ANALYSIS OF ALGORITHM

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Introduction

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Analysis of Algorithms

Concept 1

Analysing an algorithm has come to mean predicting the **resources** that the algorithm requires.

- **Time Complexity**: How long it takes the algorithm to run
- **Space Complexity**: The extra memory space the algorithm needs besides the input data itself.

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Why Analyze Algorithms?

- **Comparison**: To compare different algorithms and choose the most efficient one for a given task.
- **Optimization**: To optimize an algorithm.
- **Predictability**: To predict how an algorithm will behave with different input sizes and under various conditions.
- **Resource Management**: To ensure algorithms are efficient in terms of computational resources, such as CPU time and memory.

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Methods of Analysis

There are two methods of analysis

- Empirical analysis
- Theoretical analysis

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Time Complexity

Concept 2

The **time complexity** of an algorithm estimates how much time the algorithm will use given an input.

$$\text{time complexity} = T(\text{input size}) \tag{1}$$

Input size depends on the problem's input data.

- The input size can be the *number of items in the input*
- The input size is the *total number of bits* needed to represent the input.
- Sometimes, it is more appropriate to describe the size of the input with two or more *numbers*.

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Time Complexity (cont.)

- Time complexity isn't directly measured in seconds or minutes. It measures how many basic operations are executed.

Concept 3

Basic Operations can be

- Machine operation: assignments, arithmetic operations, ...
- Human operation: move, carry, ...
- General operation

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Space Complexity

Concept 4

The **space complexity** of an algorithm estimates the amount of extra memory space the algorithm needs to perform its operations on an input data.

$$\text{space complexity} = T(\text{input size}) \tag{2}$$

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Example

- Given an array of n numbers, our task is to calculate the **maximum subarray sum**, i.e., the largest possible sum of a sequence of consecutive values in the array.
- We assume that an empty subarray is allowed, so the maximum subarray sum is always at least 0.
- For example, in the array

-1	2	4	-3	5	2	-5	2

the following subarray produces the maximum sum 10:

-1	2	4	-3	5	2	-5	2

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Algorithm 1

- A straightforward way to solve the problem is to go through all possible subarrays, calculate the sum of values in each subarray and maintain the maximum sum.

```
int best = 0;
for (int a = 0; a < n; a++) {
    for (int b = a; b < n; b++) {
        int sum = 0;
        for (int k = a; k <= b; k++) {
            sum += array[k];
        }
        best = max(best,sum);
    }
}
cout << best << "\n";
```

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Algorithm 2

- It is easy to make Algorithm 1 more efficient by removing one loop from it. This is possible by calculating the sum at the same time when the right end of the subarray moves.

```
int best = 0;
for (int a = 0; a < n; a++) {
    int sum = 0;
    for (int b = a; b < n; b++) {
        sum += array[b];
        best = max(best,sum);
    }
}
cout << best << "\n";
```

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Algorithm 3

Consider the subproblem of finding the maximum-sum subarray that ends at position k. There are two possibilities:

1. The subarray only contains the element at position k.
2. The subarray consists of a subarray that ends at position $k-1$, followed by the element at position k.

In the latter case, since we want to find a subarray with maximum sum, the subarray that ends at position $k-1$ should also have the maximum sum. Thus, we can solve the problem efficiently by calculating the maximum subarray sum for each ending position from left to right.

```
int best = 0, sum = 0;
for (int k = 0; k < n; k++) {
    sum = max(array[k],sum+array[k]);
    best = max(best,sum);
}
cout << best << "\n";
```

Types Of Problem

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Types Of Problem

Concept 5

A **decision problem** is a problem for which the answer for every valid input is **yes** or **no**.

- For example, "deciding whether a number is prime" is a decision problem.

Concept 6

A **search problem** is a problem which requires the identification of a solution from within a potentially finite/infinite set of possible solutions.

- For example, "finding a path" is a search problem.

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Types Of Problem (cont.)

Concept 7

A **counting problem** requires a total of the solutions to a **search problem**.

- For example, "how many of the first 100 integers are prime?".

Concept 8

A **optimization problem** requires the identification of the best solution to a **search problem** from a given set of solutions.

- For example, "finding the shortest path".

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Tractable & Intractable Problems

Concept 9

If a problem has a reasonable time solution, that is to say that it can be solved in no more than polynomial time, it is said to be **tractable**.

Concept 10

A problem can only be solved with algorithms whose execution time grows too quickly in relation to their input to be solved in polynomial time. These algorithms are said to be **intractable**.

- For example, The **Travelling Salesperson Problem** is an intractable problem.

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Approximate Solutions

Concept 11

A **heuristic algorithm** is an algorithm that produces usable solutions to a problem in reasonable (polynomial) time. These solutions may not be proven to be optimal or even formally correct, they can however produce good solutions to the problem.

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Decidable/Undecidable Problems

Concept 12

There are problems that can not be solved by algorithms even with unlimited time. They are uncomputable problems.

Concept 13

If a decision problem proves to be uncomputable then it is said to be **undecidable**.

The Halting Problem

Given a description of a program (its code), and some input, is it possible to tell, without running the program, whether it will halt for the given input or loop forever?

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Complexity Classes

Concept 14

Complexity classes are problem groups based on how much time and space they require to solve problems and verify solutions.

Concept 15

The class **P** consists of all those decision problems that can be solved on a deterministic sequential machine in an amount of time that is polynomial in the size of the input.

Concept 16

The class **NP** consists of all those decision problems whose positive solutions can be verified in polynomial time given the right information.

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Complexity Classes (cont.)

Concept 17

A decision problem C is **NP-complete** if

1. C is in **NP**, and
2. Every problem in **NP** is reducible to C in polynomial time

- A problem satisfying condition 2 is said to be **NP-hard**, whether or not it satisfies condition 1

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Some NP-complete Problems

- Clique problem
- n-Queens completion
- Boolean satisfiability problem
- Subset sum problem
- Traveling salesman problem
- ...

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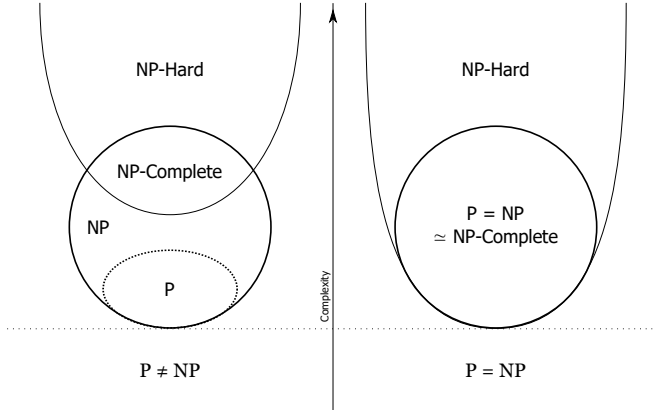
P = NP?

The unsolved problem "the question of whether P = NP?"



Asymptotic

- Big O Notation
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Big O Notation

History

- The symbol O was first introduced by number theorist Paul Bachmann in 1894
- The number theorist Edmund Landau adopted it, and was thus inspired to introduce in 1909 the notation o
- In the 1970s the big O was popularized in computer science by Donald Knuth

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Big O Notation (cont.)

Concept 18

Given function $g(n)$, we denote by $O(g(n))$ the set of functions $\{f(n)\}$, there exist positive constants c and n_0 such that

$$0 \leq f(n) \leq cg(n), \text{ for all } n \geq n_0 \tag{3}$$

- We pronounce $O(g(n))$ as "big-oh of g of n" or sometimes just "oh of g of n"
- **Big O Notation** represents the upper bound on the growth rate of an algorithm's time or space complexity.
- $T(n) = O(g(n))$ to indicate that a function $T(n)$ is a member of the set $O(g(n))$

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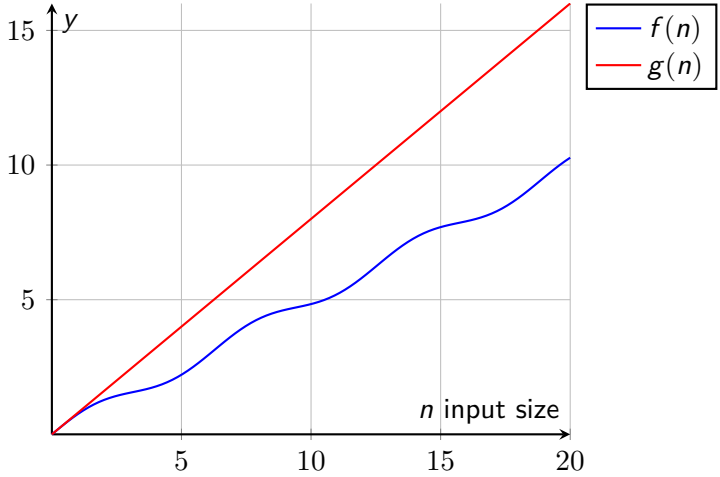
Big O Notation (cont.)



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Big O Notation (cont.)

- Common order-of-growth functions $g(n)$

Table 1: Order-of-growth function

name	Order-of-growth function
constant	1
logarithm	$\log n$
linear	n
n-log-n	$n \log n$
quadratic	n^2
cubic	n^3
exponential	2^n
permuation	$n!$

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Examples

Example 1

Prove that $T(n) = 2n^4 + 3n^3 + 5n^2 + 2n + 3 = O(n^4)$

Proof

We have

$2n^4+3n^3+5n^2+2n+3 \leq 2n^4+3n^4+5n^4+2n^4+3n^4 = (2+3+5+2+3)n^4 = 15n^4$

where $n \geq 1$ ■

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Examples (cont.)

Theorem 1

Prove that $T(n) = a_0 + a_1 n + ... + a_d n^d$ *where* $a_d > 0$ *then* $T(n) = O(n^d)$

Proof

Where $n \geq 1$, we have $1 \leq n \leq n^2 \leq ... \leq n^d$. Hence,

$$a_0 + a_1 n + a_2 n^2 ... + a_d n^d \leq (|a_0| + |a_1| + |a_2| + ... + |a_d|) n^d$$

So we have $f(n) = O(n^d)$ where $c = |a_0| + |a_1| + |a_2| + ... + |a_d|$ and $K = 1$ ■

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Examples (cont.)

Example 2

What is the order-of-growth of

$T(n)$	$O(\cdots)$
$5n^2 + 3n \log n + 2n + 5$	
$20n^3 + 10n \log n + 5$	
$3 \log n + 2$	
2^{n+2}	
$2n + 100 \log n$	

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Exercises

1. If f is in $O(g)$, what can we say about $af + b$?
2. If f_1 and f_2 are in $O(g)$, what can we say about $f_1 + f_2$?
3. If f_1 is in $O(g)$ and f_2 is in $O(h)$, what can we say about $f_1 + f_2$?
4. If f_1 is in $O(g)$ and f_2 is $O(h)$, what can we say about $f_1 \cdot f_2$?

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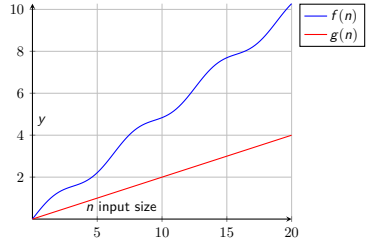
Big Omega Notation

Concept 19

Given function $g(n)$, we denote by $\Omega(g(n))$ the set of functions $\{f(n)\}$, there exist positive constants c and n_0 such that

$$f(n) \geq cg(n), \text{ for all } n \geq n_0 \tag{4}$$

- **Big Omega Notation** represents a lower bound on the growth rate, showing the minimum resources required.



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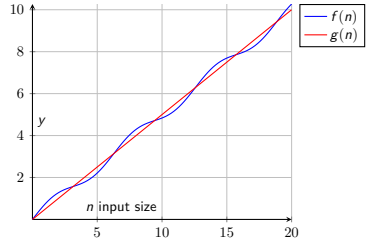
Big Theta Notation

Concept 20

Given function $g(n)$, we denote by $\Theta(g(n))$ the set of functions $\{f(n)\}$, there exist positive constants c_1, c_2, and n_0 such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n), \text{ for all } n \geq n_0 \tag{5}$$

- **Big Theta Notation** represents the growth rate, indicating both the upper and lower limits of an algorithm's complexity.



Mathematical Techniques

- Sum
- Generating Function

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Sum

Closed-form

- Series

$$1 + 2 + 3 + ... + N = \sum_{k=1}^{N} k = \frac{1}{2}(N^2 + N) \tag{6}$$

$$1^2 + 2^2 + 3^2 + ... + N^2 = \sum_{k=1}^{N} k^2 = \frac{1}{6}(2N^3 + 3N^2 + N) \tag{7}$$

$$1^3 + 2^3 + 3^3 + ... + N^3 = \sum_{k=1}^{N} k^3 = \frac{1}{4}(N^4 + 2N^3 + N^2) \tag{8}$$

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Sum (cont.)

- Geometric series

$$1 + r + r^2 + ... + r^n = \sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r} \tag{9}$$

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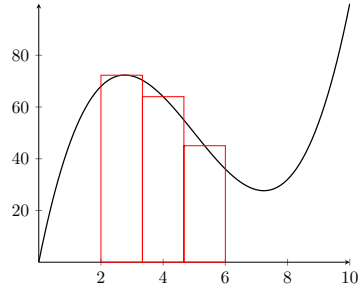
Sum (cont.)

Approximation calculation

- Use definite integral



- Harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} \approx \int_1^n \frac{1}{x} = \ln n - \ln 1 = \ln n$$

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Generating Function

Concept 21

Given a sequence of numbers $a_0, a_1, a_2, \ldots, a_k, \ldots$, the function

$$A(z) = \sum_{k \geq 0} a_k z^k$$

is called **ordinary generating function** (OGF) of the series. Denote $\left[z^k\right] A(z)$ be a_k.

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Some Generating Functions

Series	Generating function
$1, 1, 1, 1, \ldots, 1, \ldots$	$\frac{1}{1-z} = \sum_{N\geq 0} z^N$
$0, 1, 2, 3, 4, \ldots, N, \ldots$	$\frac{z}{(1-z)^2} = \sum_{N\geq 1} Nz^N$
$1, c, c^2, c^3, \ldots, c^N, \ldots$	$\frac{1}{1-cz} = \sum_{N\geq 0} c^N z^N$
$1, 1, \frac{1}{2!}, \frac{1}{3!}, \frac{1}{4!}, \ldots, \frac{1}{N!}, \ldots$	$e^z = \sum_{N\geq 0} \frac{z^N}{N!}$
$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{N}, \ldots$	$\ln \frac{1}{1-z} = \sum_{N\geq 1} \frac{z^N}{N}$

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Properties

Given two generating functions $A(z) = \sum_{k \geq 0} a_k z^k$ and $B(z) = \sum_{k \geq 0} b_k z^k$ that represent the sequences $a_0, a_1, \ldots, a_k, \ldots$ and $b_0, b_1, \ldots, b_k, \ldots$, we can perform a number of simple transformations to get generating functions for other sequences

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Properties (cont.)

Operation	Result
right shift	$zA(z) = \sum_{n\geq 1} a_{n-1} z^n$
left shift	$\frac{A(z)-a_0}{z} = \sum_{n\geq 0} a_{n+1} z^n$
differentiation	$A'(z) = \sum_{n\geq 0} (n+1) a_{n+1} z^n$
integration	$\int_0^z A(t)dt = \sum_{n\geq 1} \frac{a_{n-1}}{n} z^n$
scaling	$A(\lambda z) = \sum_{n\geq 0} \lambda^n a_n z^n$
addition	$A(z) + B(z) = \sum_{n\geq 0} (a_n + b_n) z^n$
difference	$(1-z)A(z) = a_0 + \sum_{n\geq 1} (a_n - a_{n-1}) z^n$
convolution	$A(z)B(z) = \sum_{n\geq 0} (\sum_{0\leq k\leq n} a_k b_{n-k}) z^n$
partial sum	$\frac{A(z)}{1-z} = \sum_{n\geq 0} (\sum_{0\leq k\leq n} a_k) z^n$

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Example

Example 3

Given a sequence $\{a_n\}_{n\geq 0}$ described by the recurrence

$$a_n = 5a_{n-1} - 6a_{n-2} \qquad n > 1 \text{ where } a_0 = 0 \text{ and } a_1 = 1$$

Proof

Use the generating functio $A(z) = \sum_{n\geq 0} a_n z^n$. Mutitply both sides by z^n and sum on n

$$\begin{aligned} \sum_{n\geq 2} a_n z^n &= 5\sum_{n\geq 2} a_{n-1} z^n - 6\sum_{n\geq 2} a_{n-2} z^n \\ A(z) - a_0 - a_1 z &= 5(A(z)z - a_0 z) - 6(A(z)z^2) \\ A(z) - z &= 5A(z)z - 6A(z)z^2 \end{aligned}$$

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Example (cont.)

to get the equation

$$A(z) = \frac{z}{1-5z+6z^2} = \frac{z}{(1-3z)(1-2z)} = \frac{1}{1-3z} - \frac{1}{1-2z}$$

Hence, we have $a_n = 3^n - 2^n$ ■

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Exercises

Using generating functions for solving the following recurrence relations.

1. $a_n = a_{n-1} + a_{n-2} \quad n > 1$ where $a_0 = 1$ and $a_1 = 1$.
2. $a_n = -a_{n-1} + 6a_{n-2} \quad n > 1$ where $a_0 = 0$ and $a_1 = 1$
3. $a_n = 11a_{n-2} - 6a_{n-3} \quad n > 2$ where $a_0 = 0$ and $a_1 = a_2 = 1$
4. $a_n = 3a_{n-1} - 4a_{n-2} \quad n > 1$ where $a_0 = 0$ and $a_1 = 1$
5. $a_n = a_{n-1} - a_{n-2} \quad n > 1$ where $a_0 = 0$ and $a_1 = 1$

Algorithm Analysis

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Example

Listing 1: Sum the first n integers

```
sum = 0;
for (i = 0; i < n; i++)
    sum = sum + i;
```

- Assigments: T_1
- Comparisions: T_2
- Arithmetic operation: T_3
- Time complexity $T(n) = T_1 + T_2 + T_3$

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Performance in Different Scenarios

How does the algorithm perform under different input conditions?

- **Best-case Analysis**: This considers the fastest the algorithm can possibly finish the task for a particular input.
- **Average-case Analysis**: This takes into account the average time/space complexity considering all possible inputs with equal probability.
- **Worst-case Analysis**: This identifies the slowest the algorithm can take for a specific input, representing the worst possible scenario.

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Complexities for Simple Statements

- **Sequential statement**: A sequential statement P consists of two parts, P_1 and P_2. The complexities of P, P_1 and P_2 are denoted by $T(n)$, $T_1(n)$ and $T_2(n)$, respectively.
$$T(n) = T_1(n) + T_2(n) \tag{10}$$
- **Branch statement**: A branch statement P consists two sub-statement P_1 and P_2
$$T(n) = \max(T_1(n), T_2(n)) \tag{11}$$
- **Loop statement**: A loop statement P

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Complexities for Functions

1. Simple function.
2. Function that calls other functions but does not call itself.
3. Recursive function.

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Example: Linear Search

```
1  int LinearSearch(int n, int a[], int key)
2  {
3      int i = 0;
4      while (i < n)
5      {
6          if (a[i] == key)
7              return i;
8          i++;
9      }
10     return -1;
11 }
```

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Example: Linear Search (cont.)

Consider the worst-case

lines	statements	$T(n)$
	assigment	
	loop	
	return	
Overall		

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Example: Bubble Sort

```
1  void BubbleSort(int n, int a[])
2  {
3      int i, j;
4      for (i = 0; i <= n - 2; i++)
5          for (j = n - 1; j >= i + 1; j--)
6              if (a[j] < a[j - 1])
7                  swap(a[j], a[j-1];
8  }
```

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Example: Bubble Sort (cont.)

Consider the worst-case

lines	statements	$T(n)$
Overall		

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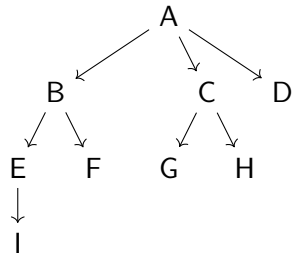
Function that calls other functions

Consider a function call tree



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Analysis

We have

1. $T(A) = T(B) + T(C) + T(D) + T_A$
2. $T(B) = T(E) + T(F) + T_B$
3. $T(C) = T(G) + T(H) + T_C$
4. $T(E) = T(I) + T_E$

Hence, $T(A) = T(D) + T(F) + +T(G) + T(H) + T(I) + T_A + T_B + T_C + T_E$

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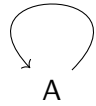
Recursive function

Concept 22

A recursive function is a function that calls itself directly or indirectly in order to solve a problem.



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Step 1: Identify the Recurrence Relation

Determine the recurrence relation that represents the recursive function. A recurrence relation is an equation or inequality that describes the runtime of a recursive function in terms of smaller inputs. You need to consider:

- **Base Case**: The simplest case that doesn't involve recursion. This often has a constant time complexity.
- **Recursive Case**: The part of the function that involves recursive calls. Identify how many times the function calls itself and with what input sizes.

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Step 2: Write the Recurrence Relation

Express the time complexity of the recursive function as a recurrence relation. For example:

- A classic recursive function like the Fibonacci sequence can be written as:

$$T(n) = \begin{cases} C_1 & n = 0, 1 \\ T(n-1) + T(n-2) + C_2 & n > 1 \end{cases}$$

- A simple recursive function like a binary search can be written as:

$$T(n) = \begin{cases} C_1 & n = 1 \\ T(n/2) + C_2 & n > 1 \end{cases}$$

- A general recursive function

$$T(n) = \begin{cases} C_1 & n = 0 \\ f(T(0)...T(n-1)) & n > 0 \end{cases}$$

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Step 3: Solve the Recurrence Relation

There are different methods to solve recurrence relations, such as:

- **Substitution**: Substitute values to determine patterns.
- **Recursion Tree**: Visualize the recursive calls as a tree, and calculate the total cost.
- **Master Theorem**: A formulaic approach to solve recurrences of a specific form, often used with divide-and-conquer algorithms.

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Factorial function

```
int factorial(int n)
{
    if (n == 0)
        return 1;
    else
        return (n * factorial(n - 1));
}
```

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Analysis

1. We have the recurrence relation

$$T(n) = \begin{cases} C_1 & n = 0 \\ T(n-1) + C_2 & n > 0 \end{cases}$$

2. Solve the recurrence relation using the subsitution method

$$\begin{aligned} T(n) &= T(n-1) + C_2 \\ T(n) &= (T(n-2) + C_2) + C_2 = T(n-2) + 2C_2 \\ T(n) &= (T(n-3) + C_2) + 2C_2 = T(n-3) + 3C_2 \\ &\dots \\ T(n) &= T(n-i) + iC_2 \\ &\dots \\ T(n) &= C_1 + nC_2 \end{aligned}$$

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Analysis (cont.)

Hence, we have

$$T(n) = C_1 + nC_2 = O(n)$$

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Merge Sort Function

```
List MergeSort(List L)
{
    List L1, L2;
    if (L.length <= 1)
        return L;
    else
    {
        Split(L, L1, L2);
        return (Merge(MergeSort(L1), MergeSort(L2)));
    }
}
```

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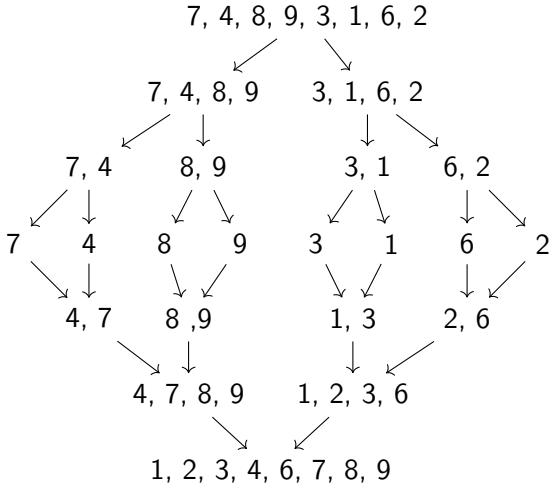
Merge Sort Function (cont.)



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Analysis

1. We have the recurrence relation

$$T(n) = \begin{cases} C_1 & n = 1 \\ 2T\left(\frac{n}{2}\right) + C_2 n & n > 1 \end{cases}$$

2. Solve the recurrence relation using the subsitution method

$$\begin{aligned} T(n) &= 2T\left(\tfrac{n}{2}\right) + C_2 n \\ T(n) &= 2\left[2T\left(\tfrac{n}{4}\right) + C_2\tfrac{n}{2}\right] + C_2 n = 4T\left(\tfrac{n}{4}\right) + 2C_2 n \\ T(n) &= 4\left[2T\left(\tfrac{n}{8}\right) + C_2\tfrac{n}{4}\right] + C_2 n = 8T\left(\tfrac{n}{8}\right) + 3C_2 n \\ &\dots \\ T(n) &= 2^i T\left(\tfrac{n}{2^i}\right) + iC_2 n \end{aligned}$$

The subsituition is ended when $\frac{n}{2^i} = 1$, so $2^i = n, i = \log n$. Hence, we have

$$T(n) = nT(1) + \log n C_2 n = C_1 n + C_2 n \log n = O(n \log n)$$

The Master Theorem

- The Master Theorem
- Multiplicative Function

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Introduction to The Master Theorem

The Master Theorem is a method used to determine the time complexity of **divide and conquer** algorithms. It provides a solution for recurrence relations of the form

$$T(n) = \begin{cases} 1 & n = 1 \\ a \cdot T\left(\frac{n}{b}\right) + f(n) & n > 1 \end{cases} \tag{12}$$

- n is the size of the input.
- $a \geq 1$ is the number of subproblems in the recursion.
- n/b $(b > 1)$ is the size of each subproblem. All subproblems are assumed to have the same size.
- $f(n)$ is the cost of the work done outside the recursive call, which includes the cost of dividing the problem and the cost of merging the solutions.

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Introduction to The Master Theorem (cont.)

The Master Theorem has the constants a, b, and the function $f(n)$

Theorem 2

The theorem has three cases:

1. ***Case 1**: If $f(n) = O(n^{\log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$. This occurs when the recursive work dominates.*
2. ***Case 2**: If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \log_b n)$. This happens when the recursive work and non-recursive work are balanced.*
3. ***Case 3**: If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some $\epsilon > 0$, and $a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$ for some $c < 1$ and sufficiently large n, then $T(n) = \Theta(f(n))$. This case arises when the non-recursive work dominates.*

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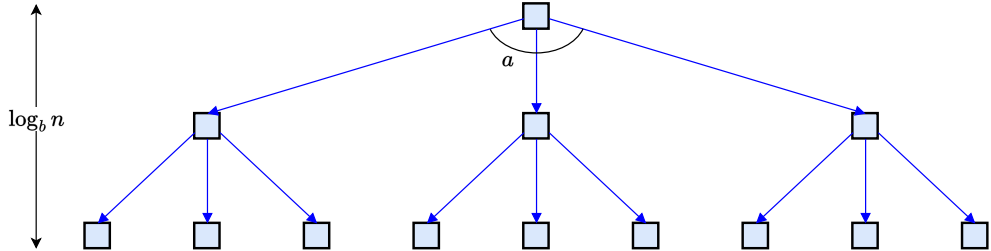
Simple Example

First, consider an algorithm with a recurrence of the form

$$T(n) = aT\left(\frac{n}{b}\right)$$

- The tree has a depth of $\log_b n$ and depth i contains a^i nodes. So there are $a^{\log_b n} = n^{\log_b a}$ leaves, and hence the runtime is $\Theta(n^{\log_b a})$.



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Simple Explanation

Apply a substitution chain

$$T(n) = aT\left(\frac{n}{b}\right) + f(n)$$
$$T(n) = a\left(aT\left(\frac{n}{b^2}\right) + f\left(\frac{n}{b}\right)\right) + f(n) = a^2 T\left(\frac{n}{b^2}\right) + af\left(\frac{n}{b}\right) + f(n)$$
$$\ldots$$

We have

$$T(n) = a^i T\left(\frac{n}{b^i}\right) + \sum_{j=0}^{i-1} a^j f\left(\frac{n}{b^j}\right)$$

Assume that $n = b^k$ or $k = \log_b n$, the substitution process is ended when $i = k$.

$$T\left(\frac{n}{b^i}\right) = T\left(\frac{n}{b^k}\right) = T\left(\frac{b^k}{b^k}\right) = T(1) = 1$$

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Simple Explanation (cont.)

Finally, we have

$$\begin{aligned} T(n) &= a^k + \sum_{j=0}^{k-1} a^j f\left(b^{k-j}\right) \\ &= n^{\log_b a} + \sum_{j=0}^{k-1} a^j f\left(b^{k-j}\right) \\ &= T_1(n) + T_2(n) \end{aligned} \tag{13}$$

where $T_1(n) = n^{\log_b a}$ is the recursive work and $T_2(n) = \sum_{j=0}^{k-1} a^j f\left(b^{k-j}\right)$ is the non-recursive work. The complexity can be

$$T(n) = \max(T_1, T_2)$$

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Multiplicative Function

Concept 23

A function $f(n)$ is a **multiplicative function** if $f(m.n) = f(m).f(n)\ \forall\, m, n$

- The function $f(n) = n^k$ is a muliplicative function
- The fucntion $f(n) = \log n$ is not a muliplicative function

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Multiplicative Function $f(n)$

Assume that $f(n)$ is a multiplicative function thì ta có đó $f(b^i) = f(b.b...b) = f(b).d(b)...d(b) = f(b)^i$. We have

$$T_2(n) = \sum_{j=0}^{k-1} a^j f\left(b^{k-j}\right)$$

$$T_2(n) = f(b)^k \sum_{j=0}^{k-1} \left(\frac{a}{f(b)}\right)^j$$

$$T_2(n) = f(b)^k \frac{\left(\frac{a}{f(b)}\right)^k - 1}{\frac{a}{f(b)} - 1}$$

$$T_2(n) = \frac{a^k - f(b)^k}{a - f(b)} f(b) \tag{14}$$

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Multiplicative Function $f(n)$ (cont.)

Three cases

- **Case 1**: $a > f(b)$. then $a^k > f(b)^k$

$$T(n) = \max(T_1, T_2) = T_1(n) = O(a^k) = O(n^{\log_b a})$$

- **Case 2**: $a < f(b)$. then $a^k < f(b)^k$

$$T(n) = \max(T_1, T_2) = T_2(n) = O(f(b)^k) = O(n^{\log_b f(b)})$$

- **Case 3**: $a = f(b)$.

$$T_2(n) = f(b)^k \sum_{j=0}^{k-1} \left(\frac{a}{f(b)}\right)^j = a^k \sum_{j=0}^{k-1} 1 = a^k k$$

$$T(n) = O\left(n^{\log_b a} \log_b n\right)$$

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Examples

Example 4

Consider a recursive function

$$T(n) = 4T\left(\frac{n}{2}\right) + n$$

Proof

- We have $a = 4, b = 2, f(n) = n$
- and $f(b) = f(2) = 2 < 4 = a$
- hence,

$$T(n) = O\left(n^{\log_b a}\right) = O\left(n^2\right)$$

■

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Examples (cont.)

Example 5

Consider a recursive function

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2$$

Proof

- We have $a = 4, b = 2, f(n) = n^2$
- and $f(b) = f(2) = 4 = a$
- hence,

$$T(n) = O\left(n^{\log_b a} \log_b n\right) = O\left(n^2 \log n\right)$$

■

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Examples (cont.)

Example 6

Consider a recursive function

$$T(n) = 4T\left(\frac{n}{2}\right) + n^3$$

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Examples (cont.)

Proof

- We have $a = 4, b = 2, f(n) = n^3$
- and $f(b) = f(2) = 8 > 4 = a$
- hence,

$$T(n) = O\left(n^{\log_b d(b)}\right) = O\left(n^3\right)$$

■

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Examples (cont.)

Example 7

Consider a recursive function

$$T(n) = 2T\left(\frac{n}{2}\right) + n \log n$$

Proof

- The recursive work

$$T_1(n) = n^{\log_b a} = n^{\log_2 2} = n$$

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Examples (cont.)

- The non-recursive work

$$T_2(n) = \sum_{j=0}^{k-1} a^j d(b^{k-j}) = \sum_{j=0}^{k-1} 2^j 2^{k-j} \log 2^{k-j}$$
$$T_2(n) = 2^k \sum_{j=0}^{k-1} (k-j) = 2^k \frac{k(k+1)}{2}$$
$$T_2(n) \sim 2^k k^2$$

- In this case, we have $n = 2^k$ và $k = \log_2 n$, so

$$T_2(n) \sim n \log^2 n$$

- In summary, we have

$$T(n) = O\left(n \log^2 n\right)$$

■

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Quiz

1. What is analysis of algorithms?

. . . . . .
. . . . . .
. . . . . .

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Exercises

Find the complexities of

1.
```
for(i = 0; i < n; i++)
    for (j = 0; j < n; j++)
        b[i][j] += c;

for(i = 0; i < n; i++)
    for (j = i+1; j < n; j++)
        b[i][j] -= c;
```

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Exercises (cont.)

2.

```
for(i = 0; i < n; i++)
    for (j = 0; j < n; j++)
        a[i][j] = b[i][j]+c[i][j];
```

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Exercises (cont.)

3.

```
for(i = 0; i<n; i++)
    for (j = 0; j<n; j++)
        for(k=a[i][j]=0; k<n; k++)
            a[i][j]+=b[i][k]+c[k][j];
```

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4. The recurrence relations

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

$$T(n) = 2T\left(\frac{n}{2}\right) + \log n$$

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Exercises (cont.)

5.

```
void HanoiTower(int n, int a, int b, int c) {
    if (n > 0) {
        HanoiTower(n - 1, a, c, b);
        cout << "move from " << a << " to " << c << endl;
        HanoiTower(n - 1, b, a, c);
    }
}
```

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Exercises (cont.)

6.

```
void Permute(int k, int n, int a[]) {
    if (k == 0) {
        for (int i = 0; i < n; i++)
            cout << a[i] << " ";
        cout << endl;
    }
    else {
        for (int i = 0; i < k; i++) {
            swap(a[i], a[k - 1]);
            Permute(k - 1, n, a);
            swap(a[i], a[k - 1]);
        }
    }
}
```

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