

## Supplement (L4DC-2020): Bayesian joint state and parameter tracking in autoregressive models

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### Derivations for GCV Node

In this section we will derive the messages ⑧, ⑩, ⑫, ⑬ and ⑭ that are around the GCV node. We start by computing the message towards  $y_t$ . This is equivalent to:

$$\vec{\nu}(y_t) \propto \int \vec{\nu}(\boldsymbol{\theta}^\top \mathbf{x}_t) \exp \left( \mathbb{E}_{q(\kappa)q(\omega)q(z_t)} [\log \mathcal{N}(y_t | \boldsymbol{\theta}^\top \mathbf{x}_t, \vartheta_t)] \right) d\boldsymbol{\theta} d\mathbf{x}_t. \quad (1)$$

We first analyze the log expectation term.

$$\mathbb{E}_{q(\kappa)q(\omega)q(z_t)} [\log \mathcal{N}(y_t | \boldsymbol{\theta}^\top \mathbf{x}_t, \vartheta_t)] \propto \mathbb{E} \left[ -0.5 \left( \kappa z_t + \omega + \left( y_t - \boldsymbol{\theta}^\top \mathbf{x}_t \right)^2 \exp(-\kappa z_t - \omega) \right) \right] \quad (2a)$$

$$\propto -0.5 \left( \mathbb{E}[\kappa] \mathbb{E}[z_t] + \mathbb{E}[\omega] + \left( y_t - \boldsymbol{\theta}^\top \mathbf{x}_t \right)^2 \mathbb{E}[\exp(-\kappa z_t - \omega)] \right) \quad (2b)$$

If we assume  $q(\kappa), q(\omega), q(z_t)$  to be Gaussian we can evaluate the means and the expectation  $\mathbb{E}[\exp(-\omega)]$  analytically, i.e.

$$\xi_3 \triangleq \mathbb{E}[\exp(-\omega)] = \exp(-\mathbb{E}[\omega] + \mathbb{V}[\omega]/2). \quad (3)$$

[Aroian \(1947\)](#) approximates the multiplication of two Normally distributed random variable. Using the results of Theorem 2.6 we approximate the expectation

$$\mathbb{E}[\exp(-\kappa z_t)] \approx \xi_2 \triangleq \exp \left( -\mathbb{E}[\kappa] \mathbb{E}[z_t] + 0.5 \underbrace{\left( \mathbb{E}[z_t]^2 \mathbb{V}[\kappa] + \mathbb{E}[\kappa]^2 \mathbb{V}[z_t] + \mathbb{V}[z_t] \mathbb{V}[\kappa] \right)}_{\xi_1} \right). \quad (4)$$

Combining (4) and (3) we can approximate

$$\exp \left( \mathbb{E}_{q(\kappa)q(\omega)q(z_t)} [\log \mathcal{N}(y_t | \boldsymbol{\theta}^\top \mathbf{x}_t, \vartheta_t)] \right) \approx \mathcal{N}(y_t | \boldsymbol{\theta}^\top \mathbf{x}_t, (\xi_2 \xi_3)^{-1}). \quad (5)$$

Then the message towards  $y_t$  can be computed as the convolution

$$\vec{\nu}(y_t) \propto \int \vec{\nu}(\boldsymbol{\theta}^\top \mathbf{x}_t) \mathcal{N}(y_t | \boldsymbol{\theta}^\top \mathbf{x}_t, (\xi_2 \xi_3)^{-1}) d\boldsymbol{\theta} d\mathbf{x}_t \quad (6a)$$

$$\propto \mathcal{N}(y_t | \mathbb{E}[\boldsymbol{\theta}]^\top \mathbf{x}_t, (\xi_2 \xi_3)^{-1}) \quad (6b)$$

The last step follows from the convolution property of Gaussians. Nevertheless, we do not add the variance to the term  $\xi_2 \xi_3$  as  $\mathbf{x}_t$  is observed and there is no uncertainty associated with it. Next we proceed to derive the message

$$\overleftarrow{\nu}(\boldsymbol{\theta}^\top \mathbf{x}_t) \propto \int \overleftarrow{\nu}(y_t) \exp\left(\mathbb{E}_{q(\kappa)q(\omega)q(z_t)}[\log \mathcal{N}(y_t | \boldsymbol{\theta}^\top \mathbf{x}_t, \vartheta_t)]\right) d\boldsymbol{\theta} dy_t. \quad (7)$$

Using the results from (3), (4) and noting that  $y_t$  is given, we again see that

$$\overleftarrow{\nu}(\boldsymbol{\theta}^\top \mathbf{x}_t) \propto \int \overleftarrow{\nu}(y_t) \mathcal{N}(y_t | \boldsymbol{\theta}^\top \mathbf{x}_t, (\xi_2 \xi_3)^{-1}) d\boldsymbol{\theta} dy_t \quad (8a)$$

$$\propto \mathcal{N}(\boldsymbol{\theta}^\top \mathbf{x}_t | y_t, (\xi_2 \xi_3)^{-1}) \quad (8b)$$

Derivation of messages towards  $z_t$  and  $\kappa$  is equivalent so we only give it for  $z_t$  and just present the result for  $\kappa$ .

$$\overleftarrow{\nu}(z_t) \propto \exp\left(\mathbb{E}_{q(\kappa)q(\omega)q(y_t, \boldsymbol{\theta}^\top \mathbf{x}_t)}[\log \mathcal{N}(y_t | \boldsymbol{\theta}^\top \mathbf{x}_t, \vartheta_t)]\right) \quad (9a)$$

$$\propto \exp\left(-0.5 \left( \mathbb{E}[\kappa]z_t + \mathbb{E}[\omega] + \mathbb{E}\left[\left(y_t - \boldsymbol{\theta}^\top \mathbf{x}_t\right)^2\right] \mathbb{E}[\exp(-\kappa z_t - \omega)] \right)\right) \quad (9b)$$

$$\propto \exp\left(-0.5 \left( \mathbb{E}[\kappa]z_t + \underbrace{(y_t - \mathbb{E}[\boldsymbol{\theta}^\top] \mathbf{x}_t)^2}_{\xi_4} \mathbb{E}[\exp(-\kappa z_t - \omega)] \right)\right) \quad (9c)$$

$$\propto \exp(-0.5 (\mathbb{E}[\kappa]z_t + \xi_3 \xi_4 \mathbb{E}[\exp(-\kappa z_t)])) \quad (9d)$$

$$\propto \exp(-0.5 (\mathbb{E}[\kappa]z_t + \xi_3 \xi_4 \exp(-\mathbb{E}[\kappa]z_t + 0.5 z_t^2 \mathbb{V}[\kappa]))) \quad (9e)$$

Similarly for  $\kappa$ ,

$$\overleftarrow{\nu}(\kappa) \propto \exp(-0.5 (\kappa \mathbb{E}[z_t] + \xi_3 \xi_4 \exp(-\mathbb{E}[z_t] \kappa + 0.5 \kappa^2 \mathbb{V}[z_t]))) . \quad (10)$$

The last message we are interested is towards  $\omega$  edge. It follows;

$$\overleftarrow{\nu}(\omega) \propto \exp\left(\mathbb{E}_{q(\kappa)q(z_t)q(y_t, \boldsymbol{\theta}^\top \mathbf{x}_t)}[\log \mathcal{N}(y_t | \boldsymbol{\theta}^\top \mathbf{x}_t, \vartheta_t)]\right) \quad (11a)$$

$$\propto \exp(-0.5 (\omega + \xi_1 \xi_4 \exp(-\omega))) \quad (11b)$$

## References

Leo A. Aroian. The Probability Function of the Product of Two Normally Distributed Variables. *The Annals of Mathematical Statistics*, 18(2):265–271, June 1947. ISSN 0003-4851, 2168-8990. doi: 10.1214/aoms/1177730442. URL <https://projecteuclid.org/euclid.aoms/1177730442>.