AE543: Homework 4

Hoang Nguyen

Table of Contents

1 Problem 1

Given the system of three masses m_1, m_2, and m_3 connected in series with one end is fixed to the ground, we have equation of motion for each of the masses as following. Assuming with $x_1(t)$, $x_2(t)$, and $x_3(t)$ is vertical displacement of the three masses from their original positions at time t

$$\begin{cases} m_1\ddot{x}_1 &= & -k_1x_1 + k_2(x_1 - x_2) & \text{at mass 1} \\ m_2\ddot{x}_2 &= & -k_2(x_1 - x_2) + k_3(x_3 - x_1) & \text{at mass 2} \\ m_4\ddot{x}_3 &= & -k_3(x_3 - x_2) & \text{at mass 3} \end{cases}$$

$$\Rightarrow \begin{cases} m_1\ddot{x}_1 + (k_1 + k_2)\mathbf{x_1} - k_2\mathbf{x_2} &= 0 \\ m_2\ddot{x}_2 - k_1\mathbf{x_1} + (k_2 + k_3)\mathbf{x_2} - k_3\mathbf{x_3} &= 0 \\ m_3\ddot{x}_3 - k_3\mathbf{x_2} + k_3\mathbf{x_3} &= 0 \end{cases}$$

We can re-write the above equation in matrix form of $[M_i]\{\ddot{x}_i\} + [K]\{x_i\} = 0$ as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = 0$$

The eigenvalue of the system can be solved as we assume the system is harmonic motion as $x_i(t) = X_i \cos(\omega t + \phi)$ for $i = 1, 2$. We have

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & -\omega^2 m_3 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = 0$$

We can substitute the given stiffness value $k_1 = 1000$ [N/m], $k_2 = 2000$ [N/m], $k_3 = 1000$ [N/m] and mass value $m_1 = 2$ [kg], $m_2 = 1$ [kg], $m_3 = 2$ [kg]. The above equation can be simplified to

$$\begin{bmatrix} -2\omega^2 + 3000 & -2000 & 0 \\ -2000 & -\omega^2 + 3000 & -1000 \\ 0 & -1000 & -2\omega^2 + 1000 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = 0$$

1.1 Part a:

For a nontrivial solution of X_1 and X_2, we can find the determinant of the above equation to obtain the eigenvalues (natural frequency)

$$\begin{aligned}
&\begin{vmatrix} -2\omega^2+3000 & -2000 & 0 \\ -2000 & -\omega^2+3000 & -1000 \\ 0 & -1000 & -2\omega^2+1000 \end{vmatrix} = 0 \\
\Rightarrow &(-2\omega^2+3000) \ \cdot [(-\omega^2+3000)(1000)-(-1000)(-2\omega^2)] \\
&-(-2000) \ \cdot [(-2000)(-2\omega^2+1000)-(-1000)(0)] \\
&+(0) \ \cdot [(-2000)(-1000)-(-\omega^2+3000)] = 0 \\
\Rightarrow &-4\omega^6+20,000\omega^4-17,000,000\omega^2+2000000000 = 0
\end{aligned}$$

Assume $\Omega = \omega^2$, the above equation becomes

$$-4\ \Omega^3 + 20000\ \Omega^2 - 17000000\ \Omega + 2000000000 = 0 \tag{1.1}$$

From here, wer can solve for the cubic equation and get the roots for Ω as

$$\begin{aligned}
\Omega_1 &= 3958.19 \Rightarrow \omega_1 = 62.914\ [\text{rad/s}] \\
\Omega_2 &= 901.72 \Rightarrow \omega_2 = 30.028\ [\text{rad/s}] \\
\Omega_3 &= 140.09 \Rightarrow \omega_3 = 11.836\ [\text{rad/s}]
\end{aligned} \tag{1.2}$$

Therefore, we have the natural frequencies ω of the system as following

$$\boxed{\omega_{\mathbf{3}} = \mathbf{62.914}\ [\text{rad/s}],\ \omega_{\mathbf{2}} = \mathbf{30.028}\ [\text{rad/s}],\ \omega_{\mathbf{1}} = \mathbf{11.836}\ [\text{rad/s}]}$$

1.2 Part b

As there are three eigenvalues, the mode shapes $u^{(i)} = \begin{pmatrix} u_1^{(i)} & u_2^{(i)} & u_3^{(i)} \end{pmatrix}^T$, the characteristic matrix equation

$$\begin{bmatrix} -2\omega^2+3000 & -2000 & 0 \\ -2000 & -\omega^2+3000 & -1000 \\ 0 & -1000 & -2\omega^2+1000 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = 0$$

Set $\lambda_i = \frac{\omega_i^2}{1000}$ and $X_1 = 1$, the above characteristic equation can be simplified as

$$\begin{bmatrix} 3-2\lambda & -2 & 0 \\ -2 & 3-\lambda & -1 \\ 0 & -1 & 1-2\lambda \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = 0$$

We have

$$\begin{aligned}(3-2\lambda)X_1 - 2X_2 = 0 \quad &\Rightarrow X_2 = \left(\frac{3-2\lambda}{2}\right)X_1\\ -X_2 + (1-\lambda)X_3 = 0 \quad &\Rightarrow X_3 = \left(\frac{1}{1-2\lambda}\right)X_2\end{aligned}$$

plugging frequency ω we have

$$\begin{aligned}\lambda_3 = \frac{\omega_3^2}{1000} \quad &= \frac{62.914^2}{1000} = 3.958\\ \lambda_2 = \frac{\omega_2^2}{1000} \quad &= \frac{30.028^2}{1000} = 0.902\\ \lambda_1 = \frac{\omega_1^2}{1000} \quad &= \frac{11.836^2}{1000} = 0.140\end{aligned}$$

- **At mode 1 $\boldsymbol{\lambda_1 = 0.140}$**

$$\begin{aligned}u_1 &= 1\\ u_2^{(1)} &= \frac{3-2(0.140)}{2} = 1.360\\ u_3^{(1)} &= \frac{1.360}{1-2(0.140)} = 1.889\end{aligned}$$

Therefore, eigenvector of mode 1 $\lambda_1 = 0.140$ is $\boxed{\mathbf{u^{(1)}} = \begin{bmatrix} 1\\ 1.360\\ 1.889\end{bmatrix}}$

- **At mode 2 $\boldsymbol{\lambda_1 = 0.902}$**

$$\begin{aligned}u_1 &= 1\\ u_2^{(2)} &= \frac{3-2(0.902)}{2} = 0.598\\ u_3^{(2)} &= \frac{1.360}{1-2(0.9002)} = -0.745\end{aligned}$$

Therefore, eigenvector of mode 2 $\lambda_2 = 0.902$ is $\boxed{\mathbf{u^{(2)}} = \begin{bmatrix} 1\\ 0.598\\ -0.745\end{bmatrix}}$

- **At mode 3 $\boldsymbol{\lambda_3 = 3.958}$**

$$\begin{aligned}u_1 &= 1\\ u_2^{(2)} &= \frac{3-2(3.958)}{2} = -2.458\\ u_3^{(2)} &= \frac{1.360}{1-2(3.958)} = 0.355\end{aligned}$$

Therefore, eigenvector of mode 3 $\lambda_3 = 3.958$ is $\boxed{\mathbf{u^{(3)}} = \begin{bmatrix} 1 \\ -2.458 \\ 0.355 \end{bmatrix}}$

2 Problem 2

The above system is two degree of freedom vibration system where $m_1 = m_2 = \frac{1}{2}$ and the stiffness $k_1 = k_2 = k_3 = \frac{1}{2}$. The general equation of motion for a 2 degree-of-freedom system

$$M\ddot{x} + C\dot{x} + Kx = f(t)$$

2.1 Part a:

For the free and undamped vibration we have $C = 0$ and $f(t) = 0$, the general 2-DOF can be simplified to

$$M\ddot{x} + Kx = 0$$

Mass matrix:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

Stiffness matrix:

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

Characteristic equation for natural frequency$\omega_\mathbf{n}$

$$\det(\mathbf{K} - \omega_n^2 \mathbf{M}) = 0$$

Substituting the matrices:

$$\det\left(\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} - \omega_n^2 \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\right) = 0$$

$$\det\begin{bmatrix} 1 - \frac{1}{2}\omega_n^2 & -1/2 \\ -1/2 & 1 - \frac{1}{2}\omega_n^2 \end{bmatrix} = 0$$

The determinant is calculated as:

$$\left(1 - \frac{1}{2}\omega_n^2\right)\left(1 - \frac{1}{2}\omega_n^2\right) - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = 0$$

$$\left(1 - \frac{1}{2}\omega_n^2\right)^2 - \frac{1}{4} = 0$$

Solving for ω_n^2:

$$\left(1 - \frac{1}{2}\omega_n^2\right)^2 = \frac{1}{4}$$

$$1 - \frac{1}{2}\omega_n^2 = \pm\sqrt{\frac{1}{4}}$$
$$1 - \frac{1}{2}\omega_n^2 = \pm\frac{1}{2}$$

- Case 1: Positive Root (+)

$$1 - \frac{1}{2}\omega_{n,1}^2 = \frac{1}{2}$$
$$\frac{1}{2}\omega_{n,1}^2 = 1 - \frac{1}{2}$$
$$\frac{1}{2}\omega_{n,1}^2 = \frac{1}{2}$$
$$\omega_{n,1}^2 = 1$$
$$\omega_{n,1} = 1$$

- Case 2: Negative Root (-)

$$1 - \frac{1}{2}\omega_{n,2}^2 = -\frac{1}{2}$$
$$\frac{1}{2}\omega_{n,2}^2 = 1 + \frac{1}{2}$$
$$\frac{1}{2}\omega_{n,2}^2 = \frac{3}{2}$$
$$\omega_{n,2}^2 = 3$$
$$\omega_{n,2} = \sqrt{3} \approx 1.732$$

Therefore, the natural frequency [rad/unit time] are $\boxed{\mathbf{w_{n,1} = 1}}$ and $\boxed{\mathbf{w_{n,2} = \sqrt{3}}}$. For a damping ratio of $\zeta = 5\%$, the damped natural frequency are

- **For natural frequency of $\mathbf{w_{n,1} = 1}$**

$$\omega_{d,1} = \omega_{n,1}\sqrt{1 - \zeta^2}$$
$$\omega_{d,2} = 1\sqrt{1 - 0.05^2} \approx 0.99875$$

- **For natural frequency of $\mathbf{w_{n,2} = 1}$**

$$\omega_{d,2} = \omega_{n,2}\sqrt{1 - \zeta^2}$$
$$\omega_{d,2} = 1.732\sqrt{1 - 0.05^2} \approx 1.72989$$

2.2 Part b:

To construct the eigenvector $X_i = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, we can solve for $(K - \omega_{ni}^2 M)X_i = 0$

- **For first mode shape $\omega^2_{\mathbf{n,1}} = 1$**

$$(K - \omega_{ni}^2 M)X_i = 0$$
$$\left(\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} - (1)\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\right)\begin{Bmatrix} X_{11} \\ X_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$
$$\begin{bmatrix} 1-1/2 & -1/2 \\ -1/2 & 1-1/2 \end{bmatrix}\begin{Bmatrix} X_{11} \\ X_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$
$$\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{Bmatrix} X_{11} \\ X_{21} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For the first row of equation, we have

$$\frac{1}{2}X_{11} - \frac{1}{2}X_{21} = 0 \quad \Longrightarrow \quad X_{11} = X_{21} \tag{2.1}$$

Therefore, eigenvector for first natural frequency is $\boxed{\mathbf{X_1} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}$

- **For second mode shape $\omega^2_{\mathbf{n,2}} = \mathbf{3}$**

$$\left(\begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix} - (3)\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}\right)\begin{Bmatrix} X_{12} \\ X_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$
$$\begin{bmatrix} 1-3/2 & -1/2 \\ -1/2 & 1-3/2 \end{bmatrix}\begin{Bmatrix} X_{12} \\ X_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$
$$\begin{bmatrix} -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix}\begin{Bmatrix} X_{12} \\ X_{22} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

For the first row of equation, we have

$$-\frac{1}{2}X_{12} - \frac{1}{2}X_{22} = 0 \quad \Longrightarrow \quad X_{12} = -X_{22}$$

Therefore, eigenvector for second natural frequency is $\boxed{\mathbf{X_2} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}}$

2.3 Part c:

We have the external force $F_1 = f_1 \cos(\omega t - \theta_1)$ on mass m_1 and $F_2 = f_2 \cos(\omega t - \theta_2)$ on mass m_2. The receptance matrix $H_{ij}(\omega)$ describes as following

$$H_{ij}(\omega) = \sum_{r=1}^{N} \frac{\phi_{ir}\phi_{jr}}{M_r(\omega_{nr}^2 - \omega^2 + 2i\zeta\omega_{nr}\omega)}$$

We have modal mass can be calculate as (M_r) is calculated using $M_r = X_r^T M X_r$, with $M = \begin{Bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{Bmatrix}$.

Modal Mass M_1:

$$M_1 = \begin{Bmatrix} 1 & 1 \end{Bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 1/2 & 1/2 \end{Bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \frac{1}{2}(1) + \frac{1}{2}(1) = 1$$

Modal Mass M_2:

$$M_2 = \begin{Bmatrix} 1 & -1 \end{Bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 1/2 & -1/2 \end{Bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = \frac{1}{2}(1) + \frac{1}{2}(-1)(-1) = \frac{1}{2} + \frac{1}{2} = 1$$

The individual modal FRFs are:

$$H_{11,Mode\ 1}(\omega) = \frac{1^2}{1(1^2 - \omega^2 + 2i(0.05)(1)\omega)}$$

$$H_{11,Mode\ 2}(\omega) = \frac{1^2}{1((\sqrt{3})^2 - \omega^2 + 2i(0.05)(\sqrt{3})\omega)}$$

$$H_{11,Total}(\omega) = H_{11,Mode\ 1}(\omega) + H_{11,Mode\ 2}(\omega)$$

As the system is symmetrical and identical the model FRF at mass 2 will be similar, we have For $H_{22}(\omega)$ (where $i = 2$ and $j = 2$):

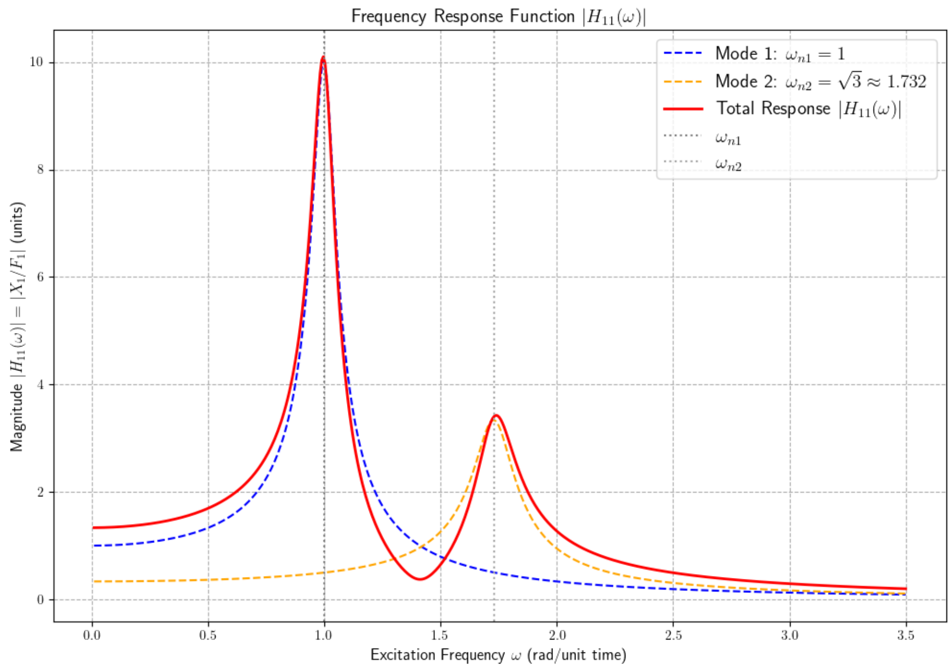
$$H_{22}(\omega) = \frac{\phi_{21}\phi_{21}}{M_1(\omega_{n1}^2 - \omega^2 + 2i\zeta\omega_{n1}\omega)} + \frac{\phi_{22}\phi_{22}}{M_2(\omega_{n2}^2 - \omega^2 + 2i\zeta\omega_{n2}\omega)}$$

Therefore,

$$H_{22,Mode\ 1}(\omega) = \frac{(1)^2}{1(1^2 - \omega^2 + 2i(0.05)(1)\omega)}$$

$$H_{22,Mode\ 2}(\omega) = \frac{(-1)^2}{1((\sqrt{3})^2 - \omega^2 + 2i(0.05)(\sqrt{3})\omega)}$$

Since the numerators are identical and the denominators only depend on the natural frequencies, ω_{n1} and ω_{n2}, which are the same for both H_{11} and H_{22}, the magnitude of the two direct receptances, $|H_{11}(\omega)|$ and $|H_{22}(\omega)|$, will be identical due to the symmetry of the system.




```python
import numpy as np
import matplotlib.pyplot as plt

plt.rcParams['text.usetex'] = True
# === System Parameters ===
# Natural Frequencies (rad/unit time)
omega_n1 = 1.0
omega_n2 = np.sqrt(3)

# Squared Natural Frequencies
omega_n1_sq = omega_n1**2
omega_n2_sq = omega_n2**2

# Damping Ratio (zeta = 5% for all modes)
zeta = 0.05

# Modal Properties (for H_11, i.e., response at m1 due to force at m1)
# Mode shapes: X1 = [1, 1], X2 = [1, -1]
# Phi_1r (displacement of m1 in mode r)
phi_11 = 1.0
phi_12 = 1.0

# Modal Mass (M_r) - calculated in thought as 1 for both modes
```

```python
M_1 = 1.0
M_2 = 1.0

# === Frequency Response Calculation ===
# Frequency range (up to 3.5 rad/unit time to clearly capture both modes)
omega = np.linspace(0.01, 3.5, 500)

def FrequencyResponse(omega, omega_n, zeta, modal_mass, phi_1r):
    """Calculates the frequency response function for a single mode (H_11 component)."""
    # Denominator D_r = M_r * (omega_nr^2 - omega^2 + 2*i*zeta*omega_nr*omega)
    # The term phi_1r^2/M_r is the Modal Participation Factor for H_11
    D_r = modal_mass * (omega_n**2 - omega**2 + 2j * zeta * omega_n * omega)
    return phi_1r**2 / D_r

# Mode 1 FRF (H_11)
H_11_mode1 = FrequencyResponse(omega, omega_n1, zeta, M_1, phi_11)

# Mode 2 FRF (H_11)
H_11_mode2 = FrequencyResponse(omega, omega_n2, zeta, M_2, phi_12)

# Total FRF (H_11) is the sum of the modal FRFs
H_11_total = H_11_mode1 + H_11_mode2

# === Plotting ===
plt.figure(figsize=(10, 6))

# Plot magnitude of individual and total FRFs
plt.plot(omega, np.abs(H_11_mode1), label=r'Mode 1: $\omega_{n1}=1$',
linestyle='--', color='blue')

plt.plot(omega, np.abs(H_11_mode2), label=r'Mode 2: $\omega_{n2}=\sqrt{3} \approx 1.732$',
  linestyle='--', color='orange')

plt.plot(omega, np.abs(H_11_total), label=r'Total Response $|H_{11}(\omega)|$',
  color='red', linewidth=2)

# Set labels and title
plt.title(r'Frequency Response Function $|H_{11}(\omega)|$', fontsize=14)
plt.xlabel(r'Excitation Frequency $\omega$ (rad/unit time)', fontsize=12)
plt.ylabel(r'Magnitude $|H_{11}(\omega)| = |X_1/F_1|$ (units)', fontsize=12)

# Add vertical lines for natural frequencies
plt.axvline(omega_n1, color='gray', linestyle=':', label=r'$\omega_{n1}$')
plt.axvline(omega_n2, color='darkgray', linestyle=':', label=r'$\omega_{n2}$')

plt.grid(True, which="both", ls="--")
```

```
plt.legend(fontsize=14)
plt.savefig('frf_plot.png')
print("frf_plot.png")
```