REU: Numerical optimal control

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Note 6

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0.1 Review

Implicit method can solve the system of equation in linear time because of the special matrix.

Our kind of applied math: solving a specific hard equation but important. Our algorithm can only give a local minimum, not all of them. The other kind of applied math is to invent another mathematical tool.

Accelerated gradient descent becomes more important because of more data and more computational power.

1 Task 1: Calculus of variations

Consider $q' = q + u$, minimize $S(u) = \int_0^T u^2 dt$ subject to $q(0) = 0, q(1) = 1$

Solution 1: Calculus of variations

Let $\mathcal{L} = (q' - q)^2$, we have:

$$\begin{aligned}
\delta S &= \int_0^1 \delta(q' - q)^2 dt \\
&= 2\int_0^1 (q' - q)^T \delta(q' - q) \\
&= 2\int_0^1 (q' - q)^T \delta q' - (q' - q)^T \delta q
\end{aligned}$$

Using integration by parts, we have:

$$\begin{aligned}
&= 2\int_0^1 -(q' - q)'^T \delta q - (q' - q)^T \delta q \\
&= 2\int_0^1 -(q'' - q')^T \delta q - (q' - q)^T \delta q \\
&= 2\int_0^1 (-q'' + q)^T \delta q
\end{aligned}$$

From the principle of least action, we have $\delta S = 0$, hence we have:

$$\int_0^1 (q'' - q)^T \delta q = 0$$

From the fundamental lemma of calculus of variations, we have:

$$q'' - q = 0$$

$$\Leftrightarrow q = c_0 e^t + c_1 e^{-t}$$

From the boundary conditions $q(0) = 0, q(1) = 1$, we have $q = \frac{e^t - e^{-t}}{e - \frac{1}{e}}$

Solution 2: Beltrami identity

From Beltrami identity we have:

$$\frac{d}{dt}\left(\mathcal{L} - q' \frac{\delta \mathcal{L}}{\delta q'}\right) = 0$$

$$\Leftrightarrow \frac{d}{dt}\left[(q' - q)^2 - q' 2(q' - q)\right] = 0$$

$$\Leftrightarrow \frac{d}{dt}\left[q^2 - q'^2\right] = 0$$

2 Task 2

Consider two dimensional dynamics satisfy $x' = u_1, y' = u_2 - g$. Minimize $\int_0^1 (u_1^2 + u_2^2) dt$ such that $x(0) = 0, y(0) = 0, x(1) = 2, y(1) = 1$
We can see that this equation is just a combination of two separate first order Euler-Lagrange equations.

A non-independent problem

Given the thrust:

$$V(x, y) = \frac{-1}{\sqrt{x^2 + y^2}}$$

We have the dynamics

$$x'' = -\frac{x}{(x^2 + y^2)^{\frac{3}{2}}} + u_x$$

$$y'' = -\frac{y}{(x^2 + y^2)^{\frac{3}{2}}} + u_y$$

3 Task 3

Simulate:

$$\delta_\tau q = \delta_t^2 q$$

Given

$$q(\tau, 0) = 0, q(\tau, 1) = 1, q(0, t) = t^3$$

Implementation

https://gist.github.com/HoangT1215/f2132c212474bf8f272f0d637d884896

4 Task

Task 1

Solve the Euler-Lagrange with implicit method.

Task 2

Research tridiagonal linear system.

Task 3

Implement accelerated gradient descent/gradient descent with momentum for the problem.

5 Appendix

Keywords

- Tridiagonal linear system