Exercise 1. *Show that*

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$$

defines a metric on $(0,\infty)$.

Proof. Clearly, $d(x,y) \geq 0$ for all $x, y \in (0,\infty)$. If $d(x,y) = 0$, then $|\frac{1}{x} - \frac{1}{y}| = 0 \Rightarrow x = y$. Moreover, we have

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{1}{y} - \frac{1}{x}\right|$$

and

$$d(x,y) + d(y,z) = \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| \geq \left|\frac{1}{x} - \frac{1}{z}\right| = d(x,z).$$

Thus d is a metric on $(0,\infty)$. □

Exercise 2. *If* d *is a metric on* M*, show that* $|d(x,z) - d(y,z)| \leq d(x,y)$ *for any* $x, y, z \in M$.

Proof. Since d is a metric on M, we have $d(x,z) \leq d(x,y) + d(y,z)$, hence $d(x,z) - d(y,z) \leq d(x,y)$. Moreover, we have $d(y,z) \leq d(y,x) + d(x,z)$, hence $-d(x,y) \leq d(x,z) - d(y,z)$. Thus $|d(x,z) - d(y,z)| \leq d(x,y)$. □

Exercise 3. *As it happens, some of our requirements for a metric are redundant. To see why this is so, let* M *be a set and suppose that* $d : M \times M \to \mathbb{R}$ *satisfies* $d(x,y) = 0$ *if and only if* $x = y$*, and* $d(x,y) \leq d(x,z) + d(y,z)$ *(*) for all* $x, y, z \in M$*. Prove that* d *is a metric.*

Proof. First, we will prove that $d(x,y) = d(y,x)$. Indeed, let $z = x$ in (*), we get

$$d(x,y) \leq d(x,x) + d(y,x) = d(y,x).$$

Similarly, we will get $d(y,x) \leq d(x,y)$. Thus $d(x,y) = d(y,x)$. Notice that

$$0 = d(x,x) \leq d(x,y) + d(x,y) = 2d(x,y),$$

thus $0 \leq d(x,y)$ for all $x, y \in M$. Thus d is a metric. □

Exercise 6. *If* d *is a metric on* M*, show that* $\rho(x,y) = \sqrt{d(x,y)}$, $\sigma(x,y) = \frac{d(x,y)}{1+d(x,y)}$*, and* $\tau(x,y) = \min\{d(x,y), 1\}$ *are also metrics on* M.

Proof.

- $\rho(x,y)$.

  Clearly, we have $\rho(x,y) \geq 0$. If $\rho(x,y) = 0$, then $\sqrt{d(x,y)} = 0$, which leads to $d(x,y) = 0$. Thus $x = y$. Otherwise, since $d(x,x) = 0$, it's not hard to check that $\rho(x,x) = 0$. Moreover, we have $\rho(x,y) = \sqrt{d(x,y)} = \sqrt{d(y,x)} = \rho(y,x)$. Also notice that for $a, b \geq 0$, we have $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$. Indeed, since $2\sqrt{xy} \geq 0$, we have

$$\begin{aligned} & x + y \leq x + y + 2\sqrt{xy} \\ \Leftrightarrow & (\sqrt{x+y})^2 \leq (\sqrt{x} + \sqrt{y})^2 \\ \Leftrightarrow & |\sqrt{x+y}| \leq |\sqrt{x} + \sqrt{y}| \\ \Leftrightarrow & \sqrt{x+y} \leq \sqrt{x} + \sqrt{y}. \end{aligned}$$

Thus

$$\rho(x,z) = \sqrt{d(x,z)} \leq \sqrt{d(x,y) + d(y,z)} \leq \sqrt{d(x,y)} + \sqrt{d(y,z)} = \rho(x,y) + \rho(y,z).$$

Thus ρ defines a metrics on M.

- $\sigma(x,y)$.

  Because $d(x,y) \geq 0$, we have $\sigma(x,y) \geq 0$ for all $x, y \in M$. If $\sigma(x,y) = 0$, then $\frac{d(x,y)}{1+d(x,y)} = 0$, thus $d(x,y) = 0$, which means $x = y$. Otherwise, since $d(x,x) = 0$, we can easily check that $\sigma(x,x) = 0$ too. Moreover,

$$\sigma(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \sigma(y,x).$$

  Now, we have a small lemma as follow.

  Lemma . *Let $a, b, c \geq 0$, if $c \leq a + b$, then*

$$\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$

  Proof. Indeed, we have

$$\begin{aligned} & \frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b} \\ \Leftrightarrow \quad & \frac{c}{1+c} \leq \frac{a+b+2ab}{1+a+b+ab} \\ \Leftrightarrow \quad & 1 - \frac{1}{1+c} \leq 1 - \frac{1-ab}{1+a+b+ab} \\ \Leftrightarrow \quad & \frac{1-ab}{1+a+b+ab} \leq \frac{1}{1+c}. \qquad (*) \end{aligned}$$

  If $1 - ab \leq 0$, then the left side is smaller or equal then 0, when the right side is larger or equal then 0. Thus the inequality is proved.

  If $1 - ab \geq 0$, then

$$\begin{aligned} (*) \Leftrightarrow \quad & 1 + c \leq \frac{1+a+b+ab}{1-ab} \\ \Leftrightarrow \quad & 1 + c \leq 1 + \frac{a+b+2ab}{1-ab} \\ \Leftrightarrow \quad & c \leq \frac{a+b+2ab}{1-ab} \\ \Leftrightarrow \quad & c - abc \leq a + b + 2ab. \end{aligned}$$

  However, we have $c \leq a + b$ and $-abc \leq 0 \leq 2ab$, thus the lemma is proved. $\square$

Now let $a = d(x,z), b = d(x,y), c = d(y,z)$ in the lemma, we have $\sigma(x,z) \leq \sigma(x,y) + \sigma(y,z)$. Thus σ defines a metric on M.

- $\tau(x,y)$.

  Since $d(x,y) \geq 0$, we have $\tau(x,y) = \min\{d(x,y),1\} \geq 0$. If $\tau(x,y) = 0$, then $\min\{d(x,y),1\} = 0$, hence $d(x,y) = 0$, which leads to $x = y$. Moreover, $\tau(x,x) = \min\{d(x,x),1\} = \min\{0,1\} = 0$. We also have $\tau(x,y) = \min\{d(x,y),1\} = \min\{d(y,x),1\} = \tau(y,x)$.

  Lemma I. *f* $a,b \geq 0$, *we have* $\min\{a+b,1\} \leq \min\{a,1\} + \min\{b,1\}$.

  Proof. Indeed, if $a,b > 1$, then the lemma becomes $a+b \leq a+b$. If $a > 1, b \leq 1$, the lemma becomes $a+b \leq a+1$, which is true, same for the case $b > 1, a < 1$. And if $a,b \leq 1$, then obviously $a+b \leq 2$. And since $1 < 2$, we have $\min\{a+b,1\} \leq 2 = \min\{a,1\} + \min\{b,1\}$. □

  Using the lemma, we have

  $$\begin{aligned}\tau(x,z) &= \min\{d(x,z),1\} \\ &\leq \min\{d(x,y)+d(y,z),1\} \\ &\leq \min\{d(x,y),1\} + \min\{d(y,z),1\} \\ &= \tau(x,y) + \tau(y,z).\end{aligned}$$

  Thus $\tau(x,y)$ defines a metric on M.

□

Exercise 9. *Recall that* 2^N *denotes the set of all sequences (or "strings") of* $0s$ *and* $1s$. *Show that* $d(a,b) = \ sum_{n=1}^{\infty} 2^{-n}|a_n - b_n|$, *where* $a = (a_n)$ *and* $b = (b_n)$ *are sequences of* $0s$ *and* $1s$, *defines a metric on* 2^N.

Proof. Clearly, $d(a,b) \geq 0$. If $d(a,b) = 0$, then $|a_n - b_n| = 0$ for all n, thus $a = b$. Otherwise, $d(a,a) = \sum_{n=1}^{\infty} 2^{-n}|a_n - a_n| = 0$. Furthermore, we have

$$d(a,b) = \ sum_{n=1}^{\infty} 2^{-n}|a_n - b_n| = \sum_{n=1}^{\infty} 2^{-n}|b_n - a_n| = d(b,a)$$

and

$$\begin{aligned}d(a,c) &= \sum_{n=1}^{\infty} 2^{-n}|a_n - c_n| \\ &\leq \sum_{n=1}^{\infty} 2^{-n}(|a_n - b_n| + |b_n - c_n|) \\ &= \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n| + \sum_{n=1}^{\infty} 2^{-n}|b_n - c_n| \\ &= d(a,b) + d(b,c).\end{aligned}$$

Thus d defines a metric on 2^N. □

Exercise 10. *The Hilbert cube H^∞ is the collection of all real sequences $x = (x_n)$ with $|x_n| \leq 1$ for $n = 1, 2, \cdots$.*

(i) *Show that $d(x, y) = \sum_{n=1}^{\infty} 2^{-n}|x_n - y_n|$ defines a metric on H^∞.*

Proof. Similar to exercise 9, d defines a metric on $\mathbb{R}^\infty$ □

(ii) *Given that $x, y \in H^\infty$ and $k \in \mathbb{N}$, let $M_k = \max\{|x_1 - y_1|, \cdots, |x_k - y_k|\}$. Show that $2^{-k}M_k \leq d(x, y) \leq M_k + 2^{1-k}$.*

Proof. For any $k \in \mathbb{N}$ and $i \leq k$, we have

$$2^{-k}|x_i - y_i| \leq 2^{-i}|x_i - y_i| \leq \sum_{i=1}^{\infty} 2^{-i}|x_i - y_i|.$$

Thus for $M_k = \max\{|x_1 - y_1|, \cdots, |x_k - y_k|\}$, we have

$$2^{-k}M_k \leq d(x, y).$$

Furthermore, we have

$$\sum_{i=1}^{k} 2^{-i}|x_i - y_i| \leq \sum_{i=1}^{k} 2^{-i}M_k = M_k(1 - 2^{-k}) \leq M_k.$$

Also notice that $|x_i - y_i| \leq 2$, we also have

$$\sum_{i=k+1}^{\infty} 2^{-i}|x_i - y_i| \leq \sum_{i=k+1}^{\infty} 2^{1-i} = 2^{1-k}.$$

Thus

$$d(x, y) \leq M_k + 2^{1-k}.$$

□

Exercise 11. *Let $\mathbb{R}^\infty$ denote the collection of all real sequences $x = (x_n)$. Show that the expression*

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|}$$

defines a metric on $\mathbb{R}^\infty$.

Proof. Since every element of this summation is larger than 0, $d(x, y) \geq 0$. It's not hard to check that $d(x, x) = 0$, let $d(x, y) = 0$, then

$$\frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|} = 0$$

for any n, thus $x_n = y_n$ for all n, which means $x = y$. Moreover, we have

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - y_n|}{1 + |x_n - y_n|} = \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|y_n - x_n|}{1 + |y_n - x_n|} = d(y,x)$$

and by exercise 6, we have

$$\begin{aligned} d(x,z) &= \sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - z_n|}{1 + |x_n - z_n|} \\ &\leq \sum_{n=1}^{\infty} \frac{1}{n!} (\frac{|x_n - y_n|}{1 + |x_n - y_n|} + \frac{|y_n - z_n|}{1 + |y_n - z_n|}) \\ &= d(x,y) + d(y,z). \end{aligned}$$

What is more, we have $\frac{|a-b|}{1+|a+b|} < 1$, thus $d(a,b) \leq \sum_{n=1}^{\infty} \frac{1}{n!}$ for all $a, b \in M$. Since $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges, $\sum_{n=1}^{\infty} \frac{1}{n!} \frac{|x_n - z_n|}{1+|x_n-z_n|}$ converges. Thus d defines a metric M. □

Exercise 12. *Check that $d(f,g) = \max_{a \leq t \leq b} |f(t) - g(t)|$ defines a metric on $C[a,b]$, the collection of all continuous, real-value functions defined on the closed interval $[a,b]$.*

Proof. Clearly, $d(f,g) \geq 0$. We have $d(f,f) = \max_{a \leq t \leq b} |f(t) - f(t)| = 0$. If $d(f,g) = 0$, then $\max_{a \leq t \leq b} |f(t) - g(t)| = 0$. Therefor, $f(t) = g(t)$ for all $a \leq t \leq b$. Moreover, we have

$$d(f,g) = \max_{a \leq t \leq b} |f(t) - g(t)| = \max_{a \leq t \leq b} |g(t) - f(t)| = d(g,f)$$

and

$$\begin{aligned} d(f,h) &= \max_{a \leq t \leq b} |f(t) - h(t)| \\ &\leq \max_{a \leq t \leq b} (|f(t) - g(t)| + |g(t) - h(t)|) \\ &\leq \max_{a \leq t \leq b} |f(t) - g(t)| + \max_{a \leq t \leq b} |g(t) - h(t)| \\ &= d(f,g) + d(g,h). \end{aligned}$$

Thus d defines a metric on $C[a,b]$. □

Exercise 14. *We said that a subset A of a metric space M is bounded if there is some $x_0 \in M$ and some constant $C < \infty$ such that $d(a, x_0) \leq C$ for all $a \in A$. Show that a finite union of bounded sets is again bounded.*

Proof. Let these bounded sets be $A_1, A_2, \cdots, A_n$ and $x_i \in A_i$ for $i = 1, 2, \cdots, n$, let $a_1, a_2, \cdots, a_n$ be n constants such that for any $1 \leq i \leq n$, we have $d(t, x_i) \leq a_i$ for any $t \in A_i$. Finally, let $A = \cup_{i=1}^{n} A_i$.

For any set A_k, an $t \in A_k$ $(1 \leq k \leq n)$, we have

$$d(x_1, t) \leq d(x_1, x_k) + d(x_k, t) \leq d(x_1, x_k) + a_k.$$

Notice that $d(x_1, x_k) + a_k$ is a constant, let $d_k = d(x_1, x_k) + a_k$, then $d = \max\{d_1, d_2, \cdots, d_n\}$ is an upper bound for A. Thus a finite union of bounded sets is bounded. □

Exercise 15. *We define the diameter of a nonempty subset A of M by $diam(A) = \sup\{d(a,b) : a, b \in A\}$. Show that A is bounded if and only if $diam(A)$ is finite.*

Proof. If A is bounded, then exist $x \in A$ and a number c such that $d(a,x) \leq c$ for all $a \in A$. Therefor, for any $a, b \in A$, we have

$$d(a,b) \leq d(a,x) + d(x,b) \leq 2c.$$

Thus the set $\{d(a,b) : a, b \in A\}$ has an upper bound, which means $diam(A) = \sup\{d(a,b) : a, b \in A\}$ is finite. Moreover, if $diam(A) = c$ a finite number, then $d(a,x) \leq c$ for all $a \in A$ and an $x \in A$. □

Exercise 16. *Let V be a vector space, and let d be a metric on V satisfying $d(x,y) = d(x-y,0)$ and $d(ax,ay) = |a|d(x,y)$ for every $x, y \in V$ and every scalar a. Show that $||x|| = d(x,0)$ defines a norm on V. Give an example of a metric on the vector space $\mathbb{R}$ that fails to be associated with a norm in this way.*

Proof. For any $x \in V$, because d defines a matrix on V, thus $||x|| = d(x,0) \geq 0$. We have $||0|| = d(0,0) = 0$ and if $||x|| = 0$, then $d(x,0) = 0$. Therefor, $x = 0$ since d is a metric on V. We also have

$$||ax|| = d(ax,0) = |a|d(x,0) = |a|||x||.$$

and

$$\begin{aligned} ||x+y|| &= d(x+y,0) \\ &= d(x,-y) \\ &\leq d(x,0) + d(0,-y) \\ &= d(x,0) + d(y,0) \\ &= ||x|| + ||y||. \end{aligned}$$

One example of a metric on $\mathbb{R}$ that fails to be associate with a norm this way is

$$d(x) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}.$$

Because if letting $||x|| = d(x,0)$, then $||ax|| = 1 \neq |a| = |a|||x||$ for any nonzero vector x and scalar a. □

Exercise 17. *Recall that for $x \in \mathbb{R}^n$ we have defined $||x||_1 = \sum_{i=1}^n |x_i|$ and $||x||_\infty = \max_{1\leq i\leq n} |x_i|$. Check that each of these is indeed a norm on $\mathbb{R}^n$.*

Proof. Obviously $||x||_1 \geq 0$. We have $||0||_1 = \sum_{i=1}^\infty 0 = 0$, whereas if $||x||_1 = 0$, then $|x_i| = 0$ for all $1 \leq i \leq n$. Thus $x = 0$. Moreover, $||ax|| = \sum_{i=1}^n |ax_i| = |a| \sum_{i=1}^n |x_i| = |a| \cdot ||x_1||$. Last but not least, we have

$$\begin{aligned} ||x+y||_1 &= \sum_{i=1}^n |x_i + y_i| \\ &\leq \sum_{i=1}^n |x_i| + \sum_{i=1}^n |y_i| \\ &= ||x||_1 + ||y||_1. \end{aligned}$$

Similarly, $||x||_\infty \geq 0$ is obvious. We have $||0||_\infty = \max_{1\leq i\leq n} |x_i| = 0$, and if $||x||_\infty = 0$, then $\max_{1\leq i\leq n} |x_i| = 0$. Therefore, $|x_i| = 0$ for all i, which means $x = 0$. Moreover, $||ax||_\infty = \max_{1\leq i\leq n} |ax_i| = |a| \max_{1\leq i\leq n} |x_i| = |a| \cdot ||x||_\infty$. Lastly,

$$\begin{aligned} ||x+y||_\infty &= \max_{1\leq i\leq n} |x_i + y_i| \\ &\leq \max_{1\leq i\leq n} (|x_i| + |y_i|) \\ &\leq \max_{1\leq i\leq n} |x_i| + \max_{1\leq i\leq n} |y_i| \\ &= ||x||_\infty + ||y||_\infty. \end{aligned}$$

Thus $||x||_1$ and $||x||_\infty$ defines two metrics on $\mathbb{R}^n$. □

Exercise 18. *Show that $||x||_\infty \leq ||x||_2 \leq ||x||_1$ for any $x \in \mathbb{R}^n$. Also check that $||x||_1 \leq n||x||_\infty$ and $||x||_1 \leq \sqrt{n}||x||_2$.*

Proof. We have

$$\begin{aligned} & ||x||_\infty \leq ||x||_2 \\ \Leftrightarrow \quad & \max_{1\leq i\leq n} |x_i| \leq \sqrt{\sum_{i=1}^n x_i^2} \\ \Leftrightarrow \quad & \max_{1\leq i\leq n} x_i^2 \leq \sum_{i=1}^n x_i^2 \text{ (square both sides)}, \end{aligned}$$

which is obvious since $x_i^2 \geq 0$ for all x_i and one of the addends on the right side equals $\max_{1\leq i\leq n} x_i^2$. Therefore, $||x||_\infty \leq ||x||_2$. Very similarly, we have

$$\begin{aligned} & ||x||_2 \leq ||x||_1 \\ \Leftrightarrow \quad & \sqrt{\sum_{i=1}^n x_i^2} \leq \sum_{i=1}^n |x_i| \\ \Leftrightarrow \quad & \sum_{i=1}^n x_i^2 \leq \sum_{i=1}^n x_i^2 + \sum_{i\neq j} |x_i||x_j|, \end{aligned}$$

which is obvious since $|x_i||x_j| \geq 0$. Hence, $||x||_\infty \leq ||x||_2 \leq ||x||_1$ for any $x \in \mathbb{R}^n$.

What is more, because $|x_i| \leq \max_{1\leq i\leq n} |x_i|$ for all $1 \leq i \leq n$, we have

$$\sum_{i=1}^{n} |x_i| \leq n \max_{1\leq i\leq n} |x_i|.$$

Hence $||x||_1 \leq n||x||_\infty$. Moreover, by Cauchy Schwarz inequality, we have

$$1|x_1| + 1|x_2| + \cdots + 1|x_n| \leq \sqrt{(1^2 + 1^2 + \cdots + 1^2)(x_1^2 + x_2^2 + \cdots + x_n^2)}.$$

Thus

$$\sum_{i=1}^{n} x_i \leq \sqrt{n \sum_{i=1}^{n} x_i^2},$$

which is synonymous to $||x||_1 \leq \sqrt{n}||x||_2$. □

Exercise 19. *Show that we have* $\sum_{i=1}^n x_i y_i = ||x||_2||y||_2$ *if and only if* x *and* y *are proportional, that is, if and only if* $x = \alpha y$ *or* $y = \alpha x$ *for some* $\alpha \leq 0$.

Proof. We will prove a stronger result, that is $\sum_{i=1}^n x_i y_i \leq ||x||_2||y||_2$ and the equality is if x and y are proportional by mathematical induction. Let x_k be a random vector in $\mathbb{R}^k$. If $n = 1$, the inequality becomes $x_1 y_1 \leq |x_1||y_1|$, which is obvious. And since both x and y are in $\mathbb{R}$, they are obviously proportional. Now assume that the result holds for $n-1$, notice that by AM-GM inequality, we have

$$\begin{aligned} & 2x_n y_n \sqrt{(x_1^2 + \cdots + x_{n-1}^2)(y_1^2 + \cdots + y_{n-1}^2)} \\ \leq \quad & 2|x_n||y_n|\sqrt{(x_1^2 + \cdots + x_{n-1}^2)(y_1^2 + \cdots + y_{n-1}^2)} \\ \leq \quad & x_n^2(y_1^2 + \cdots + y_{n-1}^2) + y_n^2(x_1^2 + \cdots + x_{n-1}^2). \end{aligned}$$

Thus

$$\begin{aligned} & (x_1^2 + \cdots + x_{n-1}^2)(y_1^2 + \cdots + y_{n-1}^2) + 2x_n y_n \sqrt{(x_1^2 + \cdots + x_{n-1}^2)(y_1^2 + \cdots + y_{n-1}^2)} + x_n^2 y_n^2 \\ \leq\ & (x_1^2 + \cdots + x_{n-1}^2)(y_1^2 + \cdots + y_{n-1}^2) + x_n^2(y_1^2 + \cdots + y_{n-1}^2) + y_n^2(x_1^2 + \cdots + x_{n-1}^2) + x_n^2 y_n^2. \end{aligned}$$

After cleaning this mess, we reach something that is not very messy.

$$\left(x_n y_n + \sqrt{(x_1^2 + \cdots + x_{n-1}^2)(y_1^2 + \cdots + y_{n-1}^2)}\right)^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2).$$

Take the square root both sides, we get

$$|x_n y_n| + \sqrt{(x_1^2 + \cdots + x_{n-1}^2)(y_1^2 + \cdots + y_{n-1}^2)} \leq \sqrt{(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2)},$$

or

$$x_n y_n + ||x_{n-1}||_2 \cdot ||y_{n-1}||_2 \leq |x_n y_n| + ||x_{n-1}||_2 \cdot ||y_{n-1}||_2 \leq ||x_n||_2 \cdot ||y_n||_2.$$

But thanks to the induction assumption, we have $\sum_{i=1}^{n-1} x_i y_i \leq ||x_{n-1}||_2 \cdot ||y_{n-1}||_2$. Thus

$$\sum_{i=1}^{n} x_i y_i \leq ||x_n||_2 ||y_n||_2.$$

The equality is when equality happens in every inequality that we used, which are $x_n y_n \leq |x_n y_n|$ and the AM-GM inequality. The first equality is equivalent to x_n and y_n have the same sign. The second one is equivalent to

$$x_n \sqrt{y_1^2 + \cdots y_{n-1}^2} = y_n \sqrt{x_1^2 + \cdots x_{n-1}^2},$$

Hence

$$\frac{x_n}{y_n} = \frac{\sqrt{x_1^2 + \cdots x_{n-1}^2}}{\sqrt{y_1^2 + \cdots y_{n-1}^2}} = \alpha$$

by the induction assumption. Thus $x = \alpha y$ for $\alpha \geq 0$. $\square$

Exercise 22. *Show that $||x||_\infty \leq ||x||_2$ for any $x \in \ell_2$, and that $||x||_2 \leq ||x||_1$ for any $x \in \ell_1$.*

Proof. Let $x \in \ell_1$, then $\sum_{i=1}^{\infty} |x_1|$ exists. Notice that $\sqrt{\sum_{i=1}^{n} x_i^2} \leq \sum_{i=1}^{n} |x_i|$ for all n, thus the sequence $\sqrt{\sum_{i=1}^{n} x_i^2}$ is both increasing and upper bounded. Thus $\sqrt{\sum_{i=1}^{n} x_i^2}$ is also exists. Thus by the inequality above, we get $||x||_2 \leq ||x||_1$.

Now let $x \in \ell_2$, we have $\sqrt{\sum_{i=1}^{\infty} x_i^2}$ exists. Similar to the upper case, we have $\max_{1\leq i\leq n} x_i$ is both increasing and upper bounded, thus $\lim_{n\to\infty}(\max_{1\leq i\leq n} x_i)$ exists. Since $\max_{1\leq i\leq n} x_i \leq \sqrt{\sum_{i=1}^{n} x_i^2}$, we have $||x||_\infty \leq ||x||_2$ for any $x \in \ell_2$. $\square$

Exercise 23. *The subset of ℓ_∞ consisting of all sequences that converge to 0 is denoted by c_0. Show that we have the following proper set inclusions: $\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty$.*

Proof. As we already checked in exercise 22, for any $x \in \ell_1$, then $||x||_2$ is finite. Therefore, $x \in \ell_2$ too, which means $\ell_1 \subset \ell_2$. Moreover $c_0 \subset \ell_\infty$ is by definition. Thus all we have to check left is $\ell_2 \subset c_0$. Notice that if $x = (x_1, x_2, \cdots) \in \ell_2$, then $\sqrt{\sum_{i=1}^{\infty} x_i^2}$ exists. Therefore, $x_n \to 0$, which leads to $x \in c_0$. Thus $\ell_1 \subset \ell_2 \subset c_0 \subset \ell_\infty$. $\square$

Exercise 24. *Prove that Holder's Inequality is also holds in the case $p = 1$ and $q = \infty$.*

Proof. For $p = 1$ and $q = \infty$, the inequality becomes $\sum_{i=1}^{\infty} |x_i y_i| \leq ||x||_1 \cdot ||y||_\infty$, which means $\sum_{i=1}^{\infty} |x_i y_i| \leq \sum_{i=1}^{\infty} |x_i| \cdot \sup_i |y_i|$. However, this result is obvious because $|y_n| \leq \sup_i |y_i|$ for any n. Thus the Holder's inequality is also holds in the case $p = 1$ and $q = \infty$. $\square$

Exercise 25. *The same techniques can be used to show that* $||f||_p = (\int_0^1 |f(t)|^p dt)^{\frac{1}{p}}$ *defines a norm on* $C[0,1]$ *for any* $1 < p < \infty$*. State and prove the analogues of lemma 3.7 (Holder's Inequality) and Theorem 3.8 (Minkowski's Inequality) in this case.*

Proof. First, notice that

$$\begin{aligned} ||f^{p-1}||_q &= \left(\int_0^1 |f(x)^{p-1}|^q\right)^{\frac{1}{q}} \\ &= \left(\int_0^1 |f(x)|^p\right)^{\frac{1}{q}} \\ &= \left(\int_0^1 |f(x)|^p\right)^{\frac{p-1}{p}} \\ &= ||f||_p^{p-1}. \end{aligned}$$

Now it's time to restate Lemma 3.7.

Lemma 3.7. *(Holder's Inequality) Let* $1 < p < \infty$ *and let* q *be defined by* $\frac{1}{p} + \frac{1}{q} = 1$*. Given* f *and* g *in* $C[0,1]$*, we have*

$$\int_0^1 |f(t)g(t)|dt \leq ||f||_p||g||_q.$$

Proof. By Young's inequality, we have

$$\frac{|f(t)| \cdot |g(t)|}{||f||_p||g||_q} \leq \frac{1}{p}\left|\frac{|f(t)|}{||f||_p}\right|^p + \frac{1}{q}\left|\frac{|g(t)|}{||g||_q}\right|^q.$$

Therefore

$$\int_0^1 \frac{|f(t)| \cdot |g(t)|}{||f||_p||g||_q} dt \leq \frac{1}{p}\int_0^1 \frac{|f(t)|^p}{||f||_p^p} dt + \frac{1}{q}\int_0^1 \frac{|g(t)|^q}{||g||_q^q} dt = \frac{1}{p} + \frac{1}{q} = 1.$$

Thus we have the Holder's inequality. □

With lemma 3.7, we can restate and prove theorem 3.8 as follow.

Lemma 3.8. *Let* $1 < p < \infty$*, then we have* $||f+g||_p \leq ||f||_p + ||g||_p$ *for* $f, g \in C[0,1]$*.*

Proof. Indeed, we have

$$\begin{aligned} ||f+g||_p^p &= \int_0^1 |f+g|^p \\ &= \int_0^1 |f+g|^{p-1}|f+g| \\ &\leq \int_0^1 |f+g|^{p-1}|f| + \int_0^1 |f+g|^{p-1}|g| \\ &\leq ||(|f+g|)^{p-1}||_q||f||_p + ||(|f+g|)^{p-1}||_q||g||_p \\ &= ||(f+g)||_p^{p-1}(||f||_p + ||g||_p). \end{aligned}$$

Thus $||f+g||_p \leq ||f||_p + ||g||_p$. □

So $||f||_p$ satisfies the triangular inequality. Moreover, it's easy to check that $||f||_p \geq 0$. If $||f||_p = 0$, then $\int_0^1 f(t)^p dt = 0$. And since f is continuous on $[0,1]$, we have $f(t) = 0$ for all $t \in [0,1]$. If $f(t) = 0$, then it's also easy to check that $||f||_p = 0$. Moreover, for a scalar a, then

$$||af||_p = \left(\int_0^1 |af(t)|^p dt\right)^{\frac{1}{p}} = |a| \left(\int_0^1 |f(t)|^p dt\right)^{\frac{1}{p}} = |a| \cdot ||f||_p.$$

Thus $||f||_p$ is a norm on $C[0,1]$. $\square$

Exercise 26. *Given $a, b > 0$, show that $\lim_{p\to\infty}(a^p + b^p)^{\frac{1}{p}} = \max\{a, b\}$. What happens as $p \to 0$? as $p \to -1$? as $p \to \infty$?*

Proof. Assume that $a < b$, let $r = \frac{a}{b}$, we have

$$\begin{aligned}(a^p + b^p)^{\frac{1}{p}} &= b \cdot \left(1 + \left(\frac{a}{b}\right)^p\right)^{\frac{1}{p}} \\ &= b \cdot (1 + r^p)^{\frac{1}{p}} \\ &= b \cdot e^{\log(1+r^p)^{\frac{1}{p}}}.\end{aligned}$$

Notice that as $p \to \infty$, we have $\log(1+r^p)^{\frac{1}{p}} \to 0 \cdot \log(1) \cdot 0 = 0$. Thus $(a^p + b^p)^{\frac{1}{p}} \to b \cdot 1 = b = \max\{a, b\}$. $\square$

Exercise 27. *Show that $diam(B_r(x)) \leq 2r$, and give an example where strict inequality occurs.*

Proof. For any $a, b \in B_r(x)$, we have $d(a, b) \leq d(a, x) + d(x, b) < 2r$. Thus $diam(B_r(x)) = \sup\{d(a, b) : a, b \in B_r(x) \leq 2r$. An example for strict inequality is the discrete space because $diam(B_1(0)) = 0 < 2$. $\square$

Exercise 28. *If $diam(A) < r$, show that $A \subset B_r(a)$ for some $a \in A$.*

Proof. For any $x \in A$, because $diam(A) < r$, we have $d(a, x) < r$. Therefore, $x \in B_r(a)$. Thus $A \subset B_r(a)$. $\square$

Exercise 30. *If $A \subset B$, show that $diam(A) \leq diam(B)$.*

Proof. Since $A \subset B$, we have $\{d(a, b) : a, b \in A\} \subset \{d(a, b) : a, b \in B\}$. Thus $diam(A) = \sup\{d(a, b) : a, b \in A\} \leq \sup\{d(a, b) : a, b \in B\} = diam(B)$. $\square$

Exercise 32. *In a normed vector space $(V, ||\cdot||)$ show that $B_r(x) = x + B_r(0) = \{x + y : ||y|| < r\}$ and that $B_r(0) = rB_1(0) = \{rx : ||x|| < 1\}$.*

Proof. For any $a \in B_r(x)$, we have $||a - x|| < r$. Therefore $a = x - (a - x) \in x + B_r(0)$. Thus $B_r(x) \subset x + B_r(0)$. And for any $a \in x + B_r(0)$, then there exists $y \in B_r(0)$ such that $a = x + y$. Thus $||a - x|| = ||y|| < r$. Thus $a \in B_r(x)$, which means $x + B_r(0) \subset B_r(x)$. Hence $B_r(x) = x + B_r(0)$.

Similarly, if $a \in B_r(0)$, then $||a|| < r$. Notice that because $r > 0$, we have $||\frac{a}{r}|| = \frac{1}{r}||a|| = 1$. And since $a = r \cdot \frac{a}{r}$, we have $a \in rB_1(0)$. Thus $B_r(0) \subset rB_1(0)$. Moreover, if $a \in rB_1(0)$, then there exists $x \in B_1(0)$ such that $a = rx$. Then $||a|| = ||rx|| = |r| \cdot ||x|| < r$. Thus $a \in B_r(0)$, which leads to $rB_1(0) \subset B_r(0)$. Thus $B_r(0) = rB_1(0)$. $\square$

Exercise 33. *Limits are unique.*

Proof. Assume that x and y are two identical limits of (x_n), then $d(x,y) > 0$. Let $\epsilon > 0$ such that $2\epsilon < d(x,y)$, there exists N such that if $n > N$, then $x_n \in B_\epsilon(x) \cap B_\epsilon(y)$. Therefore, $2\epsilon < d(x,y) \leq d(x,x_n) + d(x_n,y) < 2\epsilon$, contradiction. Thus the limit of (x_n) must be unique. □

Exercise 34. *If $x_n \to x$ in (M,d), show that $d(x_n,y) \to d(x,y)$ for any $y \in M$. More generally, if $x_n \to x$ and $y_n \to y$, show that $d(x_n,y_n) \to d(x,y)$.*

First, we will prove a small lemma.

Lemma . *For $a,b,c \in M$, we have $|d(a,b) - d(b,c)| \leq d(a,c)$.*

Proof. Indeed, since $d(a,b) \leq d(a,c)+d(b,c)$, we have $d(a,b)-d(b,c) \leq d(a,c)$. Moreover $d(b,c) \leq d(b,a) + d(a,c)$, hence $-d(a,c) \leq d(a,b) - d(b,c)$. Thus $|d(a,b) - d(b,c)| \leq d(a,c)$. □

Now back to the problem.

Proof. By the lemma, we have $|d(x_n,y) - d(x,y)| \leq d(x,x_n)$. Since $x_n \to x$, $d(x,x_n)$ can be sufficiently small. Thus $d(x_n,y) \to d(x,y)$.

Also by the lemma, we have $|d(x_n,y_n) - d(x,y)| \leq |d(x_n,y_n) - d(x_n,y)| + |d(x_n,y) - d(x,y)| \leq d(y_n,y)+d(x,x_n)$. Similarly, since we can make $d(x_n,x)$ and $d(y_n,y)$ sufficiently small, we have $d(x_n,y_n) \to d(x,y)$. □

Exercise 35. *If $x_n \to x$, then $x_{n_k} \to x$ for any subsequence (x_{n_k}) of (x_n).*

Proof. For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $d(x_n,x) < \epsilon$. Since $\{1,2,\cdots,N\}$ is finite, there exists some $k_0 \in \mathbb{N}$ such that $\{1,2,\cdots,N\} \cap \{n_{k_0+1}, n_{k_0+2}, \cdots\} = \varnothing$. Let $K = k_0$, then if $k > K$, because $n_k > N$, we have $d(x_{n_k},x) < \epsilon$. Thus $x_{n_k} \to x$. □

Exercise 36. *A convergent sequence in Cauchy, and a Cauchy sequence is bounded.*

Proof. Let $x_n \to x$, we will prove that (x_n) is Cauchy. For any $\epsilon > 0$, we can find N such that if $n > N$, then $x_n \in B_{\frac{\epsilon}{2}}(x)$. And because $diam(B_{\frac{\epsilon}{2}}(x)) < 2\frac{\epsilon}{2} = \epsilon$, for any $m,n > N$, $d(x_m,x_n) < \epsilon$. Thus (x_n) is Cauchy.

If (x_n) is Cauchy, then there exists $N \in \mathbb{N}$ such that for any $m,n > N$, $d(x_m,x_n) < 1$. Fix m, let $t = \max(\{d(x_k,x_m) : 1 \leq k \leq N\} \cup \{1\})$. Then clearly $x_n \in B_t(x_m)$ for any $n \in N$. Thus (x_n) is bounded. □

Exercise 37. *A Cauchy sequence with a convergent subsequence converges.*

Proof. Let (x_n) be Cauchy and $x_n \to x$, where (x_{n_k}) is a subsequence of (x_n). For any $\epsilon > 0$, we can find N_1 and N_2 such that for all $m,n > N_1$ and $n_p > N_2$, then $d(x_m,x_n) < \frac{\epsilon}{2}$ and $d(x,x_{n_p}) < \frac{\epsilon}{2}$. Let $N = \max\{N_1,N_2\}$ and a natural number $n_{k_0} > N$. For any $n > N$, we have

$$d(x_n,x_{n_{k_0}}) < \frac{\epsilon}{2} \text{ and } d(x_{n_{k_0}},x) < \frac{\epsilon}{2}.$$

Therefor,

$$d(x_n,x) \leq d(x_n,x_{n_{k_0}}) + d(x_{n_{k_0}},x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus $x_n \to x$. □

Exercise 40. *Given any fixed element $x \in \ell_1$, show that the sequence $x^{(k)} = (x_1, \cdots, x_k, 0, \cdots) \in \ell_1$ converges to x in ℓ_1-norm. Show that the same holds true in ℓ_2, but give an example showing that it fails in ℓ_∞.*

Proof. Since $x \in \ell_1$, we have $\sum_{i=1}^{\infty} |x_i| < \infty$, thus $\lim_{k\to\infty} \sum_{i=k}^{\infty} |x_i| = 0$. Now, for any $\epsilon > 0$, there exists K such that if $k > K$, then

$$||x - x^{(k)}||_1 = \sum_{i=k+1}^{\infty} |x_i| < \epsilon.$$

Thus $x^{(k)} \to x$ in ℓ_1.

Similarly, if $x \in \ell_2$, then $\lim_{k\to\infty}(\sum_{i=k}^{\infty} x_i^2)^{\frac{1}{2}} = 0$. Thus for any $\epsilon > 0$, there exists K such that if $k > K$, then

$$||x - x^{(k)}||_2 = \left(\sum_{i=k+1}^{\infty} x_i^2\right)^{\frac{1}{2}} < \epsilon.$$

Thus $x^{(k)} \to x$ in ℓ_2.

One example showing that ℓ_∞ false is $x = (1, 1, \cdots)$. We have $||x - x^{(k)}||_\infty = \max\{0, 1\} = 1$. Thus no matter how large k is, $d(x, x^{(k)}) = 1$, which means $x^{(k)}$ does not converge to x. □

Exercise 41. *Given $x, y \in \ell_2$, show that if $x^{(k)} \to x$ and $y^{(k)} \to y$ in ℓ_2, then $\langle x^{(k)}, y^{(k)} \rangle \to \langle x, y \rangle$.*

Proof. Assume that for some k_0, $\|x^{(k_0)} - x\|_2 < \epsilon$ and $\|x^{(k_0)} - x\|_2 < \delta$, then applying the Cauchy-Schwarz, we have

$$\begin{aligned}
|\langle x^{(k_0)}, y^{(k_0)} \rangle - \langle x, y \rangle| &= |\langle x^{(k_0)} - x, y^{(k_0)} - y \rangle + \langle x, y^{(k_0)} - y \rangle + \langle y, x^{(k_0)} - x \rangle| \\
&= |\langle x^{(k_0)} - x, y^{(k_0)} - y \rangle| + |\langle x, y^{(k_0)} - y \rangle| + |\langle y, x^{(k_0)} - x \rangle| \\
&\leq \|x^{(k_0)} - x\|_2 \|y^{(k_0)} - y\|_2 + \|x\|_2 \|y^{(k_0)} - y\|_2 + \|y\|_2 \|x^{(k_0)} - x\|_2 \\
&\leq \epsilon\delta + \delta\|x\|_2 + \epsilon\|y\|_2.
\end{aligned}$$

Because $x, y \in \ell_2$, $\|x\|_2$ and $\|y\|_2$ are finite. Moreover, because $x^{(k)} \to x$ and $y^{(k)} \to y$, ϵ and δ can be as small as possible. Therefore $|\langle x^{(k)}, y^{(k_0)} \rangle - \langle x, y \rangle|$ can be as small as possible, thus $\langle x^{(k)}, y^{(k_0)} \rangle \to \langle x, y \rangle$ □

Exercise 42. *Two metrics d and ρ on a set M is said to be equivalent if they generate the same convergent sequences; that is, $d(x_n, x) \to 0$ if and only if $\rho(x_n, x) \to 0$. If d is any metric on M, show that the metrics ρ, σ, τ are all equivalent to d.*

Proof. Assume that there exists x_n and $x \in M$ such that $d(x_n, x) \to 0$, then for any $0 < \epsilon < 1$, there exists N such that if $n > N$, then $d(x_n, x) < \epsilon$. Thus

$$\rho(x_n, x) = \sqrt{d(x_n, x)} < \sqrt{\epsilon} < \epsilon,$$

$$\sigma(x_n, x) = \frac{d(x_n, x)}{1 + d(x_n, x)} < \frac{d(x_n, x)}{1} < \epsilon,$$

and

$$\tau(x_n, x) = \min\{d(x_n, x), 1\} = d(x_n, x) < \epsilon.$$

Therefore, we have $\rho(x_n, x) \to 0, \sigma(x_n, x) \to 0$, and $\tau(x_n, x) \to 0$.

Now assume that $\rho(x_n, x) \to 0$, we have

$$d(x_n, x) = \rho(x_n, x)^2 \to 0.$$

If $\sigma(x_n, x) \to 0$, first, we will calculate $d(x_n, x)$ respect to $\sigma(x_n, x)$. We have

$$\begin{aligned}
\sigma(x_n, x) &= \frac{d(x_n, x)}{1 + d(x_n, x)} \\
\frac{1}{\sigma(x_n, x)} &= \frac{1 + d(x_n, x)}{d(x_n, x)} = 1 + \frac{1}{d(x_n, x)} \\
\frac{1}{\sigma(x_n, x)} - 1 &= \frac{1}{d(x_n, x)} \\
\frac{1}{\frac{1}{\sigma(x_x, x)} - 1} &= d(x_n, x).
\end{aligned}$$

As $\sigma(x_n, x) \to 0$, we have $\frac{1}{\sigma(x_n, x)} \to \infty$, hence $d(x_n, x) = \frac{1}{\frac{1}{\sigma(x_x, x)} - 1} \to 0$. Now, if $\tau(x_n, x) \to 0$, then $\min\{d(x_n, x), 1\} \to 0$, thus $d(x_n, x) \to 0$. $\square$

Exercise 43. *Show that the usual metric on $\mathbb{N}$ is equivalent to the discrete metric. Show that any metric on a finite set is equivalent to the discrete metric.*

Proof. For any $x_n, x \in \mathbb{N}$, if $|x_n - x| \to 0$, then let $\epsilon = \frac{1}{2}$, then there exists N such that if $n > N$, then $|x_n - x| < \frac{1}{2}$, therefore $x_n = x$. Let d be the discrete metric on N, then $d(x_n, x) = 0$ for $n > N$. Thus $d(x_n, x) \to 0$.

If $d(x_n, x) \to 0$, then for $\epsilon < 1$, there exist N such that if $n > N$, then $d(x_n, x) < \epsilon < 1$. Hence $d(x_n, x) = 0$, which means $x_n = x$. Thus $|x_n - x| \to 0$. Thus the usual metric on $\mathbb{N}$ is equivalent to the discrete metric.

Let M be a finite set and d, m defines a discrete metric and a random metric on M. Since metric is positive, let $0 < \epsilon < \min\{m(x, y) : x, y \in M, x \neq y\}$. For $x_n, x \in M$, if $m(x_n, x) \to 0$, then there exists N such that for any $n > N$, then $m(x_n, x) < \epsilon$. But by the definition of ϵ, we have $x_n = x$. Therefore, $d(x_n, x) = 0$ for any $n > N$. So $d(x_n, x) \to 0$.

If $d(x_n, x) \to 0$, then similar to the $\mathbb{N}$ case, there exists N such that for any $n > N$, we have $x_n = x$. Thus $m(x_n, x) = 0$ for $n > N$. Thus $m(x_n, x) \to 0$. $\square$

Exercise 44. *Show that the metrics induced by $\|\cdot\|_1, \|\cdot\|_2$, and $\|\cdot\|_\infty$ on $\mathbb{R}^n$ are all equivalent.*

Proof. Assume that $\|x - x_k\| \to 0$ for some $x, x_k \in \mathbb{R}^n$. By exercise 18, we have $0 \leq \|x - x_k\|_\infty \leq \|x - x_k\|_2 \leq \|x - x_k\|_1$. Thus, we also have $\|x - x_k\|_\infty \to 0$ and $\|x - x_k\|_2 \to 0$.

Now, assume that $\|x - x_k\|_2 \to 0$, then $\sqrt{n}\|x - x_k\|_2 \to 0$. Also by exercise 18, we have $\|x - x_k\| \leq \sqrt{n}\|x - x_k\|_2$. Thus $\|x - x_k\|_1 \to 0$.

Similarly, if $\|x - x_k\|_\infty \to 0$, then $n\|x - x_k\|_\infty \to 0$. By exercise 18, we have $\|x - x_k\|_1 \leq n\|x - x_k\|_\infty$. Thus $\|x - x_k\|_1 \to 0$.

Therefore, $\|\cdot\|_1, \|\cdot\|_2$, and $\|\cdot\|_\infty$ are equivalent. $\square$

Exercise 45. *We say that two norms on the same vector space X are equivalent if the metrics they induce are equivalent. Show that $\| \bullet \|$ and $|||\bullet|||$ are equivalent of X if and only if they generate the same sequences tending to 0; that is, $\|x_n\| \to 0$ if and only if $|||x_n||| \to 0$.*

Proof. Assume that $\|x_n\| \to 0$ if and only if $|||x_n||| \to 0$, we will prove that $\| \bullet \|$ and $|||\bullet|||$ are equivalent. Indeed, if $x_n, x \in X$ such that $\|x_n - x\| \to 0$, then notice that $y_n = x_n - x \in X$, we have $\|y_n\| \to 0$. However, by the assumption, we have $|||y_n||| \to 0$, which is synonymous with $|||x_n - x||| \to 0$. Similarly, if $|||x_n - x||| \to 0$, we also have $\|x_n - x\| \to 0$. Thus $\| \bullet \|$ and $|||\bullet|||$ are equivalent.

If $\| \bullet \|$ and $|||\bullet|||$ are equivalent, then obviously $\|x_n\| \to 0$ if and only if $|||x_n||| \to 0$. $\square$

Exercise 46. *Given two metric spaces (M,d) and (N,p), we can define a metric on the product $M \times N$ in a variety of ways. Our only requirement is that a sequence of pairs (a_n,x_n) in $M \times N$ should converge precisely when both coordinate sequences (a_n) and (x_n) converge. Show that each of the following define metrics on $M \times N$ that enjoy this property and that all three are equivalent:*

$$d_1((a,x),(b,y)) = d(a,b) + p(x,y),$$

$$d_2((a,x),(b,y)) = (d(a,b)^2 + p(x,y)^2)^{1/2},$$

$$d_\infty((a,x),(b,y)) = \max\{d(a,b), p(x,y)\}.$$

Proof. First we will prove that d_1, d_2, d_∞ define metrics on $M \times N$. For any $(a,x),(b,y) \in M \times N$, we have

$$d_1((a,x),(b,y)) = d(a,b) + p(x,y) \geq 0,$$
$$d_2((a,x),(b,y)) = (d(a,b)^2 + p(x,y)^2)^{1/2} \geq 0,$$
$$d_\infty((a,x),(b,y)) = \max\{d(a,b), p(x,y)\} \geq 0.$$

The equality is when $d(a,b) = d(x,y) = 0$. Thus $d_1((a,x),(b,y)) = d_2((a,x),(b,y)) = d_\infty((a,x),(b,y)) = 0$ if any only if $d(a,x) = p(b,y) = 0$. We also have

$$d_1((a,x),(b,y)) = d(a,b) + p(x,y) = d(b,a) + p(y,x) = d_1((b,y),(a,x)),$$

$$d_2((a,x),(b,y)) = (d(a,b)^2 + p(x,y)^2)^{1/2} = (d(b,a)^2 + p(y,x)^{1/2} = d_2((b,y),(a,x)),$$

$$d_\infty((a,x),(b,y)) = \max\{d(a,b), p(x,y)\} = \max\{d(b,a), p(y,x)\} = d_\infty((b,y),(a,x)).$$

Now, for any $i \geq 1$, applying the triangular inequality for d and p, and norm inequality, we have

$$\|(d(a,b), p(x,y))\|_i \leq \|(d(a,c)+d(c,b), p(x,z)+p(z,y))\|_i \leq \|(d(a,c), p(x,z))\|_i + \|(d(c,b), p(z,y))\|_i$$

Thus for $i = 1, 2, \infty$, we have three triangular inequalities in d_1, d_2, and d_∞. Thus d_1, d_2, and d_∞ defines metrics on $M \times N$. Let $(a_n, x_n) \in M \times N$ such that $a_n \to a$ and $x_n \to x$ in (M,d) and (N,p) respectively. Then $d(a_n, a) \to 0$ and $p(x_n, x) \to 0$. Thus we easily have

$$d_1((a_n,x_n),(a,x)) = d(a_n,a) + p(x_n,x) \to 0,$$
$$d_2((a_n,x_n),(a,x)) = (d(a_n,a)^2 + p(x_n,x)^2)^{1/2}, \to 0,$$

and

$$d_\infty((a_n,x_n),(a,x)) = \max\{d(a_n,a), p(x_n,x)\} \to 0.$$

Thus, d_1, d_2, d_∞ enjoy the property of this exercise. Now if $(a_n,x_n) \to (a,x)$ in $(M \times N, d_1$, that is $d_1((a_n,x_n),(a,x)) \to 0$, then we have $d(a_n,a) + p(x_n,x) \to 0$, thus $d(a_n,a)$ and $p(x_n,x)$ converge to 0. Because d_2 defines a metric on $M \times N$, we also have $d_2((a_n,a),(x_n,x)) \to 0$, thus $(a_n,x_n) \to (a,x)$ in $(M \times N, d_2)$. Similarly, if $(a_n,x_n) \to (a,x)$ in $(M \times N, d_2)$, then we have $(a_n,x_n) \to (a,x)$ in $(M \times N, d_\infty)$, and if $(a_n,x_n) \to (a,x)$ in $(M \times N, d_\infty)$, then we have $(a_n,x_n) \to (a,x)$ in $(M \times N, d_1)$. Thus d_1, d_2, and d_3 are equivalent.

□

Lemma 1. *Let $a, b, c, d \in \mathbb{R}+$, if $a \leq c$ and $b \leq d$, then for any $i \geq 1$, we have $\|(a, b)\|_i \leq \|(c, d)\|_i$.*

Proof. Indeed, if $i \geq 1$, then $a^i \leq c^i$ and $b^i \leq d^i$. And since $\frac{1}{i} > 0$, we have

$$\|(a, b)\|_i = (|a|^i + |b|^i)^{1/i} \leq (|c|^i + |d|^i)^{1/i} = \|(c, d)\|_i.$$

Thus $\|(a, b)\|_i \leq \|(c, d)\|_i$. □

Exercise Hint. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ defines by $f(x, y) = xy$, prove that f is continuous.*

Proof. For $(x_0, y_0) \in \mathbb{R}^2$, assume that $|x - x_0| < \epsilon$ and $|y - y_0| < \delta$, then we have

$$\begin{aligned}
|xy - x_0y_0| &= |(x - x_0)(y - y_0) + x_0y + y_0x - 2y_0x_0)| \\
&= |(x - x_0)(y - y_0) + x_0(y - y_0) + y_0(x - x_0)| \\
&\leq |(x - x_0)||(y - y_0)| + |x_0||(y - y_0)| + |y_0||(x - x_0)| \\
&\leq \epsilon\delta + |x_0|\delta + |y_0|\epsilon.
\end{aligned}$$

Because $|xy - x_0y_0|$ can be as small as possible by letting ϵ and δ super small, f is continuous at $(x_0, y_0$. Therefore, f is continuous everywhere. □