4}{(x-k)^2} \\ &= \frac{4\pi^2}{\sin^2(\pi x)} - \sum_{k \in \mathbb{N}} \frac{4}{(x-k)^2} \\ &= 4f(x). \end{aligned}$$

Because f is bounded on $[0, 1]$, there exists $a \in \mathbb{R}$ such that $|f(a)| = \max\{|f(x)| : x \in \mathbb{R}\}$. Then, by the formula we just proved, we have

$4|f(a)| = |f(\frac{a}{2}) + f(\frac{a+1}{2})| \leq |f(\frac{x}{2})| + |f(\frac{a+1}{2})| \leq |f(a)| + |f(a)| = 2|f(a)|$. Thus, $|f(a)| = 0$, which implies $f(x) = 0$ for all $x \in \mathbb{R}$. That is

$$\frac{\pi^2}{\sin^2(\pi x)} = \sum_{k \in \mathbb{Z}} \frac{1}{(x-k)^2}$$

for all $x \in \mathbb{R} \backslash \mathbb{Z}$. (I think all we did before is to prove this equation, but how do they know this in the first place?). And for the last equation, we have

$$\begin{aligned}
\pi^2 \tan^{(1)}(\pi x) &= \frac{\pi^2}{\cos^2(\pi x)} \\
&= \frac{\pi^2}{\sin^2(\pi x - \frac{\pi}{2}} \\
&= \frac{\pi^2}{\sin^2(\pi(x - \frac{1}{2})} \\
&= \sum_{k \in \mathbb{Z}} \frac{1}{(x - k - \frac{1}{2})^2} \\
&= 4 \sum_{k \in \mathbb{Z}} \frac{1}{(2x - 2k - 1)^2}.
\end{aligned}$$

$\square$

(iii) Prove $\sum_{k\in\mathbb{N}} \frac{1}{k^2} = \frac{\pi^2}{6}$ *by setting* $x = 0$ *is the equality in (ii) for* $\pi^2 \tan^{(1)}(\pi x)$ *and using*

$$\sum_{k\in\mathbb{N}} \frac{1}{(2k-1)^2} = \sum_{k\in\mathbb{N}} \frac{1}{k^2} - \sum_{k\in\mathbb{N}} \frac{1}{(2k)^2} = \frac{3}{4}\sum_{k\in\mathbb{N}} \frac{1}{k^2}.$$

Proof. Because all four summation in the equation above are convergent, that equation is correct. Setting $x = 0$ is the equality in (ii), we get

$$\begin{aligned} \pi^2 &= 4\sum_{k\in\mathbb{Z}} \frac{1}{(2k+1)^2} \\ &= 4\left[\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{(-2k+1)^2}\right] \\ &= 8\sum_{k\in\mathbb{N}}^{\infty} \frac{1}{(2k-1)^2} \\ &= 6\sum_{k\in\mathbb{N}} \frac{1}{k^2} \quad \text{(by the equation in (iii))}. \end{aligned}$$

So we got the mind blowing equation $\sum_{k\in\mathbb{N}} \frac{1}{k^2} = \frac{\pi^2}{6}$. □

(iv) Prove by $(2n-2)$*-fold differentiation*

$$\pi^{2n}\frac{\tan^{(2n-1)}(\pi x)}{(2n-1)!} = 2^{2n}\sum_{k\in\mathbb{Z}} \frac{1}{(2x-2k-1)^{2n}} \quad (n \in \mathbb{N}, x - \frac{1}{2} \in \mathbb{R}\setminus\mathbb{Z}).$$

In particular, for $n \in \mathbb{N}$,

$$\begin{aligned} \pi^2\frac{\tan^{(2n-1)}(0)}{(2n-1)!} &= 2^{2n}\sum_{k\in\mathbb{Z}} \frac{1}{(2k-1)^{2n}} = 2^{2n+1}\sum_{k\in\mathbb{N}} \frac{1}{(2k-1)^{2n}} \\ &= 2^{2n+1}(1-2^{-2n})\sum_{k\in\mathbb{N}} \frac{1}{k^{2n}} = 2(2^{2n}-1)\zeta(2n). \end{aligned}$$

Here we have defined $\zeta(2n) = \sum_{k\in\mathbb{N}} \frac{1}{k^{2n}}$. *Conclude that* $\zeta(2) = \frac{\pi^2}{6}$ *and* $\zeta(4) = \frac{\pi^4}{90}$.

Lemma 1. *For any $n \in \mathbb{N}$ and $n \geq 2$, we have*

$$\sum_{k\in\mathbb{Z}} \frac{1}{(2x-2k-1)^n}$$

is uniformly convergent.

Proof. Indeed, when $|k|$ is large enough, we get $\left|\frac{1}{2x-2k-1}\right| < 1$. Because $n \geq 2$, we have $\left|\frac{1}{(2x-2k-1)^n}\right| \leq \left|\frac{1}{(2n-2k-1)^2}\right| = \frac{1}{(2n-2k-1)^2}$. And since $\sum_{k\in\mathbb{Z}} \frac{1}{(2n-2k-1)^2}$ is uniformly convergent, by the Weierstrass M-test, we have $\sum_{k\in\mathbb{Z}} \frac{1}{(2x-2k-1)^n}$ is uniformly convergent. $\square$

Proof. The proof is by mathematical induction. For $n = 1$, we get the equation in (ii). Assume that the equation holds for n, that is

$$\pi^{2n}\frac{\tan^{(2n-1)}(\pi x)}{(2n-1)!} = 2^{2n}\sum_{k\in\mathbb{Z}} \frac{1}{(2x-2k-1)^{2n}}.$$

Take termwise derivative of the right side, we get:

$$\begin{aligned}
2^{2n}\sum_{k\in\mathbb{Z}}\left(\frac{1}{(2x-2k-1)^{2n}}\right)' &= 2^{2n}\sum_{k\in\mathbb{Z}}\frac{-2n(2x-2k-1)^{2n-1}\cdot 2}{(2x-2k-1)^{4n}} \\
&= 2^{2n+1}\sum_{k\in\mathbb{Z}}\frac{-2n}{(2x-2k-1)^{2n+1}} \\
&= 2^{2n+1}\cdot(-2n)\sum_{k\in\mathbb{Z}}\frac{1}{(2x-2k-1)^{2n+1}}.
\end{aligned}$$

As we can see, every term of the summation is defferentiable and by Lemma 1, their sum is uniformly convergent. Moreover, also by Lemma 1, we have $\sum_{k\in\mathbb{Z}} \frac{1}{(-2k-1)^{2n}}$ is convergent. Thus

$$\pi^{(2n+1)}\frac{\tan^{(2n)}(\pi x)}{(2n-1)!} = 2^{2n+1}\sum_{k\in\mathbb{Z}}\frac{-2n}{(2x-2k-1)^{2n+1}}.$$

Similarly, take derivative both sides again, we get

$$\begin{aligned}
\pi^{2n+2}\cdot\frac{\tan^{(2n+1)}(\pi x)}{(2n-1)!} &= 2^{2n+1}\sum_{k\in\mathbb{Z}}\frac{2n(2n+1)(2x-2k-1)^{2n}\cdot 2}{(2x-2k-1)^{4n+2}} \\
&= 2^{2n+2}\sum_{k\in\mathbb{Z}}\frac{(2n+1)(2n)}{(2x-2k-1)^{2n+2}}.
\end{aligned}$$

Thus,

$$\pi^{2(n+1)} \cdot \frac{\tan^{(2n+1)}(\pi x)}{(2n+1)!} = 2^{2n+1} \cdot \sum_{k\in\mathbb{Z}} \frac{1}{(2x-2k-1)^{2n+2}},$$

which means the equation also holds for $n+1$. Therefore, by mathematical induction, we get

$$\pi^{2n} \frac{\tan^{(2n-1)}(\pi x)}{(2n-1)!} = 2^{2n} \sum_{k\in\mathbb{Z}} \frac{1}{(2x-2k-1)^{2n}}$$

for all $n \in \mathbb{N}$. Let $x=0$, follow the instruction, we get

$$\pi^{2n} \cdot \frac{\tan^{(2n-1)}(0)}{(2n-1)!} = 2(2^{2n}-1)\zeta(2n).$$

Now let $n=1$, we get $\pi^2 \frac{\tan'(0)}{1} = 6\zeta(2)$. Thus $\zeta(2) = \frac{\pi^2}{6}$. Let $n=2$, we get the second result, that is $\zeta(4) = \frac{\pi^4}{90}$. □

(v) *Now deduce that*

$$\tan(x) = \sum_{n\in\mathbb{N}} \frac{2(2^{2n}-1)\zeta(2n)}{\pi^{2n}} x^{2n-1} \qquad \left(|x| < \frac{\pi}{2}\right).$$

The values of the $\zeta(2n) \in \mathbb{R}$, for $n>1$, can be obtained from that of $\zeta(2)$ as follows.

Proof. In part (iv), we have proved that

$$\pi^{(2n+1)} \frac{\tan^{(2n)}(\pi x)}{(2n)!} = 2^{2n+1} \sum_{k\in\mathbb{Z}} \frac{-1}{(2x-2k-1)^{2n+1}}.$$

Let $x=0$, the equation becomes

$$\begin{aligned}
\pi^{(2n+1)} \frac{\tan^{(2n)}(0)}{(2n)!} &= 2^{2n+1} \sum_{k\in\mathbb{Z}} \frac{-1}{(-2k-1)^{2n+1}} \\
&= 2^{2n+1} \left(\sum_{k=0}^{\infty} \frac{-1}{(-2k-1)^{2n+1}} + \sum_{k=1}^{\infty} \frac{1}{(-1+2k)^{2n+1}} \right) \\
&= 2^{2n+1} \left(\sum_{k=1}^{\infty} \frac{-1}{(2k-1)^{2n+1}} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n+1}} \right) \\
&= 0.
\end{aligned}$$

Thus $\tan^{(2n)}(0) = 0$ for all $n \in \mathbb{N}$. Also by part (iv), we have

$$\pi^{2n} \cdot \frac{\tan^{(2n-1)}(0)}{(2n-1)!} = 2(2^{2n}-1)\zeta(2n).$$

Thus

$$\frac{\tan^{(2n-1)}(0)}{(2n-1)!} = \frac{2(2^{2n}-1)\zeta(2n)}{\pi^{2n}}.$$

Applying the MacLaurin series for $\tan(x)$, we get

$$\begin{aligned}\tan(x) &= \tan^{(0)}(0) + \frac{\tan^{(1)}(0)}{1!}x + \frac{\tan^{(2)}(0)}{2!}x^2 + \cdots \\ &= \sum_{n\in\mathbb{N}} \frac{2(2^{2n}-1)\zeta(2n)}{\pi^{2n}} x^{2n-1} \quad \left(|x| < \frac{\pi}{2}\right).\end{aligned}$$

□

(vi) (optional) Define $g : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ *by*

$$g(k,l) = \frac{1}{kl^3} + \frac{1}{2k^2l^2} + \frac{1}{k^3l}.$$

Verify that

$$g(k,l) - g(k+l,l) - g(k,k+l) = \frac{1}{k^2l^2)} \quad (*),$$

and that summation over all $k, l \in \mathbb{N}$ *gives*

$$\zeta(2)^2 = \left(\sum_{k,l\in\mathbb{N}} - \sum_{k,l\in\mathbb{N},k>l} - \sum_{k,l\in\mathbb{N},l>k}\right) g(k,l) = \sum_{k\in\mathbb{N}} g(k,k) = \frac{5}{2}\zeta(4).$$

Conclude that $\zeta(4) = \frac{\pi^4}{90}$. *Similarly introduce, for* $n \in \mathbb{N}\setminus\{1\}$,

$$g(k,l) = \frac{1}{kl^{2n-1}} + \frac{1}{2}\sum_{2\le i\le 2n-2} \frac{1}{k^i l^{2n-i}} + \frac{1}{k^{2n-1}l} \quad (k, l \in \mathbb{N}).$$

In this case the left-hand side of (*) takes the form $\sum_{2\le i\le 2n-2,\, i \text{ even}} \frac{1}{k^i l^{2n-i}}$, *hence*

$$\zeta(2n) = \frac{2}{2n+1}\sum_{1\le i\le n-1} \zeta(2i)\zeta(2n-2i) \quad (n \in \mathbb{N}\setminus\{1\}).$$

See Exercise 0.21.(iv) for a different proof.