**Exercise 3.2.1.**

(a) *The rank of a matrix is equal to the number of its nonzero columns.*

*Proof.* False. The matrix $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ has rank 1. $\square$

(b) *The product of two matrices always has rank equal to the lesser of the ranks of the two matrices.*

*Proof.* False. We have $\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. However, $rank\left(\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right) = 2$ and $rank\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) = 0$ $\square$

(c) *The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.*

(d) *Elementary row operations preserve rank.*

(e) *Elementary column operations do not necessarily preserve rank.*

(f) *The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.*

(g) *The inverse of a matrix can be computed exclusively by means of elementary row operations.*

(h) *The rank of an $n \times n$ matrix having rank $n$ is invertible.*

**Exercise 3.2.14.** *Let $T, U : V \to W$ be linear transformations.*

(a) *Prove that $R(T+U) \subseteq R(T) + R(U)$.*

*Proof.* If $t \in R(T+U)$, then there exists $v \in V$ such that $T(v)+U(v) = t$. Thus $t \in R(T) + R(U)$. Thus $R(T+U) \subseteq R(T) + R(U)$. $\square$

(b) *Prove that if $W$ is finite-dimensional, then $rank(T+U) \leq rank(T) + rank(U)$.*

*Proof.* Since $W$ is finite-dimensional, there exist $n, m$ such that $\{v_1, v_2, \cdots, v_n\}$ is a basis for $R(T)$ and $\{u_1, u_2, \cdots, u_m\}$ is a basis for $R(U)$. Thus $rank(T) = n$ and $rank(U) = m$. Moreover, $\{v_1, v_2, \cdots, v_n, u_1, \cdots, u_m\}$ spans $R(T+U)$. Thus $rank(T+U) \leq m+n = rank(T)+rank(U)$. $\square$

(c) *Deduce from (b) that $rank(A+B) \leq rank(A)+rank(B)$ for any $m \times n$ matrices $A$ and $B$.*

*Proof.* Since a matrix represent a linear transformation, we have $rank(A+B) \leq rank(A) + rank(B)$ for any $m \times n$ matrices $A$ and $B$. $\square$

**Exercise 3.3.1.** *Label the following statements as true or false.*

(a) *Any system of linear equation has at least one solution.*

*Proof.* False, because $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ has no solution. $\square$

(b) *Any system of linear equation has at most one solution.*

*Proof.* False because $0x = 0$ has infinitely many solutions. $\square$

(c) *Any homogeneous system of linear equations has at least one solution.*

*Proof.* True, when all the unknowns equal 0. $\square$

(d) *Any system of $n$ linear equations in $n$ unknowns has at most one solution False, because $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ has infinitely many solutions.*

(e) *Any system of $n$ linear equations in $n$ unknowns has at least one solution.*

*Proof.* False. It can have no solution too. $\square$

(f) *If the homogeneous system corresponding to a given system of linear equations has a solution, then the given system has a solution.*

*Proof.* False. A homogeneous system always have a solution, but system of linear equations is not. $\square$

(g) *If the coefficient matrix of a homogeneous system of $n$ linear equations in $n$ unknowns is invertible, then the system has no nonzero solutions.*

*Proof.* True. If the coefficient matrix is invertible, then the system has exactly one solution and since it is homogeneous, all unknowns should equal to 0. $\square$

(h) *The solution set of any system of $m$ linear equations in $n$ unknowns is a subspace of $F^n$.*

*Proof.* False. This only holds for homogeneous system of linear equations. $\square$

**Exercise 3.3.9.** *Prove that the system of linear equations $Ax = b$ has a solution if and only if $b \in R(L_A)$.*

*Proof.* If $Ax = b$ has a solution, then there exists a vector $v$ such that $Av = b$ or $L_A(v) = b$. Thus $b \in R(L_A)$. Conversely, if $b \in R(L_A)$, then there exists a vector $v$ such that $L_A(v) = b$ or $Av = b$. Thus $Ax = b$ has at least one solution. $\square$

**Exercise 3.4.1.**

(a) *If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary column operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent.*

*Proof.* False. It should be row operations. □

(b) *If $(A'|b')$ is obtained from $(A|b)$ by a finite sequence of elementary row operations, then the systems $Ax = b$ and $A'x = b'$ are equivalent.*

*Proof.* True. □

(c) *If $A$ is an $n \times n$ matrix with rank $n$, then the reduced row echelon form of $A$ is $I_n$.*

*Proof.* True. Because it's rank $n$, there are $n$ 1's in each row. Thus $A$ is $I_n$. □

(d) *Any matrix can be put in reduced row echelon form by means of a finite sequence of elementary row operations.*

*Proof.* True. □

(e) *If $(A|b)$ is in reduced row echelon form, then the system $Ax = b$ is consistent.*

*Proof.* False. Any augmented matrix has a reduced row echelon form, but not all system of linear equations are consistent. □

(f) *Let $Ax = b$ be a system of $m$ linear equations in $n$ unknowns for which the augmented matrix is in reduced row echelon form. If this system is consistent, then the dimension of the solution set of $Ax = 0$ is $n - r$, where $r$ equals the number of nonzero rows in $A$.*

*Proof.* True. The dimension of the solution space i.e. *rank* is the number of nonzero rows of the reduced echelon matrix. □

(g) *If a matrix $A$ is transformed by elementary row operations into a matrix $A'$ in reduced row echelon form, then the number of nonzero rows in $A'$ equals the rank of $A$.*

*Proof.* True. That's what f said. □