REU: Numerical optimal control

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Note 2

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0.1 Review

Functional derivative

- $\delta F = F[f + \delta f] - F[f]$
- $\lim_{\epsilon \to 0} \frac{F[f+\delta f]-F[f]}{d\epsilon} F = \int_a^b \frac{\delta F}{\delta y}$
- Decomposition: Given a function $F(\rho_1, ..., \rho_n)$, we have:

$$dF = \sum_{i=1}^{n} \frac{\delta F}{\delta \rho_i} d\rho_i$$

1 Task 1: Gradient descent implementation

Gradient descent implementation for Rosenbrock function: https://github.com/HoangT1215/Numerical-methods/blob/master/Gradient

2 Task 2: Second order Euler-Lagrange equation

Find $\min_u \int_0^1 u^2 dt$, such that $u(t) = q''(t), q(0) = 0, q(1) = 1, q'(0) = 0$.

Proof. Let $S(q) = \int_0^1 u^2 dt = \int_0^1 (q''(t))^2 \, dt$, we will use functional derivative to solve this problem. Indeed, assume that q is the function such that $S(q) = \int_0^1 (q''(t))^2 \, dt$ is minimal, p is a perturbation of q. We have:

$$\delta S = S(q + \epsilon q) - S(q) = \int_0^1 (q''(t) + \epsilon q''(t))^2 \, dt - \int_0^1 (q''(t))^2 \, dt$$

$$= \epsilon^2 \int_0^1 (q''(t))^2 \, dt + 2\epsilon \int_0^1 q''(t) q''(t) dt$$

$$\Leftrightarrow \frac{S(q + \epsilon q) - S(q)}{\epsilon} = 2 \int_0^1 q''(t) \epsilon q''(t) dt + \epsilon \int_0^1 (q''(t))^2 \, dt$$

Let $\epsilon \to 0$, we have $2 \int_0^1 q''(t) \epsilon q''(t) dt = S'(q) = 0$

$$\Leftrightarrow q^{(4)} = 0$$

Which gives us q is a cubic polynomial. Using terminal conditions, we have $q(t) = \frac{3t^2 - t^3}{2}$. $\square$

3 Task 3: Proof of Euler-Lagrange equation

Let

$$S(y) = \int_a^b \mathcal{L}(x, y, y')dx$$

And h is a perturbation of y We have:

$$\delta S = S(y + \epsilon h) - S(y) = \int_a^b \mathcal{L}(x, y + \epsilon h, y' + \epsilon h')dx - \int_a^b \mathcal{L}(x, y, y')dx (3a)$$

From the decomposition rule of functional derivative, we have:

$$\frac{d\mathcal{L}}{\delta x} = \frac{\delta \mathcal{L}}{\delta x} + y' \frac{\delta \mathcal{L}}{\delta y} + y'' \frac{\delta \mathcal{L}}{y'}$$

$$(3a) \Leftrightarrow \int_a^b \frac{\delta \mathcal{L}}{\delta x} + \delta \frac{\mathcal{L}}{\delta y}(y' + \epsilon h) + \frac{\delta \mathcal{L}(x, y, y')}{\delta y'}(y' + \epsilon h')dx - \int_a^b \mathcal{L}(x, y, y')dx(*)$$

$$= \int_a^b \mathcal{L}(x, y, y') + \epsilon \frac{\delta \mathcal{L}}{\delta y} h + \frac{\delta \mathcal{L}}{\delta y'} h' dx - \int_a^b \mathcal{L}(x, y, y')$$

$$\Leftrightarrow \delta S = \epsilon \int_a^b \frac{\delta \mathcal{L}}{\delta y} h + \frac{\delta \mathcal{L}}{\delta y'} h' dx$$

$$\Leftrightarrow \delta S = \epsilon \int_a^b h \left(\frac{\delta \mathcal{L}}{\delta y} - \frac{d}{dx} \frac{\delta \mathcal{L}}{\delta y'} dx \right)$$

Choose y such that it is the function so that $S(y)$ is minimal, we have:

$$\lim_{x \to 0} \frac{\delta S}{\epsilon} = 0$$

$$\Leftrightarrow \int_a^b h \left(\frac{\delta \mathcal{L}}{\delta y} - \frac{d}{dx} \frac{\delta \mathcal{L}}{\delta y'} \right) dx = 0 \forall h$$

From the fundamental lemma of calculus of variation, we have $\frac{\delta \mathcal{L}}{\delta y} - \frac{d}{dx} \frac{\delta \mathcal{L}}{\delta y'} = 0$.
Hence proved.