DYNAMIC PROGRAMMING

Bùi Tiến Lên

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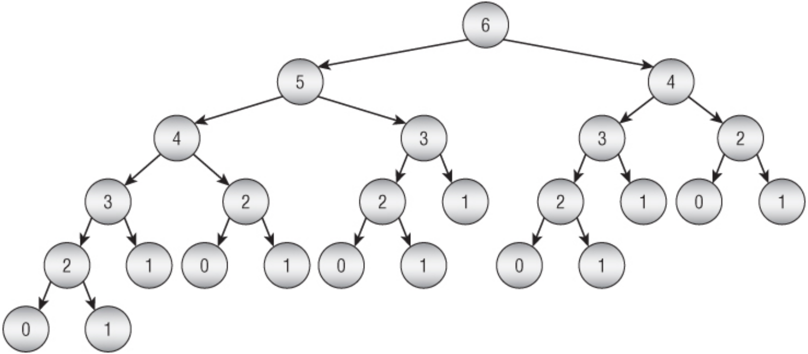
Contents

Dynamic Programming

Problems in Divide-and Conquer

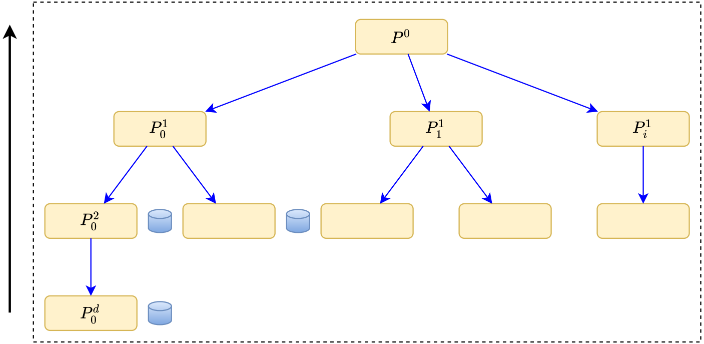
- The number of terms computed by the divide-and-conquer algorithm for determining the nth Fibonacci term is exponential in n
- The figure shows the Fibonacci algorithm's call tree when it evaluates `fibonacci(6)`



Dynamic Programming

- Dynamic programming is similar to divide-and-conquer in that an instance of a problem is divided into smaller instances.
- However, in this approach we solve small instances first, **store** the results, and later, whenever we need a result, **look it up** instead of recomputing it.
- Dynamic programming is a **bottom-up** approach

Dynamic Programming (cont.)



Dynamic Programming (cont.)

The steps in the development of a dynamic programming algorithm are as follows:

1. **Establish** a recursive property that gives the solution to an instance of the problem.
2. **Solve** an instance of the problem in a bottom-up fashion by solving smaller instances first and storing the results
   - Using an array (or sequence of arrays)

The Binomial Coefficient

- The **binomial coefficient** is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for} \quad 0 \leq k \leq n \tag{1}$$

- We have recursive formula

$$\binom{n}{k} = \begin{cases} 1 & k = 0 \text{ or } k = n \\ \binom{n-1}{k-1} + \binom{n-1}{k} & 0 < k < n \end{cases} \tag{2}$$

The Binomial Coefficient (cont.)

```
int bin(int n, int k) {
  if ( k == 0 || k == n)
    return 1;
  else
    return bin(n-1, k - 1) + bin(n - 1, k);
}
```

- This algorithm is very inefficient.

DP solution

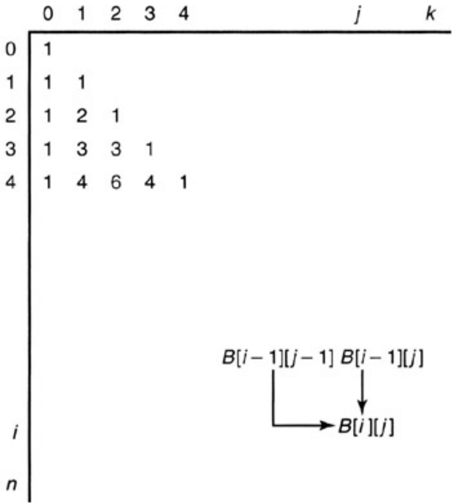
- We define an array $B[i][j]$ containing $\binom{i}{j}$

1. Establish a recursive property.

$$B[i][j] = \begin{cases} 1 & j = 0 \text{ or } j = i \\ B[i-1][j-1] + B[i-1][j] & 0 < j < i \end{cases} \tag{3}$$

2. Solve an instance of the problem in a **bottom-up** fashion by computing the rows in B in sequence starting with the first row.

DP solution (cont.)



Dynamic Programming and Optimization Problems

Principle of Optimality

Concept 1

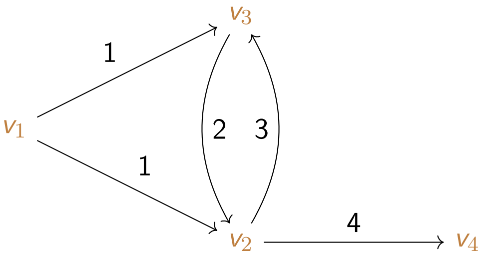
A problem is said to satisfy the **Principle of Optimality** if the subsolutions of an optimal solution of the problem are themesleves optimal solutions for their subproblems.

- In practice, it is necessary to show that the principle applies before assuming that an optimal solution can be obtained using dynamic programming.

Longest Path Problem

Finding the longest simple paths from each vertex to all other vertices.

- The optimal (longest) simple path from v_1 to v_4 is $\{v_1, v_3, v_2, v_4\}$.
- However, the subpath $\{v_1, v_3\}$ is not an optimal (longest) path from v_1 to v_3



Dynamic Programming

1. **Establish** a recursive property that gives the optimal solution to an instance of the problem.
2. **Compute** the value of an optimal solution in a bottom-up fashion.
3. **Construct** an optimal solution in a bottom-up fashion.

Chained Matrix Multiplication

- Multiply a 2×3 matrix times a 3×4 matrix, the resultant matrix is a 2×4 matrix
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 & 9 & 1 \\ 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 29 & 35 & 41 & 38 \\ 74 & 89 & 104 & 83 \end{bmatrix}$$
the total number of elementary multiplication is $2 \times 3 \times 4 = 24$.
- In general, to multiply an $m \times n$ matrix times a $n \times p$ matrix using the standard method, it is necessary to do
$$m \times n \times p \quad \text{elementary multiplications} \tag{4}$$

Chained Matrix Multiplication (cont.)

- Consider the multiplication of the following four matrices:

$$\underbrace{A}_{20\times 2} \times \underbrace{B}_{2\times 30} \times \underbrace{C}_{30\times 12} \times \underbrace{D}_{12\times 8}$$

- Matrix multiplication is an associative operation, meaning that the order in which we multiply does not matter. There are five different orders in which we can multiply four matrices, each possibly resulting in a different number of elementary multiplications.

$$\begin{array}{lll}
A(B(CD)) & 30 \times 12 \times 8 + 2 \times 30 \times 8 + 20 \times 2 \times 8 & = 3680 \\
(AB)(CD) & 20 \times 2 \times 30 + 30 \times 12 \times 8 + 20 \times 30 \times 8 & = 8880 \\
A((BC)D) & 2 \times 30 \times 12 + 2 \times 12 \times 8 + 20 \times 2 \times 8 & = 1232 \\
((AB)C)D & 20 \times 2 \times 30 + 20 \times 30 \times 12 + 20 \times 12 \times 8 & = 10320 \\
(A(BC))D & 2 \times 30 \times 12 + 20 \times 2 \times 12 + 20 \times 12 \times 8 & = 3120
\end{array}$$

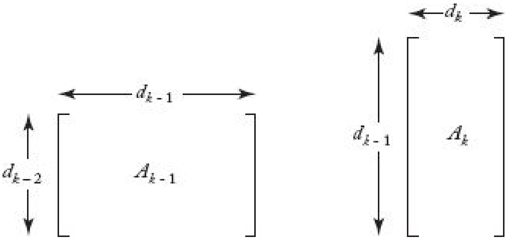
Chained Matrix Multiplication (cont.)

Multiply n matrices: $A_1, A_2, \ldots, A_n$.

- It is not hard to see that the principle of optimality applies in this problem.

Let

- d_0 be the number of rows in A_1 and
- d_k be the number of columns in A_k for $1 \le k \le n$, the dimension of A_k is $d_{k-1} \times d_k$



Solution to Chained Matrix Multiplication

- Let $M[i][j]$ = minimum number of multiplications needed to multiply A_i through A_j, if $i \leq j$.
- Principle of optimality

$$\begin{aligned} M[i][i] &= 0 \\ M[i][j] &= \min_{i \leq k \leq j-1} \left(M[i][k] + M[k+1][j] + d_{i-1} d_k d_j\right) \quad \text{if } i < j \end{aligned} \tag{5}$$

Solution to Chained Matrix Multiplication (cont.)

- Consider the multiplication of the following six matrices

$$\underbrace{A_1}_{5\times 2} \times \underbrace{A_2}_{2\times 3} \times \underbrace{A_3}_{3\times 4} \times \underbrace{A_4}_{4\times 6} \times \underbrace{A_5}_{6\times 7} \times \underbrace{A_6}_{7\times 8}$$

we have $d_0 = 5, d_1 = 2, d_2 = 3, d_3 = 4, d_4 = 6, d_5 = 7, d_6 = 8$

$$M = \begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 30 & 64 & 132 & 226 & 348 \\ 2 & & 0 & 24 & 72 & 156 & 268 \\ 3 & & & 0 & 72 & 198 & 366 \\ 4 & & & & 0 & 168 & 392 \\ 5 & & & & & 0 & 336 \\ 6 & & & & & & 0 \\ \hline \end{array}$$

Workshop

Quiz

1. What is the dynamic programming?

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Exercises

- Write a program

References

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