Case Study 1: Modeling T-33 Wing Vibrations

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1 Learning objectives

- **Connecting solid mechanics to vibration fundamentals:** This will involve determining the structural stiffness of a simplified wing structure using its geometric and material properties.

- **Conducting Systematic structural vibration Analysis:** The assignment will guide you through the typical procedure used in structural vibration analysis. You will gain an understanding of the importance of mathematically idealizing complex systems into simplified single-degree-of-freedom (SDoF) and multi-degree-of-freedom (MDoF) models prior to employing advanced finite element method (FEM) tools.

- **Realizing the advantages of energy methods over Newtonian mechanics:** You will use both Lagrangian and Newtonian mechanics to derive the equations of motion for a wing strucutre as an MDoF system. This will deepen your appreciation of Lagrangian analytical power and versatility.

1.1 Why Use Jupyter Notebook?

Jupyter Notebook serves as both a coding and reporting platform, enabling a seamless combination of computation, documentation, and visualization. It encourages clear commentary and explanation of your code in a structured, report-like format.

2 Expectations:

The project will be submitted as a Jupyter Notebook product. Throughout your notebook, briefly describe your approach, explain the underlying physics of your results. Your submission should reflect both sound technical reasoning and clarity in presentation. Figures without properly labeled axes, grids, titles, and legends will not be accepted. You will receive 25 points for completing the assignment tasks on time. You will recieve an extra point for each figure from your model beyond the assignment requirment upto five points. However, you must explain why you provided the figure with an interpretation of your engineering findings.

3 Background

The wing of a typical fighter jet, such as the Lockheed T-33 Shooting Star or T-Bird, consists of a lightweight structural airframe with attachment mechanisms at the tip for carrying a mission-related payload or an additional fuel tank attached to the wing tip, denoted as m_p. In this case study, you will perform a dynamic analysis of this aircraft wing using various idealized lumped-parameter modeling approaches. For modeling purposes, the wing may be reasonably approximated as a cantilevered closed-rectagular-cell beam (hollow beam) with a concentrated tip mass. In some

tasks you will be asked to assume that the distributed structural mass of the beam, m_w is negligible relative to the tip mass, which is given as $m_p = 100$ kg. Each wing extends 5.0 m from the fuselage.

Idealizing the airfoil closed-rectagular-cell, assume the chord and height remain constant along the span, with values of 0.70 m and 0.25 m, respectively. Accordingly, the beam's external cross-sectional area is 0.70×0.25 m ^{2}. A uniform wall thickness of 0.010 m is assumed for the box-beam structure. The wing is fabricated from aluminum alloy 6061-T6. For the purposes of this assignment, the following material properties for aluminum 6061-T6 are obtained from matweb.com: Young's modulus, E=69 GPa, density ρ=2700 kg/m ^{2}, and Poisson's ratio ν=0.33.

4 Approach 0.1: Preliminary analysis (3 points)

(d) Estimate the effective tip displacements and stiffness due to vertical, F_z, axial, F_x, and tortional, F_θ, loads at the tip. Assume the loads are static and m_w is **negligible**.

(d) Estimate the effective tip displacements and stiffness due to vertical, F_z, axial, F_x, and tortional, F_θ, loads at the tip. Assume the loads are static and m_w is **inegligible**.

(d) Estimate the undamped natural frequencies for extension, bending and torsion

of this idealized wing using the information provided above with and without m_w. Provide units in Hz and rad/s. Can w_w neglected?

4.1 Part a

As the wing box cross section is the rectangular hollow beam, the problem can be simplified as a fixed end cantilever beam with concentrated mass at tip of the beam. When we assume the concentrated loads at the tip are static and m_w is negligible, we have the free body diagram as following:

As the cross section is hollow rectangular tube, we have

$$b_{inner} = b - 2t = 0.7 - 2(0.01) = 0.68 \ [\text{m}]$$
$$h_{inner} = h - 2t = 0.25 - 2(0.01) = 0.23 \ [\text{m}]$$

We can calculate for the moment of the inertia base on the given geometry $b = 0.7$ m for chord, $h = 0.25$ m for height, and $t = 0.01$ m for uniform wall thickness. Shear modulus $G = \frac{E}{2(1+\nu)} = \frac{69 \times 10^9}{2(1+0.33)} = 25.94 \times 10^9 \ [Pa]$ Moment of inertia for bending is

$$I_y = \frac{1}{12}[hb^3 - h_{inner}b_{inner}^3] = \frac{1}{12}[(0.7)(0.25)^3 - (0.68)(0.23)^3] = 2.22 \times 10^{-4} \ [m^4]$$

Cross-section area A_{cs}

$$A_{cs} = hb - (h_{inner}b_{inner}) = (0.7)(0.25) - (0.68)(0.23) = 0.0186 \ [m]$$

Torsional constant J

$$A_m = (h - t)(b - t) = (0.69)(0.24) = 0.1656 \ [m]$$
$$P_m = 2(h - t) + 2(b - t) = 1.86 \ [m]$$
$$J = \frac{4A_m^2 t}{P_m} = \frac{4(0.1656)^2(0.01)}{1.86} = 0.000590 \ [m^4]$$

Assume vertical force F_z, axial force F_x and torsional force F_θ as unit force of 1, the effective stiffness values are

- **Vertical stiffness** k_z

$$k_z = \frac{3EI_y}{L^3} = \frac{3(69 \times 10^9)(2.22 \times 10^{-4})}{5^3} \approx 367,624 = 3.676E5 \ [N/m]$$

- **Axial stiffness** k_x

$$k_x = \frac{EA}{L} = \frac{(69 \times 10^9)(2.22 \times 10^{-4})}{5} \approx 256,680,000 = 2.57E8 \ [N/m]$$

- **Torsional stiffness k_θ**

$$k_\theta = \frac{GJ}{L} = \frac{(25.94 \times 10^9)(0.00059)}{5} = 3,060,000 = 3.06E6 \; [N/m]$$

For hooke's law $F = k\delta \Leftrightarrow \delta = \frac{F}{k}$, We can calculate the effective tip displacement

- **Vertical tip displacement δ_z**

$$\delta_z = \frac{F}{k_z} = \frac{1}{3.676 \; \times 10^5} = 2.72 \times 10^{-6} \; [m/N]$$

- **Axial tip displacement δ_x**

$$\delta_x = \frac{F}{k_x} = \frac{1}{256,680,000} = 3.9 \times 10^{-9} \; [m/N]$$

- **Torsional rotation θ**

$$\theta = \frac{1}{k_\theta} = \frac{1}{3,060,000} = 3.27 \times 10^{-7} \; [rad/N \cdot m]$$

4.2   Part b

Now having the wing mass m_w is non-negligible, $m_w = \rho A L = (2700)(0.0186)(5) = 251.1 \; [kg]$. As the calculation of the effective tip stiffness under static load is independent of mass for external static loads, the effective tip stiffness k_z, k_x, and k_θ remains unchanged. Therefore, the effective tip stiffness is

$$k_z = 3.676 \times 10^5 [N/m]$$
$$k_x = 2.57 \times 10^8 [N/m]$$
$$k_\theta = 3.06 \times 10 [rad/N \cdot m]$$

As hooke's law $F = k\delta \Leftrightarrow \delta = F/k$, we have similar effective tip displacement as

$$\delta_z = 2.72 \times 10^{-6} \; [m/N]$$
$$\delta_x = 2.9 \times 10^{-9} \; [m/N]$$
$$\theta = 3.27 \times 10^{-7} \; [rad/(N \cdot m)]$$

4.3   Part c:

As the system is single degree of fr.dom (SDOF), the natural frequency can be calculate as

$$\omega_n = \sqrt{\frac{k}{M_{eff}}} \tag{4.1}$$

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for k is the effective stiffness and M_{eff} is the total effective mass. In the case we consider the m_w is negligible, the effective mass $M_{eff} = m_p = 100 \ [kg]$. In case we consider the m_w, the effective mass $M_{eff} = m_p + m_{eq} = 100 + m_{eq}$, where m_{eq} is the equivalent lumped mass depends on the mode of vibration. To calculate for the undamped natural frequency, for the cantilever beam problem, we can solve in two case as mass of idealized wing with mass tip and mass of idealized wing without tip mass.

1. For case the idealized wing has mass tip m_p only and neglected wing mass m_w, we have the effective mass $M_{eff} = m_p = 100 \ [kg]$.

2. For case the idealized wing has both mass tip m_p and wing mass m_w, the effective mass will be $M_{eff} = m_p + m_{eq}$, which m_{eq} is the equivalent lumped mass depends on the mode of vibration

   - **Axial mode shape**, given the stiffness $k_x = \frac{EA}{L} = \delta_x \left(\frac{x}{L}\right)$,
     Potential energy:
     $$V_{max} = \frac{1}{2}k_x\delta_x^2 = \frac{1}{2}\left(\frac{EA}{L}\right)\delta_x^2$$

     Kinetic Energy:
     $$\mathcal{T}_{max} = \frac{1}{2}\omega_n^2 \int_0^L m[u(x)]^2 dx = \frac{1}{2}\omega_n^2 \frac{m_w}{L} \int_0^L \left[\delta_x\left(\frac{x}{L}\right)\right]^2 dx$$
     $$\mathcal{T}_{max} = \frac{1}{2}\omega_n^2 \left(\frac{m_w}{L}\right)\left(\frac{\delta_x^2}{L^2}\right)\left[\frac{x^3}{3}\right]_0^L$$
     $$\mathcal{T}_{max} = \frac{1}{2}\omega_n^2 \frac{m_w}{L}\left(\frac{\delta_x^2 L}{3}\right)$$
     $$\mathcal{T}_{max} = \frac{1}{2}\omega_n^2 \left(\frac{1}{3}m_w\right)\delta_x^2$$

     As the term for kinetic energy is $\mathcal{T} = \frac{1}{2}\omega_n^2 m_{eq}\delta_x^2$, the equivalent mass is

     $$\mathbf{m_{eq,x} = \frac{1}{3}m_w}$$

   - **Bending mode shape**, given the stiffness $k_z = \frac{3EI_y}{L^3}$,
     Potential energy:
     $$V = \frac{1}{2}k_z\delta_z^2 = \frac{1}{2}\left(\frac{3EI_y}{L^3}\right)\delta_z^3$$

     The static deflection of fixed-free end beam with lateral load has the static

deflection describe as

$$u(x) = \frac{F_z}{6EI_y}(6x^2 - x^2) = \delta_z \left[\frac{3L(x/L)^2 - (x/L)^3}{2L} \right] = \delta_z \left[\frac{3(x/L)^2 - (x/L)^3}{2} \right]$$

$$u(x) = \delta_z \left[\frac{3\zeta^2 - \zeta^3}{2} \right] \text{ let } \zeta = \frac{x}{L} \text{ for } 0 \le \zeta \le 1$$

Kinetic energy:

$$\mathcal{T} = \frac{1}{2}\omega_n^2 \int_0^L m[u(x)]^2 dx$$

$$\mathcal{T} = \frac{1}{2}m\omega_n^2 \int_0^L \left[\delta_z \left(\frac{3\zeta^2 - \zeta^3}{2} \right) \right]^2 d(L\zeta)$$

$$\mathcal{T} = \frac{1}{2}m\omega_n^2 L\delta_z^2 \int_0^1 \frac{1}{4} \left[\left(\frac{3\zeta^2 - \zeta^3}{2} \right) \right]^2 d\zeta$$

$$\mathcal{T} = \frac{1}{2}m\omega_n^2 L\delta_z^2 \left(\frac{1}{4} \right) \int_0^1 (9\zeta^4 - 6\zeta^5 + \zeta^6)d\zeta$$

$$\mathcal{T} = \frac{1}{2}m\omega_n^2 L\delta_z^2 \left(\frac{1}{4} \right) \left[\frac{9\zeta^5}{5} - \frac{6\zeta^6}{6} + \frac{\zeta^7}{7} \right]_0^1$$

$$\mathcal{T} = \frac{1}{2}m\omega_n^2 \delta_z^2 L \left(\frac{1}{4} \right) \left(\frac{33}{35} \right)$$

$$\mathcal{T} = \frac{1}{2}\omega_n^2 \frac{m_w}{L} \left(\frac{33}{140} L\delta_z^2 \right)$$

$$\mathcal{T} = \frac{1}{2}\omega_n^2 \left(\frac{33}{140}m_w \right) \delta_z^2$$

As the term for the kinetic energy is $\mathcal{T} = \frac{1}{2}\omega_n^2 m_{eq}\delta_z^2$, the equivalent mass is

$$\mathbf{m_{eq}} = \left(\frac{\mathbf{33}}{\mathbf{140}} \right) \mathbf{m_w}$$

- **Torsion mode shape**, we have
  Potential energy:

$$\mathcal{V} = \frac{1}{2}k_\theta \phi_{max}^2$$

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Kinetic Energy:

$$\mathcal{T} = \frac{1}{2}\omega_n^2 J \int_0^L \left[\phi_{max}\left(\frac{x}{L}\right)\right]^2 dx$$

$$\mathcal{T} = \frac{1}{2}\omega_n^2 J \int_0^L \phi_{max}^2 \frac{x^2}{L^2} dx$$

$$= \frac{1}{2}\omega_n^2 J \phi_{max}^2 \frac{1}{L^2}\left[\frac{x^3}{3}\right]_0^L$$

$$= \frac{1}{2}\omega_n^2 J \phi_{max}^2 \frac{L^3}{3L^2}$$

$$= \frac{1}{2}\omega_n^2 J \frac{1}{3} L \phi_{max}^2$$

$$= \frac{1}{2}\omega_{n,\theta}^2 \left(\frac{1}{3}J_w\right)\phi_{max}^2$$

The equivalent rotational inertia is

$$\mathbf{J_{eq,\theta}} = \frac{1}{3}\mathbf{J_w}$$

4.3.1   Undamped natural frequency in case only mass tip without mass wing m_w

- **In axial (extension) of the idealized wing**

$$\omega_{n,x} = \sqrt{\frac{k_x}{m_p}} = \sqrt{\frac{5.57 \times 10^8}{100}} = 506.64 \ [rad/s]$$

$$f_{n,x} = \frac{\omega_{n,x}}{2\pi} = \frac{506.64}{2\pi} = 80.64 \ [Hz]$$

- **In vertical (bending) of the idealized wing**

$$\omega_{n,z} = \sqrt{\frac{k_z}{m_p}} = \sqrt{\frac{3.617 \times 10^5}{100}} = 60.14 \ [rad/s]$$

$$f_{n,z} = \frac{\omega_{n,z}}{2\pi} = \frac{60.14}{2\pi} = 9.57 \ [Hz]$$

- **In torsional of the idealized wing**

$$\omega_{n,\theta} \approx \sqrt{\frac{k_\theta}{J_p}} = \sqrt{\frac{3.0590 \times 10^6 \ \text{N} \cdot \text{m/rad}}{2500 \ \text{kg} \cdot \text{m}^2}} \approx 34.95 \ [rad/s]$$

$$f_{n,\theta} = \frac{\omega_{n,\theta}}{2\pi} \approx 5.56 \ [Hz]$$

4.3.2 Undamped natural frequency in case mass tip with mass wing w_m

- **In axial (extension) of idealized wing**

$$m_{eq,x} = \frac{1}{3}m_w = \frac{251.1}{3} = 83.7\ [kg]$$

$$M_{eff,x} = 100\ \text{kg} + 83.7\ \text{kg} = 183.7\ \text{kg}$$

$$\omega_{n,x} = \sqrt{\frac{k_x}{M_{eff,x}}} = \sqrt{\frac{2.5668 \times 10^8\ \text{N/m}}{183.7\ \text{kg}}} \approx 1182.2\ \text{rad/s}$$

$$f_{n,x} = \frac{\omega_{n,x}}{2\pi} \approx 188.16\ \text{Hz}$$

- **In vertical (bending) of the idealized wing**

$$m_{eq,z} = \frac{33}{140}m_w = 0.2357(251.1\ \text{kg}) \approx 59.20\ \text{kg}$$

$$M_{eff,z} = 100\ \text{kg} + 59.20\ \text{kg} = 159.20\ \text{kg}$$

$$\omega_{n,z} = \sqrt{\frac{k_z}{M_{eff,z}}} = \sqrt{\frac{3.6167 \times 10^5\ \text{N/m}}{159.20\ \text{kg}}} \approx 47.63\ \text{rad/s}$$

$$f_{n,z} = \frac{\omega_{n,z}}{2\pi} \approx 7.58\ \text{Hz}$$

- **In torsional of idealized wing**

$$J_{eq,\theta} = \frac{J_w}{3} = \frac{2092.5\ \text{kg} \cdot \text{m}^2}{3} \approx 697.5\ \text{kg} \cdot \text{m}^2$$

$$J_{eff,\theta} = 2500\ \text{kg} \cdot \text{m}^2 + 697.5\ \text{kg} \cdot \text{m}^2 = 3197.5\ \text{kg} \cdot \text{m}^2$$

$$\omega_{n,\theta} = \sqrt{\frac{k_\theta}{J_{eff,\theta}}} = \sqrt{\frac{3.0590 \times 10^6\ \text{N} \cdot \text{m/rad}}{3197.5\ \text{kg} \cdot \text{m}^2}} \approx 30.93\ \text{rad/s}$$

$$f_{n,\theta} = \frac{\omega_{n,\theta}}{2\pi} \approx \mathbf{4.92}\ \textbf{Hz}$$

4.3.3 Comparison and result

Mode	f_n (m_w neglected)	f_n (m_w included)	Difference Δ
Axial ($f_{x,x}$)	80.64 Hz (506.64 rad/s)	188.16 Hz (1182.2 rad/s)	
Bending ($f_{n,z}$)	9.57 Hz (60.14 rad/s)	7.58 Hz (47.63 rad/s)	
Torsional ($f_{n,\theta}$)	5.56 Hz (34.95 rad/s)	4.92 Hz (30.93 rad/s)	

From the above table, we can clearly conclude that we cannot neglect the mass of the wing due the the large difference in the natural frequency of the beam vibration

5 Approach 1.0: Free and impulse vibrations using SDoF modeling. (7 points)

Model the wing as an SDOF mass-spring-damper system. This time include the wing's mass in the analysis, m_w. Don't plug in any numbers. The wing stiffness and modal damping ratio are k_w, and ζ_w, respectively. In steady level flight, the wing will typically deflect vertically due to the aerodynamic forces and its own weight.
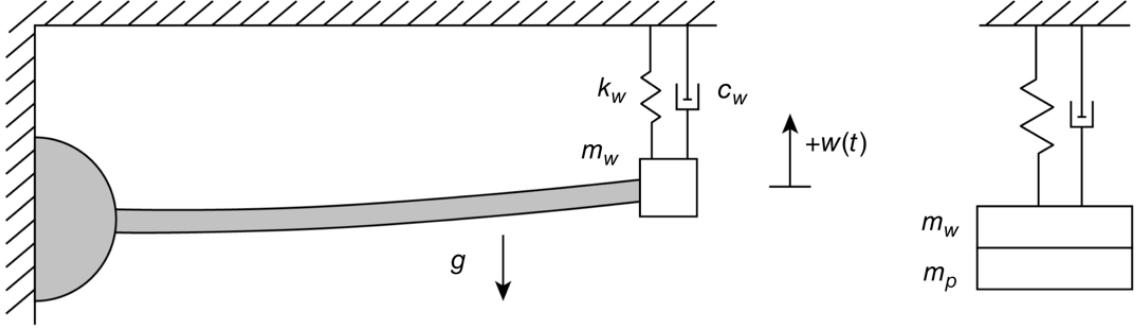
(e) Estimate the effective tip vertical deflection, $w(t)$, for $t \geq 0$, after m_p is released at $t = 0$. Assuming the aerodynamics forces do not change. Provide the damped and undamped resonance frequencies. Hint this is a free vibration problem.

(e) After releasing the payload, the aircraft encounters a gust wind at time $t = 0^+$. Idealize this change in the aerodynamic loads as an impulse force, F_i. Estimate the effective tip vertical deflection, $w(t)$, for $t \geq 0$.

(e) Plot the deflections in both cases (a and b) as a function of time, t. Assume ζ_w is 1

(e) Provide the phase plane plots (velocity vs. displacement). What do you infer from the phase plane plots that we can see in task c.

5.1 Part a:

As we consider the vibration after the m_p is released, the effective mas is $M = m_w$, and the wing stiffness will be $k = k_w$. For $t \geq 0$ is after m_p is release at $t = 0$, the governing equation of motion for a free vibration of a general SDOF system is

$$M\ddot{w} + c\dot{w} + kw = 0$$
$$(m_w)\ddot{w}(t) + (2\zeta_w\sqrt{m_w k_w})\dot{w}(t) + k_w w(t) = 0$$
$$\text{for } \zeta_w = \frac{c_w}{c_c} \Leftrightarrow c_w = \zeta_w c_c = 2\zeta_w\sqrt{m_w k_w}$$

With the damped free vibration of an underdamped system $\zeta_w < 1$ the boundary condition of the $w(0) = w_0$ and $\dot{w}(0) = 0$, we can solve the equation of motion

$$\begin{cases} (m_w)\ddot{w}(t) + (2\zeta_w\sqrt{m_w k_w})\dot{w}(t) + k_w w(t) &= 0 \\ w(0) = w_0, \dot{w}(0) = 0 \end{cases}$$

The solution will be

$$w(t) = Ae^{-\zeta_w \omega_n t}\cos(\omega_d t)$$

The amplitude A is approximately w_0 for small ζ_w. With $\dot{w}(0) = 0$, the exact amplitude is:

$$A = \frac{w_0}{\sqrt{1 - \zeta_w^2}}$$

Substituting the initial displacement $w_0 = \frac{m_p g}{k_w}$, the effective tip vertical deflection is

$$w(t) = \left(\frac{m_p g}{k_w\sqrt{1 - \zeta_w^2}}\right)e^{-\zeta_w \omega_n t}\cos(\omega_d t)$$

where ω_n is the undamped natural frequency and ω_d is the damped natural frequency.

- **Undamped natural resonance frequency $(\omega_\mathbf{n})$**

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{k_w}{m_w}} \ [\text{rad/s}]$$

- **Damped natural resonance frequency $(\omega_\mathbf{d})$**

$$\omega_d = \omega_n\sqrt{1 - \zeta_w^2} = \sqrt{\frac{k_w}{m_w}} \ [\text{rad/s}]$$

5.2 Part b:

As the wing encounters wing gust at time $t = 0^+$, An impulse force $(\mathbf{F_i})$ is modeled as a sudden change in momentum over a very short time, which is equivalent to imposing an initial velocity on the system while the displacement remains momentarily unchanged. The total response $w(t)$ for $t \geq 0$ is the solution to the homogeneous (free vibration) equation of motion, $M\ddot{w} + c\dot{w} + kw = 0$. The initial displacement is $w(0) = \frac{m_p g}{k_w}$ and initial velocity is $\dot{w} = v_0 = \frac{F_i}{m_w}$. For an underdamped system $\zeta_w < 1$ undergoing free vibration, the general solution is

$$w(t) = e^{-\zeta_w \omega_n t}\left[A_1\cos(\omega_d t) + A_2\sin(\omega_d t)\right]$$
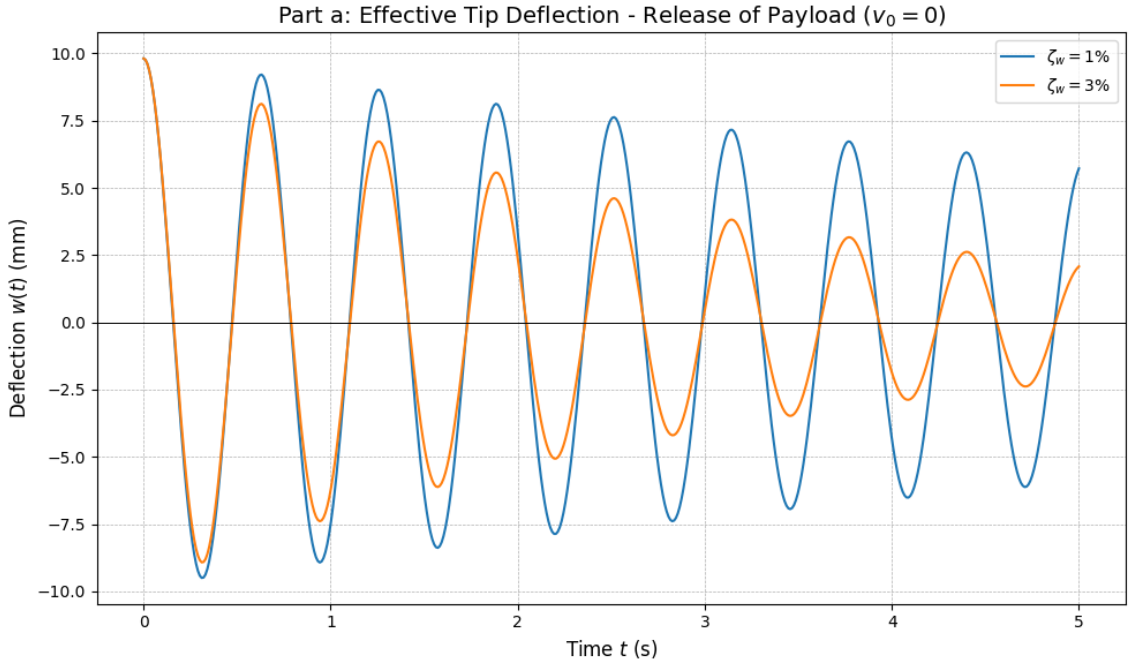
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With the initial condition:

$$w_0 = e^0[A_1\cos(0) + A_2\sin(0)] \implies A_1 = w_0 = \frac{m_p g}{k_w}$$

$$\dot{w}(0) = -\zeta_w\omega_n e^{-\zeta_w\omega_n t}[A_1\cos(\omega_d t) + A_2\sin(\omega_d t)]$$
$$+ e^{-\zeta_w\omega_n t}[-A_1\omega_d\sin(\omega_d t) + A_2\omega_d\cos(\omega_d t)]$$
$$v_0 = -\zeta_w\omega_n[A_1(1) + A_2(0)] + [0 + A_2\omega_d(1)]$$
$$v_0 = -\zeta_w\omega_n A_1 + \omega_d A_2$$

$$\Rightarrow A_2 = \frac{v_0 + \zeta_w\omega_n A_1}{\omega_d}$$

$$\Rightarrow A_2 = \frac{\frac{F_i}{m_w} + \zeta_w\omega_n\left(\frac{m_p g}{k_w}\right)}{\omega_d}$$

Therefore, the solution for the effective tip vertical displacement $w(t)$

$$w(t) = e^{-\zeta_w\omega_n t}\left[\left(\frac{m_p g}{k_w}\right)\cos(\omega_d t) + \left(\frac{\frac{F_i}{m_w} + \zeta_w\omega_n\frac{m_p g}{k_w}}{\omega_d}\right)\sin(\omega_d t)\right]$$

for $M = m_w$, $\omega_n = \sqrt{\frac{k_w}{m_w}}$, $\omega_d = \omega_n = \omega_n\sqrt{1 - \zeta_w^2}$ and F_i is the impulse magnitude.

5.3   Part c:

Part a: Effective Tip Deflection - Release of Payload ($v_0 = 0$)

Part b: Effective Tip Deflection - Release + Gust Impulse (w_0, v_0)
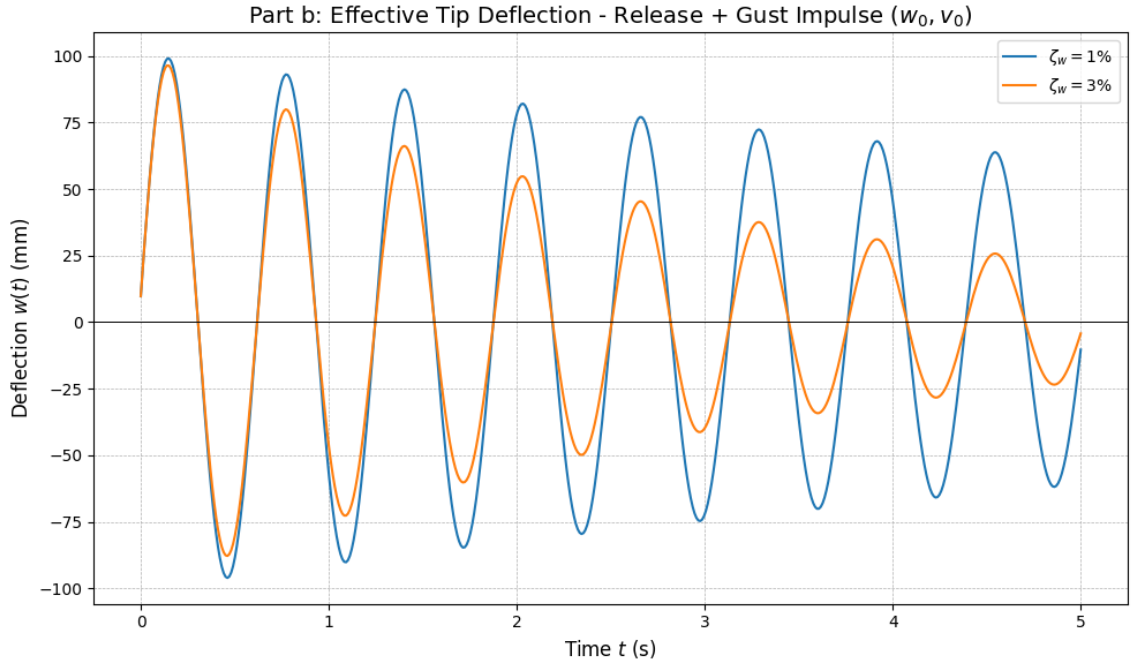
6 Approach 2.0: Equation of motion as an MDoF system (7 points)

In this approach, you will not assume that the payload m_p is a point mass. Accordingly, the contribution of m_p radius of gyration, κ, to the dynamics must be considered. The center of m_p is located at e distance from elastic axis X. Assuming the wing experiences only vertical bending and twisting deformation and neglecting its mass, m_w, formulate the equations of motion using:
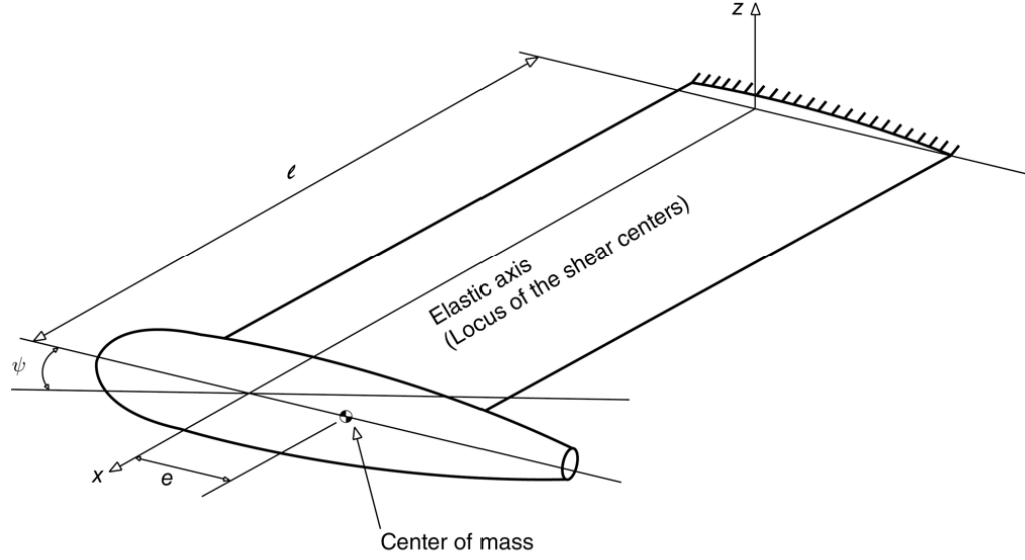
(a) Newtonian mechanics, and

(b) Lagrange's equations. Ignore damping for a and b

6.1 Part a: Using Newtonian mechanics

1. **Vertical bending**
   Assume the system is a massless elastic wing with stiffness, the vertical translation is $w_p = w(x_p, t)$ and rotation is $\theta_p = \theta(x_p, t)$. The center of mass vertical displacement is:

   $$\delta_z = w_p + e\sin(\theta_p) \approx w_p + e\theta_p \text{ assuming small angle for } \theta_p \sin\theta_p \approx \theta_p$$

   From Newton's Second Law: $\sum F_z = 0$

   $$\sum F_z = 0$$
   $$F_{\text{elastic}} + F_{\text{inertia}} = 0$$
   $$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big|_{x=x_p} + m_p\ddot{\delta}_z = 0$$
   $$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big|_{x=x_p} + m_p\frac{d^2}{dt^2}(w_p + e\theta_p) = 0$$
   $$\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big|_{x=x_p} + m_p(\ddot{w}_p + e\ddot{\theta}_p) = 0$$

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Therefore, the equation of motion for vertical bending

$$\boxed{\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{EI}\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x}^2}\right)\bigg|_{\mathbf{x}=\mathbf{x_p}} + \mathbf{m_p}(\mathbf{\ddot{w}_p} + \mathbf{e}\mathbf{\ddot{\theta}_p}) = \mathbf{0}}$$

2. **Twisting deformation** Assume the system is a massless elastic wing with internal torque in the wing is due to twisting. From Newton's second law, we have

$$\sum M_x = 0$$

$$M_{\text{elastic}} + M_{\text{inertia}} = 0$$

$$M_{\text{elastic}} + M_{\text{rotation}} + M_{\text{translation}} = 0$$

$$-\frac{\partial}{\partial x}\left(GJ\frac{\partial \theta}{\partial x}\right)\bigg|_{x=x_p} + I_p\ddot{\theta}_p - (m_p\ddot{\delta}_z) \cdot e = 0$$

$$-\frac{\partial}{\partial x}\left(GJ\frac{\partial \theta}{\partial x}\right)\bigg|_{x=x_p} + m_p\kappa^2\ddot{\theta}_p - m_p(\ddot{w}_p + e\ddot{\theta}_p)e = 0$$

$$-\frac{\partial}{\partial x}\left(GJ\frac{\partial \theta}{\partial x}\right)\bigg|_{x=x_p} - m_pe\ddot{w}_p + m_p(\kappa^2 + e^2)\ddot{\theta}_p = 0$$

Therefore, the equation of motion for twisting deformation is

$$\boxed{-\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{GJ}\frac{\partial \theta}{\partial \mathbf{x}}\right)\bigg|_{\mathbf{x}=\mathbf{x_p}} - \mathbf{m_p}\mathbf{e}\mathbf{\ddot{w}_p} + \mathbf{m_p}(\kappa^2 + \mathbf{e^2})\mathbf{\ddot{\theta}_p} = \mathbf{0}}$$

6.2   Part b: Using Lagrange's equation

Assuming the wing only experiences vertical bending and twisting deformation and neglecting wing mass m_w, we have the general solution for Lagrange's equation as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0$$

As we have two generalized coordinate $w_p(x_p, t)$ for vertical bending and $\theta_p(x_p, t)$ for twisting deformation, we can solve for the equation of motion using the Lagrange's equation as following.

16

Kinetic Energy:

$$\mathcal{T} = \mathcal{T}_{\text{Translation}} + \mathcal{T}_{\text{Rotation}}$$

$$\mathcal{T} = \frac{1}{2}m_p\dot{\delta}_z^2 + \frac{1}{2}I\dot{\theta}_p^2$$

$$\mathcal{T} = \frac{1}{2}m_p(\dot{w}_p + e\dot{\theta}_p)^2 + \frac{1}{2}(m_p\kappa^2)\dot{\theta}_p^2$$

$$\mathcal{T} = \frac{1}{2}m_p(\dot{w}_p^2 + 2e\dot{w}_p\dot{\theta}_p + e^2\dot{\theta}_p^2) + \frac{1}{2}m_p\kappa^2\dot{\theta}_p^2$$

$$\mathcal{T} = \frac{1}{2}m_p\dot{w}_p^2 + m_pe\dot{w}_p\dot{\theta}_p + \frac{1}{2}m_p(e^2 + \kappa^2)\dot{\theta}_p^2$$

Potential Energy:

$$\mathcal{V} = \mathcal{V}_{\text{Bending}} + \mathcal{V}_{\text{Torsion}}$$

$$\mathcal{V} = \int_0^L \frac{1}{2}EI\left(\frac{\partial^2 w}{\partial x^2}\right)^2 dx + \int_0^L \frac{1}{2}GJ\left(\frac{\partial\theta}{\partial x}\right)^2 dx$$

From here we can use Lagrange's equation to solve for equation of motion with respect to each generalized coordinate:

1. **Equation for vertical bending $\mathbf{w_p}$:**
   Lagrange's equation: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{w}_p}\right) - \frac{\partial T}{\partial w_p} + \frac{\partial U}{\partial w_p} = 0$

$$
\begin{cases}
\frac{\partial T}{\partial \dot{w}_p} &=& m_p\dot{w}_p + m_pe\dot{\theta}_p \\
\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{w}_p}\right) &=& \mathbf{m_p\ddot{w}_p + m_pe\ddot{\theta}_p} \\
\frac{\partial T}{\partial w_p} &=& 0 \\
\frac{\partial U}{\partial w_p} &=& -\frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big|_{x=x_p}
\end{cases}
$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{w}_p}\right) - \frac{\partial T}{\partial w_p} + \frac{\partial U}{\partial w_p} = 0$$

$$\Rightarrow (m_p\ddot{w}_p + m_pe\ddot{\theta}_p) - 0 - \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big|_{x=x_p} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right)\Big|_{x=x_p} + m_p\ddot{w}_p + m_pe\ddot{\theta}_p = 0$$

Therefore, the equation of motion for vertical bending $\mathbf{w_p}$ is

$$\boxed{\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{EI}\frac{\partial^2 \mathbf{w}}{\partial \mathbf{x^2}}\right)\Big|_{\mathbf{x=x_p}} + \mathbf{m_p\ddot{w}_p + m_pe\ddot{\theta}_p = 0}}$$

2. **Equation for twisting deformation $\theta_\mathbf{p}$:**

Lagrange's equation: $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_p}\right) - \frac{\partial T}{\partial \theta_p} + \frac{\partial U}{\partial \theta_p} = 0$

$$\begin{cases} \frac{\partial T}{\partial \dot{\theta}_p} &= m_p e \dot{w}_p + m_p(e^2 + \kappa^2)\dot{\theta}_p \\ \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_p}\right) &= m_p e \ddot{w}_p + m_p(e^2 + \kappa^2)\ddot{\theta}_p \\ \frac{\partial T}{\partial \theta_p} &= 0 \\ \frac{\partial U}{\partial \theta_p} &= -\frac{\partial}{\partial x}\left(GJ\frac{\partial \theta}{\partial x}\right)\Big|_{x=x_p} \end{cases}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_p}\right) - \frac{\partial T}{\partial \theta_p} + \frac{\partial U}{\partial \theta_p} = 0$$

$$\Rightarrow (m_p e \ddot{w}_p + m_p(e^2 + \kappa^2)\ddot{\theta}_p) - 0 - \frac{\partial}{\partial x}\left(GJ\frac{\partial \theta}{\partial x}\right)\Big|_{x=x_p} = 0$$

$$\Rightarrow -\frac{\partial}{\partial x}\left(GJ\frac{\partial \theta}{\partial x}\right)\Big|_{x=x_p} + m_p e \ddot{w}_p + m_p(e^2 + \kappa^2)\ddot{\theta}_p = 0$$

Therefore, the equation of motion for twisting deformation $\theta_{\mathbf{p}}$ is

$$\boxed{-\frac{\partial}{\partial \mathbf{x}}\left(\mathbf{GJ}\frac{\partial \theta}{\partial \mathbf{x}}\right)\Big|_{\mathbf{x}=\mathbf{x_p}} + \mathbf{m_p e \ddot{w}_p} + \mathbf{m_p(e^2 + \kappa^2)\ddot{\theta}_p} = \mathbf{0}}$$

7 Approach 3: Modal analysis of the wing as an MDoF system (8 points)

In this approach, the wing deflection is modeled as a segmented beam into %n% masses. Considering only vertical bending of the wing, formulate the equations of motion for $n=5$. Assume m_p is a point mass.

(a) Estimate the mode shapes with and without m_p.

(b) Estimate the flexural natural frequencies with and without m_p.

(c) Compare your findings with Approach 0.1.

7.1 Part a:

The total structural mass is $m_w = \bar{m}L = (50.22 \text{ kg/m})(5.0 \text{ m}) = 251.1 \text{ kg}$. The tip mass is $m_p = 100 \text{ kg}$. The total mass is $m_T = 351.1 \text{ kg}$. Assume the m_p is a point mass with the beam segment split into 5 elements for $n = 5$; the mass of each segment will be:

$$m_i = \bar{m}L = (50.22 \text{ kg/m})(1.0 \text{ m}) = 50.22 \text{ kg} \quad \text{for } i = 1, 2, 3, 4$$

The mass at the tip (m_5) includes the distributed mass contribution and the point mass m_p:

$$m_5 = \frac{1}{2}\bar{m}L + m_p = \frac{1}{2}(50.22 \text{ kg/m})(1.0 \text{ m}) + 100 \text{ kg} = 25.11 \text{ kg} + 100 \text{ kg} = 125.11 \text{ kg}$$

Derive for the stiffness matrix $[K]$:

For a cantilever beam, the stiffness matrix relationship to forces and displacement is $\{F\} = [K]\{U\}$. For C_{ij} is the flexibility matrix and assume F is a unit force equal 1, the deflection $z(x)$ due to a concentrated force P_j applied at a distance x_j from the root $(x = 0)$ is given by the standard formula:

$$z(x) = \frac{P_j x^2}{6EI}(3x_j - x) \quad \text{for } x \leq x_j$$

$$z(x) = \frac{P_j x_j^2}{6EI}(3x - x_j) \quad \text{for } x > x_j$$

As $C_{ij} = C_{ji}$ and assuming P is a unit force of 1, we have flexibility matrix C_{ij} is

$$C_{ij} = \frac{(\Delta x)^2}{6EI} \cdot (3x_j - x_i) \quad \text{for } i \leq j$$

$$[C] = \frac{1}{6EI} \begin{bmatrix} 2 & 5 & 8 & 11 & 14 \\ 5 & 16 & 28 & 40 & 52 \\ 8 & 28 & 54 & 81 & 108 \\ 11 & 40 & 81 & 128 & 175 \\ 14 & 52 & 108 & 175 & 250 \end{bmatrix}$$

With $\Delta x = 1.0$ m, we define the factor.

$$\frac{EI}{(\Delta x)^3} = \frac{15.3176 \times 10^6 \text{ N} \cdot \text{m}^2}{(1.0 \text{ m})^3} \approx 15.3176 \times 10^6 \text{ N/m}$$

We can compute for the stiffness matrix as $[K] = [C]^{-1}$,

```
1   import numpy as np
2   import matplotlib.pyplot as plt
3   # --- 1. Define Parameters ---
4   L = 5.0          # m, total length
5   b_o = 0.70       # m, outer chord
6   h_o = 0.25       # m, outer height
7   t = 0.010        # m, wall thickness
8   E = 69e9         # Pa (N/m^2), Young's modulus
9   rho = 2700       # kg/m^3, density
10  m_p = 100.0      # kg, tip payload
11  n = 5            # number of segments
```

```python
12
13   print(f"--- 1. Parameters Defined ---")
14   print(f"L={L}m, E={E/1e9} GPa, rho={rho} kg/m^3, m_p={m_p} kg, n={n} segments")
15
16   # --- 2. Calculate Beam Properties ---
17   print("\n--- 2. Calculating Beam Properties ---")
18
19   # Inner dimensions
20   b_i = b_o - 2 * t
21   h_i = h_o - 2 * t
22
23   # Area Moment of Inertia (I) for bending about the chord-wise axis
24   I = (1/12) * (b_o * h_o**3 - b_i * h_i**3)
25   EI = E * I
26
27   # Cross-sectional Area (A) and Total Wing Mass (m_w)
28   A = (b_o * h_o) - (b_i * h_i)
29   m_w = rho * A * L
30
31   # Segment properties
32   delta_L = L / n
33   m_seg = m_w / n
34
35   print(f"Area Moment of Inertia (I): {I:.4e} m^4")
36   print(f"Flexural Rigidity (EI): {EI:.4e} N*m^2")
37   print(f"Total Wing Mass (m_w): {m_w:.2f} kg")
38   print(f"Segment Mass (m_seg): {m_seg:.2f} kg")
39
40   # --- 3. Build Stiffness Matrix (K) ---
41   print("\n--- 3. Building Stiffness Matrix [K] ---")
42
43   # We build the Flexibility Matrix (C) first, then K = C^-1
44   C = np.zeros((n, n))
45   # Node positions (x=1, 2, 3, 4, 5)
46   x_nodes = np.arange(1, n + 1) * delta_L
47
48   # Use cantilever beam deflection formulas C_ij = deflection at i for force at j
49   for i in range(n):
50       for j in range(n):
51           xi = x_nodes[i]
52           xj = x_nodes[j]
53
54           if xi <= xj:
55               # Deflection at x=xi due to load P=1 at a=xj
56               C[i, j] = (xi**2 * (3*xj - xi)) / (6 * EI)
57           else: # xi > xj
```

20

```
58            # Deflection at x=xi due to load P=1 at a=xj
59            # (By reciprocity, C[i,j] = C[j,i], this is equivalent)
60            C[i, j] = (xj**2 * (3*xi - xj)) / (6 * EI)
61
62   # Stiffness Matrix K = C^-1
63   K = np.linalg.inv(C)
64
65   print("Stiffness Matrix K (N/m) (first 5x5 shown):")
66   print(np.array2string(K, formatter={'float_kind':lambda x: "%.2e" % x}))
```

The stiffness matrix will be

$$[K] = \begin{bmatrix} 2.88e+08 & -1.82e+08 & 7.31e+07 & -1.83e+07 & 3.05e+06 \\ -1.82e+08 & 2.24e+08 & -1.64e+08 & 6.40e+07 & -1.07e+07 \\ 7.31e+07 & -1.64e+08 & 2.15e+08 & -1.46e+08 & 3.96e+07 \\ -1.83e+07 & 6.40e+07 & -1.46e+08 & 1.51e+08 & -5.59e+07 \\ 3.05e+06 & -1.07e+07 & 3.96e+07 & -5.59e+07 & 2.46e+07 \end{bmatrix}$$

From the two above, assuming no damping in the system, the equation of motion for the free vibration of the system is $[M]\ddot{w}(t) + [K]w(t) = 0$. We can calculate for the eigenvalue $([K] - \omega^2[M])\Omega = 0$ for ω is the natural frequency and Ω are the mode shapes.

7.1.1   In case system with tip mass m_p

The mass matrix $[M]$

$$[M] = \begin{bmatrix} m_w^{(1)} & 0 & 0 & 0 & 0 \\ 0 & m_w^{(2)} & 0 & 0 & 0 \\ 0 & 0 & m_w^{(3)} & 0 & 0 \\ 0 & 0 & 0 & m_w^{(4)} & 0 \\ 0 & 0 & 0 & 0 & m_w^{(5)} + m_p \end{bmatrix} = \begin{bmatrix} 50.22 & 0 & 0 & 0 & 0 \\ 0 & 50.22 & 0 & 0 & 0 \\ 0 & 0 & 50.22 & 0 & 0 \\ 0 & 0 & 0 & 52.22 & 0 \\ 0 & 0 & 0 & 0 & 150.22 \end{bmatrix} [kg]$$

The mass matrix will be $m_5 = m_w + m_p = 150.22$ kg, and $m_1 = m_2 = m_3 = m_4 = 50.22$ kg. We can solve the eigenvalue problem of $([K] - \omega^2[M])\Omega = 0$

$$[K] = [M]^{-\frac{1}{2}}[K][M]^{-\frac{1}{2}}$$

```
1   # Case 2: With Payload (M2)
2   m_lumped_2 = np.full(n, m_seg)
3   m_lumped_2[-1] += m_p   # Add payload to the last node
4   M2 = np.diag(m_lumped_2)
5   print("\nMass Matrix M2 (With Payload) (kg):")
6   print(np.array2string(M2, formatter={'float_kind':lambda x: "%.2f" % x}))
7
```

```python
8   # === 5. Solve Eigenvalue Problem ===
9   print("\n=== 5. Solving Eigenvalue Problems ===")
10
11  def solve_and_print(K, M, case_name):
12      """Solves the generalized eigenvalue problem (K*phi = lambda*M*phi)
13          using only numpy and prints formatted results."""
14
15      # Convert generalized problem K*phi = w^2*M*phi
16      # to standard problem K_tilde*u = w^2*u
17
18      # 1. Get M^(-1/2)
19      # Since M is diagonal, M^(-1/2) is diag(1/sqrt(m_i))
20      M_inv_sqrt = np.diag(1.0 / np.sqrt(np.diag(M)))
21
22      # 2. Form the modified stiffness matrix K_tilde
23      # K_tilde = M^(-1/2) * K * M^(-1/2)
24      K_tilde = M_inv_sqrt @ K @ M_inv_sqrt
25
26      # 3. Solve the standard eigenvalue problem using numpy.linalg.eigh
27      # K_tilde is symmetric, so we can use eigh
28      # eigvals are w^2
29      # eigvecs_u are the eigenvectors in the {u} coordinate system
30      eigvals, eigvecs_u = np.linalg.eigh(K_tilde)
31
32      # 4. Sort eigenvalues (and corresponding eigenvectors) from low to high
33      idx = eigvals.argsort()
34      eigvals = eigvals[idx]
35      eigvecs_u_sorted = eigvecs_u[:, idx]
36
37      # 5. Transform eigenvectors {u} back to physical {phi}
38      # phi = M^(-1/2) * u
39      eigvecs = M_inv_sqrt @ eigvecs_u_sorted
40
41      # 6. Calculate frequencies
42      freqs_rad = np.sqrt(eigvals)        # omega (rad/s)
43      freqs_hz = freqs_rad / (2 * np.pi) # f (Hz)
44
45      print(f"\n=== Results: {case_name} ===")
46      print("Natural Frequencies (Hz):")
47      print(np.array2string(freqs_hz, formatter={'float_kind':lambda x: "%.2f" % x}))
48
49      print("\nMode Shapes (Eigenvectors, normalized to max component):")
50      modes = []
51      vec_formatter = {'float_kind':lambda x: "% .3f" % x} # Formatter for vectors
52
53      for i in range(n):
```

```
54        vec = eigvecs[:, i]
55        # Normalize by dividing by the component with the largest absolute value
56        normalized_vec = vec / np.max(np.abs(vec))
57        modes.append(normalized_vec)
58        print(f"  Mode {i+1} (f={freqs_hz[i]:.2f} Hz):")
59        print("    " + np.array2string(normalized_vec, formatter=vec_formatter))
60
61    return freqs_hz, modes
62
63
64 freqs2, modes2 = solve_and_print(K, M2, "With Payload (mp=100kg)")
```

The result for the mode shapes with tip mass m_p is

$$f_i = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{Bmatrix} = \begin{Bmatrix} 7.05 \\ 58.73 \\ 178.65 \\ 355.34 \\ 534.84 \end{Bmatrix} \text{ [Hz]}$$

7.1.2 In case system without tip mass m_p

The mass matrix $[M]$

$$[M] = \begin{bmatrix} m_w^{(1)} & 0 & 0 & 0 & 0 \\ 0 & m_w^{(2)} & 0 & 0 & 0 \\ 0 & 0 & m_w^{(3)} & 0 & 0 \\ 0 & 0 & 0 & m_w^{(4)} & 0 \\ 0 & 0 & 0 & 0 & m_w^{(5)} + m_p \end{bmatrix} = \begin{bmatrix} 50.22 & 0 & 0 & 0 & 0 \\ 0 & 50.22 & 0 & 0 & 0 \\ 0 & 0 & 50.22 & 0 & 0 \\ 0 & 0 & 0 & 52.22 & 0 \\ 0 & 0 & 0 & 0 & 52.22 \end{bmatrix} \text{ [kg]}$$

Using the above computing approach we have the mode shapes without the tip mass m_p is

```
1  # Case 1: Without Payload (M1)
2  m_lumped_1 = np.full(n, m_seg)
3  M1 = np.diag(m_lumped_1)
4  print("Mass Matrix M1 (Without Payload) (kg):")
5  print(np.array2string(M1, formatter={'float_kind':lambda x: "%.2f" % x}))
6
7  # === 5. Solve Eigenvalue Problem ===
8  print("\n=== 5. Solving Eigenvalue Problems ===")
9
10 def solve_and_print(K, M, case_name):
11     """Solves the generalized eigenvalue problem (K*phi = lambda*M*phi)
12         using only numpy and prints formatted results."""
```

23

```python
13
14          # Convert generalized problem K*phi = w^2*M*phi
15          # to standard problem K_tilde*u = w^2*u
16
17          # 1. Get M^(-1/2)
18          # Since M is diagonal, M^(-1/2) is diag(1/sqrt(m_i))
19          M_inv_sqrt = np.diag(1.0 / np.sqrt(np.diag(M)))
20
21          # 2. Form the modified stiffness matrix K_tilde
22          # K_tilde = M^(-1/2) * K * M^(-1/2)
23          K_tilde = M_inv_sqrt @ K @ M_inv_sqrt
24
25          # 3. Solve the standard eigenvalue problem using numpy.linalg.eigh
26          # K_tilde is symmetric, so we can use eigh
27          # eigvals are w^2
28          # eigvecs_u are the eigenvectors in the {u} coordinate system
29          eigvals, eigvecs_u = np.linalg.eigh(K_tilde)
30
31          # 4. Sort eigenvalues (and corresponding eigenvectors) from low to high
32          idx = eigvals.argsort()
33          eigvals = eigvals[idx]
34          eigvecs_u_sorted = eigvecs_u[:, idx]
35
36          # 5. Transform eigenvectors {u} back to physical {phi}
37          # phi = M^(-1/2) * u
38          eigvecs = M_inv_sqrt @ eigvecs_u_sorted
39
40          # 6. Calculate frequencies
41          freqs_rad = np.sqrt(eigvals)        # omega (rad/s)
42          freqs_hz = freqs_rad / (2 * np.pi) # f (Hz)
43
44          print(f"\n=== Results: {case_name} ===")
45          print("Natural Frequencies (Hz):")
46          print(np.array2string(freqs_hz, formatter={'float_kind':lambda x: "%.2f" % x}))
47
48          print("\nMode Shapes (Eigenvectors, normalized to max component):")
49          modes = []
50          vec_formatter = {'float_kind':lambda x: "% .3f" % x} # Formatter for vectors
51
52          for i in range(n):
53              vec = eigvecs[:, i]
54              # Normalize by dividing by the component with the largest absolute value
55              normalized_vec = vec / np.max(np.abs(vec))
56              modes.append(normalized_vec)
57              print(f"  Mode {i+1} (f={freqs_hz[i]:.2f} Hz):")
58              print("    " + np.array2string(normalized_vec, formatter=vec_formatter))
```

```
59
60      return freqs_hz, modes
61  freqs1, modes1 = solve_and_print(K, M1, "Without Payload (mp=0)")
62
```

The result for the mode shapes without tip mass m_p is

$$f_i = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{Bmatrix} = \begin{Bmatrix} 10.29 \\ 65.73 \\ 186.25 \\ 359.94 \\ 536.04 \end{Bmatrix} \text{ [Hz]}$$

7.2   Part b:

From the above code we can calculate for the eigenvector as following

7.2.1   In case system without tip mass

- Mode 1: $f_1 = 10.29$ [Hz], $w^{(1)} = \begin{bmatrix} 0.061 \\ 0.222 \\ 0.451 \\ 0.718 \\ 1.000 \end{bmatrix}$

- Mode 2: $f_2 = 65.73$ [Hz], $w^{(2)} = \begin{bmatrix} -0.395 \\ -0.963 \\ -1.000 \\ -0.289 \\ 0.896 \end{bmatrix}$

- Mode 3: $f_3 = 186.25$ [Hz], $w^{(3)} = \begin{bmatrix} -0.896 \\ -1.000 \\ 0.468 \\ 0.974 \\ -0.633 \end{bmatrix}$

- Mode 4: $f_4 = 359.94$ [Hz], $w^{(4)} = \begin{bmatrix} 1.000 \\ -0.206 \\ -0.718 \\ 0.855 \\ -0.305 \end{bmatrix}$

- Mode 5: $f_5 = 534.84$ [Hz], $w^{(1)} = \begin{bmatrix} -0.912 \\ 1.000 \\ -0.880 \\ 0.518 \\ -0.142 \end{bmatrix}$

7.2.2 In case system with tip mass

- Mode 1: $f_1 = 7.05$ [Hz], $w^{(1)} = \begin{bmatrix} 0.058 \\ 0.215 \\ 0.441 \\ 0.710 \\ 1.000 \end{bmatrix}$

- Mode 2: $f_2 = 65.73$ [Hz], $w^{(2)} = \begin{bmatrix} -0.336 \\ -0.860 \\ -1.000 \\ -0.538 \\ 0.343 \end{bmatrix}$

- Mode 3: $f_3 = 186.25$ [Hz], $w^{(3)} = \begin{bmatrix} -0.817 \\ -0.969 \\ 0.360 \\ 1.000 \\ -0.205 \end{bmatrix}$

- Mode 4: $f_4 = 359.94$ [Hz], $w^{(4)} = \begin{bmatrix} 1.000 \\ -0.177 \\ -0.744 \\ 0.853 \\ -0.100 \end{bmatrix}$

- Mode 5: $f_5 = 534.84$ [Hz], $w^{(1)} = \begin{bmatrix} -0.916 \\ 1.000 \\ -0.871 \\ 0.506 \\ -0.046 \end{bmatrix}$

7.3 Part c: Conclusion

From the above result finding, we can see the difference between the prelimary approach 0.1 and the modal analysis. While the similar assumption for tip mass and wing mass with the vertical bending gives the natural frequency of **7.58 [Hz]**, The modal analysis give a prediction natural frequency of **7.05 [Hz]** for mode 1, which is . Also, When the modal analysis give the natural frequency of **10.29 [Hz]** for the set up of wing mass only, the preliminary predict a frequency of **9.57 [Hz]**. Both case is also roughly 7.00% difference