The traveling salesman problem: A Linear programming formulation of

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P = NP: Linear Programming Formulation of the Traveling Salesman Problem

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Abstract

In this paper, we present a polynomial-sized linear programming formulation of the *Traveling Salesman Problem* (TSP). The proposed linear program is a network flow-based model with $O(n^9)$ variables and $O(n^7)$ constraints. Numerical implementation issues and results are discussed.

Keywords: Combinatorial Optimization, Traveling Salesman Problem, TSP, Scheduling, Sequencing, Computational Complexity, NP-Hardness, NP-Completeness.

1. Introduction

The Traveling Salesman Problem (TSP) is the problem of finding a least-cost sequence in which to visit a set of cities, starting and ending at the same city, and in such a way that each city is visited exactly once. This problem has received a tremendous amount of attention over the years due in part to its wide applicability in practice (see Lawler *et al.* [1985] among others, for examples). Also, since its seminal formulation as a mathematical programming problem in the 1950's (Dantzig, Fulkerson, and Johnson [1954]), the problem has been at the core of most of the developments in the area of Combinatorial Optimization (see Nemhauser and Wolsey [1988], among others). A key issue has been the question of whether there exists a polynomial-time algorithm for solving the problem (see Garey and Johnson [1979]). For example, Johnson and Papadimitriou ([1985]) wrote:

> "… *researchers have for years been attempting without success to find polynomial-time algorithms for certain problems in NP, such as the TSP. … This question is the central open problem in computer science today, and one of the most important open problems in mathematics."* (Johnson and Papadimitriou [1985, p. 56]).

Similar comments have been made by many others over the years.

In this paper, we present a polynomial-sized linear programming formulation of the *Traveling Salesman Problem* (TSP). The proposed linear program is a network flow-based model with $O(n^9)$ variables and $O(n^7)$ constraints. Numerical implementation issues and results are discussed.

The plan of the paper is as follows. The proposed linear programming formulation is developed in section 2. Numerical implementation and computational results are discussed in section 3. Conclusions are discussed in section 4.

2. Development of the Formulation

Different classical formulations of the TSP are analyzed and compared in Padberg and Sung [1991]. The approach used in this paper is different from that of any of the existing models that we know of. In this section, we first present a nonlinear integer programming (NIP) formulation of the TSP. Then, we develop an integer linear programming (ILP) reformulation of this NIP model using a network flow modeling framework. Finally, we show that the linear programming (LP) relaxation of our ILP reformulation has extreme points that correspond to TSP tours respectively.

2.1 Nonlinear Integer Programming Model

Consider the TSP defined on n nodes belonging to the set N = {1, 2, …, n}, with arc set $E = N^2$, and travel costs t_{ij} ($(i,j) \in E$; $t_{ii} = \infty$, $\forall\, i \in N$) associated with the arcs. Assume, without loss of generality, that city 1 is the starting point and the ending point of travel. Denote the set of the remaining cities as M = N \ {1}. Define S = N \ {n} as the index set for the stage of travel corresponding to the order of visit of the cities in M. Let R ≡ S \ {n-1}.

Let u_{is} ($i \in M$, $s \in S$) be a 0/1 binary variable that takes on the value “1” if city $i \in M$ is visited at stage $s \in S$. Then, in order to properly account the TSP travel costs, consecutive travel stages must be considered jointly. Hence, re-define the travel costs as:

$$c_{isj} = \begin{cases} t_{ij} + t_{1,i} & \text{for } s = 1,\ (i,j) \in M^2 \\ t_{ij} & \text{for } s \in R \setminus \{1, n-2\},\ (i,j) \in M^2 \\ t_{ij} + t_{j,1} & \text{for } s = n-2,\ (i,j) \in M^2 \end{cases} \tag{2.1}$$

Then, the cost incurred if city $i \in M$ is visited at stage $s \in R$ followed by city $j \in M$ at stage (s+1) can be expressed as $c_{isj}u_{is}u_{j,s+1}$ ($(i, j) \in M^2$, $s \in R$). For example, $c_{235}u_{23}u_{54}$ would represent the cost function associated with the situation where cities 2 and 5 are the 3rd and 4th cities to be visited (after city 1), respectively.

Note that from expression 2.1 above, $c_{i,1,j}u_{i,1}u_{j,2}$ and $c_{i,n-2,j}u_{i,n-2}u_{j,n-1}$ correctly model the costs of the travels $1 \to i \to j$ and $i \to j \to 1$, respectively. Hence, the TSP can be formulated as the following nonlinear bipartite matching problem:

Problem TSP:

Minimize

$$Z_{TSP}(\mathbf{u}) = \sum_{s\in R} \sum_{i\in M} \sum_{j\in(M\backslash\{i\})} c_{isj}u_{is}u_{j,s+1} \tag{2.2}$$

Subject to:

$$\sum_{i\in M} u_{is} = 1 \qquad s \in S \tag{2.3}$$

$$\sum_{s\in S} u_{is} = 1 \qquad i \in M \tag{2.4}$$

$$u_{is} \in \{0, 1\} \qquad i \in M;\ s \in S \tag{2.5}$$

□

The objective function 2.2 aims to minimize the total cost of all travels. Constraints 2.3 stipulate (in light of the binary requirements constraints 2.5) that only one city can be visited from city 1 and that only one city is visited at each stage of travel. Constraints 2.4 on the other hand ensure (in light of the binary requirements 2.5) that a given city is visited at exactly one stage of travel. The quadratic objective function terms (i.e., the $c_{isj}u_{is}u_{j,s+1}$'s) ensure (in light of the binary requirements constraints 2.5) that a travel cost is incurred from city i to city j iff those two cities are visited at consecutive stages of travel with i preceding j, as discussed above. Hence, *Problem TSP* accurately models the TSP.

2.2 Integer Linear Programming Model

Note that the polytope associated with *Problem TSP* is the standard assignment polytope (see Bazaraa, Jarvis, and Sherali [1990; pp. 499-513], or Evans and Minieka [1992, pp. 250-267]), and that there is a one-to-one correspondence between TSP tours and extreme points of this polytope. Our modeling consists essentially of *lifting* this polytope in higher dimension in such a way that the quadratic cost function of *Problem TSP* is correctly captured using a linear function.

To do this, we use the framework of the graph G = (V, A) illustrated in Figure 2.1, where the nodes in V correspond to (city, travel stage) pairs (i, s) $\in$ (M, S), and the arcs correspond to binary variables $x_{irj} = u_{ir}u_{j,r+1}$ ((i, j) $\in$ (M, M\{i}); r $\in$ R). Clearly, there is a one-to-one correspondence between the perfect bipartite matching solutions of *Problem TSP* (and therefore, TSP tours) and paths in this graph that simultaneously span the set of stages, S, and the set of cities, M. For simplicity of exposition we refer to such paths as "*city and stage spanning*" ("*c.a.s.s.*") paths. Also, we refer to the set of all the nodes of the graph that have a given city index in common as a "level" of the graph, and to the set of all the nodes of the graph that have a given travel stage index in common as a "stage" of the graph.

Figure 2.1 Here

The idea of our approach to reformulating *Problem TSP* is to develop constraints that "force" flow in Graph G to propagate along *c.a.s.s. paths* of the graph only. Hence, we do not deal directly with the TSP polytope *per se* (see Grötschel and Padberg [1985, pp. 256-261]) in this paper. More specifically, our approach in the paper consists of developing a reformulation of standard assignment polytope using variables that are functions of the flow variables associated with the arcs of Graph G. The correspondence between vertices of our model and TSP tours is achieved through the association of costs to the vertices of the model, much in the same way as is done in *Problem TSP*. Therefore, developments that are concerned with descriptions of the TSP polytope specifically (see Padberg and Grötschel [1985], or Yannakakis [1991] for example) are not applicable in the context of this paper.

Our model has somewhat of an analogy to a multi-commodity network flow model (see Bazaraa, Jarvis, and Sherali [1990, pp. 588-625]). This analogy comes from the fact that our model can be thought of as having "layers" of flow (like commodity flows) linked through some consistency requirements constraints (like capacity constraints in a multi-commodity flow context). However, in particular, the "commodities" flows in our framework do not necessarily originate from source nodes of the network. Rather some of the “commodities” are “created” within the network itself, at intermediate nodes. Our proposed ILP reformulation will now be developed in the following discussion.

For (i, j, u, v, k, t) $\in M^6$, (p, r, s) $\in R^3$ such that r < p < s, let $z_{irjupvkst}$ be a 0/1 binary variable that takes on the value “1” if and only if the flow on arc (i, r, j) of Graph G subsequently flows on arcs (u, p, v) and (k, s, t), respectively. Similarly, for (i, j, k, t) $\in M^4$, (s, r) $\in R^2$ such that r < s, let y_{irjkst} be a binary variable that indicates whether the flow on arc (i, r, j) subsequently flows on arc (k, s, t) (y_{irjkst} = 1) or not (y_{irjkst} = 0). Finally, denote by y_{irjirj} the binary variable that indicates whether there is flow on arc (i, r, j) or not. Then, with respect to our multi-commodity framework analogy discussed above, we liken y_{irjirj} to a “commodity” that propagates onto stages succeeding stage r in the graph through the y_{irjkst} (s > r) variables. Hence, given an instance of (***y***, ***z***), we use the term “flow layer” to refer to the sub-graph of G induced by the arc (i, r, j) corresponding to a given positive y_{irjirj} and the arcs (k, s, t) (s $\in$ R, s > r) corresponding to the corresponding y_{irjkst}’s that are positive. Hence, the flow on arc (i, r, j) also flows on arc (k, s, t) (for a given s > r) iff arc (k, s, t) belongs to the *flow layer* originating from arc (i, r, j). Also, we say that flow on a given arc (i, r, j) of Graph G “visits” a given *level* of the graph, say *level* t, if $\sum_{s \in R;\, s \le r-1} \sum_{k \in (M \setminus \{i,\, j,\, t\})} y_{tskirj} + \sum_{s \in R;\, s \ge r+1} \sum_{k \in (M \setminus \{i,\, j,\, t\})} y_{irjkst} > 0$.

Logical constraints of our model are that: 1) flow must be conserved; 2) flow must be connected; and, 3) *flow layers* must be consistent with one another. By “consistency” of the *flow layers*, we are referring to the requirement that any *flow layer* originating from a given arc (i, r, j) with r ≥ 2 must be a sub-graph of one or more *flow layers* originating from a set of arcs at any other given *stage* preceding r. More specifically, consider the arc (i, r, j) corresponding to a given positive component of (***y***), y_{irjirj} > 0. For s < r (s $\in$ R), define $F_S(i, r, j) \equiv \{(k, t) \in M^2 \mid y_{kstirj} > 0\}$.

Then, by "consistency of flow layers" we are referring to the condition that the *flow layer* originating from arc (i, r, j) must be a sub-graph of the union of the *flow layers* originating from the arcs comprising each of the F_S(i, r, j)'s, respectively. In addition to the logical constraints, the bipartite matching constraints 2.3 and 2.4 of *Problem TSP* must be respectively enforced. These ideas are developed in the following.

1) Flow Conservations:

   Any flow through Graph G must be initiated at *stage 1*. Also, for (i, j) $\in M^2$, r $\in$ R, r $\geq$ 2, the flow on arc (i, r, j) must be equal to the sum of the flows from *stage 1* that propagate onto arc (i, r, j).

$$\sum_{i \in M} \sum_{j \in M} y_{i,1,j,i,1,j} \quad = \quad 1 \tag{2.6}$$

$$y_{irjirj} - \sum_{u \in M} \sum_{v \in M} y_{u,1,virj} \quad = \quad 0 \qquad i, j \in M; \quad r \in R, r \geq 2 \tag{2.7}$$

2) Flow Connectivities:

   All flows must propagate through the graph, on to stage n-1, in a connected manner. Each *flow layer* must be a connected graph, and must conserve flow.

$$\sum_{k \in M} y_{irjkst} - \sum_{k \in M} y_{irjt,s+1,k} \quad = \quad 0 \qquad i, j, t \in M; \quad r, s \in R, r \leq n\text{-}3, r \leq s \leq n\text{-}3 \tag{2.8}$$

3) Consistency of *Flow Layers*:

   For p, s $\in$ R (1 < p < s) and (u, v, k, t) $\in M^4$, flow on (u, p, v) subsequently flows onto (k, s, t) iff for each r < p (r $\in$ R) there exists (i, j) $\in M^2$ such that flow from (i, r, j) propagates onto (k, s, t) via (u, p, v). This results in the following three types of constraints:

   i) Layering Constraints A

$$y_{irjupv} - \sum_{k \in M} \sum_{t \in M} z_{irjupvkst} \quad = \quad 0, \qquad i, j, u, v \in M; \quad p, r, s \in R, 2 \leq p \leq n\text{-}3, \quad r \leq p\text{-}1, \quad s \geq p\text{+}1 \tag{2.9}$$

   ii) Layering Constraints B

$$y_{irjkst} - \sum_{u \in M} \sum_{v \in M} z_{irjupvkst} \quad = \quad 0, \qquad i, j, k, t \in M; \quad p, r, s \in R, 2 \le p \le n\text{-}3, \quad r \le p\text{-}1, \quad s \ge p\text{+}1 \qquad (2.10)$$

ii) <u>Layering Constraints C</u>

$$y_{upvkst} - \sum_{i \in M} \sum_{j \in M} z_{irjupvkst} \quad = \quad 0,` \qquad u, v, k, t \in M; \quad p, r, s \in R, 2 \le p \le n\text{-}3, \quad r \le p\text{-}1, \quad s \ge p\text{+}1 \qquad (2.11)$$

4) "<u>Visit" *Requirements*</u>:

Flow within any *layer* must *visit* every *level* of Graph G.

$$y_{u,1,vu,1,v} - \sum_{s \in R;\, s \ge 2} \; \sum_{k \in M} y_{u,1,vkst} \quad = \quad 0 \qquad u, v \in M; \;\; t \in M\backslash\{u, v\} \qquad (2.12)$$

$$y_{u,1,virj} - \sum_{s \in R;\, s \le r-1} \; \sum_{k \in M} z_{u,1,vtskirj} - \sum_{s \in R;\, s \ge r+1} \; \sum_{k \in M} z_{u,1,virjkst} \quad = \quad 0$$

$$r \in R\backslash\{1\}; \quad u, v, i, j \in M; \;\; t \in M\backslash\{u, v, i, j\} \qquad (2.13)$$

5) "<u>Visit" *Restrictions*</u>:

Flow must be connected with respect to the stages of Graph G. There can be no flow between nodes belonging to the same *level* of the graph; No *level* of the graph can be *visited* at more than one *stage*, and vice versa.

$$\sum_{(k,t) \in M^2 | (k,t) \ne (i,j)} y_{irjkrt} \; + \sum_{(k,t) \in (M \backslash \{j\}, M) | (k, r+1, t) \in A} y_{irjk,\, r+1,\, t} \; + \sum_{s \in R; s \ge r+1} \; \sum_{k \in M} y_{irjksi} \; +$$

$$+ \sum_{s \in R; s \ge r+1} \; \sum_{k \in M} y_{irjisk} \; + \sum_{s \in R;\, s \ge r+1} \; \sum_{k \in M} y_{irjksj} \; + \sum_{s \in R;\, s \ge r+2} \; \sum_{k \in M} y_{irjjsk} \; +$$

$$+ \sum_{s \in R,\, s \ge r} \; \sum_{k \in M} \; \sum_{t \in M} y_{irikst} \; + \sum_{s \in R,\, s \le r} \; \sum_{k \in M} \; \sum_{t \in M} y_{kstjrj} \;\; = \;\; 0, \quad i, j \in M; r \in R \qquad (2.14)$$

Note that constraints 2.3 of *Problem TSP* are enforced through the combination of the "Flow Connectivities" requirements and the "'Visit' Restrictions" constraints, and that constraints 2.4 are enforced through the "'Visit' Requirements" constraints.

The complete statement of our integer linear programming model is as follows:

Problem IP:

Minimize

$$Z_{IP}(\mathbf{y}, \mathbf{z}) = \sum_{r \in R} \sum_{i \in M} \sum_{j \in M} c_{irj} y_{irjirj} \tag{2.15}$$

Subject to:

Constraints 2.6 – 2.14

$$y_{irjkst}, z_{irjupvkst} \in \{0, 1\} \qquad i, j, k, t, u, v \in M; \quad p, r, s \in R \tag{2.16}$$

□

We formally establish the equivalence between *Problem IP* and *Problem TSP* in the following proposition.

Proposition 1

Problem IP and *Problem TSP* are equivalent.

Proof:

i) For a feasible solution to *Problem TSP*, **u** = (u_{is}), let (***y***(**u**), ***z***(**u**)) be a vector with components specified as follows:

$$\begin{cases} (\boldsymbol{y}(\boldsymbol{u}))_{irjkst} = u_{ir} u_{j,r+1} u_{ks} u_{t,s+1}; & i, j, k, t \in M; r, s \in R, s \geq r \\ (\boldsymbol{z}(\mathbf{u}))_{irjapbkst} = u_{ir} u_{j,r+1} u_{ap} u_{b,p+1} u_{ks} u_{t,s+1}; & a, b, i, j, k, t \in M; \quad p, r, s \in R, \; r < p < s \end{cases}$$

It is easy to verify that (**y**(**u**), **z**(**u**)) satisfies each of the constraints of *Problem IP*.

ii) Let **(y, z)** = $(y_{irjkst}, z_{abirjkst})$ be a feasible solution to *Problem IP*. Because of constraints 2.6-2.8, and 2.16, (***y***, ***z***) must be such that there exists a set of city indices $\{i_1, i_2, \cdots, i_{n-1}\}$ with:

$$y_{i_r, r, i_{r+1}, i_r, r, i_{r+1}} = 1 \qquad \forall\, r \in R$$

Because of constraints 2.9 - 2.11, and 2.16, we must also have:

$$y_{arbcsd} = \begin{cases} 1 & \text{for } (a, b, c, d) = (i_r, i_{r+1}, i_s, i_{s+1}), \\ 0 & \text{otherwise} \end{cases} \qquad \forall\, (r, s) \in R^2 \text{ with } r < s$$

$$z_{arbcpdesf} = \begin{cases} 1 & \text{for } (a,b,c,d,e,f) = (i_r, i_{r+1}, i_p, i_{p+1}, i_s, i_{s+1}); \\ 0 & \text{otherwise} \end{cases} \quad \forall\ (r, p, s) \in R^3 \text{ with } r < p < s$$

Hence, by constraints 2.14, the i_s's must be such that:

$i_r \neq i_s$ for all $(r, s) \in R^2$ such that $s \neq r$.

Hence, a unique feasible solution to *Problem TSP* is obtained from (**y**, **z**) by setting:

$$u_{jr} = \begin{cases} 1 & \text{if } j = i_r \\ 0 & \text{otherwise} \end{cases} \quad \forall\ j \in M,\ r \in S$$

iii) Clearly, from i) and ii) above, *Problem IP* and *Problem TSP* have equivalent feasible sets. The proposition follows from this and the fact that the two problems also have equivalent objective functions.

Q.E.D.

Hence, according to Proposition 1, each feasible solution to *Problem IP* corresponds to a perfect bipartite matching solution of *Problem TSP*, and therefore, to a *c.a.s.s path* in Graph G, and a TSP tour, and conversely. Let $\varphi(\ell) = \langle \ell_1, \ell_2, \cdots, \ell_{n-1}, \ell_n \rangle$ denote the ordered set of city indices visited along a given TSP tour, Tour ℓ (i.e., with ℓ_t as the index of the city visited at stage t according to Tour ℓ). In the remainder of this paper, we will use the term "*feasible solution corresponding to (Given) Tour* ℓ" to refer to the vector $(\mathbf{y}(\varphi(\ell)), \mathbf{z}(\varphi(\ell)))$ obtained as follows:

$$(\mathbf{y}(\varphi(\ell)))_{arbcsd} = \begin{cases} 1 & \text{for } r, s \in R,\ s \geq r;\ (a,b,c,d) = (\ell_r, \ell_{r+1}, \ell_s, \ell_{s+1}); \\ 0 & \text{otherwise} \end{cases}$$

$$(\mathbf{z}(\varphi(\ell)))_{apbcrdesf} = \begin{cases} 1 & \text{for } p, r, s \in R, \quad s > r > p; \\ & \qquad (a,b,c,d,e,f) = (\ell_p, \ell_{p+1}, \ell_r, \ell_{r+1}, \ell_s, \ell_{s+1}); \\ 0 & \text{otherwise} \end{cases}$$

Our proposed linear programming model will now be developed.

2.3 Linear Programming Model

Our basic linear programming model consists of the linear programming relaxation of *Problem IP*. This problem can be stated as follows:

Problem BLP:

Minimize

$$Z_{LP}(\mathbf{y}, \mathbf{z}) = \sum_{i \in M} \sum_{r \in R} \sum_{j \in M} c_{irj} y_{irjirj} \tag{2.17}$$

Subject to:

Constraints 2.6 – 2.14

$$0 \leq y_{irjkst}, \quad z_{upvirjkst} \leq 1 \qquad u, v, i, j, k, t \in F; \quad p, r, s \in R \tag{2.18}$$

□

We begin the development of the structure of *Problem BLP* with the following result.

Lemma 1

The following constraints are valid for *Problem BLP*:

i) $y_{irjirj} - \sum_{k \in M} \sum_{t \in M} y_{irjkst} = 0 \qquad i, j \in M; \quad r, s \in R, \ s \geq r+1$

ii) $y_{irjirj} - \sum_{k \in M} \sum_{t \in M} \sum_{a \in M} \sum_{c \in M} z_{irjkst\,abc} = 0 \qquad i, j \in M; \ r, s, b \in R, r < s < b$

<u>Proof</u>:

i) $y_{irjirj} = \sum_{u \in M} \sum_{v \in M} y_{u,1,virj} \qquad i, j \in M; \quad r \in R \qquad$ (Using 2.7)

$= \sum_{u \in M} \sum_{v \in M} \sum_{k \in M} \sum_{t \in M} z_{u,1,virjkst} \qquad i, j \in M; \quad r \in R, \ s \geq r+1 \qquad$ (Using 2.9)

$= \sum_{k \in M} \sum_{t \in M} \sum_{u \in M} \sum_{v \in M} z_{u,1,virjkst} \qquad i, j \in M; \quad r, s \in R, \ s \geq r+1 \quad$ (Re-arranging)

$$= \sum_{k\in M}\sum_{t\in M} y_{irjkst} \qquad i, j \in M;\ \ r, s \in R,\ \ s \geq r+1 \quad \text{(Using 2.11)}$$

Combining the above with constraints 2.8 (for r = 1), we have:

$$y_{irjirj} = \sum_{k\in M}\sum_{t\in M} y_{irjkst} \quad i, j \in F;\ \ r, s \in R,\ \ s \geq r+1$$

ii) Condition ii) follows directly from the combination of Lemma 1-i) and constraints 2.9.

Q.E.D.

For a feasible solution $(\boldsymbol{y}, \boldsymbol{z}) = (y_{irjkst},\ z_{upvirjkst})$ to *Problem BLP*, let $G(\boldsymbol{y}, \boldsymbol{z}) = (V(\boldsymbol{y}, \boldsymbol{z}), A(\boldsymbol{y}, \boldsymbol{z}))$ be the sub-graph of *G* induced by the arcs of G corresponding to the positive components of ($\boldsymbol{y}$). For r $\in$ R, define $W_r(\mathbf{y}, \mathbf{z}) = \{(i, j) \in M^2 \mid \{(i, r, j) \in A(\boldsymbol{y}, \boldsymbol{z})\}$. Denote the arc corresponding to the ν^{th} element of $W_r(\boldsymbol{y}, \boldsymbol{z})$ $(\nu \in \{1, 2, \cdots, \chi_r(\boldsymbol{y}, \boldsymbol{z})\};\ \ 1 \leq \chi_r(\boldsymbol{y}, \boldsymbol{z}) \leq (n-1)(n-2))$ as $a_{r,\nu}(\boldsymbol{y}, \boldsymbol{z}) = (i_{r,\nu}, r, j_{r,\nu})$. Then, $W_r(\boldsymbol{y}, \boldsymbol{z})$ can be alternatively represented as $X_r(\boldsymbol{y}, \boldsymbol{z}) = \{(i_{r,\nu}, r, j_{r,\nu});\ \nu \in N_r(\boldsymbol{y}, \boldsymbol{z})\}$, where $N_r(\boldsymbol{y}, \boldsymbol{z}) = \{1, 2, \cdots, \chi_r(\boldsymbol{y}, \boldsymbol{z})\}$ is the index set for the arcs of *Graph* G($\boldsymbol{y}$, $\boldsymbol{z}$) originating at *stage* r.

For (r, s) $\in$ R^2 with s ≥ r+2, $\rho \in N_r(\boldsymbol{y}, \boldsymbol{z})$, and $\sigma \in N_s(\boldsymbol{y}, \boldsymbol{z})$ we refer to a set of arcs of G($\boldsymbol{y}$, $\boldsymbol{z}$),

$$\begin{aligned} U_{(r,\rho),(s,\sigma),t}(\boldsymbol{y}, \boldsymbol{z}) \equiv \Big\{ & a_{r,\nu_{r,(r,\rho),(s,\sigma),t}}(\boldsymbol{y},\boldsymbol{z}),\ a_{r+1,\nu_{r+1,(r,\rho),(s,\sigma),t}}(\boldsymbol{y},\boldsymbol{z}), \cdots, a_{s,\nu_{s,(r,\rho),(s,\sigma),t}}(\boldsymbol{y},\boldsymbol{z}) \Big| \\ & \nu_{r,(r,\rho),(s,\sigma),t} = \rho;\ \ \nu_{s,(r,\rho),(s,\sigma),t} = \sigma;\ \ \nu_{p,(r,\rho),(s,\sigma),\ t} \in N_p(\boldsymbol{y}, \boldsymbol{z}),\ \forall p \in (R \cap [r+1, s-1]; \\ & i_{p,\nu_{p,(r,\rho),(s,\sigma),t}} = j_{p-1,\nu_{p-1,(r,\rho),(s,\sigma),t}},\ \forall p \in (R \cap [r+1, s];\ \text{and} \\ & z_{i_{p,\nu_{p,(r,\rho),(s,\sigma),t}},p,i_{p+1,\nu_{p+1,(r,\rho),(s,\sigma),t}},\ i_{q,\nu_{q,(r,\rho),(s,\sigma),t}},q,i_{q+1,\nu_{q+1,(r,\rho),(s,\sigma),t}},\ i_{s,\sigma},s,j_{s,\sigma}} > 0, \\ & \forall (p, q) \in (R \cap [r, s-1])^2 \text{ such that } q > p \Big\} \end{aligned} \qquad (2.19)$$

as a "path in ($\boldsymbol{y}$, $\boldsymbol{z}$) from (r, ρ) to (s, σ)." Hence, for convenience, a *path in* (***y***, ***z***) *from* (*r*,ρ) *to* (*s*,σ), $U_{(r,\rho),(s,\sigma),t}(\boldsymbol{y}, \boldsymbol{z})$, can be alternatively represented as an ordered set of city indices,

$$P_{(r,\rho),(s,\sigma),t}(\boldsymbol{y},\boldsymbol{z}) = \langle\, i_{r,\nu_{r,(r,\rho),(s,\sigma),t}},\ i_{r+1,\nu_{r+1,(r,\rho),(s,\sigma),t}},\ \cdots,\ i_{s+1,\nu_{s+1,(r,\rho),(s,\sigma),t}} \rangle \qquad (2.20)$$

where $\nu_{r,(r,\rho),(s,\sigma),t} = \rho$, $\nu_{s,(r,\rho),(s,\sigma),t} = \sigma$, $i_{r+1,\nu_{r+1,(r,\rho),(s,\sigma),t}} = j_{r,\rho}$, $i_{s+1,\nu_{s+1,(r,\rho),(s,\sigma),t}} = j_{s,\sigma}$,

and $(i_{p,\nu_{p,(r,\rho),(s,\sigma),t}},\ p,\ i_{p+1,\nu_{p+1,(r,\rho),(s,\sigma),t}}) \in X_p(\boldsymbol{y},\boldsymbol{z})$, $\forall\, p \in (R \cap [r, s])$;

and $i_{p,\nu_{p,(r,\rho),(s,\sigma),t}} = j_{p-1,\nu_{p-1,(r,\rho),(s,\sigma),t}}$, $\forall\, p \in (R \cap [r+1, s])$.

Finally, we denote the set of all *paths in (y, z) from (r,ρ) to (s, σ)* as $Q_{(r,\rho),(s,\sigma)}(\boldsymbol{y},\boldsymbol{z})$, and associate to it the index set $\Psi_{(r,\rho),(s,\sigma)}(\boldsymbol{y},\boldsymbol{z}) \equiv \{1, 2, \cdots, \varphi_{(r,\rho),(s,\sigma)}(\boldsymbol{y},\boldsymbol{z})\}$, where $\varphi_{(r,\rho),(s,\sigma)}(\boldsymbol{y},\boldsymbol{z})$ is the cardinality of $Q_{(r,\rho),(s,\sigma)}(\boldsymbol{y},\boldsymbol{z})$.

We have the following.

Proposition 2

Let $(\boldsymbol{y}, \boldsymbol{z}) = (y_{irjkst}, z_{upvirjkst})$ be a feasible solution to *Problem BLP*. For $(r, s) \in R^2$ $(s \geq r+2)$, $\rho \in N_r(\boldsymbol{y},\boldsymbol{z})$, and $\sigma \in N_s(\boldsymbol{y},\boldsymbol{z})$, if $y_{i_{r,\rho}, r, j_{r,\rho}, i_{s,\sigma}, s, j_{s,\sigma}} > 0$, then we must have:

i) $Q_{(r,\rho),(s,\sigma)}(\boldsymbol{y},\boldsymbol{z}) \neq \emptyset$; and

ii) $\forall\, g \in (R \cap [r+1, s-1])$ and $\gamma \in N_g(\boldsymbol{y},\boldsymbol{z})$: $z_{i_{r,\rho}, r, j_{r,\rho}, i_{g,\gamma}, g, j_{g,\gamma}, i_{s,\sigma}, s, j_{s,\sigma}} > 0 \;\Rightarrow$

$\exists\, \iota \in \Psi_{(r,\rho),(s,\sigma)}(\boldsymbol{y},\boldsymbol{z}) \ni : (i_{g,\gamma}, j_{g,\gamma}) \in (P_{(r,\rho),(s,\sigma),\iota}(\boldsymbol{y},\boldsymbol{z}))^2$.

Proof:

First, (i) we will show that the proposition holds for all $(r, s) \in R^2$ such that $s = r+2$. Then, (ii) we will show that if the proposition holds for all $(r, s) \in R^2$ such that $s \in [r+2, r+\omega]$ for some integer $\omega \geq 2$, then the proposition must hold for all $(r, s) \in R^2$ such that $s = r+\omega+1$ (if there exists such a pair).

i) Because of constraints 2.14, constraints 2.10 for any $(r, s) \in R^2$ such that $s = r+2$ can be written as:

$$y_{i,r,j,\,u,r+2,v} - z_{i,r,j,\,j,r+1,u,\,u,r+2,v} = 0;\ \ i \in M;\ \ j \in M\backslash\{i\};\ u \in M\backslash\{i, j\};\ v \in M\backslash\{i, j, u\} \qquad (2.21)$$

It follows from 2.21 that for $\sigma \in N_{r+2}(\boldsymbol{y},\boldsymbol{z})$,

$$y_{i_{r,\rho},r,j_{r,\rho},\, i_{r+2,\sigma},r+2,j_{r+2,\sigma}} > 0 \;\Leftrightarrow\; z_{i_{r,\rho},r,j_{r,\rho},\; j_{r,\rho},r+1,i_{r+2,\sigma},\; i_{r+2,\sigma},r+2,j_{r+2,\sigma}} > 0 \tag{2.22}$$

Hence, for $\sigma \in N_{r+2}(\boldsymbol{y},\boldsymbol{z})$ such that $y_{i_{r,\rho},r,j_{r,\rho},i_{r+2,\sigma},r+2,j_{r+2,\sigma}} > 0$, we have:

$\varphi_{(r,\rho),(r+2,\sigma)}(\boldsymbol{y},\boldsymbol{z}) = 1$, so that:

$Q_{(r,\rho),(r+2,\sigma)}(\boldsymbol{y},\boldsymbol{z}) = \{ P_{(r,\rho),(r+2,\sigma),1}(\boldsymbol{y},\boldsymbol{z}) \}$, where

$$P_{(r,\rho),(r+2,\sigma),1}(\boldsymbol{y},\boldsymbol{z}) = \langle\, i_{r,\rho},\; j_{r,\rho},\; i_{r+2,\sigma},\; j_{r+2,\sigma} \,\rangle \tag{2.23}$$

Hence, the proposition holds for all $(r, s) \in R^2$ such that $s = r+2$.

ii) Suppose the proposition holds for all $(r, s) \in R^2$ such that $r+2 \le s \le r+\omega$ for some integer $\omega \ge 2$. If ω is such that there does not exist $(r, t) \in R^2$ with $t = r+\omega+1$, then the proposition is proven. Hence, assume there exist some $(r, t) \in R^2$ such that $t = r+\omega+1$. Consider one such (r, t) pair, and $\tau \in N_t(\boldsymbol{y},\boldsymbol{z})$ such that:

$$y_{i_{r,\rho},r,j_{r,\rho},\; i_{t,\tau},t,j_{t,\tau}} > 0. \tag{2.24}$$

Then, from the combination of constraints 2.10, 2.8, and 2.14 (see the first and second terms), condition 2.24 implies that there must exist a set:

$$C_{(r,\rho),(t,\tau)}(\boldsymbol{y},\boldsymbol{z}) \equiv \{ \alpha \in N_{r+1}(\boldsymbol{y},\boldsymbol{z}) \mid z_{i_{r,\rho},r,j_{r,\rho},\; j_{r,\rho},r+1,j_{r+1,\alpha},\; i_{t,\tau},t,j_{t,\tau}} > 0 \} \tag{2.25}$$

such that:

$$y_{i_{r,\rho},r,j_{r,\rho},\; i_{t,\tau},t,j_{t,\tau}} = \sum_{\alpha \in C_{(r,\rho),(t,\tau)}(\boldsymbol{y},\boldsymbol{z})} z_{i_{r,\rho},r,j_{r,\rho},\; j_{r,\rho},r+1,j_{r+1,\alpha},\; i_{t,\tau},t,j_{t,\tau}} \tag{2.26}$$

($C_{(r,\rho),(t,\tau)}(\boldsymbol{y},\boldsymbol{z})$ is the index set of the arcs at stage r+1 along which flow from arc $(i_{r,\rho}, r, j_{r,\rho})$ propagates onto arc $(i_{t,\tau}, t, j_{t,\tau})$).

By constraints 2.11, expression 2.25 implies:

$$y_{j_{r,\rho},r+1,j_{r+1,\alpha},\; i_{t,\tau},t,j_{t,\tau}} > 0 \;\; \forall \;\; \alpha \in C_{(r,\rho),(t,\tau)}(\boldsymbol{y},\boldsymbol{z}) \tag{2.27}$$

Hence, by assumption, the proposition holds for t, τ, r+1, and each $\alpha \in$

$C_{(r,\rho),(t,\tau)}(\boldsymbol{y},\boldsymbol{z})$. Combining this with 2.26 and the connectivity requirement constraints 2.8, we must have that for all h $\in$ (R $\cap$ [r+2, t-1]) and $\mu \in N_h(\boldsymbol{y},\boldsymbol{z})$:

$$z_{i_{r,\rho},r,j_{r,\rho},\ i_{h,\mu},h,j_{h,\mu},\ i_{t,\tau},t,j_{t,\tau}} > 0 \Rightarrow \exists\ \alpha \in C_{(r,\rho),(t,\tau)}(\boldsymbol{y},\boldsymbol{z}) \text{ and}$$

$$k \in \Psi_{(r+1,\alpha),(t,\tau)}(\boldsymbol{y},\boldsymbol{z}) \ni : (i_{h,\mu}, j_{h,\mu}) \in (P_{(r+1,\alpha),(t,\tau),k}(\mathbf{y},\mathbf{z}))^2. \tag{2.28}$$

Condition 2.28 combined with constraints 2.8 and 2.14, imply that:

$$\exists\ J_{(r+1,\alpha),(t,\tau)}(\boldsymbol{y},\boldsymbol{z}) \subseteq \Psi_{(r+1,\alpha),(t,\tau)}(\boldsymbol{y},\boldsymbol{z}) \ni:$$

$$z_{i_{r,\rho},r,j_{r,\rho},\ i_{p,\nu_{p,(r+1,\alpha),(t,\tau),\beta}},p,i_{p+1,\nu_{p+1,(r+1,\alpha),(t,\tau),\beta}},\ i_{t,\tau},t,j_{t,\tau}} > 0$$

$$\forall\ \alpha \in C_{(r,\rho),(t,\tau)}(\boldsymbol{y},\boldsymbol{z}),\ \beta \in J_{(r+1,\alpha),(t,\tau)}(\boldsymbol{y},\boldsymbol{z}), \text{ and } p \in (R \cap [r+1, t-1]). \tag{2.29}$$

($J_{(r+1,\alpha),(t,\tau)}(\boldsymbol{y},\boldsymbol{z})$ is the index set of the *paths in (**y**, **z**) from (r+1, α) to (t, τ)* along which flow from arc ($i_{r,\rho}$, r, $j_{r,\rho}$) propagates onto arc ($i_{t,\tau}$, t, $j_{t,\tau}$)).

Now, for $\alpha \in C_{(r,\rho),(t,\tau)}(\boldsymbol{y},\boldsymbol{z})$ and $\beta \in J_{(r+1,\alpha),(t,\tau)}(\boldsymbol{y},\boldsymbol{z})$, let:

$$T_{(r,\rho),(\alpha,\beta)(t,\tau)}(\boldsymbol{y},\boldsymbol{z}) = \{\ i_{r,\rho}\ \} \cup P_{(r+1,\alpha),(t,\tau),\beta}(\boldsymbol{y},\boldsymbol{z}), \tag{2.30}$$

(where $i_{r,\rho}$ is added to $P_{(r+1,\alpha),(t,\tau),\beta}(\boldsymbol{y},\boldsymbol{z})$ in such a way that it occupies the first position in $T_{(r,\rho),(\alpha,\beta)(t,\tau)}(\boldsymbol{y},\boldsymbol{z})$).

It is easy to verify that $T_{(r,\rho),(\alpha,\beta)(t,\tau)}(\boldsymbol{y},\boldsymbol{z})$ is a *path in (**y**, **z**) from (r, ρ) to (t, τ)*. Hence, we have $Q_{(r,\rho),(t,\tau)}(\boldsymbol{y},\boldsymbol{z}) \neq \varnothing$. Moreover, it follows directly from 2.28 above that condition ii) of the proposition must hold for r, ρ, t, and τ.

Q.E.D.

Proposition 3

Let (**y**, **z**) = (y_{irjkst}, $z_{upvirjkst}$) be a feasible solution to *Problem BLP*. Let (r, s) $\in R^2$, s $\geq$ r+2; $\rho \in N_r(\boldsymbol{y},\boldsymbol{z})$; and $\sigma \in N_s(\boldsymbol{y},\boldsymbol{z})$ be such that $y_{i_{r,\rho},r,j_{r,\rho},\ i_{s,\sigma},s,j_{s,\sigma}} > 0$. Then, we must have:

i) $Q_{(r,\rho),(s,\sigma)}(\boldsymbol{y},\boldsymbol{z}) \neq \varnothing$;

Furthermore, for each $\ell \in \Psi_{(r,\rho),(s,\sigma)}(\boldsymbol{y},\boldsymbol{z})$ we must have:

ii) $i_{q,\nu_{q,(r,\rho),(s,\sigma),\ell}} = j_{q-1,\nu_{q-1,(r,\rho),(s,\sigma),\ell}}$ for $q \in R$; $r+1 \le q \le s$;

iii) $z_{i_{p,\nu_{p,(r,\rho),(s,\sigma),\ell}},\, p,\, i_{p+1,\nu_{p+1,(r,\rho),(s,\sigma),\ell}},\ i_{q,\nu_{q,(r,\rho),(s,\sigma),\ell}},\, q,\, i_{q+1,\nu_{q+1,(r,\rho),(s,\sigma),\ell}},\ i_{s,\sigma},\, s,\, j_{s,\sigma}} > 0$

$\forall\ (p, q) \in (R \cap [r, s])^2,\ r \le p < q \le s\text{-}1$;

iv) $i_{p,\nu_{p,(r,\rho),(s,\sigma),\ell}} \neq i_{q,\nu_{q,(r,\rho),(s,\sigma),\ell}}$ $\forall\ (p, q) \in \left(S \cap [r,s+1]\right)^2$ ∍: $p \neq q$.

<u>Proof</u>:

Conditions i) –iii) follow directly from definitions and Proposition 2;

Condition iv) follows from the combination of condition iii) and the *visit* restrictions constraints 2.14.

Q.E.D.

Hence, every *path in (**y**, **z**) from (1, •) to (n-2, •)* corresponds to a *c.a.s.s. path* of Graph G (and therefore, to a TSP tours). Hence, for convenience, we refer to each $P_{(1,\rho),(n-2,\sigma),k}$ (**y, z**) simply as a "TSP tour in (**y**, **z**)," and denote it by $T_{\rho,\sigma,k}$ (**y, z**). To a *TSP tour in (**y**, **z**)*, $T_{\rho,\sigma,k}$ (**y, z**), we attach a "flow value" $\lambda_{\rho,\sigma,k}$ (***y*, *z***) defined as:

$$\lambda_{\rho,\sigma,k}(\boldsymbol{y},\boldsymbol{z}) \equiv \min_{p \in (R \cap [2, n-3])} \left\{ z_{i_{1,\rho},1,\, j_{1,\rho},\ i_{p,\nu_{p,(1,\rho),(n-2,\sigma),k}},\, p,\, i_{p+1,\nu_{p+1,(1,\rho),(n-2,\sigma),k}},\ i_{n-2,\sigma},\, n-2,\, j_{n-2,\sigma}} \right\} \quad (2.31)$$

A set of *TSP tours in (**y**, **z**)*, $\Gamma = \{T_{\rho_1,\sigma_1,k_1}(\boldsymbol{y},\boldsymbol{z}),\ T_{\rho_2,\sigma_2,k_2}(\boldsymbol{y},\boldsymbol{z}) \ldots,\ T_{\rho_m,\sigma_m,k_m}(\boldsymbol{y},\boldsymbol{z})\}$ with associated set of arc sets in *G*, $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m\}$ (where $\mathbf{e}_p$ = $\{ (i_{q,\nu_{q,(1,\rho_p),(n\text{-}2,\sigma_p),k_p}},\, q,\, i_{q+1,\nu_{q+1,(1,\rho_p),(n\text{-}2,\sigma_p),k_p}});\ 1 \le q \le n-2 \}$, for p = 1, …, m), is said to "cover" (***y***, ***z***) if $\bigcup_{1 \le p \le m} (\mathbf{e}_p) = A(\boldsymbol{y},\ \boldsymbol{z})$. Moreover, we say that (***y***, ***z***) "consists of" Γ if Γ *covers* *(**y**, **z**)*, and the following conditions hold:

i) $$y_{i_{r,\rho},r,j_{r,\rho},\,i_{r,\rho},r,j_{r,\rho}} = \sum_{p\in[1,m]\,\mid\,(i_{r,\rho},r,j_{r,\rho})\,\in\,\mathbf{e}_p} \lambda_{\rho_p,\sigma_p,k_p}(\boldsymbol{y},\boldsymbol{z})$$

for all $(r, \rho) \in (R,\ N_r(\boldsymbol{y},\ \boldsymbol{z}))$; (2.32)

ii) $$y_{i_{r,\rho},r,j_{r,\rho},\,i_{s,\sigma},s,j_{s,\sigma}} = \sum_{p\in[1,m]\,\mid\,((i_{r,\rho},r,j_{r,\rho}),\,(i_{s,\sigma},s,j_{s,\sigma}))\,\in\,\mathbf{e}_p^2} \lambda_{\rho_p,\sigma_p,k_p}(\boldsymbol{y},\boldsymbol{z})$$

for all $(r, s) \in R^2$, $s > r$; $(\rho, \sigma) \in (N_r(\boldsymbol{y},\ \boldsymbol{z}),\ N_s(\boldsymbol{y},\ \boldsymbol{z}))$; (2.33)

iii) $$z_{i_{r,\rho},r,j_{r,\rho},\,i_{s,\sigma},s,j_{s,\sigma},\,i_{t,\tau},t,j_{t,\tau}} =$$

$$= \sum_{p\in[1,m]\,\mid\,((i_{r,\rho},r,j_{r,\rho}),\,(i_{s,\sigma},s,j_{s,\sigma}),\,(i_{t,\tau},t,j_{t,\tau}))\,\in\,\mathbf{e}_p^3} \lambda_{\rho_p,\sigma_p,k_p}(\boldsymbol{y},\boldsymbol{z})$$

for all $(r, s, t) \in R^3$, $t > s > r$; $(\rho, \sigma, \tau) \in (N_r(\boldsymbol{y},\ \boldsymbol{z}),\ N_s(\boldsymbol{y},\ \boldsymbol{z}),\ N_t(\boldsymbol{y},\ \boldsymbol{z}))$. (2.34)

Clearly, if *(**y**, **z**) consists of* Γ, then (**y**, ***z***) is equal to the convex combination of the *feasible solutions corresponding to the TSP tours in (**y**, **z**)* that comprise Γ, with weights equal to the associated *flow values*, respectively. Hence, the following proposition shows that (***y***, ***z***) is a convex combination of the *feasible solutions corresponding to* (all) *the TSP tours in (**y**, **z**)*.

Proposition 4

Let (**y**, **z**) = $(y_{irjkst}, z_{irjupvkst})$ be a feasible solution to *Problem BLP*. Let Π(***y***, ***z***) denote the set of all the *TSP tours in (**y**, **z**)*.Then, (***y***, ***z***) *consists of* Π*(**y**, **z**)*.

<u>Proof</u>:

First, for convenience, associate to Π(***y***, ***z***) the index set $\pi(\boldsymbol{y},\boldsymbol{z}) \equiv \{1, 2, \ldots, \boldsymbol{m}(\boldsymbol{y}, \boldsymbol{z})\}$, where:

$$\boldsymbol{m}(\boldsymbol{y},\boldsymbol{z}) \equiv \sum_{\rho\in N_1(\boldsymbol{y},\boldsymbol{z})}\ \sum_{\sigma\in N_{n-2}(\boldsymbol{y},\boldsymbol{z})} \left(\varphi_{(1,\rho),(n-2,\sigma)}(\boldsymbol{y},\boldsymbol{z})\right), \tag{2.35}$$

Rewrite Π(***y***, ***z***) as:

$$\Pi(\boldsymbol{y},\boldsymbol{z}) = \{L_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y},\boldsymbol{z});\ p = 1, 2, \ldots, \boldsymbol{m}(\boldsymbol{y},\boldsymbol{z})\}$$

Denote the arc set associated with $L_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y}, \boldsymbol{z}) \in \Pi(\boldsymbol{y}, \boldsymbol{z})$, as:

$$\mathbf{a}_p(\boldsymbol{y}, \boldsymbol{z}) \equiv \{a_{q,\nu_{q,(1,\alpha_p),(n-2,\beta_p)},\kappa_p}(\boldsymbol{y}, \boldsymbol{z});\; q = 1, 2, \ldots, n\text{-}2\}.$$

Then, from constraints 2.7-2.11 and Proposition 3, we must have:

$$\bigcup_{p \in \pi(\boldsymbol{y},\boldsymbol{z})} \left(\mathbf{a}_p(\boldsymbol{y}, \boldsymbol{z})\right) = A(\boldsymbol{y}, \boldsymbol{z}) \tag{2.36}$$

Hence, $\Pi(\boldsymbol{y}, \boldsymbol{z})$ must *cover (**y**, **z**)*.

Note that because of constraints 2.14 and the connectivity requirements 2.8, arcs originating at the same stage of Graph G(***y***, ***z***) must belong to distinct *TSP tours in (**y**, **z**)*. Note also that a given *TSP tour in (**y**, **z**)* cannot be represented as a convex combination of other *TSP tours in (**y**, **z**).* Hence, the flows along distinct *TSP tours in (**y**, **z**)* must be additive at any given stage of Graph G(***y***, ***z***).

We will now consider conditions 2.32 - 2.34 in turn.

i) *Condition 2.32*. Constraints 2.8 combined with the additivity of the flow amounts discussed above imply that we must have:

$$y_{i_{1,\rho},1,j_{1,\rho},\, i_{1,\rho},1,j_{1,\rho}} = \sum_{\sigma \in N_s(\boldsymbol{y},\, \boldsymbol{z})} \; \sum_{p \in \pi(\boldsymbol{y},\boldsymbol{z}) \,\big|\, \alpha_p = \rho;\, (i_{s,\sigma},\, s,\, j_{s,\sigma}) \in \mathbf{a}_p(\boldsymbol{y},\, \boldsymbol{z})} (\lambda_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y}, \boldsymbol{z}))$$

$$\forall\, \rho \in N_1(\boldsymbol{y}, \boldsymbol{z});\ \text{and } s \in R\backslash\{1\} \tag{2.37}$$

From Lemma 1-i), we must also have:

$$y_{i_{1,\rho},1,j_{1,\rho},\, i_{1,\rho},1,j_{1,\rho}} = \sum_{\sigma \in N_s(\boldsymbol{y},\, \boldsymbol{z})} y_{i_{1,\rho},1,j_{1,\rho},\, i_{s,\sigma},s,j_{s,\sigma}}\,, \quad \forall\, s \in R\backslash\{1\} \tag{2.38}$$

Combining 2.37 with 2.38 and re-arranging gives:

$$\sum_{\sigma \in N_s(\boldsymbol{y},\, \boldsymbol{z})} \Bigg(y_{i_{1,\rho},1,j_{1,\rho},\, i_{s,\sigma},s,j_{s,\sigma}} + - \sum_{p \in \pi(\boldsymbol{y},\boldsymbol{z}) \,\big|\, \alpha_p = \rho;\, (i_{s,\sigma},\, s,\, j_{s,\sigma}) \in \mathbf{a}_p(\boldsymbol{y},\, \boldsymbol{z})} (\lambda_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y}, \boldsymbol{z})) \Bigg) = 0$$

$$\forall\, \rho \in N_1(\boldsymbol{y}, \boldsymbol{z}),\ \text{and } s \in R\backslash\{1\}, \tag{2.39}$$

From the additivity of the flows along distinct *TSP tours in (**y**, **z**)* at any given stage discussed above, we must also have:

$$y_{i_{1,\rho},1,j_{1,\rho},\, i_{s,\sigma},s,j_{s,\sigma}} \geq \sum_{p \in \pi(\boldsymbol{y},\boldsymbol{z}) \,\mid\, \alpha_p = \rho;\, (i_{s,\sigma},\, s,\, j_{s,\sigma}) \in \mathbf{a}_p(\boldsymbol{y},\, \boldsymbol{z})} (\lambda_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y},\boldsymbol{z}))$$

$$\forall\ \rho \in N_1(\boldsymbol{y}, \boldsymbol{z}),\ s \in R\backslash\{1\},\ \text{and } \sigma \in N_s(\boldsymbol{y}, \boldsymbol{z}) \tag{2.40}$$

Combining 2.40 and 2.39 gives:

$$y_{i_{1,\rho},1,j_{1,\rho},\, i_{s,\sigma},s,j_{s,\sigma}} = \sum_{p \in \pi(\boldsymbol{y},\boldsymbol{z}) \,\mid\, \alpha_p = \rho;\, (i_{s,\mu},\, s,\, j_{s,\mu}) \in \mathbf{a}_p(\boldsymbol{y},\, \boldsymbol{z})} (\lambda_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y},\boldsymbol{z}))$$

$$\forall\ \rho \in N_1(\boldsymbol{y}, \boldsymbol{z}),\ s \in R\backslash\{1\},\ \text{and } \sigma \in N_s(\boldsymbol{y}, \boldsymbol{z}) \tag{2.41}$$

Condition 2.32 follows from 2.37, and the combination of 2.41 and constraints 2.7.

ii) *Condition 2.33*. From Proposition 3, we have:

$$\forall\ (r, s) \in R^2,\ r < s; \text{ and } (\rho, \sigma) \in (N_r(\boldsymbol{y}, \boldsymbol{z}), N_s(\boldsymbol{y}, \boldsymbol{z})),$$

$$y_{r_{1,\rho},r,j_{r,\rho},\, i_{s,\sigma},s,j_{s,\sigma}} > 0 \iff \exists\ p \in \pi(\boldsymbol{y}, \boldsymbol{z}) \ni: ((i_{r,\rho}, r, j_{r,\rho}), (i_{s,\sigma}, s, j_{s,\sigma})) \in (\mathbf{a}_p(\boldsymbol{y}, \boldsymbol{z}))^2. \tag{2.42}$$

Combining 2.42 with Condition 2.36, we must have:

$$y_{i_{r,\rho},r,j_{r,\rho},\, i_{r,\rho},r,j_{r,\rho}} =$$

$$= \sum_{\sigma \in N_s(\boldsymbol{y},\, \boldsymbol{z})} \ \sum_{p \in \pi(\boldsymbol{y},\boldsymbol{z}) \,\mid\, ((i_{r,\rho},\, r,\, j_{r,\rho}),(i_{s,\sigma},\, s,\, j_{s,\sigma})) \in (\mathbf{a}_p(\boldsymbol{y},\, \boldsymbol{z}))^2} (\lambda_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y}, \boldsymbol{z}))$$

$$\forall\ (r, s) \in R^2,\ s > r; \text{ and } \rho \in N_r(\boldsymbol{y}, \boldsymbol{z}) \tag{2.43}$$

Also, from Lemma 1-i), we must have:

$$y_{i_{r,\rho},r,j_{r,\rho},\, i_{r,\rho},r,j_{r,\rho}} = \sum_{\sigma \in N_s(\boldsymbol{y},\, \boldsymbol{z})} y_{i_{r,\rho},r,j_{r,\rho},\, i_{s,\sigma},s,j_{s,\sigma}},$$

$$\forall\ (r, s) \in R^2,\ s > r; \text{ and } \rho \in N_r(\boldsymbol{y}, \boldsymbol{z}) \tag{2.44}$$

Combining 2.43 with 2.44 and re-arranging gives:

$$\sum_{\sigma \in N_s(\boldsymbol{y},\, \boldsymbol{z})} \Big(y_{i_{r,\rho},r,j_{r,\rho},\, i_{s,\sigma},s,j_{s,\sigma}} \quad +$$

$$- \sum_{p \in \pi(\boldsymbol{y},\boldsymbol{z}) \,\middle|\, ((i_{r,\rho},\, r,\, j_{r,\rho}),(i_{s,\sigma},\, s,\, j_{s,\sigma})) \,\in\, (\mathbf{a}_p(\boldsymbol{y},\, \boldsymbol{z}))^2} (\lambda_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y},\, \boldsymbol{z})) \;\Big) \;=\; 0$$

$$\forall\ (r, s) \in R^2,\ s > r;\ \text{ and }\ \rho \in N_r(\boldsymbol{y}, \boldsymbol{z}) \qquad (2.45)$$

From 2.42 and the additivity of the flows along distinct *TSP tours in **(y, z)*** at any given stage discussed above, we must also have:

$$y_{i_{r\rho},r,j_{r,\rho},\, i_{s,\sigma},s,j_{s,\sigma}} \geq \sum_{p \in \pi(\boldsymbol{y},\boldsymbol{z}) \,\middle|\, ((i_{r,\rho},\, r,\, j_{r,\rho}),(i_{s,\sigma},\, s,\, j_{s,\sigma})) \,\in\, (\mathbf{a}_p(\boldsymbol{y},\, \boldsymbol{z}))^2} (\lambda_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y},\, \boldsymbol{z}))$$

$$\forall\ (r, s) \in R^2,\ s > r;\ \text{ and } (\rho, \sigma) \in (N_r(\boldsymbol{y}, \boldsymbol{z}), N_s(\boldsymbol{y}, \boldsymbol{z})) \qquad (2.46)$$

Condition 2.33 follows from the combination of 2.45 and 2.46.

iii) *Condition 2.34*. From Proposition 3, we have:

$$\forall\ (r, s, t) \in R^3,\ \ r < s < t;\ \text{ and } (\rho, \sigma, \tau) \in (N_r(\boldsymbol{y}, \boldsymbol{z}), N_s(\boldsymbol{y}, \boldsymbol{z}), N_t(\boldsymbol{y}, \boldsymbol{z})),$$

$$z_{i_{r,\rho},r,j_{r,\rho},\, i_{s,\sigma},s,j_{s,\sigma},i_{t,\tau},t,j_{t,\tau}} > 0 \;\Leftrightarrow\; \exists\ p \in \pi(y, z) \quad \ni:$$

$$((i_{r,\rho}, r, j_{r,\rho}), (i_{s,\sigma}, s, j_{s,\sigma}), (i_{t,\tau}, t, j_{t,\tau})) \;\in\; ((\mathbf{a}_p(\boldsymbol{y}, \boldsymbol{z}))^3. \qquad (2.47)$$

From 2.47, Lemma 1-ii), and Condition 2.32, we must have:

$$y_{i_{r,\rho},r,j_{r,\rho},\, i_{r,\rho},r,j_{r,\rho}} =$$

$$= \sum_{\sigma \in N_s(\boldsymbol{y},\, \boldsymbol{z})} \sum_{\tau \in N_t(\boldsymbol{y},\, \boldsymbol{z})} \sum_{p \in \pi(\boldsymbol{y},\boldsymbol{z}) \,\middle|\, ((i_{r,\rho},r,j_{r,\rho}),(i_{s,\sigma},s,j_{s,\sigma}),(i_{t,\tau},t,j_{t,\tau})) \in (\mathbf{a}_p(\boldsymbol{y},\, \boldsymbol{z}))^3} (\lambda_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y},\, \boldsymbol{z}))$$

$$\forall\ (r, s, t) \in R^3,\ t > s > r;\ \text{and } \rho \in N_r(\boldsymbol{y}, \boldsymbol{z}) \qquad (2.48)$$

Also, from Lemma 1-ii), we must have:

$$y_{i_{r,\rho},r,j_{r,\rho},\, i_{r,\rho},r,j_{r,\rho}} = \sum_{\sigma \in N_s(\boldsymbol{y},\, \boldsymbol{z})} \sum_{\tau \in N_t(\boldsymbol{y},\, \boldsymbol{z})} z_{i_{r,\rho},r,j_{r,\rho},\, i_{s,\sigma},s,j_{s,\sigma},\, i_{t,\tau},t,j_{t,\tau}},$$

$$\forall\ (r, s, t) \in R^3,\ t > s > r;\ \text{and } \rho \in N_r(\boldsymbol{y}, \boldsymbol{z}) \qquad (2.49)$$

Combining 2.48 with 2.49 and re-arranging gives:

$$\sum_{\sigma\in N_s(\boldsymbol{y},\boldsymbol{z})}\ \sum_{\tau\in N_t(\boldsymbol{y},\boldsymbol{z})}\Bigg(z_{i_{r,\rho},r,j_{r,\rho},\, i_{s,\sigma},s,j_{s,\sigma},\, i_{t,\tau},t,j_{t,\tau}}\quad +$$

$$-\sum_{p\in\pi(\boldsymbol{y},\boldsymbol{z})\,\big|\,((i_{r,\rho},r,j_{r,\rho}),(i_{s,\sigma},s,j_{s,\sigma}),(i_{t,\tau},t,j_{t,\tau}))\in(\mathbf{a}_p(\boldsymbol{y},\boldsymbol{z}))^3}(\lambda_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y},\boldsymbol{z}))\ \Bigg)\ =\ 0$$

$$\forall\ (r, s, t) \in R^3,\ t > s > r;\ \text{and}\ \rho\in N_r(\boldsymbol{y},\boldsymbol{z}) \qquad (2.50)$$

From 2.47 and the additivity of the flows along distinct *TSP tours in (**y**, **z**)* discussed above, we must also have:

$$z_{i_{r,\rho},r,j_{r,\rho},\, i_{s,\sigma},s,j_{s,\sigma},\, i_{t,\tau},t,j_{t,\tau}}\ \geq$$

$$\sum_{p\in\pi(\boldsymbol{y},\boldsymbol{z})\,\big|\,((i_{r,\rho},r,j_{r,\rho}),(i_{s,\sigma},s,j_{s,\sigma}),(i_{t,\tau},t,j_{t,\tau}))\in(\mathbf{a}_p(\boldsymbol{y},\boldsymbol{z}))^3}(\lambda_{\alpha_p,\beta_p,\kappa_p}(\boldsymbol{y},\boldsymbol{z}))$$

$$\forall\ (r, s, t) \in R^3,\ t > s > r;\ \text{and}\ (\rho, \sigma, \tau) \in (N_r(\boldsymbol{y},\boldsymbol{z}),\ N_s(\boldsymbol{y},\boldsymbol{z}),\ N_t(\boldsymbol{y},\boldsymbol{z})) \qquad (2.51)$$

Condition 2.34 follows from the combination of 2.50 and 2.51.

Q.E.D.

Proposition 4 is illustrated in Figure 2.2, along with the notation discussed above.

Figure 2.2 Here

Proposition 5

The following statements are true of basic feasible solutions (BFS) of *Problem BLP* and TSP tours:

1) Every BFS of *Problem BLP* corresponds to a TSP tour;

2) Every TSP tour corresponds to a BFS of *Problem BLP*;

3) The mapping of BFS's of *Problem BLP* onto TSP tours is surjective.

Proof:

1) Correspondence of a BFS of *Problem BLP* to a TSP tour follows from the fact that every TSP tour corresponds to a feasible solution to *Problem BLP* (Proposition 1), the fact that every

feasible solution to *Problem BLP* corresponds to a convex combination of TSP tours (Proposition 4), and the fact that a BFS cannot be a convex combination of other feasible solutions.

2) Correspondence of a TSP tour to a BFS of *Problem BLP* follows from Proposition 1, Proposition 4, and the fact that a given TSP tour cannot be represented as a convex combination of other TSP tours.

3) It can be easily verified that the number of non-zero components of a *feasible solution corresponding to a given TSP tour* is less than n^3, and that the number of constraints of *Problem BLP* exceeds n^3. Hence, Statement 1) of the proposition implies that there must be basic variables that are equal to zero in any BFS of *Problem BLP*. The surjective nature of the "BFS's-to-TSP tours" mapping follows from this and the fact that BFS's of *Problem BLP* that have the same set of positive variables in common correspond to the same TSP tour.

Corollary 1

Let *Conv((•))* denote the convex hull of the feasible set of *Problem (•)*. Then, we have:

Conv(LP) = Conv(IP).

Proof:

The proof follows directly from Propositions 1 and 5.

Q.E.D.

Corollary 2

Problem BLP and *Problem IP* (and therefore, *Problem TSP*) are equivalent.

Proof:

The proof follows directly from Propositions 1 and 5.

Q.E.D.

Corollary 3

Computational complexity classes *P* and *NP* are equal.

Proof:

First, note that *Problem BLP* has $O(n^9)$ variables and $O(n^7)$ constraints. Hence, it can be explicitly stated in polynomial time. The proposition follows from this, Corollary 2, the NP-Completeness of the TSP decision problem (see Garey and Johnson [1979], or Nemhauser and Wolsey [1988], among others), and the fact that an explicitly-stated instance of *Problem BLP* can be solved in polynomial-time (see Katchiyan [1979], or Karmarkar [1984]).

Q.E.D.

3. Numerical Implementation

Note that because of Proposition 4, the upper bounds on the y_{irjkst} and $z_{irjupvkst}$ variables in constraints 2.18 are redundant in *Problem BLP*. Also, it is easy to observe that the *visits* requirements constraints 2.12 – 2.13 are not used (and hence, are not needed) in any of the proofs in section 2 of this paper. Hence, those constraints (i.e., constraints 2.12 - 2.13) are redundant in *Problem BLP* (and therefore, in *Problem IP*). Hence, in implementing the model, we discarded constraints 2.12 – 2.13 and replaced constraints 2.18 with simple non-negativity constraints on the y_{irjkst} and $z_{irjupvkst}$ variables. In addition, we did not explicitly consider constraints 2.14 and the variables they restrict to zero. This required appropriately re-writing/expanding the other constraints of the model. The resulting "streamlined" model that we implemented is as follows:

Problem SLP:

Minimize

$$Z_{LP}(\mathbf{y}, \mathbf{z}) = \sum_{r \in R} \sum_{i \in M} \sum_{j \in M \setminus \{i\})} c_{irj} y_{irjirj} \qquad (3.1)$$

Subject to:

- Flow Conservations (Corresponding to 2.6 and 2.7):

$$\sum_{i \in M} \sum_{j \in M \backslash \{i\}} y_{i,1,j,i,1,j} = 1 \qquad (3.2)$$

$$y_{i,2,ji,2,j} - \sum_{u \in (M \backslash \{i,j\})} y_{u,1,ii,2,j} = 0 \qquad i \in M;\ j \in M\backslash\{i\} \qquad (3.3)$$

$$y_{irjirj} - \sum_{u \in (M \backslash \{i,j\})} \sum_{v \in (M \backslash \{i,j,u\})} y_{u,1,virj} = 0 \qquad r \in R,\ r \geq 3;\ i \in M;\ j \in M\backslash\{i\} \qquad (3.4)$$

- Flow Connectivities (Corresponding to 2.8):

$$y_{irjirj} - \sum_{t \in (M \backslash \{i,j\})} y_{irjj,r+1,t} = 0 \qquad r \in R,\ r \leq n\text{-}3;\ i \in M;\ j \in M\backslash\{i\} \qquad (3.5)$$

$$y_{irjj,r+1,t} - \sum_{k \in (M \backslash \{i,j,t\})} y_{irjt,r+2,k} = 0 \qquad i \in M;\ j \in M\backslash\{i\};\ t \in M\backslash\{i,j\};\ r \in R,\ r \leq n\text{-}4, \qquad (3.6)$$

$$\sum_{k \in M} y_{irjkst} - \sum_{k \in M} y_{irjt,s+1,k} = 0 \qquad i, j, t \in M;\ s, r \in R,\ r \leq n\text{-}5,\ r+2 \leq s \leq n\text{-}3 \qquad (3.7)$$

- Flow Layering Consistencies A (Corresponding to 2.9):

$$y_{i,p-1,uupv} - \sum_{t \in M \backslash \{i,u,v\}} z_{i,p-1,uupvv,p+1,t} = 0 \qquad i \in M;\ u \in M\backslash\{i\};\ v \in M\backslash\{i,u\};\ p \in R,\ 2 \leq p \leq n\text{-}3 \qquad (3.8)$$

$$y_{i,p-1,uupv} - \sum_{k \in M \backslash \{i,u,v\}} \sum_{t \in M \backslash \{i,u,v,k\}} z_{i,p-1,uupvkst} = 0 \qquad i \in M;\ u \in M\backslash\{i\};\ v \in M\backslash\{i,u\};\ p, s \in R,\ 2 \leq p \leq n\text{-}4,\ s \geq p+2 \qquad (3.9)$$

$$y_{irjupv} - \sum_{t \in M \backslash \{i,j,u,v\}} z_{irjupvv,p+1,t} = 0 \qquad i \in M;\ j \in M\backslash\{i\};\ u \in M\backslash\{i,j\};\ v \in M\backslash\{i,j,u\};\ p, r \in R,\ 3 \leq p \leq n\text{-}3,\ r \leq p\text{-}2 \qquad (3.10)$$

$$y_{irjupv} - \sum_{k \in M \backslash \{i,j,u,v\}} \sum_{t \in M \backslash \{i,j,u,v,k\}} z_{irjupvkst} = 0 \qquad i \in M;\ j \in M\backslash\{i\};\ u \in M\backslash\{i, j\};\ v \in M\backslash\{i,j, u\};\ p, r, s \in R,\ 3 \leq p \leq n\text{-}4,\ r \leq p\text{-}2,\ s \geq p+2 \qquad (3.11)$$

- Flow Layering Consistencies B (Corresponding to 2.11):

$$y_{irjk,r+2,t} - z_{irjj,r+1,kk,r+2,t} = 0 \qquad i \in M;\ j \in M\backslash\{i\};\ k \in M\backslash\{i, j\};\ t \in M\backslash\{i,j,k\};$$

$r \in R,\ r \leq n\text{-}4$ (3.12)

$$y_{irjkst} - \sum_{v \in M\backslash\{i,j,k,t\}} z_{irj\,j,r+1,v\,kst} = 0 \qquad i \in M;\ j \in M\backslash\{i\};\ k \in M\backslash\{i, j\};\ t \in M\backslash\{i,j, k\};$$

$$r, s \in R,\ r \leq n\text{-}5,\ \ s \geq r+3 \qquad (3.13)$$

$$y_{irj\,kst} - \sum_{u \in M\backslash\{i,j,k,t\}} z_{irj\,u,s-1,k\,kst} = 0 \qquad i \in M;\ j \in M\backslash\{i\};\ k \in M\backslash\{i,j\};\ t \in M\backslash\{i,j,k\}$$

$$r, s \in R,\ r \leq n\text{-}5,\ s \geq r+3 \qquad (3.14)$$

$$y_{irjkst} - \sum_{u \in M\backslash\{i,j,k,t\}} \sum_{v \in M\backslash\{i,j,k,t,u\}} z_{irjupv\,kst} = 0 \qquad i \in M;\ j \in M\backslash\{i\};\ k \in M\backslash\{i, j\};\ t \in M\backslash\{i,j, k\};\ p, r, s \in R,$$

$$r \leq n\text{-}6,\ s \geq r+4\ \ r+2 \leq p \leq s\ \text{-}2 \qquad (3.15)$$

- Flow Layering Consistencies C (Corresponding to 2.11):

$$y_{upv\,v,p+1,t} - \sum_{i \in M\backslash\{u,v,t\}} z_{i,p-1,u\,upv\,v,p+1,t} = 0 \qquad u \in M;\ v \in M\backslash\{u\};\ t \in M\backslash\{u,v\};$$

$$p \in R,\ 2 \leq p \leq n\text{-}3 \qquad (3.16)$$

$$y_{upv\,v,p+1,t} - \sum_{i \in M\backslash\{u,v,t\}} \sum_{j \in M\backslash\{u,v,t,i\}} z_{irjupv\,v,p+1,t} = 0 \qquad u \in M;\ v \in M\backslash\{u\};\ t \in M\backslash\{u,v\};$$

$$p, r \in R,\ 3 \leq p \leq n\text{-}3,\ r \leq p\text{-}2 \qquad (3.17)$$

$$y_{upvkst} - \sum_{i \in M\backslash\{u,v,k,t\}} z_{i,p-1,u\,upv\,kst} = 0 \qquad u \in M;\ v \in M\backslash\{u\};\ k \in M\backslash\{u, v\};\ t \in M\backslash\{u,v,k\}$$

$$p, s \in R,\ 2 \leq p \leq n\text{-}4,\ s \geq p+2 \qquad (3.18)$$

$$y_{upvkst} - \sum_{i \in M\backslash\{u,v,k,t\}} \sum_{j \in M\backslash\{u,v,k,t,i\}} z_{irj\,upv\,kst} = 0 \qquad u \in M;\ v \in M\backslash\{u\};\ k \in M\backslash\{u, v\};\ t \in M\backslash\{u,v,k\};$$

$$p, r, s \in R,\ 3 \leq p \leq n\text{-}4;\ r \leq p\text{-}2,\ s \geq p+2 \qquad (3.19)$$

- Non-negativities (Corresponding to (2.18)):

$$y_{irjkst},\ z_{irjupvkst} \geq 0 \qquad \forall\ i, r, j, u, p, v, k, s, t \qquad (3.20)$$

□

We used the simplex method implementation of the *OSL* optimization package (IBM) to solve a set of randomly-generated 7-city and 8-city problems. The travel costs in these randomly-generated problems were taken as uniform integer numbers between 1 and 300. For each problem size, 3 instances had symmetric costs, and 3 instances had asymmetric costs. We also solved an additional set of 7-city problems we refer to as “extreme-symmetry” problems. These “extreme-symmetry” problems are labeled “*xtsp71*,” “*xtsp72*,” and “*xtsp73*,” respectively. In *Problem xtsp71*, all travel costs, t_{ij}, are equal to (-1), except for t_{12} and t_{21} which are equal to 1, respectively. In *Problem xtsp72*, all travel costs, t_{ij}, are equal to 1, except for t_{12} and t_{21} which are equal to (-100), respectively. Finally, in *Problem xtsp73*, all travel costs, t_{ij}, are equal to 0, except for t_{12} and t_{21} which are equal to 1, respectively.

We solved each of the test problems described above using their dual forms, respectively. We also solved the primal forms of the 7-city problems. The costs data and solutions obtained using the dual forms (except for *Problems xtsp71* and *xtsp73*) are shown in Appendix A of this paper. Details of the solutions from the dual forms for *Problems xtsp71* and *xtsp73*, respectively, are given in Appendices B through E. The computational results are summarized in Tables 3.1 and 3.2, for the dual and primal forms, respectively.

Using the dual forms, the averages of the numbers of iterations for the 7-city problems were 592.7, 1,825.7, and 8,433.3 for the asymmetric, symmetric, and “extreme-symmetry” problems, respectively. The corresponding average computational times were 0.0993, 0.3230, and 14.5523 CPU seconds of Sony VAIO VGN-FE 770G notebook computer (1.8 GHz Intel Core 2 Duo Processor) time, respectively. For the 8-city problems the averages of the numbers of iterations were 19,622.3 and 42,082.7 for the asymmetric and symmetric problems, respectively. The corresponding average computational times were 182.4433 and 700.9897 CPU seconds, respectively.

For the “extreme-symmetry” *Problems xtsp71* and *xtsp73*, the simplex method using the dual forms stopped with optimal, but non-extreme-point solutions, respectively. It appears from the OSL outputs for these problems (shown in Appendices B and D, respectively) that this may be due to numerical difficulties. Note also, that in both cases the solutions obtained (using the dual form) are consistent with Proposition 4 of section 2 of this paper in that they respectively are indeed convex combinations of *feasible solutions corresponding to TSP tours*. The TSP tours that comprise the solution obtained for *Problem xtsp71* are: $1\rightarrow3\rightarrow6\rightarrow2\rightarrow4\rightarrow7\rightarrow5\rightarrow1$, $1\rightarrow3\rightarrow6\rightarrow4\rightarrow2\rightarrow5\rightarrow7\rightarrow1$, $1\rightarrow6\rightarrow3\rightarrow2\rightarrow4\rightarrow5\rightarrow7\rightarrow1$, and $1\rightarrow6\rightarrow3\rightarrow4\rightarrow2\rightarrow7\rightarrow5\rightarrow1$, respectively, each with "weight" (or, *flow value*) $\lambda = 0.25$ (see Appendix C). Similarly, the solution obtained using the dual form for *Problem xtsp73 consists of* the twelve (12) *TSP tours* shown in Appendix E. Hence, we believe that these solutions further validate, somewhat, our Proposition 4 in particular. Finally, note that the TSP tours that comprise any given feasible solution to our LP model, as stipulated by Proposition 4, can be systematically identified in a straightforward manner using the "paths" defined/developed in section 2 of this paper. Hence, an optimal TSP tour can be retrieved from any given (feasible) non-extreme-point optimum in a straightforward manner.

For the primal forms, the average number of iterations was 10,862.3, 13,996.7, and 19,858.3 for the asymmetric, symmetric, and "extreme-symmetry" problems, respectively. The corresponding average computational times were 28.7080, 40.4533, and 62.2340 CPU seconds, respectively. The average number of TSP tours examined in the simplex procedure was 2.3, 1.3, and 1.0 for the asymmetric, symmetric, and "extreme-symmetry" problems, respectively.

Overall, we believe our computational experience provided the empirical validation of our theoretical developments in section 2 of this paper that we expected. The dual forms consistently outperformed the primal forms. However, the primal form appears to hold some promise with respect to future developments aimed at solving large-sized problems because of the small number of TSP tours that are examined when the primal form is used. We believe that greater overall computational performance may be achieved using the primal form if the simplex pivoting step can be customized judiciously-enough to take advantage of the special (TSP tour) nature of the basic feasible solutions of our proposed linear programming model.

4. Conclusions

We have presented a first polynomial-sized linear programming formulation of the TSP. Our approach can be used to formulate general integer programming problems as linear programs, since the general integer programming problem is *polynomially transformable* to a Hamiltonian Path problem (see Johnson and Papadimitriou [1985, pp. 61-74]. Note however, that the Hamiltonian Path problem resulting from the transformation involved is very-large-scale. Hence, we believe a key issue at this point is the question of whether the suggested modeling approach can be developed into a more general, unified framework that would extend in a more natural way to other *NP-Complete* problems (see Garey and Johnson [1979], or Nemhauser and Wolsey [1988], among others).

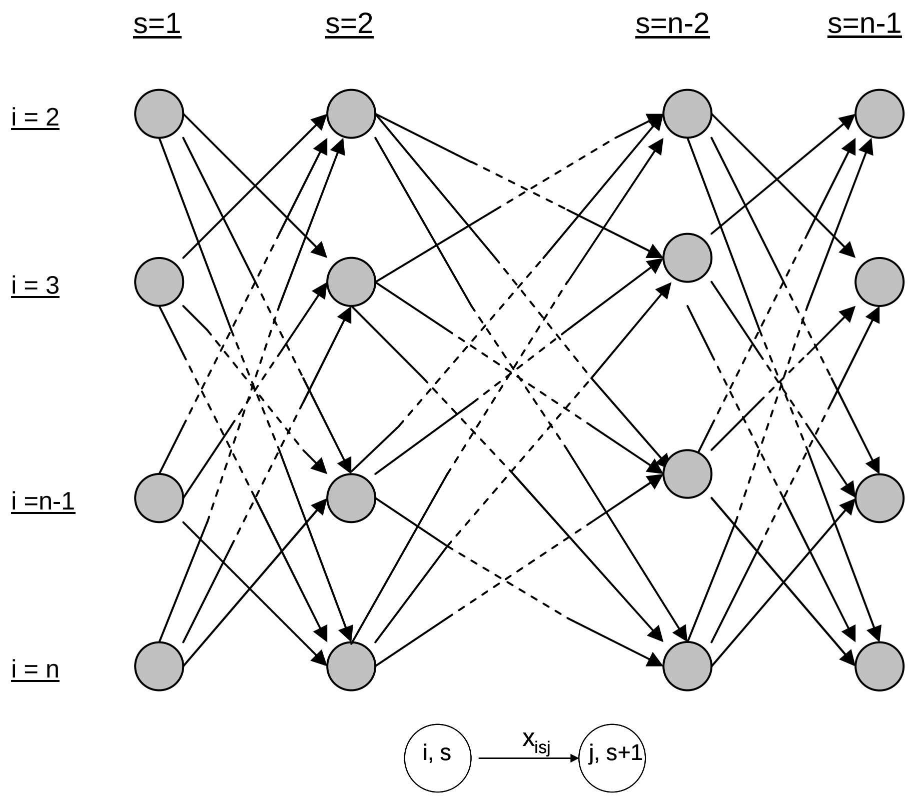
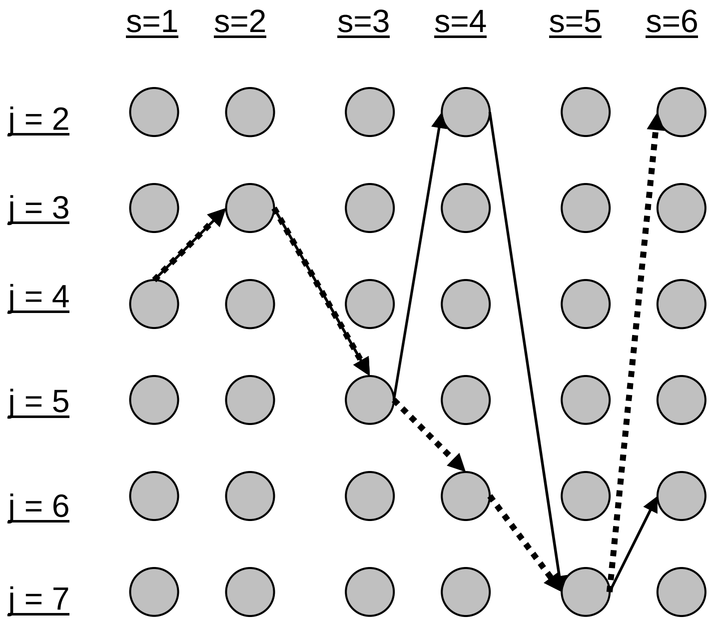


Figure 2.1: Illustration of the Network Sub-Structure of *Problem TSP*

- <u>Graphical illustration of a feasible solution</u>



- <u>Graph theoretical notation</u>

$G(\boldsymbol{y}, \boldsymbol{z}) = (V(\boldsymbol{y}, \boldsymbol{z}), A(\boldsymbol{y}, \boldsymbol{z}))$; $V(\boldsymbol{y}, \boldsymbol{z}) = \{(4,1), (3,2), (5,3), (2,4), (6,4), (7,5), (2,6), (6,6)\}$,

$A(\boldsymbol{y}, \boldsymbol{z}) = \{(4,1,3), (3,2,5), (5,3,2), (5,3,6), (2,4,7), (6,4,7), (7,5,2), (7,5,6)\}$.

$X_1(\boldsymbol{y}, \boldsymbol{z}) = \{ (4,1,3) \}$; $X_2(\boldsymbol{y}, \boldsymbol{z}) = \{ (3,2,5) \}$; $X_3(\boldsymbol{y}, \boldsymbol{z}) = \{ (5,3,1), (5,3,6 \}$;

$X_4(\boldsymbol{y}, \boldsymbol{z}) = \{ (1,4,7), (6,4,7 \}$; $X_5(\boldsymbol{y}, \boldsymbol{z}) = \{ (7,5,1), (7,5,6 \}$.

- <u>Vector $\boldsymbol{y}$ (non-zero components)</u>

$y_{413413} = y_{325325} = \lambda_{1,1,1} + \lambda_{1,2,1} = 1$; $y_{532532} = y_{247247} = y_{756756} = \lambda_{1,1,1}$;
$y_{536536} = y_{647647} = y_{752752} = \lambda_{1,2,1}$;

$y_{413325} = \lambda_{1,1,1} + \lambda_{1,2,1} = 1$

$y_{413532} = y_{413247} = y_{413756} = y_{325532} = y_{325247} = y_{325756} = y_{532247} = y_{532752} = y_{247756} = \lambda_{1,1,1}$;

$y_{413536} = y_{413647} = y_{413752} = y_{325536} = y_{325647} = y_{325752} = y_{536647} = y_{536752} = y_{647752} = \lambda_{1,2,1}$.

- <u>Vector $\boldsymbol{z}$ (non-zero components)</u>

$z_{413\ 325\ 532} = z_{413\ 325\ 247} = z_{413\ 325\ 756} = z_{413\ 532\ 247} = z_{413\ 532\ 752} = z_{413\ 247\ 756} = \lambda_{1,1,1}$;

$z_{325\ 532\ 247} = z_{325\ 532\ 756} = z_{325\ 247\ 756} = \lambda_{1,1,1}$; $z_{532\ 247\ 756} = \lambda_{1,1,1}$;

$z_{413\ 325\ 536} = z_{413\ 325\ 647} = z_{413\ 325\ 752} = z_{413\ 536\ 647} = z_{413\ 536\ 752} = z_{413\ 647\ 752} = \lambda_{1,2,1}$;

$z_{325\ 536\ 647} = z_{325\ 536\ 752} = z_{325\ 647\ 752} = \lambda_{1,2,1}$; $z_{536\ 647\ 752} = \lambda_{1,2,1}$.

- <u>*TSP Tours in* ($\boldsymbol{y}, \boldsymbol{z}$)</u>

$T_{1,1,1} = \langle 4, 3, 5, 2, 7, 6 \rangle$, *flow value* $= \lambda_{1,1,1}$; $L_{1,2,1} = \langle 4, 3, 5, 6, 7, 2 \rangle$, *flow value* $= \lambda_{1,2,1}$.

Figure 2.2: <u>Illustration of a Feasible Solution to *Problem BLP*</u>

Problem Size[1]	Problem Name[2]	Computational Performance		Problem Value
		Number of Iterations	CPU Seconds[3]	
7[a] (8,910[b]; 8881[c])	atsp71	774	0.141	414
	atsp72	605	0.094	468
	atsp73	399	0.063	354
	Average	**592.7**	**0.0993**	**---**
	stsp71	1,710	0.297	503
	stsp72	2,091	0.375	531
	stsp73	1,676	0.297	637
	Average	**1,825.7**	**0.3230**	**---**
	xtsp71	9,914	19.563	(-7) [(4)]
	xtsp72	8,801	12.172	(-94)
	xtsp73	6,585	11.922	0 [(4)]
	Average	**8,433.3**	**14.5523**	**---**
8 (63,462[b]; 40,321[c])	atsp81	23,297	265.157	331
	atsp82	20,967	212.157	371
	atsp83	14,603	70.016	608
	Average	**19,622.3**	**182.4433**	**---**
	stsp81	30,452	452.890	411
	stsp82	65,507	1,299.454	799
	stsp85	30,289	350.625	707
	Average	**42,082.7**	**700.9897**	**---**

1: ([a]): number of cities; ([b]): number of variables; ([c]): number of constraints

2: “atsp··” ⇒ asymmetric costs; “stsp··” ⇒ symmetric costs;

“xtsp··” ⇒ “extreme symmetry” problem

3: Sony VAIO VGN-FE 770G notebook computer (1.8 GHz Intel Core 2 Duo Processor)

4: Non-extreme-point optimal solution

Table 3.1: Summary of the Computational Results for the Dual Forms

Problem Size[1]	Problem Name[2]	Computational Performance			Problem Value
		Number of Tours[3]	Number of Iterations	CPU Seconds[4]	
7[a] (8,910[b]; 8,881[c])	atsp71	2	9,413	22.781	414
	atsp72	2	4,327	4.109	468
	atsp73	3	18,847	59.234	354
	Average	**2.3**	**10,862.3**	**28.7080**	**---**
	stsp71	2	8,959	21.891	503
	stsp72	1	10,542	26.859	531
	stsp73	1	22,489	72.610	637
	Average	**1.3**	**13,996.7**	**40.4533**	**---**
	xtsp71	1	8,264	17.984	(-7)
	xtsp72	1	37,698	130.828	(-94)
	xtsp73	1	13,613	37.890	0
	Average	**1.0**	**19,858.3**	**62.2340**	**---**

1: ([a]): number of cities; ([b]): number of variables; ([c]): number of constraints

2: “atsp··” ⇒ asymmetric costs; “stsp··” ⇒ symmetric costs;

“xtsp··” ⇒ “extreme symmetry” problem

3: Total number of TSP tours (including the optimal tour) examined

4: Sony VAIO VGN-FE 770G notebook computer (1.8 GHz Intel Core 2 Duo Processor)

Table 3.2: Summary of the Computational Results for the Primal Forms

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Appendix A:

Cost and Solution* Data for the

Test Problems Solved

(*: All reported solutions are from the dual-form runs, unless indicated otherwise)

	1	2	3	4	5	6	7	Cost
1	5000	181	157	91	209	153	100	100
2	180	5000	240	141	139	65	286	141
3	143	272	5000	45	127	20	121	20
4	40	63	212	5000	144	129	94	40
5	199	182	14	140	5000	281	16	14
6	293	61	79	164	287	5000	296	61
7	51	108	78	247	38	146	5000	38
Optimal Tour:	**1 - 7 - 5 - 3 - 6 - 2 - 4 - 1**							**414**

Table A.1: Cost Data and Solution Information for Problem atsp71

	1	2	3	4	5	6	7	Cost
1	5000	267	79	46	284	167	219	46
2	222	5000	213	242	155	115	157	115
3	32	107	5000	102	195	216	232	32
4	184	29	116	5000	73	281	177	73
5	190	10	174	200	5000	280	40	10
6	161	238	165	86	243	5000	72	72
7	65	227	120	220	139	163	5000	120
Optimal Tour:	**1 - 4 - 5 - 2 - 6 - 7 - 3 - 1**							**468**

Table A.2: Cost Data and Solution Information for Problem atsp72

	1	2	3	4	5	6	7	Cost
1	5000	40	15	248	68	12	258	68
2	39	5000	98	298	296	223	144	144
3	2	130	5000	202	285	178	278	2
4	197	221	71	5000	2	287	252	71
5	58	41	172	297	5000	140	174	41
6	151	218	237	23	146	5000	155	23
7	209	266	236	106	196	5	5000	5
Optimal Tour:	**1 - 5 - 2 - 7 - 6 - 4 - 3 - 1**							**354**

Table A.3: Cost Data and Solution Information for Problem atsp73

	1	2	3	4	5	6	7	Cost
1	5000	173	149	83	201	145	92	149
2	173	5000	172	232	133	131	57	133
3	149	172	5000	277	135	263	37	37
4	83	232	277	5000	119	12	113	83
5	201	133	135	119	5000	32	55	32
6	145	131	263	12	32	5000	204	12
7	92	57	37	113	55	204	5000	57
Optimal Tour:	**1 - 3 - 7 - 2 - 5 - 6 - 4 - 1**							**503**

Table A.4: Cost Data and Solution Information for Problem stsp71

	1	2	3	4	5	6	7	Cost
1	5000	48	23	257	76	20	266	20
2	48	5000	47	106	7	5	231	7
3	23	47	5000	152	10	138	211	23
4	257	106	152	5000	293	186	286	106
5	76	7	10	293	5000	205	229	10
6	20	5	138	186	205	5000	79	79
7	266	231	211	286	229	79	5000	286
Optimal Tour:	**1 - 6 - 7- 4 - 2 - 5 - 3 - 1**							**531**

Table A.5: Cost Data and Solution Information for Problem stsp72

	1	2	3	4	5	6	7	Cost
1	5000	33	8	241	60	5	251	8
2	33	5000	31	91	291	289	215	91
3	8	31	5000	137	293	123	195	31
4	241	91	137	5000	277	171	271	171
5	60	291	293	277	5000	190	213	60
6	5	289	123	171	190	5000	63	63
7	251	215	195	271	213	63	5000	213
Optimal Tour:	**1 - 3 - 2 - 4 - 6 - 7 - 5 - 1**							**637**

Table A.6: Cost Data and Solution Information for Problem stsp73

	1	2	3	4	5	6	7	Cost
1	5000	1	0	0	0	0	0	0
2	1	5000	0	0	0	0	0	0
3	0	0	5000	0	0	0	0	0
4	0	0	0	5000	0	0	0	0
5	0	0	0	0	5000	0	0	0
6	0	0	0	0	0	5000	0	0
7	0	0	0	0	0	0	5000	0
Optimal Tour:	**1 - 6 - 4 – 2 - 5 - 3 - 7 - 1**							**0**

Table A.7: Cost Data and Solution Information for Problem xtsp71 (Solution from Primal Form)

	1	2	3	4	5	6	7	Cost
1	5000	-100	1	1	1	1	1	-100
2	-100	5100	1	1	1	1	1	1
3	1	1	5000	1	1	1	1	1
4	1	1	1	5000	1	1	1	1
5	1	1	1	1	5000	1	1	1
6	1	1	1	1	1	5000	1	1
7	1	1	1	1	1	1	5000	1
Optimal Tour:	**1 - 2 - 6 - 7- 5 - 3 - 4 - 1**							**-94**

Table A.8: Cost Data and Solution Information for Problem xtsp72

	1	2	3	4	5	6	7	Cost
1	5000	1	-1	-1	-1	-1	-1	-1
2	1	5000	-1	-1	-1	-1	-1	-1
3	-1	-1	5000	-1	-1	-1	-1	-1
4	-1	-1	-1	5000	-1	-1	-1	-1
5	-1	-1	-1	-1	5000	-1	-1	-1
6	-1	-1	-1	-1	-1	5000	-1	-1
7	-1	-1	-1	-1	-1	-1	5000	-1
Optimal Tour:	**1 - 3 - 2 - 5 - 6 – 4 - 7 - 1**							**-7**

Table A.9: Cost Data and Solution Information for Problem xtsp73 (Solution from Primal Form)

	1	2	3	4	5	6	7	8	Cost
1	5000	108	84	18	136	80	27	107	27
2	167	5000	68	66	291	213	70	199	68
3	271	54	5000	247	48	266	289	139	48
4	71	56	21	5000	126	109	240	67	71
5	208	242	220	287	5000	6	91	215	6
6	223	277	35	5	174	5000	264	73	73
7	189	1	267	206	89	141	5000	144	1
8	135	163	77	37	78	253	29	5000	37
Optimal Tour:	**1 - 7 - 2 - 3 - 5 - 6 - 8 - 4 - 1**								**331**

Table A.10: Cost Data and Solution Information for Problem atsp81

	1	2	3	4	5	6	7	8	Cost
1	5000	125	101	35	153	97	44	124	35
2	184	5000	85	83	9	229	87	215	9
3	288	71	5000	263	65	283	7	156	7
4	88	73	38	5000	143	126	257	84	38
5	225	259	237	5	5000	23	108	231	23
6	240	294	52	22	191	5000	281	90	90
7	206	17	284	223	106	158	5000	161	17
8	152	180	94	54	95	270	45	5000	152
Optimal Tour:	**1 - 4 - 3 - 7 - 2 - 5 - 6 - 8 - 1**								**371**

Table A.11: Cost Data and Solution Information for Problem atsp82

	1	2	3	4	5	6	7	8	Cost
1	5000	223	198	132	250	195	142	221	132
2	281	5000	182	180	106	28	184	14	14
3	86	168	5000	62	162	81	105	253	162
4	185	170	135	5000	241	223	55	181	55
5	23	57	35	102	5000	120	206	30	57
6	38	93	149	119	289	5000	80	188	38
7	4	115	82	21	204	255	5000	259	82
8	249	278	192	151	193	68	143	5000	68
Optimal Tour:	**1 - 4 - 7 - 3 - 5 - 2 - 8 - 6 - 1**								**608**

Table A.12: Cost Data and Solution Information for Problem atsp83

	1	2	3	4	5	6	7	8	Cost
1	5000	143	118	52	170	115	62	141	62
2	143	5000	201	102	100	26	247	104	26
3	118	201	5000	233	6	88	281	82	82
4	52	102	233	5000	1	24	173	105	52
5	170	100	6	1	5000	90	55	161	6
6	115	26	88	24	90	5000	143	274	24
7	62	247	281	173	55	143	5000	101	55
8	141	104	82	105	161	274	101	5000	104
Optimal Tour:		**1 - 7 - 5 - 3 - 8 - 2 - 6 - 4 - 1**							**411**

Table A.13: Cost Data and Solution Information for Problem stsp81

	1	2	3	4	5	6	7	8	Cost
1	5000	226	201	135	254	198	145	225	225
2	226	5000	284	185	183	110	31	188	31
3	201	284	5000	17	90	172	65	165	90
4	135	185	17	5000	84	108	257	189	135
5	254	183	90	84	5000	174	139	244	84
6	198	110	172	108	174	5000	227	59	110
7	145	31	65	257	139	227	5000	185	65
8	225	188	165	189	244	59	185	5000	59
Optimal Tour:		**1 - 8 - 6 - 2 - 7 - 3 - 5 - 4 - 1**							**799**

Table A.14: Cost Data and Solution Information for Problem stsp82

	1	2	3	4	5	6	7	8	Cost
1	5000	214	190	124	242	186	133	213	213
2	214	5000	273	174	172	98	20	176	20
3	190	273	5000	6	78	160	54	154	78
4	124	174	6	5000	73	96	245	177	124
5	242	172	78	73	5000	162	127	232	73
6	186	98	160	96	162	5000	215	47	98
7	133	20	54	245	127	215	5000	173	54
8	213	176	154	177	232	47	173	5000	47
Optimal Tour:		**1 - 8 - 6 - 2 - 7 - 3 - 5 - 4 - 1**							**707**

Table A.15: Cost Data and Solution Information for Problem stsp83

Appendix B:

OSL Outputs for the Dual Form of *Problem xtsp71*

```
EKK0330I New logfile is .\TSPLP\Outputs\xtsp71D.txt
 EKK0317I               Entering OSL module ekk_importModelFree ()
 EKK0078I          1 NAME          xtsp71    FREE
 EKK0185I ekk_importModelFree will read this file in free format
 EKK0078I          2 ROWS
 EKK0078I       8885 COLUMNS
 EKK0078I      40856 RHS
 EKK0078I      40858 ENDATA

 EKK0014I         Using Matrix xtsp71
 EKK0014I         Using Objective LP_Obj
 EKK0014I         Using RHS rhs(1)

 EKK0016I         Matrix has 8881 rows, 8910 columns and 31830 entries
 EKK0019I         There are 140 objective row entries

 EKK0197I Total CPU time=0.1410000; CPU time in OSL subroutine
ekk_importModelFree=0.1410000
 EKK0317I               Entering OSL module ekk_scale (xtsp71)
 EKK0021I         The initial range of the matrix coefficients is 1.000000 - 1.000000
 EKK0023I OSL subroutine ekk_scale is creating a new copy of the matrix or is modifying an
old copy
 EKK0317I               Entering OSL module ekk_preSolve (xtsp71)
 EKK0026I         After presolving the matrix, there are 7685 active columns, 7656 active
rows, and 29105 active elements

 EKK0023I OSL subroutine ekk_preSolve is creating a new copy of the matrix or is modifying
an old copy
 EKK0317I               Entering OSL module ekk_dualSimplex (xtsp71)
 EKK0038I Iterations/Objective                  Primal/Dual Infeasibilities
 EKK0057I          0        0.000000E+000       1.0000000(    1)       1640.000(  1140)
 EKK0057I        188            -3.000000        109.8000(  196)       839.9999(   540)
 EKK0057I        376            -4.000000        141.2000(  301)       812.7499(   545)
 EKK0057I        564            -4.000000        145.1667(  420)       812.7499(   545)
 EKK0057I        752            -4.155280        307.6242(  835)       783.4906(   731)
 EKK0057I        940            -4.298487        330.8847(  870)       764.5193(   688)
 EKK0057I       1128            -4.430100        712.7988( 1185)       748.9151(   664)
 EKK0057I       1316            -5.438122        2572.205( 1240)       682.1998(   634)
 EKK0057I       1504            -5.972183        1837.227( 1361)       654.9869(   640)
 EKK0057I       1692            -6.546775        1531.542( 1608)       577.7235(   581)
 EKK0057I       1880            -6.900262        1352.695( 1698)       553.4781(   561)
 EKK0057I       2068            -7.299845        1147.476( 1828)       486.9874(   542)
 EKK0057I       2256            -7.507376        940.7844( 2016)       444.7846(   480)
 EKK0057I       2444            -7.667893        2959.317( 2218)       414.3482(   438)
 EKK0057I       2632            -7.779031        1457.246( 2250)       384.6767(   435)
 EKK0057I       2820            -7.849022        1100.049( 2328)       362.1770(   432)
 EKK0057I       3008            -7.902891        1540.380( 2333)       352.4042(   422)
 EKK0057I       3196            -8.071280        2207.014( 2611)       333.2466(   439)
 EKK0057I       3384            -8.226040        1300.982( 2729)       314.5320(   407)
 EKK0057I       3572            -8.360975        826.9534( 2851)       296.3863(   393)
 EKK0038I Iterations/Objective                  Primal/Dual Infeasibilities
 EKK0057I       3760            -8.496420        6137.426( 2932)       260.7155(   395)
 EKK0057I       3948            -8.810094        1212.428( 3028)       237.1471(   348)
 EKK0057I       4136            -9.329431        3440.589( 3320)       206.1470(   296)
 EKK0057I       4324            -9.606298        1793.925( 3283)       183.2834(   272)
 EKK0057I       4512            -9.730574        1246.524( 3305)       162.1169(   261)
 EKK0057I       4700            -9.848850        1569.965( 3402)       139.5732(   248)
 EKK0057I       4888            -9.938386        1112.217( 3392)       115.0372(   230)
 EKK0057I       5076            -9.972635        3254.101( 3485)       84.54723(   192)
 EKK0057I       5264            -9.978044        1830.853( 3476)       59.18764(   157)
 EKK0057I       5452            -9.913020        1098.767( 3474)       45.78632(   138)
 EKK0057I       5640            -9.934679        3209.067( 3533)       34.31379(    94)
 EKK0057I       5828            -10.11241        6244.435( 3555)       21.53697(    83)
 EKK0057I       6016            -10.37490        4169.223( 3655)       11.39882(    44)
 EKK0057I       6204            -10.55943        1694.127( 3663)       5.927413(    31)
 EKK0057I       6392            -10.55018        1402.171( 3646)       1.514788(     9)
 EKK0057I       6580            -10.02637        18287.54( 3692)  0.000000E+000(     0)
```

```
 EKK0057I      6768            -9.524849        4415.787( 3605)   0.000000E+000(     0)
 EKK0057I      6949            -9.105607        1363.484( 3680)   0.000000E+000(     0)
 EKK0057I      7120            -8.864572        5886.949( 3710)   0.000000E+000(     0)
 EKK0057I      7295            -8.636312        12902.63( 3733)   0.000000E+000(     0)
 EKK0038I Iterations/Objective                 Primal/Dual Infeasibilities
 EKK0057I      7457            -8.375790        8882.901( 3729)   0.000000E+000(     0)
 EKK0057I      7634            -8.077060        5406.381( 3677)   0.000000E+000(     0)
 EKK0057I      7809            -7.000000        1226.547( 3695)   0.000000E+000(     0)
 EKK0057I      7963            -7.000000        397.1772( 3725)   0.000000E+000(     0)
 EKK0057I      8122            -7.000000        236.0282( 3709)   0.000000E+000(     0)
 EKK0057I      8297            -7.000000        107.7196( 3693)   0.000000E+000(     0)
 EKK0057I      8471            -7.000000        134.6710( 3649)   0.000000E+000(     0)
 EKK0057I      8640            -7.000000        78.53908( 3563)   0.000000E+000(     0)
 EKK0057I      8828            -7.000000        110.0945( 3622)   0.000000E+000(     0)
 EKK0057I      9016            -7.000000        127.0596( 3687)   0.000000E+000(     0)
 EKK0057I      9204            -7.000000        58.69941( 3416)   0.000000E+000(     0)
 EKK0057I      9392            -7.000000        84.00530( 3507)   1.392577E-012(     1)
 EKK0057I      9580            -7.000000        80.00985( 3367)   0.000000E+000(     0)
 EKK0033I The problem has been perturbed
 EKK0057I      9768            -7.000000        80.19672( 3482)   0.000000E+000(     0)
 EKK0001I Iteration Number:     9914; Objective Value:                  -7.000000--Optimal
 EKK0197I Total CPU time=19.73500; CPU time in OSL subroutine ekk_dualSimplex=19.56300
 EKK0317I               Entering OSL module ekk_postSolve (xtsp71)
 EKK0275I ekk_postSolve: Objective value                                     -7.000000
 EKK0276I ekk_postSolve: Worst primal infeasibility                      0.000000E+000
 EKK0277I ekk_postSolve: Worst complementary slackness                   0.000000E+000
 EKK0278I ekk_postSolve: Primal objective value                              -7.000000
 EKK0276I ekk_postSolve: Worst primal infeasibility                      0.000000E+000
 EKK0277I ekk_postSolve: Worst complementary slackness                   0.000000E+000
 EKK0318I Integer control variable isolmask has been changed from 0 to 6
 EKK0317I               Entering OSL module ekk_printSolution (xtsp71)
 EKK0008I Description of Problem xtsp71
 EKK0016I         Matrix has 8881 rows, 8910 columns and 31830 entries
 EKK0009I Problem Status
 EKK0001I Iteration Number:     9914; Objective Value:                  -7.000000--Optimal
1EKK0011I    Columns Section                                  Page 1
 EKK0063I           .......Name......  Stat  ......Activity......
 EKK0064I        8  Y-3-1-6-3-1-6       BS             0.50000000
 EKK0064I       21  Y-6-1-3-6-1-3       BS             0.50000000
 EKK0064I       35  Y-3-2-2-3-2-2       BS             0.25000000
 EKK0064I       36  Y-3-2-4-3-2-4       BS             0.25000000
 EKK0064I       50  Y-6-2-2-6-2-2       BS             0.25000000
 EKK0064I       52  Y-6-2-4-6-2-4       BS             0.25000000
 EKK0064I       61  Y-2-3-4-2-3-4       BS             0.50000000
 EKK0064I       70  Y-4-3-2-4-3-2       BS             0.50000000
 EKK0064I       92  Y-2-4-5-2-4-5       BS             0.25000000
 EKK0064I       94  Y-2-4-7-2-4-7       BS             0.25000000
 EKK0064I      102  Y-4-4-5-4-4-5       BS             0.25000000
 EKK0064I      104  Y-4-4-7-4-4-7       BS             0.25000000
 EKK0064I      139  Y-5-5-7-5-5-7       BS             0.50000000
 EKK0064I      148  Y-7-5-5-7-5-5       BS             0.50000000
 EKK0064I      182  Y-3-1-6-6-2-2       BS             0.25000000
 EKK0064I      183  Y-3-1-6-6-2-4       BS             0.25000000
 EKK0064I      234  Y-6-1-3-3-2-2       BS             0.25000000
 EKK0064I      235  Y-6-1-3-3-2-4       BS             0.25000000
 EKK0064I      290  Y-3-2-2-2-3-4       BS             0.25000000
 EKK0064I      294  Y-3-2-4-4-3-2       BS             0.25000000
 EKK0064I      351  Y-6-2-2-2-3-4       BS             0.25000000
 EKK0064I      358  Y-6-2-4-4-3-2       BS             0.25000000
 EKK0064I      395  Y-2-3-4-4-4-5       BS             0.25000000
 EKK0064I      397  Y-2-3-4-4-4-7       BS             0.25000000
 EKK0064I      431  Y-4-3-2-2-4-5       BS             0.25000000
 EKK0064I      433  Y-4-3-2-2-4-7       BS             0.25000000
 EKK0064I      521  Y-2-4-5-5-5-7       BS             0.25000000
 EKK0064I      528  Y-2-4-7-7-5-5       BS             0.25000000
 EKK0064I      561  Y-4-4-5-5-5-7       BS             0.25000000
 EKK0064I      568  Y-4-4-7-7-5-5       BS             0.25000000
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 EKK0064I      918  Y-3-1-6-2-3-4       BS             0.25000000
 EKK0064I      921  Y-3-1-6-4-3-2       BS             0.25000000
 EKK0064I      931  Y-3-1-6-2-4-5       BS             0.25000000
 EKK0064I      935  Y-3-1-6-4-4-7       BS             0.25000000
 EKK0064I      950  Y-3-1-6-5-5-7       BS             0.25000000
 EKK0064I      953  Y-3-1-6-7-5-5       BS             0.25000000
 EKK0064I     1386  Y-6-1-3-2-3-4       BS             0.25000000
 EKK0064I     1389  Y-6-1-3-4-3-2       BS             0.25000000
 EKK0064I     1400  Y-6-1-3-2-4-7       BS             0.25000000
 EKK0064I     1402  Y-6-1-3-4-4-5       BS             0.25000000
 EKK0064I     1418  Y-6-1-3-5-5-7       BS             0.25000000
 EKK0064I     1421  Y-6-1-3-7-5-5       BS             0.25000000
 EKK0064I     1830  Y-3-2-2-4-4-5       BS             0.25000000
 EKK0064I     1847  Y-3-2-2-5-5-7       BS             0.25000000
 EKK0064I     1856  Y-3-2-4-2-4-7       BS             0.25000000
 EKK0064I     1876  Y-3-2-4-7-5-5       BS             0.25000000
 EKK0064I     2195  Y-6-2-2-4-4-7       BS             0.25000000
 EKK0064I     2213  Y-6-2-2-7-5-5       BS             0.25000000
 EKK0064I     2239  Y-6-2-4-2-4-5       BS             0.25000000
 EKK0064I     2258  Y-6-2-4-5-5-7       BS             0.25000000
 EKK0064I     2447  Y-2-3-4-5-5-7       BS             0.25000000
 EKK0064I     2452  Y-2-3-4-7-5-5       BS             0.25000000
 EKK0064I     2555  Y-4-3-2-5-5-7       BS             0.25000000
1EKK0011I    Columns Section                               Page 2
 EKK0063I           .......Name......  Stat  ......Activity......
 EKK0064I     2560  Y-4-3-2-7-5-5       BS             0.25000000
 EKK0064I     3120  Z(613322234)        BS             0.25000000
 EKK0064I     3123  Z(613322445)        BS             0.25000000
 EKK0064I     3132  Z(613322557)        BS             0.25000000
 EKK0064I     3180  Z(613324432)        BS             0.25000000
 EKK0064I     3184  Z(613324247)        BS             0.25000000
 EKK0064I     3194  Z(613324755)        BS             0.25000000
 EKK0064I     3990  Z(316622234)        BS             0.25000000
 EKK0064I     3994  Z(316622447)        BS             0.25000000
 EKK0064I     4004  Z(316622755)        BS             0.25000000
 EKK0064I     4125  Z(316624432)        BS             0.25000000
 EKK0064I     4128  Z(316624245)        BS             0.25000000
 EKK0064I     4137  Z(316624557)        BS             0.25000000
 EKK0064I     4674  Z(322234445)        BS             0.25000000
 EKK0064I     4678  Z(322234557)        BS             0.25000000
 EKK0064I     4694  Z(622234447)        BS             0.25000000
 EKK0064I     4700  Z(622234755)        BS             0.25000000
 EKK0064I     4715  Z(316234447)        BS             0.25000000
 EKK0064I     4716  Z(316234557)        BS             0.25000000
 EKK0064I     4734  Z(613234445)        BS             0.25000000
 EKK0064I     4737  Z(613234755)        BS             0.25000000
 EKK0064I     5432  Z(324432247)        BS             0.25000000
 EKK0064I     5437  Z(324432755)        BS             0.25000000
 EKK0064I     5449  Z(624432245)        BS             0.25000000
 EKK0064I     5454  Z(624432557)        BS             0.25000000
 EKK0064I     5470  Z(316432245)        BS             0.25000000
 EKK0064I     5473  Z(316432755)        BS             0.25000000
 EKK0064I     5491  Z(613432247)        BS             0.25000000
 EKK0064I     5492  Z(613432557)        BS             0.25000000
 EKK0064I     7235  Z(432245557)        BS             0.25000000
 EKK0064I     7245  Z(316245557)        BS             0.25000000
 EKK0064I     7281  Z(624245557)        BS             0.25000000
 EKK0064I     7354  Z(432247755)        BS             0.25000000
 EKK0064I     7381  Z(613247755)        BS             0.25000000
 EKK0064I     7386  Z(324247755)        BS             0.25000000
 EKK0064I     7832  Z(234445557)        BS             0.25000000
 EKK0064I     7857  Z(613445557)        BS             0.25000000
 EKK0064I     7873  Z(322445557)        BS             0.25000000
 EKK0064I     7951  Z(234447755)        BS             0.25000000
 EKK0064I     7973  Z(316447755)        BS             0.25000000
 EKK0064I     8005  Z(622447755)        BS             0.25000000
 EKK0317I                Entering OSL module ekk_deleteModel (xtsp71)
```

Appendix C:

Summary of the TSP Tours Comprising the Solution from the Dual Form of *Problem xtsp71*

TSP Tours			Arcs of *Graph G* in the LP Solution							
ID #	**Sequence**	**Weight**	**(3,1,6)**	**(6,1,3)**	**(3,2,2)**	**(3,2,4)**	**(6,2,2)**	**(6,2,4)**	**(2,3,4)**	**(4,3,2)**
1	1-3-6-2-4-7-5-1	0.25	1				1		1	
2	1-3-6-4-2-5-7-1	0.25	1					1		1
3	1-6-3-2-4-5-7-1	0.25		1	1				1	
4	1-6-3-4-2-7-5-1	0.25		1		1				1
Total Flow on Arc			**0.50**	**0.50**	**0.25**	**0.25**	**0.25**	**0.25**	**.50**	**.50**

Table C.1: Details of the TSP Tours in the Solution for the Dual Form of *Problem xtsp71*- Part I

TSP Tours			Arcs of *Graph G* in the LP Solution					
ID #	**Sequence**	**Weight**	**(2,4,5)**	**(2,4,7)**	**(4,4,5)**	**(4,4,7)**	**(5,5,7)**	**(7,5,5)**
1	1-3-6-2-4-7-5-1	0.25				1		1
2	1-3-6-4-2-5-7-1	0.25	1				1	
3	1-6-3-2-4-5-7-1	0.25			1		1	
4	1-6-3-4-2-7-5-1	0.25		1				1
Total Flow on Arc			**0.25**	**0.25**	**0.25**	**0.25**	**0.50**	**0.50**

Table C.2: Details of the TSP Tours in the Solution for the Dual Form of *Problem xtsp71* - Part II

Appendix D:

OSL Outputs for the Dual Form of *Problem xtsp73*

```
EKK0330I New logfile is .\TSPLP\Outputs\xtsp73D.txt
 EKK0317I                Entering OSL module ekk_importModelFree ()
 EKK0078I          1 NAME          xtsp73    FREE
 EKK0185I ekk_importModelFree will read this file in free format
 EKK0078I          2 ROWS
 EKK0078I       8885 COLUMNS
 EKK0078I      40726 RHS
 EKK0078I      40728 ENDATA

 EKK0014I         Using Matrix xtsp73
 EKK0014I         Using Objective LP_Obj
 EKK0014I         Using RHS rhs(1)

 EKK0016I         Matrix has 8881 rows, 8910 columns and 31830 entries
 EKK0019I         There are 10 objective row entries

 EKK0197I Total CPU time=0.1410000; CPU time in OSL subroutine
ekk_importModelFree=0.1410000
 EKK0317I                Entering OSL module ekk_scale (xtsp73)
 EKK0021I         The initial range of the matrix coefficients is 1.000000 - 1.000000
 EKK0023I OSL subroutine ekk_scale is creating a new copy of the matrix or is modifying an
old copy
 EKK0317I                Entering OSL module ekk_preSolve (xtsp73)
 EKK0026I         After presolving the matrix, there are 7705 active columns, 7676 active
rows, and 29365 active elements

 EKK0023I OSL subroutine ekk_preSolve is creating a new copy of the matrix or is modifying
an old copy
 EKK0317I                Entering OSL module ekk_dualSimplex (xtsp73)
 EKK0038I Iterations/Objective                Primal/Dual Infeasibilities
 EKK0057I          0         0.000000E+000        1.0000000(    1)   0.000000E+000(     0)
 EKK0057I        188         0.000000E+000         18.00000(   18)   0.000000E+000(     0)
 EKK0057I        376         0.000000E+000         35.00000(  163)   0.000000E+000(     0)
 EKK0057I        564         0.000000E+000         24.64865(  231)   0.000000E+000(     0)
 EKK0057I        752         0.000000E+000         23.44660(  454)   0.000000E+000(     0)
 EKK0057I        940         0.000000E+000         56.53779(  833)   0.000000E+000(     0)
 EKK0057I       1128         0.000000E+000         52.37572( 1011)   0.000000E+000(     0)
 EKK0057I       1316         0.000000E+000         55.31298( 1177)   0.000000E+000(     0)
 EKK0057I       1504         0.000000E+000         32.39207( 1338)   0.000000E+000(     0)
 EKK0057I       1692         0.000000E+000         51.67132( 1680)   0.000000E+000(     0)
 EKK0057I       1880         0.000000E+000         68.58153( 1793)   0.000000E+000(     0)
 EKK0057I       2068         0.000000E+000         61.22473( 1971)   0.000000E+000(     0)
 EKK0057I       2256         0.000000E+000         69.37150( 2049)   0.000000E+000(     0)
 EKK0057I       2444         0.000000E+000         86.25787( 2301)   0.000000E+000(     0)
 EKK0057I       2632         0.000000E+000         58.83881( 2317)   0.000000E+000(     0)
 EKK0057I       2820         0.000000E+000         120.1363( 2541)   0.000000E+000(     0)
 EKK0057I       3008         0.000000E+000         92.90198( 2685)   0.000000E+000(     0)
 EKK0057I       3196         0.000000E+000         92.50184( 2774)   0.000000E+000(     0)
 EKK0057I       3384         0.000000E+000         76.06881( 2937)   0.000000E+000(     0)
 EKK0057I       3572         0.000000E+000         59.77006( 2944)   0.000000E+000(     0)
 EKK0038I Iterations/Objective                Primal/Dual Infeasibilities
 EKK0057I       3760         0.000000E+000         72.17296( 3001)   0.000000E+000(     0)
 EKK0057I       3948         0.000000E+000         72.28140( 3190)   0.000000E+000(     0)
 EKK0057I       4136         0.000000E+000         70.64216( 3213)   0.000000E+000(     0)
 EKK0057I       4324         0.000000E+000         57.71482( 3258)   0.000000E+000(     0)
 EKK0057I       4512         0.000000E+000         117.7179( 3386)   0.000000E+000(     0)
 EKK0057I       4700         0.000000E+000         102.5654( 3457)   0.000000E+000(     0)
 EKK0057I       4888         0.000000E+000         85.19144( 3430)   0.000000E+000(     0)
 EKK0057I       5076         0.000000E+000         53.51423( 3543)   0.000000E+000(     0)
 EKK0057I       5264         0.000000E+000         85.42116( 3600)   0.000000E+000(     0)
 EKK0057I       5452         0.000000E+000         64.88131( 3597)   0.000000E+000(     0)
 EKK0057I       5640         0.000000E+000         55.33657( 3615)   0.000000E+000(     0)
 EKK0057I       5828         0.000000E+000         96.87463( 3708)   0.000000E+000(     0)
 EKK0057I       6016         0.000000E+000         132.2508( 3836)   0.000000E+000(     0)
 EKK0057I       6204         0.000000E+000         30.17316( 3550)   0.000000E+000(     0)
 EKK0057I       6390         0.000000E+000         63.49333( 3790)   0.000000E+000(     0)
 EKK0057I       6578         0.000000E+000         77.24236( 3692)   0.000000E+000(     0)
```

```
 EKK0001I Iteration Number:     6585; Objective Value:            0.000000E+000--Optimal
 EKK0197I Total CPU time=12.09400; CPU time in OSL subroutine ekk_dualSimplex=11.92200
 EKK0317I               Entering OSL module ekk_postSolve (xtsp73)
 EKK0275I ekk_postSolve: Objective value                              0.000000E+000
 EKK0276I ekk_postSolve: Worst primal infeasibility                   0.000000E+000
 EKK0277I ekk_postSolve: Worst complementary slackness                0.000000E+000
 EKK0278I ekk_postSolve: Primal objective value                       0.000000E+000
 EKK0276I ekk_postSolve: Worst primal infeasibility                   0.000000E+000
 EKK0277I ekk_postSolve: Worst complementary slackness                0.000000E+000
 EKK0318I Integer control variable isolmask has been changed from 0 to 6
 EKK0317I               Entering OSL module ekk_printSolution (xtsp73)
 EKK0008I Description of Problem xtsp73
 EKK0016I         Matrix has 8881 rows, 8910 columns and 31830 entries
 EKK0009I Problem Status
 EKK0001I Iteration Number:     6585; Objective Value:            0.000000E+000--Optimal
1EKK0011I    Columns Section                               Page 1
 EKK0063I          .......Name......  Stat  ......Activity......
 EKK0064I        9  Y-3-1-7-3-1-7      BS            0.25000000
 EKK0064I       14  Y-4-1-7-4-1-7      BS            0.25000000
 EKK0064I       19  Y-5-1-7-5-1-7      BS            0.25000000
 EKK0064I       24  Y-6-1-7-6-1-7      BS            0.25000000
 EKK0064I       56  Y-7-2-3-7-2-3      BS            0.25000000
 EKK0064I       57  Y-7-2-4-7-2-4      BS            0.25000000
 EKK0064I       58  Y-7-2-5-7-2-5      BS            0.25000000
 EKK0064I       59  Y-7-2-6-7-2-6      BS            0.25000000
 EKK0064I       66  Y-3-3-4-3-3-4      BS            0.08333333
 EKK0064I       67  Y-3-3-5-3-3-5      BS            0.08333333
 EKK0064I       68  Y-3-3-6-3-3-6      BS            0.08333333
 EKK0064I       71  Y-4-3-3-4-3-3      BS            0.08333333
 EKK0064I       72  Y-4-3-5-4-3-5      BS            0.08333333
 EKK0064I       73  Y-4-3-6-4-3-6      BS            0.08333333
 EKK0064I       76  Y-5-3-3-5-3-3      BS            0.08333333
 EKK0064I       77  Y-5-3-4-5-3-4      BS            0.08333333
 EKK0064I       78  Y-5-3-6-5-3-6      BS            0.08333333
 EKK0064I       81  Y-6-3-3-6-3-3      BS            0.08333333
 EKK0064I       82  Y-6-3-4-6-3-4      BS            0.08333333
 EKK0064I       83  Y-6-3-5-6-3-5      BS            0.08333333
 EKK0064I       95  Y-3-4-2-3-4-2      BS            0.25000000
 EKK0064I      100  Y-4-4-2-4-4-2      BS            0.25000000
 EKK0064I      105  Y-5-4-2-5-4-2      BS            0.25000000
 EKK0064I      110  Y-6-4-2-6-4-2      BS            0.25000000
 EKK0064I      120  Y-2-5-3-2-5-3      BS            0.25000000
 EKK0064I      121  Y-2-5-4-2-5-4      BS            0.25000000
 EKK0064I      122  Y-2-5-5-2-5-5      BS            0.25000000
 EKK0064I      123  Y-2-5-6-2-5-6      BS            0.25000000
 EKK0064I      187  Y-3-1-7-7-2-4      BS            0.08333333
 EKK0064I      188  Y-3-1-7-7-2-5      BS            0.08333333
 EKK0064I      189  Y-3-1-7-7-2-6      BS            0.08333333
 EKK0064I      207  Y-4-1-7-7-2-3      BS            0.08333333
 EKK0064I      208  Y-4-1-7-7-2-5      BS            0.08333333
 EKK0064I      209  Y-4-1-7-7-2-6      BS            0.08333333
 EKK0064I      227  Y-5-1-7-7-2-3      BS            0.08333333
 EKK0064I      228  Y-5-1-7-7-2-4      BS            0.08333333
 EKK0064I      229  Y-5-1-7-7-2-6      BS            0.08333333
 EKK0064I      247  Y-6-1-7-7-2-3      BS            0.08333333
 EKK0064I      248  Y-6-1-7-7-2-4      BS            0.08333333
 EKK0064I      249  Y-6-1-7-7-2-5      BS            0.08333333
 EKK0064I      375  Y-7-2-3-3-3-4      BS            0.08333333
 EKK0064I      376  Y-7-2-3-3-3-5      BS            0.08333333
 EKK0064I      377  Y-7-2-3-3-3-6      BS            0.08333333
 EKK0064I      379  Y-7-2-4-4-3-3      BS            0.08333333
 EKK0064I      380  Y-7-2-4-4-3-5      BS            0.08333333
 EKK0064I      381  Y-7-2-4-4-3-6      BS            0.08333333
 EKK0064I      383  Y-7-2-5-5-3-3      BS            0.08333333
 EKK0064I      384  Y-7-2-5-5-3-4      BS            0.08333333
 EKK0064I      385  Y-7-2-5-5-3-6      BS            0.08333333
 EKK0064I      387  Y-7-2-6-6-3-3      BS            0.08333333
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 EKK0064I      388  Y-7-2-6-6-3-4       BS              0.08333333
 EKK0064I      389  Y-7-2-6-6-3-5       BS              0.08333333
 EKK0064I      414  Y-3-3-4-4-4-2       BS              0.08333333
1EKK0011I     Columns Section                               Page 2
 EKK0063I           .......Name......  Stat  ......Activity......
 EKK0064I      418  Y-3-3-5-5-4-2       BS              0.08333333
 EKK0064I      422  Y-3-3-6-6-4-2       BS              0.08333333
 EKK0064I      434  Y-4-3-3-3-4-2       BS              0.08333333
 EKK0064I      438  Y-4-3-5-5-4-2       BS              0.08333333
 EKK0064I      442  Y-4-3-6-6-4-2       BS              0.08333333
 EKK0064I      454  Y-5-3-3-3-4-2       BS              0.08333333
 EKK0064I      458  Y-5-3-4-4-4-2       BS              0.08333333
 EKK0064I      462  Y-5-3-6-6-4-2       BS              0.08333333
 EKK0064I      474  Y-6-3-3-3-4-2       BS              0.08333333
 EKK0064I      478  Y-6-3-4-4-4-2       BS              0.08333333
 EKK0064I      482  Y-6-3-5-5-4-2       BS              0.08333333
 EKK0064I      530  Y-3-4-2-2-5-4       BS              0.08333333
 EKK0064I      531  Y-3-4-2-2-5-5       BS              0.08333333
 EKK0064I      532  Y-3-4-2-2-5-6       BS              0.08333333
 EKK0064I      550  Y-4-4-2-2-5-3       BS              0.08333333
 EKK0064I      551  Y-4-4-2-2-5-5       BS              0.08333333
 EKK0064I      552  Y-4-4-2-2-5-6       BS              0.08333333
 EKK0064I      570  Y-5-4-2-2-5-3       BS              0.08333333
 EKK0064I      571  Y-5-4-2-2-5-4       BS              0.08333333
 EKK0064I      572  Y-5-4-2-2-5-6       BS              0.08333333
 EKK0064I      590  Y-6-4-2-2-5-3       BS              0.08333333
 EKK0064I      591  Y-6-4-2-2-5-4       BS              0.08333333
 EKK0064I      592  Y-6-4-2-2-5-5       BS              0.08333333
 EKK0064I      958  Y-3-1-7-4-3-5       BS              0.08333333
 EKK0064I      962  Y-3-1-7-5-3-6       BS              0.08333333
 EKK0064I      964  Y-3-1-7-6-3-4       BS              0.08333333
 EKK0064I      969  Y-3-1-7-4-4-2       BS              0.08333333
 EKK0064I      972  Y-3-1-7-5-4-2       BS              0.08333333
 EKK0064I      975  Y-3-1-7-6-4-2       BS              0.08333333
 EKK0064I      978  Y-3-1-7-2-5-4       BS              0.08333333
 EKK0064I      979  Y-3-1-7-2-5-5       BS              0.08333333
 EKK0064I      980  Y-3-1-7-2-5-6       BS              0.08333333
 EKK0064I     1139  Y-4-1-7-3-3-6       BS              0.08333333
 EKK0064I     1141  Y-4-1-7-5-3-3       BS              0.08333333
 EKK0064I     1145  Y-4-1-7-6-3-5       BS              0.08333333
 EKK0064I     1149  Y-4-1-7-3-4-2       BS              0.08333333
 EKK0064I     1152  Y-4-1-7-5-4-2       BS              0.08333333
 EKK0064I     1155  Y-4-1-7-6-4-2       BS              0.08333333
 EKK0064I     1158  Y-4-1-7-2-5-3       BS              0.08333333
 EKK0064I     1159  Y-4-1-7-2-5-5       BS              0.08333333
 EKK0064I     1160  Y-4-1-7-2-5-6       BS              0.08333333
 EKK0064I     1318  Y-5-1-7-3-3-4       BS              0.08333333
 EKK0064I     1322  Y-5-1-7-4-3-6       BS              0.08333333
 EKK0064I     1324  Y-5-1-7-6-3-3       BS              0.08333333
 EKK0064I     1329  Y-5-1-7-3-4-2       BS              0.08333333
 EKK0064I     1332  Y-5-1-7-4-4-2       BS              0.08333333
 EKK0064I     1335  Y-5-1-7-6-4-2       BS              0.08333333
 EKK0064I     1338  Y-5-1-7-2-5-3       BS              0.08333333
 EKK0064I     1339  Y-5-1-7-2-5-4       BS              0.08333333
 EKK0064I     1340  Y-5-1-7-2-5-6       BS              0.08333333
 EKK0064I     1499  Y-6-1-7-3-3-5       BS              0.08333333
 EKK0064I     1501  Y-6-1-7-4-3-3       BS              0.08333333
 EKK0064I     1505  Y-6-1-7-5-3-4       BS              0.08333333
1EKK0011I     Columns Section                               Page 3
 EKK0063I           .......Name......  Stat  ......Activity......
 EKK0064I     1509  Y-6-1-7-3-4-2       BS              0.08333333
 EKK0064I     1512  Y-6-1-7-4-4-2       BS              0.08333333
 EKK0064I     1515  Y-6-1-7-5-4-2       BS              0.08333333
 EKK0064I     1518  Y-6-1-7-2-5-3       BS              0.08333333
 EKK0064I     1519  Y-6-1-7-2-5-4       BS              0.08333333
 EKK0064I     1520  Y-6-1-7-2-5-5       BS              0.08333333
 EKK0064I     2337  Y-7-2-3-4-4-2       BS              0.08333333
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 EKK0064I     2340  Y-7-2-3-5-4-2       BS             0.08333333
 EKK0064I     2343  Y-7-2-3-6-4-2       BS             0.08333333
 EKK0064I     2346  Y-7-2-3-2-5-4       BS             0.08333333
 EKK0064I     2347  Y-7-2-3-2-5-5       BS             0.08333333
 EKK0064I     2348  Y-7-2-3-2-5-6       BS             0.08333333
 EKK0064I     2361  Y-7-2-4-3-4-2       BS             0.08333333
 EKK0064I     2364  Y-7-2-4-5-4-2       BS             0.08333333
 EKK0064I     2367  Y-7-2-4-6-4-2       BS             0.08333333
 EKK0064I     2370  Y-7-2-4-2-5-3       BS             0.08333333
 EKK0064I     2371  Y-7-2-4-2-5-5       BS             0.08333333
 EKK0064I     2372  Y-7-2-4-2-5-6       BS             0.08333333
 EKK0064I     2385  Y-7-2-5-3-4-2       BS             0.08333333
 EKK0064I     2388  Y-7-2-5-4-4-2       BS             0.08333333
 EKK0064I     2391  Y-7-2-5-6-4-2       BS             0.08333333
 EKK0064I     2394  Y-7-2-5-2-5-3       BS             0.08333333
 EKK0064I     2395  Y-7-2-5-2-5-4       BS             0.08333333
 EKK0064I     2396  Y-7-2-5-2-5-6       BS             0.08333333
 EKK0064I     2409  Y-7-2-6-3-4-2       BS             0.08333333
 EKK0064I     2412  Y-7-2-6-4-4-2       BS             0.08333333
 EKK0064I     2415  Y-7-2-6-5-4-2       BS             0.08333333
 EKK0064I     2418  Y-7-2-6-2-5-3       BS             0.08333333
 EKK0064I     2419  Y-7-2-6-2-5-4       BS             0.08333333
 EKK0064I     2420  Y-7-2-6-2-5-5       BS             0.08333333
 EKK0064I     2503  Y-3-3-4-2-5-6       BS             0.08333333
 EKK0064I     2514  Y-3-3-5-2-5-4       BS             0.08333333
 EKK0064I     2527  Y-3-3-6-2-5-5       BS             0.08333333
 EKK0064I     2562  Y-4-3-3-2-5-5       BS             0.08333333
 EKK0064I     2575  Y-4-3-5-2-5-6       BS             0.08333333
 EKK0064I     2586  Y-4-3-6-2-5-3       BS             0.08333333
 EKK0064I     2623  Y-5-3-3-2-5-6       BS             0.08333333
 EKK0064I     2634  Y-5-3-4-2-5-3       BS             0.08333333
 EKK0064I     2647  Y-5-3-6-2-5-4       BS             0.08333333
 EKK0064I     2682  Y-6-3-3-2-5-4       BS             0.08333333
 EKK0064I     2695  Y-6-3-4-2-5-5       BS             0.08333333
 EKK0064I     2706  Y-6-3-5-2-5-3       BS             0.08333333
 EKK0064I     4367  Z(417723336)        BS             0.08333333
 EKK0064I     4372  Z(417723642)        BS             0.08333333
 EKK0064I     4375  Z(417723256)        BS             0.08333333
 EKK0064I     4381  Z(517723334)        BS             0.08333333
 EKK0064I     4385  Z(517723442)        BS             0.08333333
 EKK0064I     4389  Z(517723254)        BS             0.08333333
 EKK0064I     4397  Z(617723335)        BS             0.08333333
 EKK0064I     4402  Z(617723542)        BS             0.08333333
 EKK0064I     4405  Z(617723255)        BS             0.08333333
 EKK0064I     4426  Z(317724435)        BS             0.08333333
 EKK0064I     4430  Z(317724542)        BS             0.08333333
1EKK0011I    Columns Section                                Page 4
 EKK0063I           .......Name......  Stat  ......Activity......
 EKK0064I     4434  Z(317724255)        BS             0.08333333
 EKK0064I     4442  Z(517724436)        BS             0.08333333
 EKK0064I     4447  Z(517724642)        BS             0.08333333
 EKK0064I     4450  Z(517724256)        BS             0.08333333
 EKK0064I     4456  Z(617724433)        BS             0.08333333
 EKK0064I     4460  Z(617724342)        BS             0.08333333
 EKK0064I     4464  Z(617724253)        BS             0.08333333
 EKK0064I     4487  Z(317725536)        BS             0.08333333
 EKK0064I     4492  Z(317725642)        BS             0.08333333
 EKK0064I     4495  Z(317725256)        BS             0.08333333
 EKK0064I     4501  Z(417725533)        BS             0.08333333
 EKK0064I     4505  Z(417725342)        BS             0.08333333
 EKK0064I     4509  Z(417725253)        BS             0.08333333
 EKK0064I     4517  Z(617725534)        BS             0.08333333
 EKK0064I     4522  Z(617725442)        BS             0.08333333
 EKK0064I     4525  Z(617725254)        BS             0.08333333
 EKK0064I     4546  Z(317726634)        BS             0.08333333
 EKK0064I     4550  Z(317726442)        BS             0.08333333
 EKK0064I     4554  Z(317726254)        BS             0.08333333
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EKK0064I	4562	Z(417726635)	BS	0.08333333
EKK0064I	4567	Z(417726542)	BS	0.08333333
EKK0064I	4570	Z(417726255)	BS	0.08333333
EKK0064I	4576	Z(517726633)	BS	0.08333333
EKK0064I	4580	Z(517726342)	BS	0.08333333
EKK0064I	4584	Z(517726253)	BS	0.08333333
EKK0064I	5121	Z(723334442)	BS	0.08333333
EKK0064I	5125	Z(723334256)	BS	0.08333333
EKK0064I	5150	Z(517334442)	BS	0.08333333
EKK0064I	5152	Z(517334256)	BS	0.08333333
EKK0064I	5205	Z(723335542)	BS	0.08333333
EKK0064I	5208	Z(723335254)	BS	0.08333333
EKK0064I	5246	Z(617335542)	BS	0.08333333
EKK0064I	5248	Z(617335254)	BS	0.08333333
EKK0064I	5289	Z(723336642)	BS	0.08333333
EKK0064I	5293	Z(723336255)	BS	0.08333333
EKK0064I	5318	Z(417336642)	BS	0.08333333
EKK0064I	5320	Z(417336255)	BS	0.08333333
EKK0064I	5541	Z(724433342)	BS	0.08333333
EKK0064I	5544	Z(724433255)	BS	0.08333333
EKK0064I	5582	Z(617433342)	BS	0.08333333
EKK0064I	5584	Z(617433255)	BS	0.08333333
EKK0064I	5625	Z(724435542)	BS	0.08333333
EKK0064I	5629	Z(724435256)	BS	0.08333333
EKK0064I	5654	Z(317435542)	BS	0.08333333
EKK0064I	5656	Z(317435256)	BS	0.08333333
EKK0064I	5709	Z(724436642)	BS	0.08333333
EKK0064I	5712	Z(724436253)	BS	0.08333333
EKK0064I	5750	Z(517436642)	BS	0.08333333
EKK0064I	5752	Z(517436253)	BS	0.08333333
EKK0064I	5961	Z(725533342)	BS	0.08333333
EKK0064I	5965	Z(725533256)	BS	0.08333333
EKK0064I	5990	Z(417533342)	BS	0.08333333
EKK0064I	5992	Z(417533256)	BS	0.08333333
1EKK0011I	Columns Section			Page 5
EKK0063I		.......Name......	Stat	......Activity......
EKK0064I	6045	Z(725534442)	BS	0.08333333
EKK0064I	6048	Z(725534253)	BS	0.08333333
EKK0064I	6086	Z(617534442)	BS	0.08333333
EKK0064I	6088	Z(617534253)	BS	0.08333333
EKK0064I	6129	Z(725536642)	BS	0.08333333
EKK0064I	6133	Z(725536254)	BS	0.08333333
EKK0064I	6158	Z(317536642)	BS	0.08333333
EKK0064I	6160	Z(317536254)	BS	0.08333333
EKK0064I	6381	Z(726633342)	BS	0.08333333
EKK0064I	6384	Z(726633254)	BS	0.08333333
EKK0064I	6422	Z(517633342)	BS	0.08333333
EKK0064I	6424	Z(517633254)	BS	0.08333333
EKK0064I	6465	Z(726634442)	BS	0.08333333
EKK0064I	6469	Z(726634255)	BS	0.08333333
EKK0064I	6494	Z(317634442)	BS	0.08333333
EKK0064I	6496	Z(317634255)	BS	0.08333333
EKK0064I	6549	Z(726635542)	BS	0.08333333
EKK0064I	6552	Z(726635253)	BS	0.08333333
EKK0064I	6590	Z(417635542)	BS	0.08333333
EKK0064I	6592	Z(417635253)	BS	0.08333333
EKK0064I	7410	Z(433342255)	BS	0.08333333
EKK0064I	7414	Z(533342256)	BS	0.08333333
EKK0064I	7416	Z(633342254)	BS	0.08333333
EKK0064I	7427	Z(417342256)	BS	0.08333333
EKK0064I	7432	Z(517342254)	BS	0.08333333
EKK0064I	7439	Z(617342255)	BS	0.08333333
EKK0064I	7465	Z(724342256)	BS	0.08333333
EKK0064I	7466	Z(725342254)	BS	0.08333333
EKK0064I	7469	Z(726342255)	BS	0.08333333
EKK0064I	7711	Z(334442256)	BS	0.08333333
EKK0064I	7713	Z(534442253)	BS	0.08333333

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 EKK0064I     7717  Z(634442255)        BS             0.08333333
 EKK0064I     7726  Z(317442255)        BS             0.08333333
 EKK0064I     7733  Z(517442256)        BS             0.08333333
 EKK0064I     7738  Z(617442253)        BS             0.08333333
 EKK0064I     7764  Z(723442255)        BS             0.08333333
 EKK0064I     7767  Z(725442256)        BS             0.08333333
 EKK0064I     7768  Z(726442253)        BS             0.08333333
 EKK0064I     8010  Z(335542254)        BS             0.08333333
 EKK0064I     8014  Z(435542256)        BS             0.08333333
 EKK0064I     8016  Z(635542253)        BS             0.08333333
 EKK0064I     8027  Z(317542256)        BS             0.08333333
 EKK0064I     8032  Z(417542253)        BS             0.08333333
 EKK0064I     8039  Z(617542254)        BS             0.08333333
 EKK0064I     8065  Z(723542256)        BS             0.08333333
 EKK0064I     8066  Z(724542253)        BS             0.08333333
 EKK0064I     8069  Z(726542254)        BS             0.08333333
 EKK0064I     8311  Z(336642255)        BS             0.08333333
 EKK0064I     8313  Z(436642253)        BS             0.08333333
 EKK0064I     8317  Z(536642254)        BS             0.08333333
 EKK0064I     8326  Z(317642254)        BS             0.08333333
 EKK0064I     8333  Z(417642255)        BS             0.08333333
 EKK0064I     8338  Z(517642253)        BS             0.08333333
1EKK0011I    Columns Section                                Page 6
 EKK0063I           .......Name......  Stat  ......Activity......
 EKK0064I     8364  Z(723642254)        BS             0.08333333
 EKK0064I     8367  Z(724642255)        BS             0.08333333
 EKK0064I     8368  Z(725642253)        BS             0.08333333
 EKK0317I                Entering OSL module ekk_deleteModel (xtsp73)
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Appendix E:

Summary of the TSP Tours Comprising the

Solution for the Dual Form of *Problem xtsp73*

Tour #	Sequence	“Weight” (*Flow Value*)
1	1-5-7-3-4-2-6-1	0.083333
2	1-5-7-4-6-2-3-1	0.083333
3	1-5-7-6-3-2-4-1	0.083333
4	1-6-7-3-5-2-4-1	0.083333
5	1-6-7-4-3-2-5-1	0.083333
6	1-6-7-5-4-2-3-1	0.083333
7	1-3-7-4-5-2-6-1	0.083333
8	1-3-7-5-6-2-4-1	0.083333
9	1-3-7-6-4-2-5-1	0.083333
10	1-4-7-3-6-2-5-1	0.083333
11	1-4-7-5-3-2-6-1	0.083333
12	1-4-7-6-5-2-3-1	0.083333

Table E.1: Summary of the TSP tours in the Solution to *Problem xtsp73* Using the Dual Form