Answer to Real Analysis by Carothers

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Chapter 4. Open Sets and Closed Sets

Exercise 3
Some authors say that two metrics d and p on a set M are equivalent if they generate the same open sets. Prove this.

Proof. If d and p generate the same open set in M, then assume that $x_n \to x$ respect to d, we will prove that $x_n \to x$ respect to p. Indeed, for any $\delta > 0$, we have $B^p_\delta(x)$ is an open set in M, thus it is also an open set respect to d. And since x is in that open set, there exists $\epsilon > 0$ such that $B^d_\epsilon(x) \subset B^p_\delta(x)$. But because $x_n \to x$ respect to d, x_n is eventually in $B^d_\epsilon(x) \subset B^p_\delta(x)$. Therefore, $x_n \to x$ respect to p, which means d and p are equivalent. □

Exercise 5
Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\{x : f(x) > 0\}$ is an open subset of $\mathbb{R}$ and that $\{x : f(x) = 0\}$ is a closed subset of $\mathbb{R}$.

Proof. Assume that $f(x) > 0$ for some x, then because f is continuous, there exists $\delta > 0$ such that for any $y \in B_\delta(x)$, we have $f(y) > 0$. Thus $B_\delta(x) \in \{x : f(x) > 0\}$, which implies $\{x : f(x) > 0\}$ to be an open set. Similarly, we have $\{x : f(x) < 0\}$ is also an open set, which means

$$\{x : f(x) = 0\} = \mathbb{R} \setminus (\{x : f(x) > 0\} \cup \{x : f(x) < 0\})$$

is a close set. □

Exercise 7
Show that every open set in $\mathbb{R}$ is the union of (countably many) open intervals with rational endpoints. Use this to show that the collection U of all open subsets of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$ itself.

Proof. First, we will prove that for any open interval (a, b), $a, b \in \mathbb{R}$, there is countably many rational endpoint interval whose union is (a, b). Indeed, there exists an increasing sequence of rational numbers $b_n \to b$ and a decreasing sequence of rational numbers $a_n \to a$. Clearly, we have $\cup_{n=1}^{\infty}(a_n, b_n) = (a, b)$.

Therefore, by theorem 4.6, if M is an open set on $\mathbb{R}$, then M can be broken into countably many disjoint interval. We continue to break each interval into countably many unions of rational endpoint intervals. Thus any open set on $\mathbb{R}$ can be written as a union of countably many rational endpoint intervals.

Notice that the cardinality of (a, b) where $a, b \in \mathbb{Q}$ is $card(\mathbb{Q} \times \mathbb{Q}) = card(\mathbb{N}) = \aleph_0$. Therefore, the collection U of all open subsets of $\mathbb{R}$ has the cardinality equals $card(\mathcal{P}(\mathbb{N})) = card(\mathbb{R})$. Thus the two sets have the same cardinality. $\square$

Exercise 8
Show that every open interval (and hence every open set) in $\mathbb{R}$ is a countable union of closed intervals and that every closed interval in $\mathbb{R}$ is a countable intersection of open intervals.

Proof. Let (a, b) be an open interval in $\mathbb{R}$, there exists an increasing sequence (b_n) and a decreasing function (a_n) such that $b_n \to b$ and $a_n \to a$. And since $a < b$, there exists n_0 such that $a_n < b_n$ for any $n > n_0$. Therefore, without loss of generality, we can assume that $a_n < b_n$ for all n. We will claim that $\cup_{n\in\mathbb{N}}[a_n, b_n] = (a, b)$. Indeed, since $[a_n, b_n] \subset (a, b)$ for all n, we have $\cup_{n\in\mathbb{N}}[a_n, b_n] \subset (a, b)$. Now for any $x \in (a, b)$, there exists m such that $a_m < x < b_m$. Thus $x \in [a_m, b_m] \in \cup_{n\in\mathbb{N}}[a_n, b_n]$, which means $(a, b) \subset \cup_{n\in\mathbb{N}}[a_n, b_n]$. For that reason, $\cup_{n\in\mathbb{N}}[a_n, b_n] = (a, b)$.

Now, for any closed interval $[a, b]$, let a_n, b_n be increasing and decreasing sequences respectively, such that $a_n \to a$ and $b_n \to b$. We claim that $\cap_{n\in\mathbb{N}}(a_n, b_n) = [a, b]$. Well, it's kinda obvious, the proof is similar to the previous case. $\square$

Before doing exercise 10, we first prove a little lemma.

Lemma 1. *For any $x, z \in H^\infty$, if $d(x, z) < 2^{-N}t$, then $|x_k - z_k| < t$ for all $k = 1, \cdots, N$.*

Proof. notice that for any $z \in H^\infty$, we have

$$d(x, z) = \sum_{n=1}^{\infty} 2^{-n}|x_n - z_n| = \sum_{n=1}^{N} 2^{-n}|x_n - z_n| + \sum_{n=N+1}^{\infty} 2^{-n}|x_n - z_n|.$$

Because $|x_n - z_n| \geq 0$, we have

$$\sum_{n=1}^{N} 2^{-n}|x_n - z_n| \leq \sum_{n=1}^{N} 2^{-n}|x_n - z_n| + \sum_{n=N+1}^{\infty} 2^{-n}|x_n - z_n| = d(x, z) \leq 2^{-N}t.$$

Therefore, $2^{-k}|x_k - y_k| < 2^{-N}t$ for any $k = 1, \cdots, N$. That is $|x_k - y_k| < 2^{k-N}t$. But $k \leq N$, hence $2^{k-N} \leq 1$, which implies $|x_k - y_k| < t$ for all $k = 1, \cdots, N$. $\square$

Exercise 10
Given $y = (y_n) \in H^\infty, N \in \mathbb{N}$, and $\epsilon > 0$, show that $\{x = (x_n) \in H^\infty : |x_k - y_k| < \epsilon, k = 1, \cdots, N\}$ is open in H^∞.

Proof. For any $x \in H^\infty$, we will prove that there exists δ such that $B_\delta(x) \in S = \{x = (x_n) \in H^\infty : |x_k - y_k| < \epsilon, k = 1, \cdots, N\}$, so we can conclude that S is open. Indeed, by the assumption, we have $x \in S$, therefore $|x_k - y_k| < \epsilon$ for $k = 1, \cdots N$, which implies $M = \max\{|x_i - y_i| : i = 1, \cdots, N\} < \epsilon$. Using the density of real number, there exists $t > 0$ such that $M + t < \epsilon$. Now let $\delta = 2^{-N}t$, then for any $z \in H^\infty \cap B_\delta(x)$, we have $d(x, z) < 2^{-N}t$. By Lemma 1, we conclude that $|x_k - z_k| \leq t$ for all $k = 1, \cdots, N$. Notice that for such k, using the triangular inequality, we have

$$|z_k - y_k| \leq |z_k - x_k| + |x_k - y_k| < t + M < \epsilon.$$

Thus, $z \in S$, which implies $B_\delta(x) \in S$. That is S indeed open. $\square$

Exercise 11
Let $e^{(k)} = (0, \cdots, 0, 1, 0, \cdots)$, where the kth entry is 1 and the rest are 0s. Show that $\{e^{(k)} : k \geq 1\}$ is closed as a subset of ℓ_1.

Proof. One thing to notice is that for any $m, n \in \mathbb{N}$, we have

$$\|e^{(m)} - e^{(n)}\|_1 = \sum_{i=1}^{\infty} |e_i^{(m)} - e^{(n)}| = 2$$

whenever $m \neq n$. Back to the problem, assume that there exists $(x_n) \to a$ for some $x_n \in \{e^{(k)} : k \geq 1\}$. It is sufficient to prove that $a \in \{e^{(k)} : k \geq 1\}$. Indeed, by the definition of convergence, there exists $N \in \mathbb{N}$ such that $x_n \in B_{\frac{1}{2}}(a)$ for all $n \geq N$. But then, for any $m, n > N$, we have

$$\|x_m - x_n\|_1 \leq \|x_m - a\|_1 + \|a - x_n\|_1 \leq \frac{1}{2} + \frac{1}{2} = 1,$$

which implies $e^{(m)} = e^{(n)}$. Therefore, $e^{(n)}$ is a constant when $n \geq N$. That is $a = e^{(N)} \in \{e^{(k)} : k \geq 1\}$. $\square$

Exercise 12
Let F be the set of all $x \in \ell_\infty$ such that $x_n = 0$ for all but finitely many n. Is F closed? open? neither? Explain.

Proof. First, notice that $0 \in F$, but for any $\epsilon > 0$, we have $t = (\epsilon, \epsilon, \cdots)$ where $\|t - 0\|_\infty = \epsilon$, that is $t \in B_\epsilon(0)$. However, clearly $t \notin F$. So F is not open.

Second, let $x^{(i)} = (1 - \frac{1}{i}, \frac{1}{2} - \frac{1}{i}, \cdots, \frac{1}{i} - \frac{1}{i}, 0, 0, \cdots)$ and $a = (1, \frac{1}{2}, \cdots)$. For any $\epsilon > 0$, we can find $N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then, for $n > N$, we have

$$\|a - x^{(n)}\|_\infty = \left\| \left(\frac{1}{n}, \cdots, \frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \cdots \right) \right\|_\infty = \frac{1}{n} < \frac{1}{N} < \epsilon.$$

Thus $x^{(i)} \to a$. But by the definition of $x^{(i)}$ and a, we have $x^{(i)} \in F$ but $a \notin F$. Therefore F is not closed.

So F is neither closed or open. $\square$

Exercise 13
Show that c_0 is a closed subset of ℓ_∞

Proof. We will prove that $\ell_\infty \setminus c_0$ is an open set. For any $x \in \ell_\infty \setminus c_0$, we get $x \notin c_0$. Remind that $x \in c_0$ means for any $\delta > 0$, there exists $N > 0$ such that for all $n > N$, we have $|x_n| < \delta$. Therefore, $x \notin c_0$ means exists $\delta > 0$ such that for any $N > 0$, there exists $n > N$ so that $|x_n| > \delta$.

We will claim that $B_{\delta/2}(x) \cap c_0 = \varnothing$, thus $B_{\delta/2}(x) \in \ell_\infty \setminus c_0$, which leads to $\ell_\infty \setminus c_0$ be an open set.

Indeed, if $y \in B_{\delta/2}(x) \cap c_0$, then because $y \in c_0$, there exists N' such that $|y_n| < \frac{\delta}{2}$ for any $n > N'$. And because $y \in B_{\delta/2}(x)$, we get $\max\{|y_n - x_n| : n \in \mathbb{N}\} < \frac{\delta}{2}$. Thus,

$$|x_n| \leq |y_n - x_n| + |y_n| < \frac{\delta}{2} + \frac{\delta}{2} = \delta,$$

for any $n > N'$, which contradicts to the fact that there exists $n > N'$ such that $|x_n| > \delta$. So there is no such y. $\square$

Exercise 14
Show that the set $A = \{x \in \ell_2 : |x_n| \leq 1/n,\ n = 1, 2, \cdots\}$ is a closed set in ℓ_2 but that $B = \{x \in \ell_2 : |x_n| < 1/n,\ n = 1, 2, \cdots\}$ is not an open set.

Proof. Assume that $x^{(k)} \in A$ and $\|x^{(k)}\|_2 \to \|x\|_2$, then $|x_n^{(k)}| \to |x_n|$ for any $n \in \mathbb{N}$. Since $x^{(k)} \in A$, we have $|x_n^{(k)}| \leq \frac{1}{n}$ for all k, hence $|x_n| \leq \frac{1}{n}$ too. Thus $x \in A$, which implies A is a closed set.

Notice that $0 \in B$. For any $\epsilon > 0$, there exists $0 < \delta < \epsilon$ and $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta$. Let $a = (0, \cdots, \delta, 0, \cdots)$, that is $a_n = \delta$ and 0 everywhere else. Since $\|a\|_2 = \delta < \epsilon$, we have $a \in B_\epsilon(0)$. However, because $a_n = \delta > \frac{1}{n}$, we have $a_n \notin B$. Thus for any $\epsilon > 0$, we have $B_\epsilon(0) \not\subset B$. That is, B is not an open set. □

Exercise 15
The set $A = \{y \in M : d(x, y) \leq r\}$ is sometimes called the closed ball about x of radius r. Show that A is a closed set, but give an example showing that A need not equal the closure of the open ball $B_r(x)$.

Proof. We will prove that for any $a \in A$, $B_\epsilon(a) \cap A \neq \varnothing$ for all $\epsilon > 0$ implies $a \in A$. Indeed, if $a \notin A$, then $d(x, a) > r$. Let $\delta > 0$ such that $d(x, a) > r - \delta$, then $B_\delta(a) \cap A = \varnothing$. This is because if $b \in B_\delta(a) \cap A$, then

$$d(a, b) < \delta \text{ and } d(b, x) \leq r.$$

But

$$r + \delta < d(x, a) \leq d(a, b) + d(b, x) < \delta + r,$$

contradiction! Thus A is actually a close set. However, A need not equal the closure of $B_r(x)$. For example, define $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Let $r = 1$, then the closure of $B_1(x)$ is $\{x\}$ and it's not equal $\{y \in M : d(x, y) \leq 1\}$, which is M. □

Exercise 16
If $(V, \|\bullet\|)$ is any normed space, prove that the close ball $\{x \in V : \|x\| \leq 1\}$ is always the closure of the open ball $\{x \in V : \|x\| < 1\}$.

Proof. Let C be the closure of $\{x \in V : \|x\| < 1\}$. By exercise 15, we know that $A = \{x \in V : \|x\| \leq 1\}$ is a closed set. Thus $C \subset A$. Moreover, for any $x \in A$, we have $\|x\| \leq 1$. If $x < 1$, then $x \in C$. If $\|x\| = 1$, then let $x_n = \frac{n-1}{n}x$. Because

$$\|x - x_n\| = \left\|\frac{1}{n}x\right\| = \left|\frac{1}{n}\right| \cdot \|x\| = \left|\frac{1}{n}\right| \to 0,$$

we get $x_n \to x$. Moreover, because $\|x_n\| = \left\|\frac{n-1}{n}x\right\| = \left|\frac{n-1}{n}\right| \cdot \|x\| = \left|\frac{n-1}{n}\right| < 1$, we get $x_n \in \{x \in V : \|x\| < 1\}$ for any $n \in \mathbb{N}$. By Proposition 4.10, we get $x \in C$. So in any case, if $x \in A$ then $x \in C$. Thus $A \subset C$. Therefore, $A = C$. □

Exercise 17
Show that A is open if and only if $A^o = A$ and that A is closed if and only if $\bar{A} = A$.

Proof. If A is open, then because A^o is the largest open set contained in A, we must have $A^o = A$. If $A^o = A$, then because A^o is an open set, A must be open too. If A is closed, then because $\bar{A}$ is the smallest closed set containing A, we get $\bar{A} = A$. If $\bar{A} = A$, then because $\bar{A}$ is a closed set, we get A must be closed. □

Exercise 18
Given a nonempty bounded subset E of $\mathbb{R}$, show that $\sup E$ and $\inf E$ are elements of $\bar{E}$. Thus $\sup E$ and $\inf E$ are elements of E whenever E is closed.

Proof. For any nonempty subset E of $\mathbb{R}$, there exists $x_n, y_n \in E$ such that $x_n \to \sup E$ and $y_n \to \inf E$. Therefore, $\sup E$ and $\inf E$ are in $\bar{E}$. □

Exercise 19
Show that $diam(A) = diam(\bar{A})$.

Proof. Because $A \in \bar{A}$, we have $\{d(a,b) : a,b \in A\} \subset \{d(a,b) : a,b \in \bar{A}\}$. Thus $diam(A) = \sup\{d(a,b) : a,b \in A\} \leq \sup\{d(a,b) : a,b \in \bar{A}\} = diam(\bar{A})$. If $diam(A) < diam(\bar{A})$, then there exists $a', b' \in \bar{A}$ so that $d(a',b') > diam(A)$. However, because $a', b' \in \bar{A}$, there exists $a_n, b_n \in A$ such that $a_n \to a$ and $b_n \to b$. Therefore $d(a_n, b_n) \to d(a',b')$, which implies $d(a',b') \leq \sup\{d(a_n,b_n) : n \in \mathbb{N}\}$. So $d(a',b') \leq \sup\{d(a_n,b_n) : n \in \mathbb{N}\} \leq \sup\{d(a,b) : a,b \in A\} = diam(A)$. But this is contradict to the fact that $d(a',b') > diam(A)$. Thus $diam(A) = diam(\bar{A})$. □

Exercise 20
If $A \subset B$, show that $\bar{A} \subset \bar{B}$. Does $\bar{(}A) \subset \bar{(}B)$ imply $A \subset B$? Explain.

Proof. Assume that $A \subset B$, for any $a \in \bar{A}$, there exists $a_n \in A$ such that $a_n \to a$. But $A \subset B$, thus $a_n \in B$ and $a_n \to a$ implies $a \in \bar{B}$. Therefore, $\bar{A} \subset \bar{B}$. The opposite direction, however, is not true. Let $A = [0,1] \subset \mathbb{R}$ and $B = (0,1) \subset \mathbb{R}$, we have $\bar{A} = [0,1] \subset [0,1] = \bar{B}$, but $A \not\subset B$. □

Exercise 21
If A and B are any sets in M, show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. Given an example showing that this last inclusion can be proper.

Proof. Since $A \subset A \cup B$, we have $\overline{A} \subset \overline{A \cup B}$. Similarly, we get $\overline{B} \subset \overline{A \cup B}$. Thus $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. For any $x \in \overline{A \cup B}$, we have $B_\epsilon(x) \cap (A \cup B) = \varnothing$ for any $\epsilon > 0$. If $B_\epsilon(x) \cap A \neq \varnothing$ for all $\epsilon > 0$, then $x \in \overline{A} \subset \overline{A} \cup \overline{B}$. If there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \cap A = \varnothing$, then $0 < \delta < \epsilon_0$ implies $B_\delta(x) \cap A = \varnothing$, thus $B_\delta(x) \cap B \neq \varnothing$ (otherwise $B_\delta(x) \cap (A \cup B) = \varnothing$, contradiction). So $B_\delta(x) \cap B = \varnothing$ for any $\delta > 0$, which is synonymous with $x \in \overline{B} \subset \overline{A} \cup \overline{B}$. Hence $\overline{A \cup B} \subset (\overline{A} \cup \overline{B})$. Because $(\overline{A} \cup \overline{B}) \subset \overline{A \cup B}$ and $\overline{A \cup B} \subset (\overline{A} \cup \overline{B})$, we get $\overline{A} \cup \overline{B} = \overline{A \cup B}$.

Because $A \cap B \subset A$, we get $\overline{A \cap B} \subset \overline{A}$. Similarly, we get $\overline{A \cap B} \subset \overline{B}$. Thus $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$. This can be proper for example, let $A = (2,3), B = (3,4)$, then $\overline{A \cap B} = \overline{\varnothing} = \varnothing$. But $\overline{A} \cap \overline{B} = [2,3] \cap [3,4] = \{3\}$. Thus $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$. $\square$

Exercise 22
True or false? $(A \cup B)^o = A^o \cup B^o$.

Proof. This is false. A counter example is for $A = [0,1]$ and $B = [1,2]$, we have $(A \cup B)^o = [0,2]^o = (0,2)$. However, $A^o \cup B^o = (0,1) \cup (1,2) \neq (0,2)$. $\square$

Exercise 24
Show that $\bar{A} = (int(A^c))^c$ and that $A^o = (cl(A^c))^c$.

Proof. Remind that this exercise is set in a generic metric space (M, d). For the first equation, we will prove that $\overline{A} \cap int(A^c) = \varnothing$ and $\overline{A} \cup int(A^c) = M$. If $\overline{A} \cap int(A^c) \neq \varnothing$, let $a \in \overline{A} \cap int(A^c)$, because $x \in int(A^c)$, there exists $\epsilon > 0$ such that $B_\epsilon(a) \subset A^c$. Thus $B_\epsilon(a) \cap A = \varnothing$. But $a \in \overline{A}$ so for any $\epsilon > 0$, $B_\epsilon(a) \cap A \neq \varnothing$, contradiction. Thus $\overline{A} \cap int(A^c) = \varnothing$. For any $x \in M$, if $x \notin int(A^c)$, we will prove that $x \in \overline{A}$. By the definition, $x \in int(A^c)$ means for any $\epsilon > 0$, $B_\epsilon(x) \not\subset A^c$, that is $B_\epsilon(x) \cap A \neq \varnothing$, so $x \in \overline{A}$. Hence $\overline{A} = (int(A^c))^c$.

For the second equation, we will prove that $A^o \cap cl(A^c) = \varnothing$ and $A^o \cup cl(A^c) = M$. If $A^o \cap cl(A^c) \neq \varnothing$, then there exists $x \in A^o \cap cl(A^c)$. Because $x \in cl(A^c)$, we have $B_\epsilon(x) \cap A^c \neq \varnothing$ for any $\epsilon > 0$. Thus $B_\epsilon(x) \not\subset A$ for all $\epsilon > 0$, which implies $x \notin A^o$, contradiction. Therefore, $A^0 \cap cl(A^c) = \varnothing$. Next, for any $x \in M$, if $x \notin cl(A^c)$, we will prove that $x \in A^o$. Indeed, by the definition, $x]incl(A^c)$ if $B_\epsilon(x) \cap A^c \neq \varnothing$ for all $\epsilon > 0$. Thus $x \notin cl(A^c)$ if there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \cap A^c = \varnothing$. That is $B_{\epsilon_0} \subset A$, which implies $x \in A^o$. So $A^o \cup cl(A^c) = M$. Hence $A^o = (cl(A^c))^c$. □

Exercise 26
We define the distant from a point $x \in M$ to a nonempty set A in A by $d(x, A) = \inf\{d(x, a) : a \in A\}$. Prove that $d(x, A) = 0$ if and only if $x \in \bar{A}$.

Proof. If $x \in \overline{A}$, then there exist $x_n \in A$ and $x_n \to x$ for $n \in \mathbb{N}$. Therefore, $d(x_n, x) \to 0$ by the definition of convergence of sequences. Notice that $\{d(x, x_n) : n \in \mathbb{N}\} \subset \{d(x, a) : a \in A\}$, hence

$$0 \leq d(x, A) = \inf\{d(x, A) : a \in A\} \leq \inf\{d(x_n, x) : n \in \mathbb{N}\} = 0.$$

So $d(x, A) = 0$. If $d(x, A) = \inf\{d(x, a) : a \in A\} = 0$, then there exist $x_n \in A$ such that $d(x, x_n) \to \inf\{d(x, a) : a \in A\} = 0$. Therefore, $x_n \to x$, which implies $x \in \overline{A}$. □

Exercise 27
Show that $|d(x, A) - d(y, A)| \leq d(x, y)$ and conclude that the map $x \mapsto d(x, A)$ is continuous.

Proof. Without loss of generality, assume that $d(x, A) \geq d(y, A)$. Then $|d(x, A) - d(y, A)| \leq d(x, y)$ is synonymous with $d(x, A) - d(y, A) \leq d(x, y)$, or $d(x, A) \leq d(x, y) + d(y, A)$. Notice that for any $a \in A$, we have $d(x, a) \leq d(x, y) + d(y, a)$, therefore,

$$\begin{aligned} d(x, A) = \inf\{d(x, a) : a \in A\} &\leq \inf\{d(x, y) + d(y, a) : a \in A\} \\ &= d(x, y) + \inf\{d(y, a) : a \in A\} \\ &= d(x, y) + d(y, A). \end{aligned}$$

So $|d(x, A) - d(y, A)| \leq d(x, y)$. Now for any sequence $x_n \to x$, we have $d(x_n, x) \to 0$. But $|d(x_n, A) - d(x, A)| \leq d(x, y)$, thus $|d(x_n, A) - d(x, A)| \to 0$. So $d(x_n, A) \to d(x, A)$, which implies the map $x \mapsto d(x, A)$ to be continuous. □

Exercise 28
Given a set A in M and $\epsilon > 0$, show that $\{x \in M : d(x, A) < \epsilon\}$ is an open set and that $\{x \in M : d(x, A) \leq \epsilon\}$ is a closed set (and each contains A).

Proof. We will prove that $O = \{x \in M : d(x, A) < \epsilon\}$ is an open set. For any $x \in O$, since $d(x, A) < \epsilon$, there exists $\delta > 0$ such that $d(x, A) + \delta < \epsilon$. We will claim that $B_\delta(x) \subset O$. Indeed, if $y \in B_\delta(x)$, then $d(y, x) < \delta$. By exercise 27, we have

$$d(y, A) \leq d(y, x) + d(x, A) < \delta + d(x, A) < \epsilon.$$

So $y \in O$, which implies O to be an open set. Next we will prove that $C = \{x \in M : d(x, A) \leq \epsilon\}$ is a closed set. The proof is by showing that if $x \notin C$, then there exists $\delta > 0$ such that $B_\delta(x) \cap C = \varnothing$. Indeed, because $x \notin C$, we have $d(x, A) > \epsilon$, thus there exists $\delta > 0$ such that $d(x, A) > \epsilon + \delta$. We will claim that $B_\delta(x) \cap C = \varnothing$ because if $B_\delta(x) \cap C \neq \varnothing$, let $y \in B_\delta(x) \cap C$, then we get $d(x, y) < \delta$ and $d(y, A) \leq \epsilon$. So

$$\epsilon + \delta < d(x, A) \leq d(x, y) + d(y, A) < \delta + \epsilon.$$

Contradiction! Therefore, C is a closed set. $\square$

Exercise 29
Show that every closed set in M is the intersection of countably many open sets and that every open set in M is the union of countably many closed sets.

Proof. If A is a closed set, then we will claim that $A = \cap_{n=1}^{\infty}\{x \in M : d(x, A) < \frac{1}{n}\}$. Indeed, for any $n \in \mathbb{N}$, we have $A \subset \{x \in M : d(x, A) < \frac{1}{n}\}$. Thus $A \subset \cap_{n=1}^{\infty}\{x \in M : d(x, A) < \frac{1}{n}\}$. Moreover, if $a \in \cap_{n=1}^{\infty}\{x \in M : d(x, A) < \frac{1}{n}\}$, then for any $\epsilon > 0$, $B_\epsilon(a) \cap A \neq \varnothing$. Indeed, if $B_\epsilon(a) \cap A = \varnothing$, then $d(a, A) > \epsilon > \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$. Contradiction to $a \in \cap_{n=1}^{\infty}\{x \in M : d(x, A)\} \subset \{x \in M : d(x, A) < \frac{1}{n_0}\}$. Thus a is in the closure of A, which is A itself since A is a closed set. Thus $\cap_{n=1}^{\infty}\{x \in M : d(x, A) < \frac{1}{n}\} \subset A$. Therefore, $A = \cap_{n=1}^{\infty}\{x \in M : d(x, A) < \frac{1}{n}\}$. By exercise 28, we know that $\{x \in M : d(x, A) < \frac{1}{n}\}$ are open sets for all $n \in \mathbb{N}$, thus every closed set in M is the intersection of countably many open sets.

If A is an open set, then we will claim that $A = \cup_{n=1}^{\infty}\{x \in M : d(x, A^c) \geq \frac{1}{n}\}$. For any $n \in \mathbb{N}$, we get $\{x \in M : d(x, A^c) \geq \frac{1}{n}\} \subset \{x \in M : d(x, A^c) > 0\}$, which is the set of $x \in M$ and $x \notin A^c$. Therefore, $\{x \in M : d(x, A^c) \geq \frac{1}{n}\} \cap A^c = \varnothing$, which implies $\{x \in M : d(x, A^c) \geq \frac{1}{n}\} \subset A$ for all $n \in \mathbb{N}$. Thus $\cup_{n=1}^{\infty}\{x \in M : d(x, A^c) \geq \frac{1}{n}\} \subset A$. Moreover, for any $a \in A$, then because A is an open set, there exists $\epsilon > 0$ such that $B_\epsilon(a) \subset A$. Thus $B_\epsilon(a) \cap A^c = \varnothing$, which implies $d(a, A^c) \geq \epsilon > \frac{1}{n_0}$ for some $n_0 \in \mathbb{N}$. So $a \in \cup_{n=1}^{\infty}\{x \in M : d(x, A^c) \geq \frac{1}{n}\}$ for any $a \in A$, that is $A \subset \cup_{n=1}^{\infty}\{x \in M : d(x, A^c) \geq \frac{1}{n}\}$. Thus $A = \cup_{n=1}^{\infty}\{x \in M : d(x, A^c) \geq \frac{1}{n}\}$. Also by exercise 28, we have $\{x \in M : d(x, A^c) \geq \frac{1}{n}\}$ to be a closed set, thus every open set in M is the union of countably many closed set. $\square$

> **Exercise 32**
> We define the distance between two (nonempty) subsets A and B of M by $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Give an example of two disjoint closed sets A and B is $\mathbb{R}^2$ with $d(A, B) = 0$.

Proof. Let d be the Euclidean distance, $A = \{(x, y) \in \mathbb{R}^2 : x, y > 0; y \geq \frac{1}{x}\}$, and $B = \{(x, y) \in \mathbb{R}^2 : y \leq 0\}$. We will prove that A and B are two disjoint closed sets and $d(A, B) = 0$.

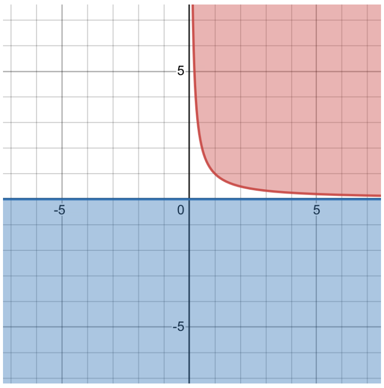

Figure 1: Set A (red) and B (blue).

If $(a_n, b_n) \in B$ and $(a_n, b_n) \to (a, b)$ for all $n \in \mathbb{N}$, then we have $b_n \to b$. Since $b_n \geq 0$ for all $n \in \mathbb{N}$, we must have $b \geq 0$. Therefore, $(a, b) \in B$, which implies B to be a closed set.

Similarly, for any $(a, b) \in \mathbb{R}^2$, if $(a_n, b_n) \in A$ and $(a_n, b_n) \to (a, b)$ for all $n \in \mathbb{N}$, then $a_n \to a$ and $b_n \to b$. We will now prove that $a \neq 0$, thus $\frac{1}{a_n} \to \frac{1}{a}$. Indeed, because $b_n \to b > 0$, there exists $\delta > 0$ such that b_n will eventually in $(b - \delta, b + \delta)$. Thus b_n will eventually smaller than $b + \delta$. That is when n is big enough, because $b_n \geq \frac{1}{a_n}$, we get

$$a_n \geq \frac{1}{b_n} \geq \frac{1}{b + \delta}.$$

Since $a_n \to a$, we also get

$$a \geq \frac{1}{b + \delta} > 0.$$

Hence $\frac{1}{a_n} \to \frac{1}{a}$.

We then prove that $b \geq \frac{1}{a}$ too. Indeed, if $b < \frac{1}{a}$, then there exists $\epsilon > 0$ such that $b - \frac{1}{a} < -\epsilon < 0$. Because $b_n \to b$ and $\frac{1}{a_n} \to \frac{1}{a}$, there exists n_0 big enough such that $|b_{n_0} - b| < \frac{\epsilon}{2}$ and $|\frac{1}{a} - \frac{1}{a_{n_0}}| < \frac{\epsilon}{2}$. Thus $b_{n_0} - b < \frac{\epsilon}{2}$ and $\frac{1}{a} - \frac{1}{a_{n_0}} < \frac{\epsilon}{2}$. But then we get a contradiction because

$$b_{n_0} - \frac{1}{a_{n_0}} = (b_{n_0} - b) + \left(b - \frac{1}{a}\right) + \left(\frac{1}{a} - \frac{1}{a_{n_0}}\right) < \frac{\epsilon}{2} - \epsilon + \frac{\epsilon}{2} = 0$$

and $(a_{n_0}, b_{n_0}) \in A$ so $b_{n_0} - \frac{1}{a_{n_0}} \geq 0$. Therefore, A is also a closed set. Since it's pretty clear that $A \cap B = \varnothing$, it's sufficient to prove that $d(A, B) = 0$. Indeed, let $x_n = (n, \frac{1}{n}) \in A$

and $y_n = (n, 0) \in B$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\le \inf\{d(a,b) : a \in A, b \in B\} \\ &\le \inf\{d(x_n, y_n) : n \in \mathbb{N}\} \\ &= \inf\{\frac{1}{n} : n \in \mathbb{N}\} \\ &= 0. \end{aligned}$$

Hence, A and B are two disjoint closed set and $d(A, B) = 0$. $\square$

33.

Proof. Assume that x is a limit point, for any $\epsilon > 0$, we will prove that $B_\epsilon(x)$ has infinite number of points. Indeed, because x is a limit point, there exists $x_1 \in B_\epsilon(x) \setminus \{x\}$. Let $0 < \epsilon_1 < d(x_1, x)$, then because $x_1 \in B_\epsilon(x)$, we get $\epsilon_1 < d(x_1, x) < \epsilon$. Thus $B_{\epsilon_1}(x) \subset B_\epsilon(x)$ and $x_1 \notin B_{\epsilon_1}(x)$. Therefore, there exists $x_2 \neq x_1$ such that $x_2 \in B_{\epsilon_1}(x) \setminus \{x\}$. In general, if $x_n \in B_{\epsilon_{n-1}}(x)$, define $0 < \epsilon_n < d(x_n, x) < \epsilon_{n-1}$. Then $x_k \notin B_{\epsilon_n}(x)$ for all $k = 1, \cdots, n$ because

$$\epsilon_n < d(x_n, x) < \epsilon_{n-1} < d(x_{n-1}, x) < \cdots < d(x_1, x).$$

Since x is a limit point, let $x_{n+1} \in B_{\epsilon_n}(x) \setminus \{x\}$. So, we have construct a sequence of distinct elements x_n and $x_n \in B_\epsilon(x)$ for all $n \in \mathbb{N}$. Therefore, every neighborhood of x contains infinitely many points of A. $\square$

34.

Proof. If x is a limit point, then $B_{\frac{1}{n}}(x) \setminus \{x\}$ is nonempty for all $n \in \mathbb{N}$. Therefore, let $x_n \in B_{\frac{1}{n}}(x) \setminus \{x\}$. Because $\frac{1}{n} \to 0$, we get $x_n \to x$ and $x_n \neq x$ for all $n \in \mathbb{N}$. If there exists a sequence $x_n \to x$ and $x_n \neq x$ for all x, then for any $\epsilon > 0$, by the definition of convergence, x_n will eventually in $B_\epsilon(x)$. Therefore, $B_\epsilon(x) \setminus \{x\} \neq \varnothing$, which means x is a limit point. $\square$

36.

Proof. Let $a_n \in A$ and $a_n \to a$, we will prove that $a \in A$. If $a = x$, then done. If $a \neq x$, let $0 < \epsilon < d(a, x)$. Because $x_n \to x$, we get $x_n \in B_\epsilon(x)$ for all but finitely many points. Let $0 < \delta < d(a, x) - \epsilon$, then $\delta + \epsilon < d(a, x)$, which implies $B_\epsilon(x)$ and $B_\delta(a)$ are distinct. That is $B_\epsilon(x) \cap B_\delta(a) = \varnothing$. Therefore, $\{a_n : a_n \in B_\delta(a)\}$ has finitely many distinct values, which means there exists $m = \min\{d(a_n, a) : a_n \in B_\delta(a), d(a_n, a) > 0\}$. Then $B_m(a)$ has only elements equal a_n, which implies $a = a_k = x_h$ for some $k, h \in \mathbb{N}$. Therefore, $a \in A$. We get A is a closed set. $\square$

40.

Proof. By definition, x is a limit point if for all $\epsilon > 0$, we have $(B_\epsilon(x) \setminus \{x\}) \cap A \neq \varnothing$. Therefore, x is **not** a limit point if there exists $\epsilon > 0$ such that $(B_\epsilon(x) \setminus \{x\}) \cap A = \varnothing$.

Let $A \subset \mathbb{R}$, we need to prove that A has at most countably many isolated points. For any isolated point a in A, by the definition, there exists ϵ_a such that $(B_{2\epsilon_a}(a) \setminus \{a\}) \cap A = \varnothing$. Then for any two isolated point a, b in A, we will claim that $B_{\epsilon_a}(a) \cap B_{\epsilon_b}(b) = \varnothing$. Indeed, if $k \in B_{\epsilon_a}(a) \cap B_{\epsilon_b}(b)$, then

$$d(a,b) \leq d(a,k) + d(k,b) < \epsilon_a + \epsilon_b < 2\max\{\epsilon_a, \epsilon_b\}.$$

Without loss of generality, assume that $\max\{\epsilon_a, \epsilon_b\} = \epsilon_a$, then the equation above gives $d(a,b) < 2\epsilon_a$, which implies $b \in B_{2\epsilon_a}(a)$, contradiction. Therefore, $B_{\epsilon_a}(a) \cap B_{\epsilon_b}(b) = \varnothing$ for any isolated points $a, b \in A$. Now, let f be a function map the set of isolated points in A to distinct intervals in $\mathbb{R}$, namely $f(a) = B_{\epsilon_a}(a)$. Because these intervals are distinct, f is an injection. Moreover, the set of open intervals of $\mathbb{R}$ is countable, therefore, the set of isolated points of A is also countable. □

41.

Proof.

(a) If $x \in bdry(A)$, then by the definition, for any $\epsilon > 0$, we have $B_\epsilon(x) \cap A = \varnothing$ and $B_\epsilon(x) \cap A^c = \varnothing$. Notice that $A = (A^c)^c$, thus $B_\epsilon(x) \cap (A^c)^c = \varnothing$. Therefore, $x \in bdry(A^c)$ too. Similarly, if $x \in bdry(A^c)$, then for any $\epsilon > 0$, we have $B_\epsilon(x) \cap A^c = \varnothing$ and $B_\epsilon(x) \cap A = B_\epsilon(x) \cap (A^c)^c = \varnothing$. Therefore $x \in bdry(A)$, which means $bdry(A) = bdry(A^c)$.

(b) Assume that $x \in cl(A)$, we need to prove that $x \in bdry(A) \cup int(A)$. Indeed, if $x \in int(A)$, then we are done. If $x \notin int(A)$, then by the definition, for any $\epsilon > 0$, we have $B_\epsilon(x) \not\subset A$. That is $B_\epsilon(x) \cap A^c \neq \varnothing$. Moreover, because $x \in cl(A)$, for any $\epsilon > 0$, we have $B_\epsilon(x) \cap A \neq \varnothing$. Therefore, $x \in bdry(A)$. So $x \in cl(A)$ implies $x \in bdry(A) \cup int(A)$. Conversely, assume that $x \in bdry(A) \cup int(A)$. If $x \in bdry(A)$, then by the definition, $B_\epsilon(x) \cap A \neq \varnothing$ for all $\epsilon > 0$. Therefore, $x \in cl(A)$. If $x \notin bdry(A)$, then $x \in int(A) \subset A \subset cl(A)$. Therefore, $x \in bdry(A) \cup int(A)$ implies $x \in cl(A)$. Thus $cl(A) = bdry(A) \cup int(A)$.

(c) We need to prove that $M = cl(A) \cup int(A^c)$, thus by part (b), we get $M = bdry(A) \cup int(A) \cup int(A^c)$. Indeed, for any $x \in M$, assume that $x \notin int(A^c)$. Notice that by definition, $x \in int(A^c)$ means there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset A^c$, that is $B_\epsilon(x) \cap A = \varnothing$. Therefore, $x \notin int(A^c)$ means for any $\epsilon > 0$, $B_\epsilon(x) \cap A \neq \varnothing$. Thus $x \in cl(A)$. So $M = cl(A) \cup int(A^c) = bdry(A) \cup int(A) \cup int(A^c)$.

□

46.

Proof. The proof is by showing that A is dense in M implies (a), (a) implies (b), (b) implies (c), (c) implies (d), and finally (d) implies A is dense in M.

(i) If A is dense in M, by the definition, we have $\overline{A} = M$. Therefore, for any $x \in M$, we get $x \in \overline{A}$. Therefore, there exist $a_n \in A$ such that $a_n \to x$.

(ii) Assume that every point in M is a limit of a sequence from A, then for any $x \in M$, there exists $a_n \in A$ and $a_n \to M$. That is, for any $\epsilon > 0$, a_n will eventually in $B_\epsilon(x)$. Therefore, $B_\epsilon(x) \cap A \neq \varnothing$ for every $x \in M$ and $\epsilon > 0$.

(iii) Assume that (b) holds. For any open set U, if $U = \varnothing$, then obviously $U \cap A = \varnothing \cap A = \varnothing$. If $U \neq \varnothing$, then there exists $x \in U$. Because U is an open set, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. But by (b), $B_\epsilon(x) \cap A \neq \varnothing$, therefore, $U \cap A \neq \varnothing$.

(iv) Assume that (c) holds and $int(A^c)$ is not empty, then there exists $x \in int(A^c)$. Because $int(A^c)$ is an open set, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset int(A^c) \subset A^c$. Thus $B_\epsilon(x) \cap A = \varnothing$, which is contradicting to (c). So (c) implies A^c is empty interior.

(v) Assume that (d) holds, in exercise 41.c, we have proved that for any $A \subset M$, $M = cl(A) \cup int(A^c)$. But $int(A^c) = \varnothing$, therefore, $M = cl(A) = \overline{A}$. Thus by the definition, A is dense in M.

□

48.

Proof. We will show that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ for any $n \in \mathbb{N}$. Indeed, for any $r = (r_1, \cdots, r_n) \in R_n$, because $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $q_i^{(k)} \in \mathbb{Q}$ such that $q_i^{(k)} \to r_i$. Therefore, $q_k = (q_1^{(k)}, \cdots, q_n^{(k)}) \in \mathbb{Q}^n$ and $q_k \to r$. By (a) exercise 46, $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. Because $\mathbb{Q}^n$ is countable, $\mathbb{R}^n$ is separable for any $n \in \mathbb{N}$. Thus both $\mathbb{R}$ and $\mathbb{R}^2$ are separable. □

49.

Proof. Let R be the set of sequences of the form $(r_1, \cdots, r_n, 0, 0, \cdots)$, where each r_k is rational. That is $R = \{r = (r_1, \cdots, r_n, 0, 0, \cdots) : n \in \mathbb{N}, r_i \in \mathbb{Q} \text{ for all } i \in \mathbb{N}\}$. We will prove that R is dense in ℓ_2 by showing that for any $x \in \ell_2$ and $\epsilon > 0$, $B_\epsilon(x) \cap R \neq \varnothing$. Let $x = (x_1, x_2, \cdots)$, because $x \in \ell_2$, we have $\sum_{i=1}^{\infty} x_i^2 < \infty$. That is, for some $N \in \mathbb{N}$ big enough, we have

$$\left| \sum_{i=1}^{\infty} x_i^2 - \sum_{i=1}^{N} x_i^2 \right| < \frac{\epsilon^2}{2} \quad \text{or} \quad \sum_{i=N+1}^{\infty} x_i^2 < \frac{\epsilon^2}{2}.$$

Now all we need to do is to choose $(r_1, \cdots, r_n, 0, \cdots) \in R$ such that $\sum_{n=1}^{N} (x_i - r_i)^2 < \frac{\epsilon^2}{2}$, then

$$\begin{aligned} \|x - r\|_2 &= \|(x_1 - r_1, \cdots, x_n - r_n, x_{n+1}, \cdots)\|_2 \\ &= \left(\sum_{i=1}^{N} (x_i - r_i)^2 + \sum_{i=N+1}^{\infty} x_i^2 \right)^{\frac{1}{2}} \\ &\leq \left(\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2} \right)^{\frac{1}{2}} \\ &= \epsilon. \end{aligned}$$

That is $r \in B_\epsilon(x)$, so $B_\epsilon(x) \neq \varnothing$. But the selection of r_i's is not hard. By the density of $\mathbb{Q}$ in $\mathbb{R}$, for any x_i, there exists an $r_i \in \mathbb{Q}$ such that $x_i - \frac{\epsilon}{\sqrt{2N}} < r_i < x_i$ for all $i \in \mathbb{N}, i \leq N$. Therefore, $0 < x_i - r_i < \frac{\epsilon}{\sqrt{2N}}$, which implies $(x_i - r_i)^2 < \frac{\epsilon^2}{2N}$ for all i. Hence

$$\sum_{i=1}^{N} (x_i - r_i)^2 < N \cdot \frac{\epsilon^2}{2N} = \frac{\epsilon^2}{2}.$$

Now we will prove that R is countable. Because $card(\mathbb{Q}) = card(\mathbb{N})$, let $N = \{(n_1, \cdots, n_k, 0, \cdots) : n_i, k \in \mathbb{N} \text{ for all } i \in \mathbb{N}\}$, then $N \sim R$. Rearrange N into this order:

$$(1, 0, \cdots), (1, 1, 0, \cdots), (2, 0, \cdots), (1, 1, 1, 0, \cdots), (1, 2, 0, \cdots), (2, 1, 0, \cdots), (3, 0, \cdots), \cdots$$

where every element is increasing by the sum of all entries and increasing by the entries by left to right. It's not hard to see that N is countable, therefore R is countable.

Similar for H^∞, let us define $S = \{(r_1, \cdots, r_n, 0, \cdots) : n \in \mathbb{N}, r_i \text{ is rational and } 0 \leq r_i \leq 1\}$. Because $S \subset R$, we get S is countable as well. Now we will prove that S is dense in H^∞. For any $x \in H^\infty$ and $\epsilon > 0$, we will show that $B_\epsilon(x) \cap R \neq \varnothing$. Because $\sum_{n=1}^{\infty} 2^{-i}$ is converges, there exists $N \in \mathbb{N}$ such that

$$\sum_{i=N+1}^{\infty} 2^{-i} < \frac{\epsilon}{2}.$$

Then because $|x_i| < 1$ for any $i \in \mathbb{N}$, we get

$$\sum_{i=N+1}^{\infty} 2^{-i}|x_i - 0| \le \sum_{i=N+1}^{\infty} 2^{-i} < \frac{\epsilon}{2}.$$

Let $r_1, \cdots, r_N$ be rational numbers in $[0,1]$ such that $|x_i - r_i| < \frac{\epsilon}{2N}$ for any $1 \le i \le N$. Such r_i exists because $x_i \in [0,1]$ too, so by the density of $\mathbb{Q}$ in $\mathbb{R}$, $|x_i - r_i|$ can be as small as possible. Then we have $s = (r_1, \cdots, r_N, 0, \cdots) \in S$ and

$$\sum_{i=1}^{N} 2^{-i}|x_i - r_i| \le \sum_{i=1}^{N} |x_i - r_i| \le N \cdot \frac{\epsilon}{2N} = \frac{\epsilon}{2}.$$

Therefore,

$$\begin{aligned} d(x,s) &= \sum_{i=1}^{\infty} 2^{-i}|x_i - r_i| \\ &= \sum_{i=1}^{N} 2^{-i}|x_i - r_i| + \sum_{i=N+1}^{\infty} 2^{-i}|x_i| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

That is $s \in B_\epsilon(x) \cap S \neq \varnothing$. So H^∞ is separable. □

50.

Proof. Let S be the set of sequences of 0's and 1's, then in chapter 3, we know that S is uncountable. For any set A a subset of ℓ_∞ and A is dense in ℓ_∞, we will prove that $card(A)$ is at least $card(S)$, thus A is uncountable. Let $0 < \epsilon < \frac{1}{2}$, we will claim that for any $a, b \in S$ and $a \neq b$, $B_\epsilon(a) \cap B_\epsilon(b) = \varnothing$. Indeed, assume that $k \in B_\epsilon(a) \cap B_\epsilon(b)$, let a_i, b_i, k_i be the ith element of the sequence a, b, k respectively. Because $a \neq b$, there exists $i \in \mathbb{N}$ such that $a_i \neq b_i$. Thus

$$1 = d(a_i, b_i) \le d(a_i, k_i) + d(k_i, b_i) < \epsilon + \epsilon < \frac{1}{2} + \frac{1}{2} = 1,$$

contradiction. Therefore, $B_\epsilon(a) \cap B_\epsilon(b) = \varnothing$. Notice that because A is dense in ℓ_∞, $B_\epsilon(a) \cap A \neq \varnothing$ for any $a \in S$. That is, there is at least one element from A in $B_\epsilon(a)$ for any $a \in S$. Since $B_\epsilon(a)$'s are distinct when a range in S, there is a one to one map from S to A. Thus $card(S) \le card(A)$, which implies A is uncountable. Therefore, ℓ_∞ is not separable. □

51.

Proof. Let M be a separable metric, and I be the set of isolated points of M, we need to prove that I is countable. Because M is separable, there exists a countable set A such that A is dense in M. For any $x \in I$, because x is an isolated point, there exists $\epsilon > 0$ such that $(B_\epsilon(x) \setminus \{x\}) \cap M = \varnothing$. Since $A \subset M$, we get $(B_\epsilon(x) \setminus \{x\}) \cap A = \varnothing$. But A is dense in M, therefore, $B_\epsilon(x) \cap A \neq \varnothing$. That is $x \in A$. So $I \subset A$. Since A is countable, I is also countable. □

61.

Proof.

(ii) Assume that $F = A \cap C$ where C is closed in (M, d). For any $x \in A$ such that $B^A_\epsilon(x) \cap F \neq \varnothing$ for all $\epsilon > 0$, we will prove that $x \in F$. Indeed, since $B^A_\epsilon(x) \subset B^M_\epsilon(x)$ and $F \subset C$, $B^A_\epsilon(x) \cap F \neq \varnothing$ implies $B^M_\epsilon(x) \cap C \neq \varnothing$ for all $\epsilon > 0$. Because C is a closed set, by the definition, we get $x \in C$. Notice that $x \in A$, thus $x \in A \cap C = F$.

Conversely, if F is a closed set in A, we will prove that $F = cl_M(F) \cap A$, thus $cl_M(F)$ is the closed set that we are looking for. Since $F \subset cl_M(F)$ and $F \subset A$, we get $F \subset cl_M(F) \cap A$. For any $x \in cl_M(F) \cap A$, by the definition of closure, $B^M_\epsilon(x) \cap F \neq \varnothing$ for all $\epsilon > 0$. But $F \subset A$, therefore $B^A_\epsilon(x) \cap F = B^M_\epsilon(x) \cap A \cap F \neq \varnothing$ for all $\epsilon > 0$. Because F is a closed set in A, we get $x \in F$. That is $cl_M(F) \cap A \subset F$. Therefore, $cl_M(F) \cap A = F$.

(iii) In the previous part, we have shown that if F is closed in A, then $F = cl_M(F) \cap A$. Let $F = cl_A(E)$, then $cl_A(E) = A \cap cl_M(cl_A(E))$. Therefore, it is sufficient to prove that $cl_M(E) = cl_M(cl_A(E))$. Because $E \subset cl_A(E)$, we get $cl_M(E) \subset cl_M(cl_A(E))$. Otherwise, if $x \in cl_M(clA(E))$, then by the definition, $B\epsilon^M(x) \cap clA(E) \neq \varnothing$ for all $\epsilon > 0$. Assume that there is $\delta > 0$ such that $B\delta^M(x) \cap E = \varnothing$, then we will claim that $B^M_{\delta/2}(x) \cap cl_M(E) = \varnothing$. Thus since $cl_A(E) \subset clM(E)$, we get $B\delta/2^M(x) \cap clA(E) = \varnothing$. Contradiction! Well indeed, for any $a \in B\delta/2^M(x)$, we get $d(a, x) < \frac{\delta}{2}$. And since $B\delta(x)^M \cap E = \varnothing$, we get $d(x, E) > \delta$. Therefore, $d(a, E) > d(x, E) - d(a, x) > \delta - \frac{\delta}{2} = \frac{\delta}{2}$. Hence $B\delta/2^M(a) \cap E = \varnothing$, which implies $a \notin clM(E)$. So $B\delta/2^M(x) \cap clM(E) = \varnothing$, which implies $B\delta^M(x) \cap E \neq \varnothing$ for all $\delta > 0$. Thus $x \in cl_M(E)$ for any $x \in cl_M(cl_A(E))$, that is $cl_M(cl_A(E)) \subset cl_M(E)$. In consumption, $cl_M(cl_A(E)) = cl_M(E)$, thus $cl_A(E) = A \cap cl_M(E)$.

□

62.

Proof. If G is open in M, then because $G \subset A$, we get $G = A \cap G$. Therefore G is also open in A. Conversely, if G is open in A, then for any $x \in G \subset A$, there exists $\epsilon > 0$ such that $B^M_\epsilon(x) \subset A$. Because G is open in A, there exists $0 < \delta < \epsilon$ such that $B^A_\delta(x) \subset G$. That is $B^M_\delta(x) \cap A \subset G$. Notice that $B^M_\delta(x) \subset B^M_\epsilon(x) \subset A$, therefore, $B^M_\delta(x) \cap A = B^m_\delta(x)$. Thus $B^M_\delta(x) \in G$ for any $x \in G$. That is G is an open set in M.

Replace "open" by "closed", the statement becomes A is closed in (M, d) and $G \subset A$, then G is closed in A if and only if G is closed in M. If G is closed in M, then because $G = G \cap A$, by exercise 61, we get G is closed in A. Conversely, if G is closed in A, then for any sequence $x_n \in G \subset A$ and $x_n \to x$ where $x \in M$, then because A is a closed set in M, we get $x \in A$. Moreover, since G is a closed set in A, we also get $x \in G$. So by the definition of closed set, G is a closed set in M. So the statement still holds. □

63.

Proof. Let A be a nonempty subset of $\mathbb{R}$, then in $\mathbb{R}^2$, $A = \{(a,0) : a \in A\}$. Clearly this is not an open set because let $(a,0) \in A$, then for any $\epsilon > 0$, $(a,\epsilon/2) \in B_\epsilon^{\mathbb{R}^2}(a,0)$ but $(a,\epsilon/2) \notin A$. Therefore $B_\epsilon^{\mathbb{R}^2}(a,0) \not\subset A$ for any $\epsilon > 0$, which means A is not open.

Let $A = [0,1]$ be a closed set in $\mathbb{R}$, then in $\mathbb{R}^2$, $A = \{(a,0) : a \in [0,2]\}$. We will claim that A is a closed set in $\mathbb{R}^2$. Indeed if $(x_n,0) \in A$ and $(x_n,0) \to (x,y)$ for some $(x,y) \in \mathbb{R}^2$, then we get $x_n \to x$ and $0 \to y$. Since A is closed in $\mathbb{R}$, we get $x \in A$ in $\mathbb{R}$. Clearly $y = 0$, therefore $(x,y) \in A$ in $\mathbb{R}^2$. Thus A is a closed set in $\mathbb{R}^2$. □

64.

Proof. The analogue of part (iii) gonna be $int_A(E) = A \cap int_M(E)$ for any subset E of A. Let $E = A = [0,2]$ in $\mathbb{R}$, then E is an open set in A, thus $int_A(E) = [0,2]$, which is not equals $int_{\mathbb{R}}(E) = (0,2)$. □

69.

Proof. If M has a countable open base, then let a be a random element in each open base. The set A of such a is therefore countable. Moreover, for any open set $U \in M$, there exists an open set of the open base such that it is a subset of U. Thus $U \cap A \neq \varnothing$ for all open set U, which means M is separable.

Conversely,if M is separable, then there exists a countable dense subset $\{x_n\}$ of M. Let $U = \{B_\epsilon(x_n) : \epsilon \in \mathbb{Q}\}$, we will prove that U is a countable open base of M. Notice that $\{x_n\}$ and $\{\epsilon\}$ have the same cardinality as $\mathbb{N}$, we get $card(U) = card(\mathbb{N} \times \mathbb{N}) = card(\mathbb{N})$. Therefore U is countable. For any open set O in U, we will claim that $O = \cup\{B_\epsilon(x_n) : B_\epsilon(x_n) \in O \cap U\}$. Indeed, since $B_\epsilon(x_n) \subset O$, there union is obviously a subset of O. Now, for any $a \in O$, there exists a rational $\delta > 0$ such that $B_\delta(a) \subset O$. Since U is dense, there exists $x_k \in B_{\frac{\delta}{2}}(a)$. Hence $a \in B_{\frac{\delta}{2}}(x_k) \subset B_\delta(a) \subset O$. (I'm not sure if this is clear yet so please tell me if you want further explanation.) Since $\frac{\delta}{2}$ is rational, we get $B_{\frac{\delta}{2}}(x_k) \subset U$, thus $a \in B_{\frac{\delta}{2}}(x_k) \subset \cup\{B_\epsilon(x_n) : B_\epsilon(x_n) \in O \cap U\}$ for all open set $O \subset M$. Thus U is a countable open base of M. □

Chapter 5. Continuous Functions

7.

Proof.

(a) Since $(-\infty, a)$ is an open set in $\mathbb{R}$ and f is continuous, we get $f^{-1}(-\infty, a) = \{x : f(x) < a\}$ is an open set. Similarly, we get $\{x : f(x) > a\}$ is an open set.

(b) For any $\epsilon > 0$, because $B_\epsilon^{\mathbb{R}}(f(a)) = \{f(x) : f(a) - \epsilon < f(x) < f(a) + \epsilon\}$, we get $f^{-1}(B_\epsilon^{\mathbb{R}}(f(a))) = \{x : f(a) - \epsilon < f(x) < f(a) + \epsilon\}$. Notice that $\{x : f(x) < f(a) + \epsilon\}$ and $\{x : f(x) > f(a) - \epsilon\}$ are open by the hypothesis, therefore, their intersection is also open, namely $f^{-1}(B_\epsilon(f(a))) = \{x : f(a) - \epsilon < f(x) < f(a) + \epsilon\}$. Since $a \in f^{-1}(B_\epsilon(f(a)))$ an open set, there exists $\delta > 0$ such that $B_\delta^M(a) \subset f^{-1}(B_\epsilon(f(a)))$. Thus f is continuous.

(c) Assume that the sets $\{x : f(x) > q\}$ and $\{x : f(x) < q\}$ are open for any $q \in \mathbb{Q}$, let $a \in \mathbb{R}$, we will prove that $\{x : f(x) > a\}$ and $\{x : f(x) < a\}$ are open too. Indeed, for any $y \in \{x : f(x) > a\}$, we have $f(y) > a$, thus there exists $a' \in \mathbb{Q}$ such that $f(y) > a' > a$. (This is by the density of $\mathbb{Q}$ in $\mathbb{R}$.) By the assumption, we get $y \in \{x : f(x) > a'\}$ an open set, therefore, there exists $\delta > 0$ such that $B_\delta^M(y) \subset \{x : f(x) > a'\} \subset \{x : f(x) > a\}$. Thus $\{x : f(x) > a\}$ is open, the same implies for $\{x : f(x) > a\}$, thus by part (b), f is continuous.

$\square$

10.

Proof. For any $\epsilon > 0$, we have $B_{\frac{1}{2}}^A(2) = \{f(2)\} \subset f^{-1}(B_\epsilon(f(2)))$. Therefore, f is continuous relative to A at 2. $\square$

11.

Proof.

(a) We will prove that this statement is true. For any $a \in A \cup B$, without loss of generality, assume that $a \in A$. Since f is continuous at each point in A, we get f to be continuous at a as well. Thus f is continuous at each point of $A \cup B$.

(b) Let

$$f(x) = \begin{cases} 0, & \text{for all } x \in A = (1,2] \\ 2, & \text{for all } x \in B = (2,3) \end{cases}.$$

It's not hard to see that f is continuous relatively to A or B. However, for any $\delta > 0$, we have $B_\delta(2) = \{2-\delta, 2+\delta\}$. But let $\epsilon = 1$, then we get $f^{-1}(B_\epsilon^{A\cup B}(f(2))) = f^{-1}(B_1^{A\cup B}(0)) = f^{-1}(0) = (1,2]$. Clearly $B_\delta(2) \not\subset f^{-1}(B_\epsilon^{A\cup B}(f(2)))$, thus f is not continuous relatively to $A \cup B$ at 2.

The modification that is necessary to make (b) true is that A and B are open in M. If so then for any $a \in A \cup B$, then without loss of generality, let $a \in A$. For any $\epsilon > 0$, because f is continuous relatively to A at a, there exists $\delta > 0$ such that $B_\delta^M(a) \cap A = B_\delta^A(a) \subset f^{-1}(B_\epsilon(f(a)))$. Notice that both $B_\delta^M(a)$ and A are open sets, we get $B_\delta^M(a) \cap A$ is an open set. Thus there exists $\gamma > 0$ such that $B_\gamma^M(a) \subset B_\delta^M(a) \cap A$. Thus $B_\gamma^{A\cup B}(a) \subset B_\gamma^M(a) \subset B_\delta^M(a) \cap A \subset f^{-1}(B_\epsilon(f(a)))$. That is, f is continuous relative to $A \cup B$.

$\square$

14.

Proof. Let C denote the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ and $c = \mathbf{card}(\mathbb{R})$. Let $R = \{f(x) : f \text{ is any function from } \mathbb{Q} \text{ to } \mathbb{R}\}$. Because a continuous function of $\mathbb{R}$ is determined by its values on $\mathbb{Q}$, there is a one to one map from C to R, namely the map preserved the value of f at any rational point. Thus $\mathbf{card}(C) \le \mathbf{card}(R)$.

Let $h : \mathbb{R} \mapsto C$ defined by $h(x) = x$ for all $x \in \mathbb{R}$. It's not hard to see that h is a one to one function, thus $c = \mathbf{card}(\mathbb{R}) \le \mathbf{card}(C) \le \mathbf{card}(R) = c^{\aleph_0} = c$. So $\mathbf{card}(C) = c$. $\square$

17.

Proof. For any $a \in M$, if $f(a) \ne g(a)$, then we get $\rho(f(a), g(a)) > 2\epsilon > 0$ for some $\epsilon > 0$. Because f and g are continuous, there exists $\delta > 0$ such that $f(B_\delta^d(a)) \subset B_\epsilon^\rho(f(a))$ and $g(B_\delta^d(a)) \subset B_\epsilon^\rho(g(a))$. Because D is dense in M, there exists $b \in B_\delta^d(a) \cap D$. Then because $b \in B_\delta^d(a)$, we get $f(b) = g(b) \in B_\epsilon^\rho(f(a)) \cap B_\epsilon^\rho(g(a))$. But then, we have

$$2\epsilon < \rho(f(a), g(a)) < \rho(f(a), f(d)) + \rho(g(d), g(a)) < \epsilon + \epsilon.$$

Contradiction! Therefore $\rho(f(a), g(a)) = 0$, that is $f(a) = g(a)$ for all $a \in M$.

Now we will prove that if f is onto, then $f(D)$ is dense in N. Also notice that all the hypotheses we need are f to be continuous and onto, and D is dense in M. This result will be reused in exercise 18. For any nonempty open set O of N, we get $f^{-1}(O)$ is an open set in M. Because f is onto, $f^{-1}(O) \ne \varnothing$. Since D is dense in M, there exists $c \in D \cap f^{-1}(O)$. Then $f(c) \in f(D) \cap O \ne \varnothing$. That is, $f(D) \cap O \ne \varnothing$ for any non empty open set O of N. Thus $f(D)$ is dense in N. $\square$

18.

Proof. Because A is separable, there exists a countable dense subset D of A. $f(D)$ is clearly countable, it is sufficient to show that $f(D)$ is also dense in $f(A)$. Notice that $f : A \mapsto f(A)$ is onto and continuous, by exercise 17, we get $f(D)$ is dense in $f(A)$. Thus $f(A)$ is also separable. $\square$

20.

Proof. Because d defines a metric on M, we get $d(y,z) \leq d(x,z) + d(x,y)$ and $d(x,z) \leq d(x,y) + d(y,z)$. That is $-d(x,y) \leq d(x,z) - d(y,z) \leq d(x,y)$ or $|d(x,z) - d(y,z)| \leq d(x,y)$.

Now, for any $a \in M$ and $\epsilon > 0$, we have $B_\epsilon(f(a)) = B_\epsilon(d(a,z)) = (d(a,z) - \epsilon, d(a,z) + \epsilon)$. We will prove that for $\delta < \epsilon$, $f(B_\delta(a)) \subset B_\epsilon(f(a))$, thus f is continuous. Indeed, for any $x \in B_\delta(a)$, we have $d(x,a) < \delta < \epsilon$. Therefore, by the previous part, we have $d(a,z) - \epsilon < d(a,z) - d(a,x) < d(x,z)$. Moreover, $d(x,z) < d(a,z) + d(x,a) < d(a,z) + \epsilon$. Thus $f(x) = d(x,z) \in (d(a,z) - \epsilon, d(a,z) + \epsilon) = B_\epsilon(f(a))$. Thus $f(B_\delta(a)) \subset B_\epsilon(f(a))$, which implies f to be continuous. $\square$

21.

Proof. If $x \neq y$, then $d(x,y) > 0$. Thus there exists $\epsilon > 0$ such that $d(x,y) > 3\epsilon$. Let $U = B_\epsilon(x)$ and $V = B_\epsilon(y)$, we will claim that $\overline{U}$ and $\overline{V}$ are disjoint. Indeed, Since the close balls radius ϵ centered at x and y are disjoint, their closures are disjoint too. $\square$

22.

Proof. For any $m > n \in \mathbb{N}$, we get $E(m) - E(n) = (0, \cdots 0, 1, \cdots, 1, 0 \cdots)$ where it $n+1$-th to m-th entries are 1's, and the rest are 0's. Therefore $\|E(m) - E(n)\|_1 = m - n$, which implies E preserves distance. So E is an isometry. $\square$

Lemma Cardinality. $\mathbf{card}(\mathbb{N} \times \mathbb{R}) = \mathbf{card}(\mathbb{R})$.

Proof. Let $f : \mathbb{N} \times \mathbb{R} \mapsto \mathbb{R}$ defined by $f(n, x) = n + \frac{\arctan(x)}{\pi}$. If there are (n, x) and (m, y) in $\mathbb{N} \times \mathbb{R}$ such that $f(n, x) = f(m, y)$, then

$$n + \frac{\arctan(x)}{\pi} = m + \frac{\arctan(y)}{\pi}$$

or

$$n - m = \frac{\arctan(x) - \arctan(y)}{\pi}.$$

Notice that because $\arctan(x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we get

$$|n - m| = \frac{|\arctan(x) - \arctan(y)|}{\pi} \leq \frac{|\arctan(x)| + |-\arctan(y)|}{\pi} < \frac{\pi/2 + \pi/2}{\pi} = 1.$$

Therefore, $m = n$, which implies $\frac{\arctan(x) - \arctan(y)}{\pi} = 0$. Since arctan is bijective, we get $x = y$. Thus $(n, x) = (m, y)$, which means f is one to one. Therefore, $\mathbf{card}(\mathbb{N} \times \mathbb{R}) \leq \mathbf{card}(\mathbb{R})$ (1).

Now let $g : \mathbb{R} \mapsto \mathbb{N} \times \mathbb{R}$ defined by $g(x) = (1, x)$. We can easily see that g is a one to one function, therefore, $\mathbf{card}(\mathbb{R}) \leq \mathbf{card}(\mathbb{N} \times \mathbb{R})$ (2). From (1) and (2), we get $\mathbf{card}(\mathbb{R}) = \mathbf{card}(\mathbb{N} \times \mathbb{R})$. □

23.

Proof. For any $x, y \in c_0$, we have

$$\|x - y\|_\infty = \sup\{|x_i - y_i| : i \in \mathbb{N}\} = \sup(|x_i - y_i| \mid i \in \mathbb{N} \cup 0) = \|S(x) - S(y)\|_\infty.$$

So S preserves distance, which means f is an isometry. □

24.

Proof. Let $f : \mathbb{R} \mapsto V$ defined by $f(\alpha) = \alpha y$ for all $\alpha \in \mathbb{R}$. If $\|y\| = 0$, then $y = 0$ and $f(\alpha) = 0$ for all $\alpha \in \mathbb{R}$. We can easily see that f in this case is continuous. If $\|y\| > 0$, then for any $\epsilon > 0$ and $\alpha \in \mathbb{R}$, let $\delta < \frac{\epsilon}{\|y\|}$. Thus for any $b \in B_\delta(\alpha)$, we have $\|f(b) - f(\alpha)\| = \|by - \alpha y\| = |b - \alpha|\|y\|$. Notice that because $b \in B_\delta(\alpha)$, we get $|b - \alpha| < \delta < \frac{\epsilon}{\|y\|}$. Therefore, $\|f(b) - f(\alpha)\| < \epsilon$, which implies $f(b) \in B_\epsilon(f(a))$. So $f(B_\delta(\alpha)) \subset B_\epsilon(f(a))$. That is f is continuous.

Let $g : V \mapsto V$ defined by $g(x) = x + y$. For any $\epsilon > 0$ and $z \in V$, let $0 < \delta < \epsilon$. Then for any $x \in B_\delta(z)$, we get $\|x - z\| < \delta$. Therefore

$$\|g(x) - g(y)\| = \|(x + y) - (z + y)\| = \|x - z\| < \delta < \epsilon.$$

That is synonymous with saying $g(x) \in B_\epsilon(g(z))$. Thus $g(B_\delta(z)) \subset B_\epsilon(g(z))$, which implies g is continuous. □

25.

Proof. For any $\epsilon > 0$ and $x \in M$, if $K = 0$, then $\rho(f(x), f(y)) \leq 0$, thus $f(x) = f(y)$ for all $x, y \in M$. Clearly f in this case is continuous. If $K \neq 0$, then let $0 < \delta < \frac{\epsilon}{K}$. Then, for any $y \in B^d_\delta(x)$, we get

$$\rho(f(x), f(y)) \leq Kd(x, y) < K\frac{\epsilon}{K} = \epsilon.$$

Therefore $f(y) \in B^\rho_\epsilon(f(x))$, which implies $f(B^d_\delta(x)) \subset B_\epsilon(f(x))$. Thus f is continuous if f is a Lipschitz mapping. $\square$

26.

Proof. For any $f, g \in C[a, b]$, because $L : C[a, b] \mapsto \mathbb{R}$, we have

$$\begin{aligned} |L(f) - L(g)| &= \left| \int_a^b (f(t) - g(t))dt \right| \\ &\leq \int_a^b |f(t) - g(t)|dt \\ &\leq \int_a^b d(f, g)dt \\ &= (b - a)d(f, g) \end{aligned}$$

because $d(f, g)$ is a constant for fixed f and g. Therefore, $L(f) = \int_a^b f(t)dt$ is Lipschitz, that is L is continuous. $\square$

27.

Proof. For any $x, y \in \ell_\infty$, we have

$$|f(x) - f(y)| = |x_k - y_k| \leq \|x - y\|_\infty.$$

Therefore, f is Lipschitz with $K = 1$. Thus f is continuous. $\square$

28.

Proof. The proof is by showing that g is Lipschitz. Let $a = (1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots)$, then $\|a\|_2 = (\sum_{n=1}^\infty \frac{1}{n^2})^{\frac{1}{2}} = (\frac{\pi^2}{6})^{\frac{1}{2}} = \frac{6\pi}{\sqrt{6}}$. Therefore, $a \in \ell_2$. Now, for any $x, y \in \ell_2$, we have

$$|g(x) - g(y)| = \left| \sum_{n=1}^\infty \frac{x_n}{n} - \sum_{n=1}^\infty \frac{y_n}{n} \right| = \left| \sum_{n=1}^\infty \frac{x_n - y_n}{n} \right| = \langle x - y, a \rangle \leq \|a\|_2 \|x - y\|_2$$

by the Cauchy-Schwarz inequality. Thus g is a $\|a\|_2$-Lipschitz image, which implies g is continuous. $\square$

29.

Proof. Because $y \in \ell_\infty$, (y_n) is bounded. Let $m = \sup\{|y_n| : n \in \mathbb{N}\}$, then for any $a, b \in \ell_1$, we get

$$\|h(a) - h(b)\|_1 = \|((a_n - b_n)y_n)_{n=1}^\infty\|_1 \leq \|(a_n - b_n)_{n=1}^\infty \cdot m\|_1 = |m| \cdot \|a - b\|_1.$$

Thus h is m-Lipschitz, which implies h is continuous. $\square$

30.

Proof. If f is continuous, for any $f(a) \in f(\overline{A})$, since $a \in \overline{A}$, there exist $a_n \in A$ such that $a_n \to a$. But f is continuous, thus $f(a_n) \to f(a)$. This implies $f(a) \in \overline{f(A)}$. So $f(\overline{A}) \subset \overline{f(A)}$. Moreover, for any $a \in f^{-1}(\operatorname{Int} B)$, we get $f(a) \in \operatorname{Int} B$. Thus there exists $\epsilon > 0$ such that $B_\epsilon^\rho(f(a)) \subset B$. But f is continuous, thus there exists $\delta > 0$ such that $B_\delta^d(a) \subset f^{-1}(B_\epsilon^\rho(f(a))) \subset f^{-1}(B)$. So $a \in \operatorname{Int}(f^{-1}(B))$ for all $a \in f^{-1}(\operatorname{Int} B)$, which implies $f^{-1}(\operatorname{Int} B) \subset \operatorname{Int}(f^{-1}(B))$.

If $f(\overline{A}) \subset \overline{f(A)}$ for all $A \subset M$, then we will prove that B closed in N will imply $f^{-1}(B)$ closed in M. Indeed, if $f^{-1}(B)$ is not closed, then there exists $a \in \overline{f^{-1}(B)} \setminus f^{-1}(B)$. Since $a \in \overline{f^{-1}(B)}$, by the hypothesis, we get

$$f(a) \in f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \overline{B} = B.$$

However, because $a \notin f^{-1}(B)$, we get $f(a) \notin B$. Contradiction. Thus $f^{-1}(B)$ is closed in M whenever B is closed in N, which is synonymous with f being continuous.

Similarly, assume that $f^{-1}(\operatorname{Int} B) \subset \operatorname{Int}(f^{-1}(B))$ for all $B \subset N$, we will prove that B open in N will imply $f^{-1}(B)$ open in M. Indeed, if $f^{-1}(B)$ is not open, then there exists $a \in f^{-1}(B) \setminus \operatorname{Int}(f^{-1}(B))$. Because $a \notin \operatorname{Int}(f^{-1}(B))$, using the hypothesis, we get $a \notin f^{-1}(\operatorname{Int} B)$ too, which is contradict to the definition of a. Therefore $f^{-1}(B)$ must be open in M whenever B is open in N, which is synonymous with f is continuous.

So f is continuous if and only if $f(\overline{A}) \subset \overline{f(A)}$ for every $A \subset M$ if and only if $f^{-1}(\operatorname{Int} B) \subset \operatorname{Int}(f^{-1}(B))$ for every $B \subset N$.

One example such that $f(\overline{A}) \neq \overline{f(A)}$ is let $f : \mathbb{Q} \mapsto \mathbb{R}$ defined by $f(x) = x$ and $A = \mathbb{Q}$. It's not hard to see that this map is 1-Lipschitz, therefore f is continuous. However $f(\overline{A}) = f(\mathbb{Q}) = \mathbb{Q}$, which does not equal $\overline{f(A)} = \overline{\mathbb{Q}} = \mathbb{R}$ since $\mathbb{Q}$ is dense in $\mathbb{R}$. $\square$

31.

Proof.

(i) For any $\epsilon > 0$ and $a \in M$, there exists $n \in \mathbb{N}$ such that $a \in U_n$. Since U_n is open, there exists $\gamma > 0$ such that $B_\gamma^d(a) \in U_n$. And because f is continuous on U_n, there exists $0 < \delta < \gamma$ such that $d(a, b) < \delta$ implies $\rho(f(a), f(b)) < \epsilon$ for all $b \in M$. Thus f is continuous on M.

(ii) For any $\epsilon > 0$ and $a \in M$, since M is a union of a finite number of E_n, a is in some finite number of closed sets. Without loss of generality, assume that $a \in E_i$ for all $1 \leq i \leq k$ where k is fixed in $\mathbb{N}$. Because f is continuous on each E_i, there exists $\delta_i > 0$ such that $d(a, b) < \delta_i$ implies $\rho(f(a), f(b)) < \epsilon$ for any $b \in E_i$ and for each $1 \leq i \leq k$. Now let

$$0 < \delta < \min(\{\delta_i : 1 \leq i \leq k\} \cup \{d(a, E_i) : k + 1 \leq i \leq n\}).$$

Then for any $b \in M$ and $d(a, b) < \delta$ implies $b \in \cup_{i=1}^k E_i$. And since $\delta < \delta_i$ for any i, we get $\rho(f(a), f(b)) < \epsilon$. That is, there exists $\delta > 0$ such that $d(a, b) < \delta$ implies $\rho(f(a), f(b)) < \epsilon$ for any $\epsilon > 0$ and $a \in M$. Thus f is continuous on M.

(iii) Let $f : [0, 1] \mapsto \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, & \text{for } x = 0 \\ 1, & \text{for } x \in (0, 1] \end{cases} .$$

Clearly f is not continuous on $[0, 1]$. However, let $E_1 = \{0\}$, $E_n = [\frac{1}{n}, 1]$ for $n > 1$, then E_i is closed for each $i \in \mathbb{N}$. Also, one can easily see that f is continuous on each E_i. However, since $\cup_{n=1}^\infty E_n = [0, 1]$, f is not continuous on their union.

□

34.

Proof. Let $(x_n, y_n) \to (x, y)$ in $M \times M$, it is sufficient to show that $d(x_n, y_n) \to (x, y)$. By exercise 3.46, for any metric on $M \times M$, $(x_n, y_n) \to (x, y)$ implies $x_n \to x$ and $y_n \to y$. Therefore, $d(x_n, y_n) \to d(x, y)$. □

35.

Proof. If $f : M \to \mathbb{R}$ is continuous and V is an open set in $\mathbb{R}$, then $f^{-1}(V) = U$ is also open. Conversely, if U is an open set in M, then $N = U^c$ is closed in M. Let $f : M \to \mathbb{R}$ defined by $f(x) = d(x, N)$, then f is continuous. Notice that N is closed, we get $f^{-1}(0) = N$. Thus $f^{-1}(\mathbb{R} \setminus \{0\}) = M \setminus N = U$. So $\mathbb{R} \setminus \{0\}$ is the open set that satisfy the exercise's conditions. □

36.

Proof. If $f(a_n) \to f(a)$ for every continuous real-value function, then because $f : M \to \mathbb{R}$ defined by $f(x) = d(a, x)$ is continuous, we get $d(a_n, a) \to d(a, a) = 0$. Therefore, $a_n \to a$. □

39.

Proof. Let U be an open set in M, and let $U_n = \{x \in M : d(x, U^c) \geq \frac{1}{n}\}$. We will claim that U_n's are closed and $U = \cup_{n=1}^{\infty} U_n$; then U can be written as a union of countably many closed set. Indeed, for any fixed $n \in \mathbb{N}$, let $x_n, x \in U_n$ such that $x_n \to x$. If $d(x, U^c) < \frac{1}{n}$, then there exists $\epsilon > 0$ such that $d(x, U^c) < \frac{1}{n} + \epsilon$. Because $x_n \to x$, there exists $k \in \mathbb{N}$ such that $d(x_k, x) < \epsilon$. But then we have

$$\frac{1}{n} < d(x_k, U^c) \leq d(x, x_k) + d(x, U^c) < \epsilon + d(x, U^c) < \frac{1}{n},$$

contradiction!. Therefore $d(x, U^c) \geq \frac{1}{n}$, which means $x \in U_n$ too. Thus U_n is closed for any $n \in \mathbb{N}$.

Moreover, for any $n \in \mathbb{N}$, if $a \in U_n$, then $d(a, U^c) \geq \frac{1}{n} > 0$, which implies $a \notin U^c$ or $a \in U$. Thus $U_n \subset U$ for any $n \in \mathbb{N}$, which implies $\cup_{n=1}^{\infty} U_n \subset U$ (1). For any $a \in U$, because U is open, there exists $\epsilon > \frac{1}{n} > 0$ such that $B_\epsilon(a) \subset U$. Therefore, $B_\epsilon(a) \cap U^c = \varnothing$ or $d(a, U^c) > \epsilon > \frac{1}{n}$. This implies $a \in U_n \subset \cup_{n=1}^{\infty} U_n$. Thus $U \subset \cup_{n=1}^{\infty} U_n$ (2). From (1) and (2), we get $U = \cup_{n=1}^{\infty} U_n$.

Similarly, for any closed set E in M, let $E_n = \{x \in M : d(x, E) < \frac{1}{n}\}$ be open sets in M, then $E = \cap_{n=1}^{\infty} E_n$. Thus every closed set is the intersection of countably many open sets. □

41.

Proof. For any $x \notin C$, we defined a_x and b_x be the largest number that is smaller than x and the smallest number that is larger than x respectively. If such min or max doesn't exists, then a_n and b_n equal 0. Specifically, $a_x = \max\{a \in C : a < x\}$ and $b_x = \min\{b \in C : x < b\}$ if $\{a \in C : a < x\}$ and $\{b \in C : x < b\}$ are not $\varnothing$. Otherwise, $a_n = b_n = 0$. Notice that min and max exists because C is closed. We will defined $g(x) : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x), \quad \text{for } x \in C \\ (f(b_x) - f(a_x))(x - a_x) + f(a_x), \quad \text{for } x \in \mathbb{R} \setminus C \end{cases}.$$

Notice that $g(x)$ simply "connect the boundary points" of C, one can easily see this function is continuous. Indeed, if $a \in C^o$, then $g(a)$ is continuous. If $a \in \mathbb{R} \setminus C$, then $g(a)$ is also continuous since $\mathbb{R} \setminus C$ is open. Lastly, assume that a is a boundary point of C, if the right limit of $g(a)$ is defined by f, then $g(a)$ is continuous since $g(a) = f(a) = \lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x)$. If it is defined by $(f(b_x) - f(a_x))(x - a_x) + f(a_x)$, then $a = a_x$ for any x closed enough to a. Therefore

$$g(a) = g(a_n) = f(a_n) = \lim_{x \to a_n^+} (f(b_x) - f(a_x))(x - a_x) + f(a_x) = \lim_{x \to a_n^+} g(x).$$

Hence $g(x)$ is a continuous function where $g(x) = f(x)$ for any $x \in C$. □

43.

Proof. By the definition, d and ρ are equivalent means $x_n \xrightarrow{d} x$ if and only if $x_n \xrightarrow{\rho} x$. Therefore, by theorem 5.5, the identity map $i : (M, d) \mapsto (M, \rho)$ is a homeomorphism. □

44.

Proof. Let $(M, d), (N, \rho), (A, h)$ be metric spaces. Obviously (M, d) is homeomorphic to itself. By theorem 5.5, we know that if (M, d) is homeomorphic to (N, ρ), then (M, d) is homeomorphic to (N, ρ). Now if (M, d) is homeomorphic to (N, ρ), then there exists a homeomorphism $f : (M, d) \mapsto (N, \rho)$. Similarly, if (N, ρ) is homeomorphic to (A, h), then there exists a homeomorphism $g : (N, \rho) \mapsto (A, h)$. We will claim that $g \circ f$ is a homeomorphism from (M, d) to (A, h). Indeed, because both f and g are one to one and onto, we get $g \circ f$ is one to one and onto. Moreover, we have

$$x_n \xrightarrow{d} x \text{ if and only if } f(x_n) \xrightarrow{\rho} f(x) \text{ if and only if } g(f(x_n)) \xrightarrow{h} g(f(x)).$$

Thus by theorem 5.5, we have $g \circ f$ is a homeomorphism from (M, d) to (A, h). Thus (M, d) is homeomorphic to (A, h). So "is homeomorphic to" is an equivalent relation. $\square$

45.

Proof. Let $\mathbb{N}^{-1} = \{(1/n) : n \geq 1\}$ and $f : \mathbb{N} \mapsto \mathbb{N}^{-1}$ defined by $f(n) = \frac{1}{n}$. We will prove that f is a homeomophism. For any $a_n, a \in \mathbb{N}$, if $a_n \to a$, then a_n is eventually equals a. Therefore $f(a_n) = \frac{1}{a_n} \to \frac{1}{a} = f(a)$. Conversely, for any $\frac{1}{a} \in \mathbb{N}^{-1}$, if $\frac{1}{a_n} \to \frac{1}{a}$ then $\frac{1}{a_n}$ is eventually equal $\frac{1}{a}$ (notice that $0 \notin \mathbb{N}^{-1}$. Therefore a_n will eventually equal a, or $a_n \to a$. By theorem 5.5, f is a homeomorphism, which imply $\mathbb{N}$ is homeomorphic to $\mathbb{N}^{-1}$. $\square$

46.

Lemma 2. *If* (M, d) *is a metric space, then* $k(a, b) = \arctan(d(a, b))$ *for any* $a, b \in M$ *defines a metric on* M*.*

Proof. Because $d(a, b) \geq 0$ for all $a, b \in M$ we have $k(a, b) = \arctan(d(a, b)) \geq 0$ for all $a, b \in M$. If $k(a, b) = 0$, then we get $\arctan(d(a, b)) = 0$. Therefore $d(a, b) = 0$, which implies $a = b$. And obviously $k(a, a) = \arctan(d(a, a)) = \arctan(0) = 0$. Thus $a = b$ in M if and only if $k(a, b) = 0$. Also because $k(a, b) = \arctan(d(a, b)) = \arctan(d(b, a)) = k(b, a)$, if k satisfy the triangular inequality, k defines a metric on M.

For any $a, b \geq 0$, we have $1 - ab < 1$. Thus

$$a + b \leq \frac{a + b}{1 - ab}.$$

Therefore,

$$\begin{aligned}
\tan(\arctan(a + b)) &\leq \frac{\tan(\arctan(a)) + \tan(\arctan(b))}{1 - \tan(\arctan(a))\tan(\arctan(b))} \\
&= \tan(\arctan(a) + \arctan(b)).
\end{aligned}$$

If both $\arctan(a)+\arctan(b), \arctan(a+b) \in [0, \frac{\pi}{2})$, then because tan is increasing in $[0, \frac{\pi}{2})$, we get $\arctan(a+b) \leq \arctan(a)+\arctan(b)$. If not, then $\arctan(a+b) \geq \frac{\pi}{2} > \arctan(a+b)$. So in any case, if $a, b \geq 0$, then $\arctan(a + b) \leq \arctan(a) + \arctan(b)$. Now, for any $x, y, z \in M$, we have

$$\begin{aligned}
k(x, z) &= \arctan(d(x, z)) \\
&\leq \arctan(d(x, y) + d(y, z)) \\
&\leq \arctan(d(x, y)) + \arctan(d(y, z)) \\
&= k(x, y) + k(y, z).
\end{aligned}$$

So k satisfy the triangular inequality, which implies k defines a metric on M. ∎

Proof. For any metric space (M, d), we will show that $(M, d) \cong (M, \arctan(d))$. Then because *arctan* is bounded, (M, d) is homeomorphic to a finite diameter metric space. Indeed, $d(x_n, x) \to 0$ if and only if $\arctan^{-1}(d(x_n, x)) \to 0$ since arctan is continuous. Thus $x_n \xrightarrow{d} x$ if and only if $x_n \xrightarrow{\arctan(d)} x$, which means (M, d) is homeomorphic to $(M, \arctan(d))$. ∎

47.

Proof. For any $n > m \in \mathbb{N}$, we have

$$n - m = \|(0, \cdots, 1, \cdots, 1, 0, \cdots)\|_1$$

where the $m + 1$-th to the n-th entries are 1 and the rest are 0. This equals

$$\|E(n) - E(m)\|_1.$$

Therefore E is an isometry. ∎

48.

Proof. Since $\tan : \mathbb{R} \mapsto (\frac{-\pi}{2}, \frac{\pi}{2})$, we have $f(x) = \frac{\tan(x)}{\pi} + \frac{1}{2} : \mathbb{R} \mapsto (0,1)$. Because f is continuous, if $x_n \to x$ in $\mathbb{R}$, then $f(x_n) \to f(x)$ in $(0,1)$. Notice that f is a bijection, so $f(x_n) \to f(x)$ in $(0,1)$ implies $f^{-1}(f(x_n)) \to f^{-1}(f(x))$ or $x_n \to x$ in $\mathbb{R}$. Thus, by theorem 5.5, we have $\mathbb{R}$ is homeomorphic to $(0,1)$.

Let $f : (0,1) \mapsto (0,\infty)$ defined by $f(x) = \frac{1}{x} - 1$, we will prove that f is a homeomorphism. Indeed, for $x_n, x \in (0,1)$, $x_n \to x$ if and only if $\frac{1}{x_n} \to \frac{1}{x}$, which is synonymous with $f(x_n) \to f(x)$. Thus f is a homeomorphism, which implies $(0,1)$ is homeomorphic to $(0,\infty)$.

However, $\mathbb{R}$ is not isometric to $(0,1)$ because $|4-2| = 2$ in $\mathbb{R}$ and there isn't exists $a, b \in (0,1)$ such that $|a-b| = 2$. Also, $\mathbb{R}$ is not isometric to $(0,\infty)$ because if $\mathbb{R}$ is isometric to $(0,\infty)$, then there exists an isometry $g : \mathbb{R} \mapsto (0,\infty)$. Let $a \in \mathbb{R}$ such that $f(a) = 1$, then because $|a-(a-1)| = |a-(a+1)| = 1$, we get $|f(a) - f(a-1)| = |f(a) - f(a+1)| = 1$ or $|1 - f(a-1)| = |1 - f(a+1)| = 1$ in $(0,\infty)$. So $f(a-1) = f(a+1) = 2$, contradiction since f is a injective. Therefore, $\mathbb{R}$ is not isometric to $(0, infty)$. $\square$

49.

Proof. It is not hard to see that $f(x,y) = x+y$ is a bijection. Moreover, for any $a, b \in V$, we have

$$\|f(a) - f(b)\| = \|(a+y) - (b+y)\| = \|a-b\|.$$

So f is an isometry on V. Also, since $\alpha \neq 0$, it's not hard to see that $g(x) = \alpha x$ is a bijection on V. Moreover, we have

$$\|g(x_n) - g(x)\| = \|\alpha x_n - \alpha x\| = |\alpha|\|x_n - x\|.$$

So $x_n \to x$ if and only if $|\alpha|\|x_n - x\| \to 0$, which is the same as $\|g(x_n) - g(x)\| \to 0$ or $g(x_n) \to g(x)$. By theorem 5.5, we get g is a homeomorphism. $\square$

51.

Proof.

(i) For any $x, y \in M$, if $f(x) = f(y)$, then we have $\rho(x, x_n) = \rho(y, x_n)$ for all $n \in \mathbb{N}$. Now since $\{x_n, n \in \mathbb{N}\}$ is dense in M, there exists a subsequence $x_{k_n} \to x$. But because $\rho(x, x_n) = \rho(y, x_n)$, we get $x_{k_n} \to y$ too. Therefore, $x = y$, which implies f is one to one. Moreover, let d be the metric of H^∞, then for any $x, y \in M$, we have

$$d(f(x), f(y)) = \sum_{n=1}^{\infty} \frac{1}{2^n} |\rho(x, x_n) - \rho(y, x_n)| \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \rho(x, y) = \rho(x, y).$$

Thus f is 1-Lipschitz, which implies f is continuous.

(ii) For any fixed $\epsilon > 0$ and $x \in M$, because $\{x_n, n \in \mathbb{N}\}$ is dense in M, there exists $m \in \mathbb{N}$ such that $\rho(x, x_m) < \frac{\epsilon}{2}$. Now let $\delta = \frac{1}{2^m} \cdot (\epsilon - 2\rho(x, x_m))$, then if $d(f(x), f(y)) < \delta$, then we have

$$\frac{1}{2^m} |\rho(x, x_m) - \rho(y, x_m)| \leq d(f(x), f(y)) < \delta = \frac{1}{2^m} \cdot (\epsilon - 2\rho(x, x_m)).$$

Therefore, $|\rho(x, x_m) - \rho(y, x_m)| < \epsilon - 2\rho(x, x_m)$. But then we would have

$$\begin{aligned} \rho(x, y) &\leq \rho(x, x_m) + \rho(y, x_m) \\ &= -\rho(x, x_m) + \rho(y, x_m) + 2\rho(x, x_m) \\ &\leq |\rho(x, x_m) - \rho(y, x_m)| + 2\rho(x, x_m) \\ &\leq \epsilon - 2\rho(x, x_m) + 2\rho(x, x_m) \\ &= \epsilon \end{aligned}$$

This means both f and f^{-1} are continuous, therefore $x_n \xrightarrow{\rho} x$ if and only if $f(x_n) \xrightarrow{d} f(x)$. Hence f is a homeomorphism

□

Exercise 52
Prove theorem 5.5.

Proof. If f is a homeomorphism then f is continuous. Hence $x_n \xrightarrow{d} x$ implies $f(x_n) \xrightarrow{\rho} f(x)$ (ii), $f(G)$ is open (closed) in N implies $G = f^{-1}(f(G)) = G$ open (closed) since f is one to one and onto ((iii) and (iv)).

Also because f is homeomorphism, thus f^{-1} is continuous. Thus $f(x_n) \xrightarrow{\rho} f(x)$ implies $x_n \xrightarrow{d} x$ and G is open (closed) in M implies $f(G) = f^{-1})^{-1}(G)$ is open (closed) since f is both one to one and onto.

So (i) implies (ii), (iii), and (iv). Conversely, any one of (ii), (iii), (iv) will imply f and f^{-1} to be continuous, thus f is a homeomorphism.

What is more, if $\hat{d}(x, y) = \rho(f(x), f(y))$ is equivalent to d, then $x_n \xrightarrow{d} x$ if and only if $x_n \xrightarrow{\hat{d}} x$ if and only if $f(x_n) \xrightarrow{\rho} f(x)$. Thus (v) is equivalent to (ii). So Theorem 5.5 is proved. □

53.

Proof. Let $f : M \mapsto \mathbb{R}$ defined by $f(a) = d(x, a)$ for every $a \in M$, thus f is continuous. Using the hypothesis, we get $f(x_n) \to f(x)$, that is $d(x, x_n) \to d(x, x) = 0$. Therefore, $x_n \to x$ in M. □

54.

Proof. If f is homeomorphism, thus $f : M \to N$ and $f^{-1} : N \to M$ are continuous. Using lemma 5.7, $g : N \to \mathbb{R}$ continuous implies $g \circ f : M \to \mathbb{R}$ continuous and $g \circ f : M \to \mathbb{R}$ continuous implies $(g \circ f) \circ f^{-1} = g$ continuous (since f is one to one and onto). Thus (i) implies (ii).

Conversely, assume that (ii) is true, for any $x_n, x \in M$ and $x_n \xrightarrow{d} x$, let $g : N \to \mathbb{R}$ defined by $g(a) = \rho(a, f(x))$. Hence g is continuous, which implies $g \circ f(a) = \rho(f(a), f(x))$ is continuous. Therefore, $x_n \xrightarrow{d} x$ implies $\rho(f(x_n), f(x)) \to \rho(f(x), f(x)) = 0$. So $f(x_n) \xrightarrow{\rho} f(x)$. Also, for any $f(x_n), f(x) \in N$, and $f(x_n) \xrightarrow{\rho} f(x)$, let $g : N \to \mathbb{R}$ defined by $g(a) = d(f^{-1}(a), x)$ (since f is one to one and onto, $f^{-1}(a)$ is defined). Thus $g \circ f(a) = d(f^{-1}(f(a)), x) = d(a, x)$, which is continuous. Using the hypothesis, we get g be continuous too. That is, since $f(x_n) \xrightarrow{\rho} f(x)$, then $g(f(x_n)) \xrightarrow{g} (f(x))$ or $d(x_n, x) \to d(x, x) = 0$. Thus $x_n \to x$.

So (ii) implies $x_n \to x$ if and only if $f(x_n) \to f(x)$. Using theorem 5.5, (ii) implies (i). So (i) and (ii) are equivalent. □

55.

Proof. Assume that M is separable, then there exists a countable dense subset X of M. We will prove that $f(X)$ is a countable dense subset of M. Indeed, for any open set E in N, since f is a homeomorphism, $f^{-1}(E)$ is open in M. Therefore $f^{-1}(E) \cap X \neq \varnothing$. This implies $E \cap f(X) \neq \varnothing$. So $f(X)$ is dense in N. Also, because X is countable, thus $f(X)$ is countable. So N is separable. Moreover, we have $f^{-1} : n \to M$ is a homeomorphism. Similarly, we get N being separable implies M being separable. Hence M is separable if and only if N is separable. □

56.

Proof.

(i) Let $f : S^1 \to [0, 2\pi)$ maps $(\cos(x), \sin(x)) \to x$. We can see that f is not continuous at $(1, 0)$ since $x_n = (\cos(2\pi - \frac{1}{n}), \sin(2\pi - \frac{1}{n})) \to (\cos(0), \sin(0)) = (1, 0)$. However, $f(x_n) = 2\pi - \frac{1}{n}$ doesn't converge to $f(1, 0) = 0$. So f is not continuous. But $f^{-1}(x)$ is continuous and f is a bijection, thus f is an open map. So f is an open map yet not continuous.

Let $f : \mathbb{R} \to \mathbb{R}$ maps $x \to |x|$. It's not hard to see that f is continuous, however, $f(-1, 1) = (1, 0]$ which is not open.

(ii) We know that $f : \mathbb{Q} \to \mathbb{R}$ map $f(x) \to x$ is continuous. However, $f([0, 1]) = \{x \in [0, 1] : x \in \mathbb{Q}\}$ which is not closed since its closure contain irrational numbers. So f is continuous yet not closed.

Reconsider the first function in part (i). We know that f is not continuous. But f^{-1} is continuous and bijective, thus f maps closed set to closed set. So f is closed yet not continuous.

□

57.

Proof.

1. (i)→(ii)

   If $f : M \to N$ is open, then for any closed set $U \in M$, since U^c is open, we get $f(U^c)$ being open. But f is one to one and onto, thus $f(U)^c = f(U^c)$ is open. Therefore $f(U)$ is closed.

2. (ii)→(iii)

   If $f : M \to N$ is closed, then for any U closed in M, we get $f(U)$ closed in N. Now, for any closed set $E \in M$, since f is one to one and onto, we have $(f^{-1})^{-1}(E) = f(E)$ closed in N. Therefore, f is continuous.

3. (iii)→(i)

   If $f^{-1} : N \to M$ is continuous, then for any open set U in M, we have $(f^{-1})^{-1}(U) = f(U)$ is open since f is one to one and onto. But this means f is open, thus f^{-1} continuous implies f to be open.

□

58.

Proof. Assume that f is homeomorphism. For any $x \in \overline{A}$, there exists $x_n \in A$ such that $x_n \to x$. Using the hypothesis, we get f continuous, thus $f(x_n) \to f(x)$. But since $f(x_n) \in f(A)$, we get $f(x) \in \overline{f(A)}$. So $f(\overline{A}) \subset \overline{f(A)}$ (1). What is more, f being a homeomorphism also implies f^{-1} to be continuous. Since f is onto, for any set $f(A)$ in N, if $f(x) \in \overline{f(A)}$, then there exists $f(x_n) \in f(A)$ such that $f(x_n) \to f(x)$. Thus $f^{-1}(f(x_n)) \to f^{-1}(f(x))$ or $x_n \to x$. So $f(x) \in f(\overline{A})$, which implies $\overline{f(A)} \subset f(\overline{A})$ (2). From (1) and (2), we get $f(\overline{A}) = \overline{f(A)}$.

Conversely, if $f(\overline{A}) = \overline{f(A)}$ for any subset A of M, then for any close set E of M, we have $f(E) = f(\overline{E}) = \overline{f(E)}$. Thus $f(E)$ is closed for any closed set E in M. That is f is closed. Since f is one to one and onto, for any closed set $f(A)$ in N, we have $f^{-1}(f(A)) = f^{-1}(\overline{f(A)}) = f^{-1}(f(\overline{A})) = \overline{A}$ which is closed. Therefore f^{-1} is closed. Since both f and f^{-1} are closed, we get f is a homeomorphism. □

60.
(i): (M, d) and (M, τ) are homeomorphism.
(ii) Every subset of M is open in (M, d).
(iii) Every function $f : (M, d) \to \mathbb{R}$ is continuous.

Proof.

(i) (i)$\to$ (ii)

Because $f : (M, d) \to (M, \tau)$ is a homeomorphism, for any subset E of M, E is open in (M, d) if and only if $f(E)$ is open in (M, τ). But any subset set in (M, τ) is open, thus E is open in (M, d) for any $E \subset M$. (But since E is closed in (M, τ) for any $E \subset M$, E is also closed in (M, d) right?)

(ii) (ii)$\to$ (iii)

For any E open in $\mathbb{R}$, we have $f^{-1}(E)$ is also open. Thus $f : (M, d) \to \mathbb{R}$ is continuous.

(iii) (iii)$\to$ (i)

Let f be the identity map in M, if $x_n \xrightarrow{d} x$ in M, then let $g : M \to \mathbb{R}$ maps $a \to \tau(f(a), f(x))$. Using the hypothesis, we get g continuous, thus $g(x_n) \to g(x)$, which is the same as $\tau(f(x_n), f(x)) \to \tau(f(x), f(x)) = 0$. So $f(x_n) \xrightarrow{\tau} f(x)$. Conversely, if $f(x_n) \xrightarrow{\tau} f(x)$, then $x_n \xrightarrow{\tau} x$, which means x_n will eventually equal x. Therefore $x_n \xrightarrow{d} x$. And since f is one to one and onto, we get f is a homeomorphism.

□

61.

Proof. First, notice that if $m \neq n$, then

$$\|e^{(m)} - e^{(n)}\|_1 = 2, \quad \|e^{(m)} - e^{(n)}\|_2 = \sqrt{2}, \quad \|e^{(m)} - e^{(n)}\|_\infty = 1.$$

Thus for any $e^{(m)} \in E \subset \{e^{(n)} : n \in \mathbb{N}\}$, we have $B^{c_0}_{\frac{1}{2}}(e^{(m)}) = B^{\ell_1}_{\frac{1}{2}}(e^{(m)}) = B^{\ell_2}_{\frac{1}{2}}(e^{(m)}) = B^{\ell_\infty}_{\frac{1}{2}}(e^{(m)}) = \{e^{(m)}\} \subset E$. So E is open in $c_0, \ell_1, \ell_2, \ell_\infty$ for any subset E of $\{e^{(n)} : n \in \mathbb{N}\}$.

Now, let $f : \mathbb{N} \to \{e^{(n)} : n \in \mathbb{N}\}$ map $n \to e^{(n)}$. Clearly f is open since every subset of $\{e^{(n)} : n \in \mathbb{N}\}$ is open. Thus f^{-1} is continuous. Moreover, $f^{-1}(V)$ is open for any open set $V \subset \{e^{(n)} : n \in \mathbb{N}\}$ because any subset of $\mathbb{N}$ is open. Therefore, f is continuous. Since f is also one to one and onto, we get f is a homeomorphism.

If we take the discrete metric on $\mathbb{N}$, then if $m \neq n \in \mathbb{N}$, then we have

$$d(m, n) = 1 = \|e^{(m)} - e^{(n)}\|_\infty.$$

Ihus f is an isometry. $\square$

63.

Proof.

(i) Because $[a, b]$ in an interval, thus $a \neq b$. Since $\frac{d\sigma}{dt} = (b-a)$, which is a constant, we get σ is a bijection. It's not hard to see that $\sigma(t)$ is continuous. Moreover, we have $\sigma^{-1}(t) = \frac{t-a}{b-a}$ which is also continuous. Thus σ is a homeomorphism.

(ii) Assume that $f \in C[a, b]$, then $f : [a, b] \to \mathbb{R}$ is continuous. Because $\sigma : [0, 1] \to [a, b]$ is also continuous, we have $f \circ \sigma : [0, 1] \to \mathbb{R}$ to be continuous. Therefore, $f \circ \sigma \in C[0, 1]$.

Assume that $f \circ \sigma \in C[0, 1]$, then $f \circ \sigma : [0, 1] \to \mathbb{R}$ is continuous. Because $\sigma^{-1} : [a, b] \to [0, 1]$ is continuous, we get $f \circ \sigma \circ \sigma^{-1} : [a, b] \to \mathbb{R}$, (which is $f : [a, b] \to \mathbb{R}$) is continuous. So $f \in C[a, b]$

(iii) For any $f, g \in C[a, b]$, we have

$$\begin{aligned} \|f \circ \sigma - g \circ \sigma\|_\infty &= \max_{0 \leq t \leq 1} |f \circ \sigma(t) - g \circ \sigma(t)| \\ &= \max_{0 \leq t \leq 1} |f(a + t(b-a)) - g(a + t(b-a))| \\ &= \max_{a \leq x \leq b} |f(x) - g(x)| \\ &= \|f - g\|_\infty. \end{aligned}$$

So the map $f \mapsto f \circ \sigma$ is an isometry from $C[a, b]$ to $C[0, 1]$.

(iv) For any $\alpha, \beta \in \mathbb{R}$ and $f, g \in C[a, b]$, we have

$$T(\alpha f + \beta g) = (\alpha f + \beta g) \circ \sigma = \alpha f \circ \sigma + \beta g \circ \sigma = \alpha T(f) + \beta T(g).$$

(v) For any $f, g \in C[a, b]$, we have

$$T(fg) = (fg) \circ \sigma = f \circ \sigma \cdot g \circ \sigma = T(f)T(g).$$

(vi) If $T(f) \leq T(g)$, then $f(\sigma(x)) \leq g(\sigma(x))$ for all $x \in [0, 1]$. Since σ is onto, we get $f(x) \leq g(x)$ for all $x \in [a, b]$. Conversely, if $f \leq g$, then $f(x) \leq g(x)$ for all $x \in [a, b]$. This implies $f(\sigma(x)) \leq g(\sigma(x))$ for all $x \in [0, 1]$. So $T(f) \leq T(g)$ if and only if $f \leq g$.

□

Exercise 1
Supply the missing details in the proof of Lemma 6.3.

Proof. If U and V are trivial subsets, then we can set $A = U$ and $B = V$. If U and V are not trivial, all we need to prove is the claim. Indeed, if y is in $B_{\epsilon_x}(x)$, then $y \in V$, which is contradict to U and V are disjoint. So $y \notin U$ or $\epsilon_x \leq d(x, y)$. Similarly, we get $\delta_y \leq d(x, y)$, thus

$$\frac{\epsilon_x}{2} + \frac{\delta_y}{2} \leq \frac{d(x, y)}{2} + \frac{d(x, y)}{2} = d(x, y).$$

So $B_{\frac{\epsilon_x}{2}}(x) \cap B_{\frac{\delta_y}{2}}(y) = \varnothing$. □

Chapter 6. Connectedness

Exercise 2
Show that the only nonempty connected subsets of Δ are singletons.

Proof. Let E be a nonempty connected subset of Δ. If E contains 2 distinct elements $x < y$, then there exists $z \in \mathbb{R}$ such that $z \notin \Delta$ and $x < z < y$. So $[x, y] \notin E$, contradiction. Therefore E has less than 2 elements, which means E is singleton since E is nonempty. $\square$

Exercise 5
If E and F are connected subsets of M with $E \cap F = \varnothing$, show that $E \cup F$ is connected.

Proof. Since $E \cap F \neq \varnothing$, let $x \in E \cap F$. Assume that $E \cup F$ is not connected, then there exists C a nontrivial clopen subset of $E \cup F$. Therefore, C^c is also a nontrivial clopen subset of $E \cup F$. So either $x \in C$ or $x \in C^c$. Without loss of generality, assume that $x \in C$, so $C \neq \varnothing$. Also because C is nontrivial in $E \cup F$, we get $C \neq E \cup F$. Thus either $E \not\subset C$ or $F \not\subset C$. Also without loss of generality, we assume that $E \not\subset C$. So C is a nontrivial clopen subset of E relatively to E. This implies E is disconnect, contradiction. So $E \cup F$ is connected. $\square$

Exercise 6
More generally, if C is a collection of connected subsets of M, all having a point in common, prove that $\cup C$ is connected. Use this to give another proof that $\mathbb{R}$ is connected.

Proof. Let $x \in \cap C$. If $\cup C$ is disconnected, then there exists $V \subset \cup C$ such that V is nontrivial clopen in $\cup C$. Without loss of generality, assume $x \in V$ (or else $x \in V^c$). Because V is nontrivial in $\cup C$, there exists $C_1 \in C$ such that $C_1 \not\subset V$. Thus V is a nontrivial clopen subset of C_1, which implies C_1 is disconnected. Contradiction. Hence $\cup C$ is connected. $\square$

Exercise 7
If every pair of points in M is contained in some connected set, show that M is itself connected.

Proof. Fix $a \in M$, and let A_b denote the connected set containing $a, b \in M$. Then using exercise 6, we get $\cup_{b \in M} A_b$ is connected. It's not hard to see that $\cup_{b \in M} A_b = M$. Therefore, M is connected. $\square$

Exercise 9
If $A \subset B \subset \overline{A} \subset M$ and if A is connected, show that B is connected. In particular, show that $\overline{A}$ is connected.

Proof. If B is disconnected, there exists a continuous function f from B onto $\{0,1\}$. However, because A is connected, $f(A)$ is a singleton. Without loss of generality, assume that $f(A) = 0$, then for any $x \in B \subset \overline{A}$, there exists $x_n \in A$ such that $x_n \to x$. Because f is continuous, we get $0 = f(x_n) \to f(x)$. Hence $f(x) = 0$ for all $x \in B$, which is contradict to the fact that f is onto. So B is connected. And since $\overline{A} \subset \overline{A}$, we have $\overline{A}$ is also connected. $\square$

Exercise 12
If M is connected and has at least two points, show that M is uncountable.

Proof. Let x and y be two distinct points in (M, d), we will claim that for any $0 \leq t \leq d(x, y)$, there exists $z \in M$ such that $d(x, z) = t$. Indeed, if there exists $0 \leq k \leq d(x, y)$ such that there is no $z \in M$ and $d(x, z) = k$, then consider $B_k(x)$. Number one, this set is obviously open. Number two, for any $t_n \in B_k(x)$ and $t_n \to t$, then because $d(x, t_n) < k$ and $d : M \to \mathbb{R}$ is continuous, we get $d(x, t) \leq k$. Since $d(x, t) \neq k$, we get $d(x, t) < k$ or $t \in B_k(x)$. So $B_k(x)$ is also closed. This ball is nontrivial since $x \in B_k(x) \neq \varnothing$ and $y \notin B_k(x)$ so $B_k(x) \neq M$. Because this ball is clopen, M is disconnected. Contradiction! So the claim is proved. Let $g : [0, d(x, y)] \to M$ map $t \mapsto x_t$ where x_t is a random point in M such that $d(x, x_t) = t$. It's not hard to see that g is one to one, hence the cardinality of M is larger than the cardinality of $[0, d(x, y)]$. But $[0, d(x, y)]$ is uncountable, M is also uncountable. □

Exercise 26
Let $f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that although f is not continuous, the graph of f is a connected subset of $\mathbb{R}^2$.

Proof. We know that f is discontinuous at 0, it is sufficient to show that the set $\{(x.f(x)) : x \in [0, 1]\}$ is connected in $\mathbb{R}^2$. Let $A = \{(x, f(x)) : x \in (0, 1]\}$ and $g : (0, 1] \to A$ maps $x \mapsto (x, f(x))$. For any $x_n, x \in (0, 1]$, if $x_n \to x$ then $f(x_n) \to f(x)$ (for f is continuous). This implies $g(x_n) = (x_n, f(x_n)) \to (x, f(x)) = g(x)$, so g is continuous. Notice that $(0, 1]$ is connected, thus A is connected.

Because $\frac{1}{2\pi n} \in (0, 1]$ for all $n \in \mathbb{N}$, we have $(\frac{1}{2\pi n}, f(\frac{1}{2\pi n})) \in A$. But $f(\frac{1}{2\pi n}) = \sin(2\pi n) = 0$, thus $(\frac{1}{2\pi n}, 0) \in A$. Since $(\frac{1}{2\pi n}, 0) \to (0, 0)$, we get $(0, 0) \in \overline{A}$. Using exercise 9, because $A \subset (A \cup (0, 0)) \subset \overline{A}$ and A is connected, we get $(A \cup (0, 0)$ is connected, or the graph of $f : [0, 1] \to \mathbb{R}$ is connected in $\mathbb{R}^2$. □

Exercise 27
Let V be a normed vector space and let $x \neq y \in V$. Show that the map $f(t) = x + t(y - x)$ is a homeomorphism from $[0, 1]$ into V. The range of f is the line segment joining x and y, and often written $[x, y]$ (since f is a homeomorphism, this interval notation is justified).

Proof. It's not hard to see that f is a bijection. If $t_n, t \in [0, 1]$ and $t_n \to t$, then $x + t_n(y - x) \to x + t(y - x)$. This is synonymous with $f(t_n) \to f(t)$. Conversely, if $f(t_n) \to f(t)$, then we have $x + t_n(y - x) \to x + t(y - x)$. Since $y - x$ doesn't equal the vector 0, we get $t_n \to t$. Thus f is a homeomorphism. □

P/S: I don't understand the text in the brackets. Why do we need f to be homeomorphism in oder to define the interval?

Exercise 28.
Deduce from exercise 7 and exercise 27 that any normed vector space is connected.

Proof. For any $x, y \in V$, let $f(t) = x + t(y - x)$, then by exercise 27, f is a continuous function, which maps $[0, 1]$ onto $[x, y]$. Since $[0, 1]$ is connected, we get $[x, y]$ is connected. So $\{x, y\}$ is contained in a connected set for all pair x, y in V. Using exercise 7, we get V is connected. $\square$

Chapter 7. Completeness

Exercise 1
If $A \subset B \subset M$, and B is totally bounded, show that A is totally bounded.

Proof. If B is totally bounded, then there exists $x_1, \cdots, x_n \in M$ such that $A \subset B \subset \cup_{i=1}^{n} B_\epsilon(x_i)$. So by the definition, we also get A is totally bounded. $\square$

Exercise 2
Show that a subset A of $\mathbb{R}$ is totally bounded if and only if it is bounded.

Proof. If A is bounded, then there exists $x, d \in \mathbb{R}$ and $A \subset B_d(x)$. Without loss of generality, let $d = 1$ and $x = 0$. Then for any $\epsilon > 0$, we have $\frac{1}{\epsilon}$ is finite. We define $a_i = i\epsilon$ and $b_i = -i\epsilon$, then

$$A \subset \left(\cup_{i=0}^{\frac{1}{\epsilon}} B_\epsilon(a_i)\right) \cup \left(\cup_{i=0}^{\frac{1}{\epsilon}} B_\epsilon(b_i)\right).$$

Thus A is totally bounded in $\mathbb{R}$. Conversely, if A is totally bounded in $\mathbb{R}$, then there exists $x_1, \cdots, x_n$ such that $A \subset \cup_{i=1}^{n} B_1(x_i)$. But then, let $d = \max\{d(x_1, x_i) : 1 \le i \le n\}$, using the triangular inequality, we get

$$A \subset B_{d+1}(x_1).$$

Thus A is bounded in $\mathbb{R}$. $\square$

Exercise 4
Show that A is totally bounded if and only if A can be covered by finitely many closed sets of diameter at most ϵ for every $\epsilon > 0$.

Proof. If A can be covered by finitely many closed sets, then obviously A is totally bounded. If A is totally bounded, then for $\epsilon > 0$, there exists $x_1, x_2, \cdots, x_n$ such that

$$A \subset \cup_{i=1}^{n} B_{\frac{\epsilon}{2}}(x_i) \subset \cup_{i=1}^{n} \overline{B_{\frac{\epsilon}{2}}(x_i)}$$

where $\overline{B_{\frac{\epsilon}{2}}(x_i)}$ is the closure of $B_{\frac{\epsilon}{2}}(x_i)$. Because for any $a, b \in B_{\frac{\epsilon}{2}}(x_i)$, using the triangular inequality, we know that the diameter of this set is less than ϵ. So $\overline{B_{\frac{\epsilon}{2}}(x_i)}$ are the sets we are looking for. $\square$

Exercise 5
Prove that A is totally bounded if and only if $\overline{A}$ is totally bounded.

Proof. Because $A \subset \overline{A}$, if $\overline{A}$ is bounded, then we get A is bounded. Conversely, if A is bounded, then for any $\epsilon > 0$, there exists $x_1, \cdots, x_n$ such that

$$A \subset \cup_{i=1}^{n} B_{\frac{\epsilon}{2}}(x_i).$$

We claim that $\overline{A} \subset \cup_{i=1}^{n} B_{\epsilon}(x_i)$. Indeed, for any $a \in \overline{A}$, there exists $a' \in A$ such that $d(a, a') < \frac{\epsilon}{2}$. Without loss of generality, assume that $a' \in B_{\frac{\epsilon}{2}}(x_1)$. Then using the triangular inequality, we get $d(a, x_1) \leq d(a, a') + d(a', x_1) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore $a \in B_{\epsilon}(x_1) \subset \cup_{i=1}^{n} B_{\epsilon}(x_i)$. So $\overline{A} \subset \cup_{i=1}^{n} B_{\epsilon}(x_i)$, which means $\overline{A}$ is totally bounded. $\square$

Exercise 8
If A is not totally bounded, show that A has an infinite subset B that is homeomorphic to a discrete space (where B is supplied with its relative metric).

Proof. If A is not totally bounded, then there exists $\epsilon > 0$ such that there exists no subset C of A where C is ϵ-dense in A. Let $x_1 \in A$, because $B_\epsilon(x_1)$ doesn't cover A, exists $x_2 \in A \setminus B_\epsilon(x_1)$. Using induction, let $x_n = A \setminus (\cup_{i=1}^{n-1} B_\epsilon(x_i)$. Thus $d(x_n, x_m) > \epsilon$ for all $m < n$. But $d(x_m, x_n) = d(x_n, x_m)$, so $d(x_m, x_n) > \epsilon$ for all $m \neq n$. Let $B = \{x_i : i \in \mathbb{N}\}$, we will show that B is homeomorphic to the discrete space $N = \{1, 2, \cdots\}$. Let $f : B \to N$ maps $x_i \mapsto i$, it's not hard to see f is a bijection. If $x_{n_k} \to x_m$, then because $d(x_{n_k}, x_m) > \epsilon$ for all $n_k \neq m$, we get n_k eventually equal m. Thus $n_k \to m$ in the discrete space. If $n_k \to m$ in N, then n_k will eventually equal m. Thus $x_{n_k} \to x_m$. So f is a homeomorphism, or B is homeomorphic to a dicrete space. □

Exercise 9
Give an example of a closed bounded subset of ℓ_∞ that is not totally bounded.

Proof. Let $x_n = (0, \cdots, 1, 0, \cdots)$ where the n-th entry is 1 and the rest are 0's. Because $\|x_m - x_n\|_\infty = 2 > 0$ for all $m \neq n$, the set $\{x_n : n \in \mathbb{N}\}$ is closed and bounded. However, there is no finite subset B of $\{x_n : n \in \mathbb{N}\}$ such that B is $\frac{1}{2}$-dense in $\{x_n : n \in \mathbb{N}\}$. Hence this set is not totally bounded. □

Exercise 10
Prove that a totally bounded metric space M is separable.

Proof. Assume that M is separable, then there exists a subset $\{x_{1,1}, \cdots, x_{1,n_1}\}$ of M, which is 1-dense in M. Similarly, there exists a $\frac{1}{k}$-dense subset of M $\{x_{k,1}, \cdots, x_{k,n_k}\}$ for all $k \in \mathbb{N}$. Since $A = \{x_{z,t} : z, t \in \mathbb{N}\, t \leq n_z\}$ is countable, it is sufficient to show that A is dense in M. Indeed, for any $a \in M$ and $\epsilon > 0$, let $k \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$. Because $\{x_{k,1}, \cdots, x_{k,n_k}\}$ is k-dense in M, there exists $x_{k,h}$ such that $a \in B_{\frac{1}{k}}(x_{h,k})$. Thus $a \in B_{\frac{1}{k}}(a) \subset B_\epsilon(a)$. So $B_\epsilon(a) \cap A \neq \varnothing$ for all $a \in M$ and $\epsilon > 0$. Thus A is dense in M. So M is separable by A. □

Exercise 12
Let A be a subset of an arbitrary metric space (M, d). If (A, d) is complete, show that A is closed in M.

Proof. Assume that A is complete in (M, d), then for any $x_n \in A$ and $x_n \to x$, we have (x_n) is Cauchy. Since A is complete, $x \in A$. Therefore A is closed. □

Exercise 15
Prove or disprove: If M is complete and $f : (M, d) \to (N, \rho)$ is continuous then $f(M)$ is complete.

Proof. Let $f : ((0,1), \text{discrete})) \to ((0,1), \text{normal})$ maps $x \mapsto x$. Then f is continuous, $((0,1), \text{discrete})$ is complete, yet $((0,1), \text{normal})$ is not complete. $\square$

P/S: Because completeness involve Cauchy sequences, to my understanding, it is not a topological property. However, I can't find an example where f is a homeomorphism. The closest function I found is the function above, where f is continuous, one to one and onto. However, f^{-1} is discontinuous.

Exercise 16
Prove that $\mathbb{R}^n$ is complete under any of the norms $\|\bullet\|_1$, $\|\bullet\|_2$, or $\|\bullet\|_\infty$.

Proof. Let (f_k) be a Cauchy sequence in $(\mathbb{R}^n, \|\bullet\|_1)$, then for any $\epsilon > 0$, there exists $N > 0$ such that $k, h > N$ implies $\|f_k - f_h\|_1 < \epsilon$, or

$$\sum_{i=1}^{n} |f_k(i) - f_h(i)| < \epsilon.$$

Notice that for any $j \in \mathbb{N}$ and $1 \le j \le n$, we have

$$|f_k(i) - f_h(i)| \le \sum_{i=1}^{n} |f_k(i) - f_h(i)| = \|f_k - f_h\|_1,$$

thus (f_n) Cauchy implies $(f_n(i))$ Cauchy for any $1 \le i \le n$. Thus $(f_n(i))$ is convergent respect to n. Let $f(i) = \lim_{n\to\infty} f_n(i) < \infty$, then we have $f \in \mathbb{R}^n$ and $f_n \to f$ under $\|\bullet\|_1$. This means $\mathbb{R}^n$ is complete under $\|\bullet\|_1$. Similarly, we have $\mathbb{R}^n$ is complete under $\|\bullet\|_2$ and $\|\bullet\|_\infty$. (Please let me know if you want further explanation.) □

Exercise 17
Given metric spaces M and N, show that $M \times N$ is complete if and only if both M and N are complete.

Proof. Let (M, d) and (N, ρ) be the metric spaces, and let $d_1 : M \times N \to \mathbb{R}$ maps $d_1((a, x), (b, y)) \mapsto d(a, b) + \rho(x, y)$ defines a metric on $M \times N$.

Assume that $M \times N$ is complete, then for any Cauchy sequence $(a_n) \in M$, by the definition, for any $\epsilon > 0$, there exists $N > 0$ such that $m, n > N$ implies $d_1((a_m, x), (a_n, x)) = d(a_m, a_n) < \epsilon$ for $x \in N$. So (a_n, x) is Cauchy in $M \times N$, which implies (a_n, x) converges. So (a_n) converges in M, or M is complete. Similarly, we get N is complete.

Assume that M and N are complete. Let (a_n, x_n) be a Cauchy sequence in $M \times N$, then for any $\epsilon > 0$, there exists N such that $m, n > N$ implies

$$d(a_m, a_n) < d_1((a_m, x_m), (a_n, x_n)) < \epsilon.$$

So (a_n) is Cauchy in M. But M is complete, thus (a_n) is convergent in M. Similarly, we get (x_n) is convergent in N. Thus (a_n, x_n) is convergent, or $M \times N$ is complete.

By exercise 3.46, all the metrics on $M \times N$ are equivalent to d_1, $M \times N$ is complete if and only if both M and N are complete. □

Exercise 18
Fill in the details of the proofs that ℓ_1 and ℓ_∞ are complete.

Proof. For any Cauchy sequence $f_n \in \ell_1$, using the definition, for any $\epsilon > 0$, there exists $N > 0$ such that $m, n > N$ implies

$$|f_n(i) - f_m(i)| \le \sum_{i=1}^{\infty} |f_n(i) - f_m(i)| = \|f_n - f_m\|_1 < \epsilon$$

for some fixed $i \in \mathbb{N}$. Thus $(f_n(i))$ is a Cauchy sequence respect to n. Since $f_n(i) \in \mathbb{R}$, we get $f_n(i) \to f(i)$. Next we will prove that $f \in \ell_1$. Because (f_n) is Cauchy, there exists $N_0 > 0$ such that $m, n > N_0$ implies $\|f_n - f_m\|_1 < 1$. Fix $n > N_0$, let $M = \max\{\|f_m - f_n\|_1 : m \leq N\} \cup \{1\}$, then for any $m \in \mathbb{N}$, we have

$$\|f_m\|_1 \leq \|f_m - f_n\|_1 + \|f_n\|_1 \leq M + \|f_n\|_1.$$

So f_n is bounded under $\|\bullet\|_1$. Let B be the upper bound of f_n, then

$$\sum_{i=1}^{N} f(i) = \lim_{n\to\infty} \sum_{i=1}^{N} f_n(i) \leq B$$

for all $N \in \mathbb{N}$. Thus $\|f\|_1 = \sum_{i=1}^{\infty} f(i) \leq B$, which means $f \in \ell_1$. Next we will prove that $f_n \to f$ respect to $\|\bullet\|_1$. Indeed, because (f_n) is Cauchy in ℓ_1, for any $\epsilon > 0$, there exists $n_0 > 0$ such that $m, n > n_0$ implies

$$\sum_{i=1}^{N} |f_m(i) - f(i)| = \lim_{n\to\infty} \sum_{i=1}^{N} |f_m(i) - f_n(i)| < \epsilon$$

for all $N \in \mathbb{N}$. Thus $\|f_m - f\|_1 = \sum_{i=1}^{\infty} |f_m(i) - f(i)| < \epsilon$ for all $m > n_0$. This means $f_m \to f$ in ℓ_1. So ℓ_1 is complete. Very similarly, we get ℓ_∞ is complete. □

Exercise 19
Prove that c_0 is complete by showing that c_0 is closed in ℓ_∞.

Proof. Let $f_n \in c_0$ and $f_n \to f$ in ℓ_∞, then it is sufficient to show that $f \in c_0$, that is $\lim_{i\to\infty} f(i) = 0$. Indeed, for any $\epsilon > 0$, because $f_n \to f$, there exists n_0 such that $n > n_0$ implies $\|f_n - f\|_\infty < \frac{\epsilon}{2}$. So $|f_n(i) - f(i)| < \frac{\epsilon}{2}$ for all $n > n_0$. Fix $n > n_0$, since $f_n \in c_0$, there exists $I > 0$ such that $|f_n(i)| < \frac{\epsilon}{2}$ for all $i > I$. Hence, we have

$$|f(i)| \leq |f_n(i)| + |f(i) - f_n(i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

For all $i > I$. But this is synonymous with $\lim_{i\to\infty} f(i) = 0$, or $f \in c_0$. So c_0 is closed. Because ℓ_∞ is complete, using theorem 7.9, we get c_0 is complete. □

Lemma . *Let (M, d) be a metric space and $a, b, c, d \in M$, then*

$$|d(a,b) - d(c,d)| \leq d(a,c) + d(b,d).$$

Proof. We have $d(a,b) \leq d(a,c) + d(c,d) + d(d,b)$, thus $d(a,b) - d(c,d) \leq d(a,c) + d(b,d)$. Moreover, we have $d(c,d) \leq d(a,c) + d(a,b) + d(b,d)$, thus $-d(a,c) - d(b,d) \leq d(a,b) - d(c,d)$. Therefore,

$$|d(a,b) - d(c,d)| \leq d(a,c) + d(b,d).$$

□

Exercise 20
If (x_n) and (y_n) are Cauchy in (M,d), show that $(d(x_n,y_n))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$.

Proof. Assume that $(x_n), (y_n)$ are Cauchy sequences in (M,d). For any $\epsilon > 0$, there exists $n_0 > 0$ such that $m, n > n_0$ implies $d(x_n, x_m) < \frac{\epsilon}{2}$ and $d(y_n, y_m) < \frac{\epsilon}{2}$. Using the Lemma, we get

$$|d(x_n, y_n) - d(x_m, y_m)| \leq d(x_n, x_m) + d(y_n, y_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So $(d(x_n, y_n))_{n=1}^{\infty}$ is Cauchy. $\square$

Exercise 21
If (M, d) is complete, prove that two Cauchy sequences (x_n) and (y_n) have the same limit if and only if $d(x_n, y_n) \to 0$.

Proof. Assume that (x_n) and (y_n) are Cauchy sequences in (M, d), because (M, d) is complete, there exists $x, y \in M$ such that $x_n \to x$ and $y_n \to y$. If $x = y$, then for any $\epsilon > 0$, there exists n_0 such that $n > n_0$ implies $d(x_n, x) < \frac{\epsilon}{2}$ and $d(y_n, y) < \frac{\epsilon}{2}$. Hence

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n) < \frac{\epsilon}{2} + 0 + \frac{\epsilon}{2} = \epsilon.$$

So $d(x_n, y_n) \to 0$.

Assume that $d(x_n, y_n) \to 0$. Because $x_n \to x$ and $y_n \to y$, we have $d(x_n, y_n) \to d(x, y)$. By the assumption, we get $d(x, y) = 0$, thus $x = y$. So (x_n) and (y_n) have the same limit if and only if $d(x_n, y_n) \to 0$. □

Exercise 24
Prove that the Hilbert cube H^∞ is complete.

Proof. For any Cauchy sequences $f_n \in (H^\infty, d)$ and $i \in \mathbb{N}$, we will show that $f_n(i)$ is Cauchy. For any $\epsilon > 0$, because (f_n) is Cauchy, there exists $n_0 > 0$ such that $m, n > n_0$ implies $d(f_m, f_n) < \epsilon$. Let $M_i = \max\{|f_n(1) - f_m(1)|, \cdots, |f_n(i) - f_m(i)|\}$, then using exercise 3.10, we get

$$|f_n(i) - f_m(i)| \leq M_i \leq 2^i d(f_n, f_m) \leq 2^i \epsilon.$$

Since $2^i\epsilon$ can be sufficiently small, $(f_n(i))_{i=1}^\infty$ is Cauchy, which implies $(f_n(i))_{i=1}^\infty$ is convergent. Define $f : \mathbb{N} \to \mathbb{R}$ such that $f_n(i) \to f(i)$ for all $i \in \mathbb{N}$. Because $|f_n(i)| \leq 1$, we get $|f(i)| \leq 1$. Hence $f \in H^\infty$.

Now, we will show that $f_n \to f$ in H^∞. Indeed, for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $2^{1-k} < \epsilon$. Using exercise 3.10, we know that

$$d(f_n, f) < M_{k,n} + 2^{1-k}$$

for any $n \in \mathbb{N}$, where $M_{k,n} = \max\{|f_n(1) - f_m(1)|, \cdots, |f_n(k) - f_m(k)|\}$. But we can make $M_{k,n}$ sufficiently small when n is big enough. More specifically, let $0 < \epsilon_1 < \epsilon - 2^{1-k}$, then because $f_n(i) \to f(i)$ for all $1 \leq i \leq k$, there exists $n_1 > 0$ such that $n > n_1$ implies $|f_n(i) - f(i)| < \epsilon_1$ for all $n > n_1$. Hence $M_{k,n} < \epsilon_1 < \epsilon - 2^{1-k}$, that is $d(f_n, f) < M_{n,k} + 2^{1-k} < \epsilon$ for all $n > n_1$. So $f_n \to f$, when ever (f_n) is Cauchy in H^∞. Hence H^∞ is complete. □

Exercise 30
If (M, d) is complete, prove that every open subset G of M is homeomorphic to a complete metric space. [Hint: Let $F = M \setminus G$ and consider the metric $\rho(x, y) = d(x, y) + \left|\frac{1}{d(x,F)} - \frac{1}{d(y,F)}\right|$ on G.]

Proof. Let $F = M \setminus G$. Because F is closed, thus $d(x, F) > 0$ for all $x \in G$. Let $\rho(x, y) = d(x, y) + \left|\frac{1}{d(x,F)} - \frac{1}{d(y,F)}\right|$ on G. Firstly, we will show that ρ defines a metric on G. For $x, y, z \in G$, because $d(x, y) > 0$, it's not hard to see that $\rho(x, y) > 0$. Moreover, $\rho(x, y) = \rho(y, x)$ is obvious. Notice that $0 < d(x, y) \le \rho(x, y)$, thus if $\rho(x, y) = 0$, then $d(x, y) = 0$, which implies $x = y$. If $x = y$, then $\rho(x, y) = 0$. What is more, we have

$$\begin{aligned}\rho(x, z) &= d(x, z) + \left|\frac{1}{d(x, F)} - \frac{1}{d(z, F)}\right| \\ &\le d(x, y) + d(y, z) + \left|\frac{1}{d(x, F)} - \frac{1}{d(y, F)}\right| + \left|\frac{1}{d(y, F)} - \frac{1}{d(z, F)}\right| \\ &= \rho(x, y) + \rho(y, z).\end{aligned}$$

So ρ defines a metric on G. Next we will show that (G, d) is homeomorphic to (G, ρ). Define $f : (G, d) \to (G, \rho)$ maps $x \mapsto x$. Obviously, f is a one to one and onto. If $x_n \xrightarrow{d} x$, then $d(x_n, x) \to 0$ and $\left|\frac{1}{d(x_n,F)}\right| \to \left|\frac{1}{d(x,F)}\right|$. Therefore, $\rho(x_n, x) \to 0$, which means $x_n \xrightarrow{\rho} x$. Conversely, if $x_n \xrightarrow{\rho} x$, then we have $\rho(x_n, x) \to 0$. But $0 < d(x_n, x) \le \rho(x_n, x)$, using the comparison test, we get $d(x_n, x) \to 0$, which means $x_n \xrightarrow{d} x$. So f is a homeomorphism from (G, d) to (G, ρ).

Lastly, we will prove that (G, ρ) is complete. Indeed, for any Cauchy sequence $x_n \in (G, \rho)$, notice that for $m, n \in \mathbb{N}$, we have

$$d(x_m, x_n) \le \rho(x_m, x_n).$$

So (x_n) is Cauchy in (M, d). But (M, d) is complete, thus there exists $x \in M$ such that $x_n \xrightarrow{d} x$. If $x \in G$, then because d and ρ are equivalent, we have $x_n \xrightarrow{\rho} x$. Because $x_n \in G$, x can be either in G or in its boundary. If $x \in G$, then we are done. If $x \in bdry(G)$, then $x_n \to x$ implies $d(x_n, F) \to 0$ or $\frac{1}{d(x_n,F)} \to \infty$. So for any $N > 0$, fix $n > N$, then we can always find $m \in \mathbb{N}$ big enough such that $m > N$ and

$$1 < \left|\frac{1}{d(x_m, F)} - \frac{1}{d(x_n, F)}\right| \le d(x_m, x_n) + \left|\frac{1}{d(x_m, F)} - \frac{1}{d(x_n, F)}\right| = \rho(x_m, x_n).$$

So (x_n) is not Cauchy in (G, ρ), contradiction. So $x \in G$ and thus (G, ρ) is complete. $\square$

Exercise 31
If $\sum_{n=1}^{\infty} x_n$ is a convergent series in a norm vector space X, show that $\|\sum_{n=1}^{\infty} x_n\| \le \sum_{n=1}^{\infty} \|x_n\|$.

Proof. Because $\sum_{n=1}^{\infty} x_n$ is converges, let $\sum_{n=1}^{\infty} x_n = x \in X$, then $\| \sum_{n=1}^{\infty} x_n\| = \|x\| < \infty$. If $\sum_{n=1}^{\infty} \|x_n\|$ is not a convergent series, then $\sum_{n=1}^{\infty} \|x_n\|$ doesn't exists??? loosely speaking, $\| \sum_{n=1}^{\infty} x_n\| = \|x\| < \infty = \sum_{n=1}^{\infty} \|x_n\|$. If $\sum_{n=1}^{\infty} \|x_n\|$ is convergent, let $\sum_{n=1}^{\infty} \|x_n\| = L$. For any $k \in \mathbb{N}$, using the triangular inequality, we have

$$\| \sum_{n=1}^{k} x_n\| \leq \sum_{n=1}^{k} \|x_n\| \leq \sum_{n=1}^{\infty} \|x_n\| = L.$$

Therefore, $\sum_{n=1}^{\infty} \|x_n\| \leq L$. $\square$

Exercise 32
Use Theorem 7.12 to prove that ℓ_1 is complete.

Proof. Assume that $f_n \in \ell_1$ and $\sum_{n=1}^{\infty} \|f_n\|_1 < \infty$, then $\sum_{n,i\in\mathbb{N}} |f_n(i)|$ is converges in $\mathbb{R}$. Therefore, $\sum_{n,i\in\mathbb{N}} f_n(i)$ is absolutely converges, so it is converges. In other word, we get $\sum_{n=1}^{\infty} f_n < \infty$. Using Theorem 7.12, ℓ_1 is complete. $\square$

Exercise 33
Let s denote the vector space of all finitely nonzero real sequences; that is, $x = (x_n) \in s$ if $x_n = 0$ for all but finitely many n. Show that s is not complete under the sup norm $\|x\|_\infty = \sup_n |x_n|$.

Proof. Let $f_n : \mathbb{N} \to \mathbb{R}$ be sequences in s, where $f_n(n) = \frac{1}{2^n}$ and $f_n(i) = 0$ for all $i \neq n$. Hence, $\|f_n\|_\infty = \frac{1}{2^n}$ for all $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \|f_n\|_\infty = \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

However, $\sum_{n=1}^{\infty} f_n$ doesn't converge, because, assume that $\sum_{n=1}^{\infty} f_n \to f \in s$, then because $f(i) \neq 0$ for a finite number of i, there exists $K \in \mathbb{N}$ such that $f(i) = 0$ for all $i > K$. But then, for any $n \in \mathbb{N}$, we would have

$$\left\| \sum_{n=1}^{\infty} f_n - f \right\| = \sup \left\{ \left| \frac{1}{2^i} - f(i) \right| : i \leq K \right\} \cup \left\{ \frac{1}{2^i} : I > K \right\} \geq \frac{1}{2^K},$$

which contradict to $\sum_{n=1}^{\infty} f_n \to f$. So $\sum_{n=1}^{\infty} f_n$ doesn't converge in s. By theorem 7.12, s is not complete. $\square$

Exercise 36

Proof. Let $\delta = \frac{1}{2}$. Then $|x - p_0| < \delta$ is synonymous with $|x| < \frac{1}{2}$. Thus

$$|f(x) - p_0| = |x^2| = |x|^2 < \frac{1}{2}|x| \leq |x| = |x - p_0| < \frac{1}{2}.$$

But because $|f(x)| = |f(x) - p_0| < \frac{1}{2}$, we have $|f^2(x) - p_0| < \frac{1}{2}|f(x)| < \frac{1}{4}|x|$. By mathematical induction, we get $|f^n(x) - p_0| < \frac{1}{2^n}|x| \to 0$. Thus $f^n(x) \to p_0$.

Also let $\delta = \frac{1}{2}$, then $|x - 1| < \frac{1}{2}$ implies $x > 0$. Therefore, $|x + 1| > 1$. Multiply both sides by $|x - 1$, we get $|(x - 1)(x + 1)| > |x - 1|$. Thus $|f(x) - p_1| = |x^2 - 1| > |x - 1| = |x - p_1|$. We will claim that $f^n(x) \not\to 1$. Indeed, if $f^n(x) \to 1$, then there exists $N > 0$ such that $n > N$ implies $|f^n(x) - 1| < \frac{1}{2}$. Fix n, using the result above, we have $|f^n(x) - 1| < |f^{n+1}(x) - 1| < \frac{1}{2}$. By mathematical induction, for any $m > n$, we get $|f^n(x) - 1| < |f^m(x) - 1| < \frac{1}{2}$, contradict to $f^n(x) \to 1$. Thus $f^n \not\to 1$. $\square$

Exercise 37
Suppose that $f : (a,b) \to (a,b)$ has a fixed point p in (a,b) and that f is differentiable at p. If $|f'(p)| < 1$, prove that p is an attracting fixed point for f. If $|f'(p)| > 1$, prove that p is a repelling fixed point for f.

Proof. Assume that $|f'(p)| = t < 1$, then by the $\epsilon - \delta$ definition, there exists $\delta > 0$ such that $|x-p| < \delta$ and $x \neq p$ imply

$$\left| \left| \frac{f(x)-f(p)}{x-p} \right| - t \right| < \frac{1-t}{2}.$$

Hence

$$\left| \frac{f(x)-f(p)}{x-p} \right| < t + \frac{1-t}{2} = \frac{1+t}{2} < 1.$$

Let $\frac{1+t}{2} = k$, then $|f(x)-f(p)| < k|x-p|$. But p is a fixed point, hence $|f(x)-p| < k|x-p|$ whenever $|x-p| < \delta$ and $x \neq p$. Similar to exercise 36, we get $f^n(x) \to p$. Thus p is an attracting fixed point (let me know if you want further explanation).

Assume that $|f'(p)| = t > 1$, then using the $\epsilon - \delta$ definition, there exists $\delta > 0$ such that $|x-p| < \delta$ and $x \neq p$ imply

$$\left| \left| \frac{f(x)-f(p)}{x-p} \right| - t \right| < \frac{t-1}{2}.$$

Hence

$$\frac{1-t}{2} < \left| \frac{f(x)-f(p)}{x-p} \right| - t.$$

Adding t both sides, we get

$$1 < \frac{1+t}{2} = \frac{1-t}{2} + t < \left| \frac{f(x)-f(p)}{x-p} \right|.$$

Multiply both sides by $|x-p|$, we get

$$|x-p| < |f(x)-f(p)| = |f(x)-p|.$$

So p is a rebelling fixed point of f. $\square$

Exercise 40
Extend the result in Example 7.15 as follows: Suppose that $F : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable in (a, b), and satisfies $F(a) < 0, F(b) > 0$, and $0 < K_1 \leq F'(x) \leq K_2$. Show that there is a unique solution to the equation $F(x) = 0$.

Proof. Because $K_1 > 0$, by the density of $\mathbb{R}$, there exists $\lambda > 0$ such that $0 < \lambda < \frac{1}{K_2}$. But $0 < F'(x) \leq K_2$, thus

$$0 < \lambda < \frac{1}{K_2} < \frac{2}{K_2} \leq \frac{2}{F'(x)}.$$

Multiply the distribution by $F'(x)$, we get $0 < \lambda F'(x) < 2$. Minus 1 in every term, we get $-1 < \lambda F'(x) - 1 < 1$, hence $|\lambda F'(x) - 1| < 1$. Let $f(x) = x - \lambda F(x)$, then $|f'(x)| = |1 - \lambda F'(x)| < 1$.

What is more, we have $\lambda < \frac{1}{K_2} \leq \frac{1}{F'(x)}$. Hence $0 < \lambda F'(x) < 1$. Because $f(x) = x - \lambda F(x)$, we have $f'(x) = 1 - \lambda F'(x) > 0$. Hence $f(x)$ is monotone on $[a, b]$. Because $F(a) < 0$ and $F(b) > 0$, we get

$$a < a - \lambda F(a) = f(a) \leq f(b) = b - \lambda F(b) < b.$$

So $f(x) \in [a, b]$ for all $x \in [a, b]$. Similar to Example 7.15, there is a unique fixed point $p \in [a, b]$, that is $p = f(p) = p - \lambda F(p)$, so $F(p) = 0$. That is, F has a unique zero in $[a, b]$. □

Exercise 43
Show that each of the hypotheses of the contraction mapping principle is necessary by finding examples of a space M and a map $f : M \to M$ having no fixed point where:

(a) M is incomplete (but f is still a strict contraction).

(b) f satisfies only $d(f(x), f(y)) < d(x, y)$ for all $x \neq y$ (but M is complete).

Proof.

(a) Let $f : (0, \frac{1}{4}) \to (0, \frac{1}{4})$ map $x \mapsto x^2$. Notice that, for any $x > y \in (0, \frac{1}{4})$, we have $x + y < \frac{1}{4} + \frac{1}{4} < \frac{1}{2}$. Therefore $(x - y)(x + y) < \frac{1}{2}(x - y)$. Expanding the left side, we get $x^2 - y^2 < \frac{1}{2}(x - y)$ or $|f(x) - f(y)| < \frac{1}{2}|x - y|$. So f is a contraction and by exercise 36, its fixed points can only be 0 or 1. Unfortunately, $0, 1 \notin (0, \frac{1}{4})$, so f has no fixed point.

(b) Let $f : [0, \infty) \to [0, \infty)$ maps $x \to \log(e^x + 1)$. For any $x < y \in [0, \infty)$, because log and e^x are increasing, we have $e^x < e^y$, thus $f(x) = \log(e^x + 1) < \log(e^y + 1) = f(y)$. So $d(f(x), f(y)) = f(y) - f(x) = \log(e^y + 1) - \log(e^x + 1)$. We will show that $\log(e^y + 1) - \log(e^x + 1) < d(x, y) = y - x$. Let $g(x) = x - \log(e^x + 1)$, then we have $g'(x) = 1 - \frac{e^x}{e^x + 1} > 0$. So g is increasing, which yields $x - \log(e^x + 1) = g(x) < g(y) = y - \log(e^y + 1)$, or

$$d(f(x), f(y)) = \log(e^y + 1) - \log(e^x + 1) < y - x = d(x, y).$$

However, because $f(x) = \log(e^x + 1) > \log(e^x) = x$, f has no fixed point.

$\square$

Exercise 44
Given any set M, check that $\ell_\infty(M)$ is a complete normed vector space.

Proof. For any Cauchy sequence $f_n \in \ell_\infty(M)$, using the definition, for any $\epsilon > 0$, there exists $N > 0$ such that $m, n > N$ implies

$$|f_n(i) - f_m(i)| \leq \sup_{x \in M} |f_n(x) - f_m(x)| = \|f_n - f_m\|_\infty < \epsilon$$

for some fixed $i \in M$. Thus $(f_n(i))$ is a Cauchy sequence respect to n. Since $f_n(i) \in \mathbb{R}$, we get $f_n(i)$ converges to some number, say $f_n(i) \to f(i)$.

Next, we will prove that $f \in \ell_\infty(M)$. Because $f_n(x)$ is bounded for all $x \in M$ and $n \in \mathbb{N}$, $S = \sup_{x \in M, n \in \mathbb{N}} |f_n(x)|$ exists. It's not hard to see that $|f(x)| \leq S$ for all $x \in M$, thus f is bounded, which yields $f \in \ell_\infty(M)$.

Finally, we will show that $f_n \to f$ respect to $\| \bullet \|_\infty$. For any $\epsilon > 0$, because (f_n) is Cauchy, there exists $N_0 > 0$ such that $n, m > N_0$ implies $\|f_m - f_n\|_\infty < \frac{\epsilon}{2}$. We will claim that $\|f_n - f\|_\infty < \epsilon$ for all $n > N_0$. Indeed, for any fixed $m > N_0$ and $n > N_0$, we have $\sup_{x \in M} |f_n(x) - f_m(x)| = \|f_n - f_m\|_\infty < \frac{\epsilon}{2}$, thus for any $x_0 \in M$, $|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$ for all $n > N_0$. Notice that

$$|f(x) - f_m(x)| < |f(x) - f_n(x)| + |f_n(x) - f_m(x)| < |f(x) - f_n(x)| + \frac{\epsilon}{2}$$

for all $n > N_0$ and $|f(x) - f_n(x)|$ can be sufficiently small since $f_n(x) \to f(x)$, we get $|f(x) - f_m(x)| \leq \frac{\epsilon}{2}$ for all $x \in M$. But this just means

$$\|f - f_m\|_\infty = \sup_{x \in M} |f(x) - f_m(x)| \leq \frac{\epsilon}{2} < \epsilon.$$

So the claim is proved, which implies $f_n \to f$ in $\ell_\infty(M)$. So $\ell_\infty(M)$ is complete. $\square$

Exercise 45
If M and N are equivalent sets, show that $\ell_\infty(M)$ and $\ell_\infty(N)$ are isometric.

Proof. If M and N are equivalent, then there exists a bijection $g : N \to M$. For all $f \in \ell_\infty(M)$, consider the map $f \mapsto f \circ g$. It's not hard to see that $f \circ g \in \ell_\infty(N)$, it's sufficient to show that $\|f - f'\|_\infty = \|f \circ g - f' \circ g\|_\infty$ for any $f, f' \in \ell_\infty(M)$. But notice that because g is a bijection, for any $x \in M$, we have $y = g^{-1}(x) \in N$ and $|f(x) - f'(x)| = |f(g(g^{-1}(x))) - f'(g(g^{-1}(x)))| = |f \circ g(y) - f' \circ g(y)|$. So,

$$\|f - f'\|_\infty = \sup_{x \in M} |f(x) - f'(x)| \leq \sup_{y \in N} |f \circ g(y) - f' \circ g(y)| = \|f \circ g - f' \circ g\|_\infty.$$

Similarly, we get $\|f \circ g - f' \circ g\|_\infty \leq \|f - f'\|_\infty$. So, $\|f - f'\|_\infty = \|f \circ g - f' \circ g\|_\infty$, which yields $\ell_\infty(M)$ and $\ell_\infty(N)$ are isometric. $\square$

Exercise 46
If A is a dense subset of a metric space (M, d), show that (A, d) and (M, d) have the same completion (isometrically).

Proof. Let $(\hat{M}, \hat{d})$ be the completion of (M, d), for simplicity of notation, let's suppose that $A \subset M \subset \hat{M}$ under the metric d. For any $x \in \hat{M}$ and $\epsilon > 0$, because M is dense in $\hat{M}$, we have $B_{\frac{\epsilon}{x}} \cap M \neq \varnothing$. Let $a \in B_{\frac{\epsilon}{x}} \cap M$, then because A is dense in M and $a \in M$, we have $B_{\frac{\epsilon}{2}}(x) \cap A \neq \varnothing$. But $a \in B_{\frac{\epsilon}{2}}(x)$, thus $d(a, x) < \frac{\epsilon}{2}$. Hence $B_{\frac{\epsilon}{2}}(a) \subset B_{\epsilon}(x)$. This yields

$$\varnothing \neq B_{\frac{\epsilon}{2}} \cap A \subset B_{\epsilon}(x) \cap A$$

for all $x \in \hat{M}$. So A is dense in $\hat{M}$. And since $\hat{M}$ is complete, it is the completion of A. So (A, d) and (M, d) have the same completion. $\square$

Chapter 8. Compactness

Exercise 1
If K is a nonempty compact subset of $\mathbb{R}$, show that $\sup K$ and $\inf K$ are elements of K.

Proof. Because there are sequences $x_n, y_n \in K$ such that $x_n \to \sup K$ and $y_n \to \inf K$, and because K is compact, thus closed, we get $\sup K, \inf K \in K$. $\square$

Exercise 2
Let $E = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$, considered as a subset of $\mathbb{Q}$. Show that E is closed and bounded but not compact.

Proof. First, notice that E can be written as

$$E = \{x \in \mathbb{Q} : -\sqrt{3} < x < -\sqrt{2} \text{ or } \sqrt{2} < x < \sqrt{3}\}.$$

If $x_n \in E$ and $x_n \to x$ where $x \in \mathbb{Q}$. Because x_n is convergent, without loss of generality, assume that eventually, $\sqrt{2} < x_n < \sqrt{3}$. Thus, we get $\sqrt{2} \leq x \leq \sqrt{3}$. But $x \in \mathbb{Q}$, so $\sqrt{2} < x < \sqrt{3}$, which means $x \in E$. So E is closed. The fact that E is bounded is very clear, specifically by $-\sqrt{3}$ and $\sqrt{3}$. However, E is not compact because the sequence

$$x_1 = 1.41, x_2 = 1.4142, x_3 = 1.414213, \cdots$$

are in E, yet it converges to $\sqrt{2}$, which is not even in $\mathbb{Q}$. $\square$

Exercise 3
If A is compact in M, prove that $diam(A)$ is finite. Moreover, if A is nonempty, show that there exits points x and y in A such that $diam(A) = d(x, y)$.

Proof. Because A is compact, A is totally bounded, thus bounded. Therefore, its diameter is finite. Because $diam(A) = \sup\{d(a,b) : a,b \in A\}$, there exists $a_n, b_n \in A$ such that $d(a_n, b_n) \to diam(A)$. Since A is compact, a_n has a convergent subsequence a_{n_k} that converge to a. Also because A is compact, b_{n_k} has a convergent subsequence $b_{n_{k_l}}$, which converges to b. So $d(a,b) = \lim_{l\to\infty} d(a_{n_{k_l}} b_{n_{k_l}}) = \lim_{n\to\infty} d(a_n, b_n) = diam(A)$, but A is closed, thus $a, b \in A$. So there exist a, b such that $d(a,b) = diam(A)$. □

Exercise 4
If A and B are compact sets in M, show that $A \cup B$ is compact.

Proof. For any sequence (x_n) in $A \cup B$, there would be infinitely many element in either A or B (or both). Without loss of generality, assume that there is a subsequence $x_{n_k} \in A$, then because A is compact, it has a subsequence that converges to a point of $A \subset A \cup B$. So (x_n) has a subsequence that converge to a point of $A \cup B$, which means $A \cup B$ is compact. □

Exercise 6
If A is compact in M and B is compact in N, show that $A \times B$ is compact in $M \times N$.

Proof. For any sequence $(a_n, b_n) \in A \times B$, because A is compact, there exists a convergent subsequence $a_{n_k} \to a$, in A. And because B is compact, there exists a convergent subsequence $b_{n_{k_l}} \to b$ in B. Then (a_n, b_n) has a subsequence $(a_{n_{k_l}}, b_{n_{k_l}}) \to (a,b)$ in $A \times B$. Therefore, $A \times B$ is compact. □

Exercise 7
If K is a compact subset of $\mathbb{R}^2$, show that $K \subset [a, b] \times [c, d]$ for some pair of compact intervals $[a, b]$ and $[c, d]$.

Proof. If K is compact in $\mathbb{R}^2$, then K is bounded, thus K is bounded respect to the x-axis and K is bounded respect to the y-axis. So there are a, b, c, d such that for any $(x, y) \in K$, $x \in [a, b]$ and $y \in [c, d]$. So $K \subset [a, b] \times [c, d]$. □

Exercise 8
Prove that the set $\{x \in \mathbb{R}^n : \|x\|_1 = 1\}$ is compact in $\mathbb{R}^n$ under the Euclidean norm.

Proof. Let $A = \{x \in \mathbb{R}^n : \|x\|_1 = 1\}$, for any $f_n \in \mathbb{R}^n$ and $f_n \to f$ in $\mathbb{R}^n$, we have $f_n(i) \to f(i)$ for all $1 \leq i \leq n$. Thus $\sum_{i=1}^n |f_n(i)| \to \sum_{i=1}^n |f(i)|$. But $f_n \in A$, thus $\|f_n\|_1 = \sum_{i=1}^n |f_n(i)| = 1$ for all n, thus $\|f\|_1 = \sum_{i=1}^n |f(i)| = 1$. So $f \in A$, which means A is closed. Moreover, if $f \in A$, then we have

$$\|f\|_2 \leq \|f\|_1 = 1.$$

So A is bounded under the Euclidean norm. And since A is closed, it is compact. □

Exercise 11
Prove that compactness is not a relative property. That is, if K is compact in M, show that K is compact in any metric space that contains it (isometrically).

Proof. If K is compact in M, then any sequence in K has a convergent subsequence. Since the convergence only depends on the metric, K is compact in any metric space that contains it. So compactness is not a relative property. □

Exercise 14
Show that the Hilbert cube H^∞ is compact.

Proof. In exercise 7.24, we have showed that H^∞ is complete, so it's prerequisite to show that H^∞ is totally bounded. For any $\epsilon > 0$, there exists $N > 0$ such that $\sum_{n=N}^\infty 2^{-n} < \frac{\epsilon}{2}$. Now we just look at the first N digits of elements in H^∞. Notice that the set $A = \{x \in \mathbb{R}^N : |x(i)| \leq 1 \text{ for all } 1 \leq i \leq N\}$ is bounded in $(\mathbb{R}^N, \|\bullet\|_1)$ because $\|x\|_1 \leq$ for all $x \in A$. So this set is totally bounded, which yields the existence of a $\frac{\epsilon}{2}$-net $\{x_1, \cdots, x_k\}$ of A. But for $x, y \in A$, we have

$$d(x - y) = \sum_{n=1}^N 2^{-n}|x(n) - y(n)| \leq \sum_{n=1}^N |x(n) - y(n)| = \|x - y\|_1.$$

Therefore, the set $\{x_1, \cdots, x_k\}$ is also $\frac{\epsilon}{2}$-dense in (A, d). We will claim that $\{x_1, \cdots, x_k\}$ is ϵ-dense in H^∞. Indeed, for any $y \in H^\infty$, there exists x_i such that

$$\sum_{n=1}^N 2^{-n}|x_i(n) - y(n)| < \frac{\epsilon}{2}.$$

But $x_i \in A$, thus $x_i(n) = 0$ for all $n > N$. So

$$d(x_i, y) = \sum_{n=1}^{N} 2^{-n}|x_i(n) - y(n)| + \sum_{n=N+1}^{\infty} 2^{-n}|x_i(n) - y(n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So H^∞ is totally bounded and complete, which imply H^∞ is compact. □

Exercise 15
If A is totally bounded subset of a complete metric space M, show that $\overline{A}$ is compact in M. For this reason, totally bounded sets are sometimes called precompact or conditionally compact.

Proof. Because $\overline{A}$ is a closed subset of a complete metric space, $\overline{A}$ is complete. Moreover, because A is totally bounded, using exercise 7.5, we have $\overline{A}$ is totally bounded. Thus $\overline{A}$ is compact. □

Exercise 16
Show that a metric space M is totally bounded if and only if its completion $\hat{M}$ is compact.

Proof. Without loss of generality, assume that $M \subset \hat{M}$. Then $\hat{M}$ is just equals $\overline{M}$. If M is totally bounded, using exercise 15, $\hat{M}$ is compact. In the opposite direction, if $\hat{M}$ is compact, then $\hat{M}$ is totally bounded. And since $M \subset \hat{M}$, M is also totally bounded. □

Exercise 17
If M is compact, show that M is also separable.

Proof. For any $n \in \mathbb{N}$, because M is totally bounded, there exists a finite $\frac{1}{n}$-dense subset $E_n = \{x_n(1), x_n(2), \cdots, x_n(k_n)\}$ of M. Let $E = \cup_{n=1}^{\infty} E_n$. Because E_n is finite for any $n \in \mathbb{N}$, E is thus countable. For any $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Since E_n is $\frac{1}{n}$-dense in M, we have $\varnothing \neq B_\epsilon(x) \cap E_n \subset B_\epsilon(x) \cap E$ for all $x \in M$. So E is a countable dense subset of M, which yields M is separable. □

Exercise 19
Prove that M is separable if and only if M is homeomorphic to a totally bounded metric space (specifically, a subset of the Hilbert cube).

Proof. If M is homeomorphic to a subset of the Hilbert cube, then because H^∞ is separable and it is a topological property, M is also separable. For the other direction, if (M, ρ) is separable, then there exists a countable dense subset $\{x_1, \cdots\}$ of M. Define $f : M \to H^\infty$ maps $x \mapsto (\rho(x, x_n))_{n=1}^{\infty}$. By exercise 5.51, f is a homeomorphism to $f(M) \subset H^\infty$. And since H^∞ is a totally bounded metric space, $f(M)$ is also totally bounded. So M is homeomorphic to a totally bounded metric space. □

Exercise 21
Prove Corollary 8.6.

Proof. If $f[a,b]$ is disconnected, it can be break into two disjoint open sets A and B. But because f is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are open. Moreover, if $a \in f^{-1}(A) \cap f^{-1}(B)$, then $f(a) \in A \cap B = \varnothing$, contradiction. Thus $f^{-1}(A) \cap f^{-1}(B) = \varnothing$. This implies $f^{-1}(f[a,b]) = [a,b]$ is disconnected, which is not true. So $f[a,b]$ must be connected. Now, because $[a,b]$ is compact, $f[a,b]$ is bounded and f attains its maximum and minimum values (Corollary 8.5). Let $c = \min f[a,b]$ and $d = \max f[a,b]$, then since $f[a,b]$ is connected, we have $f[a,b] = [c,d]$. $\square$

Exercise 22
If M is compact and $f : M \to N$ is continuous, prove that f is a closed map.

Proof. For any closed set C in M, we will show that $f(C)$ is closed. For any sequence $f(x_n) \to y$, because $f(M)$ is compact, we have $y \in f(C)$. Notice that because M is compact, thus x_n has a convergent subsequence $x_{n_k} \to x$. But f is continuous, thus $f(x_{n_k}) \to f(x)$. So $f(x_n) \to f(x)$, thus $y = f(x) \in f(C)$. Hence $f(C)$ is closed, which means f is a closed map. $\square$

Exercise 23
Suppose that M is compact and that $f : M \to N$ is continuous, one-to-one, and onto. Prove that f is a homeomorphism.

Proof. By exercise 22, f is closed, so f^{-1} is continuous. But f is also continuous, one to one and onto. Thus f is a homeomorphism. $\square$

Exercise 25
Let V be a normed vector space, and let $x \neq y \in V$. Show that the map $f(t) = x+t(y-x)$ is a homeomorphism from $[0,1]$ into V. The range of f is the line segment joining x and y; it is often written as $[x, y]$.

Proof. First, if $t_n \to t$ in $[0,1]$, then it's not hard to see that $x+t_n(y-x) \to x+t(y-x)$. So f is continuous. If $f(t) = f(t')$, then we would have $x + t(y - x) = x + t'(y - x)$. Minus x both sides, we get $t(y - x) = t'(y - x)$. But $y - x \neq 0$, thus $t = t'$. So f is one to one. Using exercise 23, f is a homeomorphism from $[0,1]$ to $f[0,1]$. $\square$

Exercise 29
Let M be a compact metric space and suppose that $f : M \to M$ satisfies $d(f(x), f(y)) < d(x, y)$ whenever $x \neq y$. Show that f has a fixed point.

Proof. Because f is 1-Lipschitz, f is continuous. So for any $x_n \to x$ in M, we also have $f(x_n) \to f(x)$. Hence, $d(x_n, f(x_n)) \to d(x, f(x))$. Let $g(x) = d(x, f(x))$, we have showed that $g : M \to \mathbb{R}$ is continuous. Since M is compact, $g(M)$ is also compact, thus it has a minimum $g(m)$. But notice that, if $m \neq f(m)$, then we would have $g(f(m)) = d(f(m), f^2(m)) < d(m, f(m)) = g(m)$, which contradict to the assumption that $g(m)$ is the minimum of g. Thus $f(m) = m$, which means m is a fixed point of f. $\square$

Exercise 30
Prove lemma 8.8.

Proof. Assume that we have (a), we will prove that (b) is true. Assume that $\cap_{i=1}^n F_i \neq \varnothing$ for all choices of finitely many $F_1, \cdots, F_n \in \mathcal{F}$. If $\cap\{F : F \in \mathcal{F}\}$, then take the complement both sides, we get $M \subset \cup\{F^c : F \in \mathcal{F}\}$. But by (a), there are finite open sets $F_1^c, \cdots, F_n^c \in \mathcal{F}$ such that $M \subset \cup_{i=1}^n F_i^c$. Taking the complement both sides again, we would get $\cap_{i=1}^n F_i = \varnothing$, contradiction. So $\cap\{F : F \in \mathcal{F}\} \neq \varnothing$.

Assume that (b) is true and $\cup\{G : G \in \mathcal{G}\} \supset M$. If there don't exist $G_1, \cdots, G_n$ such that $\cup_{i=1}^n G_i \supset M$, then for any finite closed sets $G_1^c, \cdots, G_n^c$, we get $\cap_{i=1}^n G_i^c \neq \varnothing$. But by (b), we get $\cap\{G^c : G \in \mathcal{G}\} \neq \varnothing$, or $\cup\{G : G \in \mathcal{G}\} \not\supset M$. Contradiction! So there exist $G_1, \cdot, G_n \in \mathcal{G}$ such that $\cup_{i=1}^n G_i \supset M$. $\square$

Exercise 31
Given an arbitrary metric space M, show that a decreasing sequence of nonempty compact sets in M has nonempty intersection.

Proof. Because these sets are compact, thus closed. Using corollary 8.10, we get this sequence has nonempty intersection. □

Exercise 32
Prove Corollary 8.11 by showing that the following two statements are equivalent.

(i) Every decreasing sequence of nonempty closed sets in M has nonempty intersection.

(ii) Every countable open cover of M admits a finite subcover; that is, if (G_n) is a sequence of open sets in M satisfying $\bigcup_{n=1}^{\infty} G_n \supset M$, then $\bigcup_{n=1}^{N} G_n \supset M$ for some (finite) N.

Proof. If (i) is true, then M is compact. Thus by Lemma 8.8, if M is covered by countably many open sets, it can be covered by a finite number of open sets. In the opposite direction, if (ii) is true, then for any decreasing sequence of nonempty closed sets $F_1, \cdots \subset M$, we will show that it has nonempty intersection. Assume that $\bigcap_{i=1}^{\infty} F_i = \varnothing$, then taking the completion both sides, we get $\bigcup_{i=1}^{\infty} F_i^c = M$. Using (ii), there exists $N \in \mathbb{N}$ such that $\bigcup_{i=1}^{N} F_i^c \supset M$, so $\bigcap_{n=1}^{N} F_i = \varnothing$. But F_n is a decreasing sequence of sets, thus we have $F_{N+1} \subset \bigcap_{n=1}^{N} F_i = \varnothing$, which yields $F_{N+1} = \varnothing$. Contradiction. So $\cap_{i=1}^{\infty} F_i \neq \varnothing$. $\square$

P/S: I feel like my proof is repeating what I did in Lemma 8.8. I tried to utilize lemma 8.8 to shorten my proof but no hope. Did I missed something?

Exercise 36
Let F and K be disjoint, nonempty subsets of a metric space M with F closed and K compact. Show that $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} > 0$. Show that this may fail if we assume only that F and K are disjoint closed sets.

Proof. If $d(F, K) = \inf\{d(x, y) : x \in F, y \in K\} = 0$, then there exists $d(x_n, y_n) \to 0$ where $x_n \in F$ and $y_n \in K$. But K is compact, thus there is a subsequence $y_{n_k} \to y$ in K. Thus we have $d(x_{n_k}, y_{n_k}) \to 0$, hence $d(x_{n_k}, y) \to 0$. But this just means $x_{n_k} \to y$, which implies $y \in F$ since F is closed. So $y \in F \cap K = \varnothing$, contradiction. So $d(F, K) > 0$.

Notice that F and K are only disjoint closed set will not be enough to imply $d(F, K) > 0$. For example, $F = [0, 1)$ and $K = (1, 2]$ are closed in $\mathbb{R} \setminus \{1\}$ and disjoint, yet $d(F, K) = 0$. $\square$

Exercise 44
Show that any Lipschitz map $f : (M, d) \to (N, \rho)$ is uniformly continuous. In particular, any isometry is uniformly continuous.

Proof. Assume that f is K-Lipschitz, that is $\rho(f(x), f(y)) < Kd(x, y)$ for all $x, y \in M$. So for any $\epsilon > 0$, choose $\delta = \frac{\epsilon}{K}$. Then, $d(x, y) < \delta = \frac{\epsilon}{K}$ will imply $\rho(f(x), f(y)) < Kd(x, y) < K\frac{\epsilon}{K} = \epsilon$. So f is uniformly continuous. Since an isometry is 2-Lipschitz, any isometry is uniformly continuous. $\square$

Exercise 45
Prove that every map $f : \mathbb{N} \to \mathbb{R}$ is uniformly continuous.

Proof. For any $\epsilon > 0$, just choose $\delta = \frac{1}{2}$. Then, for $x, y \in \mathbb{N}$, $|x - y| < \delta = \frac{1}{2}$ will imply $x = y$. Thus $|f(x) - f(y)| = 0 < \epsilon$. So f is uniformly continuous. $\square$

Exercise 48
Prove that uniformly continuous map sends Cauchy sequences to Cauchy sequences.

Proof. Assume that $f : (M, d) \to (N, \rho)$ is uniformly continuous and (x_n) is a Cauchy sequence in M, then for any $\epsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \epsilon$. But (x_n) is a Cauchy sequence in M, thus there exists $N > 0$ such that $d(x_n, x_m) < \delta$ whenever $n, m > N$. Thus $n, m > N$ implies $\rho(f(x_n), f(x_m)) < \epsilon$, which means $(f(x_n))$ is a Cauchy sequence in N. So uniformly continuous map sends Cauchy sequences to Cauchy sequences. $\square$

Exercise 51
If $f : (0, 1) \to \mathbb{R}$ is uniformly continuous, show that $\lim_{x\to 0^+} f(x)$ exists. Conclude that f is bounded on $(0, 1)$.

Proof. We know that $(\frac{1}{n})$ is Cauchy in $(0, 1)$. Because f is uniformly continuous, it maps Cauchy sequences to Cauchy sequences, thus $(f(\frac{1}{n}))$ is Cauchy in $\mathbb{R}$. So there exists $y \in \mathbb{R}$ such that $\lim_{n\to\infty} f(\frac{1}{n}) = y$. Now for any sequence $(x_n) \to 0^+$, we have $|x_n - \frac{1}{n}| \leq |x_n| + |\frac{1}{n}$ which can be sufficiently small. Thus $|x_n - \frac{1}{n}| \to 0$. Using exercise 56, we get $|f(x_n) - f(\frac{1}{n})| \to 0$. But $|f(x_n) - y| \leq |f(x_n) - f(\frac{1}{n})| + |f(\frac{1}{n}) - y|$ which can be sufficiently small too. Thus $f(x_n) \to y$. So $lim_{x\to 0^+} f(x)$ exists, and similarly, $lim_{x\to 1^-} f(x)$ exists. So f is defined on $[0, 1]$, a compact set. Thus f is bounded and have minimum and maximum. $\square$

Exercise 53
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $f(x) \to 0$ as $x \to \pm\infty$. Prove that f is uniformly continuous.

Proof. For any $\epsilon > 0$, because $\lim_{x\to\pm\infty} f(x) = 0$, there exists $N > 0$ such that $|x| > N$ implies $|f(x)| < \frac{\epsilon}{2}$. Notice that $[-N - 1, N + 1]$ is compact in $\mathbb{R}$, thus there exists $\delta > 0$ such that $x, y \in [-N-1, N+1]$ and $|x-y| < \delta$ imply $|f(x)-f(y)| < \epsilon$. Let $\delta' = \min\{\delta, \frac{1}{2}\}$, then for any $x, y \in \mathbb{R}$, whenever $|x - y| < \delta'$, we will prove that $|f(x) - f(y)| < \epsilon$. Indeed, if $|x|$ and $|y|$ are smaller than $N + 1$, then $|f(x) - f(y)| < \epsilon$ because $|x - y| < \delta$. If $|x| > N + 1$, then $|x - y| < \frac{1}{2}$ implies $|y| > |x| - |x - y| > N + 1 - \frac{1}{2} > N$. So $|f(x)|, |f(y)| < \frac{\epsilon}{2}$. Hence $|f(x) - f(y)| < |f(x)| + |f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So f is uniformly continuous. $\square$

Exercise 56
Prove that $f : (M, d) \to (N, \rho)$ is uniformly continuous if and only if $\rho(f(x_n), f(y_n)) \to 0$ for any pair of sequences (x_n) and (y_n) in M satisfying $d(x_n, y_n) \to 0$.

Proof. If f is uniformly continuous, then for $x_n, y_n \in M$ satisfying $d(x_n, y_n) \to 0$, we will prove that $\rho(f(x_n), f(y_n)) \to 0$. Indeed, for any $\epsilon > 0$, because f is uniformly continuous, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \epsilon$. But $d(x_n, y_n) \to 0$, there exists $N > 0$ such that $n > N$ implies $d(x_n, y_n) < \delta$, thus $\rho(f(x_n), f(y_n)) < \epsilon$. So $\rho(f(x_n), f(y_n)) \to 0$.

For the other direction, assume that $\rho(f(x_n), f(y_n)) \to 0$ for any pair of sequences $(x_n), (y_n)$ in M satisfying $d(x_n, y_n) \to 0$. If f is not uniformly continuous, there exists $\epsilon > 0$ such that for any $\delta > 0$, there exists $x, y \in M$ such that $d(x, y) < \delta$ but $\rho(f(x), f(y)) > \epsilon$. Thus let x_n and y_n be two points in M such that $d(x_n, y_n) < \frac{1}{n}$ yet $\rho(f(x_n), f(y_n)) > 1$. So $d(x_n, y_n) \to 0$, but $\rho(f(x_n), f(y_n)) > \epsilon$ for all $n \in \mathbb{N}$, thus $\rho(f(x_n), f(y_n)) \not\to 0$. Contradiction. Therefore, f is uniformly continuous. $\square$

Exercise 57
A function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy a Lipschitz condition of order α, where $\alpha > 0$, if there is a constant $K < \infty$ such that $|f(x) - f(y)| \leq K|x - y|^\alpha$ for all x, y. Prove that such a function is uniformly continuous.

Proof. If $K = 0$, then $|f(x) - f(y)| \leq 0$ for all x, y, thus f is a constant, which means f is uniformly continuous. If $K \neq 0$, for any $\epsilon > 0$, choose $\delta = \sqrt[\alpha]{\frac{\epsilon}{K}}$, then $|x - y| < \delta$ implies $|f(x) - f(y)| \leq K|x - y|^\alpha < K(\sqrt[\alpha]{\frac{\epsilon}{K}})^\alpha = \epsilon$. So f is uniformly continuous. $\square$

Exercise 58
Show that any function $f : \mathbb{R} \to \mathbb{R}$ having a bounded derivative is Lipschitz of order 1.

Proof. Assume that $f'(x)$ exists and $|f'(x)| < N$ for all $x \in \mathbb{R}$, then for any $x \neq y \in \mathbb{R}$, we would have

$$\left| \frac{f(x) - f(y)}{x - y} \right| < \mathbb{N}.$$

Thus

$$|f(x) - f(y)| < N|x - y|$$

for all $x, y \in \mathbb{R}$. So f is Lipschitz of order 1. $\square$

Exercise 61
Two metric spaces (M, d) and (N, ρ) are said to be uniformly homeomorphic if there is a one to one and onto map $f : M \to N$ such that both f and f^{-1} are uniformly continuous. In this case we say that f is uniform homeomorphism. Prove that completeness is preserved by uniform homeomorphisms.

Proof. Assume that (M, d) and (N, ρ) are uniformly homeomorphic and (M, d) is complete, we will show that (N, ρ) is also complete. Indeed, because (M, d) and (N, ρ) are uniformly homeomorphic, there exists a one to one and onto function $f : M \to N$ such that both f and f^{-1} are uniformly continuous. For any sequence $f(x_n)$ in N that is Cauchy, because f^{-1} is uniformly continuous, we have $f^{-1}(f(x_n))$ is Cauchy or (x_n) is Cauchy in M. But M is complete, there exists $x \in M$ such that $x_n \to x$. Notice that f is continuous, thus we have $f(x_n) \to f(x)$ in N. So (N, ρ) is complete, which means completeness is preserved by uniform homeomorphism. $\square$

Exercise 62
Two metrics d and ρ on a set M are said to be uniformly equivalent if the identity map between (M, d) and (M, ρ) is uniformly continuous in both directions. If there are constants $0 < c, C < \infty$ such that $c\rho(x, y) \leq d(x, y) \leq C\rho(x, y)$ for every pair of points $x, y \in M$, prove that d and ρ are uniformly equivalent.

Proof. First, notice that $0 \leq d(x, y) \leq C\rho(x, y)$ and $\rho(x, y) > 0$ for all $x, y \in M$. Thus $0 \leq C < \infty$. Now let $I : (M, d) \to (M, \rho)$ be the identity map. Because $d(x, y) \leq C\rho(x, y)$, I is C-Lipschitz, which yields I uniformly continuous. Moreover, we have $\rho(x, y) \leq \frac{1}{c}d(x, y)$, thus I^{-1} is $\frac{1}{c}$-Lipschitz. So I^{-1} is uniformly continuous too, which means d and ρ are uniformly equivalent. $\square$

Exercise 67
Define $f : \ell_2 \to \ell_1$ by $f(x) = (x_n/n)_{n=1}^{\infty}$. Show that f is uniformly continuous.

Proof. Let $t \in \mathbb{R}^{\infty}$ where $t = (\frac{1}{n})_{n=1}^{\infty}$. We know that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi}{6}$. For any $x \in \ell_2$, using the Cauchy Schwartz inequality, we get

$$\|f(x)\|_1 = \langle x, t\rangle \leq \|x\|_2\|t\|_2 = \sqrt{\frac{\pi}{6}}\|x\|_2$$

So $f(x) \in \ell_1$ for all $x \in \ell_2$ and f is $\frac{\pi}{6}$-Lipschitz. Therefore, f is uniformly continuous. $\square$

Exercise 68
Fix $y \in \ell_{\infty}$ and define $g : \ell_1 \to \ell_1$ by $g(x) = (x_n y_n)_{n=1}^{\infty}$. Show that g is uniformly continuous.

Proof. Because $y \in \ell_{\infty}$, the sequence $(y_n)_{n=1}^{\infty}$ is bounded. Let $K = \sup\{y_n : n \in \mathbb{N}\}$, then we have

$$\|g(x)\|_1 = \sum_{n=1}^{\infty} |x_n y_n| \leq K \sum_{n=1}^{\infty} x_n = K\|x\|_1.$$

So g is K-Lipschitz, which yields g is uniformly continuous. $\square$

Exercise 70
Let $K = \{x \in \ell_{\infty} : \lim x_n = 1\}$. Prove

(a) K is a closed (and hence complete) subset of ℓ_{∞}.

Proof. Assume that $x_n \in K$ and $x_n \to x$ in ℓ_{∞}, we will show that $x \in K$. For any $\epsilon > 0$ because $x_n \to x$, there exists $n \in \mathbb{N}$ such that $\|x_n - x\|_{\infty} < \frac{\epsilon}{2}$. But $x_n(i) \to 1$ when $i \to \infty$, thus there exists $I > 0$ such that $i > I$ implies $|x_n(i) - 1| < \frac{\epsilon}{2}$. Therefore, when $i > I$, we get

$$|x(i) - 1| \leq |x(i) - x_n(i)| + |x_n(i) - 1| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So $x(i) \to 1$, which yields $x \in K$. So K is closed. $\square$

(b) If $T : \ell\infty \to \ell\infty$ is given by $T(x) = (0, x_1, x_2, \cdots)$ for $x = (x_1, x_2, \cdots)$ in ℓ_{∞}, that is, if T shifts the entries forward and plus 0 in the empty slot, then $T(K) \subset K$.

Proof. If $x \in K$, then we would have $x(i) \to 1$ when $i \to \infty$. Therefore, the sequence $(0, x(1), x(2), \cdots)$ also converges to 1. Thus $T(x) \in K$ for all $x \in K$. In other words, we have $T(K) \subset K$. $\square$

(c) T is an isometry on K, but T has no fixed point in K.

Proof. For any $x, y \in K$, we have

$$\begin{aligned}\|T(x) - T(y)\|_\infty &= \sup\{0, |x(1) - y(1)|, |x(2) - y(2)|, \cdots\} \\ &= \sup\{|x(1) - y(1)|, |x(2) - y(2)|, \cdots\} \\ &= \|x - y\|_\infty.\end{aligned}$$

So T is an isometry. However, if $x \in \ell_\infty$ and $T(x) = x$, then we have $(0, x(1), x(2), \cdots) = (x(1), x(2), x(3), \cdots)$. So $0 = x(1) = x(2) = \cdots$ or $x = 0$ in ℓ_∞. But then, $x \notin K$ because $\lim_{i\to\infty} x(i) = 0 \neq 1$. So T has no fixed point. □

Exercise 72
Let D be dense in M. Show that M is isometric to a subset of $\ell_\infty(D)$.

Proof. Let $\hat{D} \subset \ell_\infty(D)$ be a completion of D, so there is an isometry function $f : D \to f(D)$ where $f(D)$ is a subset of $\ell_\infty(D)$. Using Theorem 8.16, because D is dense in M and $\ell_\infty(D)$ is complete, and f is an isometry, there is a unique isometry extension $F : M \to \hat{D}$. So M is isometric to a subset of $\ell_\infty(D)$. □

Exercise 74
Let $d(x, y) = \|x - y\|_2$ be the usual (Euclidean) metric on $\mathbb{R}^2$, and define a second metric ρ on $\mathbb{R}^2$ by

$$\rho(x, y) = \frac{\|x - y\|_2}{(1 + \|x\|_2^2)^{1/2}(1 + \|y\|_2^2)^{1/2}}.$$

Show that d and ρ are equivalent but not uniformly equivalent.

Proof. Let $x_n = (n, n) \in \mathbb{R}^2$ where $n \in \mathbb{N}$ and $y_n = x_n + \left(\frac{1}{\|x_n\|_2}\right) x_n$, then we have $\|x_n - y_n\|_2 = \| \left(\frac{1}{\|x_n\|_2}\right) x_n\|_2 = \left|\frac{1}{\|x_n\|_2}\right| \|x_n\|_2 = 1$. However, it's not hard to see that $\|x_n\|_2 = \sqrt{2}n \to \infty$. Moreover, since $\|y_n\|_2 = \left(1 + \frac{1}{\|x_n\|_2}\right) x_n \geq \|x_n\|_2$, we get $\|y_n\| \to \infty$. So

$$\rho(x_n, y_n) = \frac{\|x_n - y_n\|_2}{(1 + \|x_n\|_2^2)^{1/2}(1 + \|y_n\|_2^2)^{1/2}} = \frac{1}{(1 + \|x_n\|_2^2)^{1/2}(1 + \|y_n\|_2^2)^{1/2}} \to 0$$

as $n \to \infty$. Now consider the inverse identity function $I^{-1} : (\mathbb{R}^2, \rho) \to (\mathbb{R}^2, d)$, we have $\rho(x_n, y_n) \to 0$ while $d(x_n, y_n) \to 1$, so I^{-1} is not uniformly continuous, which means d and ρ are not uniformly equivalent.

Next we will show that d and ρ are indeed equivalent. Let $I : (\mathbb{R}^2, d) \to (\mathbb{R}^2, \rho)$ be the identity map. Because

$$\rho(x, y) = \frac{\|x - y\|_2}{(1 + \|x\|_2^2)^{1/2}(1 + \|y\|_2^2)^{1/2}} \leq \frac{\|x - y\|_2}{1 \cdot 1} = \|x - y\|_2 = d(x, y),$$

we get I is 1-Lipschitz, which means I is continuous. Next we will show that I^{-1} is continuous, that is for any $x \in \mathbb{R}^2$ and $\epsilon > 0$, there exists $\delta > 0$ such that $\rho(x, y) < \delta$

implies $d(x,y) < \epsilon$. Or for any $x \in \mathbb{R}^2$ and $\epsilon > 0$, there exists $\delta > 0$ such that $d(x,y) \geq \epsilon$ implies $\rho(x,y) \geq \delta$. We will prove such δ exists by showing that $\min\{\rho(x,y) : d(x,y) \geq \epsilon\}$ exists and bigger than 0.

Let $g : \{y \in \mathbb{R}^2 : d(x,y) \geq \epsilon\} \to \mathbb{R}$ maps $y \mapsto \rho(x,y)$. Because $\{y \in \mathbb{R}^2 : d(x,y) \geq \epsilon\}$ is closed in $\mathbb{R}^2$, it is compact. It's not hard to see that g is continuous, thus g is bounded. That is $\min\{\rho(x,y) : d(x,y) \geq \epsilon\} = \min\{g(y) : d(x,y) \geq \epsilon\}$ exists. Let $m = \min\{\rho(x,y) : d(x,y) \geq \epsilon\}$, if $m = 0$, then there exists $y_0 \in \mathbb{R}^2$ such that $d(x,y_0) \geq \epsilon > 0$, thus $x \neq y_0$ and

$$\rho(x,y_0) = \frac{\|x-y_0\|_2}{(1+\|x\|_2^2)^{1/2}(1+\|y_0\|_2^2)^{1/2}} = 0,$$

which implies $\|x-y_0\|_2 = 0$ or $x = y_0$. Contradiction. Therefore, $m \neq 0$, but $\rho(x,y) \leq 0$ for all $x,y \in \mathbb{R}^2$, so $m > 0$. Let $m = \delta$, we get $\rho(x,y) \geq \delta$ for all $d(x,y) \geq \epsilon$. So I^{-1} is continuous. That is d and ρ are equivalent. □

Exercise 76
Fix $y \in \mathbb{R}^n$ and define a linear map $L : \mathbb{R}^n \to \mathbb{R}$ by $L(x) = \langle x, y\rangle$. Show that L is continuous and compute $\|L\| = \sup_{x\neq 0} |L(x)|/\|x\|_2$.

Proof. Because L is linear, for any $a, b \in \mathbb{R}^n$, applying the Cauchy Schwartz theorem, we get

$$|L(a) - L(b)| = |L(a-b)| = \langle a-b, y\rangle \leq \|a-b\|_2 \cdot \|y\|_2.$$

So L is $\|y\|_2$-Lipschitz, which implies L is continuous. For any $x \in \mathbb{R}^n$, and $x \neq 0$, applying the Cauchy Schwartz, we have

$$\frac{|L(x)|}{\|x\|_2} \leq \frac{\|x\|_2\|y\|_2}{\|x\|_2} = \|y\|_2.$$

Moreover, we have $\frac{|L(y)|}{\|y\|_2} = \frac{\langle y,y\rangle}{\|y\|_2} = \|y\|_2$. So $\|L\| = \sup_{x\neq 0} |L(x)|/\|x\|_2 = \max_{x\neq 0} |L(x)|/\|x\|_2 = \|y\|_2$. □

Exercise 77
Fix $k \geq 1$ and define $f : \ell_\infty \to \mathbb{R}$ by $f(x) = x_k$. Show that f is linear and has $\|f\| = 1$.

Proof. For any $x, y \in \ell_\infty$ and $\alpha \in \mathbb{R}$, we have

$$f(\alpha x + y) = (\alpha x + y)_k = \alpha x_k + y_k = \alpha f(x) + f(y).$$

So f is linear. Moreover, we have

$$\frac{|f(x)|}{\|x\|_\infty} = \frac{|x_k|}{\sup_{i\in\mathbb{N}} x_i} \leq 1.$$

Because $\frac{|f(1)|}{\|1\|_\infty} = 1$ where $1 = (1, 1, \cdots)$, we have $\|f\| = \sup \frac{|f(x)|}{\|x\|_\infty} = \max \frac{|f(x)|}{\|x\|_\infty} = 1$. □

Exercise 78
Define a linear map $f : \ell_2 \to \ell_1$ by $f(x) = (x_n/n)_{n=1}^{\infty}$. Is f bounded? If so, what is $\|f\|$?

Proof. Let $k \in \mathbb{R}^{\infty}$ defined by $k = (1, \frac{1}{2}, \cdots, \frac{1}{n}, \cdots)$. For any $x \in \ell_2$, using the Cauchy Schwartz inequality, we get

$$\|f(x)\|_1 = \langle x, k\rangle \leq \|x\|_2 \cdot \|k\|_2 = \sqrt{\frac{\pi}{6}}\|x\|_2.$$

So f is bounded, and since

$$\frac{\|f(k)\|_1}{\|k\|_2} = \frac{\langle k, k\rangle}{\|k\|_2} = \|k\|_2 = \sqrt{\frac{\pi}{6}},$$

we get $\|f\| = \sup_{x\neq 0} \frac{\|f(x)\|_1}{\|x\|_2} = \sqrt{\frac{\pi}{6}}$. $\square$

Exercise 79
If $S, T \in B(V,W)$, show that $S+T \in B(V,W)$ and that $\|S+T\| \leq \|S\| + \|T\|$. Using this, complete the proof that $B(V,W)$ is a normed space under the operation norm.

Proof. Because the sum of two continuous functions is again a continuous function, we get $S+T$ continuous. Moreover, for any $x, y \in V$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}
(S+T)(\alpha x + y) &= S(\alpha x + y) + T(\alpha x + y) \\
&= \alpha S(x) + S(y) + \alpha T(x) + T(y) \\
&= \alpha(S(x) + T(x)) + (S(y) + T(y)) \\
&= \alpha(S+T)(x) + (S+T)(y).
\end{aligned}$$

So $S+T$ is linear, which means $S+T \in B(V,W)$. What is more, by applying the triangular inequality and $\sup A + B \leq \sup A + \sup B$ (where $A, B \subset \mathbb{R}$), we have

$$\begin{aligned}
\|S+T\| &= \sup_{x \neq 0} \frac{\|(S+T)(x)\|_2}{\|x\|_2} \\
&= \sup_{x \neq 0} \frac{\|S(x) + T(x)\|_2}{\|x\|_2} \\
&\leq \sup_{x \neq 0} \frac{\|S(x)\|_2 + \|T(x)\|_2}{\|x\|_2} \\
&\leq \sup_{x \neq 0} \frac{\|S(x)\|_2}{\|x\|_2} + \sup_{x \neq 0} \frac{\|T(x)\|_2}{\|x\|_2} \\
&= \|S\| + \|T\|.
\end{aligned}$$

In addition, we have $\|S\| = \sup_{x \neq 0} \frac{\|S(x)\|_2}{\|x\|_2} \geq 0$. If $\|S\| = 0$, then $\|S(x)\|_2 = 0$ for all $x \in V$ or $S = 0$. If $S = 0$, then we obviously have $\|S\| = \sup_{x \neq 0} \frac{\|S(x)\|_2}{\|x\|_2} = 0$. Lastly, for $c \in \mathbb{R}$, we have

$$\begin{aligned}
\|cS\| &= \sup_{x \neq 0} \frac{\|cS(x)\|_2}{\|x\|_2} \\
&= \sup_{x \neq 0} |c| \cdot \frac{\|S(x)\|_2}{\|x\|_2} \\
&= |c| \cdot \sup_{x \neq 0} \frac{\|S(x)\|_2}{\|x\|_2} \\
&= |c| \cdot \|S\|.
\end{aligned}$$

So the operation norm defines a normed on $B(V,W)$. $\square$

Exercise 80
Show that the definite integral $I(f) = \int_a^b f(t)dt$ is continuous from $C[a,b]$ into $\mathbb{R}$. What is $\|I\|$?

Proof. For any $f \in C[a,b]$, we have

$$|I(f)| = \left| \int_a^b f(t)dt \right| \leq |b-a| \cdot \sup\{f(t) : t \in [a,b]\} = |b-a| \cdot \|f\|_\infty.$$

So I is Lipschitz, thus continuous. Moreover, for $f(x) = 1$ when $x \in [a,b]$, we have

$$|I(f)| = \left| \int_a^b 1dt \right| = |b-a| \cdot 1 = |b-a| \cdot \|f\|_\infty.$$

Thus $\|I\| = |b-a|$. $\square$

Exercise 81
Prove that the indefinite integral, defined by $T(f)(x) = \int_a^x f(t)dt$, is continuous as a map from $C[a,b]$ into $C[a,b]$. Estimate $\|T\|$.

Proof. For any $T, f \in C[a,b]$, we have

$$\|T(f)\|_\infty = \sup_{x \in [a,b]} \left| \int_a^x f(t)dt \right| \leq \int_a^b |f(t)|dt \leq |b-a| \cdot \|f\|_\infty.$$

So T is $|b-a|$-Lipschitz, which means T is continuous and because for $f(x) = 1$ for all $x \in [a,b]$, we have

$$\|T(f)\|_\infty = \sup_{x \in [a,b]} \left| \int_a^x f(t)dt \right| = |b-a|,$$

Therefore, $\|T\| = |b-a|$. $\square$

Exercise 82
For $T \in B(V,W)$, prove that $\|T\| = \sup\{|||Tx||| : \|x\| = 1\}$.

Proof. Since we already know that $\|T\| = \sup_{x \neq 0} \left\{ \frac{|||T(x)|||}{\|x\|} : x \in V \right\}$, it is sufficient to show that $\left\{ \frac{|||T(x)|||}{\|x\|} : x \in V, x \neq 0 \right\} = \{|||Tx||| : \|x\| = 1\}$. It is not hard to see that the latter set is a subset of the former, we will show that the first set is a subset of the latter. Indeed, for any $x \in V, x \neq 0$, because T is linear, we have

$$\frac{|||T(x)|||}{\|x\|} = \left|\left|\left| \frac{T(x)}{\|x\|} \right|\right|\right| = \left|\left|\left| T\left(\frac{x}{\|x\|}\right) \right|\right|\right| \in \{|||Tx||| : \|x\| = 1\}$$

because $\left\| \frac{x}{\|x\|} \right\| = 1$. So $\|T\| = \sup_{x \neq 0} \left\{ \frac{|||T(x)|||}{\|x\|} : x \in V \right\} = \sup\{|||Tx||| : \|x\| = 1\}$. $\square$

Exercise 84
Prove that $B(V, W)$ is complete whenever W is complete.

Proof. Assume that $\sum_{n=1}^{\infty} \|T_n\|$ is finite, where $T_n \in W$, we will show that $\sum_{n=1}^{\infty} T_n$ is convergent. For any $x \in V$, we have $\sum_{n=1}^{\infty} |||T_n(x)||| \le \sum_{n=1}^{\infty} \|T_n\| < \infty$. But $|||T_n(x)||| \in W$ and W is complete, thus $\sum_{n=1}^{\infty} T_n(x) \to T(x)$ for some $T(x) \in W$.

For any $x, y \in V$ and $\alpha \in \mathbb{R}$, by the definition of the function T, we have $\sum_{n=1}^{\infty} T_n(\alpha x + y) \to T(\alpha x+y)$. But T_n's are linear, thus $\sum_{n=1}^{\infty} T_n(\alpha x + y) = \sum_{n=1}^{\infty} \alpha T_n(x) + T_n(y) = \alpha \sum_{n=1}^{\infty} T_n(x)+ \sum_{n=1}^{\infty} T_n(y) \to \alpha T(x) + T(y)$. By the uniqueness of convergence, we get $T(\alpha x + y) = \alpha T(x) + T(y)$. So T is linear.

For all $n \in \mathbb{N}$, we define $\|T_n\| = C_n$. Because $T_n \in B(V, W)$, we have T_n is continuous, or $|||T_n(x)||| \le C_n\|x\|$. Let $C = \sum_{n=1}^{\infty} C_n = \sum_{n=1}^{\infty} \|T_n\| < \infty$, we will show that for any $x \in V$, we have $|||T(x)||| \le C\|x\|$, so T is continuous. Using the triangular inequality, we have

$$|||T(x)||| = \lim_{N\to\infty} \left|\left|\left| \sum_{n=1}^{N} T_n(x) \right|\right|\right| \le \lim_{N\to\infty} \sum_{n=1}^{N} |||T_n(x)||| \le \lim_{N\to\infty} \sum_{n=1}^{N} C_n\|x\| = C\|x\|.$$

So T is not only linear but also continuous, which implies $T \in B(V, W)$.

For $n, m \in N$, we have

$$\left|\left|\left| \sum_{i=n}^{m} T_i(x) \right|\right|\right| \le \sum_{i=n}^{m} |||T_i(x)||| \le \sum_{i=n}^{m} \|T_i\|$$

For all $x \in V$. But $\sum_{i=n}^{m} \|T_i\|$ can be sufficiently small because $\sum_{i=1}^{\infty} \|T_i\| < \infty$. So the partial sum sequence $(\sum_{i=1}^{n} T_i)_{n=1}^{\infty}$ is Cauchy. Therefore, for any $\epsilon > 0$, there exists $N_0 > 0$ such that $m, n > N_0$ implies

$$\left\| \sum_{i=1}^{n} T_i - \sum_{i=1}^{m} T_i \right\| < \frac{\epsilon}{2}.$$

Fix $m > N_0$ and let $n \to \infty$, for any $x \in W$, we get

$$\left|\left|\left| \sum_{i=1}^{m} T_i(x) - T(x) \right|\right|\right| = \lim_{n\to\infty} \left|\left|\left| \sum_{i=1}^{m} T_i(x) - \sum_{i=1}^{n} T_i(x) \right|\right|\right| \le \lim_{n\to\infty} \left\| \sum_{i=1}^{n} T_i - \sum_{i=1}^{m} T_i \right\| \le \frac{\epsilon}{2} < \epsilon.$$

So ϵ is an upper bound for $|||\sum_{i=1}^{m} T_i(x) - T(x)|||$, where x runs in V. Therefore, we get $\|\sum_{i=1}^{m} T_i - T\| \le \epsilon$. So $\sum_{i=1}^{\infty} T_i \to T$ in $B(V, W)$. This concludes that $B(V, W)$ is complete. $\square$

Exercise 85
Fill in the missing details in the proof of Theorem 8.22. That is, let V be an n-dimensional vector space with basis $x_1, \cdots, x_n$. Define a norm on V by

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| = \sum_{i=1}^{n} |\alpha_i| = \left\| \sum_{i=1}^{n} \alpha_i e_i \right\|_1 .$$

Prove that the unit sphere $S = \{x \in V : \|x\| = 1\}$ is compact in $(V, \|\bullet\|)$ because the corresponding set in $\mathbb{R}^n$ is compact.

Proof. Let's first remind that in the proof of Theorem 8.22, we have showed that the basis-to-basis map $T : V \to \mathbb{R}^n$ is a linear isometry between $(V, \|\bullet\|)$ and $(\mathbb{R}^n, \|\bullet\|_1)$.

For any sequence $x_n \in S = \{x \in V : \|x\| = 1\}$, because T is an isometry, we get $\|T(x_n)\|_1 = \|x_n\| = 1$. So $T(x_n) \in S' = \{y \in \mathbb{R}^n : \|y\| = 1\}$ for all $n \in \mathbb{N}$. But we know that the unit sphere of $\mathbb{R}^n$, $S' = \{y \in \mathbb{R}^n : \|y\| = 1\}$, is compact, therefore, there exists a convergent subsequence $T(x_{n_k}) \to T(x)$ in S'. And since T is an isometry between $(V, \|\bullet\|)$ and $(\mathbb{R}^n, \|\bullet\|_1)$, T is also an isometry between S and S'. Hence, $x_{n_k} \to x$ in S. So S is compact. $\square$

Exercise 86
If $(V, \|\bullet\|)$ is an n-dimensional normed vector space, show that there is a norm $|||\bullet|||$ on $\mathbb{R}^n$ such that $(\mathbb{R}^n, |||\bullet|||)$ is linearly isometric to $(V, \|\bullet\|)$.

Proof. Let $v_1, \cdots, v_n$ be a basis on V, we define $T : V \to \mathbb{R}^n$ by $T(v_i) = e_i$ and

$$T\left(\sum_{i=1}^{n} \alpha_i v_i\right) = \sum_{i=1}^{n} \alpha_i T(v_i).$$

Clearly, T is a one to one and onto linear map from V to $\mathbb{R}^n$. Now let $|||T(v)||| = \|v\|$ for all $v \in V$, we will show that $|||\bullet|||$ defines a metric on $\mathbb{R}^n$. Notice that T is linear and $\|\bullet\|$ defines a norm on V, we have:

1. $|||T(v)||| = \|v\| \geq 0$ for all $v \in V$.
2. $|||T(v)||| = 0$ if and only if $\|v\| = 0$ if and only if $v = 0$.
3. $|||\alpha T(v)||| = |||T(\alpha v)||| = \|\alpha v\| = |\alpha| \cdot \|v\|$.
4. $|||T(v) + T(u)||| = |||T(v + u)||| = \|v + u\| \leq \|v\| + \|u\| = |||T(v)||| + |||u|||$.

So $|||\bullet|||$ defines a norm on $\mathbb{R}^n$, hence $(R^n, |||\bullet|||)$ is linearly isometric to $(V, \|\bullet\|)$. $\square$

Exercise 87
Prove Corollary 8.24.

Proof. Let $(V, \|\bullet\|)$ and $(W, |||\bullet|||)$ be two n-dimensional normed vector spaces, where $v_1, \cdots, v_n$ and $w_1, \cdots, w_n$ are the bases of V and W respectively. We will define a new norm d in W such that $(V, \|\bullet\|)$ and (W, d) are isometric by $d(w_i) = \|v_i\|$ for all $1 \leq i \leq n$ and d is linear. Then the map $T : V \to W$ maps $\sum_{i=1}^n \alpha_i v_i \mapsto \sum_{i=1}^n \alpha_i w_i$ is an isometric between $(V, \|\bullet\|)$ and (W, d). But by Theorem 8.22, we have (W, d) and $(W, |||\bullet|||)$ are equivalent thus uniformly equivalent. So $(V, \|\bullet\|)$ and $(W, |||\bullet|||)$ are uniformly equivalent. $\square$

Exercise 88
Prove Corollary 8.25.

Proof. For any finite dimensional vector V, let $n = \dim(V)$. Using Corollary 8.24, we get V uniformly homeomorphic with $(\mathbb{R}^n, d)$ where d is the Euclidean distance because they have the same dimension. But $(\mathbb{R}^n, d)$ is complete and completeness is preserved under uniform homeomorphism, we get V is complete. $\square$

Exercise 89
Show that $\{x \in \ell_1 : x_n = 0 \text{ for all but finitely many } n\}$ is a proper dense linear subspace of ℓ_1.

Proof. Let $L = \{x \in \ell_1 : x_n = 0 \text{ for all but finitely many } n\}$, we will first show that L is a linear subspace of ℓ_1. Because $0 = (0, \cdots) \in L$, we have $L \neq \varnothing$. If $x, y \in L$, let t_x and t_y be the number of nonzero entries of x and y respectively, then it is not hard to see that $t_{x+y} \leq t_x + t_y$ where t_{x+y} is the number of nonzero entries of $x + y$. So $x + y \in L$. Moreover, for any $\alpha \in \mathbb{R}$ and, let $t_{\alpha x}$ be the number of nonzero entries of αx, then we have $t_{\alpha x} \leq t_x$. So αx also have finitely many nonzero entries, which implies $\alpha x \in L$. So L is a linear subspace of ℓ_1.

Now, we will show that L is dense in ℓ_1. For any $x \in \ell_1$, we have $\sum_{i=1}^\infty |x(i)| < \infty$. Define $x_n = (x(1), \cdots, x(n), 0, 0, \cdots)$ where the first n entries of x_n and x are the same and the rests are 0's. Because for any $n \in \mathbb{N}$, x_n has at most n nonzero entries, we have $x_n \in L$. Moreover, we have

$$\|x - x_n\|_1 = \sum_{i=1}^{\infty} |x(i) - x_n(i)| = \sum_{i=n+1}^{\infty} |x(i) - x_n(i)| = \sum_{i=n+1}^{\infty} |x(i)|.$$

Notice that because $\sum_{i=1}^\infty |x(i)| < \infty$, as $n \to \infty$, the term $\sum_{i=n+1}^\infty |x(i)|$ can be sufficiently small. So $x_n \to x$ in ℓ_1. This implies L is dense in ℓ_1. $\square$

Exercise 74
(continue)

Proof. Assume that $x_n \xrightarrow{\rho} x$, we will show that $x_n \xrightarrow{d} x$, thus $I^{-1} : (\mathbb{R}^2, \rho) \to (\mathbb{R}^2, d)$ is continuous. For any subsequence x_{n_k} of x_n, because $x_n \xrightarrow{\rho} x$, we have

$$\rho(x_{n_k}, x) = \left\| \frac{x_{n_k} - x}{(1 + \|x_{n_k}\|_2^2)^{1/2}(1 + \|x\|_2^2)^{1/2}} \right\|_2 \to 0.$$

But $(1+\|x\|_2^2)^{1/2}$ is a constant, thus

$$\left\|\frac{x_{n_k}}{(1+\|x_{n_k}\|_2^2)^{1/2}}-\frac{x}{(1+\|x_{n_k}\|_2^2)^{1/2}}\right\|_2=\left\|\frac{x_{n_k}-x}{(1+\|x_{n_k}\|_2^2)^{1/2}}\right\|_2\to 0. \quad (1)$$

Because $\left\|\frac{x_{n_k}}{(1+\|x_{n_k}\|_2^2)^{1/2}}\right\|_2<\left\|\frac{x_{n_k}}{(\|x_{n_k}\|_2^2)^{1/2}}\right\|_2=1$ and the set $\{x\in\mathbb{R}^2:\|x\|\le 1\}$ is compact, there exists a subsequence $x_{n_{k_h}}$ such that $\frac{x_{n_{k_h}}}{(1+\|x_{n_k}\|_2^2)^{1/2}}\to a$ in $\mathbb{R}^2$. By (1), we also have $\frac{x}{(1+\|x_{n_k}\|_2^2)^{1/2}}\to a$. Because

$$\|x_{n_{k_h}}-x\|_2\le\|x_n-a\sqrt{1+\|x_n\|_2^2}\|_2+\|a\sqrt{1+\|x_n\|_2^2}-x\|_2$$

and the left side can be sufficiently small, we get $x_{n_{k_h}}\xrightarrow{d}x$. So every subsequence of x_n has a further subsequence that converges to x, we get $x_n\xrightarrow{d}x$. This implies I^{-1} is continuous. $\square$

Chapter 9. Category

Exercise 1
If f is increasing, show that $w_f(a)=f(a+)-f(a-)$.

Proof. Because f is increasing, we get

$$\begin{aligned}w_f(a)&=\lim_{h\to 0^+}\sup\{|f(x)-f(y)|:x,y\in B_h(a)\}\\&=\lim_{h\to 0^+}\sup\{f(x)-f(y):x\ge y\in B_h(a)\}\end{aligned}$$

But $x,y\in[a-h,a+h]$ for all $x,y\in B_h(a)$, therefore

$$f(x)-f(y)\le f(a+h)-f(a+h),$$

which implies

$$\begin{aligned}w_f(a)&=\lim_{h\to 0^+}\sup\{f(x)-f(y):x\ge y\in B_h(a)\}\\&\le\lim_{h\to 0^+}f(a+h)-f(a-h)\\&=f(a+)-f(a-).\end{aligned}$$

Moreover, $[a-h/2,a+h/2]\subset B_h(a)$, which yields

$$\sup\{f(x)-f(y):x\ge y\in B_h(a)\}\ge f\left(a+\frac{h}{2}\right)-f\left(a-\frac{h}{2}\right).$$

Take $h\to 0^+$ similar to the above, we get $w_f(a)\ge f(a+)-f(a-)$. Therefore, $w_f(a)=f(a+)-f(a-)$. $\square$

Exercise 2
Prove that f is continuous at a if and only if $w_f(a) = 0$.

Exercise 3
Given $f : \mathbb{R} \to \mathbb{R}$, show that $g(x) = \arctan f(x)$ satisfies $D(g) = D(f)$. Thus, in any discussion of $D(f)$, we may assume that f is bounded.

Exercise 12
More generally, in any metric space, show that every open set is an F_a and that every closed set is a G_δ.

Exercise 14
Prove that A has an empty interior in M if and only if A^c is dense in M.

Exercise 15
If G is open and dense in $\mathbb{R}$, show that the same is true of $G/\{x\}$ for any $x \in \mathbb{R}$. Is this true in any metric space? Explain.

Exercise 16
Show that $\{x\}$ is nowhere dense in M if and only if x is not an isolated point of M.

Exercise 17
Prove that a complete metric space without any isolated points is uncountable. In particular, this gives another proof that δ is uncountable.

Exercise 19
Show that each of the following is equivalent to the statement that A is nowhere dense in M:

(a) $\tilde{A}$ contains no nonempty open set.

(b) Each nonempty open set in M contains a nonempty open subset that is disjoint from A.

(c) Each nonempty open set in M contains an open ball that is disjoint from A.