REU: Numerical optimal control

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Note 8

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1 Task 1: Two-dimensional dynamics

Problem

Minimize $\int_0^1 (u_1^2 + u_2^2)$ given that $u_1 = x', u_2 = y' - g$ with the initial conditions of $x(0) = 0, x(1) = 2, y(0) = 0, y(1) = 1$.

Analytical solution

For $u_1 = x'$, we can quickly solve that $x(t) = 2t$ is the optimal solution given our initial condition.
For $u_2 = y' - g$, we assume that $g = 1$ for the sake of convenience. Let $\mathcal{L} = (y' - 1)^2$, we have:

$$\delta S = \int_0^1 \delta (y' - 1)^2 dt$$

$$= 2 \int_0^1 (y' - 1)^T \delta (y' - 1)$$

$$= 2 \int_0^1 (y' - 1)^T \delta y' - (y' - 1)^T \delta 1$$

Using integration by parts, we have:

$$= 2 \int_0^1 -(y' - 1)'^T \delta y = 0$$

From the fundamental lemma, we have $y'' = 0$. Solve for y using our initial condition, we have $y(t) = t$

Finite difference modeling

WLOG assume that g = 1, we have:

$$\int_0^1 (u_1^2 + u_2^2) = \int_0^1 \left[x'^2 + (y' - 1)^2 \right]$$

We have:

$$\delta u_1 = x'', \delta u_2 = y''$$

Hence we can use fictitious time method to model the dynamics as following:

$$d_\tau x = d_t^2 x$$

$$d_\tau y = d_t^2 y$$

2 Task 2: One-dimensional Newtonian dynamics

Problem

Minimize $\int_0^1 q''^2 dt$ given $q(0) = 0, q(1) = 1, q'(0) = 0$

Using calculus of variations, we have that $q^{(4)} = 0$. This gives us:

$$d_\tau q = d_t^4 q$$

$$\Leftrightarrow q_{i,j+1} = q_{i,j} - h\frac{q_{i+2,j} - 4q_{i+1,j} + 6q_{i,j} - 4q_{i-1,j} + q_{i-2,j}}{(\Delta t)^4}$$

We can solve for q using implicit method as below:

$$q_{j+1} = \left(\frac{(N-2)I}{h} - L\right)\left(\frac{q_j}{h} + b\right)$$

Where I is the identity matrix.

Stability analysis

Lemma: A finite difference scheme is stable if the growth factor G satisfies $|G| \leq 1$.

Explicit method

Back to our finite difference model, let $u_{l,j} = e^{ik\Delta tj}$, we have:

$$q_{l,j+1} = \left(1 - h(e^{2ik\Delta t} + e^{-2ik\Delta t} - 4(e^{ik\Delta t} + e^{-ik\Delta t}) + 6)\right) e^{ik\Delta tj}$$

$$= (1 - 2h(\cos 2k\Delta t - 4\cos k\Delta t + 3))e^{ik\Delta tj}$$

$$= (1 - 2h(2\cos k\Delta t^2 - 4\cos k\Delta t + 2))e^{ik\Delta tj}$$

$$= (1 - 4h(\cos k\Delta t - 1)^2)e^{ik\Delta tj}$$

Thus, we have the growth factor $G = 1 - 4h(\cos k\Delta t - 1)^2$
Since $h > 0, (\cos k\Delta t - 1)^2 \geq 0$, we have $G < 1 \forall h > 0$. To have $|G| \leq 1$, we need to have $G = 1 - 4h(\cos k\Delta t - 1)^2 \geq -1 \Leftrightarrow h \leq \frac{1}{2(\cos k\Delta t - 1)^2}$.
To ensure the inequality holds $\forall k$, we must have $h \leq \frac{k^4}{8}$

Implicit method

Let G be the growth factor, we have the equation:

$$\frac{q_{l,j+1} - q_{l,j}}{\Delta\tau} = -\frac{h}{2}\left(\frac{q_{i+2,j} - 4q_{i+1,j} + 6q_{i,j} - 4q_{i-1,j} + q_{i-2,j}}{(\Delta t)^4} + \frac{q_{i+2,j+1} - 4q_{i+1,j+1} + 6q_{i,j+1} - 4q_{i-1,j+1} + q_{i-2,j+1}}{(\Delta t)^4}\right)$$

$$\Leftrightarrow \frac{G-1}{\Delta\tau} = -\frac{h(G+1)}{2}\left(\frac{4(\cos k\Delta t - 1)^2}{(\Delta t)^4}\right)$$

$$\Leftrightarrow \frac{G-1}{\Delta\tau} = -2h(G+1)\left(\frac{(\cos k\Delta t - 1)^2}{(\Delta t)^4}\right)$$

$$\Leftrightarrow G = \frac{1 - 2h(\cos k\Delta t - 1)^2}{1 + 2h(\cos k\Delta t - 1)^2}$$

$$\Leftrightarrow |G| < 1$$

Hence we have the implicit method is unconditionally stable $\forall h$.

3 Research directions

Approach 1: Solving nonlinear and complicated problems

We can continue to use simple gradient descent method to solve complicated PDE, such as the complex Newtonian mechanics problem with energy cost $\int_0^T (q'' + f(q))^2 dt$.

Approach 2: Apply novel methods

Some methods that we can look at:

- Langevin equation and gradient descent with momentum, Algorithm: $\delta_t q = p, \delta_t p = -\nabla V(q) - \gamma p$
- Accelerated gradient descent

4 Appendix

Changelog

- 20/06/2018 Finished two-dimensional dynamics model
- 21/06/2018 Added von Neumann stability analysis
- 22/06/2018 Fix errors and redo von Neumann analysis

Keywords

- Lyapunov stability method
- Forward Euler stability analysis
- Langevin equation

Reference

- `https://www.colorado.edu/engineering/CAS/courses.d/FSI.d/FSI.Ch03.d/FSI.Ch03.pdf`
- `https://ocw.mit.edu/courses/mathematics/18-336-numerical-methods-for-partial-differential-equation lecture-notes/MIT18_336S09_lec14.pdf`
- Relevant methods on binomial options pricing: `http://wwwf.imperial.ac.uk/~mdavis/FDM11/LECTURE_SLIDES2.PDF`