UNIVERSITY OF TECHNOLOGY VNUHCM FACULTY OF AS	MIDTERM EXAM	Semester/ Academic year		221	2022 - 2023
		Date	October 2022		
	Course title	Calculus 1			
	Course ID	MT1003			
	Duration	50 mins	Question sheet code	1254	

Intructions to students: - There are 6 pages in the exam
-This is a closed book exam. Only your calculator is allowed. Total available score: 10.

Question 01. Find the domain of function $y = \sqrt{\ln\left(1+\frac{1}{x}\right)}$

(A) $x \leqslant -1$ (B) None of them (C) $x > 0$ or $x \leqslant -1$ (D) $x > 0$

Question 02. Find real values of a such that $f(x) = \begin{cases} ax-2, & x \leqslant 1 \\ \arctan(\sqrt{x-1}), & x > 1 \end{cases}$ is continuous at $x = 1$.

(A) None of them (B) 2 (C) 1 (D) -2

Question 03. Find real values of a such that $f(x) = \begin{cases} \arctan \frac{x^3-2}{x-1}, & x \neq 1 \\ a, & x = 1 \end{cases}$ is continuous at $x = 1$.

(A) $-\frac{\pi}{2}$ (B) $\frac{\pi}{2}$ (C) $-\frac{\pi}{4}$ (D) a does not exist

Question 04. Find ALL points on the graph of $f(x) = x^3 - 3x$ where the tangent line is parallel to the $x-$axis.

(A) None of them (B) $(1,-2)$ and $(-1,2)$ (C) $(1,-2)$ (D) $(1,2)$ and $(-1,-2)$

Question 05. Find ALL points on the graph of $f(x) = \frac{1}{x}$ where the tangent line is perpendicular to the line $y = 4x - 3$.

(A) $\left(1,\frac{1}{2}\right)$ and $\left(-1,-\frac{1}{2}\right)$ (B) $\left(2,\frac{1}{2}\right)$

(C) None of them (D) $\left(2,\frac{1}{2}\right)$ and $\left(-2,-\frac{1}{2}\right)$

Question 06. Find constant A, B, C so that $y = Ax^3 + Bx + C$ satisfies the equation $y''' + 2y'' - 3y' + y = x$.

(A) None of them (B) $A = 1, B = 1, C = 3$ (C) $A = 0, B = 1, C = 3$ (D) $A = 0, B = 2, C = 3$

Question 07. For what values of A and B does $y = Ax\cos x + Bx\sin x$ satisfy $y'' + y = -3\cos x$?

(A) $A = 1, B = 2$ (B) $A = 0, B = 1$ (C) $A = 0, B = -\frac{3}{2}$ (D) None of them

Question 08. Determine ALL values of x such that $f(x) = \frac{x^2-1}{x^2-4}$ is **not differentiable**.

(A) $x \neq \pm 2$ (B) None of them (C) $x \neq 1$ (D) $x = \pm 2$

Question 09. Determine ALL values of x such that $f(x) = \sqrt[3]{(x^2-1)^2}$ is **not differentiable**.

(A) $x = \pm 2$ (B) $x = \pm 1$ (C) None of them (D) $x \neq 1$

Question 10. Find a, b such that function f is differentiable at $x = 0$, if $f(x) = \begin{cases} x^2 + 2x, & x \leqslant 0 \\ \ln(ax+b), & x > 0 \end{cases}$

(A) None of them (B) $a = 1, b = 1$ (C) $a = 2, b = 1$ (D) $a = 0, b = 2$

Question 11. A particle moving on the $x-$axis has position $s(t) = t^3 - 9t^2 + 24t + 20$ after an elapsed time of t seconds. What is the total distance travelled by the particle during the first 8 seconds?

(A) None of them (B) 126 (C) 116 (D) 136

Question 12. A particle moving on the $x-$axis has position $s(t) = 2t^3 + 3t^2 - 36t + 40$ after an elapsed time of t seconds. What is the total distance travelled by the particle during the first 3 seconds?

(A) 62 (B) None of them (C) 60 (D) 61

Question 13. An importer of Rwandan coffee estimates that local consumers will buy approximately $D(p) = \frac{4374}{p^2}$ pounds of the coffee per week when the price is p dollars per pound. It is estimated that t weeks from now, the price of Rwandan coffee will be $p(t) = 0.02t^2 + 0.01t + 6$ dollars per pound. At what rate will the weekly demand for the coffee be changing with respect to time 10 weeks from now?

(A) None of them (B) $-\frac{1640}{243}$ (C) $-\frac{1643}{243}$ (D) $-\frac{1647}{243}$

Question 14. When electric blenders are sold for p dollars apiece, local consumers will buy approximately $D(p) = \frac{8000}{p}$ blenders per month. It is estimated that t months from now, the price of the blenders will be $p(t) = 0.04\sqrt{t^3} + 15$ dollars. Compute the rate at which the monthly demand for the blenders will be changing with respect to time 25 months from now.

(A) -4 (B) None of them (C) -2 (D) -6

Question 15. A ball of ice melts so that its radius decreases from 5 cm to 4.92 cm. By approximately how much does the volume of the ball decrease?

(A) $\Delta V \approx -4\pi\ cm^3$ (B) $\Delta V \approx -6\pi\ cm^3$ (C) $\Delta V \approx -8\pi\ cm^3$ (D) None of them

Question 16. Find $df(1)$ if $f(x) = x^{2x+8}$

(A) None of them (B) $10dx$ (C) $2dx$ (D) dx

Question 17. If $f(x) = x + (x-1)\arctan\sqrt{\frac{x+5}{9x^2+8x+1}}$, then calculate $f'(1)$.

(A) $f'(1) = 1$ (B) $f'(1) = 0$ (C) None of them (D) $f'(1) = 1 + \frac{\pi}{6}$

Question 18. If $f(x) = |x^2 - 3x + 2| + 4$, the find the domain of $f'(x)$.

(A) $\mathbb{R}$ (B) $\mathbb{R}\backslash\{1,2\}$ (C) None of them (D) $\mathbb{R}\backslash\{1\}$

Question 19. If $y = \sin(e^{f(x)})$, then find y'

(A) $f'(x)\cos(e^{f(x)})$ (B) None of them (C) $e^{f(x)}\cos(e^{f(x)})$ (D) $f'(x)e^{f(x)}\cos(e^{f(x)})$

Question 20. Find $df(1)$ if $f(x) = x^2\arctan(x)$.

(A) $\frac{2+\pi}{2}dx$ (B) $(\pi+2)dx$ (C) None of them (D) $\frac{1+\pi}{2}dx$

Question 21. Find $df(1)$ if $f(x) = \ln(3 + x^4)$.

(A) $(\ln 3)dx$ (B) $2dx$ (C) dx (D) None of them

Question 22. Find $d^2y(0)$ if $y = \cos^3(5x)$.

(A) $d^2y(0) = -75dx^2$ (B) None of them (C) $d^2y(0) = -35dx^2$ (D) $d^2y(0) = 75dx^2$

Question 23. For which values of constants E and F is it true that $\lim_{x\to 0}(x^{-3}\sin 7x + Ex^{-2} + F) = -2$.

(A) None of them (B) $E = -5; F = 2$ (C) $E = -6; F = 3$ (D) $E = -7; F = \frac{331}{6}$

Question 24. For a certain value of D, the limit $\lim_{x\to\infty}(x^4 + 5x^3 + 3)^D - x$ is finite and nonzero. Find D.

(A) $D = \frac{1}{4}$ (B) None of them (C) $D = \frac{3}{4}$ (D) $D = \frac{1}{2}$

Question 25. Find $\lim_{x\to 0}(1 + 3x^2e^{2x})^{\frac{1}{4x^2}}$

(A) $I = 1$ (B) None of them (C) $I = 0$ (D) $I = e^{\frac{3}{4}}$

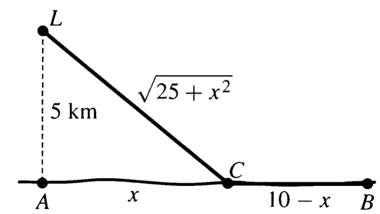
Question 26. Studying the local extreme values of the function $f(x) = \arctan\left(\dfrac{2x-1}{x^2+2}\right)$, which statement is always true?

(A) f has local minimum when $x = -1$, and no local maximum
(B) f has local minimum when $x = -1$, and local maximum when $x = 2$
(C) None of them
(D) f has no local minimum, and local maximum when $x = 2$

Question 27. If $y = \dfrac{\ln x^2}{x}$, then which statement is always true?

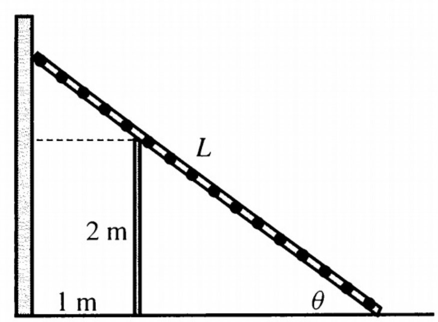
(A) The function y is increasing on $(-e, e)$
(B) The function y is increasing on $\mathbb{R}$
(C) The function y is increasing on $(-e, 0) \cup (0, e)$
(D) None of them

Question 28. A lighthouse L is located on a small island 5 km north of a point A on a straight east-west shoreline. A cable is to be laid from L to point B on the shoreline 10 km east of A. The cable will be laid through the water in a straight line from L to a point C on the shoreline between A and B, and from there to B along the shoreline. The part of the cable lying in the water costs 5000 dollars/km, and the part along the shoreline costs 3000 dollars/km. Where should C be chosen to minimize the total cost of the cable?



(A) $x = 5.75\ km$ (B) $x = 3.75\ km$ (C) None of them (D) $x = 4.75\ km$

Question 29. Find the length of the shortest ladder that can extend from a vertical wall, over a fence 2 m high located 1 m away from the wall, to a point on the ground outside the fence.



(A) $5.1619\ m$
(B) $6.1619\ m$
(C) $4.1619\ m$
(D) None of them

Question 30. If $y = x(e^{\frac{1}{x}} - 1)$, then the number of asymptotes of y is

(A) 1 (B) 0 (C) 3 (D) 2

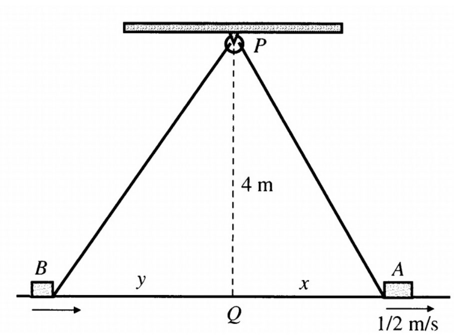
Question 31. If $y = x^3 e^x$, then the number of point(s) of inflection of the graph of y is

(A) 2 (B) 3 (C) 1 (D) 0

Question 32. A spherical balloon is inflated so that its radius increases from 20 cm to 20.2 cm in 1 min. By approximately how much has its volume increased in that minute?

(A) $220\pi\ cm^3/min$ (B) $420\pi\ cm^3/min$ (C) $320\pi\ cm^3/min$ (D) None of them

Question 33. Two crates, A and B, are on the floor of a warehouse. The crates are joined by a rope $15m$ long, each crate being hooked at floor level to an end of the rope. The rope is stretched tight and pulled over a pulley P that is attached to a rafter $4m$ above a point Q on the floor directly between the two crates. If crate A is $3m$ from Q and is being pulled directly away from Q at a rate of $0.5m/s$, how fast is crate B moving toward Q?



(A) $\dfrac{\sqrt{21}}{14}$
(B) $-\dfrac{\sqrt{21}}{14}$
(C) None of them
(D) $-\dfrac{\sqrt{23}}{14}$

Question 34. Recall that volume of a right cylinder is $V = \pi.r^2.h$, where r is the radius of the base and h is the height. Given a right cylinder with the radius of the base of 10 cm and the height of 10 cm. The radius of the base is increasing at a rate of 5 cm/s, and its height is increasing at a rate of 4 cm/s. How fast is the volume of cylinder increasing?

(A) $1800\pi\ cm^3/s$ (B) None of them (C) $1600\pi\ cm^3/s$ (D) $1400\pi\ cm^3/s$

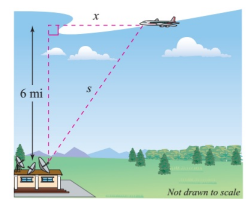
Question 35. Boyle's law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy equation $PV = C = const$. Suppose that at a certain instant the volume is $600cm^3$, the pressure is $150kPA$, and the pressure is increasing at a rate of $20kPA/min$. At what rate is the volume decreasing at this instant?

(A) None of them (B) $8 \times 10^{-5}\ m^3/min$ (C) $4 \times 10^{-5}\ m^3/min$ (D) $6 \times 10^{-5}\ m^3/min$

Question 36. If a truck factory employs x workers and has daily operating expenses of y dollars, it can produce $P = \frac{1}{3}x^{0.6}y^{0.4}$ trucks per year. How fast are the daily expenses decreasing when they are 10000 dollars and the number of workers is 40, if the number of workers is increasing at 1 per day and production is remaining constant?

(A) 275 dollars (B) 475 dollars (C) 375 dollars (D) None of them

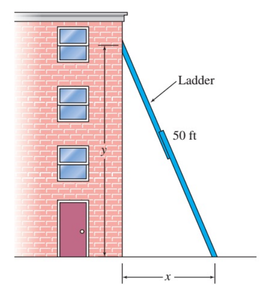
Question 37. An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown in the figure below. If s is decreasing at a rate of 400 miles per hour when $s = 10$ miles, what is the speed of the plane?



An airplane is flying at an altitude of 6 miles, s miles from the station.

(A) 400 miles per hour (B) None of them
(C) 500 miles per hour (D) 600 miles per hour

Question 38. A 50−ft ladder is placed against a large building. The base of the ladder is resting on an oil spill, and it slips to the right at the rate of 3 ft per minute. Find the rate of change of the height of the top of the ladder above the ground at the instant when the base of the ladder is 30 ft from the base of the building.

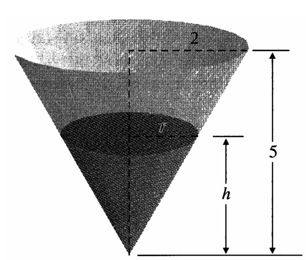


(A) None of them (B) $-\frac{5}{4}$ ft per minute
(C) $-\frac{3}{4}$ ft per minute (D) $-\frac{9}{4}$ ft per minute

Question 39. When air expands adiabatically (that is, with no change in heat), the pressure P and the volume V satisfy the relationship $PV^{1.4} = C$, where C is a constant. At a certain instant, the pressure is $20lb/inch^2$ and the volume is $280inch^3$. If the volume is decreasing at the rate of $5inch^3/s$ at this instant, what is the rate of change of the pressure?

(A) $0.5\ lb/inch^2/s$ (B) None of them (C) $0.4\ lb/inch^2/s$ (D) $0.6\ lb/inch^2/s$

Question 40. A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $\frac{1}{12}(m^3/min)$. How fast is the water level in the tank dropping at that time?

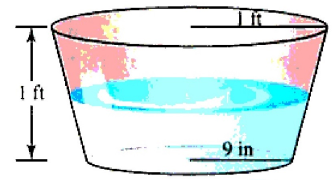


(A) $-\frac{25}{768\pi}$ (m/min) (B) $-\frac{23}{768\pi}$ (m/min)
(C) $-\frac{7}{256\pi}$ (m/min) (D) None of them

Question 41. A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

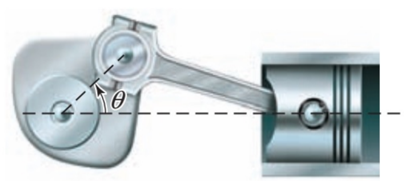
(A) $6\pi\ (ft^2/s)$ (B) $10\pi\ (ft^2/s)$ (C) None of them (D) $8\pi\ (ft^2/s)$

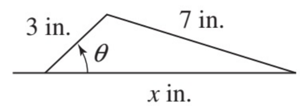
Question 42. Suppose a water bucket is modeled by the frustum of a cone with height $1ft$ and upper and lower radii of $1ft$ and $9in.$, respectively. Note that $1ft = 12in.$ If water is leaking from the bottom of the bucket at the rate of $8in.^3/min$, at what rate is the water level falling when the depth of water in the bucket is $6in.$?



(A) $-\dfrac{11}{147\pi}\ (in./min)$ (B) $-\dfrac{32}{441\pi}\ (in./min)$ (C) $-\dfrac{35}{441\pi}\ (in./min)$ (D) None of them

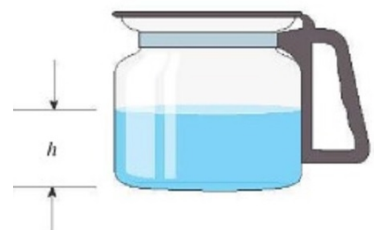
Question 43. A piston is attached to a crankshaft of radius 3 in. by means of a 7 in. connecting rod. Let x denote the position of the piston. If the crankshaft rotates counterclockwise at a constant rate of 60 revolution/second, what is the velocity of the piston $\left(\dfrac{dx}{dt}\right)$ when $\theta = \dfrac{\pi}{3}$?





(A) $-\dfrac{2882\sqrt{3}\times\pi}{13}$ (B) $-\dfrac{2881\sqrt{3}\times\pi}{13}$ (C) $-\dfrac{2880\sqrt{3}\times\pi}{13}$ (D) None of them

Question 44. A coffee pot in the form of a circular cylinder of radius 4 in. is being filled with water flowing at a constant rate. If the water level is rising at the rate of $0.4in./sec$, what is the rate at which water is flowing into the coffee pot?



(A) None of them (B) $\dfrac{33\pi}{5}$ (in./sec)

(C) $\dfrac{31\pi}{5}$ (in./sec) (D) $\dfrac{32\pi}{5}$ (in./sec)

Question 45. A study prepared for National Association of Realtors estimates that the number of housing starts in the southwest, $N(t)$ (in units of a million), over the next 5 years is related to the mortgage rate $r(t)$ (percent per year) by the equation $9N^2 + r = 36$, where t is measured in years. What is the rate of change of the number of housing starts with respect to time when the mortgage rate is 11% per year and is increasing at the rate of 1.5% per year?

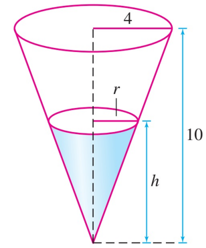
(A) -0.06 (B) None of them (C) -0.04 (D) -0.05

Question 46. Texar Inc., a manufacturer of disk drives is willing to make x thousand 1GB USB flash drives available in the marketplace each week when the wholesale price is p dollars per drive. It is known that the relationship between x and p is governed by the supply equation $x^2 - 3xp + p^2 = 5$. How fast is the supply of drives changing when the price per drive is 11 dollars, the quantity supplied is 4000 drives, and the wholesale price per drive is increasing at the rate of 0.10 dollars per drive each week?

(A) None of them (B) 0.06 (C) 0.05 (D) 0.04

Question 47. Water pours into a conical tank of height $10m$ and radius $4m$

at a rate of $6m^3/min$. At what rate is the water level rising when the level is $5m$ high?



(A) $\frac{3}{2\pi}$ (m/min) (B) None of them (C) $\frac{1}{2\pi}$ (m/min) (D) $\frac{5}{2\pi}$ (m/min)

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ANSWER

01. D
02. B
03. D
04. B
05. D
06. C
07. C
08. D
09. B
10. C
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26. B
27. D
28. B
29. C
30. D
31. B
32. C
33. B
34. D
35. B
36. C
37. C
38. D
39. A
40. A
41. D
42. B
43. C
44. D
45. D
46. D
47. A

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MIDTERM EXAM AND THE ANSWER

Question 01. Find the domain of function $y = \sqrt{\ln\left(1 + \frac{1}{x}\right)}$

(A) $x \leqslant -1$ (B) None of them (C) $x > 0$ or $x \leqslant -1$ (D) $x > 0$

Solution. The correct answer (D). $\ln\left(1+\frac{1}{x}\right) \geqslant 0 \Rightarrow 1 + \frac{1}{x} \geqslant 1 \Rightarrow x > 0.$ □

Question 02. Find real values of a such that $f(x) = \begin{cases} ax - 2, & x \leqslant 1 \\ \arctan(\sqrt{x-1}), & x > 1 \end{cases}$ is continuous at $x = 1$.

(A) None of them (B) 2 (C) 1 (D) -2

Solution. The correct answer (B). The function f is continuous at $x = 1$ if

$$\lim_{x\to 1^+} f(x) = 0 = \lim_{x\to 1^-} f(x) = f(1) = a - 2 \Rightarrow a = 2$$

□

Question 03. Find real values of a such that $f(x) = \begin{cases} \arctan\dfrac{x^3-2}{x-1}, & x \neq 1 \\ a, & x = 1 \end{cases}$ is continuous at $x = 1$.

(A) $-\frac{\pi}{2}$ (B) $\frac{\pi}{2}$ (C) $-\frac{\pi}{4}$ (D) a does not exist

Solution. The correct answer (D). The function f is continuous at $x = 1$ if

$$\lim_{x\to 1^+} f(x) = -\frac{\pi}{2}$$

$$\lim_{x\to 1^-} f(x) = \frac{\pi}{2}$$

Therefore, f is not continuous at $x = 1$. □

Question 04. Find ALL points on the graph of $f(x) = x^3 - 3x$ where the tangent line is parallel to the $x-$axis.

(A) None of them (B) $(1,-2)$ and $(-1,2)$ (C) $(1,-2)$ (D) $(1,2)$ and $(-1,-2)$

Solution. The correct answer (B). The tangent line is parallel to the $x-$axis when its slope is equal to 0. It means that

$$f'(x) = 3x^2 - 3 = 0 \Leftrightarrow x = \pm 1.$$

Therefore, all points on the graph of $f(x)$ where the tangent line is parallel to the $x-$axis are $(1,-2)$ and $(-1,2)$.
□

Question 05. Find ALL points on the graph of $f(x) = \frac{1}{x}$ where the tangent line is perpendicular to the line $y = 4x - 3$.

Ⓐ $\left(1, \frac{1}{2}\right)$ and $\left(-1, -\frac{1}{2}\right)$

Ⓑ $\left(2, \frac{1}{2}\right)$

Ⓒ None of them

Ⓓ $\left(2, \frac{1}{2}\right)$ and $\left(-2, -\frac{1}{2}\right)$

Solution. The correct answer Ⓓ. The tangent line is perpendicular to the line $y = 4x - 3$ when its slope is equal to $-\frac{1}{4}$. It means that

$$f'(x) = -\frac{1}{x^2} = -\frac{1}{4} \Leftrightarrow x = \pm 2.$$

Therefore, all points on the graph of $f(x)$ where the tangent line is perpendicular to the line $y = 4x - 3$, are $\left(2, \frac{1}{2}\right)$ and $\left(-2, -\frac{1}{2}\right)$. □

Question 06. Find constant A, B, C so that $y = Ax^3 + Bx + C$ satisfies the equation $y''' + 2y'' - 3y' + y = x$.

Ⓐ None of them Ⓑ $A = 1, B = 1, C = 3$ Ⓒ $A = 0, B = 1, C = 3$ Ⓓ $A = 0, B = 2, C = 3$

Solution. The correct answer Ⓒ. $y = Ax^3 + Bx + C \Rightarrow \begin{cases} y' &= 3Ax^2 + B \\ y'' &= 6Ax \\ y''' &= 6A \end{cases}$

Substituting these derivatives into the equation $y''' + 2y'' - 3y' + y = x$, we receive

$$6A + 2 \times 6Ax - 3 \times (3Ax^2 + B) + (Ax^3 + Bx + C) = x$$

$$\Leftrightarrow Ax^3 - 9Ax^2 + (12A + B - 1)x + (6A - 3B + C) = 0$$

$$\Leftrightarrow \begin{cases} A = 0 \\ -9A = 0 \\ 12A + B - 1 = 0 \\ 6A - 3B + C = 0 \end{cases} \Leftrightarrow \begin{cases} A = 0 \\ B = 1 \\ C = 3 \end{cases}$$

□

Question 07. For what values of A and B does $y = Ax\cos x + Bx\sin x$ satisfy $y'' + y = -3\cos x$?

Ⓐ $A = 1, B = 2$ Ⓑ $A = 0, B = 1$ Ⓒ $A = 0, B = -\frac{3}{2}$ Ⓓ None of them

Solution. The correct answer Ⓒ. $y = Ax\cos x + Bx\sin x$

$$\Rightarrow \begin{cases} y' &= A\cos x - Ax\sin x + B\sin x + Bx\cos x \\ y'' &= -A\sin x - A\sin x - Ax\cos x + B\cos x + B\cos x - Bx\sin x \end{cases}$$

Substituting these derivatives into the equation $y'' + y = -3\cos x$, we receive

$$-2A\sin x + 2B\cos x = -3\cos x \Leftrightarrow \begin{cases} -2A = 0 \\ 2B = -3 \end{cases} \Leftrightarrow \begin{cases} A = 0 \\ B = -\frac{3}{2} \end{cases}$$

□

Question 08. Determine ALL values of x such that $f(x) = \frac{x^2 - 1}{x^2 - 4}$ is **not differentiable**.

Ⓐ $x \neq \pm 2$ Ⓑ None of them Ⓒ $x \neq 1$ Ⓓ $x = \pm 2$

Solution. The correct answer (D). The domain of $f(x) = \dfrac{x^2-1}{x^2-4}$ is $D = \mathbb{R}\backslash\{\pm 2\}$.

$$\Rightarrow y' = \frac{-6x}{(x^2-4)^2}$$

So the function f is not differentiable when $x = \pm 2$. □

Question 09. Determine ALL values of x such that $f(x) = \sqrt[3]{(x^2-1)^2}$ is **not differentiable**.

(A) $x = \pm 2$ (B) $x = \pm 1$ (C) None of them (D) $x \neq 1$

Solution. The correct answer (B). The domain of $f(x) = \sqrt[3]{(x^2-1)^2}$ is $D = \mathbb{R}$.

$$\Rightarrow y' = \frac{2}{3} \times (x^2-1)^{-1/3} \times 2x = \frac{4x}{3\sqrt[3]{(x+1)(x-1)}}$$

So the function f is not differentiable when $x = \pm 1$. □

Question 10. Find a, b such that function f is differentiable at $x = 0$, if $f(x) = \begin{cases} x^2+2x, & x \leqslant 0 \\ \ln(ax+b), & x > 0 \end{cases}$

(A) None of them (B) $a = 1, b = 1$ (C) $a = 2, b = 1$ (D) $a = 0, b = 2$

Solution. The correct answer (C). In order to be differentiable, function has to be continuous

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) \Rightarrow \ln b = 0 \Rightarrow b = 1.$$

Function f is differentiable at $x = 0$ if and only if

$$f'_+(0) = f'_-(0) \Rightarrow \frac{a}{b} = 2 \Rightarrow a = 2b = 2.$$

□

Question 11. A particle moving on the x−axis has position $s(t) = t^3 - 9t^2 + 24t + 20$ after an elapsed time of t seconds. What is the total distance travelled by the particle during the first 8 seconds?

(A) None of them (B) 126 (C) 116 (D) 136

Solution. The correct answer (D). $s(t) = t^3 - 9t^2 + 24t + 20 \Rightarrow v(t) = s'(t) = 3t^2 - 18t + 24.$

$$v(t) > 0 \Leftrightarrow \left[\begin{array}{l} 0 < t < 2 \\ 4 < t < 8 \end{array}\right. \quad \text{and} \quad v(t) < 0 \Leftrightarrow 2 < t < 4$$

So the total distance travelled by the particle during the first 8 seconds is

$$|s(2) - s(0)| + |s(4) - s(2)| + |s(8) - s(4)| = 136.$$

□

Question 12. A particle moving on the x−axis has position $s(t) = 2t^3 + 3t^2 - 36t + 40$ after an elapsed time of t seconds. What is the total distance travelled by the particle during the first 3 seconds?

(A) 62 (B) None of them (C) 60 (D) 61

Solution. The correct answer (D). $s(t) = 2t^3 + 3t^2 - 36t + 40 \Rightarrow v(t) = s'(t) = 6t^2 + 6t - 36.$

$$v(t) > 0 \Leftrightarrow 2 < t < 3 \quad \text{and} \quad v(t) < 0 \Leftrightarrow 0 < t < 2$$

So the total distance travelled by the particle during the first 8 seconds is

$$|s(2) - s(0)| + |s(3) - s(2)| = 44 + 17 = 61.$$

□

Question 13. An importer of Rwandan coffee estimates that local consumers will buy approximately $D(p) = \dfrac{4374}{p^2}$ pounds of the coffee per week when the price is p dollars per pound. It is estimated that t weeks from now, the price of Rwandan coffee will be $p(t) = 0.02t^2 + 0.01t + 6$ dollars per pound. At what rate will the weekly demand for the coffee be changing with respect to time 10 weeks from now?

(A) None of them (B) $-\dfrac{1640}{243}$ (C) $-\dfrac{1643}{243}$ (D) $-\dfrac{1647}{243}$

Solution. The correct answer (B). $D(t) = D(p(t)) \Rightarrow D'(t) = D'(p).p'(t) = 4374 \times \left(-\dfrac{2}{p^3}\right) \times (0.04t + 0.01)$

When $t = 10 \Rightarrow p(10) = 8.1$ and

$$D'(10) = 4374 \times \left(-\frac{2}{8.1^3}\right) \times (0.04 \times 10 + 0.01) = -\frac{1640}{243} \approx -6.7490 < 0.$$

The demand will be decreasing. □

Question 14. When electric blenders are sold for p dollars apiece, local consumers will buy approximately $D(p) = \dfrac{8000}{p}$ blenders per month. It is estimated that t months from now, the price of the blenders will be $p(t) = 0.04\sqrt{t^3} + 15$ dollars. Compute the rate at which the monthly demand for the blenders will be changing with respect to time 25 months from now.

(A) -4 (B) None of them (C) -2 (D) -6

Solution. The correct answer (D). $D(t) = D(p(t)) \Rightarrow D'(t) = D'(p).p'(t) = 8000 \times \left(-\dfrac{1}{p^2}\right) \times \left(0.04 \times \dfrac{3}{2}t^{1/2}\right)$

When $t = 25 \Rightarrow p(25) = 20$ and

$$D'(25) = 8000 \times \left(-\frac{1}{20^2}\right) \times (0.06 \times \sqrt{25}) = -6 < 0.$$

The demand will be decreasing. □

Question 15. A ball of ice melts so that its radius decreases from 5 cm to 4.92 cm. By approximately how much does the volume of the ball decrease?

(A) $\Delta V \approx -4\pi\ cm^3$ (B) $\Delta V \approx -6\pi\ cm^3$ (C) $\Delta V \approx -8\pi\ cm^3$ (D) None of them

Solution. The correct answer (C). The volume of the ball of ice is $V = \dfrac{4}{3}\pi R^3$

$$\Rightarrow \Delta V \approx dV = V'(R)dR = V'(R)\Delta R = 4\pi R^2 \Delta R$$

Since $\Delta R = 4.92 - 5 = -0.08\ cm$ and $R = 5\ cm$, so

$$\Delta V \approx 4\pi \times 5^2 \times (-0.08) = -8\pi \approx -25.13\ cm^3.$$

□

Question 16. Find $df(1)$ if $f(x) = x^{2x+8}$

(A) None of them (B) $10dx$ (C) $2dx$ (D) dx

Solution. The correct answer (B). We have $f(x) = x^{2x+8} = e^{(2x+8)\ln x} \Rightarrow f'(x) = e^{(2x+8)\ln x}\left(2\ln x + \dfrac{2x+8}{x}\right) \Rightarrow f'(1) = 10 \Rightarrow df(1) = 10dx.$ □

Question 17. If $f(x) = x + (x-1)\arctan\sqrt{\dfrac{x+5}{9x^2+8x+1}}$, then calculate $f'(1)$.

(A) $f'(1) = 1$ (B) $f'(1) = 0$ (C) None of them (D) $f'(1) = 1 + \frac{\pi}{6}$

Solution. The correct answer (D). Let $u(x) = \arctan\sqrt{\dfrac{x+5}{9x^2+8x+1}}$. Then

$$f(x) = x + (x-1)u(x) \Rightarrow f'(x) = 1 + u(x) + (x-1)u'(x) \Rightarrow f'(1) = 1 + u(1) = 1 + \arctan\sqrt{\frac{6}{18}} = 1 + \frac{\pi}{6}$$

$\square$

Question 18. If $f(x) = |x^2 - 3x + 2| + 4$, the find the domain of $f'(x)$.

(A) $\mathbb{R}$ (B) $\mathbb{R}\backslash\{1,2\}$ (C) None of them (D) $\mathbb{R}\backslash\{1\}$

Solution. The correct answer (B). We have

$$f(x) = \begin{cases} x^2 - 3x + 2 + 4, & \text{if } x^2 - 3x + 2 \geqslant 0 \\ -(x^2 - 3x + 2) + 4, & \text{if } x^2 - 3x + 2 < 0 \end{cases} \Rightarrow f(x) = \begin{cases} 2x - 3, & \text{if } x > 2 \vee x < 1 \\ -2x + 3, & \text{if } 1 < x < 2 \end{cases}$$

Therefore, $f'_+(1) = 1 \neq f'_-(1) = -1$ and $f'_+(2) = 1 \neq f'_-(2) = -1$. So, $f'(1)$ and $f'(2)$ do not exist. $\square$

Question 19. If $y = \sin(e^{f(x)})$, then find y'

(A) $f'(x)\cos(e^{f(x)})$ (B) None of them (C) $e^{f(x)}\cos(e^{f(x)})$ (D) $f'(x)e^{f(x)}\cos(e^{f(x)})$

Solution. The correct answer (D). We have $y = \sin(e^{f(x)}) \Rightarrow y' = \cos(e^{f(x)}) \times \left(e^{f(x)}\right) \times f'(x)$. $\square$

Question 20. Find $df(1)$ if $f(x) = x^2\arctan(x)$.

(A) $\dfrac{2+\pi}{2}dx$ (B) $(\pi + 2)dx$ (C) None of them (D) $\dfrac{1+\pi}{2}dx$

Solution. The correct answer (D). We have $f(x) = x^2\arctan(x) \Rightarrow f'(x) = 2x\arctan(x) + x^2\dfrac{1}{1+x^2} \Rightarrow f'(1) = \dfrac{\pi}{2} + \dfrac{1}{2} \Rightarrow df(1) = \dfrac{1+\pi}{2}dx$. $\square$

Question 21. Find $df(1)$ if $f(x) = \ln(3 + x^4)$.

(A) $(\ln 3)dx$ (B) $2dx$ (C) dx (D) None of them

Solution. The correct answer (C). We have $f(x) = \ln(3 + x^4) \Rightarrow f'(x) = \dfrac{4x^3}{3 + x^4} \Rightarrow f'(1) = \dfrac{4}{3+1} = 1 \Rightarrow df(1) = 1dx$. $\square$

Question 22. Find $d^2y(0)$ if $y = \cos^3(5x)$.

(A) $d^2y(0) = -75dx^2$ (B) None of them (C) $d^2y(0) = -35dx^2$ (D) $d^2y(0) = 75dx^2$

Solution. The correct answer (A). We have

$$y' = 3\cos^2(5x)(-\sin(5x)) \times 5 = -15\cos^2(5x)\sin(5x)$$

$$\Rightarrow y'' = -15 \times 2\cos(5x)(-\sin(5x)) \times 5\sin(5x) - 15 \times 5\cos^2(5x)\cos(5x)$$

Thus

$$y''(0) = -15 \times 5 = -75 \Rightarrow d^2y(0) = -75dx^2.$$

$\square$

Question 23. For which values of constants E and F is it true that $\lim\limits_{x\to 0}(x^{-3}\sin 7x + Ex^{-2} + F) = -2$.

(A) None of them (B) $E = -5; F = 2$ (C) $E = -6; F = 3$ (D) $E = -7; F = \dfrac{331}{6}$

Solution. The correct answer (D). $I = \lim_{x\to 0}(x^{-3}\sin 7x + Ex^{-2} + F) = \lim_{x\to 0}\dfrac{\sin 7x + Ex + Fx^3}{x^3} = \dfrac{0}{0}$. Using L'Hospital Rule, we have

$$I = \lim_{x\to 0}\frac{7\cos 7x + E + 3Fx^2}{3x^2} = \frac{7+E}{0}$$

The limit is a finite number, so $7 + E = 0 \Rightarrow E = -7$. Using L'Hospital Rule, we also receive

$$I = \lim_{x\to 0}\frac{-49\sin 7x + 6Fx}{6x} = \lim_{x\to 0}\left(-\frac{49\times 7}{6}\times\frac{\sin 7x}{7x} + F\right) = -\frac{49\times 7}{6} + F = -2 \Rightarrow F = \frac{331}{6}$$

□

Question 24. For a certain value of D, the limit $\lim_{x\to\infty}(x^4 + 5x^3 + 3)^D - x$ is finite and nonzero. Find D.

(A) $D = \dfrac{1}{4}$ (B) None of them (C) $D = \dfrac{3}{4}$ (D) $D = \dfrac{1}{2}$

Solution. The correct answer (A). Let $t = \dfrac{1}{x}$, then

$$I = \lim_{x\to\infty}\left[(x^4 + 5x^3 + 3)^D - x\right] = \lim_{t\to 0}\left[\left(\frac{1}{t^4} + \frac{5}{t^3} + 3\right)^D - \frac{1}{t}\right] = \lim_{t\to 0}\frac{(1 + 5t + 3t^4)^D - t^{4D-1}}{t^{4D}}$$

If $4D - 1 \neq 0$, then $I = \infty$. Therefore, $D = \dfrac{1}{4}\cdot$ Then using L'Hospital Rule, we have

$$I = \lim_{t\to 0}\frac{(1 + 5t + 3t^4)^{1/4} - 1}{t} = \frac{0}{0} = \lim_{t\to 0}\frac{1/4\times(1 + 5t + 3t^4)^{-3/4}\times(5 + 12t^3)}{1} = \frac{5}{4}$$

□

Question 25. Find $\lim_{x\to 0}(1 + 3x^2e^{2x})^{\frac{1}{4x^2}}$

(A) $I = 1$ (B) None of them (C) $I = 0$ (D) $I = e^{\frac{3}{4}}$

Solution. The correct answer (D). Let $y = (1 + 3x^2e^{2x})^{\frac{1}{4x^2}} \Rightarrow \ln y = \dfrac{\ln(1 + 3x^2e^{2x})}{4x^2}\cdot$ Then

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1 + 3x^2e^{2x})}{4x^2} = \lim_{x\to 0}\frac{\ln(1 + 3x^2e^{2x})}{3x^2e^{2x}}\times\lim_{x\to 0}\frac{3x^2e^{2x}}{4x^2} = 1\times\frac{3}{4}$$

Therefore, $\lim_{x\to 0} y = \lim_{x\to 0} e^{\ln y} = e^{\frac{3}{4}}$ □

Question 26. Studying the local extreme values of the function $f(x) = \arctan\left(\dfrac{2x-1}{x^2+2}\right)$, which statement is always true?

(A) f has local minimum when $x = -1$, and no local maximum
(B) f has local minimum when $x = -1$, and local maximum when $x = 2$
(C) None of them
(D) f has no local minimum, and local maximum when $x = 2$

Solution. The correct answer (B). $f(x) = \arctan\left(\dfrac{2x-1}{x^2+2}\right) \Rightarrow f'(x) = \dfrac{\dfrac{-2x^2+2x+4}{(x^2+2)^2}}{1 + \left(\dfrac{2x-1}{x^2+2}\right)^2} = \dfrac{-2x^2+2x+4}{(x^2+2)^2 + (2x-1)^2}$

$\Rightarrow f'(x) = 0 \Leftrightarrow x = -1 \vee x = 2$

x	$-\infty$		-1		2		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$	$-\frac{\pi}{2}$	$\searrow$	$f(-1)$	$\nearrow$	$f(2)$	$\searrow$	$\frac{\pi}{2}$

□

Question 27. If $y = \dfrac{\ln x^2}{x}$, then which statement is always true?

(A) The function y is increasing on $(-e, e)$
(B) The function y is increasing on $\mathbb{R}$
(C) The function y is increasing on $(-e, 0) \cup (0, e)$
(D) None of them

Solution. The correct answer (D). Domain: $\mathbb{D} = \mathbb{R} \backslash \{0\}$. We have

$$y = \frac{\ln x^2}{x} = \frac{2\ln|x|}{x} \Rightarrow y' = \frac{2(1 - \ln|x|)}{x^2}$$

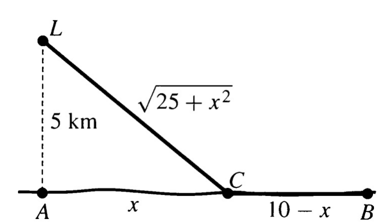
Thus

$$y' = 0 \Leftrightarrow |x| = e \Leftrightarrow x = \pm e.$$

x	$-\infty$		$-e$		0		e		$+\infty$
$f'(x)$		$-$	0	$+$	$\|$	$+$	0	$-$	
$f(x)$	0	$\searrow$	$-\frac{2}{e}$	$\nearrow$	$+\infty$ $\|$ $-\infty$	$\nearrow$	$\frac{2}{e}$	$\searrow$	0

Therefore, y is increasing on $(-e, 0)$ and $(0, e)$. □

Question 28. A lighthouse L is located on a small island 5 km north of a point A on a straight east-west shoreline. A cable is to be laid from L to point B on the shoreline 10 km east of A. The cable will be laid through the water in a straight line from L to a point C on the shoreline between A and B, and from there to B along the shoreline. The part of the cable lying in the water costs 5000 dollars/km, and the part along the shoreline costs 3000 dollars/km. Where should C be chosen to minimize the total cost of the cable?



(A) $x = 5.75\ km$
(B) $x = 3.75\ km$
(C) None of them
(D) $x = 4.75\ km$

Solution. The correct answer (B). The total cost of the cable is

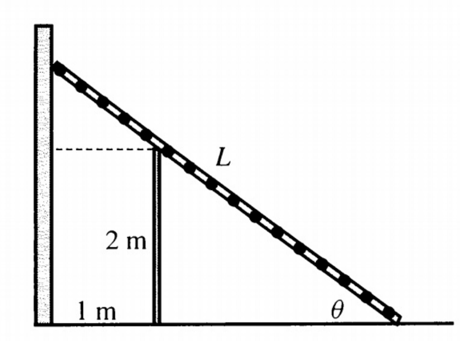
$$C(x) = 5000\sqrt{25 + x^2} + 3000(10 - x), \quad x \in [0, 10].$$

$$\Rightarrow C'(x) = \frac{5000x}{\sqrt{25 + x^2}} - 3000 = 0 \Leftrightarrow 5x = 3\sqrt{x^2 + 25}$$

$$\Leftrightarrow \left[\begin{array}{r} x = \dfrac{15}{4} = 3.75 \quad \text{(accepted)} \Rightarrow C(3.75) = 50000 \\ x = -\dfrac{15}{4} = -3.75 \quad \text{(rejected)} \end{array}\right.$$

Comparing $C(3.75)$ with $C(0) = 55000$, and $C(10) = 55901.69944$, we conclude that the minimum total cost of the cable is 50000 dollars when $x = 3.75$ km. □

Question 29. Find the length of the shortest ladder that can extend from a vertical wall, over a fence 2 m high located 1 m away from the wall, to a point on the ground outside the fence.



Ⓐ 5.1619 m Ⓑ 6.1619 m
Ⓒ 4.1619 m Ⓓ None of them

Solution. The correct answer Ⓒ. The length of the ladder is

$$L(\theta) = \frac{1}{\cos\theta} + \frac{2}{\sin\theta}, \quad \theta \in \left(0, \frac{\pi}{2}\right).$$

$$\Rightarrow L'(\theta) = \frac{\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta} = 0 \Leftrightarrow \frac{\sin^3\theta - 2\cos^3\theta}{\sin^2\theta\cos^2\theta} = 0$$

$$\Leftrightarrow \tan^3\theta = 2 \Leftrightarrow \theta = \arctan\sqrt[3]{2}$$

The length of the shortest ladder is $L(\arctan\sqrt[3]{2}) \approx 4.1619$ □

Question 30. If $y = x(e^{\frac{1}{x}} - 1)$, then the number of asymptotes of y is

Ⓐ 1 Ⓑ 0 Ⓒ 3 Ⓓ 2

Solution. The correct answer Ⓓ. Domain: $\mathbb{D} = \mathbb{R}\backslash\{0\}$

$$\lim_{x\to0^+} y = \lim_{x\to0^+} x(e^{\frac{1}{x}} - 1) = \lim_{x\to0^+} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} = \lim_{t=\frac{1}{x}\to+\infty} \frac{e^t - 1}{t} = \lim_{t\to+\infty} \frac{e^t}{1} = +\infty.$$

Therefore, $x = 0$ is the vertical asymptote from the right.

$$\lim_{x\to0^-} y = \lim_{x\to0^-} x(e^{\frac{1}{x}} - 1) = \lim_{x\to0^-} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} = \lim_{t=\frac{1}{x}\to-\infty} \frac{e^t - 1}{t} = 0.$$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} x(e^{\frac{1}{x}} - 1) = \lim_{x\to\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} = \lim_{t=\frac{1}{x}\to0} \frac{e^t - 1}{t} = 1.$$

Thus, $y = 1$ is the horizontal asymptote. So there is no slant asymptote. □

Question 31. If $y = x^3e^x$, then the number of point(s) of inflection of the graph of y is

Ⓐ 2 Ⓑ 3 Ⓒ 1 Ⓓ 0

Solution. The correct answer Ⓑ. Domain: $\mathbb{D} = \mathbb{R}$

$$y' = 3x^2e^x + x^3e^x \Rightarrow y'' = 6xe^x + 3x^2e^x + 3x^2e^x + x^3e^x = e^x x(x^2 + 6x + 6).$$

$$y'' = 0 \Leftrightarrow x = 0 \vee x = -3 - \sqrt{3} \vee x = -3 + \sqrt{3}.$$

x	$-\infty$		$-3-\sqrt{3}$		$-3-\sqrt{3}$		0		$+\infty$
$f''(x)$		$-$	0	$+$	0	$-$	0	$+$	
$f(x)$		CD	PI	CU	PI	CD	PI	CU	

□

Question 32. A spherical balloon is inflated so that its radius increases from 20 cm to 20.2 cm in 1 min. By approximately how much has its volume increased in that minute?

(A) $220\pi\ cm^3/min$ (B) $420\pi\ cm^3/min$ (C) $320\pi\ cm^3/min$ (D) None of them

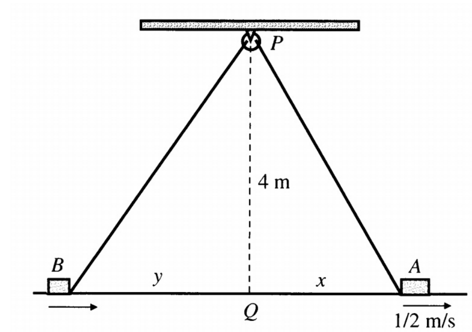
Solution. The correct answer (C). The volume of the spherical balloon is $V = \frac{4}{3}\pi R^3 \Rightarrow \frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt}$

Since $\frac{dR}{dt} = \frac{20.2-20}{1} = 0.2\ cm/min$ and $R = 20$, so

$$\frac{dV}{dt} = 4\pi R^2 \frac{dR}{dt} = 4\pi \times 20^2 \times 0.2 = 320\pi \approx 1005.31\ cm^3/min.$$

□

Question 33. Two crates, A and B, are on the floor of a warehouse. The crates are joined by a rope $15m$ long, each crate being hooked at floor level to an end of the rope. The rope is stretched tight and pulled over a pulley P that is attached to a rafter $4m$ above a point Q on the floor directly between the two crates. If crate A is $3m$ from Q and is being pulled directly away from Q at a rate of $0.5m/s$, how fast is crate B moving toward Q?



(A) $\frac{\sqrt{21}}{14}$ (B) $-\frac{\sqrt{21}}{14}$

(C) None of them (D) $-\frac{\sqrt{23}}{14}$

Solution. The correct answer (B). The Pythagorean theorem can give us

$$AP + BP = 15 \Leftrightarrow \sqrt{x^2+4^2} + \sqrt{y^2+4^2} = 15$$

Differentiating both sides of the equation, we receive

$$\frac{x}{\sqrt{x^2+16}} \cdot \frac{dx}{dt} + \frac{y}{\sqrt{y^2+16}} \cdot \frac{dy}{dt} = 0 \Rightarrow y'(t) = -\frac{x}{\sqrt{x^2+16}} \cdot x'(t) \cdot \frac{\sqrt{y^2+16}}{y}$$

When $x = 3 \Rightarrow y = 2\sqrt{21}$ and $x'(t) = 0.5$, so

$$y'(t) = -\frac{3}{\sqrt{3^2+16}} \times 0.5 \times \frac{\sqrt{84+16}}{2\sqrt{21}} = -\frac{\sqrt{21}}{14} \approx -0.3273\ m/s$$

□

Question 34. Recall that volume of a right cylinder is $V = \pi.r^2.h$, where r is the radius of the base and h is the height. Given a right cylinder with the radius of the base of 10 cm and the height of 10 cm. The radius of the base is increasing at a rate of 5 cm/s, and its height is increasing at a rate of 4 cm/s. How fast is the volume of cylinder increasing?

(A) $1800\pi\ cm^3/s$ (B) None of them (C) $1600\pi\ cm^3/s$ (D) $1400\pi\ cm^3/s$

Solution. The correct answer (D). The formula for the volume of a right cylinder is

$$V = \pi.r^2.h, \qquad (1)$$

where r is the radius of the base and h is the height.

In this problem, V, r and h are functions of the time t in seconds. Taking the derivative of both sides of equation (1) with respect to time yields

$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + h.2r\frac{dr}{dt}\right) \tag{2}$$

Since the radius of the base is increasing at a rate of 5 cm/s and the height is increasing at a rate of 4 cm/s,

$$\frac{dr}{dt} = 5, \qquad \frac{dh}{dt} = 4.$$

Substituting these, as well as $r = 10$ and $h = 10$, into equations (2) yields

$$\frac{dV}{dt} = \pi\left[10^2 \times 4 + 10 \times 2 \times 10 \times 5\right] = 1400\pi \approx 4396$$

Because the sign of $\frac{dV}{dt}$ is positive, the volume of the right cylinder is increasing at a rate of 4396 cm^3/s. □

Question 35. Boyle's law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy equation $PV = C = const$. Suppose that at a certain instant the volume is $600cm^3$, the pressure is $150kPA$, and the pressure is increasing at a rate of $20kPA/min$. At what rate is the volume decreasing at this instant?

(A) None of them (B) $8 \times 10^{-5}\ m^3/min$ (C) $4 \times 10^{-5}\ m^3/min$ (D) $6 \times 10^{-5}\ m^3/min$

Solution. The correct answer (B). Since the pressure P and volume V satisfy equation

$$PV = C = const, \tag{3}$$

where P and V are functions of the time t in minutes, so we can differentiate both sides of equation (3) with respect to time and receive

$$\frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = 0. \tag{4}$$

Since the pressure P is increasing at a rate of 20 kPA/min, so

$$\frac{dP}{dt} = 20$$

Substituting these, as well as $V = 600cm^3 = 600 \times 10^{-6}m^3$ and $P = 150kPA$, into equations (4) yields

$$20 \times 600 \times 10^{-6} + 150 \cdot \frac{dV}{dt} = 0$$

$$\Rightarrow \frac{dV}{dt} = -\frac{20 \times 600 \times 10^{-6}}{150} = -\frac{1}{12500} = -8 \times 10^{-5}.$$

Because the sign of $\frac{dV}{dt}$ is negative, the volume is decreasing at a rate of $8 \times 10^{-5}\ m^3/min$. □

Question 36. If a truck factory employs x workers and has daily operating expenses of y dollars, it can produce $P = \frac{1}{3}x^{0.6}y^{0.4}$ trucks per year. How fast are the daily expenses decreasing when they are 10000 dollars and the number of workers is 40, if the number of workers is increasing at 1 per day and production is remaining constant?

(A) 275 dollars (B) 475 dollars (C) 375 dollars (D) None of them

Solution. The correct answer (C). Differentiating both sides of the following equation with respect to time, we get

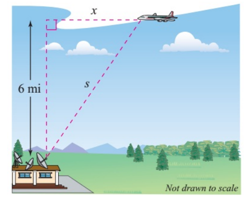
$$P = \frac{1}{3}x^{0.6}y^{0.4} = Const \Rightarrow 0.6 \times x^{-0.4}y^{0.4}\frac{dx}{dt} + 0.4 \times x^{0.6}y^{-0.6}\frac{dy}{dt} = 0$$

In this constant, $\frac{dx}{dt} = 1; x = 40; y = 10000$. Therefore,

$$\frac{dy}{dt} = -\frac{0.6 \times 40^{-0.4} \times 10000^{0.4} \times 1}{0.4 \times 40^{0.6} \times 10000^{-0.6}} = -375.$$

The daily expenses are decreasing at 375 dollars. □

Question 37. An airplane is flying on a flight path that will take it directly over a radar tracking station, as shown in the figure below. If s is decreasing at a rate of 400 miles per hour when $s = 10$ miles, what is the speed of the plane?

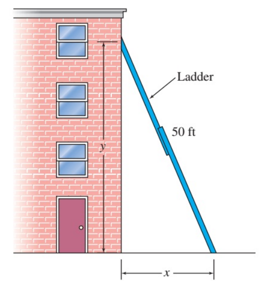


An airplane is flying at an altitude of 6 miles, s miles from the station.

(A) 400 miles per hour
(B) None of them
(C) 500 miles per hour
(D) 600 miles per hour

Solution. The correct answer (C). We have $x^2 + 6^2 = s^2 \Rightarrow \frac{dx}{dt} = \frac{s}{x} \cdot \frac{ds}{dt} = \frac{10}{8} \times (-400) = -500$ □

Question 38. A 50−ft ladder is placed against a large building. The base of the ladder is resting on an oil spill, and it slips to the right at the rate of 3 ft per minute. Find the rate of change of the height of the top of the ladder above the ground at the instant when the base of the ladder is 30 ft from the base of the building.



(A) None of them
(B) $-\frac{5}{4}$ ft per minute
(C) $-\frac{3}{4}$ ft per minute
(D) $-\frac{9}{4}$ ft per minute

Solution. The correct answer (D). We have $x^2 + y^2 = 50^2 \Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{9}{4}$ □

Question 39. When air expands adiabatically (that is, with no change in heat), the pressure P and the volume V satisfy the relationship $PV^{1.4} = C$, where C is a constant. At a certain instant, the pressure is $20 lb/inch^2$ and the volume is $280 inch^3$. If the volume is decreasing at the rate of $5 inch^3/s$ at this instant, what is the rate of change of the pressure?

(A) $0.5\ lb/inch^2/s$
(B) None of them
(C) $0.4\ lb/inch^2/s$
(D) $0.6\ lb/inch^2/s$

Solution. The correct answer (A). Differentiating both sides with respect to t, we receive

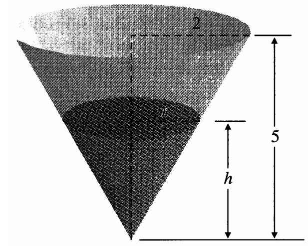
$$1.4PV^{0.4}\frac{dV}{dt} + V^{1.4}\frac{dP}{dt} = 0.$$

The specific situation: At the instant in question, we have $P = 20$, $V = 280$, and $\frac{dV}{dt} = -5$ (negative because the volume is decreasing). The goal is to find $\frac{dP}{dt}$

$$\frac{dP}{dt} = \frac{5 \times 1.4 \times 20 \times 280^{0.4}}{280^{1.4}} = 0.5$$

□

Question 40. A leaky water tank is in the shape of an inverted right circular cone with depth 5 m and top radius 2 m. When the water in the tank is 4 m deep, it is leaking out at a rate of $\frac{1}{12}(m^3/min)$. How fast is the water level in the tank dropping at that time?



(A) $-\frac{25}{768\pi}$ (m/min) (B) $-\frac{23}{768\pi}$ (m/min)

(C) $-\frac{7}{256\pi}$ (m/min) (D) None of them

Solution. The correct answer (A). Let r and h denote the surface radius and depth of water in the tank at time t (both measured in metres). Thus, the volume V (in m^3) of water in the tank at time t is

$$V = \frac{1}{3}\pi r^2 h.$$

Using similar triangles, we can find a relationship between r and h:

$$\frac{r}{h} = \frac{2}{5} \Rightarrow r = \frac{2h}{5}; \quad V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 h = \frac{4\pi h^3}{75}$$

Differentiating this equation with respect to t we obtain

$$\frac{dV}{dt} = \frac{4\pi}{25} \times h^2 \times \frac{dh}{dt}$$

Since $\frac{dV}{dt} = -\frac{1}{12}$ when $h = 4$, we have

$$-\frac{1}{12} = \frac{4\pi}{25} \times 4^2 \times \frac{dh}{dt} \Rightarrow \frac{dh}{dt} = -\frac{25}{768\pi} \approx 0.0104(m/min)$$

When the water in the tank is 4 m deep, its level is dropping at a rate of $0.0104(m/min)$. □

Question 41. A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius r of the outer ripple is increasing at a constant rate of 1 foot per second. When the radius is 4 feet, at what rate is the total area A of the disturbed water changing?

(A) $6\pi\ (ft^2/s)$ (B) $10\pi\ (ft^2/s)$ (C) None of them (D) $8\pi\ (ft^2/s)$

Solution. The correct answer (D). The variables r and A are related by $A = \pi r^2$. The rate of change of the radius r is $\frac{dr}{dt} = 1$.

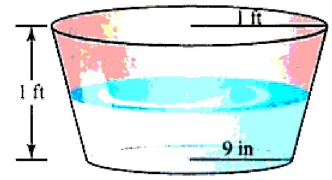
$$\frac{dA}{dt} = \frac{d(\pi r^2)}{dt} = 2\pi r\frac{dr}{dt}$$

Substituting 4 for r and 1 for $\frac{dr}{dt}$, we obtain

$$\frac{dA}{dt} = 2\pi \times 4 \times 1 = 8\pi.$$

When radius is 4 feet, the area is changing at a rate of 8π square feet per second. □

Question 42. Suppose a water bucket is modeled by the frustum of a cone with height $1ft$ and upper and lower radii of $1ft$ and $9in.$, respectively. Note that $1ft = 12in$. If water is leaking from the bottom of the bucket at the rate of $8in.^3/min$, at what rate is the water level falling when the depth of water in the bucket is $6in.$?



(A) $-\dfrac{11}{147\pi}\,(in./min)$ (B) $-\dfrac{32}{441\pi}\,(in./min)$ (C) $-\dfrac{35}{441\pi}\,(in./min)$ (D) None of them

Solution. The correct answer (B). The volume of water in the bucket

$$V = \frac{1}{3}\pi h(r^2 + rR + R^2)$$

where $r = 9, 9 \leqslant R \leqslant 12; 0 \leqslant h \leqslant 12; R = \dfrac{12h}{12} + \dfrac{9}{12}(12-h) = 9 + \dfrac{h}{4}$. Therefore,

$$V = \frac{1}{3}\pi h\left[81 + 9\left(9 + \frac{h}{4}\right) + \left(9 + \frac{h}{4}\right)^2\right]$$

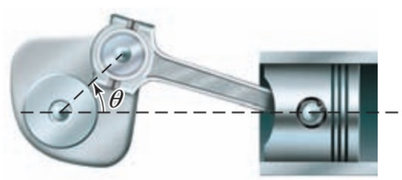
$$\Rightarrow \frac{dV}{dt} = \frac{1}{3}\pi\left[81 + 81 + \frac{9h}{2} + 81 + 9h + \frac{3h^2}{16}\right]\frac{dh}{dt}$$

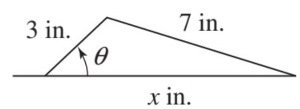
With $h = 6$, $\dfrac{dV}{dt} = -8$, we have

$$\Rightarrow \frac{dh}{dt} = -\frac{3\times 8}{\pi\left(243 + \dfrac{27}{2}\times 6 + \dfrac{3\times 6^2}{16}\right)} = -\frac{32}{441\pi} \approx -0.0231(in./min)$$

□

Question 43. A piston is attached to a crankshaft of radius 3 in. by means of a 7 in. connecting rod. Let x denote the position of the piston. If the crankshaft rotates counterclockwise at a constant rate of 60 revolution/second, what is the velocity of the piston $\left(\dfrac{dx}{dt}\right)$ when $\theta = \dfrac{\pi}{3}$?





(A) $-\dfrac{2882\sqrt{3}\times\pi}{13}$ (B) $-\dfrac{2881\sqrt{3}\times\pi}{13}$ (C) $-\dfrac{2880\sqrt{3}\times\pi}{13}$ (D) None of them

Solution. The correct answer (C). Using the law of cosines we obtained

$$7^2 = 3^2 + x^2 - 2\times 3\times x\times\cos\theta \Leftrightarrow x^2 - 6x\cos\theta = 40$$

Differentiating both sides of the equation we have

$$2x\frac{dx}{dt} - 6\frac{dx}{dt}\cos\theta + 6x\sin\theta\frac{d\theta}{dt} = 0 \Leftrightarrow \frac{dx}{dt}(2x - 6\cos\theta) = -6x\sin\theta\frac{d\theta}{dt}$$

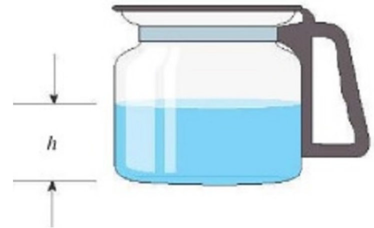
When $\theta = \dfrac{\pi}{3}$ we have $x^2 - 6x\times\dfrac{1}{2} = 40 \Leftrightarrow \left[\begin{array}{l} x = 8(\text{accepted}) \\ x = -5(\text{rejected})\end{array}\right.$

Substituting $\dfrac{\pi}{3}$ for θ and $60\times 2\pi$ for $\dfrac{d\theta}{dt}$, and 8 for x, we obtain

$$\frac{dx}{dt} = \frac{-6\times 8\times\sqrt{3}/2\times 60\times 2\pi}{2\times 8 - 6\times 1/2} = -\frac{1440\sqrt{3}\times 2\pi}{13} \approx -1205.4790$$

□

Question 44. A coffee pot in the form of a circular cylinder of radius 4 in. is being filled with water flowing at a constant rate. If the water level is rising at the rate of $0.4 in./sec$, what is the rate at which water is flowing into the coffee pot?



(A) None of them
(B) $\frac{33\pi}{5}$ (in./sec)
(C) $\frac{31\pi}{5}$ (in./sec)
(D) $\frac{32\pi}{5}$ (in./sec)

Solution. The correct answer (D). Let $h(in)$ and $V(in^3)$ denote the water level and the volume of water in the tank at time t. A coffee pot is in the form of a circular cylinder of constant radius $4in$. Thus, the volume V of water in the tank at time t is

$$V = \pi R^2 h = 16\pi h.$$

Differentiating this equation with respect to t we obtain

$$\frac{dV}{dt} = 16\pi \times \frac{dh}{dt}$$

Since $\frac{dh}{dt} = 0.4$, we have

$$\frac{dV}{dt} = 16\pi \times 0.4 = \frac{32\pi}{5} \approx 20.11(in./sec)$$

Therefore, the water is flowing into the coffee pot at the rate of $\frac{32\pi}{5} \approx 20.11(in./sec)$ □

Question 45. A study prepared for National Association of Realtors estimates that the number of housing starts in the southwest, $N(t)$ (in units of a million), over the next 5 years is related to the mortgage rate $r(t)$ (percent per year) by the equation $9N^2 + r = 36$, where t is measured in years. What is the rate of change of the number of housing starts with respect to time when the mortgage rate is 11% per year and is increasing at the rate of 1.5% per year?

(A) -0.06 (B) None of them (C) -0.04 (D) -0.05

Solution. The correct answer (D). Differentiating both sides of the equation $9N^2 + r = 36$ with respect to time, we receive

$$18N \times \frac{dN}{dt} + \frac{dr}{dt} = 0$$

where $r = 11 \Rightarrow N = \frac{\sqrt{36 - r}}{3} = \frac{5}{3}; \frac{dr}{dt} = 1.5$. Therefore,

$$18 \times \frac{5}{3} \times \frac{dN}{dt} + 1.5 = 0 \Rightarrow \frac{dN}{dt} = -\frac{1.5}{30} = -0.05$$

□

Question 46. Texar Inc., a manufacturer of disk drives is willing to make x thousand 1GB USB flash drives available in the marketplace each week when the wholesale price is p dollars per drive. It is known that the relationship between x and p is governed by the supply equation $x^2 - 3xp + p^2 = 5$. How fast is the supply of drives changing when the price per drive is 11 dollars, the quantity supplied is 4000 drives, and the wholesale price per drive is increasing at the rate of 0.10 dollars per drive each week?

(A) None of them (B) 0.06 (C) 0.05 (D) 0.04

Solution. The correct answer (D). Differentiating the equation $x^2 - 3xp + p^2 = 5$ with respect to t, we obtain

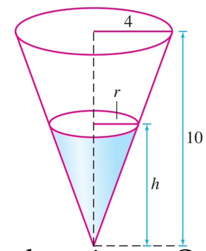
$$2x \times \frac{dx}{dt} - 3\left(x \times \frac{dp}{dt} + p \times \frac{dx}{dt}\right) + 2p \times \frac{dp}{dt} = 0$$

Since $x = 4; p = 11; \dfrac{dp}{dt} = 0.1$, we have

$$2 \times 4 \times \frac{dx}{dt} - 3\left(4 \times 0.1 + 11 \times \frac{dx}{dt}\right) + 2 \times 11 \times 0.1 = 0 \Rightarrow \frac{dx}{dt} = 0.04$$

□

Question 47. Water pours into a conical tank of height $10m$ and radius $4m$ at a rate of $6m^3/min$. At what rate is the water level rising when the level is $5m$ high?



(A) $\dfrac{3}{2\pi}$ (m/min) (B) None of them (C) $\dfrac{1}{2\pi}$ (m/min) (D) $\dfrac{5}{2\pi}$ (m/min)

Solution. The correct answer (A). Let V and h be the volume and height of the water in the tank at time t. Our problem, in terms of derivatives, is:

Compute $\dfrac{dh}{dt}$ at $h = 5$ given that $\dfrac{dV}{dt} = 6m^3/min$.

When the water level is h, the volume of water in the cone is $V = \dfrac{1}{3}\pi h r^2$, where r is the radius of the cone at height h, but we cannot use this relation unless we eliminate the variable r. Using similar triangles, we see that

$$\frac{r}{h} = \frac{4}{10} \Rightarrow r = 0.4h \Rightarrow V = \frac{1}{3}\pi h(0.4h)^2 = \frac{0.16}{3}\pi h^3 \Rightarrow \frac{dV}{dt} = 0.16\pi h^2 \frac{dh}{dt} = 6 \Rightarrow \frac{dh}{dt} = \frac{6}{0.16\pi h^2}$$

When $h = 5$, the level is rising at a rate of $\dfrac{dh}{dt} = \dfrac{6}{0.16\pi 5^2} = \dfrac{3}{2\pi} \approx 0.48m/min$. □