

AE543: Final Exam

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1 Section 1: Understanding Fundamentals

1. [1 point] What is the advantage of describing the physical coordinates of a dynamic system in terms of generalized coordinates? (one sentence only)

⇒ The primary advantage is that generalized coordinates define the configuration of a system using the minimum number of independent coordinates necessary, thereby eliminating the need to explicitly solve for constraint forces. (Ref: [Rao] Chapter 1, [Meirovitch] Chapter 1).

2. [1 point] What is the advantage of describing the generalized coordinates of a dynamic system in terms of modal coordinates? (one sentence only)

⇒ Using modal coordinates decouples the equations of motion (diagonalizes the mass and stiffness matrices), allowing a multi-degree-of-freedom system to be solved as a set of independent single-degree-of-freedom equations. (Ref: [Inman] Chapter 5).

3. [1 point] What are the properties of virtual displacement?

⇒ Virtual displacements are imaginary, infinitesimal changes in coordinates that are consistent with the system's geometric constraints and occur instantaneously (time is held fixed, $dt = 0$). (Ref: [Ginsberg] Chapter 3).

4. [1 point] What is the difference between Lagrange's equation and Lagrangian in vibration analysis? (one sentence only)

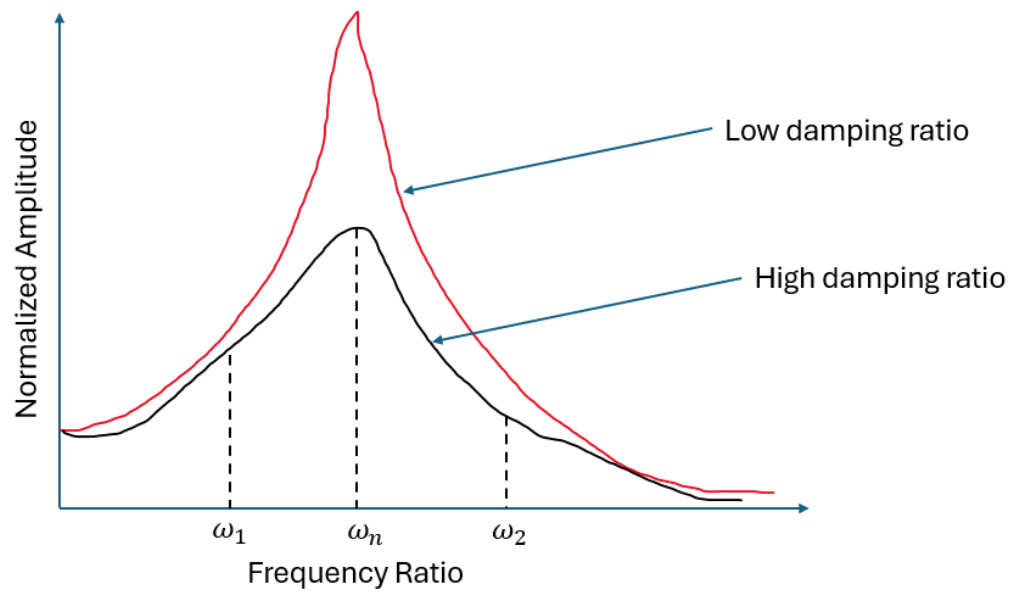
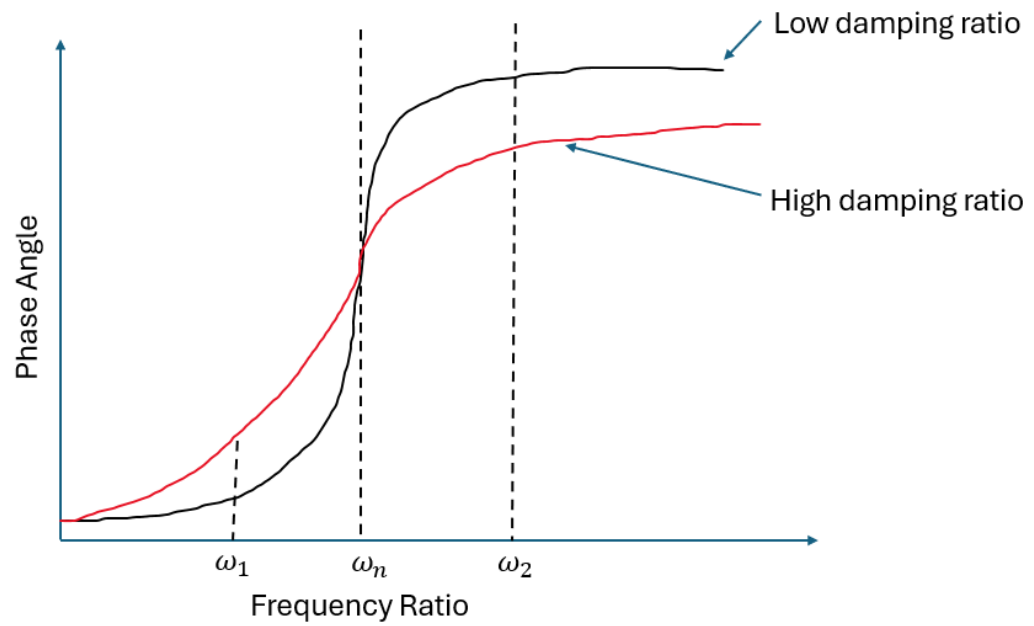
⇒ The Lagrangian is a scalar function defined as the difference between kinetic and potential energy ($L = T - V$), whereas Lagrange's equation is the differential equation of motion derived by applying operations to that function. (Ref: [Rao] Chapter 6).

5. [1 point] What are the advantages of using virtual displacement and virtual work?

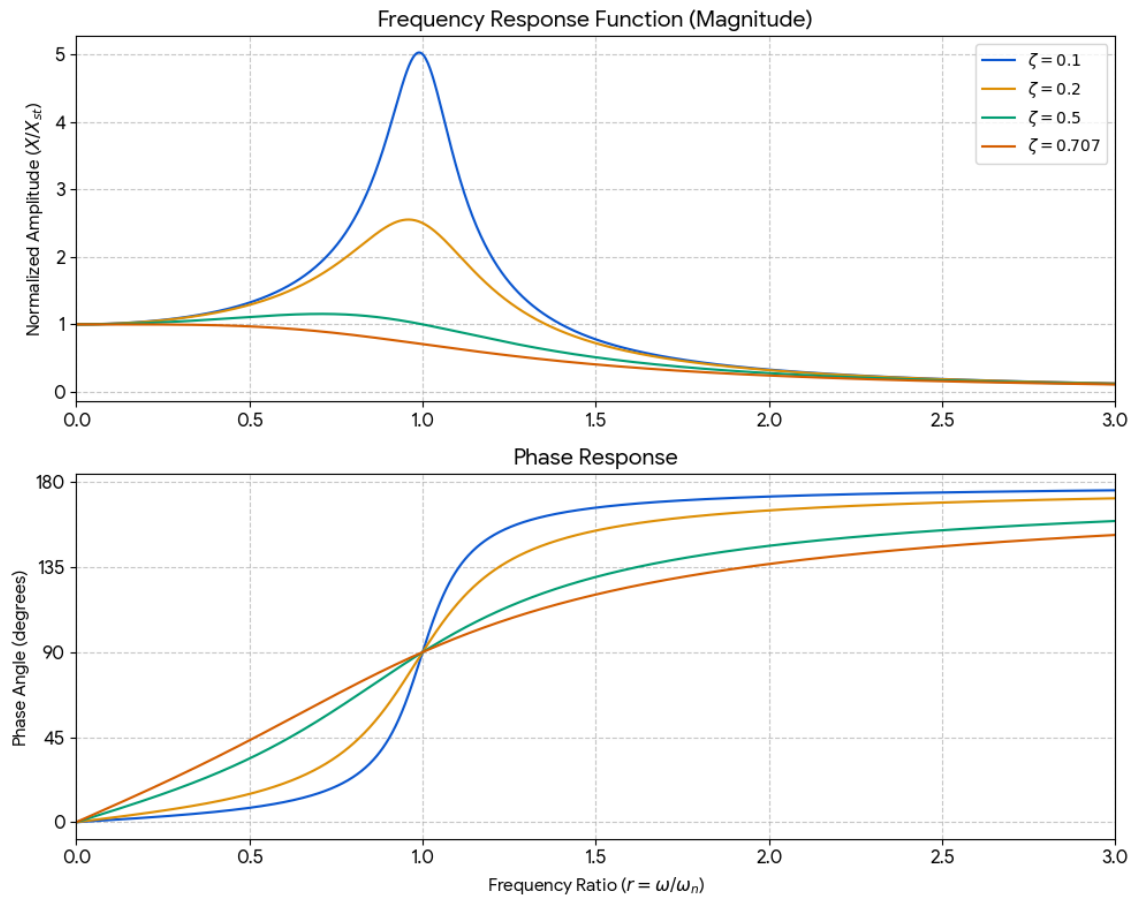
⇒ Here are the advantages of virtual displacement and virtual work approach:

- Elimination of constraint forces: constraint forces do not appear in the formulation which helps opting out solving the internal reaction forces to establish equation of motions.

- Use of generalized coordinates: As we choose a set of independent coordinates that matches the system's degree of freedoms naturally, we can reduce the number of equations to the minimum necessary number of degree of freedom
 - Applicability to multi-body system: as we can treat the entire interconnected system as a whole, we can opt out solving each individual component and interaction forces between them like Newtonian approach
6. **[1 point] What is the definition of virtual work? (one sentence only)**
 \Rightarrow Virtual work is the total work done by all applied forces acting through the virtual displacements of the system. (Ref: [Meirovitch] Chapter 1).
 7. **[1 point] For the virtual work approach, when is equilibrium attained? (one sentence only)**
 \Rightarrow Equilibrium is attained when the total virtual work done by all active forces and inertial forces vanishes (equals zero) for any arbitrary virtual displacement. (Ref: [Ginsberg] Chapter 3).
 8. **[1 point] How many normal modes of vibration are in the Boeing 777 wing? (one sentence only)**
 \Rightarrow As a continuous physical structure, the wing possesses an infinite number of normal modes of vibration, though in engineering practice, it is approximated by a finite but large number of degrees of freedom. (Ref: [Inman] Chapter 6 - Continuous Systems).
 9. **[3 point] Sketch by hand the FRF and phase response of a single degree of freedom spring-mass system with a damper. Plot the effect of increasing and decreasing the damping ratio, ζ , on the natural frequency? \Rightarrow**



We can plot as following



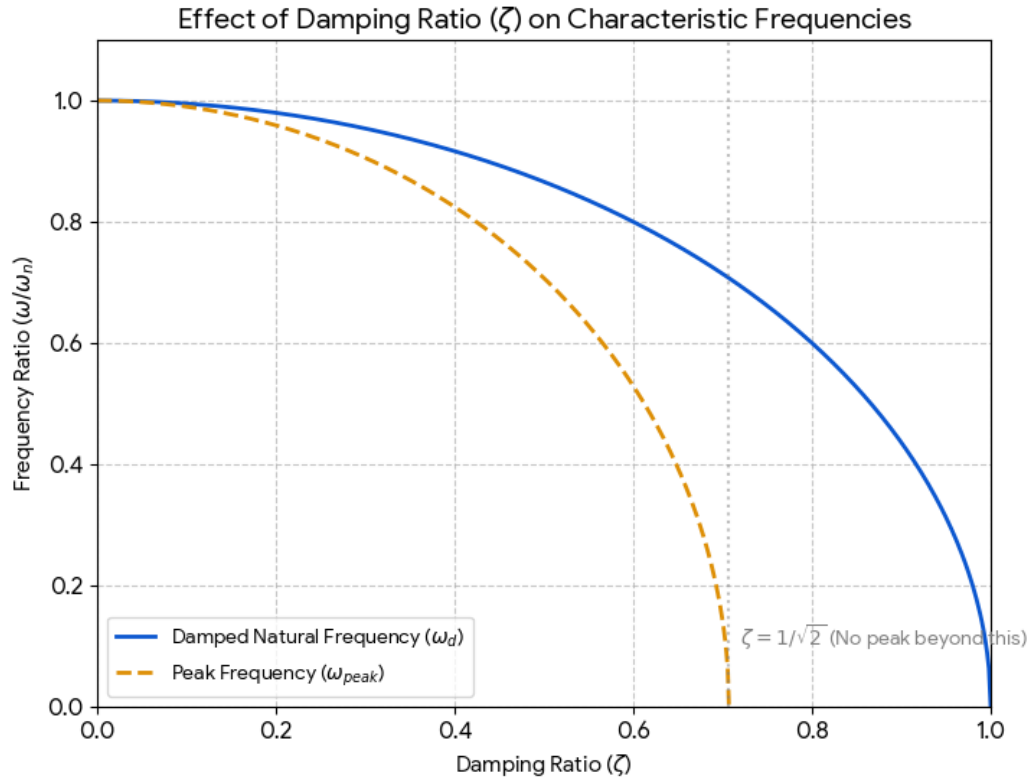


Figure 1: Effect of increasing and decreasing the damping ratio

10. [2 point] Show on FRF plots how to obtain stiffness, damping, and mass of a 1-DoF system without knowing anything about the specifications of the structure

⇒ We have the receptance FRF

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{k - m^2\omega^2 + i\omega c}$$

To extract parameters from an experimental FRF (Bode plot) of Compliance (X/F):

Stiffness (k): Look at the low-frequency asymptote (where $\omega \rightarrow 0$). The magnitude of the response approaches $1/k$. Thus, $k = 1/|H(\omega)|_{\omega \approx 0}$.

Mass (m): Look at the high-frequency asymptote (mass-controlled region). The slope is -40 dB/decade. Pick a point on this line; the magnitude is

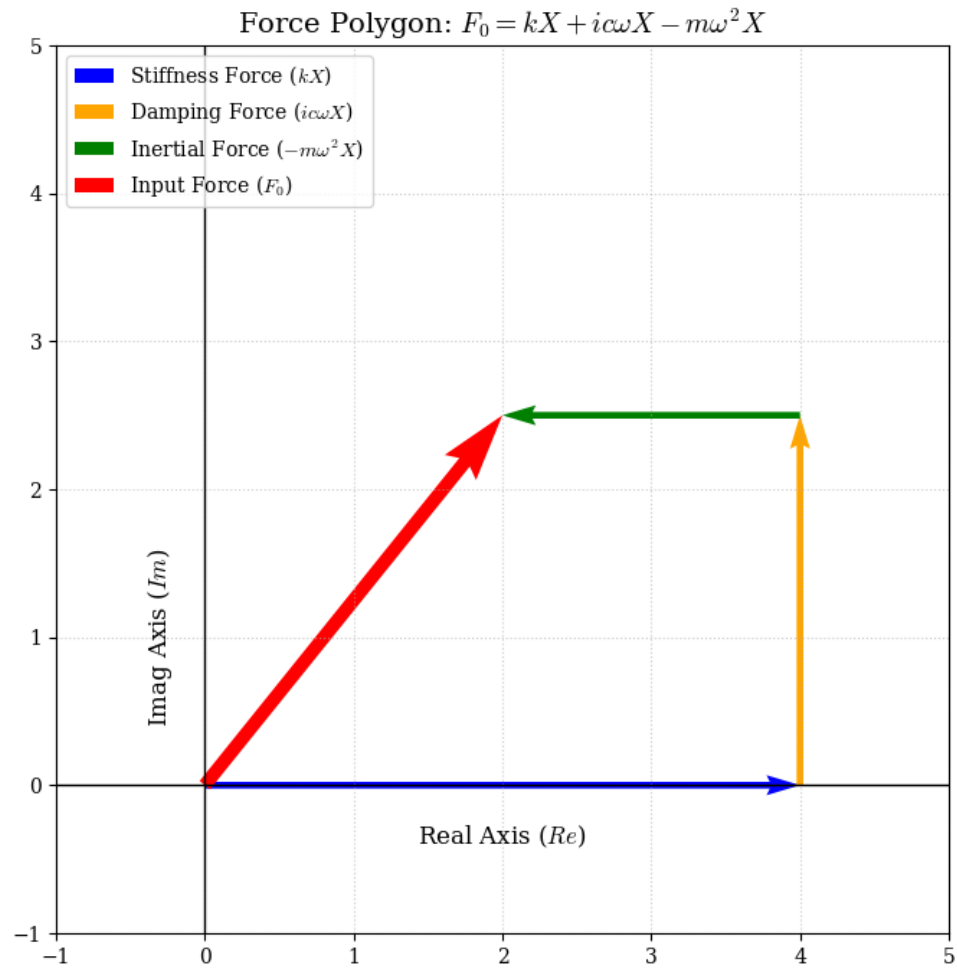
$1/(m\omega^2)$. Thus, $m = 1/(|H(\omega)| \cdot \omega^2)$.

Damping (c or ζ): Look at the resonance peak. Use the "Half-Power Bandwidth" method. Identify the peak frequency ω_n and the frequencies ω_1, ω_2 on either side where the amplitude is $0.707(-3\text{dB})$ of the peak.

$$\zeta \approx \frac{\omega_2 - \omega_1}{2\omega_n}$$

Once ζ , k , and m are known, $c = 2\zeta\sqrt{km}$.

11. **[1 point] What is the difference between the Complex Response Function and the Frequency Response Function? (one sentence only)**
 \implies The Complex Response Function is generally a function of the complex variable s (Laplace domain), covering transient and steady states, while the Frequency Response Function is the specific evaluation of that function along the imaginary axis ($s = i\omega$) describing only the steady-state sinusoidal response. (Ref: [Inman] Chapter 3).
12. **[4 point] In the complex domain, show the inertial, damping, stiffness, and input forces graphically**



13. [2 point] The figure below shows an aircraft with left and right structural symmetry points. Each point has six degrees of freedom. For example, the left point has three translations: x, y, z , and three rotations: θ (rotation about x), θ (rotation about y) and θ (rotation about z).

Let the movements on coordinate system on each side is describe as following

$$Q_L = \begin{bmatrix} x_L \\ y_L \\ z_L \\ \theta_{xL} \\ \theta_{yL} \\ \theta_{zL} \end{bmatrix}$$

$$Q_R = \begin{bmatrix} x_R \\ y_R \\ z_R \\ \theta_{xR} \\ \theta_{yR} \\ \theta_{zR} \end{bmatrix}$$

- (a) **For symmetric modes, write an expression relating the right and left points' degrees of freedom.**

\implies Symmetric Modes (e.g., wing bending up simultaneously): Left and right points move in mirror images across the plane of symmetry.

$$x_L = -x_R \text{ (Lateral motion is opposite)}$$

$$y_L = y_R \text{ (Longitudinal motion is identical)}$$

$$z_L = z_R \text{ (Vertical motion is identical)}$$

$$\theta_{xL} = -\theta_{xR} \text{ (Bank/Roll is opposite)}$$

$$\theta_{yL} = \theta_{yR} \text{ (Pitch is identical)}$$

$$\theta_{zL} = -\theta_{zR} \text{ (Yaw is opposite)}$$

- (b) **For unsymmetric modes, write an expression relating the right and left points' degrees of freedom.**

\implies Unsymmetric (Antisymmetric) Modes (e.g., aircraft rolling): Left and right points move opposite to the mirror image.

$$x_L = x_R$$

$$y_L = -y_R$$

$$z_L = -z_R$$

$$\theta_{xL} = \theta_{xR}$$

$$\theta_{yL} = -\theta_{yR}$$

$$\theta_{zL} = \theta_{zR}$$

14. [1 point] How long did it take you to complete this exam? 2 days

1.1 Reference:

Ginsberg, J. H. (2001). Mechanical and Structural Vibrations. John Wiley & Sons.
Inman, D. J. (2013). Engineering Vibration (4th ed.). Pearson.
Rao, S. S. (2016). Mechanical Vibrations (6th ed.). Pearson.
Meirovitch, L. (2010). Fundamentals of Vibrations. Waveland Press. (Original work published 2001 by McGraw-Hill).

2 Section 2: Short Analytical Problem

2.1 Problem 1:

2.1.1 Part a:

Given $y(x) = y_{tip} \frac{x^2}{L^2}$, we have

$$\frac{dy}{dx} = y_{tip} \frac{2x}{L^2}, \text{ and } \frac{d^2y}{dx^2} = y_{tip} \frac{2}{L^2}$$

From Rayleigh's quotient, we have the ratio of maximum potential energy and reference kinetic defined as

$$\omega^2 = \frac{PE_{max}}{KE_{ref}} = \frac{\int_0^L EI \left(\frac{d^2y}{dx^2} \right)^2 dx}{\int_0^L m [y(x)]^2 dx}$$

- **Potential energy:**

$$\int_0^L EI \left(\frac{d^2y}{dx^2} \right)^2 dx = \int_0^L EI \left(y_{tip} \frac{2}{L^2} \right)^2 dx = \frac{4EI y_{tip}^2}{L^4} [x]_0^L = \frac{4EI y_{tip}^2 L}{L^3}$$

- **Kinetic energy:**

$$\int_0^L m [y(x)]^2 dx = \int_0^L m \left[y_{tip} \frac{x^2}{L^2} \right]^2 dx = m \frac{y_{tip}^2}{L^4} \left[\frac{x^5}{5} \right]_0^L = \frac{m y_{tip}^2 L}{5}$$

Therefore, we can solve for the natural frequency as following

$$\omega^2 = \frac{\frac{4EI y_{tip}^2 L}{L^3}}{\frac{m y_{tip}^2 L}{5}} = \frac{20EI}{mL^4}$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{20EI}{mL^4}} \approx 4.472 \sqrt{\frac{EI}{mL^4}}}$$

2.1.2 Part b:

For a uniform Euler-Bernoulli beam equation we have the characteristic equation as

$$\cos(\beta L) \cosh(\beta L) = -1$$

and for the first root of the equation is $\beta_1 L = 1.8751$. For the exact fundamental frequency, we have:

$$\omega_1 = (\beta_1 L)^2 \sqrt{\frac{EI}{mL^4}} = (1.8751)^2 \sqrt{\frac{EI}{mL^4}} = 3.516 \sqrt{\frac{EI}{mL^4}}$$

We can observe that the Rayleigh's quotient overestimate the natural frequency by

$$\text{Error} = \frac{4.472 - 3.516}{3.516} \times 100\% = \boxed{27.2\%}$$

2.2 Problem 2:

2.2.1 Part a:

We have

$$\omega_n = 2\pi f_n = 2\pi(10) = 20\pi \text{ [rad/s]}$$

$$\zeta = 0.05 \Rightarrow c = 2\zeta\omega_n = 2(0.05)(20\pi) = 2\pi$$

$$m = 1$$

Assume the motion is harmonic, the general solution is $e^{i\omega t}$. We have the governing dynamic stiffness is $Z(\omega) = k - \omega^2 + i c \omega$ for $m = 1$, and $\omega^2 = \frac{k}{m} \Rightarrow k = \omega^2 = (20\pi)^2 = 400\pi^2$. Therefore, we have

$$Z(\omega) = 400\pi^2 - \omega^2 + i(0.05)\omega$$

- Complex receptance (displacement per unit force)

$$\alpha(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{Z(\omega)}$$

$$\alpha(\omega) = \frac{1}{400\pi^2 - \omega^2 + i(0.05)\omega}$$

- Complex mobility (velocity per unit force)

$$Y(\omega) = \frac{V(\omega)}{F(\omega)} = i\omega\alpha(\omega)$$

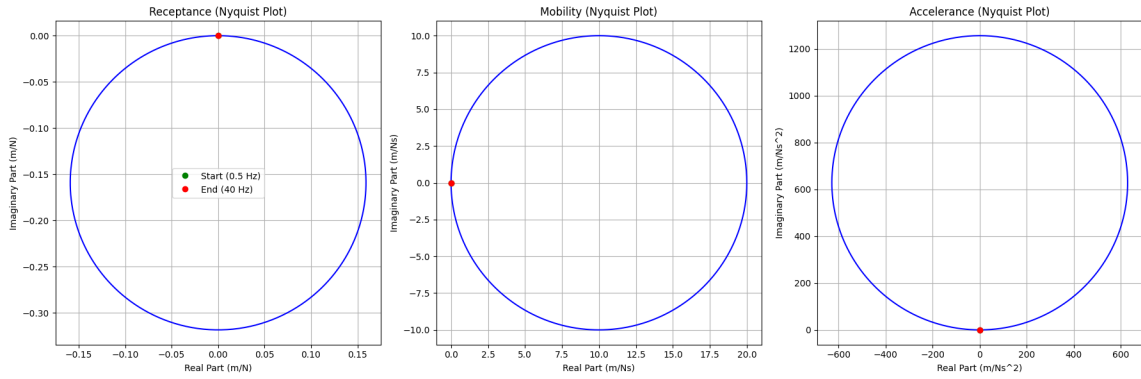
$$Y(\omega) = \frac{i\omega}{400\pi^2 - \omega^2 + i(0.05)\omega}$$

- Complex acceleration (acceleration per unit force)

$$A(\omega) = \frac{A(\omega)}{F(\omega)} = (i\omega)^2\alpha(\omega)$$

$$A(\omega) = \frac{-\omega^2}{400\pi^2 - \omega^2 + i(0.05)\omega}$$

2.2.2 Part b:



```

1  import numpy as np
2  import matplotlib.pyplot as plt
3
4  # System Parameters
5  m = 1.0 # kg
6  fn = 10.0 # Hz
7  c = 0.05 # Ns/m
8
9  # Derived Parameters
10 wn = 2 * np.pi * fn # rad/s
11 k = m * wn**2 # N/m
12
13 # Frequency Range
14 f_start = 0.5
15 f_end = 40.0
16 f = np.linspace(f_start, f_end, 1000000)
17 omega = 2 * np.pi * f
18
19 # Denominator (Dynamic Stiffness)
20 #  $D(\omega) = (k - m\omega^2) + i(c\omega)$ 
21 D = (k - m * omega**2) + 1j * (c * omega)
22
23 # Frequency Response Functions
24 # Receptance (Displacement / Force)
25 alpha = 1.0 / D
26
27 # Mobility (Velocity / Force)
28 mobility = 1j * omega * alpha
29
30 # Accelerance (Acceleration / Force)
31 accelerance = -omega**2 * alpha
32
33 # Plotting
34 plt.figure(figsize=(18, 6))
35
36 # Receptance Plot
37 plt.subplot(1, 3, 1)
38 plt.plot(np.real(alpha), np.imag(alpha), 'b-')
39 plt.title('Receptance (Nyquist Plot)')
40 plt.xlabel('Real Part (m/N)')
41 plt.ylabel('Imaginary Part (m/N)')
42 plt.grid(True)
43 # Mark start and end points for direction
44 plt.plot(np.real(alpha[0]), np.imag(alpha[0]), 'go', label='Start (0.5 Hz)')
45 plt.plot(np.real(alpha[-1]), np.imag(alpha[-1]), 'ro', label='End (40 Hz)')

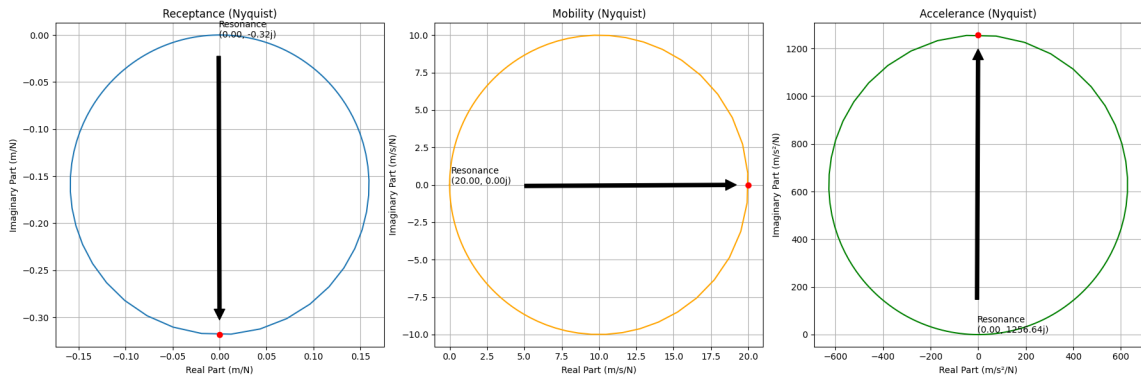
```

```

46 plt.legend()
47
48 # Mobility Plot
49 plt.subplot(1, 3, 2)
50 plt.plot(np.real(mobility), np.imag(mobility), 'b-')
51 plt.title('Mobility (Nyquist Plot)')
52 plt.xlabel('Real Part (m/Ns)')
53 plt.ylabel('Imaginary Part (m/Ns)')
54 plt.grid(True)
55 plt.plot(np.real(mobility[0]), np.imag(mobility[0]), 'go')
56 plt.plot(np.real(mobility[-1]), np.imag(mobility[-1]), 'ro')
57
58
59 # Accelerance Plot
60 plt.subplot(1, 3, 3)
61 plt.plot(np.real(accelerance), np.imag(accelerance), 'b-')
62 plt.title('Accelerance (Nyquist Plot)')
63 plt.xlabel('Real Part (m/Ns^2)')
64 plt.ylabel('Imaginary Part (m/Ns^2)')
65 plt.grid(True)
66 plt.plot(np.real(accelerance[0]), np.imag(accelerance[0]), 'go')
67 plt.plot(np.real(accelerance[-1]), np.imag(accelerance[-1]), 'ro')
68
69 plt.tight_layout()
70 plt.savefig('nyquist_plots.png')

```

2.2.3 Part c:



```

1 import numpy as np
2 import matplotlib.pyplot as plt
3
4 # System Parameters

```

```

5  m = 1.0 # kg
6  fn = 10.0 # Hz
7  zeta = 0.05
8  wn = 2 * np.pi * fn # rad/s
9  k = m * wn**2 # N/m
10 c = 0.05
11 # Frequency Range for the curve
12 f = np.linspace(0.5, 40, 100000)
13 w = 2 * np.pi * f
14
15 # Calculate FRFs for the range
16 alpha = 1 / (k - m * w**2 + 1j * c * w)
17 Y = 1j * w * alpha
18 A = -w**2 * alpha
19
20 # Calculate FRFs specifically at Natural Frequency
21 w_n_point = wn
22 alpha_n = 1 / (k - m * w_n_point**2 + 1j * c * w_n_point)
23 Y_n = 1j * w_n_point * alpha_n
24 A_n = -w_n_point**2 * alpha_n
25
26 # Plotting
27 fig, axs = plt.subplots(1, 3, figsize=(18, 6))
28
29 # Receptance Plot
30 axs[0].plot(np.real(alpha), np.imag(alpha), label='Receptance Path')
31 axs[0].scatter(np.real(alpha_n), np.imag(alpha_n), color='red', zorder=5, label='Natural Freq')
32 axs[0].annotate(f'Resonance\n({np.real(alpha_n):.2f}, {np.imag(alpha_n):.2f}j)',
33                xy=(np.real(alpha_n), np.imag(alpha_n)), xytext=(-0.001, -0.003),
34                arrowprops=dict(facecolor='black', shrink=0.05))
35 axs[0].set_title('Receptance (Nyquist)')
36 axs[0].set_xlabel('Real Part (m/N)')
37 axs[0].set_ylabel('Imaginary Part (m/N)')
38 axs[0].grid(True)
39 axs[0].axis('equal')
40
41 # Mobility Plot
42 axs[1].plot(np.real(Y), np.imag(Y), color='orange', label='Mobility Path')
43 axs[1].scatter(np.real(Y_n), np.imag(Y_n), color='red', zorder=5, label='Natural Freq')
44 axs[1].annotate(f'Resonance\n({np.real(Y_n):.2f}, {np.imag(Y_n):.2f}j)',
45                xy=(np.real(Y_n), np.imag(Y_n)), xytext=(0.1, 0.05),
46                arrowprops=dict(facecolor='black', shrink=0.05))
47 axs[1].set_title('Mobility (Nyquist)')
48 axs[1].set_xlabel('Real Part (m/s/N)')
49 axs[1].set_ylabel('Imaginary Part (m/s/N)')
50 axs[1].grid(True)

```

```

51  axs[1].axis('equal')
52
53  # Accelerance Plot
54  axs[2].plot(np.real(A), np.imag(A), color='green', label='Accelerance Path')
55  axs[2].scatter(np.real(A_n), np.imag(A_n), color='red', zorder=5, label='Natural Freq')
56  axs[2].annotate(f'Resonance\n({np.real(A_n):.2f}, {np.imag(A_n):.2f}j)',
57                  xy=(np.real(A_n), np.imag(A_n)), xytext=(-5, 8),
58                  arrowprops=dict(facecolor='black', shrink=0.05))
59  axs[2].set_title('Accelerance (Nyquist)')
60  axs[2].set_xlabel('Real Part (m/s2/N)')
61  axs[2].set_ylabel('Imaginary Part (m/s2/N)')
62  axs[2].grid(True)
63  axs[2].axis('equal')
64
65  plt.tight_layout()
66  plt.savefig('nyquist_plots_resonance.png')
67

```

2.2.4 Part d:

The Mobility plot is the best choice for this because, for a viscous damped system, it forms a circle passing through the origin.

1. Identify the Natural Frequency (f_n) Locate the point with the Maximum Real Amplitude (the right-most point of the circle). The frequency at this point is your natural frequency, f_n . From your data: $f_n = 10$ Hz.
2. Identify the Half-Power Points (f_1 and f_2) On the Mobility Nyquist circle, the half-power points are geometrically distinct:
 - f_1 : The frequency at the Maximum Imaginary value (the very top of the circle).
 - f_2 : The frequency at the Minimum Imaginary value (the very bottom of the circle).

(Note: At these points, the real part is exactly half of the maximum real amplitude).

3. Apply the Half-Power Bandwidth Formula Once you have the frequencies for the top and bottom of the circle, use this formula:

$$\zeta = \frac{f_2 - f_1}{2f_n}$$

Example Walkthrough with Your Data: If we look at the Mobility plot generated in Part B: Resonance (f_n): We know this is 10 Hz. Bandwidth Points: We know $\zeta = 0.05$. Using the formula in reverse, the bandwidth ($f_2 - f_1$) should be:

$$f_2 - f_1 = 2 \cdot \zeta \cdot f_n$$

$$f_2 - f_1 = 2 \cdot 0.05 \cdot 10 = 1 \text{ Hz}$$

Therefore, if you hovered over the top of the circle, you would find $f_1 \approx 9.5$ Hz. If you hovered over the bottom of the circle, you would find $f_2 \approx 10.5$ Hz.

$$\zeta = \frac{10.5 - 9.5}{2(10)} = \frac{1}{20} = \mathbf{0.05}$$

2.3 Problem 3:

Since the mode shapes are orthonormal such that the mass matrix is the identity matrix ($[I]$), the system is decoupled in the modal domain. We can construct the equations of motion in terms of the modal coordinates (q). The general equation of motion in modal coordinates is:

$$[I]\{\ddot{q}\} + [\hat{C}]\{\dot{q}\} + [\hat{K}]\{q\} = \{0\}$$

Where:

- $[I]$ is the Modal Mass matrix (Identity).
- $[\hat{C}]$ is the Modal Damping matrix (diagonal, with terms $2\zeta_i\omega_i$).
- $[\hat{K}]$ is the Modal Stiffness matrix (diagonal, with terms ω_i^2).

The given frequencies f (Hz) must be converted to angular natural frequencies ω_n (rad/s) using $\omega = 2\pi f$.

Mode 1: $\omega_1 = 2\pi(10) = 20\pi \approx 62.83$ rad/s

Mode 2: $\omega_2 = 2\pi(15) = 30\pi \approx 94.25$ rad/s

Mode 3: $\omega_3 = 2\pi(25) = 50\pi \approx 157.08$ rad/s

Modal Stiffness Elements (ω_n^2):

$$k_1 = (20\pi)^2 = 400\pi^2 \approx 3,948$$

$$k_2 = (30\pi)^2 = 900\pi^2 \approx 8,883$$

$$k_3 = (50\pi)^2 = 2500\pi^2 \approx 24,674$$

Modal Damping Elements ($2\zeta_n\omega_n$):

$$c_1 = 2(0.015)(20\pi) = 0.6\pi \approx 1.88$$

$$c_2 = 2(0.020)(30\pi) = 1.2\pi \approx 3.77$$

$$c_3 = 2(0.030)(50\pi) = 3.0\pi \approx 9.42$$

Since the physical mode shape vectors (Φ) are not provided, we cannot transform back to physical coordinates (x). The equations of motion are expressed in the modal coordinates vector $\{q\} = [q_1, q_2, q_3]^T$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} 1.88 & 0 & 0 \\ 0 & 3.77 & 0 \\ 0 & 0 & 9.42 \end{bmatrix} \begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{Bmatrix} + \begin{bmatrix} 3,948 & 0 & 0 \\ 0 & 8,883 & 0 \\ 0 & 0 & 24,674 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

2.4 Problem 4:

2.4.1 Part a:

Mass per unit length (m):

$$m = \rho \times A$$

$$m = 7,850 \times 1.501 \times 10^{-5} \approx 0.1178 \text{ kg/m}$$

Cross-section area (A)

$$A = \frac{\pi}{4}(d_o^2 - d_i^2) = \frac{\pi}{4}(0.008^2 - 0.0067^2) \approx 1.501 \times 10^{-5} \text{ m}^2$$

Second moment of area (I)

$$I = \frac{\pi}{64}(r_o^4 - r_i^4) = \frac{\pi}{64}(0.008^4 - 0.0067^4) = 1.021 \times 10^{-10} [\text{m}^4]$$

As we assume the hydraulic pipe is a simply supported uniform beam, the first bending natural frequency is

$$\omega_1 = \frac{\pi^2}{L^2} \sqrt{\frac{EI}{mL^4}} = \frac{\pi^2}{0.432^2} \sqrt{\frac{(205 \times 10^9)(1.021 \times 10^{-10})}{(0.1178)(0.432)^4}} \approx \pi^2 \sqrt{5101.1} = 704.9 \text{ [rad/s]}$$

$$f_n = \frac{\omega_n}{2\pi} \approx \frac{704.9}{6.283} \approx 112.2 \text{ Hz}$$

Check: The calculated frequency (112.2 Hz) falls within the excitation range of 50 to 500 Hz. Therefore, the pipe will excite resonance, and we must calculate the displacement at this worst-case frequency.

For mode shape $\phi(x) = \sin\left(\frac{\pi x}{L}\right)$, and participation Factor (Γ): For a simply supported beam, $\Gamma = \frac{4}{\pi} \approx 1.273$. We have maximum relative displacement amplitude (Z_{peak}) at the center of the beam at resonance is:

$$Z_{peak} = \Gamma \times \frac{A_{base}}{2\zeta\omega_n^2}$$

Where:

- Γ : $4/\pi$ (Modal participation factor)
- A_{base} : Base acceleration amplitude = $5.0g = 5.0 \times 9.81 = 49.05 \text{ m/s}^2$
- ζ : Damping ratio = 0.02
- ω_n : Natural frequency $\approx 704.9 \text{ rad/s}$

Static Equivalent Deflection (A_{base}/ω_n^2):

$$\frac{49.05}{704.9^2} = \frac{49.05}{496884} \approx 9.871 \times 10^{-5} \text{ m}$$

Apply Damping ($1/2\zeta$):

$$\frac{1}{2 \times 0.02} = \frac{1}{0.04} = 25$$

(This is the Q-factor or dynamic amplification factor) Apply Modal Factor ($\Gamma = 4/\pi = 1.273$): Total Displacement (Z_{peak}):

$$Z_{peak} = 1.273 \times 25 \times (9.871 \times 10^{-5})$$

$$Z_{peak} \approx 31.825 \times (9.871 \times 10^{-5})$$

$$Z_{peak} \approx 0.00314 \text{ m}$$

Therefore, the vertical single-peak displacement at the center of the pipe span relative to the supporting structure is 3.14 [mm]

2.4.2 Part b

From the maximum deflection of $\delta_{max} = 3.14$ [mm]. For a beam, the bending stress (σ) is related to the bending moment (M) and the distance from the neutral axis (c) by the flexure formula:

$$\sigma = \frac{Mc}{I}$$

Since the problem assumes the fundamental mode of a simply supported beam, the mode shape is sinusoidal: $y(x) = Z_{peak} \sin\left(\frac{\pi x}{L}\right)$

The bending moment is proportional to the curvature (second derivative of deflection):

$$M(x) = EI \frac{d^2 y}{dx^2}$$

$$\frac{d^2 y}{dx^2} = -Z_{peak} \left(\frac{\pi}{L}\right)^2 \sin\left(\frac{\pi x}{L}\right)$$

Substituting the maximum curvature (at the center, $x = L/2$, where $\sin = 1$) into the stress formula:

$$\sigma_{max} = \frac{EI \left| \frac{d^2 y}{dx^2} \right|_{max} c}{I} = E \cdot c \cdot Z_{peak} \left(\frac{\pi}{L}\right)^2$$

where:

- Young's Modulus (E): 205×10^9 Pa
- Pipe Outer Radius (c): $d_o/2 = 8.0 \text{ mm}/2 = 4.0 \text{ mm} = 0.004 \text{ m}$
- Span Length (L): 0.432 m
- Max Deflection (Z_{peak}): 0.00314 m .

Substitute the values into the derived equation:

$$\sigma_{max} = (205 \times 10^9) \times (0.004) \times (0.00314) \times \left(\frac{\pi}{0.432}\right)^2$$

Geometric term $(\pi/L)^2$:

$$\left(\frac{3.14159}{0.432}\right)^2 \approx (7.272)^2 \approx 52.88 \text{ m}^{-2}$$

Combine constants:

$$\sigma_{max} = (205 \times 10^9) \times (0.004) \times (0.00314) \times (52.88)$$

Multiply:

$$\sigma_{max} \approx 205 \times 10^9 \times 6.642 \times 10^{-4}$$

$$\sigma_{max} \approx 136,161,000 \text{ Pa}$$

$$\sigma_{max} \approx 136.2 \text{ [MPa]}$$

3 Section 3: Critical Thinking

3.1 Case I: Response to turbulence

3.1.1 Part a:

The input forces P_1 and P_2 are correlated because they originate from the same upstream turbulent airflow. This correlation means the "cross-talk" between the forces significantly affects the total energy at the tip. a. Acceleration Spectral Density (ASD) of the tip To write the Acceleration Spectral Density (S_{aa}) at the wing tip, we treat this as a Multiple-Input Single-Output (MISO) system.

1. Define the Frequency Response Functions (FRFs): Let $H_1(\omega)$ be the FRF relating force P_1 to the tip acceleration. Let $H_2(\omega)$ be the FRF relating force P_2 to the tip acceleration. Let ω be the angular frequency.

2. Define the Input Spectra: $S_{P_1 P_1}(\omega)$ and $S_{P_2 P_2}(\omega)$ are the Auto-Power Spectral Densities (PSD) of forces P_1 and P_2 . $S_{P_1 P_2}(\omega)$ and $S_{P_2 P_1}(\omega)$ are the Cross-Power Spectral Densities (CPSD) between the two forces.

3. The ASD Equation: Because the forces are correlated (meaning $S_{P_1 P_2} \neq 0$), the total output spectrum is the sum of the individual responses plus the interaction (interference) terms between them. The equation for the Acceleration Spectral Density at the tip ($S_{aa}(\omega)$) is:

$$S_{aa}(\omega) = \underbrace{|H_1(\omega)|^2 S_{P_1 P_1}(\omega)}_{\text{Contribution from } P_1} + \underbrace{|H_2(\omega)|^2 S_{P_2 P_2}(\omega)}_{\text{Contribution from } P_2} + \underbrace{H_1^*(\omega) H_2(\omega) S_{P_1 P_2}(\omega) + H_2^*(\omega) H_1(\omega) S_{P_2 P_1}(\omega)}_{\text{Interaction terms due to Correlation}}$$

Since $S_{P_2 P_1}(\omega)$ is the complex conjugate of $S_{P_1 P_2}(\omega)$, the last two terms are complex conjugates of each other. Their sum is twice the real part. The equation is often written as:

$$S_{aa}(\omega) = |H_1(\omega)|^2 S_{P_1 P_1}(\omega) + |H_2(\omega)|^2 S_{P_2 P_2}(\omega) + 2 \cdot \text{Re} [H_1^*(\omega) H_2(\omega) S_{P_1 P_2}(\omega)]$$

3.1.2 Part b:

The variance (σ_d^2) is the integral of the displacement power spectral density ($S_{dd}(\omega)$) over the frequency range. Since acceleration is the second derivative of displacement ($a = \ddot{x}$), the relationship in the frequency domain involves dividing the acceleration spectrum by ω^4 .

$$\sigma_d^2 = \int_{-\infty}^{\infty} S_{dd}(\omega) d\omega = \int_{-\infty}^{\infty} \frac{S_{aa}(\omega)}{\omega^4} d\omega$$

Substituting the $S_{aa}(\omega)$ from part (a):

$$\sigma_d^2 = \int_{-\infty}^{\infty} \frac{1}{\omega^4} \left[|H_1(\omega)|^2 S_{P_1 P_1}(\omega) + |H_2(\omega)|^2 S_{P_2 P_2}(\omega) + 2 \operatorname{Re} (H_1^*(\omega) H_2(\omega) S_{P_1 P_2}(\omega)) \right] d\omega$$

3.2 Case II: Forces from two different jet engines

3.2.1 Part a:

In this case, the forces are uncorrelated. This means the cross-power spectral density is zero ($S_{P_1 P_2}(\omega) = S_{P_2 P_1}(\omega) = 0$).

a. Acceleration Spectral Density (ASD) of the tip Because the inputs are uncorrelated, the cross-terms vanish. The total output ASD is simply the superposition of the ASDs generated by each independent force.

$$S_{aa}(\omega) = |H_1(\omega)|^2 S_{P_1 P_1}(\omega) + |H_2(\omega)|^2 S_{P_2 P_2}(\omega)$$

Where:

- $S_{aa}(\omega)$ is the Acceleration Spectral Density at the wing tip.
- $|H_1(\omega)|^2$ is the squared magnitude of the FRF relating force P_1 to the tip acceleration.
- $S_{P_1 P_1}(\omega)$ is the Auto-Spectral Density of force P_1 .
- $|H_2(\omega)|^2$ is the squared magnitude of the FRF relating force P_2 to the tip acceleration.
- $S_{P_2 P_2}(\omega)$ is the Auto-Spectral Density of force P_2 .

3.2.2 Part b:

Variance of the displacement at the wing tip Similar to Case I, we integrate the displacement PSD. However, we use the simplified uncorrelated spectrum derived above.

$$\sigma_d^2 = \int_{-\infty}^{\infty} S_{dd}(\omega) d\omega = \int_{-\infty}^{\infty} \frac{S_{aa}(\omega)}{\omega^4} d\omega$$

$$\sigma_d^2 = \int_{-\infty}^{\infty} \frac{1}{\omega^4} [|H_1(\omega)|^2 S_{P_1 P_1}(\omega) + |H_2(\omega)|^2 S_{P_2 P_2}(\omega)] d\omega$$

3.3 Case III: Dynamic properties of aircraft wing

3.3.1 Part a:

Let $F(\omega)$ be the Fourier Transform of the known shaker force $F(t)$. Let $A_i(\omega)$ be the Fourier Transform of the acceleration response at location i (where $i = 1, 2, 3, 4$). Let $N_i(\omega)$ be the response caused by the unknown turbulent wind forces (noise). The total response measured at any accelerometer i is the sum of the response to the shaker and the response to the wind:

$$A_i(\omega) = H_i(\omega)F(\omega) + N_i(\omega)$$

To isolate $H_i(\omega)$, we multiply both sides of the equation by $F^*(\omega)$, the complex conjugate of the known input force.

$$F^*(\omega)A_i(\omega) = F^*(\omega)[H_i(\omega)F(\omega) + N_i(\omega)]$$

$$F^*(\omega)A_i(\omega) = H_i(\omega)[F^*(\omega)F(\omega)] + F^*(\omega)N_i(\omega)$$

We take the expected value $E[\cdot]$ (ensemble average) of the terms. This converts the raw Fourier products into Power Spectral Densities (PSD).

$$E[F^*(\omega)A_i(\omega)] = H_i(\omega)E[F^*(\omega)F(\omega)] + E[F^*(\omega)N_i(\omega)]$$

Using standard PSD notation (S_{xy}): $S_{Fa_i}(\omega) = E[F^*(\omega)A_i(\omega)]$ (Cross-PSD of Force and Acceleration) $S_{FF}(\omega) = E[F^*(\omega)F(\omega)]$ (Auto-PSD of the Force) $S_{FN_i}(\omega) = E[F^*(\omega)N_i(\omega)]$ (Cross-PSD of Force and Wind Noise) The equation becomes:

$$S_{Fa_i}(\omega) = H_i(\omega)S_{FF}(\omega) + S_{FN_i}(\omega)$$

This is the critical step. Because the shaker force $F(t)$ is controlled and the wind turbulence is random and external, they are uncorrelated. Therefore, the Cross-PSD between the shaker force and the wind noise is zero:

$$S_{FN_i}(\omega) \approx 0$$

The equation simplifies to:

$$S_{Fa_i}(\omega) = H_i(\omega)S_{FF}(\omega)$$

Rearranging to solve for the FRF:

$$H_i(\omega) = \frac{S_{Fa_i}(\omega)}{S_{FF}(\omega)}$$

Final Answer: The Frequency Response Functions for the four locations ($i = 1, 2, 3, 4$) are:

$$H_1(\omega) = \frac{S_{Fa_1}(\omega)}{S_{FF}(\omega)}$$

$$H_2(\omega) = \frac{S_{Fa_2}(\omega)}{S_{FF}(\omega)}$$

$$H_3(\omega) = \frac{S_{Fa_3}(\omega)}{S_{FF}(\omega)}$$

$$H_4(\omega) = \frac{S_{Fa_4}(\omega)}{S_{FF}(\omega)}$$

Where: $S_{Fa_i}(\omega)$ is the Cross-Power Spectral Density between the known force and the i -th accelerometer. $S_{FF}(\omega)$ is the Auto-Power Spectral Density of the known force.

3.3.2 Part b:

Once you have calculated the FRFs from part (a), you can extract the dynamic properties (Natural Frequencies, Damping, and Mode Shapes) using standard modal analysis techniques:

1. **Natural Frequencies** (ω_n) The natural frequencies correspond to the resonances of the structure.
 - Procedure: Plot the magnitude of the FRF, $|H_i(\omega)|$, versus frequency ω for one or more of the sensors.

- Identification: Look for sharp peaks in the plot. The frequency at which a peak occurs is a natural frequency (ω_n) of the wing.
- Verification: These peaks should appear at roughly the same frequency across all four accelerometers (though the amplitude will vary depending on the location).

2. **Damping Ratios** (ζ) The damping ratio indicates how quickly vibrations decay. A common method to estimate this from an FRF is the Half-Power Bandwidth Method (also known as the -3dB method).

- Procedure: Focus on a single resonance peak at ω_n .
- Identify the Peak Amplitude: Let the maximum amplitude at resonance be A_{max} .
- Find Half-Power Points: Find the two frequencies, ω_1 (lower) and ω_2 (upper), on either side of the peak where the amplitude drops to $\frac{A_{max}}{\sqrt{2}}$ (or roughly $0.707 \times A_{max}$).
- Calculation: Calculate the damping ratio (ζ) using the formula:

$$\zeta \approx \frac{\omega_2 - \omega_1}{2\omega_n}$$

3. **Mode Shapes** (ϕ) The mode shape describes the deformation pattern of the wing at a specific natural frequency.

- Procedure: Select a specific natural frequency ω_n identified in Step 1.
- Extract Values: Record the complex value (Magnitude and Phase) of the FRF for all four sensors at that exact frequency: $H_1(\omega_n), H_2(\omega_n), H_3(\omega_n), H_4(\omega_n)$.
- Construct the Shape:
 - The Magnitude $|H_i(\omega_n)|$ tells you the relative displacement at that location (how much it moves).
 - The Phase tells you the direction of motion relative to the force. Typically, points are either in-phase (0°) or out-of-phase (180°).
- Result: The vector $\begin{Bmatrix} H_1 & H_2 & H_3 & H_4 \end{Bmatrix}^T$ represents the mode shape of the wing for that specific mode.