HCMUTE

Computation of Tangent, Euler, and Bernoulli Numbers*

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Abstract

Some elementary methods are described which may be used to calculate tangent numbers, Euler numbers, and Bernoulli numbers much more easily and rapidly on electronic computers than the traditional recurrence relations which have been used for over a century. These methods have been used to prepare an accompanying table which extends the existing tables of these numbers. Some theorems about the periodicity of the tangent numbers, which were suggested by the tables, are also proved.

Introduction

The tangent numbers T_n, Euler numbers E_n, and Bernoulli numbers B_n, are defined to be the coefficients in the following power series:

$$(1)\ \tan z = T_0/0! + T_1 z/1! + T_2 z^2/2! + \cdots = \sum_{n\geq 0} T_n z^n/n!\ ,$$

$$(2)\ \sec z = E_0/0! + E_1 z/1! + E_2 z^2/2! + \cdots = \sum_{n\geq 0} E_n z^n/n!\ ,$$

$$(3)\ z/(e^z - 1) = B_0/0! + B_1 z/1! + B_2 z^2/2! + \cdots = \sum_{n\geq 0} B_n z^n/n!\ .$$

Formulas for Computation

$$T_{0k} = \partial_{1k}\ ; \qquad T_{n+1,k} = (k - 1)T_{n,k-1k} + (k+1)T_{n,k+1}\ .$$

$$E_{0k} = \partial_{0k}\ ; \qquad E_{n+1,k} = kE_{n,k-1k} + (k+1)E_{n,k+1}\ .$$

$$B_{2n} = C_{2n} - \sum_{p \text{ prime };(p-1) \backslash 2n} \frac{1}{p}$$

C_{2n} is an integer. The table appended to this paper expresses Bn in this form.

Details of the Computation

6	.	1	6	,	4	0	,	2	4	.	,	8	,
		↑ P									↑ Q		

Here P and Q represent variables in the program that point to the current places of interest in the memory; P points to the number that will be accessed next, and Q points to the place where the next value is to be written. Only locations from P to Q contain information that will be used subsequently by the program. The symbols "." and "," represent special negative codes in the table which delimit the numbers in an obvious fashion. As we begin the calculation for $n = 5$, we set area A to zero and a variable k to 1. The basic cycle is then

- **(a)** Set area B to k times the next value indicated by P, and move P to the right.
- **(b)** Store the value of $A + B$ into the locations indicated by Q, and move Q to the right.
- **(c)** Transfer the contents of B to area A.
- **(d)** Increase fc by *2*.

6	.	1	6	,	4	0	,	2	4	.	1	6	,
↑ Q					↑ P								

$k = 3$ *$A = 16$* *$B = 16$*

Value table Bernoulli Numbers Tangent, Euler

n	T_n
1	1.
3	2.
5	16.
7	272.
9	7936.

n	E_n
0	1.
2	1.
4	5.
6	272.
8	1385.

n	B_n
2	1 – {2,3}
4	1 – {2,3,5}
6	1 – {2,3,7}
8	1 – {2,3,5}
10	1 – {2,3,11}

Periodicity of the Sequences

Examination of the tables produced by the computer program shows that the unit's digits of the nonzero tangent numbers repeat endlessly in the pattern 2, 6, 2, 6, 2, 6, starting with T_3 ; furthermore the two least significant digits ultimately form a repeating period of length 10: 16, 72, 36, 92, 56, 12, 76, 32, 96, 52, 16, 72, ... The three least significant digits have a period of length 50, and for four digits the period-length is 250. These empirical observations suggest that theoretical investigation of period-length might prove fruitful.

References

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