# AE543: Homework 2

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# 1 Problem 1

We have position vector as following:

$$\begin{aligned} \vec{r}_1(t) &= (x+d)\ \mathbf{i} + y_1\ \mathbf{j} \\ \vec{r}_2(t) &= (x_1+x_2)\ \mathbf{i} + y\ \mathbf{j} \end{aligned} \tag{1.1}$$

For $x_2 = s\cos\beta$, $y_2 = s\sin\beta$ and $\dot{x}_2 = \dot{s}\cos\beta$, $\dot{y}_2 = \dot{s}\sin\beta$, we have

$$\begin{aligned} v_1 &= \dot{x}\ \mathbf{i} \\ v_2 &= (\dot{x}_1 + \dot{s}\cos\beta)\ \mathbf{i} + \dot{y}_2\ \mathbf{j} \end{aligned} \tag{1.2}$$

**Kinetic Energy**

$$\begin{aligned} T &= \frac{1}{2}m_1\vec{v_1}\cdot\vec{v_1} + \frac{1}{2}m_2\vec{v_2}\cdot\vec{v_2} \\ T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\left[(\dot{x}_1+\dot{s}\cos\beta)^2 + (\dot{s}\sin\beta)^2\right] \\ T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2(\dot{x}^2 + \dot{s}^2 - 2\dot{x}\dot{s}\cos\beta) \end{aligned} \tag{1.3}$$

**Potential Energy**

$$U = -m_2 g s\sin\beta \tag{1.4}$$

**The Lagrangian**

$$\begin{aligned} \mathcal{L} &= T - V \\ \mathcal{L} &= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2(\dot{x}^2 + \dot{s}^2 - 2\dot{x}\dot{s}\cos\beta) + m_2 g s\sin\beta \\ \mathcal{L} &= \frac{1}{2}(m_1+m_2)\dot{x}_1^2 + m_2\dot{x}_1\dot{s}\cos\beta + \frac{1}{2}m_2\dot{s}^2 + m_2 g s\sin\beta \end{aligned} \tag{1.5}$$

Therefore, with the **Hamilton's principle** $\int_{t_1}^{t_2} \delta\mathcal{L}dt = 0$, we have

$$\begin{aligned}
\delta\mathcal{L} &= m_1\dot{x}_1\ \delta\dot{x}_1 + m_2\dot{x}_1\ \delta\dot{x}_1 + m_2\dot{s}\cos\beta\ \delta\dot{x}_1 \\
&\quad + m_2\dot{x}\cos\beta\ \delta\dot{s} + m_2\dot{s}\ \delta\dot{s} - m_2 g\sin\beta\ \delta s \\
\int_{t_1}^{t_2}\delta\mathcal{L}dt &= \int_{t_1}^{t_2}\left[m_1\dot{x}_1\ \delta\dot{x}_1 + m_2\dot{x}_1\ \delta\dot{x}_1 + m_2\dot{s}\cos\beta\ \delta\dot{x}_1\right]dt \\
&\quad + \int_{t_1}^{t_2}\left[m_2\dot{x}\cos\beta\ \delta\dot{s} + m_2\dot{s}\ \delta\dot{s} - m_2 g\sin\beta\ \delta s\right]dt = 0 \\
&\quad \text{Integrate by part: } \int udv = uv - \int vdu \\
&\quad \int_{t_1}^{t_2}\left[(m_1+m_2)\ddot{x}_1 + m_2\ddot{s}\cos\beta\right]\delta x dt - (m_1+m_2)\dot{x}\ \delta x\ |_{t_1}^{t_2} \\
&\quad + \int_{t_1}^{t_2}\left[m_2\cos\beta\ddot{x}_1 + m_2\ddot{s} - m_2 g s\sin\beta\right]\delta s dt - (m_2\dot{x}\delta s\cos\beta + m_2\dot{s}\delta s)\ |_{t_1}^{t_2} = 0
\end{aligned}$$

With $(m_1+m_2)\dot{x}\ \delta x\ |_{t_1}^{t_2} = 0$ and $(m_2\dot{x}\delta s\cos\beta + m_2\dot{s}\delta s)\ |_{t_1}^{t_2} = 0$, The above equation of motion will be

$$\boxed{(\mathbf{m_1} + \mathbf{m_2})\ddot{\mathbf{x}}_\mathbf{1} + \mathbf{m_2}\ddot{\mathbf{s}}\cos\beta = \mathbf{0}} \tag{1.6}$$

and

$$\boxed{\mathbf{m_2}\ddot{\mathbf{x}}\cos\beta + \mathbf{m_2}\ddot{\mathbf{s}} = \mathbf{m_2}\mathbf{g}\sin\beta} \tag{1.7}$$

We can express the equation in matrix form $[M]\{\ddot{x}\} + [K]\{x\} = 0$

$$\begin{bmatrix} m_1+m_2 & m_2\cos\beta \\ m_2\cos\beta & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{s} \end{Bmatrix} = \begin{Bmatrix} 0 \\ m_2 g\sin\beta \end{Bmatrix} \tag{1.8}$$

# 2 Problem 2

## 2.1 Part a

with $u_{rot} = u\ \mathbf{i}$, we have

$$\begin{aligned}
r(t) &= u\cos(\omega)\ \mathbf{i} + u\sin(\omega)\ \mathbf{j} \\
\dot{r}(t) &= \dot{u}\left[\cos(\omega t)\ \mathbf{i} + \sin(\omega t)\ \mathbf{j}\right] + u\left[-\omega\sin(\omega t)\ \mathbf{i} + \omega\cos(\omega t)\ \mathbf{j}\right] \\
|\dot{r}(t)|^2 &= \dot{r}\cdot\dot{r} \\
\Rightarrow |\dot{r}(t)|^2 &= \dot{u}^2 + u^2\omega
\end{aligned} \tag{2.1}$$

**Kinetic Energy**

$$T = \frac{1}{2}m(\dot{u}^2 + u^2\omega) \tag{2.2}$$

**Potential Energy**

$$V = \frac{1}{2}ku^2 \tag{2.3}$$

**The Lagrangian**

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{u}^2 + u^2\omega) - \frac{1}{2}ku^2 \tag{2.4}$$

Applying the Lagrange Equation's $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) - \frac{\partial \mathcal{L}}{\partial \dot{u}} = 0$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot{u}} = m\dot{u} \\ \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{u}}\right) = m\ddot{u} \\ \dfrac{\partial \mathcal{L}}{\partial u} = \dfrac{1}{2}m(2u\omega^2) - \dfrac{1}{2}k(2(u - L_0)) = mu\omega^2 - ku \end{cases} \tag{2.5}$$

Substituting into Lagrange's Equation

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{u}}\right) - \frac{\partial \mathcal{L}}{\partial \dot{u}} = 0 \\ m\ddot{u} + (k - m\omega^2)u = 0 \end{aligned} \tag{2.6}$$

Therefore, the equation of motion for the system is

$$\boxed{\mathbf{m\ddot{u} + (k - m\omega^2)u = 0}} \tag{2.7}$$

### 2.2 Part b

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```python
import sympy as sp
from sympy import Function, symbols, diff
#===============================================
#      1. Define Symbols and the Generalized Coordinate
#===============================================
t = symbols('t')
m, k, R, omega = symbols('m k R omega', positive=True)
u = Function('u')(t)

u_dot = diff(u, t)
u_ddot = diff(u_dot, t)
#===============================================
#      2. Define Kinetic and Potential Energy
#===============================================

T = sp.Rational(1, 2) * m * (u_dot**2 + u**2 * omega**2)
V = sp.Rational(1, 2) * k * (u)**2
L = T - V
#===============================================
#      3. Define the Lagrangian
#===============================================
```

```
L = T - V
#===============================================
#     4. Lagrange's Equation
#===============================================
dL_du_dot = diff(L, u_dot)
d_dt_dL_du_dot = diff(dL_du_dot, t)
dL_du = diff(L, u)

# Lagrange's Equation of Motion (EOM)
EOM = d_dt_dL_du_dot - dL_du
#===============================================
#     5. Simplify and Display Result
#===============================================
EOM_simplified = EOM.subs(u_ddot, symbols('u_ddot'))
EOM_simplified = sp.simplify(EOM_simplified)
EOM_final = sp.Eq(EOM_simplified, 0)
EOM_rearranged = sp.Eq(m * u_ddot + (k - m * omega**2) * u,0)
```

We will get the answer similarly to the hand solving above in part a as

$$m\frac{d^2}{dt^2}u(t) + (k - m\omega^2)u(t) = 0 \tag{2.8}$$

# 3 Problem 3

Assuming the rolling without slipping, the displacement vector is $\{x\} = \{x_1\ x_2\}^T$. With $R$ is the disc radius the relative velocity is

$$\dot{x}_2 - \dot{x}_1 = -R\dot{\theta} \Rightarrow \dot{\theta} = \frac{\dot{x}_1 - \dot{x}_2}{R} \tag{3.1}$$

**Kinetic Energy**

$$\begin{aligned}
&T = T_{\text{cart}} + T_{\text{disc}}\\
&T = \frac{1}{2}m(\dot{x}_1)^2 + \frac{1}{2}m_1(\dot{x}_2)^2 + \frac{1}{2}I\dot{\theta}^2\\
&T = \frac{1}{2}m(\dot{x}_1)^2 + \frac{1}{2}m_1(\dot{x}_2)^2 + \frac{1}{2}\left(\frac{1}{2}m_1R^2\right)\left(\frac{\dot{x}_1 - \dot{x}_2}{R}\right)^2\\
&\quad \text{for } I = \frac{1}{2}m_1R^2\\
&T = \frac{1}{2}m(\dot{x}_1)^2 - \frac{1}{2}m_1\dot{x}_1\dot{x}_2 + \frac{3}{4}m_1(\dot{x}_2)^2 + \frac{1}{4}m_1(\dot{x}_1)^2
\end{aligned} \tag{3.2}$$

**Potential Energy**

$$V = \frac{1}{2}k_1(x_1)^2 + k_2(x_2 - x_1)^2 \tag{3.3}$$

**The Lagrangian**

$$\mathcal{L} = T - V = \left(\frac{1}{2}m + \frac{1}{4}m_1\right)\dot{x}_1^2 - \frac{1}{2}m_1\dot{x}_1\dot{x}_2 + \frac{3}{4}m_1\dot{x}_2^2 - \left[\frac{1}{2}k_1(x_1)^2 + k_2(x_2 - x_1)^2\right] \tag{3.4}$$

**Apply the Lagrange's Equation**

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = 0 \tag{3.5}$$

- Equation for generative coordinate $x_1$

$$\left\{\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{x}_1} &= \left(m + \frac{1}{2}m_1\right)\dot{x}_1 - \frac{1}{2}m_1\dot{x}_2 \\ \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) &= \left(m + \frac{1}{2}m_1\right)\ddot{x}_1 - \frac{1}{2}m_1\ddot{x}_2 \\ \frac{\partial \mathcal{L}}{\partial x_1} &= -k_1x_1 + 2k_2(x_2 - x_1) \\ &= -(k_1 + 2k_2)x_1 + 2k_2x_2 \end{aligned}\right. \tag{3.6}$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = 0 \\ \left(m + \frac{1}{2}m_1\right)\ddot{x}_1 - \frac{1}{2}m_1\ddot{x}_2 + (k_1 + 2k_2)x_1 - 2k_2x_2 = 0 \end{aligned} \tag{3.7}$$

so, equation of motion in generative coordinate $x_1$ is

$$\boxed{\left(\mathbf{m} + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{m_1}\right)\ddot{\mathbf{x}}_\mathbf{1} - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{m_1}\ddot{\mathbf{x}}_\mathbf{2} + (\mathbf{k_1} + \mathbf{2k_2})\mathbf{x_1} - \mathbf{2k_2x_2} = \mathbf{0}} \tag{3.8}$$

- Equation for generative coordinate $x_2$ is

$$\left\{\begin{aligned} \frac{\partial \mathcal{L}}{\partial \dot{x}_2} &= -\frac{1}{2}m_1\dot{x}_1 + \frac{3}{2}m_1\dot{x}_2 \\ \Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) &= -\frac{1}{2}m_1\ddot{x}_1 + \frac{3}{2}m_1\ddot{x}_2 \\ \frac{\partial \mathcal{L}}{\partial x_2} &= -2k_2(x_2 - x_1) \\ &= 2k_2x_1 - 2k_2x_2 \end{aligned}\right. \tag{3.9}$$

Therefore,

$$\begin{gathered}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \ddot{x}_2}\right) - \frac{\partial \mathcal{L}}{\partial x_2} = 0 \\ -\frac{1}{2}m_1\ddot{x}_1 + \frac{3}{2}m_1\ddot{x}_2 - 2k_2x_1 + 2k_2x_2 = 0\end{gathered} \tag{3.10}$$

so, equation of motion in generative coordinate $x_2$ is

$$\boxed{-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{m_1\ddot{x}_1} + \frac{\mathbf{3}}{\mathbf{2}}\mathbf{m_1\ddot{x}_2} - \mathbf{2k_2x_1} + \mathbf{2k_2x_2} = \mathbf{0}} \tag{3.11}$$

With the matrix form $[M]\{\ddot{x}\} + [K]\{x\} = 0$, we have

$$\begin{bmatrix} m + \frac{1}{2}m_1 & -\frac{1}{2}m_1 \\ -\frac{1}{2}m_1 & \frac{3}{2}m_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{pmatrix} k_1 + 2k_2 & -2k_2 \\ -2k_2 & 2k_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{3.12}$$

For $m = 10$ [kg], $m_1 = 1$ [kg], and $k_1 = k_2 = 1$ [N/m], we can substitute numerical value in matrix, the equation of matrix will be

$$\boxed{\begin{bmatrix} 10.5 & -0.5 \\ -0.5 & 1.5 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}} \tag{3.13}$$