# AE543: Homework 1

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# 1 Problem 1

Given the general solution as

$$u(t) = e^{i\zeta\omega_n t}\left(C_1 e^{\lambda t} + C_2 e^{-\lambda t}\right) \tag{1.1}$$

in which $\lambda$ is the root to the characteristic equation, $\zeta = \frac{c}{2m\omega_n}$ and $\omega_n^2 = \frac{k}{m}$. Therefore, we have the roots of the characteristic equation as

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{1.2}$$

## 1.1 Part a:

Consider the $\lambda_{1,2}$ for critical damping, $\zeta = 1.0$, we have the equation 1.2 becomes

$$\lambda_1 = \lambda_2 = \lambda = -\zeta\omega_n = -\omega_n \tag{1.3}$$

As the $\zeta = 1.0$, the specific solution will have a single repeated root. The general equation 1.1 will become

$$\begin{aligned} u(t) &= e^{i\zeta\omega_n t}\left(C_1 e^{-\zeta\omega_n t} + C_2 e^{\zeta\omega_n t}\right) \\ u(t) &= (C_1 + C_2 t)e^{-\omega_n t} \\ u(t) &= (C_1 + C_2 t)e^{-\omega_n t} \\ \text{with } \omega_n^2 &= \frac{k}{m} \end{aligned} \tag{1.4}$$

so

$$\boxed{u(t) = \ (C_1 + C_2 t)e^{-\omega_n t} \text{ with } \omega_n^2 = \frac{k}{m}} \tag{1.5}$$

## 1.2 Part b:

With boundary condition $u(t = 0) = u_0$ and $\dot{u}(t = 0) = \dot{u}_0$,

$$\begin{aligned} u(0) &= u_0 = C_1 \\ \dot{u}(t) &= \frac{d}{dt}\left[(u_0 + C_2 t)e^{-\omega_n t}\right] \\ \dot{u}(t) &= u_0 \frac{d}{dt}\left[e^{-\omega_n t}\right] + C_2 \frac{d}{dt}\left[t e^{\omega_n t}\right] \\ \dot{u}(t) &= u_0 - \omega_n e^{-\omega_n t} + C_2(e^{-\omega_n t} - \omega_n t e^{-\omega_n t}) \\ \dot{u}(t = 0) &= -u_0\omega_n + C_2 = \dot{u}_0 \\ \dot{u}_0 &= -\ u_0\omega_n + C_2 \\ C_2 &= \dot{u}_0 + u_0\omega_n \\ C_2 &= v_0 + u_0\omega_n \end{aligned} \tag{1.6}$$

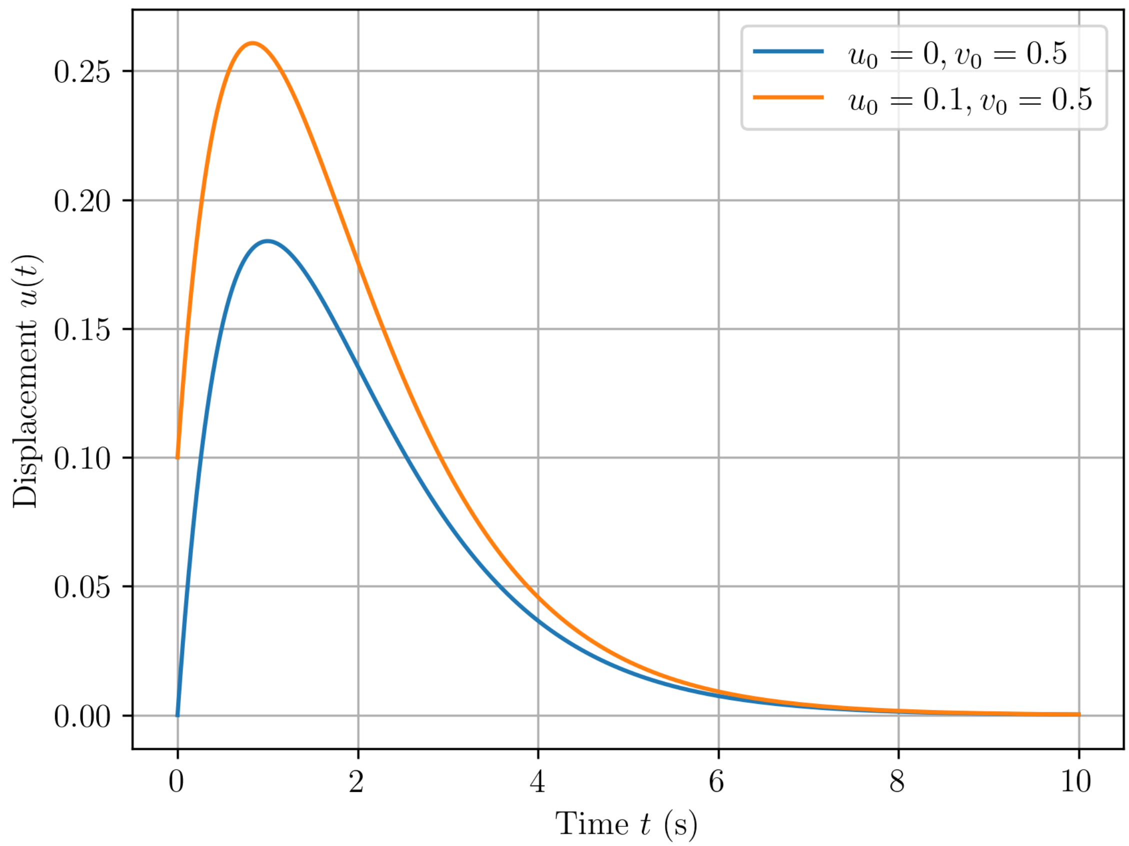
So, $\boxed{C_1 = u_0}$ and $\boxed{C_2 = v_0 + u_0 - \omega_n}$ for the general solution become

$$\boxed{u(t) = \left[u_0 + (v_0 + u_0\omega_n)t\right] e^{-\omega_n t}} \tag{1.7}$$

with $\zeta = 1$ and $\omega_n = \sqrt{\frac{k}{m}}$

### 1.3 *Part c:*

Assuming $v_0 = 1$ and $\omega_n = 1$, we can plot the displacement $u(t)$ in a function of time as following.




```python
import numpy as np
import matplotlib.pyplot as plt
u0,v0,omega = 0,0.5,1
t = np.linspace(0,10,1000)
u = (u0 + (v0 + u0*omega)*t)*np.exp(-omega*t)
u0 = 0.1
u2 = (u0 + (v0 + u0*omega)*t)*np.exp(-omega*t)
plt.grid(True)
plt.plot(t,u,t,u2)
plt.xlabel('Time $t$ (s)')
```

```python
plt.ylabel('Displacement $u(t)$')
plt.legend([r'$u_0=0, v_0 = 0.5$', r'$u_0=0.1, v_0 = 0.5$'])
plt.savefig('plot.png', dpi=300, bbox_inches='tight')
plt.show()
```

## 1.4 Part d:

When initial displacement of the landing gear is $u_0 = 0$, we have the new boundary equation as following $u(t = 0) = 0$ and $\dot{u}(t) = v_0$

$$\begin{aligned}
u(0) &= u_0 = 0 = C_1 \\
\dot{u}(t) &= \frac{d}{dt}\left[(C_2 t)e^{-\omega_n t}\right] \\
\dot{u}(t) &= C_2 \frac{d}{dt}\left[t e^{-\omega_n t}\right] \\
\dot{u}(t) &= C_2(e^{-\omega_n t} - \omega_n t e^{-\omega_n t}) \\
\dot{u}(t=0) &= C_2 = \dot{u}_0 \\
C_2 &= v_0
\end{aligned} \tag{1.8}$$

So, $\boxed{C_1 = u_0 = 0}$ and $\boxed{C_2 = v_0}$ for the general solution become

$$\boxed{u(t) = v_0 t e^{-\omega_n t}} \tag{1.9}$$

## 1.5 Part e:

To stop the vertical motion of the aircraft, the velocity $\dot{u} = v_0$ must reach zero. Therefore, we can use the general equation from part b: $u(t) = [u_0 + (v_0 + u_0\omega_n)t]\, e^{-\omega_n t}$. With $C_1 = 0$ and $C_2 = v_0$

$$\begin{aligned}
u(t) &= [u_0 + (v_0 + u_0\omega_n)t]\, e^{-\omega_n t} \\
\dot{u}(t) &= [v_0 + (v_0 - \omega_n(0 + v_0 t)]\, e^{-\omega_n t} \\
\dot{u}(t) &= v_0(1 - \omega_n t)e^{-\omega_n t}
\end{aligned} \tag{1.10}$$

We can find the time when the motion of the aircraft stops, by setting $\dot{u}(t) = 0$

$$\begin{aligned}
v_0(1 - \omega_n t)e^{-\omega_n t} &= 0 \\
1 - \omega_n t &= 0 \text{ as } v_0 \neq 0 \\
t &= \frac{1}{\omega_n}
\end{aligned} \tag{1.11}$$

The distance require at stop time is

$$
\begin{aligned}
u\left(t=\frac{1}{\omega_n}\right) &= v_0\left(\frac{1}{\omega_n}\right)e^{-\omega_n\frac{1}{\omega_n}} \\
u\left(t=\frac{1}{\omega_n}\right) &= \frac{v_0}{\omega_n e} \text{ for } e \approx 2.718
\end{aligned} \tag{1.12}
$$

Therefore, to stop the vertical motion of the aircraft, we need $\boxed{t=\frac{1}{\omega_n}}$ to stop the motion and the require distance is $\boxed{u(t_{\text{stop}})=\frac{v_0}{\omega_n e}}$

# 2 Problem 2

Given the Duffing oscillation as following

$$
m\ddot{x}+c\dot{x}+kx+k_c x^3 = f(t) \tag{2.1}
$$

Assuming the harmonic balance method (HBM) and applying stepped sine excitation, the trial solution $x(t)=X_0\sin(\omega t)$ and $F=F_0\sin(\omega t)$. By applying the stepped-sine excitation, we have

$$
f(t)=F\sin(\omega t-\phi)=F\sin(\omega t)\cos(\phi)-F\cos(\omega t)\sin(\phi) \tag{2.2}
$$

.

## 2.1 Part a:

Using the trial solution, we have

$$
\begin{cases}
x(t) &= X_0\sin(\omega t) \\
\dot{x}(t) &= X_0\omega\cos(\omega t) \\
\ddot{x}(t) &= -X_0\omega^2\sin(\omega t)
\end{cases} \tag{2.3}
$$

Therefore, from 2.3 and 2.2, the Duffing oscillator equation 2.1 becomes:

$$
\begin{aligned}
& m\left[-X_0\omega^2\sin(\omega t)\right]+c\left[X_0\omega\cos(\omega t)\right]+k\left[X_0\sin(\omega t)\right]+k_c\left[X_0\sin(\omega t)\right]^3 \\
& = F\sin(\omega t)\cos(\phi)-F\cos(\omega t)\sin(\phi) \\
& m\left[-X_0\omega^2\sin(\omega t)\right]+c\left[X_0\omega\cos(\omega t)\right]+k\left[X_0\sin(\omega t)\right] \\
& + k_c(X_0)^3\left[\frac{3}{4}\sin(\omega t)-\frac{1}{4}\sin(3\omega t)\right] \\
& = F\sin(\omega t)\cos(\phi)-F\cos(\omega t)\sin(\phi) \\
& \text{As HBM consider only fundamental response we can ignore } \sin(3\omega t) \text{ term.}
\end{aligned} \tag{2.4}
$$

The equation can be simplify as following

$$(-mX_0\omega^2 + kX_0 + \frac{3}{4}k_c(X_0)^3)\sin(\omega t) + cX_0\omega\cos(\omega t) = F\cos(\phi)\sin(\omega t) - F\sin(\phi)\cos(\omega t) \tag{2.5}$$

Equating the coefficients of $\sin(\omega t)$ and $\cos(\omega t)$, we have

$$\begin{aligned} &=> \begin{cases} (-mX_0\omega^2 + kX_0 + \frac{3}{4}k_c(X_0)^3)\sin(\omega t) & = & F\cos(\phi)\sin(\omega t) \\ cX_0\omega\cos(\omega t) & = & -F\sin(\phi)\cos(\omega t) \end{cases} \\ &=> \begin{cases} -m\omega^2 X_0 + kX_0 + \frac{3}{4}k_c(X_0)^3 & = & F\cos(\phi) \\ cX_0\omega & = & -F\sin(\phi) \end{cases} \end{aligned} \tag{2.6}$$

Therefore, the characteristic equation for this system is

$$\boxed{\begin{cases} -m\omega^2 X_0 + kX_0 + \frac{3}{4}k_c(X_0)^3 & = & F\cos(\phi) \\ cX_0\omega & = & -F\sin(\phi) \end{cases}} \tag{2.7}$$

## 2.2 *Part b:*

$$\begin{aligned} &=> \begin{cases} -m\omega^2 X_0 + kX_0 + \frac{3}{4}k_c(X_0)^3 & = & F\cos(\phi) \\ cX_0\omega & = & -F\sin(\phi) \end{cases} \\ &=> \begin{cases} \left[-m\omega^2 X_0 + kX_0 + \frac{3}{4}k_c(X_0)^3\right]^2 & = & F^2\cos^2(\phi) \\ \left[cX_0\omega\right]^2 & = & F^2\sin^2(\phi) \end{cases} \\ &=> \left[-m\omega^2 X_0 + kX_0 + \frac{3}{4}k_c(X_0)^3\right]^2 + \left[cX_0\omega\right]^2 = F^2\cos^2(\phi) + F^2\sin^2(\phi) \\ &\text{for } \cos^2(\phi) + \sin^2(\phi) = 1 \\ &=> \left[-m\omega^2 X_0 + kX_0 + \frac{3}{4}k_c(X_0)^3\right]^2 + \left[cX_0\omega\right]^2 = F^2 \end{aligned} \tag{2.8}$$

We can simplify as following

$$\boxed{(X_0)^2\left[-m\omega^2 + k + \frac{3}{4}k_c(X_0)^2\right]^2 + \left[cX_0\omega\right]^2 = F^2} \tag{2.9}$$

or

$$|H(\omega)| = \frac{F^2}{X^2} = \left(k - m\omega^2 + \frac{3}{4}k_cX^2\right)^2 + (c\omega)^2 \tag{2.10}$$

with $F$ is the forcing amplitude, $X$ is the displacement amplitude and frequency $\omega$

## 2.3 *Part c:*

As the system is nonlinear dynamic, the backbone analysis assumes the system's natural frequency and amplitude of its response to be undamped and unforced. Therefore, at undamped and free vibration, we have $c = 0$ and $F = 0$ for that the 2.9 becomes

$$(X_0)^2 \left[ -m\omega^2 + k + \frac{3}{4} k_c (X_0)^2 \right]^2 = 0 \tag{2.11}$$

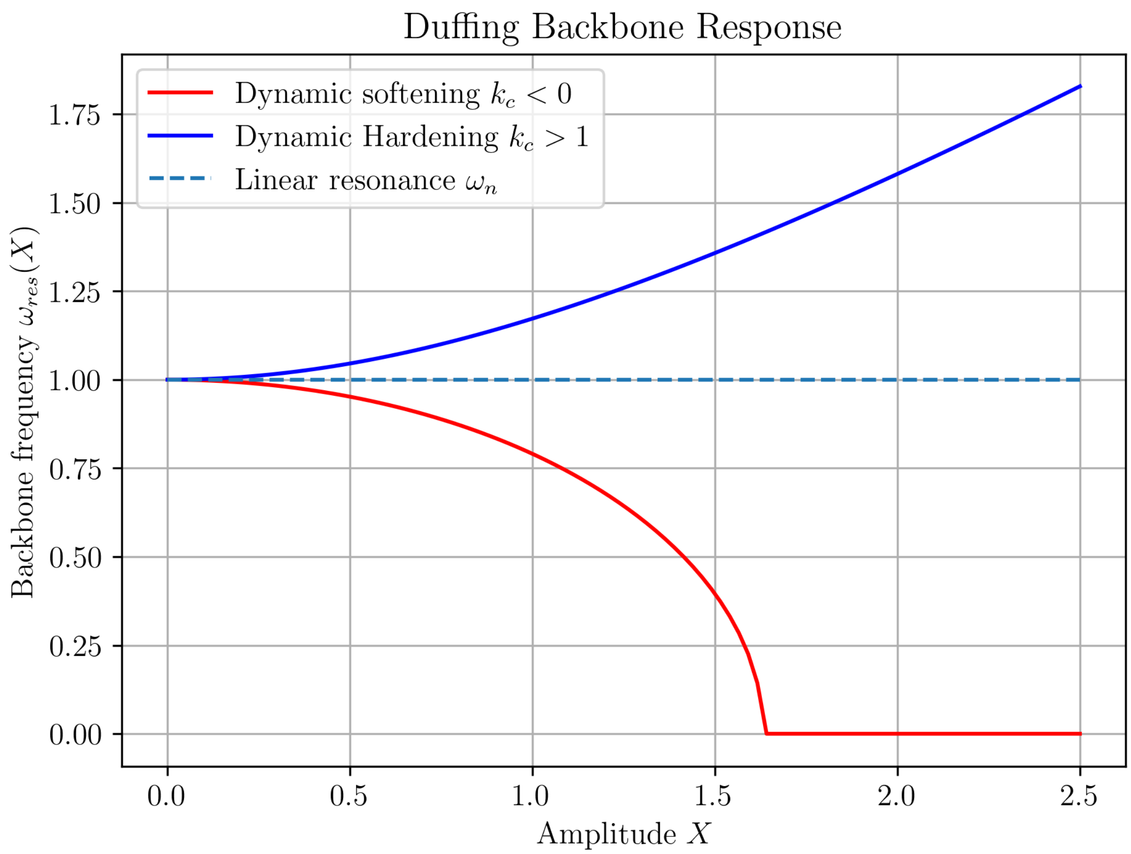
While $X_0 \neq 0$ due to non-zero displacement amplitude, the equation can be simplified as

$$\begin{aligned}
\left[ -m\omega^2 + k + \frac{3}{4} k_c (X_0)^2 \right]^2 &= 0 \\
-m\omega^2 + k + \frac{3}{4} k_c (X_0)^2 &= 0 \\
m\omega^2 &= k + \frac{3}{4} k_c (X_0)^2 \\
\omega^2 &= \frac{k}{m} + \frac{3}{4m} k_c (X_0)^2
\end{aligned} \tag{2.12}$$

Let natural frequency as $\omega_n^2 = \frac{k}{m}$ and resonance frequency as $\omega = \omega_{\text{res}}$, we have the backbone equation is

$$\begin{aligned}
\omega_{\text{res}}^2 &= \omega_n^2 + \frac{3k_c}{4m} X^2 \\
\omega_{\text{res}} &= \sqrt{\omega_n^2 + \frac{3k_c}{4m} X^2} \\
\omega_{\text{res}} &= \omega_n \sqrt{1 + \frac{3k_c}{4k} X^2} \text{ for } \omega_n^2 = \frac{k}{m}
\end{aligned} \tag{2.13}$$

Plotting $\omega_{\text{res}} = \omega_n \sqrt{1 + \frac{3k_c}{4k} X^2}$ for two case of $k_c > 0$ (dynamic hardening) and $k_c < 0$ (dynamic softening)




```
#================= Parameter =================
m,k,X = 1,1,2.5
kc_hardening = 0.5
kc_softening = -0.5
#================ Computing ==================
X =np.linspace(0,X,100)
w_n = [np.sqrt(k/m)]*len(X)
w_res_hard = w_n * np.sqrt(np.maximum(0.0, 1 + ((3*kc_hardening*X**2)/(4*k))) )
w_res_soft = w_n * np.sqrt(np.maximum(0.0, 1 + ((3*kc_softening*X**2)/(4*k))) )
#================= Plotting =================
plt.figure(figsize = (7,5))
plt.grid(True)
plt.plot(X,w_res_soft, color ='red')
plt.plot(X,w_res_hard, color ='blue')
plt.plot(X,w_n, linestyle = '--')

plt.xlabel('Amplitude $X$')
plt.ylabel(r'Backbone frequency $\omega_{res}(X)$')
plt.title('Duffing Backbone Response')


```

```
plt.legend([r'Dynamic softening $k_c<0$'
    , r'Dynamic hardening $k_c>1$'
    ,r'Linear resonance $\omega_n$'])
plt.savefig('2c.png', dpi=300, bbox_inches='tight')
plt.show()
```

# 3 Problem 3

yes I have watch and successfully install the python environment and library package

# 4 Problem 4

I have a full-time job during the day, so I spread out the homework and do 2-3 hours a day. Total hours are aroundd 10 to 12 hours.