Trees

Bùi Tiến Lên

2021

KHOA CÔNG NGHỆ THÔNG TIN
TRƯỜNG ĐẠI HỌC KHOA HỌC TỰ NHIÊN

Contents

Trees and Their Applications

- Trees
- M-ary trees
- Parental trees
- Visualsing trees
- Applications

Introduction

Trees are a mathematical abstraction that play a central role in the design and analysis of algorithms because

- Trees are used to describe dynamic properties of algorithms.
- Trees are fundamental data storage structures that combine advantages of an array and a linked list.
  - **Searching** as fast as array.
  - **Insertion** and **deletion** as fast as linked list.

Trees

Concept 1

A **tree** is a nonlinear collection. It consists of

- A set of **nodes** that often represent entities.
- A set of **edges/links** that represent the relationship between nodes.

A tree T (**rooted tree**)

- is **empty tree**

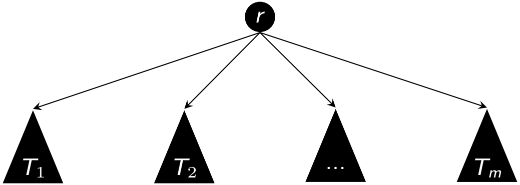
$$T = \emptyset \tag{1}$$

- is a node r (called the **root)** connected to a sequence of of disjoint trees $\{T_1, T_2, ..., T_m\}$ (called the **subtrees**)

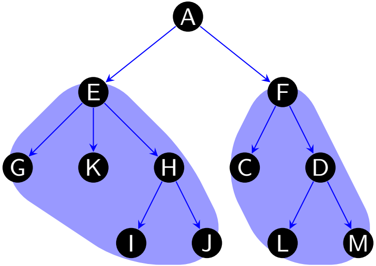
$$T = \{r \rightarrow \{T_1,, T_2, ..., T_m\}\} \tag{2}$$

Trees (cont.)

- Tree vs. Subtree



- Node A has two subtrees



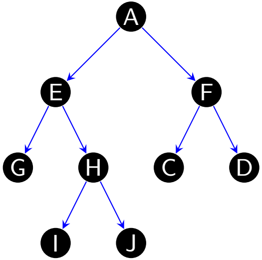
Terminology

In a tree

- **Node**: a simple object
- **Edge/Link/Branch**: a connection between two nodes

In a connection

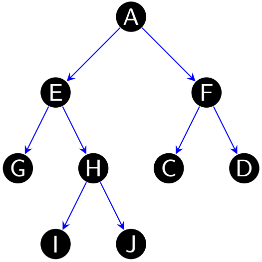
- **Parent node**: above a node
- **Child node**: below a node



Terminology (cont.)

In a tree or subtree

- **Root node**: node doesn't have parent
- **Leaf node/External node**: node doesn't have children
- **Internal node**: node has children
- **Sibling nodes**: nodes have the same parent



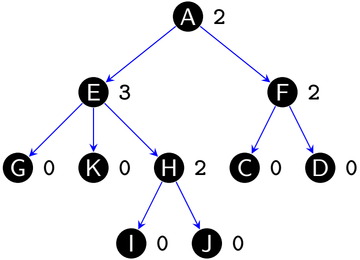
Terminology (cont.)

- **Degree of node** p

$$deg(p) = \text{the number of children of } p \tag{3}$$

- **Degree of tree** T

$$\deg(T) = \max\left(\deg(p_i), p_i \in T\right) \tag{4}$$



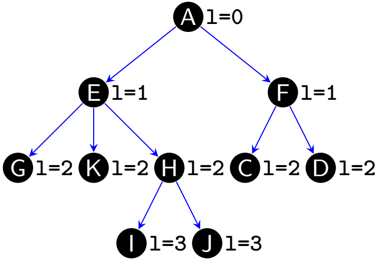
Terminology (cont.)

- **Level/depth of node** p:

$$level\,(p) = \begin{cases} 0 & p = root \\ level\,(parent\,(p)) + 1 & p \neq root \end{cases} \tag{5}$$

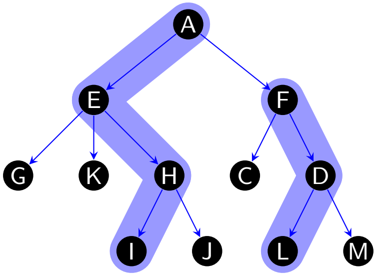
- **Height of tree** T:

$$height\,(T) = \max\,(level\,(p_i) + 1, p_i \in T) \tag{6}$$



Terminology (cont.)

- **Path**: A path in a tree is a list of distinct nodes in which successive nodes are connected by edges in the tree. In a path $p_1 - p_2 - ... - p_k$ is path, node p_1 is the **ancestor** and p_k is the **descendant**.



Terminology (cont.)

Concept 2

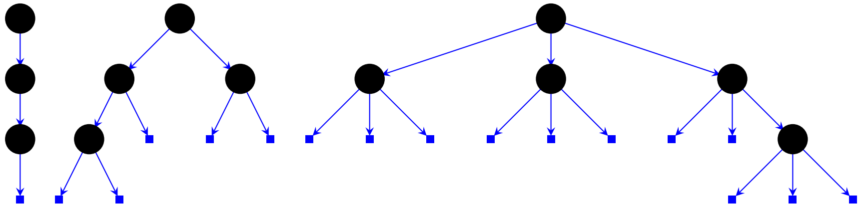
- The **path length of a tree** is the sum of the levels of all the tree's nodes.
- The **internal path length of a tree** is the sum of the levels of all the tree's internal nodes.
- The **external path length of a tree** is the sum of the levels of all the tree's external nodes

M-ary tree

Concept 3

An *M*-ary tree is each node connected to an ordered sequence of *M* trees that are also *M*-ary trees

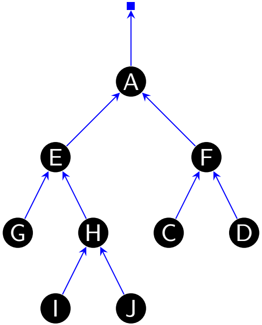
- **linear tree/linked list**: each node has only 1 subtree
- **binary tree**: each node has 2 subtrees
- **ternary tree**: each node has 3 subtrees



Parental Trees

Concept 4

A **parental tree** is a tree where each node only keeps a reference to its parent node



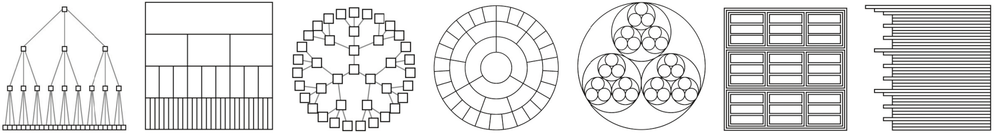
Parental Trees (cont.)

The parental tree representation is used in numerous places:

- Prim's algorithm: storing a minimum spanning trees of a weighted graph
- Dijkstra's algorithm: storing the minimum paths in a weighted graph
- Tree search based AI algorithms in general

Visualsing Trees

- Seven visual representations showing the same tree dataset



(a) (b) (c) (d) (e) (f) (g)

Applications

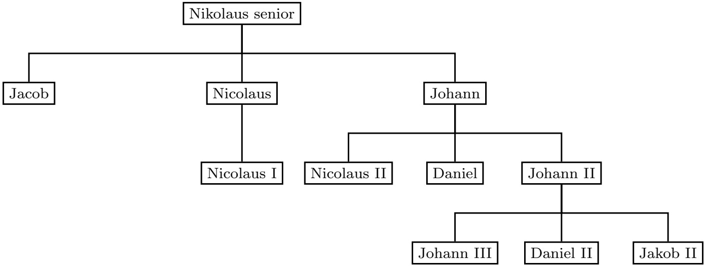


Figure 1: The Bernoulli

Applications (cont.)

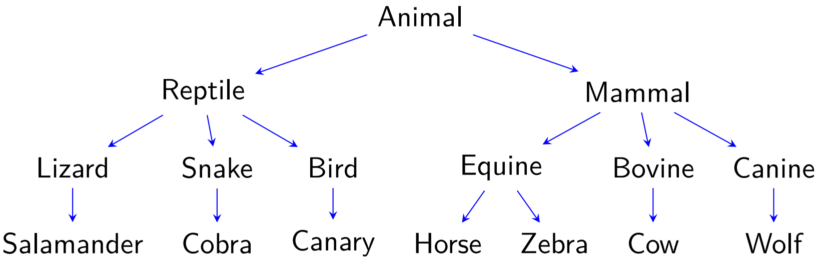


Figure 2: Animal tree

Applications (cont.)

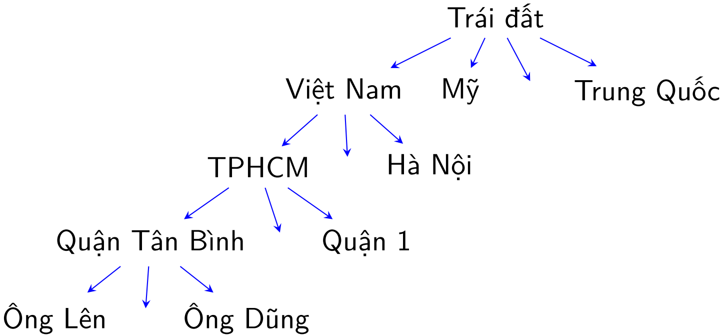


Figure 3: Management tree

Applications (cont.)

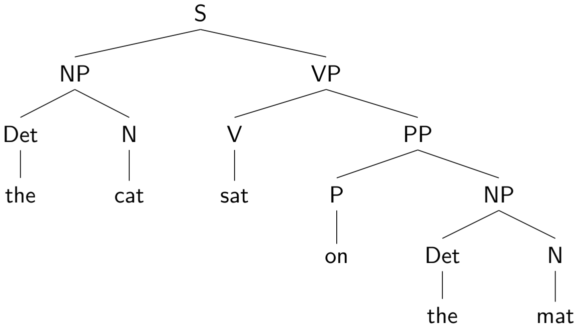


Figure 4: Syntax tree of the sentenece “the cat sat on the mat”

Applications (cont.)

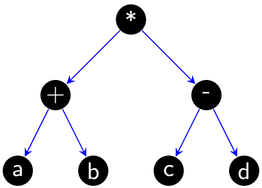


Figure 5: Tree of the algebra expression $(a + b) * (c - d)$

Applications (cont.)

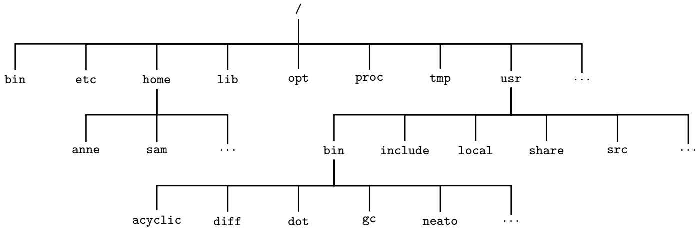


Figure 6: A file directory on Linux OS

Applications (cont.)

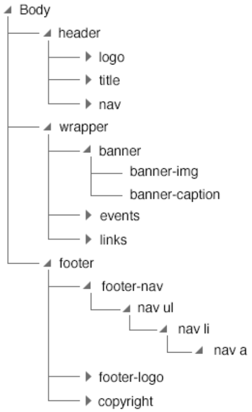
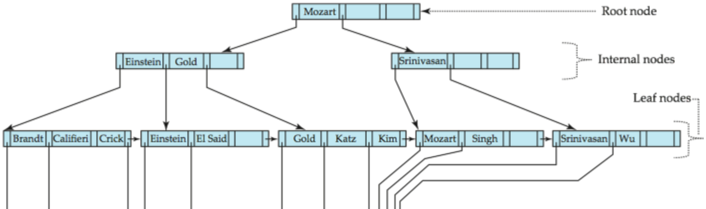


Figure 7: Structure of html file

Applications (cont.)



10101	Srinivasan	Comp. Sci.	65000
12121	Wu	Finance	90000
15151	Mozart	Music	40000
22222	Einstein	Physics	95000
32343	El Said	History	80000
33456	Gold	Physics	87000
45565	Katz	Comp. Sci.	75000
58583	Califieri	History	60000
76543	Singh	Finance	80000
76766	Crick	Biology	72000
83821	Brandt	Comp. Sci.	92000
98345	Kim	Elec. Eng.	80000

Figure 8: Database

Binary Trees

- Concepts
- Binary Tree API

Binary Trees

Concept 5

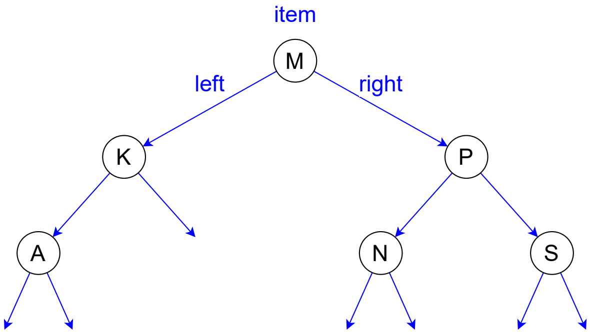
A binary tree is each node connected to a pair of binary trees, which are called the **left subtree** and the **right subtree** of that node

- Each node may have up to two successors, a **left child node** or a **right child node**.

- The concrete representation that we use most often is a structure with two links (a left link and a right link) for each node.

Binary-tree representation

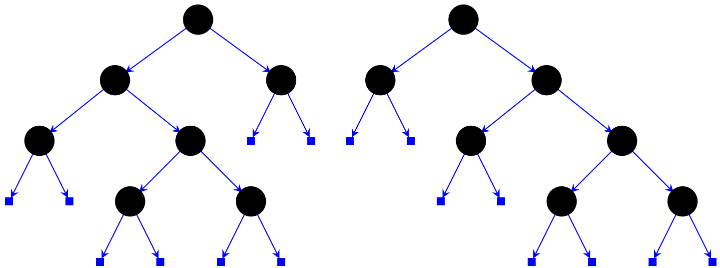
```
struct Node {
  Item item;
  Node *left, *right;
};
typedef Node *link;
```



Special Binary Trees

Concept 6

A **full binary tree** is binary in which each internal node has two children.

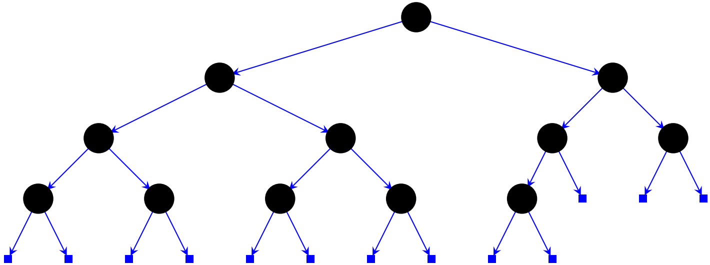


Special Binary Trees (cont.)

Concept 7

A **complete binary tree** is a binary tree in which

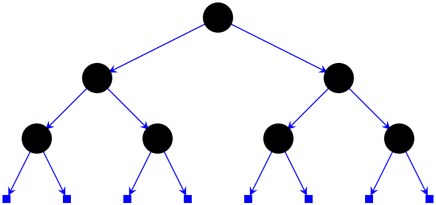
- From level 0 to $h-1$: the tree is completely full (maximum number of nodes)
- At the last level h: nodes are filled from left to right.



Special Binary Trees (cont.)

Concept 8

A **perfect binary tree** in which all internal nodes have two children and all leaf nodes are at the same level.



Number of nodes in binary tree

- **size** is the number of nodes in a binary tree/subtree T

$$size(T) = size(T \rightarrow leftSubtree) + size(T \rightarrow rightSubtree) + 1 \tag{7}$$

Level	Maximum number of nodes at each level
0	$2^0 = 1$
1	$2^1 = 2$
2	$2^2 = 4$
3	$2^3 = 8$
...	
10	$2^{10} = 1024$
...	
l	2^l

Height in binary tree

- A binary tree/subtree T

$$height(T) = \max(height(T \rightarrow leftSubtree), height(T \rightarrow rightSubtree)) + 1 \quad (8)$$

Properties of Binary Trees

Theorem 1

A binary tree T

1. *The number of nodes at level l is*
   - *at least* 1 *and*
   - *at most* 2^l.
2. *The of nodes in a binary tree of height h is*
   - *at least* h *and*
   - *at most* $2^h - 1$.
3. *The number of leaf nodes in a binary tree of height h is*
   - *at least* 1 *and*
   - *at most* 2^{h-1}.

Properties of Binary Trees (cont.)

Theorem 1

1. *The height of a binary tree with N nodes is*
   - *at least* $\log_2(N+1)$ *and*
   - *at most* N.
2. *A binary tree with N nodes has $N+1$ null links and $N-1$ not null links.*

Representing a Binary Tree

- A binary tree is represented by a reference to its **root node**.
  `link root;`
- An empty binary tree is represented with a reference whose value is null.

```
template <class Item>
class BinaryTree {
public:
  struct Node {
    Item item;
    Node *left, *right;
    Node(Item val) {
      item = val;
      left = nullptr;
      right = nullptr;
    }
    Node(Item val,Node *leftChild
        , Node *rightChild) {
        item = val;
        left = leftChild;
        right = rightChild;
      }
    };
    typedef Node *link;
private:
  link root;
public:
  ...
};
```

Traversal of Binary Trees

- A traversal of a binary tree is a systematic method of visiting each node in the binary tree. There are three binary tree traversal techniques:
  - Preorder traversal
  - Inorder traversal
  - Postorder traversal

Preorder Traversal

- Preorder traversal visits the root first, and then recursively traverses the left and right subtrees.

```
void preorder(link root) {
  if (root != nullptr) {
    visit(root);
    preorder(root->left);
    preorder(root->right);
  }
}
```

Inorder Traversal

- Inorder traversal recursively traverses the left subtree, then visits the root, and then traverses the right subtree.

```
void inorder(link root) {
  if (root != nullptr) {
    inorder(root->left);
    visit(root);
    inorder(root->right);
  }
}
```

Postorder Traversal

- Postorder traversal recursively traverses the left and right subtrees, and then visits the root.

```
void postorder(link root) {
  if (root != nullptr) {
    postorder(root->left);
    postorder(root->right);
    visit(root);
  }
}
```

Draw tree

- This recursive program keeps track of the tree height and uses that information for indentation in printing out a representation of the tree that we can use to debug tree-processing programs

```
void printNode(Item x, int h) {
  for (int i = 0; i < h; i++)
    cout << "  ";
  cout << x << endl;
}
void printTree(link t, int h) {
  if (t == nullptr) {
    for (int i = 0; i < h; i++)
      cout << "  ";
    cout << "* " << endl;
    return;
```

Draw tree (cont.)

```
    }
    printTree(t->left, h + 1);
    printNode(t->item, h);
    printTree(t->right, h + 1);
}
void printTree() {
    printTree(root, 0);
}
```

Binary Search Trees

- Concepts
- Tree API

Binary Search Trees

- Binary search trees are binary trees that organize their nodes to allow a form of binary search.
- Binary search trees work with values such as strings or numbers, that can be sorted.
- The idea is to store values in nodes so that small values are stored in the left subtree, and larger values are stored in the right subtree.

Binary Search Trees (cont.)

Concept 9

A **binary search tree** (BST) is a binary tree, at each node p,

- Every key stored in the left subtree of p is less than the key stored at p.

$$\forall q \in \textbf{\textit{LeftSubtree}}\,(p) : q.key < p.key$$

- Every key stored in the right subtree of p is greater than key stored at p.

$$\forall q \in \textbf{\textit{RightSubtree}}\,(p) : q.key > p.key$$

Binary Search Trees (cont.)

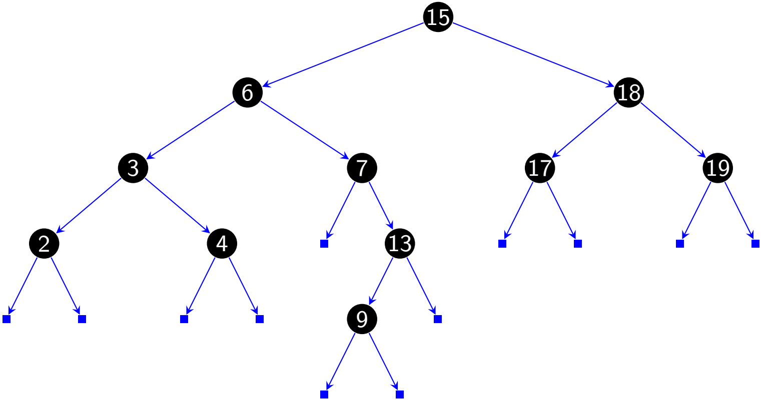


Figure 9: A binary search tree

Binary Search Trees (cont.)

```
template <class Key, class Value>
class BST {
public:
  struct Node {
    Key key;
    Value value;
    int N, h;
    Node *left, *right;
    Node(Key key, Value value) {
      this->key = val;
      this->value = value;
      N = 1; h = 1;
      left = nullptr;
      right = nullptr;
    }
  };
  typedef Node *link;
private:
  link root;
public:
  ...
};
```

Size & Height

```
int size() {
    return size(root);
}
int size(link x) {
    if (x == nullptr) { return 0; }
    else return x.N;
}
int height() {
    return height(root);
}
int height(link x) {
    if (x == nullptr) { return 0; }
    else return x.height;
}
```

Search

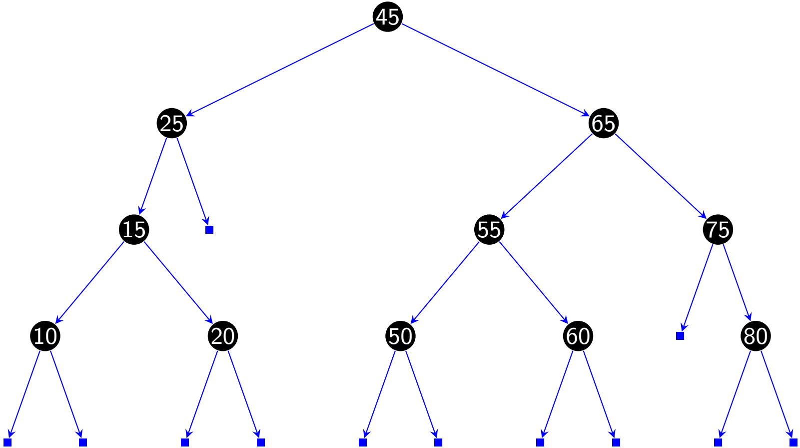
Search. If less, go left; if greater, go right; if equal, search hit.

The strategy for checking if a binary search tree contains a key value *key* is recursive.

- **Base case**: if the tree is empty, search miss.
- **Non base case**: Compare *key* to the key in the **root node** x
  - If *key* equals the value in the root, search hit and return *value*.
  - If *key* is less, recursively check if the left subtree contains *key*.
  - If *key* is greater, recursively check if the right subtree contains *key*.

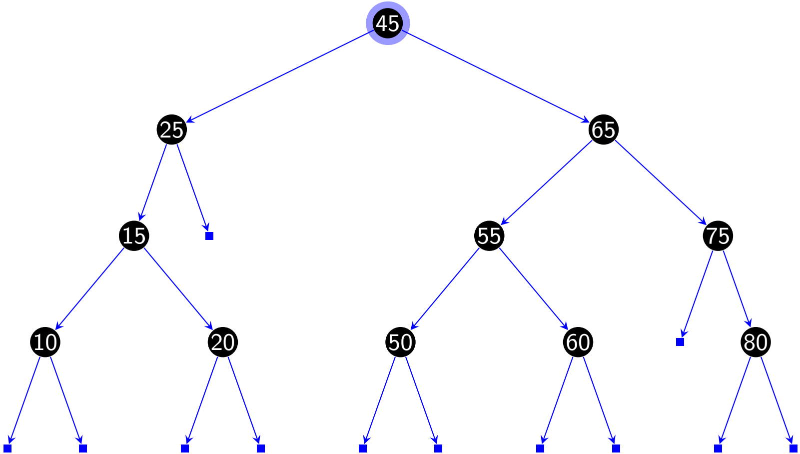
Illustration

Figure 10: Searching 20 (search hit)



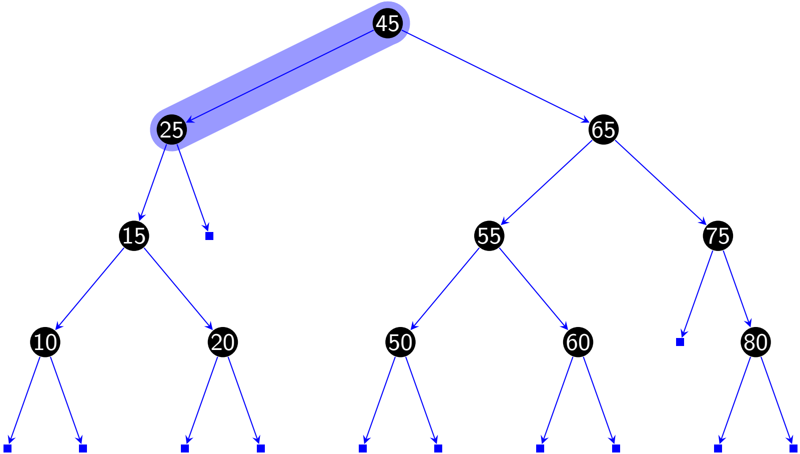
Illustration

Figure 10: Searching 20 (search hit)



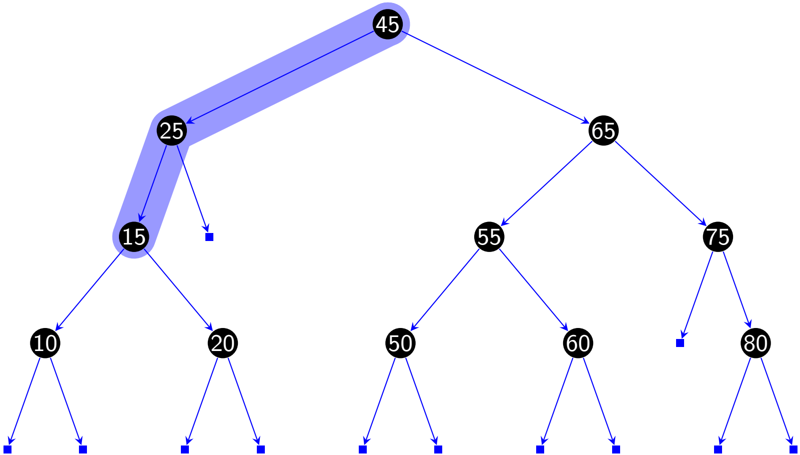
Illustration

Figure 10: Searching 20 (search hit)



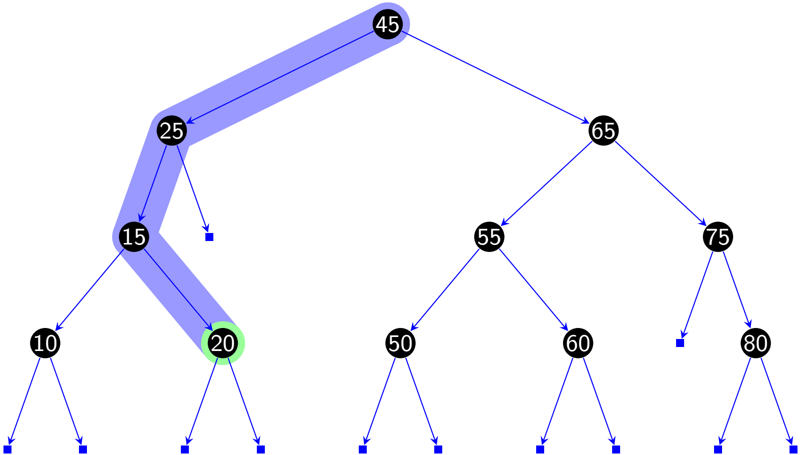
Illustration

Figure 10: Searching 20 (search hit)



Illustration

Figure 10: Searching 20 (search hit)



Insert

Insert. If less, go left; if greater, go right; if null, insert.

The strategy for adding $(key, value)$ to a binary search tree is recursive.

- **Base case**: if the tree is empty, create a tree with a single node containing $(key, value)$.
- **Non base case**: Compare *key* to the key value of the **root node** x
  - If *key* is less, recursively add *key* to the left subtree.
  - If *key* is greater, recursively add *key* to the right subtree.

Illustration

- Built BST from keys $\{4, 3, 5, 1, 2, 7, 9, 8\}$. The initial is a empty tree.

Illustration (cont.)

Figure 11: Insert 4

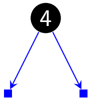


Illustration (cont.)

Figure 12: Insert 3

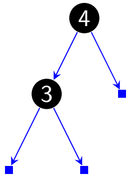


Illustration (cont.)

Figure 13: Insert 5

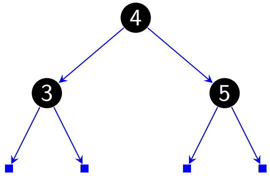


Illustration (cont.)

Figure 14: Insert 1

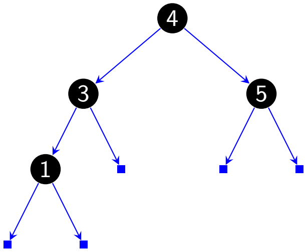


Illustration (cont.)

Figure 15: Insert 2

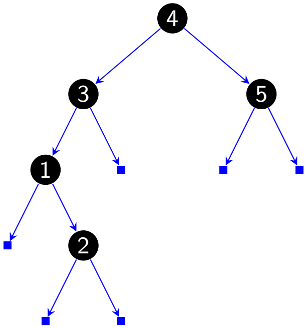


Illustration (cont.)

Figure 16: Insert 7

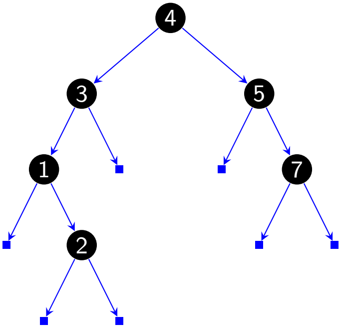


Illustration (cont.)

Figure 17: Insert 9

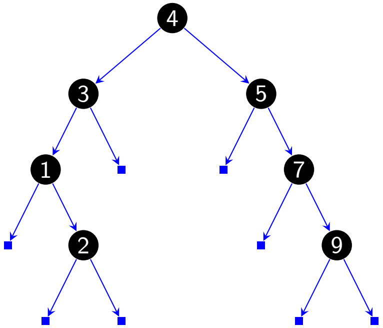
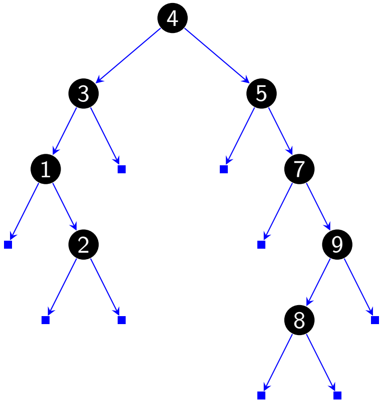


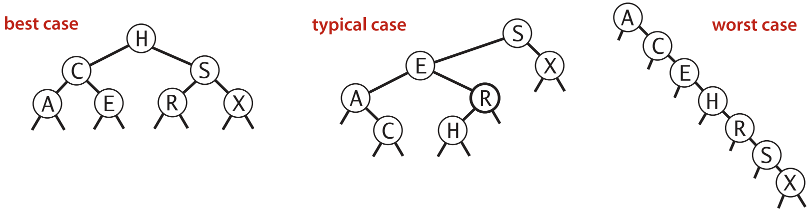
Illustration (cont.)

Figure 18: Insert 8



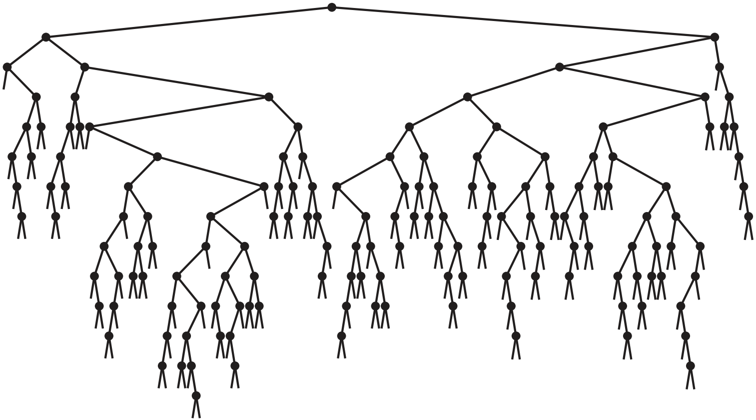
Tree shape

- Many BSTs correspond to same set of keys.
- Tree shape depends on order of insertion.
- Number of comparisons for search/insert is equal to 1 + depth of node.



Tree shape (cont.)

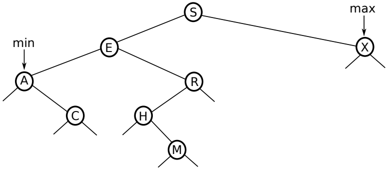
- If N distinct keys are inserted into a BST in random order, the expected number of comparisons for a search/insert is $\sim 2 \ln N$
- Typical BST, built from 256 random keys



Minimum and maximum

Minimum. Smallest key in BST.

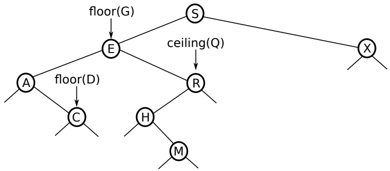
Maximum. Largest key in BST.



Floor

Computing the floor of key k

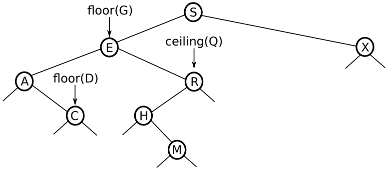
- **Case 1** (k equals the key in the node): the floor of k is k
- **Case 2** (k is less than the key in the node): the floor of k is in the left subtree
- **Case 3** (k is greater than the key in the node): the floor of k is in the right subtree if there is any key $\leq k$ in there; otherwise, it is the key in the node



Ceiling

Computing the ceiling of key k

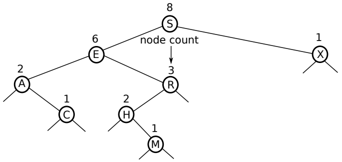
- **Case 1** (k equals the key in the node): the ceiling of k is k
- **Case 2** (k is greater than the key in the node): the ceiling of k is in the right subtree
- **Case 3** (k is less than the key in the node): the ceiling of k is in the left subtree if there is any key $\geq k$ in there; otherwise, it is the key in the node



Rank and selection

Rank. How many keys $<k$?

Select. Key has rank k ?



Delete min or max

To delete the minimum (maximum) key

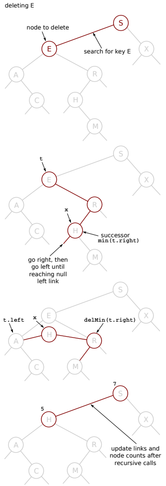
- Go left (right) until you find a node with null left (right) link
- Replace that node by its right (left) link
- Update subtree counts

Delete (Hibbard deletion)

To delete a node with key k, search for the node t containing key k

- **Case 1** (0 children): delete t by setting parent link to null
- **Case 2** (1 child): delete t by replacing parent link
- **Case 3** (2 children): find successor x of t (x has no left child); delete the minimum in t's right subtree; and put x in t's spot

and update subtree counts



Illustration

- Deleting leaf node 4: Before deletion

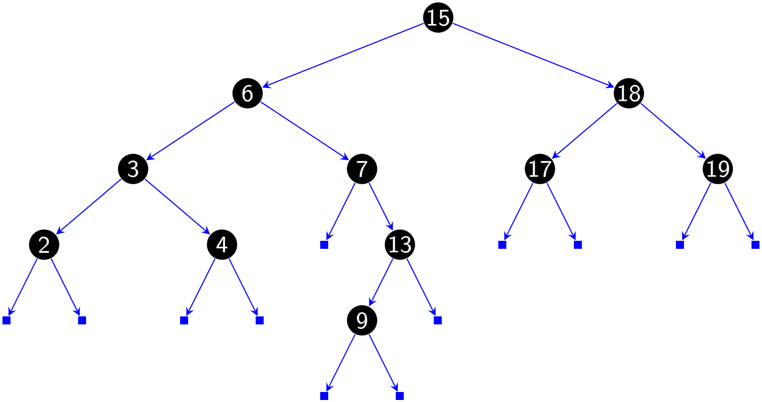


Illustration (cont.)

- Deleting leaf node 4: After deletion

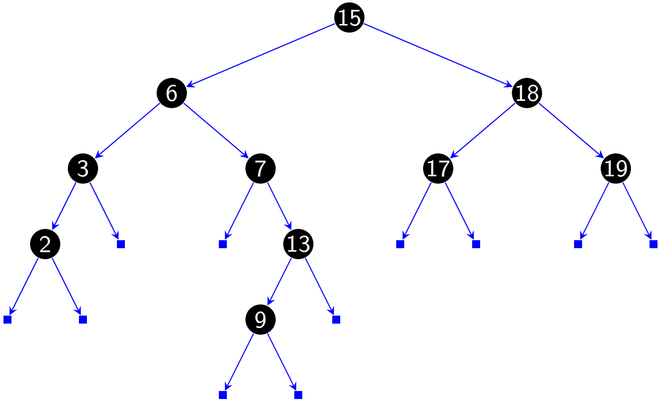


Illustration (cont.)

- Deleting node 7: Before deletion

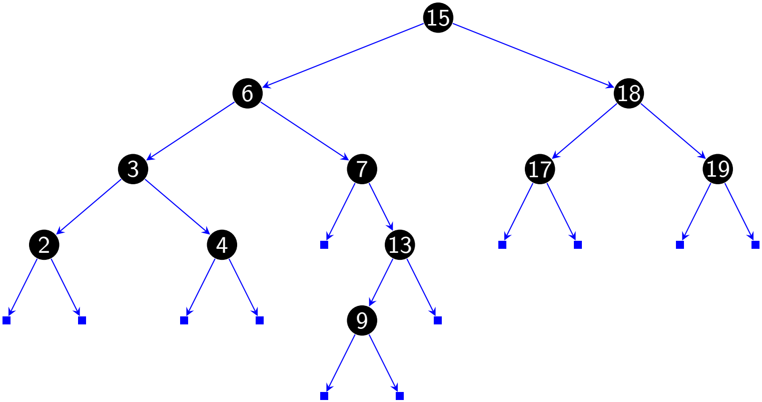


Illustration (cont.)

- Deleting node 7: After deletion

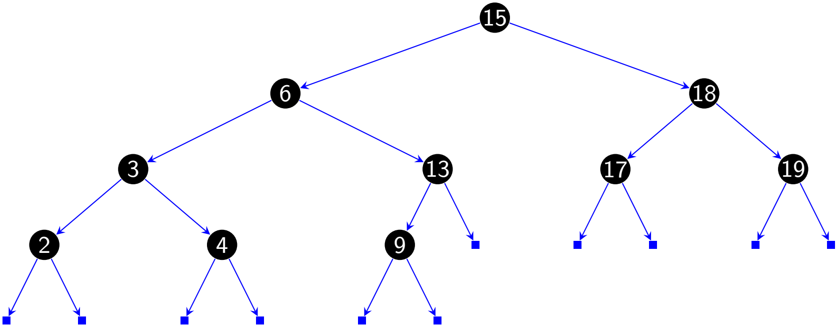


Illustration (cont.)

- Deleting node 15: Before deletion

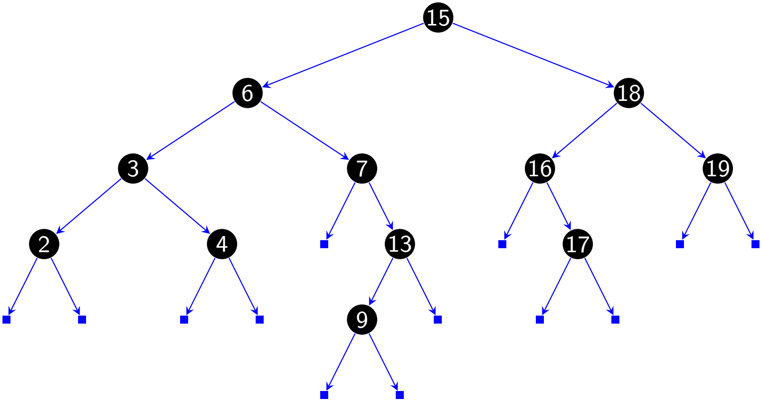
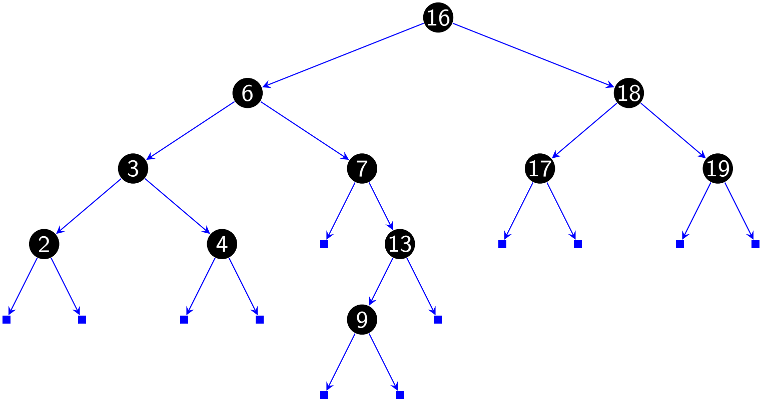


Illustration (cont.)

- Deleting node 15: After deletion



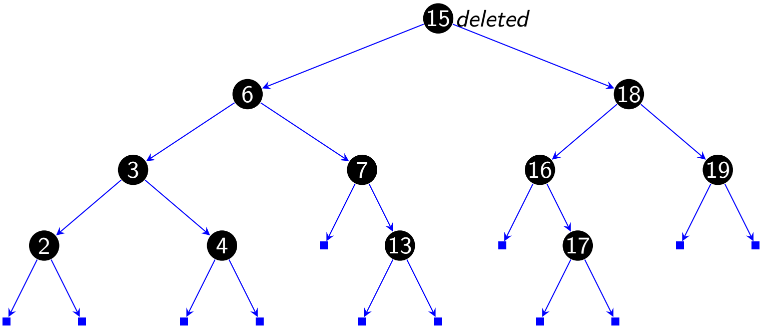
Hibbard deletion: analysis

- Unsatisfactory solution. Not symmetric.
- Surprising consequence. Trees not random $\rightarrow \sqrt{N}$ per op.
- Longstanding open problem. Simple and efficient delete for BSTs.

Deletion: lazy approach

To remove a node with a given key:

- Set its value to null.
- Leave key in tree to guide search (but don't consider it equal in search).
- Deleting node 15



Performance Characteristics

- Summary of Operations

operation	BST
search	h
insert	h
delete	$\sqrt{N}$
min/max	h
floor/ceiling	h
rank	h
select	h
ordered iteration	N

Cost summary for symbol-table implementations

implementation	worst case			average case			ordered	key
	search	insert	remove	search hit	insert	remove	iteration	
unordered list	N	1	N	$N/2$	1	$N/2$	no	equal
ordered list	N	N	N	$N/2$	$N/2$	$N/2$	yes	compare
ordered array	$\log_2 N$	N	N	$\log_2 N$	$N/2$	$N/2$	yes	compare
BST	N	N	N	$c \log_2 N$	$c \log_2 N$	$\sqrt{N}$	yes	compare
goal?								

Note: $c = 1.39$

Workshop

Quiz

1. What is a tree?

   ....

   ....

   ....

2. What is a binary sreach tree?

   ....

   ....

   ....

Exercises

- Programming exercises in [Cormen, 2009, Sedgewick, 2002]

References

Cormen, T. H. (2009).
Introduction to algorithms.
MIT press.

Sedgewick, R. (2002).
Algorithms in Java, Parts 1-4, volume 1.
Addison-Wesley Professional.

Walls and Mirrors (2014).
Data Abstraction And Problem Solving with C++.
Pearson.