Functional Analysis, Sobolev Spaces and Partial Differential Equations - Haim Brezis: Exercise Solutions

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3 Weak Topologies. Reflexive Spaces. Separable Spaces. Uniform Convexity

Exercise 3.1
Let E be a Banach space and let $A \subset E$ be a subset that is compact in the weak topology $\sigma(E, E^*)$. Prove that A is bounded.

Proof. For any $f \in E^*$, because A is compact and f is continuous, we get $f(A)$ compact in $\mathbb{R}$. So $f(A)$ is bounded for all $f \in E^*$. Applying Corollary 2.4, we get A is bounded. $\square$

Exercise 3.3
Let E be a Banach space. Let $A \subset E$ be a convex subset. Prove that the closure of A in the strong topology and that in the weak topology $\sigma(E, E^*)$ are the same.

Proof. Let $\bar{A}$ be the closure of A in the strong topology, and A' is the closure of A in the weak topology. Because A' is also closed in the strong topology and $A \subset A'$, we get $A' \subset \bar{A}$.

Conversely, it is not hard to see that a closure of a convex subset is convex, thus $\bar{A}$ is convex. Thus $\bar{A}$ is closed in the weak topology. So $\bar{A} \subset A'$. So $\bar{A} = A'$. $\square$

Exercise 3.8
Let E be an infinite-dimensional Banach space. Our purpose is to show that E equipped with the weak topology is not metrizable. Suppose, by contradiction, that there is a metric $d(x, y)$ on E that induces on E the same topology as $\sigma(E, E^*)$.

1. For every integer $k \geq 1$ let V_k denote a neighborhood of 0 in the topology $\sigma(E, E^*)$, such that
$$V_k \subset \left\{ x \in E; d(x, 0) < \frac{1}{k} \right\}.$$
Prove that there exists a sequence (f_n) in E^* such that every $g \in E^*$ is a finite linear combination of the f_n's.

2. Deduce that E^* is finite-dimensional.

3. Conclude.

4. Prove by a similar method that E^* equipped with the weak* topology $\sigma(E^*, E)$ is not metrizable.

Proof. Assume that $d(x, y)$ generates the same topology as $\sigma(E, E^*)$. Because $\{x \in E; d(x, 0) < 1/k\}$ is open with respect to the metric d, so there exist $f_1^{(k)}, \cdots, f_{t_k}^{(k)}$ such that
$$V_k := \{x \in E : |\langle f_i^{(k)}, x \rangle| < \varepsilon, 1 \leq i \leq t_k\} \subset \{x \in E; d(x, 0) < 1/k\}.$$
Let g_n be the sequence of $f_i^{(j)}$, we will show that for any $g \in E^*$, we can write g is a linear combination of finitely many g_i's. Let
$$V = \{x \in E : |\langle g, x \rangle| < 1\},$$

then V is open in $\sigma(E^*, E)$. Therefore, there is exists $m \in \mathbb{N}$ such that

$$V_m \subset \{x \in E : d(x, 0) < 1/m\} \subset V.$$

For any $u \in E$, if $\langle f_i^{(m)}, u\rangle = 0$ for all $1 \leq i \leq t_m$, then $u \in V_m \subset V$. So $|\langle g, u\rangle| < 1$, which implies $\langle g, u\rangle = 0$. So Lemma 3.2 yields that g is a finite linear combination of $f_i^{(m)}$.

Let $E_n \subset E^*$ be the space that is generated by $g_1, \cdots, g_n$. Then our previous argument implies that

$$\bigcup_{i \in N} E_n = E^*.$$

Because E_n are finite dimensional, they are closed. By Baire's Theorem, there exists $n_0 \in \mathbb{N}$ such that E_{n_0} has nonempty interior. Thus $E_{n_0} = E^*$ and that E^* is finite dimensional. Let $g_i = \langle x, z_i\rangle$ be the Reisz representations for $1 \leq i \leq n_0$. We will show that $z_1, \cdots, z_{n_0}$ generates E. Indeed, for any $u \in E$, we have

$$\langle x, u\rangle = \sum \lambda_i \langle x, z_i\rangle = \left\langle x, \sum \lambda_i z_i \right\rangle.$$

But this is true for all x, we get $u = \sum \lambda_i z_i$. So $\dim(E) \leq n_0$, which gives a contradiction. $\square$

Exercise 3.17

1. Let (x^n) be a sequence in ℓ^p with $1 \leq p \leq \infty$. Assuming $x^n \rightharpoonup x$ in $\sigma(\ell^p, \ell^{p'})$ prove that:
   (a) (x^n) is bounded in ℓ^p,
   (b) $x_i^n \to x_i$ for every i, where $x^n = (x_1^n, \cdots, x_i^n, \cdots)$ and $x = (x_1, \cdots, x_i, \cdots)$.
2. Conversely, suppose (x^n) is a sequence in ℓ^p with $1 < p \leq \infty$. Assume that (a) and (b) hold (for some limit denoted by x_i). Prove that $x \in \ell^p$ and that $x^n \rightharpoonup x$ in $\sigma(\ell^p, \ell^{p'})$.

Proof. 1. For any $f \in \ell^{p'}$, the weak convergence implies that $\langle f, x^n\rangle$ converges as $n \to \infty$. Thus $\{|\langle f, x^n\rangle| : n \in \mathbb{N}\}$ is bounded. So by the Uniform bounded principle, there exists a $c > 0$ such that

$$\langle f, x^n\rangle < c\|f\|_{p'}$$

for all $n \in \mathbb{N}$. This implies that (x_n) is bounded in ℓ^p by c. Let $\pi_1 \colon \ell^p \to \mathbb{R}$ maps $(x_1, \cdots) \mapsto x_1$ be the projection of the first entry. Clearly π_1 is linear and bounded by 1. The weak convergence of (x^n) implies that $\pi_1(x^n) \to \pi_1(x)$, or $x_1^n \to x_1$ as $n \to \infty$. Similarly, we get $x_i^n \to x_i$ for all $i \in \mathbb{N}$. (So in ℓ^p, weak convergence implies strong convergence!)

2. Conversely, assume that there exist x^n and x that satisfy (a) and (b). If $p = \infty$, then (a) implies that $\|x^n\|$ bounded by M, which means $|x_i^n| \leq M$ for all $i, n \in \mathbb{N}$. So (b) implies that

$$|x_i| = \lim_{n\to\infty} |x_i^n| \leq M.$$

Thus

$$\|x\|_\infty = \sup\{|x_i| : i \in \mathbb{N}\} \leq M,$$

or equivalently $x \in \ell^\infty$. Now let f be any function in $\ell^{\infty'}$, we will show that $\langle f, x^n\rangle \to \langle f, x\rangle$ using DCT.

Let $X = \mathbb{N}$ and B_X be the set of all subsets of N. Define a measure μ on B_X such that $\mu(n) = \langle f, e_n\rangle$. This is indeed a measure by Exercise 3.D [Bartle]. Consider any $a \in \ell^\infty$ as a function from $\mathbb{N}$ to $\mathbb{R}$, we get

$$\int a \,\mathrm{d}\mu = \sum_{i\in\mathbb{N}} a_i \langle f, e_i\rangle = \left\langle f, \sum_{i\in\mathbb{N}} a_i \cdot e_i \right\rangle = \langle f, a\rangle.$$

Because x^n and x are in ℓ^∞, and f is a functional on ℓ^∞, $\langle f, x^n\rangle$ and $\langle f, x\rangle$ are finite. Hence, x^n and x are measurable. Condition (b) implies that $x^n \to x$ pointwise, and condition (a) says that x^n is dominated by some constant sequence, which is integrable because any constant sequence is in ℓ^∞. By Lebesgue DCT, we get

$$\lim_{n\to\infty} \langle f, x^n\rangle = \lim_{n\to\infty} \int x^n \,\mathrm{d}\mu = \int x \,\mathrm{d}\mu = \langle f, x\rangle.$$

So $x^n \rightharpoonup x$ in $\sigma(\ell^\infty, \ell^{\infty'})$.

If $p < \infty$, then ℓ^p is reflexive. Since (x_n) is bounded, by Theorem 3.18, there is a subsequence (x^{n_k}) that converges in the weak topology $\sigma(\ell^p, \ell^{p'})$ to some $x' \in \ell^p$. By part 1, x^{n_k} converges pointwise to x', thus $x' = x$. So $x \in \ell^p$.

Now we will show that $x^n \rightharpoonup x$ by contradiction. Assume the opposite, then there is $g \in \ell^{p'}$ such that $\langle g, x^n\rangle \not\to \langle g, x\rangle$. That is, there exists $\varepsilon > 0$ such that for all $N > 0$, there is some $n_k > N$ such that

$$|\langle g, x^{n_k}\rangle - \langle g, x\rangle| > \varepsilon.$$

So we can extract a subsequence x^n (no relabelling lol) such that

$$|\langle g, x^n\rangle - \langle g, x\rangle| > \varepsilon$$

for all $n \in \mathbb{N}$. But this new sequence x^n is also bounded by (a), thus we can extract a (sub)subsequence x^{n_k} that converges to x in weak topology $\sigma(\ell^p, \ell^{p'})$. But this means

$$|\langle g, x^{n_k}\rangle - \langle g, x\rangle| \to 0$$

as $k \to \infty$, contradiction. So $x^n \rightharpoonup x$ in $\sigma(\ell^p, \ell^{p'})$.

$\square$

Exercise 3.18
For every integer $n \geq 1$ let

$$e^n = (0, 0, \cdots, 1, 0, \cdots).$$

1. Prove that $e^n \rightharpoonup 0$ in ℓ^p weakly in $\sigma(\ell^p, \ell^{p'})$ with $1 < p \leq \infty$.
2. Prove that there is no subsequence (e^{n_k}) that converges in ℓ^1 with respect to $\sigma(\ell^1, \ell^\infty)$.
3. Construct an example of a Banach space E and a sequence (f_n) in E^* such that $\|f_n\| = 1$ for all n and such that (f_n) has no subsequence that converges in $\sigma(E^*, E)$. Is there a contradiction with the compactness of B_{E^*} in the topology $\sigma(E^*, E)$?

Proof. 1. Because $e_i^n \to 0$ as $i \to \infty$ for each $n \in \mathbb{N}$ and $\|e_i^n\|_p = 1$, thus bounded, using part 2 of Exercise 3.17, we conclude that $e^n \rightharpoonup 0$ in $\sigma(\ell^p, \ell^{p'})$ for all $1 < p \leq \infty$.

2. Let $f \colon \ell^1 \to \mathbb{R}$ maps $(e_i) \mapsto \sum_{i \in \mathbb{N}} e_i$. It is not hard to see that f is a linear functional, which is bounded by 1. For any $n \in \mathbb{N}$, we have $\langle f, e^n \rangle = 1$. But $\langle f, 0 \rangle = 0$, thus there is no subsequence of e^n that converges to 0 in the weak topology $\sigma(\ell^1, \ell^\infty)$.

3. Let $E = \ell^\infty$, and $f_n \colon E \to \mathbb{R}$ defined by

$$f_n(x) = \langle e^n, x \rangle.$$

It is not hard to see that $f_n(x)$ is a linear functional and

$$\|f_n\| = \|\langle e^n, x \rangle\| = \|e_n\| = 1.$$

Assume that (f_n) has a convergent subsequence, say (f_{n_k}), that converges weak*, then for any $x \in E$, we would have

$$\langle e^{n_k}, x \rangle = f_{n_k}(x) \to f(x) = \langle e, x \rangle$$

for some $e \in E^*$. Because $e^m \in \ell^\infty$, for any $m \in \mathbb{N}$, we have

$$\langle e^{n_k}, e^m \rangle \to \langle e, e^m \rangle,$$

which shows that $\langle e, e^m \rangle = 0$ for all $m \in \mathbb{N}$ (because we can let n_k be sufficiently large and the left hand side become 0). So for any $e = 0$. Notice that for $a = (1, 1, \cdots) \in \ell^\infty$, we have

$$\langle e^{n_k}, a \rangle = 1 \not\to 0 = \langle e, a \rangle.$$

So e^{n_k} does not converge to e in the weak* topology.

This is not a contradiction because compactness does not necessarily imply sequential compactness.

□

Exercise 3.20
Let E be a Banach space.

1. Prove that there exist a compact topological space K and an isometry from E into $C(K)$ equipped with its usual norm.

2. Assuming that E is separable, prove that there exists an isometry from E into ℓ^∞.

Proof. 1. Let $K = B_{E^*}$ with the weak* topology $\sigma(E^*, E)$. By Theorem 3.16, K is compact. Let $J\colon E \to C(K) = C(B_{E^*})$ that maps $x \mapsto \langle Jx, f\rangle = \langle f, x\rangle$. (Clearly $\langle f, x\rangle$ is a linear continuous function from K to $\mathbb{R}$.) Therefore, we have

$$\|\langle f, x\rangle\| = \sup\{\langle f, x\rangle : f \in B_{E^*}\} = \|x\|,$$

which means that J is an isometry.

2. Using part 1, there is an isometry φ from E to B_{E^*}. We will construct an isometry ψ from B_{E^*} to ℓ^∞, then $\psi \circ \varphi$ is an isometry from E to ℓ^∞. Assume that E is separable, then $B_E = \{x \in E : \|x\| = 1\}$ is also separable. Let $\{e_1, e_2, \cdots\}$ be a countable dense subset of B_E. Let ψ be the map from $B_{E^*} \to \mathbb{R}^\infty$ that maps

$$f \mapsto (\langle f, e_i\rangle)_{i\in\mathbb{N}}$$

It is not hard to see that this map is linear, and $\langle f, e_i\rangle \le \|f\|$, thus $\mathrm{Im}(\psi) \subset \ell^\infty$. Moreover,

$$\|f\| = \sup\{\langle f, x\rangle : \|x\| = 1\} = \sup\{\langle f, e_i\rangle : i \in \mathbb{N}\} = \|\psi\|_\infty.$$

The second equality is because $\{e_i\}$ dense in B_E. So ψ is an isometry, which complete our proof.

$\square$

Exercise 3.25
Let K be a compact metric space that is not finite. Prove that $C(K)$ is not reflexive.

Proof. Because K is compact, any function $f \in C(K)$ is bounded. So $\|f\| < \infty$, which shows that $C(K)$ is indeed a norm vector space.

Because K is a compact metric space, K is sequential compact. Since K has infinitely many elements, let (a_n) be a sequence with no duplicate. This sequence has a subsequence that converge to some $a \in K$. Since a appears at most once in this sequence (a_n), there is a subsequence that doesn't have a. Without relabeling, we constructed a subsequence $a_n \to a$ such that $a_n \neq a$ for all $n \in \mathbb{N}$.

Now assume that $C(K)$ is reflexive. Let $E = \{u \in C(K) : u(a) = 0\}$, we will show that E is a closed linear subspace of $C(K)$. Clearly E is a linear subspace. For any $u_n \in E$ and $u_n \to u$, then $0 = u_n(a) \to u(a)$, which implies that $u \in E$. So E is a closed linear subspace of a reflexive space, by Proposition 3.20, E is reflexive.

Let us define $f : E \to \mathbb{R}$ by

$$f(u) = \sum_{n=1}^{\infty} \frac{1}{2^n} u(a_n).$$

Notice that because u is continous and $a_n \to a$, we get $u(a_n) \to u(a) = 0$. Using Exercise 1.4, we get f being a continuous linear functional on E where $\|f\| = 1$. And there is no $u \in E$ such that $\|u\| = 1$ and $f(u) = \|f\|$.

But E is reflexive, we can use Hahn-Banach Theorem to construct $\varphi \in E^{**}$ such that $\varphi(f) = \|f\|$ and $\|\varphi\| = \|f\| = 1$. But E is reflexive, thus there exists $u \in E$ such that $f(u) = \varphi(f) = \|f\|$ and $\|u\| = \|\varphi\| = \|f\| = 1$. This contradicts to what we get out of Exercise 1.4. So our assumption is false, that is, $C(K)$ is not reflexive. $\square$

4 L^p Spaces

Exercise 4.2
Assume $|\Omega| < \infty$ and let $1 \leq p \leq q \leq \infty$. Prove that $L^q(\Omega) \subset L^p(\Omega)$ with continuous injection. More percisely, show that

$$\|f\|_p \leq |\Omega|^{\frac{1}{p} - \frac{1}{q}} \|f\|_q \quad \forall f \in L^q(\Omega).$$

Proof. Let $f \in L^q$. Because $\frac{p}{q} + (1 - \frac{p}{q}) = 1$, applying the Holder's inequality, we get

$$\int |f|^p \leq \left(\int (|f|^p)^{\frac{q}{p}} \right)^{\frac{p}{q}} \left(\int 1 \right)^{1 - \frac{p}{q}} = \left(\int |f|^q \right)^{\frac{p}{q}} |\Omega|^{1 - \frac{p}{q}}.$$

Take both sides to the $\frac{1}{p}$ power, we get

$$\|f\|_p \leq |\Omega|^{\frac{1}{p} - \frac{1}{q}} \|f\|_q.$$

So $L^q(\Omega) \subset L^p(\Omega)$. $\square$

Exercise 4.3

1. Let $f, g \in L^p(\Omega)$ with $1 \leq p \leq \infty$. Prove that

$$h(x) = \max\{f(x), g(x)\} \in L^p(\Omega).$$

2. Let (f_n) and (g_n) be two sequences in $L^p(\Omega)$ with $1 \leq p \leq \infty$ such that $f_n \to f$ in $L^p(\Omega)$ and $g_n \to g$ in $L^p(\Omega)$. Set $h_n = \max\{f_n, g_n\}$ and prove that $h_n \to h$ in $L^p(\Omega)$.

3. Let (f_n) be a sequence in $L^p(\Omega)$ with $1 \leq p < \infty$ and let (g_n) be a bounded sequence in $L^\infty(\Omega)$. Assume $f_n \to f$ in $L^p(\Omega)$ and $g_n \to g$ a.e. Prove that $f_n g_n \to fg$ in $L^p(\Omega)$.

Proof. 1. Because $f, g \in L^p$, we get $|f|, |g| \in L^p$. Thus $|f| + |g| \in L^p$. Since $h(x) = \max\{f(x), g(x)\}$ is smaller than $|f| + |g|$, we get

$$\int |h(x)|^p \, \mathrm{d}\mu \leq \int ||f| + |g||^p \, \mathrm{d}\mu < \infty.$$

So $h \in L^p$.

2. Because $f_n, g_n, f, g \in L^p$, part 1 implies that $h_n, h \in L_p$. We will prove that $\|h_n - h\|_p \to 0$ using contradiction. Assume that this is not the case, then there is some $\varepsilon > 0$ and a subsequence h_n (no relabelling) such that $\|h_n - h\|_p > \varepsilon$ for all $n \in \mathbb{N}$. But $f_n \to f$ in L^p, so we can construct a subsubsequence such that $f_n \to f$ almost everywhere (and no relabelling again) where f_n is dominated by a function $\varphi_1 \in L^p$. This construction is possible by Theorem 4.9. With this new subsubsequence, because $g_n \to g$ in L^p, we can construct a subsubsubsequence $g_n \to g$ almost everywhere (and no relabelling for clean purpose lol), such that $g_n(x)$ is dominated by a function $\varphi_2 \in L^p$.

So what we get in the end are $f_n \to f$ and $g_n \to g$ both a.e. and in L^p. And $\varphi_1, \varphi_2 \in L^p$ that dominate f_n and g_n respectively. These implies that $h_n = \max\{f_n, g_n\} \to h = \max\{f, g\}$ almost everywhere. Indeed, the subset of Ω consists of x such that $f_n(x) \not\to f(x)$ or $g_n(x) \not\to g(x)$ is a union of two measure 0 subsets, thus has measure 0. For other x, we have $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$. One can easily check that $\max\{f_n(x), g_n(x)\} \to \max\{f(x), g(x)\}$ in this case.

However, this means $h_n \to h$ almost everywhere and h_n being dominated by $\varphi_1 + \varphi_2 \in L^p$, which by Lebesgue DCT, yields $h_n \to h$ in L_p. This contradicts to our assumption at the beginning that $\|h_n - h\|_p > \varepsilon$ for all n.

3. For any $g_n \in L^\infty$, we have

$$\int |f_n g_n|^p \, \mathrm{d}\mu \leq \|g_n\|_\infty \int |f_n|^p \, \mathrm{d}\mu < \infty.$$

So $f_n g_n \in L^p$ and so is fg. With the same technique as part 2, we only need to show that there is subsequence of $f_n g_n$ that converge to fg. Ideed, let f_n, g_n be a subsequences that converge almost everywhere to f and g, that is dominated by ψ_1 and ψ_2 respectively. Then clearly $f_n g_n \to fg$ almost everywhere and $f_n g_n$ is dominated by $\psi_1\psi_2 \in L^p$. So Lebesgue DCT implies that $f_n g_n \to fg$ in L^p, which complete our proof.

□

Exercise 4.4

1. Let $f_1, f_2, \cdots, f_k$ be k functions such that $f_i \in L^{p_i}(\Omega)$ for all i with $1 \le p_i \le \infty$ and $\sum_{i=1}^k \frac{1}{p_i} \le 1$. Set
$$f(x) = \prod_{i=1}^k f_i(x).$$
Prove that $f \in L^p(\Omega)$ with $\frac{1}{p} = \sum_{i=1}^k \frac{1}{p_i}$ and that
$$\|f\|_p \le \prod_{i=1}^k \|f_i\|_{p_i}.$$

2. Deduce that if $f \in L^p(\Omega) \cap L^q(\Omega)$ with $1 \le p \le \infty$ and $1 \le q \le \infty$, then $f \in L^r(\Omega)$ for every r between p and q. More precisely, write
$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-a}{q} \quad \text{with } \alpha \in [0,1]$$
and prove that
$$\|f\|_r \le \|f\|_p^\alpha \|f\|_q^{1-\alpha}.$$

Proof. 1. The proof is by mathematical induction on k. When $k = 2$, then the statement becomes
$$\|f\|_p \le \|f_1\|_{p_1} \|f_2\|_{p_2}.$$
Notice that with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, we get $\frac{p}{p_1} + \frac{p}{p_2} = 1$. Applying the Holder's inequality, we get
$$\int |f|^p \le \left(\int (|f_1|^p)^{\frac{p_1}{p}}\right)^{\frac{p}{p_1}} \left(\int (|f_2|^p)^{\frac{p_2}{p}}\right)^{\frac{p}{p_2}}.$$
Therefore, $\|f\|_p \le \|f_1\|_{p_1} \|f_2\|_{p_2}$. Assume that the statement holds for $k-1$ numbers. With the same reasoning,
$$\left(\frac{p}{p_1} + \cdots + \frac{p}{p_{k-1}}\right) + \frac{p}{p_k} = 1.$$
So the Holder's inequality implies that
$$\|f\|_p \le \|f_1 \cdots f_{k-1}\|_{1/(\frac{1}{p_1} + \cdots + \frac{1}{p_{k-1}})} \|f_k\|_{p_k} \le \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}$$
where the last inequality is by our induction hypothesis. This completes our proof.

2. Because r is between p and q, $\frac{1}{r}$ is between $\frac{1}{p}$ and $\frac{1}{q}$. So there is an $\alpha \in [0,1]$ such that
$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}.$$
Let $p_1 = \frac{p}{\alpha}, p_2 = \frac{q}{1-\alpha}$ and $f = f^\alpha \cdot f^{1-\alpha}$, then part 1 implies that
$$\|f\|_r \leq \|f^\alpha\|_{p/\alpha} \|f^{1-\alpha}\|_{q/(1-\alpha)} = \|f\|_p^\alpha \|f\|_q^{1-\alpha}.$$

$\square$

Exercise 4.6
Assume $|\Omega| < \infty$.

1. Let $f \in L^\infty(\Omega)$. Prove that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.
2. Let $f \in \bigcap_{1\leq p<\infty} L^p(\Omega)$ and assume that there is a constant C such that
$$\|f\|_p \leq C \quad \forall 1 \leq p < \infty.$$
Prove that $f \in L^\infty(\Omega)$.
3. Construct an example of a function $f \in \bigcap$

Exercise 4.14
Assume $|\Omega| < \infty$. Let (f_n) be a sequence of measurable functions such that $f_n \to f$ a.e. (with $|f| < \infty$ a.e.).

1. Let $\alpha > 0$ be fixed. Prove that
$$\mathrm{meas}[|f_n - f| > \alpha] \to 0.$$
2. More precisely, let
$$S_n(\alpha) = \bigcup_{k\geq n} [|f_k - f| > \alpha].$$
Prove that $|S_n(\alpha)| \to 0$.
3. Prove that
$$\begin{cases} \forall \delta > 0 \quad \exists A \subset \Omega \quad \text{measurable such that} \\ |A| < \delta \text{ and } f_n \to f \text{ uniformly on } \Omega \setminus A. \end{cases}$$
4. Let (f_n) be a sequence in $L^p(\Omega)$ with $1 \leq p < \infty$. Assume that
   (i) $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\int_A |f_n|^p < \varepsilon\ \forall n$ and $\forall A \subset \Omega$ measurable with $|A| < \delta$.

(ii) $f_n \to f$ a.e.

Prove that $f \in L^p(\Omega)$ and that $f_n \to f$ in $L^p(\Omega)$.

Proof. 1. Let

$$S_n(\alpha) = \bigcup_{k \geq n} [|f_k - f| > \alpha],$$

then S_n is a decreasing sequence. Notice that if $x \in \bigcap_{n \in \mathbb{N}} S_n(\alpha)$, then $f_n(x) \not\to f(x)$ as $n \to \infty$, therefore $\bigcap_{n \in \mathbb{N}} S_n(\alpha)$ has measure 0. Since Ω has measure 0, we get $\mu(S_n(\alpha)) \to 0$ as $n \to \infty$. Since

$$\text{meas}[|f_n - f| > \alpha] < S_n(\alpha),$$

we got the conclusion.

2. Like part 1.

3. For any $\delta > 0$, part 2 suggests that $|S_n(\frac{1}{m})| \to 0$, so there is an $n_m > 0$ such that $|S_n(\frac{1}{m})| < \frac{\delta}{2^m}$ for all $n \geq n_m$. Let

$$A = \bigcup_{m \in \mathbb{N}} S_{n_m}\left(\frac{1}{m}\right).$$

Clearly $|A| \leq \delta$. We will show that $f_n \to f$ uniformly on $\Omega \setminus A$. For any $\varepsilon > 0$, there is an $m' \in \mathbb{N}$ such that $\frac{1}{n_{m'}} < \varepsilon$. So for any $n > n_{m'}$, because

$$A^c \subset S_{n_{m'}}\left(\frac{1}{n'_m}\right)^c = \left(\bigcup_{k \geq n_{m'}} \left[|f_k - f| > \frac{1}{n_{m'}}\right]\right)^c = \bigcap_{k \geq n_{m'}} \left[|f_k - f| \leq \frac{1}{n_{m'}}\right].$$

Therefore, $|f_k(x) - f(x)| \leq \frac{1}{n_{m'}} < \varepsilon$ for all $k \geq n_{m'}$, which means f_n is uniformly convergent on A^c.

4. For any $\varepsilon > 0$, we apply (i) to get δ. With this δ, we use part 3 to construct a subset A such that $f_n \to f$ uniformly on $\Omega \setminus A$. So for any $m, n \in \mathbb{N}$, we have

$$\begin{aligned}\left(\int_\Omega |f_m - f_n|^p\right)^{\frac{1}{p}} &\leq \left(\int_A |f_m - f_n|^p\right)^{\frac{1}{p}} + \left(\int_{\Omega \setminus A} |f_m - f_n|^p\right)^{\frac{1}{p}} \\ &\leq \left(\int_A |f_m|^p\right)^{\frac{1}{p}} + \left(\int_A |f_n|^p\right)^{\frac{1}{p}} + \left(\int_{\Omega \setminus A} |f_m - f_n|^p \, \mathrm{d}\mu\right)^{\frac{1}{p}}.\end{aligned}$$

Notice that when m and n are sufficiently large, the first two terms are sufficiently small by (i) and the last term is sufficiently small by part 3. So f_n is Cauchy in L^p, which means there is a subsequence f_{n_k} that converge a.e. to f. This implies that $f \in L^p$ and $f_n \to f$ in L^p.

$\square$

Exercise 4.16
Let $1 < p < \infty$. Let (f_n) be a sequence in $L^p(\Omega)$ such that

(i) f_n is bounded in $L^p(\Omega)$.

(ii) $f_n \to f$ a.e. on Ω.

1. Prove that $f_n \rightharpoonup f$ weakly $\sigma(L^p, L^{p'})$.
2. Same conclusion if assumption (ii) is replaced by

$$(ii') \quad \|f_n - f\|_1 \to 0.$$

3. Assume now (i), (ii), and $|\Omega| < \infty$. Prove that $\|f_n - f\|_q \to 0$ for every q with $1 \le q < p$.

Proof. 1. Because f_n is bounded in $L^p(\Omega)$, applying the Fatou's Lemma, we get

$$\int |f|^p \,\mathrm{d}\mu \le \liminf \int |f_n|^p \,\mathrm{d}\mu < \infty.$$

So $f \in L^p(\Omega)$. Because L^p is reflexive and f_n is bounded in L^p, Theorem 3.18 implies that there is a subsequence $f_{n_k} \rightharpoonup \widetilde{f}$ in $\sigma(L^p, L^{p'})$. But $f_n \to f$ almost everywhere, we get $f_n \rightharpoonup f$ in $\sigma(L^p, L^{p'})$.

2. Because $\|f_n - f\|_1 \to 0$, there is a subsequence $f_n \to f$ almost everywhere (without relabelling). So part 1 implies that $f_n \rightharpoonup f$ in $\sigma(L^p, L^{p'})$. Notice that this means any subsequence of f_n has a subsubsequence that weakly converges f, we conclude that $f_n \rightharpoonup f$.

3. Because $|\Omega| < \infty$, Exercise 4.2 implies that $L^p \subset L^q$. Therefore, $f_n, f \in L^p \subset L^q$, and so is $f_n - f$. Moreover, also from Exercise 4.2, we have

$$\|f_n\|_q \le |\Omega|^{\frac{1}{q} - \frac{1}{p}} \|f\|_p.$$

Since $\|f\|_p$ is bounded, we get $\|f_n\|_q$ is bounded by some $M > 0$. Notice that for any measurable subset $A \subset \Omega$, we have

$$\int_\Omega |f_n - f|^q \,\mathrm{d}\mu = \int_A |f_n - f|^q \,\mathrm{d}\mu + \int_{A^c} |f_n - f|^q \,\mathrm{d}\mu \le |A| \cdot M + \int_{A^c} |f_n - f|^q \,\mathrm{d}\mu.$$

Notice that by Egorov's Theorem, we can construct a measurable subset A with measure sufficiently small and $f_n \to f$ uniformly on A^c. So the right hand side can be sufficiently small as $n \to \infty$. We conclude that $\|f_n - f\|_q \to 0$.

$\square$

Exercise 4.20
Assume $|\Omega| < \infty$. Let $1 \le p < \infty$ and $1 \le q < \infty$. Let $a\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that

$$|a(t)| \le C\{|t|^{p/q} + 1\} \quad \forall t \in \mathbb{R}.$$

Consider the (nonlinear) map $A \colon L^p(\Omega) \to L^q(\Omega)$ defined by

$$(Au)(x) = a(u(x)), x \in \Omega.$$

1. Prove that A is continuous from $L^p(\Omega)$ strong into $L^q(\Omega)$ strong.
2. Take $\Omega = (0,1)$ and assume that for every sequence (u_n) such that $u_n \rightharpoonup u$ weakly $\sigma(L^p, L^{p'})$ then $Au_n \rightharpoonup Au$ weakly $\sigma(L^q, L^{q'})$. Prove that a is an affine function.

Exercise 4.22

1. Let (f_n) be a sequence in $L^p(\Omega)$ with $1 < p \leq \infty$ and let $f \in L^p(\Omega)$. Show that the following properties are equivalent:

   (A) $f_n \rightharpoonup f$ in $\sigma(L^p, L^{p'})$.

   (B) $\begin{cases} \|f_n\|_p \leq C, \\ \int_E f_n \to \int_E f \quad \forall E \subset \Omega, E \text{ measurable and } |E| < \infty. \end{cases}$

2. If $p = 1$ and $|\Omega| < \infty$ prove that (A) $\iff$ (B).
3. Assume $p = 1$ and $|\Omega| = \infty$. Prove that (A)$\Rightarrow$(B). Construct an example showing that in general, (B)$\not\Rightarrow$(A).
4. Let (f_n) be a sequence in $L^1(\Omega)$ and let $f \in L^1(\Omega)$ with $|\Omega| = \infty$. Assume that

   (i) $f_n \geq 0 \quad \forall n$ and $f \geq 0$ a.e. on Ω,

   (ii) $\int_\Omega f_n \to \int_\Omega f$,

   (iii) $\int_E f_n \to \int_E f \quad \forall E \subset \Omega$, E measurable and $|E| < \infty$.

   Prove that $f_n \rightharpoonup f$ in $L^1(\Omega)$ weakly $\sigma(L^1, L^\infty)$.

Proof. Let $q := p'$.

1. If $f_n \rightharpoonup f$ in $\sigma(L^p, L^q)$, then Proposition 3.5 (iii) implies that $\|f_n\|_p$ is bounded. For any E measurable such that $|E| < \infty$, we get $\chi_E \in L^q(\Omega)$. Therefore

   $$\int_E f_n = \langle \chi_E, f_n \rangle \to \langle \chi_E, f \rangle = \int_E f.$$

   Conversely, assume that $\|f_n\| \leq C$ and $\int_E f_n \to \int_E f$ for all $E \subset \Omega, E$ measurable and $|E| < \infty$. For any $\varphi \in L^q$ it is sufficient to show that $\int f_n \varphi \to \int f \varphi$ or $\int (f_n - f)\varphi \to 0$. Because $\varphi \in L^q$, there exists $E \subset \Omega$, such that $|E| < \infty$ and

$\left(\int_{E^c} |\varphi|^q\right)^{1/q} < \varepsilon$. Therefore by Holder's inequality, we have

$$\begin{aligned}\int_{E^c} (f_n - f)\varphi &\le \left(\int_{E^c} |f_n - f|^p\right)^{\frac{1}{p}} \left(\int_{E^c} |\varphi|^{\frac{1}{q}}\right)^q \\ &\le \|f_n - f\|_p \cdot \varepsilon \\ &\le (\|f_n\|_p + \|f\|_p)\varepsilon \\ &\le (C + \|f\|_p)\varepsilon.\end{aligned}$$

So this term is sufficiently small. What is more, because $|E| < \infty$, Exercise 4.2 implies that $\varphi \in L^\infty(E)$ or φ is bounded almost everywhere by a constant D. So

$$\int_E (f_n - f)\varphi \le \int_E (f_n - f) \cdot D \to 0$$

because $\int_E f_n \to \int_E f$. Therefore

$$\int (f_n - f)\varphi \to 0$$

as $n \to \infty$ or $f_n \rightharpoonup f$ as desired.

2. The proof for (A) to (B) is the same as part 1. To prove the converse, notice that for any $\varphi \in L^\infty(\Omega)$, φ is bounded almost everywhere by some constant D and $\int f_n \to \int f$ since $|\Omega| < \infty$. Thus

$$\int (f_n - f)\varphi \le D \cdot \int (f_n - f) \to 0.$$

3. Similar to part 1, (A) implies (B). For a counterexample, let $\Omega = \mathbb{R}$ and $f_n = \chi_{[n,n+1]}$. It is not hard to see that $\|f_n\|_1 \le 1$ and for any E bounded, $\int_E f_n \to 0$. But for the constant function $1 \in L^\infty(\mathbb{R})$, we have $\int 1 \cdot f_n = 1 \not\to 0 = \int 1 \cdot 0$. So (A) is false.

4. We first show that for any measurable subset F of Ω, then $\int_F f_n \to \int_F f$. Because $f \in L^1$, there is some $E \subset \Omega$, $|E| < \infty$ and $\int_{E^c} f < \varepsilon$. Decompose F into $F \cap E$ and $F \setminus E$, we can see that $|F \cap E| \le |E| < \infty$, thus $\int_{F\cap E} f_n \to \int_{F\cap E} f$ by (iii). Moreover, because

$$\int_{E^c} f_n = \int f_n - \int_E f_n \to \int f - \int_{E^c} f = \int_{E^c} f$$

as $n \to \infty$, when n is sufficiently large, we would have $0 \le \int_{E^c} f_n < 2\varepsilon$. So

$$\left|\int_{F\setminus E} f_n - f\right| \le \int_{F\setminus E} |f_n - f| \le \int_{E^c} |f_n| + |f| \le 3\varepsilon.$$

Therefore $\int_F f_n \to \int_F f$ for any measurable subset F of Ω. For any $\varphi \in L^\infty$, it is sufficient to show that $\int f_n\varphi \to \int f\varphi$ or $\int (f_n - f)\varphi \to 0$ as $n \to \infty$. Let

$$F = \{x \in \Omega : (f_n - f) \ge 0\},$$

and assume that $|\varphi| \leq C$, then

$$\left|\int_F (f_n - f)\varphi\right| \leq \int_F |f_n - f||\varphi| \leq C\int_F (f_n - f) \to 0.$$

And similarly we have

$$\left|\int_{F^c} (f_n - f)\varphi\right| \leq \int_{F^c} |f_n - f||\varphi| \leq C\int_F (f - f_n) \to 0.$$

Combining the two previous calculations and the trianglular inequality, we get

$$\left|\int (f_n - f)\varphi\right| \leq \left|\int_F (f_n - f)\varphi\right| + \left|\int_{F^c} (f_n - f)\varphi\right| \to 0$$

as $n \to \infty$. So $f_n \rightharpoonup f$ in $\sigma(L^1, L^\infty)$.

□

Exercise 4.23
Let $f\colon \Omega \to \mathbb{R}$ be a measurable function and let $1 \leq p \leq \infty$. The purpose of this exercise is to show that the set

$$C = \{u \in L^p(\Omega) : u \geq f \text{ a.e.}\}$$

is closed in $L^p(\Omega)$ with respect to the topology $\sigma(L^p, L^{p'})$.

1. Assume first that $1 \leq p < \infty$. Prove that C is convex and closed in the strong L^p topology. Deduce that C is closed in $\sigma(L^p, L^{p'})$.

2. Taking $p = \infty$, prove that

$$C = \left\{u \in L^\infty(\Omega) : \int u\varphi \geq \int f\varphi \forall\varphi \in L^1(\Omega) \text{ with } f\varphi \in L^1(\Omega) \text{ and } \varphi \geq 0 \text{ a.e}\right\}$$

3. Deduce that when $p = \infty$, C is closed in $\sigma(L^\infty, L^1)$.

4. Let $f_1, f_2 \in L^\infty(\Omega)$ with $f_1 \leq f_2$ a.e. Prove that the set

$$C = \{u \in L^\infty(\Omega); f_1 \leq u \leq f_2 \text{ a.e.}\}$$

is compact in $L^\infty(\Omega)$ with respect to the topology $\sigma(L^\infty, L^1)$.

Exercise 4.24
Let $u \in L^\infty(\mathbb{R}^N)$. Let (ρ_n) be a sequence of mollifiers. Let (ζ_n) be a sequence in $L^\infty(\mathbb{R}^N)$ such that

$$\|\zeta_n\|_\infty \leq 1 \quad \forall n \quad \text{and} \quad \zeta_n \to \zeta \text{ a.e. on } \mathbb{R}^N.$$

Set

$$v_n = \rho_n * (\zeta_n u) \quad \text{and} \quad v = \zeta u.$$

1. Prove that $v_n \overset{*}{\rightharpoonup} v$ in $L^\infty(\mathbb{R}^N)$ weak* $\sigma(L^\infty, L^1)$.

2. Prove that $\int_B |v_n - v| \to 0$ for every ball B.

Proof. 1. For any $x \in L^1(\mathbb{R}^N)$, it is sufficient to prove that

$$\left| \int (v_n - v) x \, \mathrm{d}x \right| = |\langle v_n - v, x \rangle| \to 0.$$

Indeed, using the triangular inequality, we have

$$\begin{aligned}
\left| \int (v_n - v) x \right| &= \left| \int (\rho_n * (\zeta_n u) - \zeta u) x \right| \\
&= \left| \int (\rho_n * (\zeta_n u) - \zeta_n u) x + \int (\zeta_n u - \zeta u) x \right| \\
&\le \left| \int (\rho_n * (\zeta_n u) - \zeta_n u) x \right| + \left| \int (\zeta_n u - \zeta u) x \right|.
\end{aligned}$$

We will show that each term of the last summation converges to 0, thus complete our proof. Let $\check{\rho}_n(x) = \rho(-x)$, by Proposition 4.16, we get

$$\int (\rho_n * (\zeta_n u)) x = \int \zeta_n u (\check{\rho}_n * x).$$

Therefore,

$$\begin{aligned}
\left| \int (\rho_n * (\zeta_n u) - \zeta_n u) x \right| &= \left| \int (\rho_n * (\zeta_n u)) x - \zeta_n u x \right| \\
&= \left| \int \zeta_n u (\check{\rho}_n * x) - \zeta_n u x \right| \\
&= \left| \int \zeta_n u \cdot ((\check{\rho}_n * x) - x) \right| \\
&\le \|\zeta_n u\|_\infty \|\check{\rho}_n * x - x\|_1
\end{aligned}$$

where the last inequality is by the Holder's inequality. Notice that because $x \in L^1$, by Theorem 4.22, we get

$$\|\zeta_n u\|_\infty \|\check{\rho}_n * x - x\|_1 \le \|\zeta_n\|_\infty \|u\|_\infty \|\check{\rho}_n * x - x\|_1 \le \|u\|_\infty \|\check{\rho}_n * x - x\|_1 \to 0.$$

For the second term, because $\zeta_n \to \zeta$ almost everywhere, thus $\zeta_n u \to \zeta u$ almost everywhere. But $\|\zeta_n\|_\infty \le 1$, the function $\zeta_n u$ is bounded almost everywhere by $|u| \in L^\infty$. So by Holder's inequality and DCT, we get

$$\left| \int (\zeta_n u - \zeta u) x \right| \le \|\zeta_n u - \zeta u\|_\infty \|x\|_1 \to 0.$$

So $v_n \overset{*}{\rightharpoonup} v$ weak* $\sigma(L^\infty, L^1)$ as desired.

2. For any ball $B \subset \mathbb{R}^N$, it is not hard to see that $\chi_B \in L^1(\mathbb{R}^N)$. Using part 1, we get

$$\int_B |v_n - v| = \int |v_n - v| \chi_B \to 0.$$

□

Exercise 4.28
Let $\rho \in L^1(\mathbb{R}^N)$ with $\int \rho = 1$. Set $\rho_n(x) = n^N \rho(nx)$. Let $f \in L^p(\mathbb{R}^N)$ with $1 \le p < \infty$. Prove that $\rho_n * f \to f$ in $L^p(\mathbb{R}^N)$.

A small remark here: this exercise is a generalization of Theorem 4.22. If ρ_n is a sequence of mollifiers, then we are done. But we have lost lots of information. The $\operatorname{supp} \rho_n$ are not compact closed balls, ρ_n is not necessarily positive nor continuous (thus not smooth at all).

Proof. By changing of variable, we can easily check that $\int \rho_n = \int \rho = 1$. Therefore $\|\rho\|_1 = \int |\rho| < \infty$. Using Young's inequality, we get

$$\|\rho_n * f\|_p \le \|\rho\|_1 \|f\|_p \le \infty.$$

So $\rho_n * f \in L^p(\mathbb{R}^N)$. Because $C_c(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N)$, we can choose $f' \in C_c(\mathbb{R}^N)$ such that

$$\|f' - f\|_p < \varepsilon.$$

Using the triangular inequality, we get

$$\|\rho_n * f - f\|_p \le \|\rho_n * f - \rho_n * f'\|_p + \|\rho_n * f' - f'\|_p + \|f' - f\|_p. \tag{1}$$

So we have two terms to deal with (i.e. prove that it is sufficiently small as n large enough), because we got the last term for free by our construction of f'.

For the first term, we have

$$\int |\rho_n| = \int \rho_n^+ + \int \rho_n^- = \int \rho^+ + \int \rho^- = \int |\rho| < \infty.$$

So

$$\|\rho_n * f - \rho_n * f'\|_p = \|\rho_n * (f - f')\|_p \le \|\rho_n\|_1 \|f - f'\|_p = \|\rho\|_1 \|f - f'\|_p < \varepsilon \|\rho\|_1.$$

Since $\|\rho\|_1$ is a constant, our first term can be sufficiently small.

Now for the second term. Because $\int_{B(0,M)} |\rho| \to \int |\rho| := T < \infty$ as $M \to \infty$, we can choose M big enough such that $\operatorname{supp} f' \subset B(0, M)$ and

$$\left| \int_{B(0,M)} |\rho(x)| \,\mathrm{d}x - T \right| < \varepsilon. \tag{2}$$

Using the change of variable, we can prove that

$$\left| \int_{B(0,\frac{M}{n})} |\rho_n(x)| \,\mathrm{d}x - T \right| = \left| \int_{B(0,M)} |\rho(x)| \,\mathrm{d}x - T \right| < \varepsilon.$$

So

$$\left| \int_{B(0,\frac{M}{n})} \rho_n(x) \,\mathrm{d}x \right| \le \varepsilon + T.$$

Because $f' \in C_c(\mathbb{R}^N)$, f' is uniformly continuous on $\mathbb{R}^N$. So there is a $\delta > 0$ such that

$$|f'(x-y) - f'(x)| < \varepsilon$$

for all $x \in \mathbb{R}^N$ and $y \in B(0,\delta)$. Notice that $\frac{M}{n} \to 0$ as $n \to \infty$. So we can choose n large enough such that $\frac{M}{n} < \delta$. In this case, we have

$$\begin{aligned} \left| \int_{B(0,\frac{M}{n})} \rho_n(x-y)(f'(y) - f'(x)) \,\mathrm{d}y \right| &\leq \varepsilon \int_{B(0,\frac{M}{n})} |\rho_n(x-y)| \,\mathrm{d}y \\ &\leq \varepsilon(\varepsilon + T). \end{aligned}$$

So for letting $\rho_n^0 = \rho_n \chi_{B(0,\frac{M}{n})}$, the previous inequality implies that

$$|(\rho_n^0 * f')(x) - f'(x)| = \left| \int_{\mathbb{R}^N} \rho_n^0(x-y)(f'(y) - f'(x)) \,\mathrm{d}y \right| \leq \varepsilon(\varepsilon + T).$$

for all $x \in \mathbb{R}^n$, or $\rho_n * f' \to f'$ uniformly. Notice that ρ_n^0 has compact support, thus $\rho_n^0 * f'$ has compact support. Therefore

$$\|\rho_n^0 * f' - f'\|_p \to 0.$$

Also notice that by (2), we have

$$\|\rho_n - \rho_n^0\|_1 = \left| T - \int_{B(0,M)} |\rho(x)| \,\mathrm{d}x \right| < \varepsilon.$$

So in the end, we can control the second term of (1) by

$$\begin{aligned} \|\rho_n * f' - f'\|_p &\leq \|\rho_n * f' - \rho_n^0 * f'\|_p + \|\rho_n^0 * f' - f'\|_p \\ &\leq \|\rho_n - \rho_n^0\|_1 \|f'\|_p + \|\rho_n^0 * f' - f'\|_p \to 0. \end{aligned}$$

So all three terms of (1) are (finally!) sufficiently small as $n \to \infty$. This implies that $\rho_n * f \to f$ in $L^p(\mathbb{R}^N)$.

□

Exercise 4.31
Let $f \in L^p(\mathbb{R}^N)$ with $1 \leq p < \infty$. For every $r > 0$ set

$$f_r(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) \,\mathrm{d}y, \quad x \in \mathbb{R}^N.$$

1. Prove that $f_r \in L^p(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and that $f_r(x) \to 0$ as $|x| \to \infty$ (r being fixed).

2. Prove that $f_r \to f$ in $L^p(\mathbb{R}^N)$ as $r \to 0$.

Proof. 1. Let

$$\rho_\varepsilon(x) = \frac{\chi_{B(0,\varepsilon)}}{|B(0,\varepsilon)|}.$$

Since $\operatorname{supp}\rho_\varepsilon = \overline{B(0,\varepsilon)}$ is compact, we deduce that $\rho_\varepsilon \in L^1$. Moreover,

$$\|\rho_\varepsilon\|_1 = \int \frac{\chi_{B(0,\varepsilon)}}{|B(0,\varepsilon)|} = \frac{|B(0,\varepsilon)|}{|B(0,\varepsilon)|} = 1.$$

The bad news is that ρ is not continuous on $\partial B(0,\varepsilon)$, so it is not a mollifier. We have

$$\begin{aligned}
\rho_\varepsilon * f(x) &= \int \rho_\varepsilon(x-y)f(y)\,\mathrm{d}y \\
&= \int \frac{\chi_{B(0,\varepsilon)}(x-y)}{|B(0,\varepsilon)|} f(y)\,\mathrm{d}y \\
&= \frac{1}{|B(0,\varepsilon)|}\int \chi_{B(x,\varepsilon)}(y)f(y)\,\mathrm{d}y \\
&= \frac{1}{|B(x,\varepsilon)|}\int_{B(x,\varepsilon)} f(y)\,\mathrm{d}y \\
&= f_\varepsilon(x).
\end{aligned}$$

With this, we can easily check that $f_\varepsilon \in L^p$ following this inequality

$$\|f_\varepsilon\|_p = \|\rho_\varepsilon * f\|_p \le \|\rho_\varepsilon\|_1\|f\|_p = \|f\|_p < \infty.$$

Checking f_ε continuous is a little trickier. The proof is by showing that if $x_n \to x$, then $f_\varepsilon(x_n) \to f_\varepsilon(x)$. The key takeaway is that if $a \notin \partial B(x,\varepsilon)$, then $\chi_{B(x_n,\varepsilon)}(a) \to \chi_{B(x,\varepsilon)}(a)$. Indeed, if $a \notin \partial B(x,\varepsilon)$, then two cases can happen: (A) $a \in B(x,\varepsilon)$, and (B) $a \notin \overline{B(x,\varepsilon)}$.

(A) If $a \in B(0,\varepsilon)$ then $\|a-x\| < \varepsilon$ or $\varepsilon - \|a-x\| > 0$. Since $x_n \to x$, eventually $\|x_n - x\| < \varepsilon - \|a-x\|$. This implies

$$\|x_n - a\| \le \|x_n - x\| + \|x-a\| < \varepsilon.$$

So eventually $\chi_{B(x_n,\varepsilon)}(a) = \chi_{B(x,\varepsilon)}(a) = 1$.

(B) If $a \notin \overline{B(x,\varepsilon)}$, then $\|a-x\| - \varepsilon > 0$. Eventually, we have $\|x_n - x\| < \|a-x\| - \varepsilon$. In this case, the triangular inequality implies

$$\|x_n - a\| \ge \|x-a\| - \|x-x_n\| > \varepsilon.$$

So eventually $\chi_{B(x_n,\varepsilon)}(a) = \chi_{B(x,\varepsilon)}(a) = 0$.

With that in mind, we get

$$\chi_{B(x_n,\varepsilon)}(y)f(y) \to \chi_{B(x,\varepsilon)}(y)f(y)$$

for all $y \notin \partial B(x,\varepsilon)$. Notice that this boundary has measure 0, we deduce that

$$\chi_{B(x_n,\varepsilon)}(y)f(y) \to \chi_{B(x,\varepsilon)}(y)f(y)$$

almost everywhere. But $\chi_{B(x_n,\varepsilon)}(y)f(y)$ is dominated by $f(y) \in L^1$, thus

$$\int \chi_{B(x_n,\varepsilon)}(y)f(y)\,\mathrm{d}y \to \int \chi_{B(x,\varepsilon)}(y)f(y)\,\mathrm{d}y.$$

Therefore, $f_\varepsilon(x_n) \to f_\varepsilon(x)$ for any $x_n \to x$ and $\varepsilon > 0$. We conclude that $f_\varepsilon \in L^p(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ for all $\varepsilon > 0$.

Because $f \in L^1$, for any $\delta > 0$, there exists $M > 0$ such that

$$\left| \int_{B(0,M)} f(y)\,\mathrm{d}y - \|f\|_1 \right| < \delta.$$

So whenvere $|x| > M + \varepsilon$, the ball $B(x,\varepsilon)$ fall out of the ball $B(0,M)$. This means $\int_{B(x,\varepsilon)} f(y)\,\mathrm{d}y < \delta$ and thus

$$f_\varepsilon(x) = \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f(y)\,\mathrm{d}y < \frac{\delta}{|B(x,\varepsilon)|},$$

which can be sufficiently small as $\delta \to 0$. So $f_\varepsilon(x) \to 0$ as $|x| \to \infty$.

2. Since

$$\rho_\varepsilon(x) = \frac{1}{|B(x,\varepsilon)|} \int_{B(x,\varepsilon)} f(y)\,\mathrm{d}y = \frac{1}{\varepsilon^N |B(x,1)|} \int_{B(x,1)} f\left(\frac{y}{\varepsilon}\right)\mathrm{d}y = \frac{1}{\varepsilon^N} \rho_1\left(\frac{x}{\varepsilon}\right).$$

So applying Exercise 28, we get $f_\varepsilon = \rho_\varepsilon(x) * f \to f$ in $L^p(\mathbb{R}^N)$.

$\square$