

# AE543: Final Exam

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# 1 Section 1: Understanding Fundamentals

1. [1 point] What is the advantage of describing the physical coordinates of a dynamic system in terms of generalized coordinates? (one sentence only)

⇒ The primary advantage is that generalized coordinates define the configuration of a system using the minimum number of independent coordinates necessary, thereby eliminating the need to explicitly solve for constraint forces. (Ref: [Rao] Chapter 1, [Meirovitch] Chapter 1).

2. [1 point] What is the advantage of describing the generalized coordinates of a dynamic system in terms of modal coordinates? (one sentence only)

⇒ Using modal coordinates decouples the equations of motion (diagonalizes the mass and stiffness matrices), allowing a multi-degree-of-freedom system to be solved as a set of independent single-degree-of-freedom equations. (Ref: [Inman] Chapter 5).

3. [1 point] What are the properties of virtual displacement?

⇒ Virtual displacements are imaginary, infinitesimal changes in coordinates that are consistent with the system's geometric constraints and occur instantaneously (time is held fixed,  $dt = 0$ ). (Ref: [Ginsberg] Chapter 3).

4. [1 point] What is the difference between Lagrange's equation and Lagrangian in vibration analysis? (one sentence only)

⇒ The Lagrangian is a scalar function defined as the difference between kinetic and potential energy ( $L = T - V$ ), whereas Lagrange's equation is the differential equation of motion derived by applying operations to that function. (Ref: [Rao] Chapter 6).

5. [1 point] What are the advantages of using virtual displacement and virtual work?

⇒ Here are the advantages of virtual displacement and virtual work approach:

- Elimination of constraint forces: constraint forces do not appear in the formulation which helps opting out solving the internal reaction forces to establish equation of motions.

- Use of generalized coordinates: As we choose a set of independent coordinates that matches the system's degree of freedoms naturally, we can reduce the number of equations to ten minimum necessary number of degree of freedom
- Applicability to multi-body system: as we can treat the entire interconnected system as a whole, we can opt out solving each individual component and interaction forces between them like Newtonian approach

6. [1 point] What is the definition of virtual work? (one sentence only)

⇒ Virtual work is the total work done by all applied forces acting through the virtual displacements of the system. (Ref: [Meirovitch] Chapter 1).

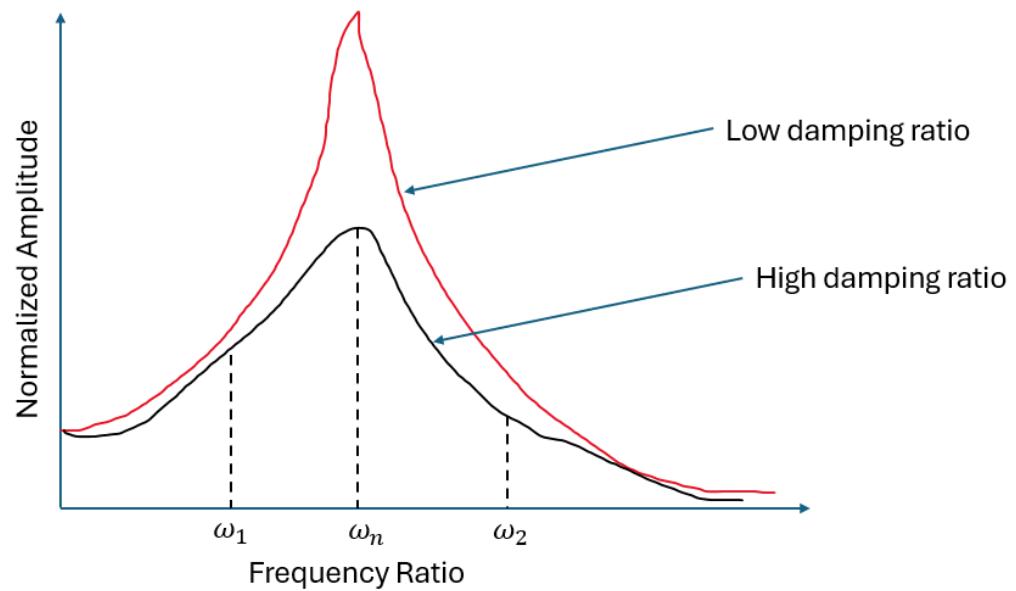
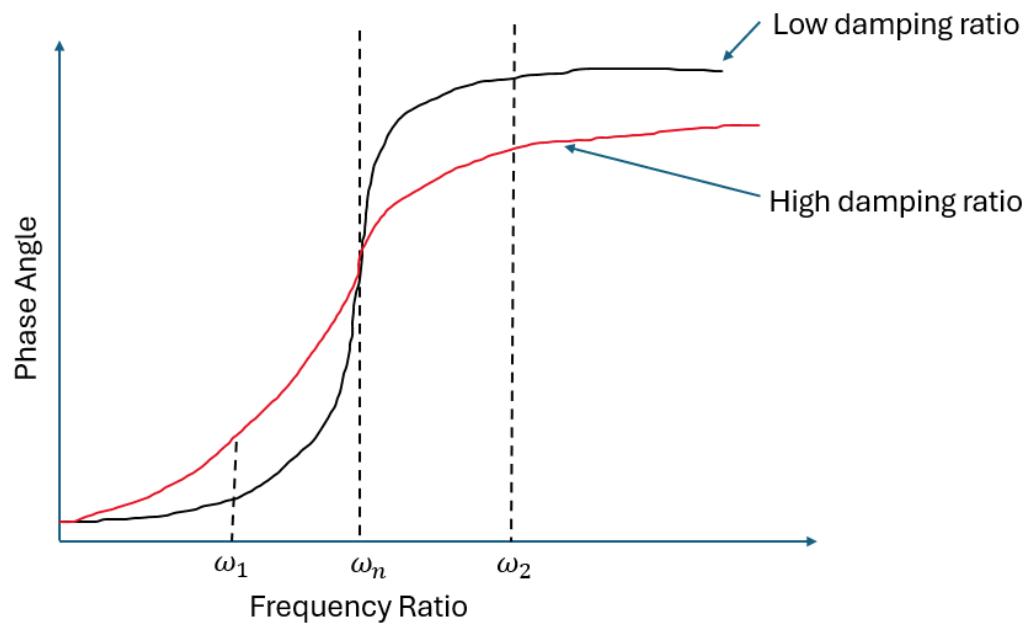
7. [1 point] For the virtual work approach, when is equilibrium attained? (one sentence only)

⇒ Equilibrium is attained when the total virtual work done by all active forces and inertial forces vanishes (equals zero) for any arbitrary virtual displacement. (Ref: [Ginsberg] Chapter 3).

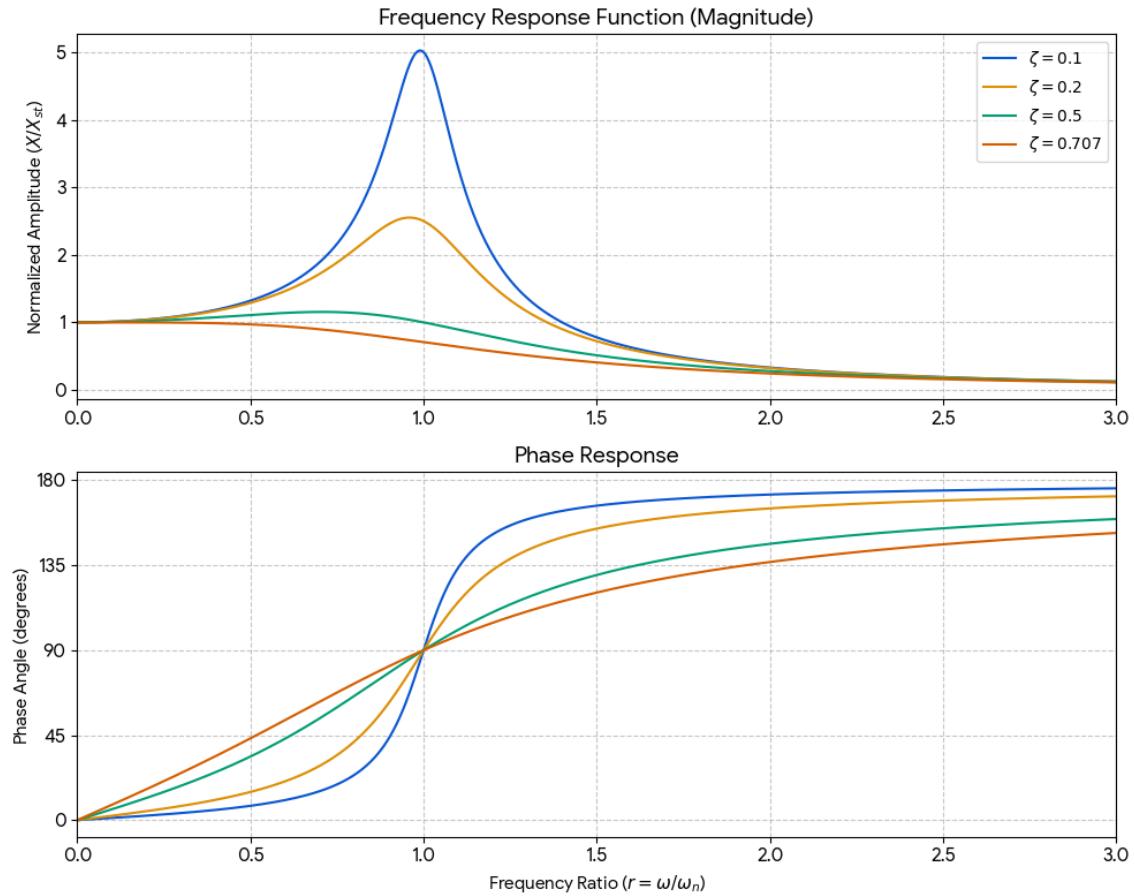
8. [1 point] How many normal modes of vibration are in the Boeing 777 wing? (one sentence only)

⇒ As a continuous physical structure, the wing possesses an infinite number of normal modes of vibration, though in engineering practice, it is approximated by a finite but large number of degrees of freedom. (Ref: [Inman] Chapter 6 - Continuous Systems).

9. [3 point] Sketch by hand the FRF and phase response of a single degree of freedom spring-mass system with a damper. Plot the effect of increasing and decreasing the damping ratio,  $\zeta$ , on the natural frequency? ⇒



We can plot as following



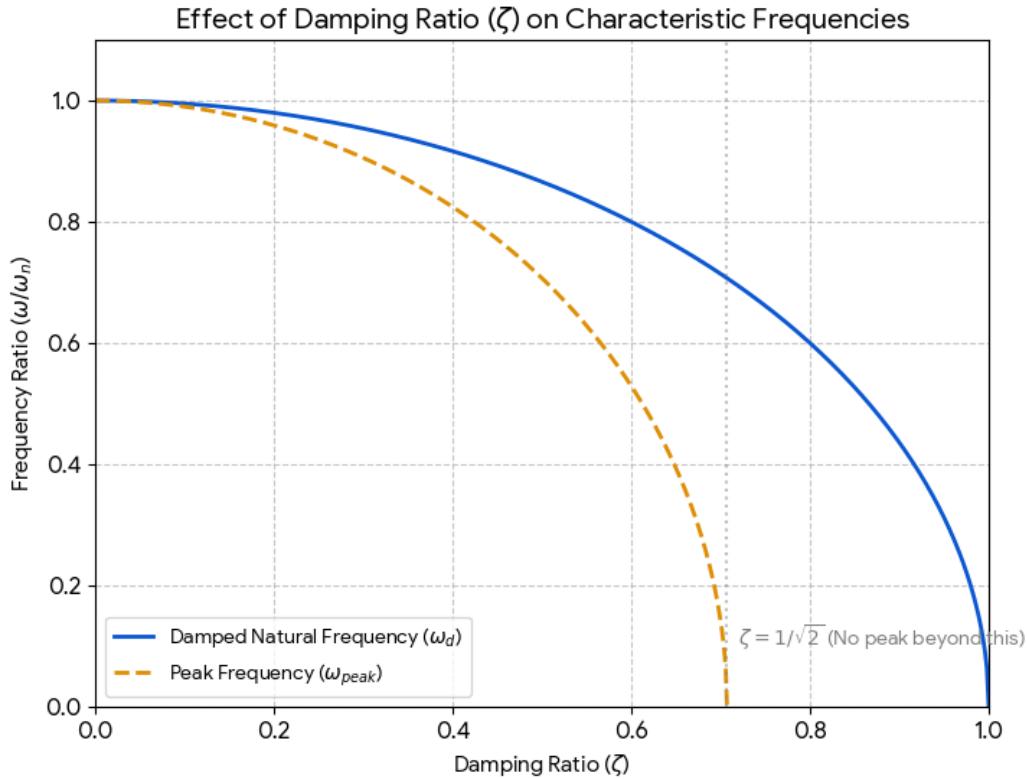


Figure 1: Effect of increasing and decreasing the damping ratio

10. [2 point] Show on FRF plots how to obtain stiffness, damping, and mass of a 1-DoF system without knowing anything about the specifications of the structure

⇒ We have the receptance FRF

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{k - m^2\omega^2 + i\omega c}$$

To extract parameters from an experimental FRF (Bode plot) of Compliance ( $X/F$ ):

**Stiffness ( $k$ ):** Look at the low-frequency asymptote (where  $\omega \rightarrow 0$ ). The magnitude of the response approaches  $1/k$ . Thus,  $k = 1/|H(\omega)|\omega \approx 0$ .

**Mass ( $m$ ):** Look at the high-frequency asymptote (mass-controlled region). The slope is  $-40$  dB/decade. Pick a point on this line; the magnitude is

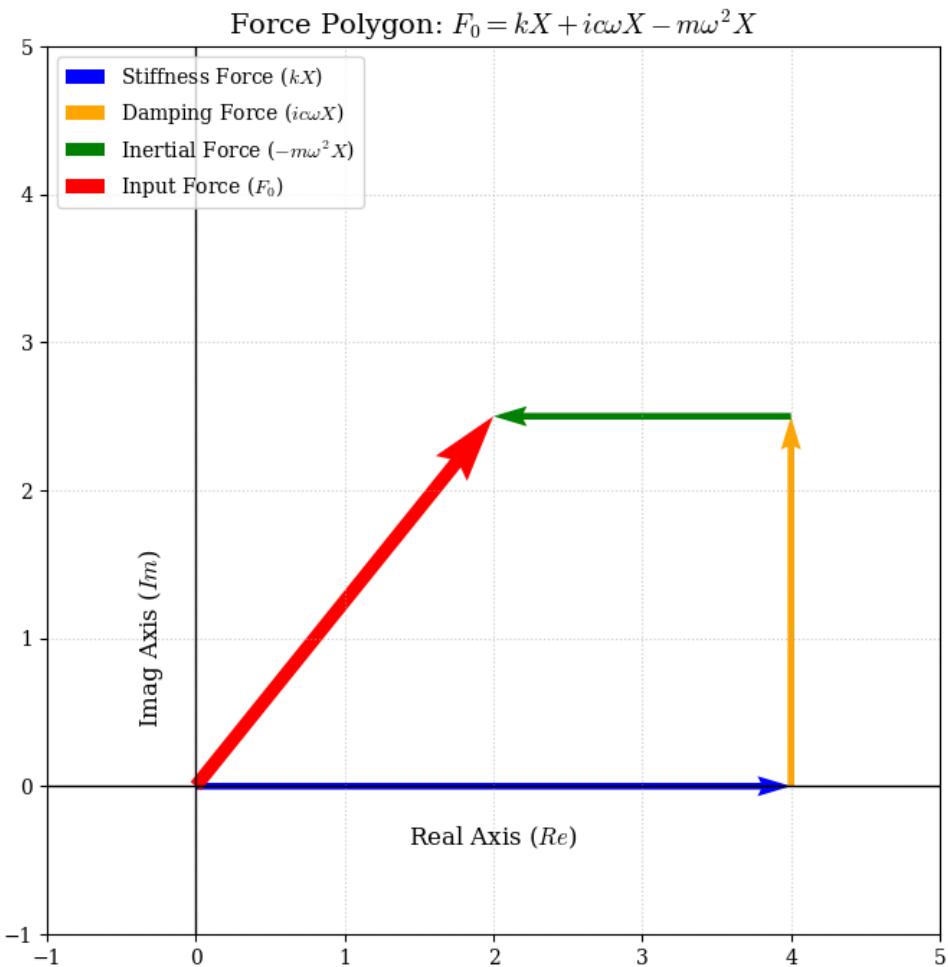
$1/(m\omega^2)$ . Thus,  $m = 1/(|H(\omega)| \cdot \omega^2)$ .

**Damping** ( $c$  or  $\zeta$ ): Look at the resonance peak. Use the "Half-Power Bandwidth" method. Identify the peak frequency  $\omega_n$  and the frequencies  $\omega_1, \omega_2$  on either side where the amplitude is  $0.707(-3\text{dB})$  of the peak.

$$\zeta \approx \frac{\omega_2 - \omega_1}{2\omega_n}$$

Once  $\zeta$ ,  $k$ , and  $m$  are known,  $c = 2\zeta\sqrt{km}$ .

11. [1 point] What is the difference between the Complex Response Function and the Frequency Response Function? (one sentence only)  
⇒ The Complex Response Function is generally a function of the complex variable  $s$  (Laplace domain), covering transient and steady states, while the Frequency Response Function is the specific evaluation of that function along the imaginary axis ( $s = i\omega$ ) describing only the steady-state sinusoidal response. (Ref: [Inman] Chapter 3).
12. [4 point] In the complex domain, show the inertial, damping, stiffness, and input forces graphically



13. [2 point] The figure below shows an aircraft with left and right structural symmetry points. Each point has six degrees of freedom. For example, the left point has three translations:  $x, y, z$ , and three rotations:  $\theta$  (rotation about  $x$ ),  $\theta$  (rotation about  $y$ ) and  $\theta$  (rotation about  $z$ ).

Let the movements on coordinate system on each side is describe as following

$$Q_L = \begin{bmatrix} x_L \\ y_L \\ z_L \\ \theta_{xL} \\ \theta_{yL} \\ \theta_{zL} \end{bmatrix}$$

$$Q_R = \begin{bmatrix} x_R \\ y_R \\ z_R \\ \theta_{xR} \\ \theta_{yR} \\ \theta_{zR} \end{bmatrix}$$

- (a) **For symmetric modes, write an expression relating the right and left points' degrees of freedom.**

$\implies$  Symmetric Modes (e.g., wing bending up simultaneously): Left and right points move in mirror images across the plane of symmetry.

$x_L = -x_R$  (Lateral motion is opposite)

$y_L = y_R$  (Longitudinal motion is identical)

$z_L = z_R$  (Vertical motion is identical)

$\theta_{xL} = -\theta_{xR}$  (Bank/Roll is opposite)

$\theta_{yL} = \theta_{yR}$  (Pitch is identical)

$\theta_{zL} = -\theta_{zR}$  (Yaw is opposite)

- (b) **For unsymmetric modes, write an expression relating the right and left points' degrees of freedom.**

$\implies$  Unsymmetric (Antisymmetric) Modes (e.g., aircraft rolling): Left and right points move opposite to the mirror image.

$$x_L = x_R$$

$$y_L = -y_R$$

$$z_L = -z_R$$

$$\theta_{xL} = \theta_{xR}$$

$$\theta_{yL} = -\theta_{yR}$$

$$\theta_{zL} = \theta_{zR}$$

14. [1 point] How long did it take you to complete this exam? 2 days

### 1.1 Reference:

- Ginsberg, J. H. (2001). Mechanical and Structural Vibrations. John Wiley & Sons.
- Inman, D. J. (2013). Engineering Vibration (4th ed.). Pearson.
- Rao, S. S. (2016). Mechanical Vibrations (6th ed.). Pearson.
- Meirovitch, L. (2010). Fundamentals of Vibrations. Waveland Press. (Original work published 2001 by McGraw-Hill).

## 2 Section 2: Short Analytical Problem

### 2.1 Problem 1:

#### 2.1.1 Part a:

Given  $y(x) = y_{tip} \frac{x^2}{L^2}$ , we have

$$\frac{dy}{dx} = y_{tip} \frac{2x}{L^2}, \text{ and } \frac{d^2y}{dx^2} = y_{tip} \frac{2}{L^2}$$

From Rayleigh's quotient, we have the ratio of maximum potential energy and reference kinetic defined as

$$\omega^2 = \frac{PE_{max}}{KE_{ref}} = \frac{\int_0^L EI(\frac{d^2y}{dx^2})^2 dx}{\int_0^L m[y(x)]^2 dx}$$

- Potential energy:

$$\int_0^L EI \left( \frac{d^2y}{dx^2} \right)^2 dx = \int_0^L EI \left( y_{tip} \frac{2}{L^2} \right)^2 dx = \frac{4EIy_{tip}^2}{L^4} [x]_0^L = \frac{4EIy_{tip}^2}{L^3}$$

- Kinetic energy:

$$\int_0^L m[y(x)]^2 dx = \int_0^L m \left[ y_{tip} \frac{x^2}{L^2} \right]^2 dx = m \frac{y_{tip}^2}{L^4} \left[ \frac{x^5}{5} \right]_0^L = \frac{my_{tip}^2 L}{5}$$

Therefore, we can solve for the natural frequency as following

$$\omega^2 = \frac{\frac{4EIy_{tip}^2}{L^3}}{\frac{my_{tip}^2 L}{5}} = \frac{20EI}{mL^4}$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{20EI}{mL^4}} \approx 4.472\sqrt{\frac{EI}{mL^4}}}$$

### 2.1.2 Part b:

For a uniform Euler-Bernoulli beam equation we have the characteristic equation as

$$\cos(\beta L) \cosh(\beta L) = -1$$

and for the first root of the equation is  $\beta_1 L = 1.8751$ . For the exact fundamental frequency, we have:

$$\omega_1 = (\beta_1 L)^2 \sqrt{\frac{EI}{mL^4}} = (1.8751)^2 \sqrt{\frac{EI}{mL^4}} = 3.516 \sqrt{\frac{EI}{mL^4}}$$

We can observe that the Rayleigh's quotient overestimate the natural frequency by

$$\text{Error} = \frac{4.472 - 3.516}{3.516} \times 100\% = \boxed{27.2\%}$$

### 2.2 Problem 2:

#### 2.2.1 Part a:

We have

$$\omega_n = 2\pi f_n = 2\pi(10) = 20\pi \text{ [rad/s]}$$

$$\zeta = 0.05 \implies c = 2\zeta\omega_n = 2(0.05)(20\pi) = 2\pi$$

$$m = 1$$

Assume the motion is harmonic, the general solution is  $e^{i\omega t}$ . We have the governing dynamic stiffness is  $Z(\omega) = k - \omega^2 + i\zeta\omega$  for  $m = 1$ , and  $\omega^2 = \frac{k}{m} \implies k = \omega^2 = (20\pi)^2 = 400\pi^2$ . Therefore, we have

$$Z(\omega) = 400\pi^2 - \omega^2 + i(0.05)\omega$$

- Complex receptance (displacement per unit force)

$$\alpha(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{Z(\omega)}$$

$$\alpha(\omega) = \frac{1}{400\pi^2 - \omega^2 + i(0.05)\omega}$$

- Complex mobility (velocity per unit force)

$$Y(\omega) = \frac{V(\omega)}{F(\omega)} = i\omega\alpha(\omega)$$

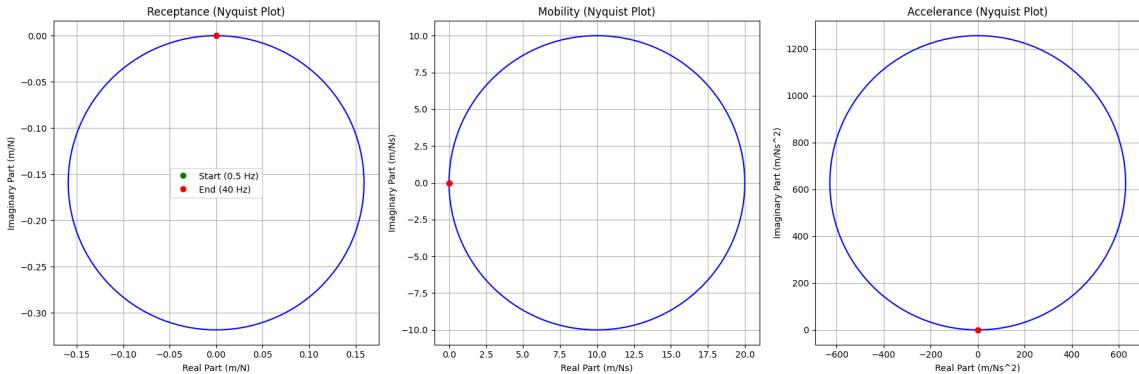
$$Y(\omega) = \frac{i\omega}{400\pi^2 - \omega^2 + i(0.05)\omega}$$

- Complex acceleration (acceleration per unit force)

$$A(\omega) = \frac{A(\omega)}{F(\omega)} = (i\omega)^2\alpha(\omega)$$

$$A(\omega) = \frac{-\omega^2}{400\pi^2 - \omega^2 + i(0.05)\omega}$$

### 2.2.2 Part b:



---

```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 # System Parameters
5 m = 1.0 # kg
6 fn = 10.0 # Hz
7 c = 0.05 # Ns/m
8
9 # Derived Parameters
10 wn = 2 * np.pi * fn # rad/s
11 k = m * wn**2 # N/m
12
13 # Frequency Range
14 f_start = 0.5
15 f_end = 40.0
16 f = np.linspace(f_start, f_end, 1000000)
17 omega = 2 * np.pi * f
18
19 # Denominator (Dynamic Stiffness)
20 #  $D(w) = (k - mw^2) + i(cw)$ 
21 D = (k - m * omega**2) + 1j * (c * omega)
22
23 # Frequency Response Functions
24 # Receptance (Displacement / Force)
25 alpha = 1.0 / D
26
27 # Mobility (Velocity / Force)
28 mobility = 1j * omega * alpha
29
30 # Accelerance (Acceleration / Force)
31 accelerance = -omega**2 * alpha
32
33 # Plotting
34 plt.figure(figsize=(18, 6))
35
36 # Receptance Plot
37 plt.subplot(1, 3, 1)
38 plt.plot(np.real(alpha), np.imag(alpha), 'b-')
39 plt.title('Receptance (Nyquist Plot)')
40 plt.xlabel('Real Part (m/N)')
41 plt.ylabel('Imaginary Part (m/N)')
42 plt.grid(True)
43 # Mark start and end points for direction
44 plt.plot(np.real(alpha[0]), np.imag(alpha[0]), 'go', label='Start (0.5 Hz)')
45 plt.plot(np.real(alpha[-1]), np.imag(alpha[-1]), 'ro', label='End (40 Hz)')

```

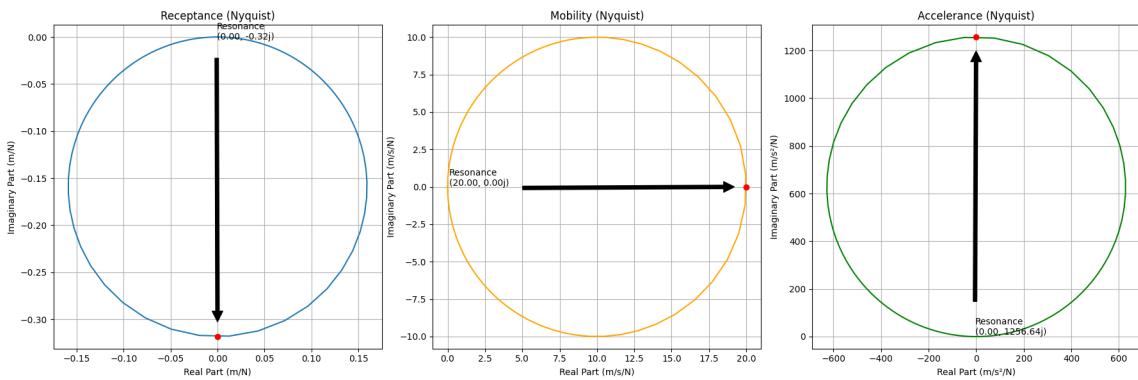
```

46 plt.legend()
47
48 # Mobility Plot
49 plt.subplot(1, 3, 2)
50 plt.plot(np.real(mobility), np.imag(mobility), 'b-')
51 plt.title('Mobility (Nyquist Plot)')
52 plt.xlabel('Real Part (m/Ns)')
53 plt.ylabel('Imaginary Part (m/Ns)')
54 plt.grid(True)
55 plt.plot(np.real(mobility[0]), np.imag(mobility[0]), 'go')
56 plt.plot(np.real(mobility[-1]), np.imag(mobility[-1]), 'ro')
57
58
59 # Accelerance Plot
60 plt.subplot(1, 3, 3)
61 plt.plot(np.real(accelerance), np.imag(accelerance), 'b-')
62 plt.title('Accelerance (Nyquist Plot)')
63 plt.xlabel('Real Part (m/Ns^2)')
64 plt.ylabel('Imaginary Part (m/Ns^2)')
65 plt.grid(True)
66 plt.plot(np.real(accelerance[0]), np.imag(accelerance[0]), 'go')
67 plt.plot(np.real(accelerance[-1]), np.imag(accelerance[-1]), 'ro')
68
69 plt.tight_layout()
70 plt.savefig('nyquist_plots.png')

```

---

### 2.2.3 Part c:



```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 # System Parameters

```

---

```

5   m = 1.0 # kg
6   fn = 10.0 # Hz
7   zeta = 0.05
8   wn = 2 * np.pi * fn # rad/s
9   k = m * wn**2 # N/m
10  c = 0.05
11  # Frequency Range for the curve
12  f = np.linspace(0.5, 40, 100000)
13  w = 2 * np.pi * f
14
15  # Calculate FRFs for the range
16  alpha = 1 / (k - m * w**2 + 1j * c * w)
17  Y = 1j * w * alpha
18  A = -w**2 * alpha
19
20  # Calculate FRFs specifically at Natural Frequency
21  w_n_point = wn
22  alpha_n = 1 / (k - m * w_n_point**2 + 1j * c * w_n_point)
23  Y_n = 1j * w_n_point * alpha_n
24  A_n = -w_n_point**2 * alpha_n
25
26  # Plotting
27  fig, axs = plt.subplots(1, 3, figsize=(18, 6))
28
29  # Receptance Plot
30  axs[0].plot(np.real(alpha), np.imag(alpha), label='Receptance Path')
31  axs[0].scatter(np.real(alpha_n), np.imag(alpha_n), color='red', zorder=5, label='Natural Freq')
32  axs[0].annotate(f'Resonance\n{np.real(alpha_n):.2f}, {np.imag(alpha_n):.2f}j',
33                  xy=(np.real(alpha_n), np.imag(alpha_n)), xytext=(-0.001, -0.003),
34                  arrowprops=dict(facecolor='black', shrink=0.05))
35  axs[0].set_title('Receptance (Nyquist)')
36  axs[0].set_xlabel('Real Part (m/N)')
37  axs[0].set_ylabel('Imaginary Part (m/N)')
38  axs[0].grid(True)
39  axs[0].axis('equal')
40
41  # Mobility Plot
42  axs[1].plot(np.real(Y), np.imag(Y), color='orange', label='Mobility Path')
43  axs[1].scatter(np.real(Y_n), np.imag(Y_n), color='red', zorder=5, label='Natural Freq')
44  axs[1].annotate(f'Resonance\n{np.real(Y_n):.2f}, {np.imag(Y_n):.2f}j',
45                  xy=(np.real(Y_n), np.imag(Y_n)), xytext=(0.1, 0.05),
46                  arrowprops=dict(facecolor='black', shrink=0.05))
47  axs[1].set_title('Mobility (Nyquist)')
48  axs[1].set_xlabel('Real Part (m/s/N)')
49  axs[1].set_ylabel('Imaginary Part (m/s/N)')
50  axs[1].grid(True)

```

```

51 axs[1].axis('equal')
52
53 # Accelerance Plot
54 axs[2].plot(np.real(A), np.imag(A), color='green', label='Accelerance Path')
55 axs[2].scatter(np.real(A_n), np.imag(A_n), color='red', zorder=5, label='Natural Freq')
56 axs[2].annotate(f'Resonance\n({np.real(A_n):.2f}, {np.imag(A_n):.2f}j)', 
57                 xy=(np.real(A_n), np.imag(A_n)), xytext=(-5, 8),
58                 arrowprops=dict(facecolor='black', shrink=0.05))
59 axs[2].set_title('Accelerance (Nyquist)')
60 axs[2].set_xlabel('Real Part (m/s^2/N)')
61 axs[2].set_ylabel('Imaginary Part (m/s^2/N)')
62 axs[2].grid(True)
63 axs[2].axis('equal')
64
65 plt.tight_layout()
66 plt.savefig('nyquist_plots_resonance.png')
67

```

---

#### 2.2.4 Part d:

The Mobility plot is the best choice for this because, for a viscous damped system, it forms a circle passing through the origin.

1. Identify the Natural Frequency ( $f_n$ ) Locate the point with the Maximum Real Amplitude (the right-most point of the circle).The frequency at this point is your natural frequency,  $f_n$ .From your data:  $f_n = 10$  Hz.
2. Identify the Half-Power Points ( $f_1$  and  $f_2$ )On the Mobility Nyquist circle, the half-power points are geometrically distinct:
  - $f_1$ : The frequency at the Maximum Imaginary value (the very top of the circle).
  - $f_2$ : The frequency at the Minimum Imaginary value (the very bottom of the circle).

(Note: At these points, the real part is exactly half of the maximum real amplitude).

3. Apply the Half-Power Bandwidth Formula Once you have the frequencies for the top and bottom of the circle, use this formula:

$$\zeta = \frac{f_2 - f_1}{2f_n}$$

Example Walkthrough with Your Data: If we look at the Mobility plot generated in Part B: Resonance ( $f_n$ ): We know this is 10 Hz. Bandwidth Points: We know  $\zeta = 0.05$ . Using the formula in reverse, the bandwidth ( $f_2 - f_1$ ) should be:

$$f_2 - f_1 = 2 \cdot \zeta \cdot f_n$$

$$f_2 - f_1 = 2 \cdot 0.05 \cdot 10 = 1 \text{ Hz}$$

Therefore, if you hovered over the top of the circle, you would find  $f_1 \approx 9.5 \text{ Hz}$ . If you hovered over the bottom of the circle, you would find  $f_2 \approx 10.5 \text{ Hz}$ .

$$\zeta = \frac{10.5 - 9.5}{2(10)} = \frac{1}{20} = \mathbf{0.05}$$

### 2.3 Problem 3:

Since the mode shapes are orthonormal such that the mass matrix is the identity matrix ( $[I]$ ), the system is decoupled in the modal domain. We can construct the equations of motion in terms of the modal coordinates ( $q$ ). The general equation of motion in modal coordinates is:

$$[I]\{\ddot{q}\} + [\hat{C}]\{\dot{q}\} + [\hat{K}]\{q\} = \{0\}$$

Where:

- $[I]$  is the Modal Mass matrix (Identity).
- $[\hat{C}]$  is the Modal Damping matrix (diagonal, with terms  $2\zeta_i\omega_i$ ).
- $[\hat{K}]$  is the Modal Stiffness matrix (diagonal, with terms  $\omega_i^2$ ).

The given frequencies  $f$  (Hz) must be converted to angular natural frequencies  $\omega_n$  (rad/s) using  $\omega = 2\pi f$ .

Mode 1:  $\omega_1 = 2\pi(10) = 20\pi \approx 62.83 \text{ rad/s}$

Mode 2:  $\omega_2 = 2\pi(15) = 30\pi \approx 94.25 \text{ rad/s}$

Mode 3:  $\omega_3 = 2\pi(25) = 50\pi \approx 157.08 \text{ rad/s}$

Modal Stiffness Elements ( $\omega_n^2$ ):

$$k_1 = (20\pi)^2 = 400\pi^2 \approx 3,948$$

$$k_2 = (30\pi)^2 = 900\pi^2 \approx 8,883$$

$$k_3 = (50\pi)^2 = 2500\pi^2 \approx 24,674$$

Modal Damping Elements ( $2\zeta_n\omega_n$ ):

$$c_1 = 2(0.015)(20\pi) = 0.6\pi \approx 1.88$$

$$c_2 = 2(0.020)(30\pi) = 1.2\pi \approx 3.77$$

$$c_3 = 2(0.030)(50\pi) = 3.0\pi \approx 9.42$$

Since the physical mode shape vectors ( $\Phi$ ) are not provided, we cannot transform back to physical coordinates ( $x$ ). The equations of motion are expressed in the modal coordinates vector  $\{q\} = [q_1, q_2, q_3]^T$ .

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} 1.88 & 0 & 0 \\ 0 & 3.77 & 0 \\ 0 & 0 & 9.42 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} + \begin{bmatrix} 3,948 & 0 & 0 \\ 0 & 8,883 & 0 \\ 0 & 0 & 24,674 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

## 2.4 Problem 4:

### 2.4.1 Part a:

Mass per unit length ( $m$ ):

$$m = \rho \times A$$

$$m = 7,850 \times 1.501 \times 10^{-5} \approx 0.1178 \text{ kg/m}$$

Cross-section area ( $A$ )

$$A = \frac{\pi}{4}(d_o^2 - d_i^2) = \frac{\pi}{4}(0.008^2 - 0.0067^2) \approx 1.501 \times 10^{-5} \text{ m}^2$$

Second moment of area ( $I$ )

$$I = \frac{\pi}{64}(r_o^4 - r_i^4) = \frac{\pi}{64}(0.008^4 - 0.0067^4) = 1.021 \times 10^{-10} [\text{m}^4]$$

As we assume the hydraulic pipe is a simply supported uniform beam, the first bending natural frequency is

$$\omega_1 = \frac{\pi^2}{L^2} \sqrt{\frac{EI}{mL^4}} = \frac{\pi^2}{0.432^2} \sqrt{\frac{(205 \times 10^9)(1.021 \times 10^{-10})}{(0.1178)(0.432)^4}} \approx \pi^2 \sqrt{5101.1} = 704.9 \text{ [rad/s]}$$

$$f_n = \frac{\omega_n}{2\pi} \approx \frac{704.9}{6.283} \approx 112.2 \text{ Hz}$$

Check: The calculated frequency (112.2 Hz) falls within the excitation range of 50 to 500 Hz. Therefore, the pipe will excite resonance, and we must calculate the displacement at this worst-case frequency.

For mode shape  $\phi(x) = \sin(\frac{\pi x}{L})$ , and participation Factor ( $\Gamma$ ): For a simply supported beam,  $\Gamma = \frac{4}{\pi} \approx 1.273$ . We have maximum relative displacement amplitude ( $Z_{peak}$ ) at the center of the beam at resonance is:

$$Z_{peak} = \Gamma \times \frac{A_{base}}{2\zeta\omega_n^2}$$

Where:

- $\Gamma$ :  $4/\pi$  (Modal participation factor)
- $A_{base}$ : Base acceleration amplitude =  $5.0g = 5.0 \times 9.81 = 49.05 \text{ m/s}^2$
- $\zeta$ : Damping ratio = 0.02
- $\omega_n$ : Natural frequency  $\approx 704.9 \text{ rad/s}$

Static Equivalent Deflection ( $A_{base}/\omega_n^2$ ):

$$\frac{49.05}{704.9^2} = \frac{49.05}{496884} \approx 9.871 \times 10^{-5} \text{ m}$$

Apply Damping ( $1/2\zeta$ ):

$$\frac{1}{2 \times 0.02} = \frac{1}{0.04} = 25$$

(This is the Q-factor or dynamic amplification factor) Apply Modal Factor ( $\Gamma = 4/\pi = 1.273$ ): Total Displacement ( $Z_{peak}$ ):

$$Z_{peak} = 1.273 \times 25 \times (9.871 \times 10^{-5})$$

$$Z_{peak} \approx 31.825 \times (9.871 \times 10^{-5})$$

$$Z_{peak} \approx 0.00314 \text{ m}$$

Therefore, the vertical single-peak displacement at the center of the pipe span relative to the supporting structure is 3.14 [mm]

## 2.4.2 Part b

From the maximum deflection of  $\delta_{max} = 3.14$  [mm]. For a beam, the bending stress ( $\sigma$ ) is related to the bending moment ( $M$ ) and the distance from the neutral axis ( $c$ ) by the flexure formula:

$$\sigma = \frac{Mc}{I}$$

Since the problem assumes the fundamental mode of a simply supported beam, the mode shape is sinusoidal:  $y(x) = Z_{peak} \sin\left(\frac{\pi x}{L}\right)$

The bending moment is proportional to the curvature (second derivative of deflection):

$$M(x) = EI \frac{d^2y}{dx^2}$$

$$\frac{d^2y}{dx^2} = -Z_{peak} \left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right)$$

Substituting the maximum curvature (at the center,  $x = L/2$ , where  $\sin = 1$ ) into the stress formula:

$$\sigma_{max} = \frac{EI \left| \frac{d^2y}{dx^2} \right|_{max} c}{I} = E \cdot c \cdot Z_{peak} \left(\frac{\pi}{L}\right)^2$$

where:

- Young's Modulus ( $E$ ):  $205 \times 10^9$  Pa
- Pipe Outer Radius ( $c$ ):  $d_o/2 = 8.0 \text{ mm}/2 = 4.0 \text{ mm} = 0.004 \text{ m}$
- Span Length ( $L$ ): 0.432 m
- Max Deflection ( $Z_{peak}$ ): 0.00314 m.

Substitute the values into the derived equation:

$$\sigma_{max} = (205 \times 10^9) \times (0.004) \times (0.00314) \times \left(\frac{\pi}{0.432}\right)^2$$

Geometric term  $(\pi/L)^2$ :

$$\left(\frac{3.14159}{0.432}\right)^2 \approx (7.272)^2 \approx 52.88 \text{ m}^{-2}$$

Combine constants:

$$\sigma_{max} = (205 \times 10^9) \times (0.004) \times (0.00314) \times (52.88)$$

Multiply:

$$\sigma_{max} \approx 205 \times 10^9 \times 6.642 \times 10^{-4}$$

$$\sigma_{max} \approx 136,161,000 \text{ Pa}$$

$\sigma_{max} \approx 136.2 \text{ [ MPa]}$

### 3 Section 3: Critical Thinking

#### 3.1 Case I: Response to turbulence

##### 3.1.1 Part a:

The input forces  $P_1$  and  $P_2$  are correlated because they originate from the same upstream turbulent airflow. This correlation means the "cross-talk" between the forces significantly affects the total energy at the tip.a. Acceleration Spectral Density (ASD) of the tipTo write the Acceleration Spectral Density ( $S_{aa}$ ) at the wing tip, we treat this as a Multiple-Input Single-Output (MISO) system.

- 1. Define the Frequency Response Functions (FRFs):** Let  $H_1(\omega)$  be the FRF relating force  $P_1$  to the tip acceleration. Let  $H_2(\omega)$  be the FRF relating force  $P_2$  to the tip acceleration. Let  $\omega$  be the angular frequency.
- 2. Define the Input Spectra:**  $S_{P_1 P_1}(\omega)$  and  $S_{P_2 P_2}(\omega)$  are the Auto-Power Spectral Densities (PSD) of forces  $P_1$  and  $P_2$ .  $S_{P_1 P_2}(\omega)$  and  $S_{P_2 P_1}(\omega)$  are the Cross-Power Spectral Densities (CPSD) between the two forces.
- 3. The ASD Equation:** Because the forces are correlated (meaning  $S_{P_1 P_2} \neq 0$ ), the total output spectrum is the sum of the individual responses plus the interaction (interference) terms between them. The equation for the Acceleration Spectral Density at the tip ( $S_{aa}(\omega)$ ) is:

$$S_{aa}(\omega) = \underbrace{|H_1(\omega)|^2 S_{P_1 P_1}(\omega)}_{\text{Contribution from } P_1} + \underbrace{|H_2(\omega)|^2 S_{P_2 P_2}(\omega)}_{\text{Contribution from } P_2} + \underbrace{H_1^*(\omega) H_2(\omega) S_{P_1 P_2}(\omega) + H_2^*(\omega) H_1(\omega) S_{P_2 P_1}(\omega)}_{\text{Interaction terms due to Correlation}}$$

Since  $S_{P_2 P_1}(\omega)$  is the complex conjugate of  $S_{P_1 P_2}(\omega)$ , the last two terms are complex conjugates of each other. Their sum is twice the real part. The equation is often written as:

$$S_{aa}(\omega) = |H_1(\omega)|^2 S_{P_1 P_1}(\omega) + |H_2(\omega)|^2 S_{P_2 P_2}(\omega) + 2 \cdot \text{Re}[H_1^*(\omega) H_2(\omega) S_{P_1 P_2}(\omega)]$$

### 3.1.2 Part b:

The variance ( $\sigma_d^2$ ) is the integral of the displacement power spectral density ( $S_{dd}(\omega)$ ) over the frequency range. Since acceleration is the second derivative of displacement ( $a = \ddot{x}$ ), the relationship in the frequency domain involves dividing the acceleration spectrum by  $\omega^4$ .

$$\sigma_d^2 = \int_{-\infty}^{\infty} S_{dd}(\omega) d\omega = \int_{-\infty}^{\infty} \frac{S_{aa}(\omega)}{\omega^4} d\omega$$

Substituting the  $S_{aa}(\omega)$  from part (a):

$$\sigma_d^2 = \int_{-\infty}^{\infty} \frac{1}{\omega^4} [|H_1(\omega)|^2 S_{P_1 P_1}(\omega) + |H_2(\omega)|^2 S_{P_2 P_2}(\omega) + 2 \operatorname{Re}(H_1^*(\omega) H_2(\omega) S_{P_1 P_2}(\omega))] d\omega$$

## 3.2 Case II: Forces from two different jet engines

### 3.2.1 Part a:

In this case, the forces are uncorrelated. This means the cross-power spectral density is zero ( $S_{P_1 P_2}(\omega) = S_{P_2 P_1}(\omega) = 0$ ).

a. Acceleration Spectral Density (ASD) of the tip  
Because the inputs are uncorrelated, the cross-terms vanish. The total output ASD is simply the superposition of the ASDs generated by each independent force.

$$S_{aa}(\omega) = |H_1(\omega)|^2 S_{P_1 P_1}(\omega) + |H_2(\omega)|^2 S_{P_2 P_2}(\omega)$$

Where:

- $S_{aa}(\omega)$  is the Acceleration Spectral Density at the wing tip.
- $|H_1(\omega)|^2$  is the squared magnitude of the FRF relating force  $P_1$  to the tip acceleration.
- $S_{P_1 P_1}(\omega)$  is the Auto-Spectral Density of force  $P_1$ .
- $|H_2(\omega)|^2$  is the squared magnitude of the FRF relating force  $P_2$  to the tip acceleration.
- $S_{P_2 P_2}(\omega)$  is the Auto-Spectral Density of force  $P_2$ .

### 3.2.2 Part b:

Variance of the displacement at the wing tipSimilar to Case I, we integrate the displacement PSD. However, we use the simplified uncorrelated spectrum derived above.

$$\sigma_d^2 = \int_{-\infty}^{\infty} S_{dd}(\omega) d\omega = \int_{-\infty}^{\infty} \frac{S_{aa}(\omega)}{\omega^4} d\omega$$

$$\sigma_d^2 = \int_{-\infty}^{\infty} \frac{1}{\omega^4} [|H_1(\omega)|^2 S_{P_1 P_1}(\omega) + |H_2(\omega)|^2 S_{P_2 P_2}(\omega)] d\omega$$

## 3.3 Case III: Dynamic properties of aircraft wing

### 3.3.1 Part a:

Let  $F(\omega)$  be the Fourier Transform of the known shaker force  $F(t)$ . Let  $A_i(\omega)$  be the Fourier Transform of the acceleration response at location  $i$  (where  $i = 1, 2, 3, 4$ ). Let  $N_i(\omega)$  be the response caused by the unknown turbulent wind forces (noise). The total response measured at any accelerometer  $i$  is the sum of the response to the shaker and the response to the wind:

$$A_i(\omega) = H_i(\omega)F(\omega) + N_i(\omega)$$

To isolate  $H_i(\omega)$ , we multiply both sides of the equation by  $F^*(\omega)$ , the complex conjugate of the known input force.

$$F^*(\omega)A_i(\omega) = F^*(\omega)[H_i(\omega)F(\omega) + N_i(\omega)]$$

$$F^*(\omega)A_i(\omega) = H_i(\omega)[F^*(\omega)F(\omega)] + F^*(\omega)N_i(\omega)$$

We take the expected value  $E[\cdot]$  (ensemble average) of the terms. This converts the raw Fourier products into Power Spectral Densities (PSD).

$$E[F^*(\omega)A_i(\omega)] = H_i(\omega)E[F^*(\omega)F(\omega)] + E[F^*(\omega)N_i(\omega)]$$

Using standard PSD notation ( $S_{xy}$ ): $S_{Fa_i}(\omega) = E[F^*(\omega)A_i(\omega)]$  (Cross-PSD of Force and Acceleration) $S_{FF}(\omega) = E[F^*(\omega)F(\omega)]$  (Auto-PSD of the Force) $S_{FN_i}(\omega) = E[F^*(\omega)N_i(\omega)]$  (Cross-PSD of Force and Wind Noise)The equation becomes:

$$S_{Fa_i}(\omega) = H_i(\omega)S_{FF}(\omega) + S_{FN_i}(\omega)$$

This is the critical step. Because the shaker force  $F(t)$  is controlled and the wind turbulence is random and external, they are uncorrelated. Therefore, the Cross-PSD between the shaker force and the wind noise is zero:

$$S_{FN_i}(\omega) \approx 0$$

The equation simplifies to:

$$S_{Fa_i}(\omega) = H_i(\omega)S_{FF}(\omega)$$

Rearranging to solve for the FRF:

$$H_i(\omega) = \frac{S_{Fa_i}(\omega)}{S_{FF}(\omega)}$$

Final Answer: The Frequency Response Functions for the four locations ( $i = 1, 2, 3, 4$ ) are:

$$H_1(\omega) = \frac{S_{Fa_1}(\omega)}{S_{FF}(\omega)}$$

$$H_2(\omega) = \frac{S_{Fa_2}(\omega)}{S_{FF}(\omega)}$$

$$H_3(\omega) = \frac{S_{Fa_3}(\omega)}{S_{FF}(\omega)}$$

$$H_4(\omega) = \frac{S_{Fa_4}(\omega)}{S_{FF}(\omega)}$$

Where:  $S_{Fa_i}(\omega)$  is the Cross-Power Spectral Density between the known force and the  $i$ -th accelerometer.  $S_{FF}(\omega)$  is the Auto-Power Spectral Density of the known force.

### 3.3.2 Part b:

Once you have calculated the FRFs from part (a), you can extract the dynamic properties (Natural Frequencies, Damping, and Mode Shapes) using standard modal analysis techniques:

1. **Natural Frequencies ( $\omega_n$ )** The natural frequencies correspond to the resonances of the structure.
  - Procedure: Plot the magnitude of the FRF,  $|H_i(\omega)|$ , versus frequency  $\omega$  for one or more of the sensors.

- Identification: Look for sharp peaks in the plot. The frequency at which a peak occurs is a natural frequency ( $\omega_n$ ) of the wing.
- Verification: These peaks should appear at roughly the same frequency across all four accelerometers (though the amplitude will vary depending on the location).

2. **Damping Ratios ( $\zeta$ )** The damping ratio indicates how quickly vibrations decay. A common method to estimate this from an FRF is the Half-Power Bandwidth Method (also known as the -3dB method).

- Procedure: Focus on a single resonance peak at  $\omega_n$ .
- Identify the Peak Amplitude: Let the maximum amplitude at resonance be  $A_{max}$ .
- Find Half-Power Points: Find the two frequencies,  $\omega_1$  (lower) and  $\omega_2$  (upper), on either side of the peak where the amplitude drops to  $\frac{A_{max}}{\sqrt{2}}$  (or roughly  $0.707 \times A_{max}$ ).
- Calculation: Calculate the damping ratio ( $\zeta$ ) using the formula:

$$\zeta \approx \frac{\omega_2 - \omega_1}{2\omega_n}$$

3. **Mode Shapes ( $\phi$ )** The mode shape describes the deformation pattern of the wing at a specific natural frequency.

- Procedure: Select a specific natural frequency  $\omega_n$  identified in Step 1.
- Extract Values: Record the complex value (Magnitude and Phase) of the FRF for all four sensors at that exact frequency:  $H_1(\omega_n), H_2(\omega_n), H_3(\omega_n), H_4(\omega_n)$ .
- Construct the Shape:
  - The Magnitude  $|H_i(\omega_n)|$  tells you the relative displacement at that location (how much it moves).
  - The Phase tells you the direction of motion relative to the force. Typically, points are either in-phase ( $0^\circ$ ) or out-of-phase ( $180^\circ$ ).
- Result: The vector  $\begin{Bmatrix} H_1 & H_2 & H_3 & H_4 \end{Bmatrix}^T$  represents the mode shape of the wing for that specific mode.