REU: Numerical optimal control

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Note 4

June 11, 2018

1 Task 1: Functional gradient descent

The pseudo-code can be found here `https://gist.github.com/HoangT1215/e807dbcb7a51ba8de9f40fe5675e927b`.

2 Task 2: Extended second-order Euler-Lagrange equation

Given the same boundary and initial conditions, find

$$\min_q \int_0^1 \left[q''(t) + f(q(t))\right]^2 dt$$

Solution
Let $S(q) = \int_0^1 \left[q''(t) + f(q(t))\right]^2 dt$, using ideas of calculus of variation, we have:

$$\frac{\delta S}{\epsilon} = \frac{S(q+\epsilon p) - S(q)}{\epsilon}$$

where p is the perturbation of q

$$= \frac{\int_0^1 [q'' + \epsilon p'' + f(q+\epsilon p)]^2 - [q'' + f(q)]^2 dt}{\epsilon}$$

$$= \frac{\int_0^1 \left[(q'' + \epsilon p'')^2 - q''^2\right] dt + 2\int_0^1 \left[(q'' + \epsilon p'')f(q+\epsilon p) - q'' f(q)\right] dt + \int_0^1 \left[f(q+\epsilon p)^2 - f(q)^2\right] dt}{\epsilon}$$

$$= \int_0^1 pq^{(4)} dt + \frac{2\int_0^1 \left[(q'' + \epsilon p'')f(q+\epsilon p) - q'' f(q)\right] dt}{\epsilon} + \frac{\int_0^1 \left[f(q+\epsilon p)^2 - f(q)^2\right] dt}{\epsilon}$$

(from the double integration by parts in the original problem)

$$= \int_0^1 pq^{(4)} dt + \int_0^1 \frac{d}{dt} f(q)^2 dt + \frac{2\int_0^1 \left[(q'' + \epsilon p'')f(q+\epsilon p) - q'' f(q)\right] dt}{\epsilon}$$

$$= \int_0^1 pq^{(4)} dt + f(q)^2\Big|_0^1 + \frac{2\int_0^1 q'' \left[f(q+\epsilon p) - f(q)\right] + \epsilon p'' f(q+\epsilon p) dt}{\epsilon}$$

$$= f(1)^2 - f(0)^2 + \int_0^1 pq^{(4)} dt + 2\int_0^1 q'' \frac{d}{dt} f(q) dt + \int_0^1 p'' f(q+\epsilon p) dt$$

$$= \int_0^1 pq^{(4)} dt + 2\int_0^1 p'' f(q+\epsilon p) dt$$

Since $f(q)$ is independent of t.

$$= \int_0^1 pq^{(4)} dt + 2\int_0^1 p'' f(q) dt$$

For $\epsilon \to 0$. By taking double integration by parts, we have $2\int_0^1 p'' f(q)dt = 2\int_0^1 p f''(q)dt$, hence:

$$\int_0^1 pq^{(4)}dt + 2\int_0^1 p'' f(q)dt = \int_0^1 p(q^{(4)} + 2f'')dt = 0$$

From the fundamental lemma of calculus of variations, we have $q^{(4)} + 2f'' = 0$, thus $q(t) = 2\int\int f(t) + Q(t)$ where $Q(t)$ is a cubic polynomial.

3 Appendix

- $http://www.techscience.com/doi/10.3970/cmes.2008.034.155.pdf$ (Original FTIM paper by Liu and Atluri)
- $http://vision.ucla.edu/\ ganeshs/dsp_course/lect5.pdf$