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Survey of State-Dependent Riccati Equation in Nonlinear Optimal Feedback Control Synthesis

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I. Introduction

AEROSPACE engineering applications greatly stimulated the development of optimal control theory during the 1950s and 1960s, where the objective was to drive the system states in such a way that some defined cost was minimized. This turned out to have very useful applications in the design of regulators (where some steady state is to be maintained) and in tracking control strategies (where some predetermined state trajectory is to be followed). Among such applications was the problem of optimal flight trajectories for aircraft and space vehicles. Linear optimal control theory in particular has been very well documented and widely applied, where the plant that is controlled is assumed linear and the feedback controller is constrained to be linear with respect to its input. However, the availability of powerful low-cost microprocessors has spurred great advantages in the theory and applications of nonlinear control. The competitive era of rapid technological change, particularly in aerospace exploration, now demands stringent accuracy and cost requirements in nonlinear control systems. This has motivated the rapid development of nonlinear control theory for application to challenging, complex, dynamical real-world problems, particularly those that bear major practical significance in aerospace, marine, and defense industries.

Infinite-time horizon nonlinear optimal control (ITHNOC) presents a viable option for synthesizing stabilizing controllers for nonlinear systems by making a state-input tradeoff, where the objective is to minimize the cost given by a performance index. The original theory of nonlinear optimal control dates from the 1960s. Various theoretical and practical aspects of the problem have been addressed in the literature over the decades since. In particular, the continuous-time nonlinear deterministic optimal control problem associated with autonomous (time-invariant) nonlinear regulator systems that are affine (linear) in the controls has been studied by many authors. The long-established theory of optimal control offers quite mature and well-documented techniques for solving this control-affine nonlinear optimization problem, based on dynamic programming or calculus of variations, but their application is generally a very tedious task. Bellman's dynamic programming approach reduces to solving a nonlinear first-order partial differential equation (PDE), expressed by the Hamilton–Jacobi–Bellman (HJB) equation. The solution to the HJB equation gives the optimal performance/cost value (or storage) function and determines an optimal control in feedback form under some smoothness assumptions. Alternatively, in the classical calculus of variations, optimal control problems can be characterized locally in terms of the Hamiltonian dynamics arising from Pontryagin's minimum principle. These are the characteristic equations of the HJB PDE, which result in a nonlinear, constrained two-point boundary value problem (TPBVP) that, in general, can only be solved by successive approximation of the optimal control input using iterative numerical techniques for each set of initial conditions. Numerically, even though the nonlinear TPBVP is somewhat easier to solve than the HJB PDE, control signals can only be determined offline and are thus best suited for feedforward control of plants for which the state trajectories are known a priori. Therefore, contrary to the dynamic programming approach, the resultant control law is not generally in feedback form. Open-loop control, however, is sensitive to random disturbances and requires that the initial state be on the optimal trajectory. In contrast, nonlinear optimal feedback has inherent robustness properties (inherent in the sense that it is obtained by ignoring uncertainty and disturbances).

The potential difficulty with the HJB approach is that no efficient algorithm is available to solve the PDE when it is nonlinear and the problem dimension is high, making it impossible to derive exact expressions for optimal controls for most nontrivial problems of interest. The optimal can only be computed in special cases, such as linear dynamics and quadratic cost, or very low-dimensional systems. In particular, if the plant is linear time invariant (LTI) and the (infinite-time) performance index is quadratic, then the corresponding HJB equation for this infamous linear-quadratic regulator (LQR) problem reduces to an algebraic Riccati equation (ARE). Contrary to the well-developed and widely applied theory and computational tools for the Riccati equation (for example, see [1]), the HJB equation is difficult, if not impossible, to solve for most practical applications. The exact solution for the optimal control policies is very complex

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and almost impossible to implement. As a consequence, the practicing engineer often seeks a control law that is close to optimal with respect to the particular performance index and has attractive features such as feedback, small computations, etc. For reasons of practical implementation, designers of closed-loop control algorithms have been concerned with approximately optimal (suboptimal) control algorithms: algorithms that use some of the apparatus of optimal control theory but sacrifice some performance by introducing approximations that facilitate ease of implementation. Such suboptimal control laws are considered a tradeoff between achieving true optimality, which is expensive and complicated to implement, and achieving a system performance that is not optimal but acceptable and inexpensive with ease to implement.

The fundamental goal in the research of (sub)optimal control synthesis methods for nonlinear systems is to provide control practitioners with a unified approach that avoids directly solving nonlinear TPBVPs or HJB PDEs and yields attractive stability, optimality, robustness, and computational properties so as to make real-time implementation in feedback form feasible. The method described next provides such an approach, and it is the topic of this Survey Paper.

State-dependent Riccati equation (SDRE) control synthesis is a highly promising and very attractive practical tool for obtaining approximate solutions to ITHNOC problems in feedback form. The SDRE method provides an appealing alternative to solving the HJB PDE, allowing for the systematic and effective design of nonlinear feedback controllers for a broad class of nonlinear systems under very mild conditions. This method also possesses many of the capabilities of other nonlinear design methods, at least collectively, addressing all the aforementioned features of stability, optimality, real-time implementability, and inherent robustness with respect to parametric uncertainties and unmodeled dynamics, as well as disturbance rejection.

The goal of this Survey Paper is to reflect the rapid growth and strong interest in the field of SDRE paradigm by providing control theoreticians and practitioners with a good balance between the theoretical developments, systematic design tools, and real-time implementation prospects of SDRE control methodology. Successful efforts on the application of various SDRE design strategies to a variety of platforms across all industries are also reviewed. However, the abundant number and broad spectrum of successful simulation, experimental, and practical real-world applications of the various SDRE design strategies have outpaced the available theoretical results scattered in the literature. Therefore, the theoretical developments on SDRE nonlinear regulation for solving continuous-time ITHNOC problems are the main contribution of the Survey Paper, as are the clarification of some common misconceptions or myths revolving around this method and discussion of issues that are still open for investigation. The existence of solutions as well as optimality and stability properties associated with SDRE controllers are examined in detail in the hope of providing a sound theoretical basis for SDRE control of nonlinear systems. For traceability of results, proofs of all theorems concerning SDRE nonlinear regulator theory are reported in the Survey Paper. However, in cases where space does not permit, only a sketch of the proof is presented, providing readers with appropriate references, should additional details be required. The application and validity of the developed theories are also illustrated on several pedagogical examples throughout the Survey Paper to fill the gap between theory and practice. While these examples are just mathematical, the difficulties they present for nonlinear design frequently arise in more physically motivated examples. The theories developed are then used in addressing practical aspects of SDRE design, providing insights into design conditions and practical rules toward actual implementation. A summary of the main results and recommendations for further research are also provided before concluding the Survey Paper.

II. State-Dependent Riccati Equation Paradigm

In 1962, Pearson [2] proposed a time- and state-dependent formulation of nonlinear and nonautonomous (time-varying) systems associated with finite-time horizon nonlinear optimal control problems, and he suggested treating it as an instantaneously LTI system to approximate the nonlinear optimal control problem at each instant of time. The resultant problem was then solved using the LQR technique pointwise in state space. The infinite-time problem has also been considered in this framework as the steady-state solution of the state-dependent matrix Riccati differential equation. This approach, with its foundations now dating back to half a century ago, is the celebrated SDRE paradigm, which gained immense popularity, particularly in the aerospace community, as a result of ground-breaking contributions in the field by Cloutier et al. [3–5], Mracek and Cloutier [6], and Cloutier and Stansbery [7] over the past two decades.

The SDRE method entails factorization (that is, parameterization) of the nonlinear dynamics into the product of a matrix-valued function (which depends on the state) and the state vector. In so doing, the SDRE algorithm fully captures the nonlinearities of the system, bringing the nonlinear system to a nonunique linear structure having state-dependent coefficient (SDC) matrices and minimizing a nonquadratic performance index (in the state) having a quadraticlike structure. The nonuniqueness of the parameterization creates extra degrees of freedom, which can be used to enhance controller performance. An ARE using the SDC matrices is then solved online to give the suboptimum (and in some cases optimum) control law. The coefficients of this ARE vary with the given point in state space. The algorithm thus involves solving, at a given point in state space, an algebraic SDRE, for which the pointwise stabilizing solution during state evolution yields the SDRE nonlinear feedback control law. As the SDRE depends only on the current state, the computation can be carried out either online, in which case the SDRE is defined along the state trajectory, or offline if an algebraic solution is available. Obviously, the advantage of this method is that the characterization of the resulting feedback controller has a similar structure to the LQR problem, and it is obtained by solving the corresponding SDRE instead of the HJB equation. Since the computational complexity of an ARE is only of polynomial growth rate with the state dimension, the SDRE algorithm provides the possibility of dealing with high-dimensional nonlinear systems. This is clearly desirable and makes implementation in real time in closed-loop form using state feedback feasible [8].

The rest of this section provides historical reviews on the developments of the SDRE paradigm. Theoretical developments of SDRE nonlinear regulators are presented in Sec. II.A. Methodical and practical impacts of the SDRE paradigm on control science and engineering are then discussed in Secs. II.B and II.C, with respective reviews of the various SDRE design methodologies developed to date and successful applications of these methods to a variety of platforms across all industries.

A. Theoretical Developments

Optimality, suboptimality, and stability properties of SDRE nonlinear regulators for ITHNOC problems have been studied by several authors over the past decades. Wernli and Cook [9] extended the applicability of the control law proposed by Pearson [2] to a more general class of nonlinear systems that may be time-varying nonlinear in the state and nonaffine in the control. Furthermore, asymptotic stability of the control law was shown to hold within a sufficiently small region around the origin under mild restrictions; however, conditions for large-scale asymptotic stability could not be established. Ehrler and Vadali [10] analyzed the mathematical structure of SDRE-controlled systems for autonomous control-affine nonlinear regulator problems with quadratic performance criteria. They showed that solving an ARE as it evolved over time provided one means of obtaining a suboptimal solution of the ITHNOC problem. In essence, the SDRE was treated as being time dependent and its state dependency was not explicitly acknowledged, addressed, or analyzed. Banks and Mhana [11] applied the feedback from a differential SDRE, which is shown to be stable under certain bounds on the derivatives of the SDC matrix functions defining the nonlinear dynamics, and then studied the feedback obtained from an algebraic SDRE, but in this latter case, they mistakenly claimed that the application of the SDRE feedback control to the nonlinear regulation problem would result in a solution that is both optimally and globally stabilizing. Unfortunately, simple counterexamples in the literature are available that contradict these claims (see [12,13]). Cloutier et al. [3] and Mracek and Cloutier [6] later precisely showed that the SDRE feedback scheme for the autonomous ITHNOC problem (with control terms that appear affine in the dynamics and quadratic in the cost) in the multivariable case is both large-scale asymptotically stabilizing and large-scale asymptotically optimal, and in the scalar case is globally asymptotically stabilizing and globally optimal. Furthermore, in the general multivariable case, the SDRE feedback is shown to possess the desirable suboptimality property that Pontryagin's necessary condition for optimality is always satisfied, while the costate condition is satisfied asymptotically at a quadratic rate by the algorithm under asymptotic stability as the states are driven to the origin. Additional results were also derived in [3,14–17] to determine the instances when global asymptotic stability can be concluded for multivariable problems.

B. Methodical Impact

Inspired by the SDRE framework, several methods that are characterized by solving SDREs have been suggested in the literature [5]. First and foremost, the SDRE method can be readily extended to the state-feedback nonlinear H_∞-control problem as posed by van der Schaft [18,19]. This has been studied by Cloutier et al. [3], where a general autonomous nonlinear minimum-energy (nonlinear H_∞) problem has been considered, and solved using the SDRE method. The SDRE H_2 method has also been briefly identified in this work. In the case of partial-state information, the proposed SDRE nonlinear H_∞ and SDRE nonlinear H_2 design methodologies involve the solution of two coupled SDREs at each point along the trajectory. Internal stability properties of the SDRE nonlinear H_∞ controller have also been addressed in [3] for the case of full-state information. Under mild conditions of stabilizability and detectability, it is shown that the SDRE nonlinear H_∞ controller is internally locally asymptotically stable. The method is therefore readily applicable to full-state-feedback nonlinear H_∞ suboptimal control and is viable for output-feedback nonlinear H_2 and H_∞ controls. Aside from these continuous-time methods, discrete-time SDRE nonlinear regulators have also been investigated in [20–22].

Following the duality between linear-quadratic (LQ) optimal regulation and LQ Gaussian estimation, derivative-free SDRE nonlinear observers and filters have naturally been suggested in the literature for both continuous-time and discrete-time nonlinear systems [23–26]. Theoretical considerations in [27,28] have shown that the asymptotic behavior of the SDRE observer/filter yields much improved performance and convergence properties compared with local approximations, such as the usual linearized and extended Kalman observers/filters, and it is capable of tracking the true values of the states with little sensitivity to the selection of the statistics or even to severe differences in the initial state estimates, unlike local methods. Theoretical investigation of the continuous-time robust H_∞ SDRE nonlinear filter has also been pursued in [29].

All the aforementioned SDRE techniques are defined by their linearlike structures, which have SDC matrices. The SDRE nonlinear regulator has the same structure as the infinite-time horizon LQR. SDRE H_2 and SDRE H_∞ controls have the same structures as linear H_2 and linear H_∞ controls, respectively. The continuous-time SDRE nonlinear observer/filter has the same structure as the continuous-time steady-state linear Kalman observer/filter. These SDRE methods require the solution of an algebraic SDRE, or two in the case of partial-state information SDRE H_2 and SDRE H_∞ controls. The differential SDRE method [24,30] has the same structure as the finite-time horizon LQR, and the differential SDRE observer/filter [25] has the same structure as the linear Kalman observer/filter. These latter techniques require the integration of a differential (as opposed to an algebraic) SDRE. On the other hand, discrete-time counterparts of SDRE regulators, observers, and filters all require the solution of difference equations.

Apart from these SDRE design methods, other hybrid approaches, combining several nonlinear design methods with the SDRE approach, have also been proposed in the literature in order to overcome some of the shortcomings of either existing methodologies or the SDRE method, or both. For example, a suboptimal regulator for nonlinear input-affine systems is proposed in [31] based upon the combination of receding-horizon and SDRE techniques. The proposed controller is shown to be almost globally optimal, provided that a good approximation to the storage function in the original HJB equation is known in a neighborhood of the origin. The controller is capable of globally asymptotically stabilizing the plant, provided that enough computational power is available to solve a finite-time horizon optimization problem online. Since solving the HJB equation is very difficult, the authors introduce a finite-time approximation to this infinite-time horizon optimization problem. Alternatively, combining the SDRE method with backstepping would allow the designer to design adaptive and robust controls for cascaded nonlinear systems with guaranteed stability and performance so that fictitious controls are made to be suboptimal or even optimal. Qu and Cloutier [32] proposed such a suboptimal control design technique based on a forward recursive design rather than a backstepping design, which eliminates differentiation of fictitious controls or their functions, thus making the design much simpler in applications. Optimality is then achieved for the individual subsystems, whereas suboptimality, semiglobal stability, and tracking performance are established for the overall closed-loop system. Additionally, Curtis and Beard [33] combined the SDRE approach with satisficing, and the stability of the system is shown by a control Lyapunov function. The SDRE technique has also been used in combination with neural networks [20], dynamic inversion [34], variable structure control with time-varying sliding sector [35], and sliding mode control design with moving and adaptive sliding surfaces [36,37].

C. Practical Impact

The theoretical contributions on SDRE control for autonomous ITHNOC problems have initiated an increasing use of SDRE and SDRE-based design methodologies in a wide variety of nonlinear applications in very diverse fields of study in control science and engineering, demonstrating the feasibility of applying the technique to realistic problems. These include (and are by no means limited to the references provided herein) the following.

1. Missiles

Practical contributions associated with missile applications are as follows:

a) The first contribution is the full-envelope tail-controlled missile longitudinal (pitch) autopilot design using output-feedback nonlinear H_2 control, which is robust to large parameter variations while tracking highly dynamic maneuvers [38,39].

b) The second contribution is autopilot design for agile missiles using full-state information nonlinear H_∞ control [40].

c) The third contribution is full-envelope hybrid bank-to-turn/skid-to-turn autopilot design for a generic airbreathing air-to-air missile [41] with dynamic conversion of flight-path angle commands to body attitude (angle of attack and bank angle) commands for real-time trajectory optimization using integral servomechanism tracking control [42].

d) The fourth contribution is autopilot design for a dual-controlled (tail and canard) missile using state-dependent weightings [43].

e) The fifth contribution is the development of a generic approach for inherently robust high-performance full-envelope three-axis nonlinear missile autopilot design, independent of any flight or trim conditions [44].

f) The sixth contribution is guidance law development for acceleration-limited [45] and impact-angle constrained [46] trajectories.

g) The seventh contribution is fully integrated guidance and control design of agile missiles to significantly improve the mean and standard deviation of the final miss distance against stressing threats using both algebraic SDRE [47] and differential SDRE [30] schemes.

h) The eighth contribution is guidance filter design for an exoatmospheric, terminal homing intercontinental-ballistic-missile intercept problem using passive and active sensor information of azimuth, elevation, and range [48].

2. *Aircraft*

Practical contributions associated with aircraft applications are as follows:

a) The first contribution is the control of a fighter aircraft during a high angle of attack to extend its range of recovery from stall [11].

b) The second contribution is flutter suppression of aeroelastic wing sections using full-state and/or partial-state feedback [49–51].

c) The third contribution is enhancement of existing aircraft flight control system performance through an adaptive control scheme by augmenting it with a SDRE controller implemented in a parallel fashion [52].

3. *Unmanned Aerial Vehicles*

Practical contributions associated with unmanned aerial vehicle (UAV) applications are as follows:

a) The first contribution is trajectory-tracking control and experimental testing of a small helicopter and a micro-ducted-fan rotorcraft [53].

b) The second contribution is attitude and velocity control of a quadrotor UAV for near-area surveillance and search-and-rescue missions [54].

c) The third contribution is control and real-time flight testing of small autonomous helicopters for autonomous operations through a broad spectrum of maneuvers, which include aggressive trajectories and model-parameter mismatch [55].

d) The fourth contribution is inertial navigation system/Global Positioning System sensor fusion based on SDRE nonlinear filtering for the UAV localization problem [56].

4. *Satellites and Spacecraft*

Practical contributions associated with satellite and spacecraft applications are as follows:

a) The first contribution is position and/or large-angle attitude control of rigid spacecraft [57–59].

b) The second contribution is hypersonic guidance of space vehicles during the midcourse phase of flight [60].

c) The third contribution is angular-rate estimation of satellites using either vector measurements obtained from sensors, such as magnetometers and sun or horizon sensors [61], or delayed quaternion measurements obtained from a cluster of star trackers [62] with application to real data obtained from the Rossi X-Ray Timing Explorer satellite.

d) The fourth contribution is formation flying control of multiple satellites/spacecraft [63].

5. *Ships*

Practical contributions associated with ship applications are as follows:

a) The first contribution is nonlinear observer design for dynamic positioning of ships to obtain accurate estimates of the low-frequency ship motions for use in stationkeeping so as to prevent unnecessary actuator usage [64].

b) The second contribution is track-keeping autopilot design for real-time waypoint guidance of constrained nonlinear oil tanker motion in restricted waterways [65].

6. *Autonomous Underwater Vehicles*

Practical contributions associated with autonomous underwater vehicle applications are as follows:

a) The first contributions are fault-tolerant and robust steering motion and dive-plane control [66].

b) The second contribution is integrated design of a real-time obstacle avoidance system [67].

c) The third contribution is optimal and robust motion control for homing and docking tasks [68].

7. *Automotive Systems*

Practical contributions associated with automotive applications are as follows:

a) The first contribution is stabilization of the lateral motion dynamics of a vehicle model [69,70].

b) The second contribution is obstacle-avoiding feedforward-feedback path-tracking steering control for a vehicle using sensitivity-based gain parameterization of controller gains [71].

8. *Biological and Biomedical Systems*

Practical contributions associated with biological and biomedical applications are as follows:

a) The first contribution is control of the artificial human pancreas [72].

b) The second contribution is human immunodeficiency virus feedback control for combined drug/immune response using full-state feedback as well as partial-state measurements [73].

c) The third contribution is optimal administration of chemotherapy in cancer treatment [74], with combined control and state constraints [8] due to chemotherapy drug dose level and threshold for immune cell level, in order to keep patients healthy.

9. *Process Control*

Practical contributions associated with process control applications are as follows:

a) The first contribution is temperature estimation in rapid thermal processing systems for performing thermal manufacturing operations involved in integrated circuit fabrication, such as annealing, formation of thin dielectric films, and chemical vapor deposition [75].

b) The second contribution is speed sensorless observer design for current and voltage source-inverter-fed induction motor drives [76,77] for separate state and parameter estimations of rotor flux vector and load torque parameters.

c) The third contribution is control of the nonlinear nonminimum-phase dynamics of a continuously stirred tank reactor [78].

d) The fourth contribution is regulation of the growth of thin films in a high-pressure chemical vapor deposition thin-film reactor for real-time applications in the microelectronic industry [79].

e) The fifth contribution is control of the tandem cold rolling process of the metal strip with improved tolerance in mill exit thickness [80].

10. *Robotics*

Practical contributions associated with robotic applications are as follows:

a) The first contribution is manipulator path control of a vertically articulated five-axis robot with experimental validation [81].

b) The second contribution is real-time experimental control of two-link underactuated nonlinear nonminimum-phase robot dynamics [82].

c) The third contribution is experimental control of a planar three-link manipulator with two flexible links [83].

d) The fourth contribution is robust full-state-feedback nonlinear H_∞ control of the tip position of a single-link flexible manipulator [84].

e) The fifth contribution is locomotion control with experimental validation of a three-link snakelike robot based on friction force, under holonomic and nonholonomic constraints [85].

11. *Various Benchmark Problems*

Practical contributions associated with various benchmark problems are as follows:

a) The first contribution is control of a translational oscillator with an eccentric rotational proof-mass actuator in the presence of parametric variations, sinusoidal disturbances, and actuator saturation [6].

b) The second contribution is control of the toy nonlinear optimal control problem [86] that was posed as a counterexample to the SDRE design method during the Nonlinear Control Workshop conducted at the 1997 American Control Conference [87].

c) The third contribution is ducted-fan control [31].

d) The fourth contribution is experimental control of the unstable, nonlinear, electromechanical magnetic levitation system [17].

e) The fifth contribution is real-time experimental and/or simulation swingup/stabilization control of the underactuated linear or rotary single-inverted pendulum system using either a switching control or a single controller for both modes [8,35,88–93].

12. *Other Nonlinear Studies*

Practical contributions associated with other nonlinear studies are as follows:

a) The first contribution is adaptive SDRE control design [52,94,95].

b) The second contribution is design of control systems with parasitic effects of friction and backlash [96].

c) The third contribution is control of stable and unstable nonlinear nonminimum-phase systems [38,39,43,44,78,82].

d) The fourth contribution is robust global delay-dependent and delay-independent stabilizations of nonlinear time-delay systems [97].

e) The fifth contribution is active structural control of slewing beams [98].

f) The sixth contribution is synchronization of nonlinear chaotic systems [99] for secure communications by chaotic masking [100].

g) The seventh contribution is data assimilation based on the global ionosphere–thermosphere model to predict space weather [101].

III. Background Material

This section presents a brief overview of the required background material and mathematical preliminaries for the SDRE control methodology. Section III.A introduces the notation adopted in the Survey Paper. The formulation of the ITHNOC problem for nonlinear regulation is presented in Sec. III.B, with the corresponding HJB equation in Sec. III.C. A brief overview of the infinite-time horizon LQR approach is pursued in Sec. III.D. The concept of extended linearization, also known as SDC parameterization, is reviewed in Sec. III.E. Finally, the nonuniqueness of the SDC parameterization is examined in Sec. III.F.

A. Notation

The notation used in the Survey Paper is as follows. $\mathbb{R}$ is the set of real numbers. The set of nonnegative real numbers is represented by $\mathbb{R}^+ = [0, \infty) \subset \mathbb{R}$. For any positive integer n, $\mathbb{R}^n$ is n-dimensional real Euclidean space; $\|\cdot\|$ denotes the Euclidean norm of a vector in $\mathbb{R}^n$ Euclidean space. For B_r, it is understood to be the open ball in some Euclidean space with some radius $r > 0$, which is measured by the Euclidean norm. Ω is the state set that is a bounded open subset of some Euclidean space that contains the origin, and $\mathbf{0} \in \Omega \subseteq \mathbb{R}^n$. A vector field $\mathbf{f}\colon \Omega \to \mathbb{R}^n$ is an n-dimensional column vector. The Lebesgue space $L_2(\mathbb{R}^+)$ consists of measurable square-integrable (vector-valued) functions $\mathbf{u}\colon \mathbb{R}^+ \to \mathbb{R}^m$, such that

$$\int_{\mathbb{R}^+} \|\mathbf{u}(t)\|^2 \,\mathrm{d}t < \infty$$

$\mathbb{R}^{n\times m}$ is the set of all $n \times m$ real matrices. The transpose of some matrix $\mathbf{M} \in \mathbb{R}^{n\times m}$ is denoted by $\mathbf{M}^T$. The set of eigenvalues of a square matrix $\mathbf{N} \in \mathbb{R}^{n\times n}$ is denoted by $\lambda_i(\mathbf{N})$. For some symmetric matrix $\mathbf{P}$, $\mathbf{P} \geq \mathbf{0}$ (or $\mathbf{P} > \mathbf{0}$) is used to mean that the matrix is nonnegative (or positive) definite. A function is said to be of class $C^k(\Omega)$ (or simply C^k) if it is continuously differentiable k times in Ω, such that $C^0(\Omega)$ (or C^0) stands for the class of continuous functions in Ω. For a C^1 function $V\colon \mathbb{R}^n \to \mathbb{R}$, $\partial V(\mathbf{x})/\partial \mathbf{x}$ denotes the row vector of partial derivatives and $\partial V^T(\mathbf{x})/\partial \mathbf{x}$ the corresponding column vector.

B. Infinite-Time Horizon Nonlinear Optimal Control

Consider the continuous-time deterministic full-state-feedback infinite-time horizon nonlinear optimal regulation (stabilization) problem, where the system is autonomous (time-invariant), nonlinear in the state, and affine (linear) in the input, represented in the form

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \sum_{j=1}^{m} \mathbf{g}_j(\mathbf{x}) u_j(t) = \mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

with state vector $\mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, and (unconstrained) input vector $\mathbf{u}(t) = [u_1(t), \ldots, u_m(t)]^T \in \mathbb{R}^m$ $(1 \leq m \leq n)$ for each $t \in \mathbb{R}^+$. The vector fields $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{g}_j\colon \mathbb{R}^n \to \mathbb{R}^n$ are C^k mappings with $k \geq 0$, such that they are at least continuous vector-valued functions of $\mathbf{x}$, where $\mathbf{g}_j(\mathbf{x})$ corresponds to the jth column of the matrix $\mathbf{B}\colon \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $\mathbf{B}(\mathbf{x}) \neq \mathbf{0}\ \forall\ \mathbf{x}$. In this context, the minimization of an infinite-time performance criterion with a convex integrand, nonquadratic in $\mathbf{x}$ but quadratic in $\mathbf{u}$, is considered, given by

$$J(\mathbf{x}_0, \mathbf{u}(\cdot)) = \frac{1}{2}\int_0^\infty \{\mathbf{x}^T(t)\mathbf{Q}(\mathbf{x})\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}(\mathbf{x})\mathbf{u}(t)\}\,\mathrm{d}t \tag{2}$$

The respective state and input weighting matrices (design parameters) $\mathbf{Q}\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ and $\mathbf{R}\colon \mathbb{R}^n \to \mathbb{R}^{m\times m}$ are also state-dependent C^k functions with $k \geq 0$ so that they are at least continuous matrix-valued functions of $\mathbf{x}$. For a nonnegative-definite integrand, $\mathbf{Q}(\mathbf{x})$ may be factored as $\mathbf{Q}(\mathbf{x}) = \mathbf{C}^T(\mathbf{x})\mathbf{C}(\mathbf{x})$, forming the penalty variable $\mathbf{z} \in \mathbb{R}^n$, defined as $\mathbf{z}(t) \triangleq \mathbf{C}(\mathbf{x})\mathbf{x}$, such that

$$J = \frac{1}{2}\int_0^\infty \{\mathbf{z}^T(t)\mathbf{z}(t) + \mathbf{u}^T(t)\mathbf{R}(\mathbf{x})\mathbf{u}(t)\}\,\mathrm{d}t$$

Define the set of control functions by $\psi = \{\mathbf{u}\colon \mathbb{R}^+ \to \mathbb{R}^m\colon \mathbf{u}(\cdot) \in L_2(\mathbb{R}^+)\}$ so that the control $\mathbf{u}(\cdot) \in \psi$ is some appropriate bounded and measurable scheme on $t \in \mathbb{R}^+$. Then, given a bounded open set containing the origin $\mathbf{0} \in \Omega \subseteq \mathbb{R}^n$ and an initial point $\mathbf{x}_0 \in \Omega$, the ITHNOC problem on the set Ω is to minimize Eq. (2) with respect to $\mathbf{u}(\cdot) \in U$, where $U = \{\mathbf{u} \in \psi\colon \mathbf{x}(\cdot) \in \Omega\ \forall\ t \geq 0\}$ is the set of admissible controls in Ω such that the unique solution $\mathbf{x}(\cdot; \mathbf{u})$, or simply $\mathbf{x}(\cdot)$, to Eq. (1), corresponding to the choice of control $\mathbf{u}(\cdot) \in \psi$, stays in Ω for all t and tends to the origin as $t \to \infty$ for all $\mathbf{x}_0 \in \Omega$.

Under the specified conditions, a stabilizing feedback control law

$$\mathbf{u}(\mathbf{x}) = -\mathbf{k}(\mathbf{x}), \qquad \mathbf{k}(\mathbf{0}) = \mathbf{0} \tag{3}$$

with $\mathbf{k}(\cdot) \in C^k(\Omega)$, $k \geq 0$, is then sought that will (approximately) minimize Eq. (2) subject to the input-affine nonlinear differential constraint [Eq. (1)] while regulating the system to the origin $\forall\ \mathbf{x} \in \Omega$, such that

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$$

This problem forms the basis of the SDRE method for the unconstrained minimization problem for nonlinear regulation.

C. Hamilton–Jacobi–Bellman Equation

From the outset of Sec. III.B, the ITHNOC problem on the set $\Omega \subseteq \mathbb{R}^n$ is to minimize Eq. (2) with respect to $\mathbf{u}(\cdot) \in U$. In particular, a solution to this problem is said to exist on the set Ω if there exists a finite continuous nonnegative-definite value function $V\colon \Omega \to \mathbb{R}^+$ defined by

$$V(\mathbf{x}) \triangleq \inf_{\mathbf{u}(\cdot)\in U} J(\mathbf{x}, \mathbf{u}(\cdot)) \tag{4}$$

for all $\mathbf{x} \in \Omega$, the infimum being over the given set of admissible controls U. Ideally, the desired value function V is a stationary solution to the Cauchy problem for the associated dynamic programming (or Bellman's) equation, represented by the first-order nonlinear Hamilton–Jacobi PDE $\partial V(\mathbf{x})/\partial t + H(\mathbf{x}, \partial V(\mathbf{x})/\partial \mathbf{x}, \mathbf{u}) = 0$. Here, H is the Hamiltonian, and $\partial V/\partial \mathbf{x}$ denotes the row vector of partial derivatives of V with respect to $\mathbf{x}$. The Hamiltonian for the corresponding ITHNOC problem [Eqs. (1) and (2)] is given by

$$H = \inf_{\mathbf{u}(\cdot)\in U}\left\{\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\left[\mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}\right] + \frac{1}{2}\left[\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x} + \mathbf{u}^T\mathbf{R}(\mathbf{x})\mathbf{u}\right]\right\} \tag{5}$$

For the infinite-time formulation, V is assumed stationary ($\partial V/\partial t = 0$), so that the HJB equation becomes

$$\inf_{\mathbf{u}(\cdot)\in U}\left\{\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\left[\mathbf{f}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u}\right] + \frac{1}{2}\left[\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x} + \mathbf{u}^T\mathbf{R}(\mathbf{x})\mathbf{u}\right]\right\} = 0 \tag{6}$$

with boundary condition $V(\mathbf{0}) = 0$, since

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$$

which calls for closed-loop stability. In particular, note that $\partial^2 H/\partial \mathbf{u}^2|_{\mathbf{u}=\mathbf{u}_*} = \mathbf{R}(\mathbf{x})$; thus, for $\mathbf{R}(\mathbf{x}) > \mathbf{0}\ \forall\ \mathbf{x}$, the optimal control that minimizes Eq. (6) satisfies

$$\mathbf{0} = \left.\frac{\partial H^T}{\partial \mathbf{u}}\right|_{\mathbf{u}=\mathbf{u}_*} = \mathbf{B}^T(\mathbf{x})\frac{\partial V^T(\mathbf{x})}{\partial \mathbf{x}} + \mathbf{R}(\mathbf{x})\mathbf{u}_* \tag{7}$$

D. Linear-Quadratic Regulator

As a consequence of the well-established theory of LQ optimal control and its practical engineering significance, linear systems have always been very heavily emphasized in the design of controllers, since they are much easier to handle mathematically. The first step in dealing with a nonlinear system is usually to linearize it around some nominal operating point, if this is in fact possible. Assuming that the deviations from this condition are not too large in practice, the linearized approximation may well be adequate as a basis for analysis and design over a limited range of operations. In fact, most control systems in practice are often designed initially this way. The effects of departures from linearity are then investigated through simulations.

Application of the LQR formulation to the ITHNOC problem [Eqs. (1) and (2)] involves a first-order Taylor-series approximation (Jacobian linearization) around the origin. The nonlinear dynamics [Eq. (1)] then takes the linear form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{8}$$

which has stationary dynamics with constant time-invariant matrices $\mathbf{A} = \partial \mathbf{f}(\mathbf{0})/\partial \mathbf{x}$ and $\mathbf{B} = \mathbf{B}(\mathbf{0})$. In addition, the infinite-time performance functional [Eq. (2)] takes a quadratic form with constant time-invariant weighting matrices $\mathbf{Q} = \mathbf{Q}(\mathbf{0})$ and $\mathbf{R} = \mathbf{R}(\mathbf{0})$, with $\mathbf{Q} = \mathbf{C}^T\mathbf{C}$. It is well known that the optimal cost (or value function) [Eq. (4)] for the linearized problem, if it exists, must be of the form $V(\mathbf{x}) = J_*(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x}$ for some unique, symmetric nonnegative-definite matrix $\mathbf{P} \in \mathbb{R}^{n\times n}$. The corresponding HJB equation [Eq. (6)] simply reduces to finding the solution of the ARE:

$$\mathbf{P}\mathbf{A} + \mathbf{A}^T\mathbf{P} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0} \tag{9}$$

for $\mathbf{P} \geq \mathbf{0}$, which is constant. The optimal feedback control policy [Eq. (7)] associated with this LQ optimization problem then becomes time-invariant, and it is given by

$$\mathbf{u}_* = -\mathbf{K}\mathbf{x} = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x} \tag{10}$$

Therefore, the infinite-time horizon solution is particularly appealing for the regulation of processes that are well modeled by LTI dynamics because the solution comprises a set of static gains $\mathbf{K}$ that are calculated once (offline) and are implemented causally thereafter. Provided that the pair $\{\mathbf{A}, \mathbf{B}\}$ is stabilizable and the pair $\{\mathbf{C}, \mathbf{A}\}$ is detectable, Kalman [102] has shown that a unique nonnegative-definite (stabilizing) solution to the ARE exists and is finite. The optimal state feedback then generates an asymptotically stable closed-loop system for this LQ problem.

Jacobian linearization still maintains its popularity among practitioners in the design of linear control laws for nonlinear systems. However, the resulting controller remains valid only close to the chosen equilibrium or operating point. If the system is required to operate under a wide range of conditions, in order to apply linear design techniques such as LQR, it may then be necessary to obtain a set of approximate models by linearizing around different operating points. This will then generate a sequence of controllers for scheduled control, which are either brought into operation successively as the system passes through conditions where the corresponding models are approximately valid or combined by continuously interpolating the point designs in order to obtain a controller for the overall (nonlinear) system. Jacobian linearization used in conjunction with gain-scheduling indeed presents a very effective solution to this problem without requiring severe structural assumptions on the plant model [103]. This method also preserves well-understood linear intuition in the formulation process. However, for wide variations in operating conditions, gain-scheduling can be a very tedious process and can consume an enormous amount of time and effort. Moreover, for complex higher-order systems, it becomes a liability because control design methods based on LTI models ignore the fundamental (nonlinear) nature of the plant. Controllers so designed may offer unsatisfactory performance and are severely limited, especially when the dynamics is highly nonlinear. The SDRE framework constructs the bridge that is required for the practicing control engineer between the familiar results of LQ optimal control and those of nonlinear optimal control. First, let us review the concept of extended linearization.

E. Extended Linearization (State-Dependent Coefficient Parameterization)

Extended linearization [104], apparent linearization [9], or SDC parameterization [3,6] is the process of factorizing a nonlinear system into a linearlike structure that contains SDC matrices. Under the assumptions $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{f}(\cdot) \in C^1(\Omega)$, there always exists at least C^0 nonlinear matrix-valued function $\mathbf{A}(\mathbf{x})$ on Ω such that

$$\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x} \tag{11}$$

where $\mathbf{A}\colon \Omega \to \mathbb{R}^{n\times n}$ is found by mathematical factorization and is, clearly, nonunique when $n > 1$. It is important to note that the aforementioned assumptions on $\mathbf{f}(\mathbf{x})$ guarantee the existence of a global SDC parameterization of $\mathbf{f}(\mathbf{x})$ on Ω, with $\mathbf{A}(\cdot) \in C^0(\Omega)$ [105]. The following proposition, due to Bass [106], further shows that $\mathbf{f}(\mathbf{x})$ can then be recast in the form given by Eq. (11) (for the proof, readers may also refer to [8]).

Proposition 1: Let $\mathbf{f}\colon \Omega \to \mathbb{R}^n$ be such that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{f}(\cdot) \in C^k(\Omega)$, $k \geq 1$. Then, for all $\mathbf{x} \in \Omega$, a SDC parameterization [Eq. (11)] of $\mathbf{f}(\mathbf{x})$ always exists for some C^{k-1} matrix-valued function $\mathbf{A}\colon \Omega \to \mathbb{R}^{n\times n}$. One such parameterization, which is guaranteed to exist under the above conditions, is given by

$$\mathbf{A}(\mathbf{x}) = \int_0^1 \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\lambda\mathbf{x}} \mathrm{d}\lambda$$

where λ is a dummy variable introduced in the integration.

The following lemma now reveals a general property of the SDC parameterization [Eq. (11)] at the origin.

Lemma 1: For any choice of SDC parameterization $\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}$, $\mathbf{A}(\mathbf{0})$ is the linearization of $\mathbf{f}(\mathbf{x})$ at the origin; that is, $\mathbf{A}(\mathbf{0}) = \partial\mathbf{f}(\mathbf{0})/\partial\mathbf{x}$.

Proof: Suppose $\mathbf{A}_1(\mathbf{x})$ and $\mathbf{A}_2(\mathbf{x})$ are two distinct SDC matrices, such that $\mathbf{f}(\mathbf{x}) = \mathbf{A}_1(\mathbf{x})\mathbf{x} = \mathbf{A}_2(\mathbf{x})\mathbf{x}$, and let $\tilde{\mathbf{A}}(\mathbf{x}) = \mathbf{A}_1(\mathbf{x}) - \mathbf{A}_2(\mathbf{x})$. Then, $\tilde{\mathbf{A}}(\mathbf{x})\mathbf{x} = \mathbf{0}$ for all $\mathbf{x}$; thus,

$$\partial/\partial\mathbf{x}[\tilde{\mathbf{A}}(\mathbf{x})\mathbf{x}] = \tilde{\mathbf{A}}(\mathbf{x}) + [\partial\tilde{\mathbf{A}}(\mathbf{x})/\partial\mathbf{x}]\mathbf{x} = \mathbf{0}$$

Since $[\partial\tilde{\mathbf{A}}(\mathbf{x})/\partial\mathbf{x}]\mathbf{x} = \mathbf{0}$ at $\mathbf{x} = \mathbf{0}$, it follows that $\tilde{\mathbf{A}}(\mathbf{0}) = \mathbf{0}$. This implies $\mathbf{A}_1(\mathbf{0}) = \mathbf{A}_2(\mathbf{0})$, and hence the uniqueness of the parameterization evaluated at $\mathbf{x} = \mathbf{0}$. Without loss of generality, consider now the parameterization given by $\mathbf{A}_1(\mathbf{x})$. Noting that

$$\partial\mathbf{f}(\mathbf{x})/\partial\mathbf{x} = \partial/\partial\mathbf{x}[\mathbf{A}_1(\mathbf{x})\mathbf{x}] = \mathbf{A}_1(\mathbf{x}) + [\partial\mathbf{A}_1(\mathbf{x})/\partial\mathbf{x}]\mathbf{x}$$

the linearization of $\mathbf{f}(\mathbf{x})$ at the origin gives $\partial\mathbf{f}(\mathbf{0})/\partial\mathbf{x} = \mathbf{A}_1(\mathbf{0})$. Since $\mathbf{A}_1(\mathbf{0})$ has been shown to be unique for all parameterizations, this completes the proof. □

Remark 1: Lemma 1 implies that, for any choice of SDC matrix $\mathbf{A}$, $\mathbf{A}(\mathbf{x}) \to \partial\mathbf{f}(\mathbf{0})/\partial\mathbf{x}$ as $\mathbf{x} \to \mathbf{0}$; that is, $\mathbf{A}$ converges to its conventional Taylor-series linearization at the origin, meaning that $\mathbf{A}(\mathbf{x})$ evaluated at $\mathbf{x} = \mathbf{0}$ [that is, $\mathbf{A}(\mathbf{0})$] is unique, regardless of the choice of $\mathbf{A}(\mathbf{x})$.

The following illustrative example from [13] is now used for demonstrating the above results.

Example 1: Consider the two-dimensional unforced nonlinear system $\dot{x}_1 = -x_1$ and $\dot{x}_2 = -x_2 + x_1 x_2^2$, with

$$\frac{\partial\mathbf{f}(\mathbf{x})}{\partial\mathbf{x}} = \begin{bmatrix} -1 & 0 \\ x_2^2 & -1 + 2x_1x_2 \end{bmatrix}, \qquad \mathbf{A}_1(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ x_2^2 & -1 \end{bmatrix},$$
$$\mathbf{A}_2(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 + x_1x_2 \end{bmatrix}, \qquad \mathbf{A}_3(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ \frac{1}{3}x_2^2 & -1 + \frac{2}{3}x_1x_2 \end{bmatrix}$$

Thus, $\mathbf{A}_i(\mathbf{x})$, $i = 1, 2, 3$, results in three equivalent SDC parameterizations in the form of Eq. (11), where $\mathbf{A}_3(\mathbf{x})$ corresponds to the parameterization obtained from $\partial\mathbf{f}(\mathbf{x})/\partial\mathbf{x}$ using Proposition 1. Now, local inspection of these three choices of SDC matrices together with the gradient of $\mathbf{f}(\mathbf{x})$ at the origin yields

$$\partial\mathbf{f}(\mathbf{0})/\partial\mathbf{x} = \mathbf{A}_1(\mathbf{0}) = \mathbf{A}_2(\mathbf{0}) = \mathbf{A}_3(\mathbf{0}) = -\mathbf{I}_{2\times 2}$$

Therefore, using extended linearization, any input-affine nonlinear system [Eq. (1)] satisfying the conditions on $\mathbf{f}(\mathbf{x})$ stated in Proposition 1 can always be represented in SDC form

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{x})\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{12}$$

which has the linear structure in Eq. (8), with at least C^0 SDC matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$. The following system-theoretic concept definitions, pointwise in $\mathbf{x}$, are then associated with the SDC form in Eq. (12).

Definition 1: The SDC parameterization [Eq. (12)] is a stabilizable (or controllable) parameterization of the nonlinear system [Eq. (1)] in a region Ω if the pair $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ is pointwise stabilizable (or controllable) in the linear sense for all $\mathbf{x} \in \Omega$.

Definition 2: The SDC parameterization [Eq. (12)] is a detectable (or observable) parameterization of the nonlinear system [Eq. (1)] in a region Ω if, for $\mathbf{C}^T(\mathbf{x})\mathbf{C}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})$, the pair $\{\mathbf{C}(\mathbf{x}), \mathbf{A}(\mathbf{x})\}$ is pointwise detectable (or observable) in the linear sense for all $\mathbf{x} \in \Omega$.

Definition 3: The SDC parameterization [Eq. (12)] is pointwise Hurwitz in a region Ω if the eigenvalues of $\mathbf{A}(\mathbf{x})$ are in the open left half of the complex plane (that is, have negative real parts), such that $Re[\lambda_i(\mathbf{A}(\mathbf{x}))] < 0$ for all $\mathbf{x} \in \Omega$.

The application of any linear control synthesis method (pointwise) to the linearlike SDC structure [Eq. (12)], where $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are treated as constant matrices, forms an extended linearization control method [104,107]. These represent a rather broad class of nonlinear control design methods, leading to nonlinear control laws of the form in Eq. (3) with

$$\mathbf{k}(\mathbf{x}) = \mathbf{K}(\mathbf{x})\mathbf{x} \tag{13}$$

that render the closed-loop dynamics (or SDC) matrix

$$\mathbf{A}_{\mathrm{CL}}(\mathbf{x}) \triangleq \mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x}) \tag{14}$$

pointwise Hurwitz; that is, $Re[\lambda_i(\mathbf{A}_{\mathrm{CL}}(\mathbf{x}))] < 0 \ \forall \ \mathbf{x}$. From Eq. (3), since $\mathbf{k}(\mathbf{0}) = \mathbf{0}$ by assumption, $\mathbf{k}(\mathbf{x})$ can be represented nonuniquely in SDC form [Eq. (13)]. Additionally, from Proposition 1, if $\mathbf{k}(\cdot) \in C^1(\Omega)$, then there always exists a global SDC parameterization of $\mathbf{k}(\mathbf{x})$ on Ω with at least a C^0 SDC matrix $\mathbf{K}(\cdot)$.

F. Nonuniqueness of the State-Dependent Coefficient Parameterization

Consider the following two technical lemmas concerning SDC parameterization [Eq. (11)] for scalar and multivariable systems, respectively [3,107–109].

Lemma 2: For scalar systems, the SDC parameterization is unique for all $x \neq 0$ and is given by $a(x) = f(x)/x$. Additionally, if $a(x)$ is a C^0 function, then it is unique for all x.

Proof: Let $a_1(x)$ and $a_2(x)$ be two SDC parameterizations of Eq. (12). Then, from Eq. (11), $f(x) = a_1(x)x = a_2(x)x$, which implies $[a_1(x) - a_2(x)]x = 0$. Hence, for all $x \neq 0$, $a_1(x) = a_2(x)$, and uniqueness follows from Eq. (11) with $f(x) = a(x)x$. The result is extended to the case when $x = 0$ if $a(x)$ is C^0, which readily follows if $f(x)$ is C^1. □

Lemma 3: Suppose the function $\mathbf{f}$: $\Omega \to \mathbb{R}^n$ with $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ is represented in SDC form as $\mathbf{f}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\mathbf{x}$ for a given continuous (C^0) matrix-valued function $\mathbf{A}$: $\Omega \to \mathbb{R}^{n\times n}$, $n > 1$. Then, any other SDC parameterization $\mathbf{f}(\mathbf{x}) = \mathbf{A}_0(\mathbf{x})\mathbf{x}$ can be written in the form $\mathbf{A}_0(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x})$, where $\mathbf{E}$: $\Omega \to \mathbb{R}^{n\times n}$ satisfies $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{0}$ $\forall \ \mathbf{x} \in \Omega$.

Proof: Let us write $\mathbf{A}_0(\mathbf{x})$ as $\mathbf{A}(\mathbf{x}) + [\mathbf{A}_0(\mathbf{x}) - \mathbf{A}(\mathbf{x})]$ and let $\mathbf{E}(\mathbf{x}) = [\mathbf{A}_0(\mathbf{x}) - \mathbf{A}(\mathbf{x})]$. Since $\mathbf{A}_0(\mathbf{x})\mathbf{x} = \mathbf{A}(\mathbf{x})\mathbf{x} = \mathbf{f}(\mathbf{x})$, then $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{0}$; thus,

$$\mathbf{A}_0(\mathbf{x})\mathbf{x} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{A}(\mathbf{x})\mathbf{x} = \mathbf{f}(\mathbf{x})$$

□

Hence, in the multivariable case, the SDC parameterization [Eq. (11)] is not unique. In fact, there are an infinite number of ways to bring the nonlinear system [Eq. (1)] to SDC form [Eq. (12)], as shown by the following proposition [3,8].

Proposition 2: For multivariable problems, let $\mathbf{A}(\mathbf{x}, \alpha) = \alpha\mathbf{A}_1(\mathbf{x}) + (1 - \alpha)\mathbf{A}_2(\mathbf{x})$ for two distinct SDC matrices $\mathbf{A}_1(\mathbf{x})$ and $\mathbf{A}_2(\mathbf{x})$, where $\alpha \in \mathbb{R}$. Then, for any $\alpha \in \mathbb{R}$, $\mathbf{A}(\mathbf{x}, \alpha)$ represents an infinite number of SDC matrices corresponding to a hyperplane.

Remark 2: In general, using Proposition 2, a complete characterization of the possible factorizations of $\mathbf{f}(\mathbf{x})$ that span the space of all valid and distinct SDC parameterizations can be obtained in the form $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})\mathbf{x}$ by constructing a SDC matrix $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})$, which is the parametric representation of a hypersurface, and $\boldsymbol{\alpha}$ is a vector of dimension k. Each free design parameter in $\boldsymbol{\alpha}$ can be selected to have any finite real value but is often constrained to be within the range $[0, 1]$ to form a convex hypersurface of all possible distinct SDC matrices.

Example 1 Revisited: Reconsider the nonlinear system in Example 1 together with the three given distinct SDC matrices $\mathbf{A}_i(\mathbf{x})$ for $i = 1, 2, 3$. Using Lemma 3 and Proposition 2, let us construct

$$\mathbf{E}_1(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ -x_2^2 & x_1x_2 \end{bmatrix}, \qquad \mathbf{E}_2(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ -\frac{2}{3}x_2^2 & \frac{2}{3}x_1x_2 \end{bmatrix},$$
$$\mathbf{A}(\mathbf{x}, \alpha) = \begin{bmatrix} -1 & 0 \\ \alpha x_2^2 & -1 + (1 - \alpha)x_1x_2 \end{bmatrix}$$

such that $\mathbf{E}_1(\mathbf{x})\mathbf{x} = \mathbf{E}_2(\mathbf{x})\mathbf{x} = \mathbf{0}$ and $\alpha \in [0, 1]$. Then, $\mathbf{A}_1(\mathbf{x}) + \mathbf{E}_1(\mathbf{x}) = \mathbf{A}_2(\mathbf{x})$ and $\mathbf{A}_1(\mathbf{x}) + \mathbf{E}_2(\mathbf{x}) = \mathbf{A}_3(\mathbf{x})$, whereas the convex hyperplane of SDC matrices $\mathbf{A}(\mathbf{x}, \alpha)$, constructed from $\mathbf{A}_1(\mathbf{x})$ and $\mathbf{A}_2(\mathbf{x})$, yields $\mathbf{A}_i(\mathbf{x})$ by setting $\alpha \in [0, 1]$ to 1, 0, and $\frac{1}{3}$.

IV. State-Dependent Riccati Equation Nonlinear Regulation

In this section, the SDRE method for nonlinear regulation is reviewed, outlining the basic SDRE controller structure and conditions [8].

The SDRE methodology uses extended linearization as the key design concept in formulating the ITHNOC problem [Eqs. (1) and (2)]. The underlying linear control synthesis method in this case is the LQR synthesis method reviewed in Sec. III.D. Motivated by the LQR problem, which is characterized by the ARE [Eq. (9)], SDRE feedback control is an extended linearization control method that provides a similar approach to this nonlinear regulation problem. First, the following conditions from [8] must be met (or imposed) before controller design, where Condition 1 in [8] (restated here as Condition 2) has now been relaxed.

Condition 1: Without any loss of generality, the origin $\mathbf{x} = \mathbf{0} \in \Omega$ is an equilibrium point of the system with $\mathbf{u} = \mathbf{0}$, such that $\mathbf{f}(\mathbf{0}) = \mathbf{0}$, whereas $\mathbf{B}(\mathbf{x}) \neq \mathbf{0}\ \forall\ \mathbf{x} \in \Omega$.

Condition 2: $\mathbf{A}(\cdot)$ is a continuous matrix-valued function of $\mathbf{x}$ on Ω [that is, $\mathbf{A}(\cdot) \in C^0(\Omega)$] and the vector-valued functions $\mathbf{g}_j(\cdot) \in C^0(\Omega)$, such that $\mathbf{B}(\cdot) \in C^0(\Omega)$.

Condition 3: The design parameters are $C^0(\Omega)$ matrix-valued functions satisfying $\mathbf{Q}(\mathbf{x}) = \mathbf{Q}^T(\mathbf{x}) \geq \mathbf{0}$ and $\mathbf{R}(\mathbf{x}) = \mathbf{R}^T(\mathbf{x}) > \mathbf{0}$ $\forall\ \mathbf{x} \in \Omega$.

Condition 4: The respective pairs $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ and $\{\mathbf{C}(\mathbf{x}), \mathbf{A}(\mathbf{x})\}$ are pointwise stabilizable and detectable SDC parameterizations of the nonlinear system [Eq. (1)] $\forall\ \mathbf{x} \in \Omega$, where $\mathbf{C}^T(\mathbf{x})\mathbf{C}(\mathbf{x}) = \mathbf{Q}(\mathbf{x})$.

Under Condition 1, using extended linearization, the input-affine nonlinear system [Eq. (1)] is first represented in SDC form [Eq. (12)], with SDC matrices $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ satisfying the requirements set forth in Condition 2. By mimicking the LQR formulation, the continuous-time algebraic SDRE

$$\mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) = \mathbf{0} \tag{15}$$

is solved for unique, symmetric nonnegative-definite $\mathbf{P}(\mathbf{x})$ for each $\mathbf{x} \in \Omega$, which corresponds to the pointwise stabilizing solution under Condition 3 and Condition 4. The state-feedback controller is then obtained in the form

$$\mathbf{u}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x})\mathbf{x} \tag{16}$$

so that the resulting SDRE-controlled trajectory becomes the solution of the state-dependent closed-loop dynamics

$$\dot{\mathbf{x}}(t) = [\mathbf{A}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x})]\mathbf{x}(t) \tag{17}$$

Therefore, SDRE control is an extended linearization control method, which means that the nonlinear control law [Eq. (16)] of the form in Eq. (3) renders the closed-loop SDC matrix $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$ defined in Eq. (14) pointwise Hurwitz $\forall\ \mathbf{x}$. The nonlinear state-feedback gain in Eqs. (13) and (14) for (approximately) minimizing Eq. (2) thus becomes

$$\mathbf{K}(\mathbf{x}) = \mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) \tag{18}$$

Some key remarks regarding the aforementioned SDRE controller structure and conditions are now in order.

Remark 3: Under the assumption $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ (Condition 1), for $\mathbf{A}(\cdot) \in C^0(\Omega)$ in Condition 2 to hold, it is sufficient that $\mathbf{f}(\mathbf{x})$ be a continuously differentiable vector-valued function of $\mathbf{x}$ on $\Omega \subseteq \mathbb{R}^n$; that is, $\mathbf{f}(\cdot) \in C^1(\Omega)$. This will guarantee that a SDC parameterization [Eq. (11)] of $\mathbf{f}(\mathbf{x})$ always globally exists on Ω for some $\mathbf{A}\colon \Omega \to \mathbb{R}^{n\times n}$, such that $\mathbf{A}(\cdot)$ is at least a continuous matrix-valued function of $\mathbf{x}$ on Ω (see Proposition 1), which is a necessary condition for large-scale asymptotic stability to hold (see Sec. VII). When the nonlinear dynamics $\mathbf{f}(\mathbf{x})$ is not a C^1 function of $\mathbf{x}$, $\mathbf{f}(\mathbf{x})$ may not yield a C^0 SDC matrix $\mathbf{A}(\cdot)$. In this case, the function $\mathbf{f}(\mathbf{x})$ must be approximated by a C^1 function, as discussed in [8]. However, if $\mathbf{f}(\mathbf{x})$ is nondifferentiable but continuous, and a C^0 SDC matrix $\mathbf{A}(\cdot)$ can be constructed from it, then it is not a requirement that $\mathbf{f}(\cdot) \in C^1(\Omega)$. Consequently, the requirement $\mathbf{f}(\cdot) \in C^1(\Omega)$ in Condition 1 of [8] has been replaced with the milder one of Condition 2 above; that is, $\mathbf{A}(\cdot) \in C^0(\Omega)$.

Remark 4: Condition 4 is essential in obtaining a legitimate solution $\mathbf{P}(\mathbf{x}) \geq \mathbf{0}$ of the SDRE [Eq. (15)] pointwise. Sufficient tests for stabilizability and detectability in Condition 4 require checking that the $n \times nm$ state-dependent controllability matrix

$$\mathbf{M}_C = [\,\mathbf{B}(\mathbf{x}) \quad \mathbf{A}(\mathbf{x})\mathbf{B}(\mathbf{x}) \quad \cdots \quad \mathbf{A}^{n-1}(\mathbf{x})\mathbf{B}(\mathbf{x})\,]$$

has $\mathrm{rank}(\mathbf{M}_C) = n\ \ \forall\ \mathbf{x} \in \Omega$, and the $n \times n^2$ state-dependent observability matrix

$$\mathbf{M}_O = [\,\mathbf{C}^T(\mathbf{x}) \quad \mathbf{A}^T(\mathbf{x})\mathbf{C}^T(\mathbf{x}) \quad \cdots \quad (\mathbf{A}^T(\mathbf{x}))^{n-1}\mathbf{C}^T(\mathbf{x})\,]$$

has $\mathrm{rank}(\mathbf{M}_O) = n\ \forall\ \mathbf{x} \in \Omega$, respectively [8]. Clearly, the detectability requirement can be guaranteed for the entire domain of interest by ensuring that $\mathbf{Q}(\mathbf{x})$ is positive definite (as opposed to nonnegative definite) $\forall\ \mathbf{x} \in \Omega$, which also ensures $\mathbf{P}(\mathbf{x}) > \mathbf{0}$.

Remark 5: For any choice of $\mathbf{A}(\mathbf{x})$ satisfying Eq. (11), recall from Lemma 1 that $\mathbf{A}(\mathbf{x}) \to \partial\mathbf{f}(\mathbf{0})/\partial\mathbf{x}$ as $\mathbf{x} \to \mathbf{0}$; that is, $\partial\mathbf{f}(\mathbf{0})/\partial\mathbf{x} = \mathbf{A}(\mathbf{0})$. Consequently, it is obvious that if the Jacobian linearization of Eq. (1) is unstabilizable, there is no SDC matrix $\mathbf{A}(\mathbf{x})$ to satisfy Condition 4 such that the pair $\{\mathbf{A}(\mathbf{0}), \mathbf{B}(\mathbf{0})\}$ is stabilizable. In such cases, the system must be converted to a conforming one so that the SDRE technique can be applied, as discussed next.

Remark 6: Condition 1 through Condition 4 are required for the direct application of the SDRE technique. However, there are many systems that do not conform to the structure or conditions specified here, and the SDRE technique cannot be directly applied. In these cases, the given system must be converted to a system that is conforming, so that an effective SDRE design can be performed. An overview of the various methods available for systematically handling numerous systems that do not meet these basic structure and conditions has been presented in [8]. These nonconforming cases include the existence of 1) nondifferentiable dynamics, 2) state-independent terms, 3) state-dependent terms that do not go to zero as the state vector goes to zero, 4) uncontrollable and unstable but bounded state dynamics, 5) nonlinearity (such as hard constraints) in the controls, and 6) state constraints. The SDRE nonlinear regulator with integral servomechanism action for tracking or command following and the SDRE controller design with partial-state feedback, as opposed to full-state information, are also reviewed therein.

Remark 7: The implications of the nonuniqueness of the SDC parameterization [Eq. (12)] for representing the original system [Eq. (1)] have been considered in [107–109] and will be discussed in the Survey Paper. It is important to realize at this point that the issue of nonuniqueness plays a major role in not only recovering the global optimal control but also achieving global asymptotic stability. By Proposition 2 (or Remark 2), if a hyperplane (or hypersurface) of SDC matrices $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})$ is formed with $\boldsymbol{\alpha} \in \mathbb{R}$ (or $\boldsymbol{\alpha} \in \mathbb{R}^k$, $k > 1$), the solution of the SDRE [Eq. (15)] will be of the form $\mathbf{P}(\mathbf{x}, \boldsymbol{\alpha})$. This results in the nonlinear feedback controller being parameterized by $\boldsymbol{\alpha}$. Alternatively, Lemma 3 implies that, for some $\mathbf{E}(\mathbf{x})$ satisfying $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{0}$, $\dot{\mathbf{x}}(t) = [\mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x})]\mathbf{x}(t) + \mathbf{B}(\mathbf{x})\mathbf{u}(t)$ is also a representation of Eq. (1). Hence, the optimal feedback control law [Eq. (3)] can be given by the state-feedback SDRE controller [Eq. (16)], where $\mathbf{P}(\mathbf{x})$ is now the nonnegative-definite solution to

$$\mathbf{P}(\mathbf{x})[\mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x})] + [\mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x})]^T\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) = \mathbf{0} \tag{19}$$

Consequently, in general, the solution provided by the SDRE control [Eqs. (15) and (16)] does not recover global optimality with respect to the performance index [Eq. (2)] for some arbitrary choice of the SDC matrix $\mathbf{A}(\mathbf{x})$. Moreover, a proper choice of $\mathbf{A}(\mathbf{x})$ also plays a significant role in affecting the controllability of the resulting parameterized pair $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$. Note, however, that the presence or lack of controllability of this pair need not have any implication on the controllability of the original dynamics given in Eq. (1). These issues have been considered in [58], where a general nonequivalence relationship between true and factored controllabilities has been established, even though local equivalence between true nonlinear controllability and pointwise (or factored) controllability of SDC parameterizations of nonlinear systems holds. Thus, the two concepts are generally different, and yet there are particular cases where both types of controllability hold, one of which corresponds to full-rank constant **B** matrices.

V. Existence of Solutions

In [110], a detailed discussion has been pursued on the development of geometric existence theory for classical (smooth) solutions of the HJB PDE associated with general ITHNOC problems for nonlinear regulation of input-affine systems. The link between the HJB equation and Lagrangian manifolds is also emphasized, using stable manifold theory, with their relation to optimal control laws characterized by Riccati equations. This section reviews the main results. Necessary and sufficient conditions for the existence of solutions to the HJB equation are first reviewed in Sec. V.A. The motivation for characterization of solutions to nonlinear optimal control problems by Riccati equations (in particular, by symmetric nonnegative-definite solutions) is then justified in Sec. V.B, providing the theoretical basis for existence of solutions of SDRE controls under a very mild, necessary, and sufficient condition. Finally, existence of an optimum positive-definite SDRE solution by SDRE feedback control is examined in Sec. V.C.

A. Existence of Solutions to the Hamilton–Jacobi–Bellman Equation

Let us first make some comments about the existence of the dynamic programming solution to the ITHNOC problem described in Sec. III.C. Suppose $\mathbf{f}(\cdot)$, $\mathbf{B}(\cdot)$, $\mathbf{Q}(\cdot)$, and $\mathbf{R}(\cdot)$ are sufficiently smooth functions of class $C^k(\Omega)$ for some $k \geq 1$, so that the value function defined by Eq. (4) is continuously differentiable (C^1 function).

Hypothesis 1: The linearization of Eqs. (1) and (2) at the equilibrium [that is, the triple $\{\partial\mathbf{f}(\mathbf{0})/\partial\mathbf{x}, \mathbf{B}(\mathbf{0}), \mathbf{C}(\mathbf{0})\}$] is stabilizable and detectable.

Under Hypothesis 1, it is well known that the linearization of the ITHNOC problem has a classical (C^2) solution on a small neighborhood Ω_0 of $\mathbf{x} = \mathbf{0}$ in state space. Clearly, taking $V(\mathbf{0}) = 0$ and letting $\mathbf{P} = \partial^2 V(\mathbf{0})/\partial\mathbf{x}^2$, then on Ω_0, $V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{x}$, where $\mathbf{P}$ satisfies the ARE [Eq. (9)]. An optimal feedback control then exists in the form $\mathbf{u}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{0})\mathbf{B}^T(\mathbf{0})\partial V^T/\partial\mathbf{x}$. The existence of this stationary solution is proved directly by showing that the value functions for the corresponding sequence of linearized finite-time horizon problems converge to an explicit limit as $t \to \infty$. This idea is extended to ITHNOC of the type considered in Eqs. (1) and (2) by Brunovsky [111] and Lukes [112] and to nonlinear H_∞ control by van der Schaft [18,19]. Under the same conditions given by Hypothesis 1, a smooth (C^2) solution to the full nonlinear problem exists on a region Ω larger than Ω_0 containing the equilibrium point $\mathbf{x} = \mathbf{0}$. The key to this analysis is the link between stationary solutions to the HJB PDE [Eq. (6)] and stable Lagrangian manifolds for the corresponding Hamiltonian dynamics

$$\dot{\mathbf{x}} = \frac{\partial H_*^T(\mathbf{x}, \mathbf{p})}{\partial\mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H_*^T(\mathbf{x}, \mathbf{p})}{\partial\mathbf{x}} \tag{20}$$

on $\mathbb{R}^{2n}$ phase space, which is the real $2n$-dimensional vector space with coordinates $(\mathbf{x}, \mathbf{p}) = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ for the dynamics of $\mathbf{x} \in \mathbb{R}^n$ and the adjoint variable (or costate) $\mathbf{p} \in \mathbb{R}^n$ arising from Pontryagin's minimum principle, with the Hamiltonian function $H_*(\mathbf{x}, \mathbf{p}) \triangleq H(\mathbf{x}, \mathbf{p}, \mathbf{u}_*)$.

Associated with H is a Hamiltonian vector field X_H defined, in coordinate terms, by Eq. (20). The vector field X_H is called hyperbolic in the origin if its linearization at $\mathbf{x} = \mathbf{p} = \mathbf{0}$ does not have purely imaginary eigenvalues. Thus, by Hypothesis 1, X_H is hyperbolic. Now, Hypothesis 1 can be used to construct a smooth $V(\mathbf{x})$ geometrically in a neighborhood of the origin. By the stable manifold theorem, the existence of a solution to the linearized problem at the origin, by this assumption, implies the existence of a stable n-dimensional Lagrangian manifold M of $\mathbb{R}^{2n}$ phase space through the origin. M is in fact an invariant manifold with respect to the Hamiltonian flow corresponding to H. The term Hamiltonian (or phase) flow refers to the transformation of phase space given by the solution to Eq. (20), where $(\mathbf{x}, \mathbf{p}) \in M$. The characteristic curves for the Cauchy problem are the trajectories of the Hamiltonian system [Eq. (20)] corresponding to the HJB equation [Eq. (6)]. This fact forms the basis of the classical method of characteristics for constructing local solutions to the Cauchy problem involving the HJB equation. There exists a simply connected region L of the stable manifold M that contains the point $\mathbf{x} = \mathbf{p} = \mathbf{0}$, and M locally (in the region L around the origin) has a well-defined projection onto a region Ω in state space containing the point $\mathbf{x} = \mathbf{0}$. Letting π: $\mathbb{R}^{2n} \to \mathbb{R}^n$ denote the canonical projection of phase space onto state space, this means that $\pi|_L$ is nonsingular, and so $\mathbf{p}$ can be expressed as a function of $\mathbf{x}$ for $(\mathbf{x}, \mathbf{p}) \in L$. Furthermore, the corresponding value function $V(\mathbf{x})$ to the HJB equation [Eq. (6)] for the ITHNOC problem is a smooth (C^2) stationary solution on Ω. $V(\mathbf{x})$ is in fact the generating function for the stable submanifold L. This means that, defining $\mathbf{p} = \partial S(\mathbf{x})/\partial\mathbf{x}$, L is the graph constructed from the set of points $(\mathbf{x}, \mathbf{p})$ in phase space, and S satisfies $\mathrm{d}S(\mathbf{x}) = \mathbf{p}\,\mathrm{d}\mathbf{x}$ for $\mathbf{x} \in \Omega$ along trajectories of the Hamiltonian flow lying on L, with $S(\mathbf{0}) = 0$ (for details, see [18]). It follows that $V(\mathbf{x})$ defined by Eq. (4) exists on Ω and equals $S(\mathbf{x}) = \int \mathbf{p}\,\mathrm{d}\mathbf{x}$, such that the submanifold

$$L = \left\{(\mathbf{x}, \mathbf{p}) \colon \mathbf{p} = \frac{\partial V^T(\mathbf{x})}{\partial\mathbf{x}}\right\} \subset T^*M \tag{21}$$

is the invariant manifold of the Hamiltonian system [Eq. (20)], where T^*M denotes the cotangent bundle of M. It also follows that an optimal feedback control exists and is obtained from the necessary optimality condition [Eq. (7)] for $\mathbf{u}$ to minimize H, giving the feedback

$$\mathbf{u}_*(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{p}(\mathbf{x}) = -\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\frac{\partial V^T(\mathbf{x})}{\partial\mathbf{x}} \tag{22}$$

This is the nonlinear extension of the feedback that solves the linear problem. In addition, the class of solutions in Eqs. (21) and (22) satisfies the following standard result (for the proof, see Lemma 2.2 in [113]).

Lemma 4: Suppose a vector-valued function $\mathbf{p}$: $\Omega \to \mathbb{R}^n$ is of class C^k for some integer $k \geq 1$, and let $\mathbf{p}(\mathbf{x}) = [p_1(\mathbf{x}), \ldots, p_n(\mathbf{x})]^T$ for $\mathbf{x} \in \Omega$. Then, there exists a C^{k+1} function V: $\Omega \to \mathbb{R}$ such that $\partial V^T(\mathbf{x})/\partial\mathbf{x} = \mathbf{p}(\mathbf{x})$ if and only if

$$\frac{\partial p_i(\mathbf{x})}{\partial x_j} = \frac{\partial p_j(\mathbf{x})}{\partial x_i} \quad \forall\ \mathbf{x} \in \Omega \quad \text{and} \quad i, j = 1, 2, \ldots, n \tag{23}$$

Moreover, if Eq. (23) holds, then V with $V(\mathbf{0}) = 0$ is given by

$$V(\mathbf{x}) = \mathbf{x}^T \int_0^1 \mathbf{p}(t\mathbf{x})\,\mathrm{d}t$$

Remark 8: The interpretation of the symmetry (or curl) condition [Eq. (23)] is that the partial derivative of $\mathbf{p}$ with respect to $\mathbf{x}$ is a symmetric matrix; that is, $\partial\mathbf{p}(\mathbf{x})/\partial\mathbf{x} = \partial\mathbf{p}^T(\mathbf{x})/\partial\mathbf{x}$. Since $V(\mathbf{x})$ is a scalar function and $\partial V^T(\mathbf{x})/\partial\mathbf{x} = \mathbf{p}(\mathbf{x})$, the second-order partial derivatives of V with respect to $\mathbf{x}$ must satisfy $\partial^2 V/\partial\mathbf{x}^2 = \partial^2 V^T/\partial\mathbf{x}^2$, corresponding to the requirement that the Hessian matrix of $V(\mathbf{x})$ must be symmetrical.

The well-known difficulty in formulating a Cauchy problem is that, even with smooth initial data, the solution to the HJB equation is generally nonsmooth. The existence of a smooth solution to the HJB PDE [Eq. (6)] by the method of characteristics discussed above breaks down when the asymptotically stable optimal trajectories (that is, the characteristic curves) start to cross one another (going backward in time) at a finite distance from the initial manifold. At such points, singularities develop in $\pi|_M$ (that is, in the projection of

M onto state space) and $\int \mathbf{p}\, d\mathbf{x}$ no longer gives a well-defined function of $\mathbf{x}$. However, the manifold M exists globally in phase space and, in general, covers a region $\tilde{\Omega}$ of state space strictly larger than Ω, with the corresponding graph $\tilde{L} \subset M$, where $L \subset \tilde{L}$. On this larger region $\tilde{\Omega}$, the solution becomes multivalued when thought of as a section of the cotangent bundle T^*M of M over state space. The resolution is that the HJB equation [Eq. (6)] does not in general have a smooth solution V existing for all times $t \geq 0$. Consequently, classical analysis of the first-order HJB problem by the method of characteristics is limited to only local considerations, owing to the crossing of characteristics.

Viscosity solutions provide a powerful tool for global analysis of first-order nonlinear HJB problems by constructing generalized (weak) solutions to the equations [114]. The basic idea is to approximate a nonsmooth solution by smooth functions at a given point and then require the smooth functions to all satisfy a HJB inequality. It is well known that the theory of viscosity solutions can be used to obtain approximate solutions of the HJB equation. For example, the finite-difference scheme [115] provides one approach to numerically solving this PDE for obtaining viscosity solutions. Unfortunately, this method works only for problems with low dimensions because the computational complexity increases exponentially with the dimension of the state. On the other hand, since the computational complexity of an ARE is only of polynomial growth rate with the state dimension, the SDRE approach provides the possibility of dealing with high-dimensional nonlinear systems. This is the main motivation for characterizing solutions by Riccati equations, which is the subject of the subsequent discussion.

B. Existence of Solutions Characterized by Riccati Equations

In the region Ω where $V(\mathbf{x})$ is a smooth (C^2) nonnegative solution to Eq. (6), the minimum is achieved by Eq. (22), so that by substitution, the HJB equation becomes

$$\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}) - \frac{1}{2}\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\frac{\partial V^T(\mathbf{x})}{\partial \mathbf{x}} + \frac{1}{2}\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x} = 0 \tag{24}$$

with $\partial V^T(\mathbf{x})/\partial \mathbf{x} = \mathbf{p}(\mathbf{x})$, and the corresponding optimal cost $V(\mathbf{x})$ is the solution to Eq. (24) with $V(\mathbf{0}) = 0$. Since $\partial V(\mathbf{0})/\partial \mathbf{x} = \mathbf{0}$ [18], analogous to $\mathbf{f}(\mathbf{x})$ by Condition 1 (see Proposition 1), $\partial V(\mathbf{x})/\partial \mathbf{x}$ can be written in SDC form

$$\frac{\partial V^T(\mathbf{x})}{\partial \mathbf{x}} = \mathbf{p}(\mathbf{x}) = \mathbf{P}(\mathbf{x})\mathbf{x} \tag{25}$$

for some matrix-valued function $\mathbf{P}$: $\Omega \to \mathbb{R}^{n\times n}$. In the subsequent analyses, the implications of Eq. (25) are explicitly examined, treating and clarifying some issues related to this equation. First, let us determine the infinite-time version of the nonlinear regulator, such that the optimal feedback control is given by Eq. (16).

Consider the most general case in which $\mathbf{P}(\mathbf{x})$ is not necessarily nonnegative definite, or even symmetric. Since $\mathbf{f}(\mathbf{x})$ and $\partial V(\mathbf{x})/\partial \mathbf{x}$ both can be written in the SDC form of Eqs. (11) and (25), respectively, substituting for $\mathbf{f}(\mathbf{x})$ and $\partial V(\mathbf{x})/\partial \mathbf{x}$, and noting that

$$2\mathbf{x}^T\mathbf{P}^T(\mathbf{x})\mathbf{A}(\mathbf{x})\mathbf{x} = \mathbf{x}^T[\mathbf{P}^T(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x})]\mathbf{x}$$

Eq. (24) becomes

$$\mathbf{x}^T[\mathbf{P}^T(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{P}^T(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x})]\mathbf{x} = 0 \tag{26}$$

In the linear case, the ARE [Eq. (9)] is obtained directly from this vector Riccati equation [Eq. (26)]. However, since $\mathbf{A}$ is a matrix-valued function of $\mathbf{x}$, setting the quantity inside the parenthesis in Eq. (26) to zero is not equivalent to solving the HJB equation [Eq. (24)]. This is due to the fact that, for an arbitrary factorization of the nonlinear vector function $\mathbf{f}(\mathbf{x})$, the additional degrees of freedom resulting from the nonuniqueness of the SDC matrix $\mathbf{A}(\mathbf{x})$ result in additional $\mathbf{P}(\mathbf{x})$ functions satisfying the quantity inside the parenthesis in Eq. (26), many of which do not satisfy Eq. (25) on $V(\mathbf{x})$. Recalling that the Hessian matrix of $V(\mathbf{x})$ must be symmetrical (see Lemma 4 and Remark 8), $\mathbf{P}(\mathbf{x})$ must additionally satisfy the symmetry condition $\partial^2 V/\partial \mathbf{x}^2 = \partial^2 V^T/\partial \mathbf{x}^2$ where, using Eq. (25),

$$\frac{\partial^2 V^T(\mathbf{x})}{\partial \mathbf{x}^2} = \frac{\partial}{\partial \mathbf{x}}[\mathbf{P}(\mathbf{x})\mathbf{x}] = \mathbf{P}(\mathbf{x}) + \frac{\partial \mathbf{P}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x} \tag{27}$$

Then, from Eq. (26), the following alternative form of the SDRE needs to be solved for unsymmetrical solution $\mathbf{P}(\mathbf{x})$:

$$\mathbf{P}^T(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{P}^T(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) + \mathbf{Q}(\mathbf{x}) = \mathbf{0} \tag{28}$$

This equation is of the same form as the standard SDRE [Eq. (15)], except that the required symmetry of $\mathbf{P}(\mathbf{x})$ has been relaxed [32,107]. It too has $n(n+1)/2$ equations, but contrary to Eq. (15), it has n^2 unknowns. In terms of the nonlinear SDC parameterization [Eq. (25)], using Eq. (27), the symmetry condition translates into the $n(n-1)/2$ scalar equations

$$P_{ij}(\mathbf{x}) + \sum_{k=1}^{n}\frac{\partial P_{ik}(\mathbf{x})}{\partial x_j}x_k = P_{ji}(\mathbf{x}) + \sum_{k=1}^{n}\frac{\partial P_{jk}(\mathbf{x})}{\partial x_i}x_k \tag{29}$$

which must be solved in conjunction with Eq. (28). Solving Eq. (28) with Eq. (29) is indeed equivalent to solving the HJB equation [Eq. (24)], such that, if $\mathbf{P}(\mathbf{x})$ is a solution to Eqs. (28) and (29), then $\mathbf{P}(\mathbf{x})\mathbf{x}$ is a solution to Eq. (24). This approach, referred to as the nonsymmetric-SDRE method, provides an alternative structured (Riccati equation) characterization to the HJB equation. However, the solution is as computationally difficult as solving Eq. (24).

Remark 9: Even though the nonsymmetric-SDRE method is based on extended linearization of the system, note that this approach is not an extended linearization control method since it does not guarantee that the closed-loop SDC matrix $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$, given by Eq. (14) with Eq. (18), is pointwise Hurwitz.

Let us now illustrate these theoretical results on an academic problem from [31,107] and demonstrate how the nonsymmetric-SDRE method provides an alternative approach to solving the HJB equation.

Example 2: Consider the analytical second-order nonlinear regulation problem $\dot{x}_1 = x_2$ and $\dot{x}_2 = -x_1 e^{x_1} + \frac{1}{2}x_2^2 + e^{x_1}u$ with the cost functional in Eq. (2), where $\mathbf{Q} = \mathrm{diag}\{0, 1\}$ and $r = 1$. It is easy to verify that the optimal control law for this example is given by $u_* = -x_2$ with the corresponding optimal cost (value) function $V(\mathbf{x}) = \frac{1}{2}(x_1^2 + x_2^2 e^{-x_1})$, which is evaluated with the initial state $\mathbf{x}(0) = \mathbf{x}_0$. For this simple problem, a closed-form analytical solution of the SDRE in Eq. (28) can be obtained. Selecting the obvious SDC parameterization, four symmetric $\mathbf{P}(\mathbf{x})$ solutions of Eq. (28) are obtained, only one of which is pointwise nonnegative definite, and represents the solution of SDRE [Eq. (15)] for the standard SDRE method. Additionally, two nonsymmetric $\mathbf{P}(\mathbf{x})$ solutions of Eq. (28) are obtained. The unique, symmetric nonnegative-definite solution and the two nonsymmetric solutions are given by

$$\mathbf{P}_1(\mathbf{x}) = \begin{bmatrix} e^{x_1} & 0 \\ 0 & 1 \end{bmatrix} p(\mathbf{x})$$

$$\mathbf{P}_2(\mathbf{x}) = \begin{bmatrix} c_1 & -\frac{1}{2}(c_1 x_2 e^{-x_1} + 1 - c_1^2) \\ -2e^{-x_1} & x_2 e^{-2x_1} - c_1 e^{-x_1} \end{bmatrix}$$

$$\mathbf{P}_3(\mathbf{x}) = \begin{bmatrix} c_1 & -\frac{1}{2}(c_1 x_2 e^{-x_1} + 1 - c_1^2) \\ 0 & c_1 e^{-x_1} \end{bmatrix}$$

where $p(\mathbf{x}) \triangleq \frac{1}{2}[x_2 + (x_2^2 + 4e^{2x_1})^{1/2}]e^{-2x_1}$, and c_1 is an arbitrary constant. However, the symmetric nonnegative-definite solution $\mathbf{P}_1(\mathbf{x})$ does not satisfy the symmetry condition in Eq. (29). Substituting $\mathbf{P}_2(\mathbf{x})$ into Eq. (29), the symmetry condition again cannot be satisfied for any c_1. However, upon substituting $\mathbf{P}_3(\mathbf{x})$ into Eq. (29), the symmetry condition is satisfied with $c_1 = 1$. Thus, $\mathbf{P}_3(\mathbf{x})$ with $c_1 = 1$ yields the optimal controller $u_* = -r^{-1}\mathbf{B}^T(\mathbf{x})\mathbf{P}_3(\mathbf{x})\mathbf{x} = -x_2$. Note also that $\mathbf{P}_3(\mathbf{x})\mathbf{x} = \partial V^T(\mathbf{x})/\partial\mathbf{x}$, which is easily verified by taking the partial derivatives of $V(\mathbf{x})$ with respect to $\mathbf{x}$.

Unfortunately, the complexity of the HJB equation [Eq. (24)] or the nonsymmetric-SDRE equations [Eqs. (28) and (29)] prevents any solution except in some very simple low-dimensional systems, such as that considered in Example 2. Basically, to compute the optimal controller associated with Eqs. (1) and (2), the HJB equation [Eq. (24)] must be solved with $V(\mathbf{0}) = 0$, and the feedback controller is then constructed from Eq. (22). However, to make real-time implementation possible, one has to avoid solving any PDE. This has prompted control design engineers to search for alternative, suboptimal approaches to the problem, such as the standard SDRE technique. This SDRE approach provides an approximation to the solution of Eq. (26), and thus the HJB equation [Eq. (24)], and yields a suboptimal feedback control law for the ITHNOC problem defined by Eqs. (1) and (2). Contrary to Eqs. (28) and (29), application of the SDRE algorithm as an approximation to the solution of Eq. (26) involves ignoring the requirement that $\mathbf{P}(\mathbf{x})\mathbf{x}$ be the gradient of some function $V(\mathbf{x})$ and assumes instead that $\mathbf{P}(\mathbf{x})$ is symmetric. Then, at any given $\mathbf{x}$, the SDRE algorithm consists of simply finding the symmetric nonnegative-definite solution $\mathbf{P}(\mathbf{x})$ to the algebraic SDRE [Eq. (15)] and applying, at that $\mathbf{x}$, the control [Eq. (16)]. This Riccati equation characterization is much more appealing and computationally efficient than solving the HJB equation [Eq. (24)] or the symmetry condition [Eq. (29)].

In the Riccati equation characterization discussed above, the solution $\mathbf{P}(\mathbf{x})$ is not required to be nonnegative definite, or even symmetric. However, the solution can be chosen as nonnegative definite, which is the case of interest for the standard SDRE method. Let us now justify, under a very mild condition, the existence and uniqueness of nonnegative-definite (and continuous) solutions to the SDRE in Eq. (28), and thus the standard SDRE [Eq. (15)].

Consider the SDRE in Eq. (28) and define a state-dependent Hamiltonian matrix $\mathbf{H}\colon \Omega \to \mathbb{R}^{2n\times 2n}$ as

$$\mathbf{H}(\mathbf{x}) \triangleq \begin{bmatrix} \mathbf{A}(\mathbf{x}) & -\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x}) \\ -\mathbf{C}^T(\mathbf{x})\mathbf{C}(\mathbf{x}) & -\mathbf{A}^T(\mathbf{x}) \end{bmatrix} \tag{30}$$

for which the eigenvalues are symmetric with respect to the imaginary axis. Now, by hyperbolicity under Hypothesis 1, the Hamiltonian matrix [Eq. (30)] has no eigenvalues on the imaginary axis. Therefore, an invariant manifold of the form in Eq. (21) exists, and Eq. (30) will have two complementary n-dimensional invariant eigenspaces $N_-(\mathbf{H}(\mathbf{x}))$ and $N_+(\mathbf{H}(\mathbf{x}))$: the former associated with all n eigenvalues with negative real parts and the latter associated with all n eigenvalues with positive real parts. The stable invariant eigenspace $N_-(\mathbf{H})$ can be expressed in the form

$$N_-(\mathbf{H}) = \operatorname{span}\begin{bmatrix} \mathbf{I} \\ \mathbf{P} \end{bmatrix}$$

In other words, $\mathbf{P}(\mathbf{x})$ spans the stable invariant eigenspace of $\mathbf{H}(\mathbf{x})$ [18,116]. Alternatively, denoting the function $\mathbf{H}\mapsto\mathbf{P}$ by $\Re$, and rephrasing in the terminology of Doyle et al. [117], means that an invariant manifold of the form in Eq. (21) exists if the state-dependent Hamiltonian matrix [Eq. (30)] of the Hamiltonian vector field [Eq. (20)] belongs to the domain of $\Re$, denoted $\operatorname{dom}(\Re)$; it follows that $\mathbf{P}(\mathbf{x}) \triangleq \Re(\mathbf{H}(\mathbf{x}))$. The following well-known result, which is essentially from Lemma 2.4 of Glover and Doyle [118], then gives the required uniqueness property of $\mathbf{P}(\mathbf{x})$.

Lemma 5: Consider the SDRE in Eq. (28) and the state-dependent Hamiltonian matrix defined by Eq. (30). The SDRE has a unique nonnegative-definite solution $\mathbf{P}(\mathbf{x}) \geq \mathbf{0}$ if and only if the state-dependent Hamiltonian defined by Eq. (30) is in $\operatorname{dom}(\Re)$; that is, $\mathbf{H}(\mathbf{x}) \in \operatorname{dom}(\Re)$ for each $\mathbf{x} \in \Omega$. Moreover, $\mathbf{P}(\mathbf{x}) = \Re(\mathbf{H}(\mathbf{x})) \geq \mathbf{0}$ is such a solution; the feedback control defined in Eq. (16) is stabilizing pointwise. In addition, if for each $\mathbf{x} \in \Omega$ the pair $\{\mathbf{C}(\mathbf{x}), \mathbf{A}(\mathbf{x})\}$ is observable, such that

$$\cap_{i=0}^{n-1} \ker(\mathbf{C}(\mathbf{x})\mathbf{A}^i(\mathbf{x})) = \emptyset$$

then this solution is positive definite; that is, $\mathbf{P}(\mathbf{x}) = \Re(\mathbf{H}(\mathbf{x})) > \mathbf{0}$.

The condition $\mathbf{H}(\mathbf{x}) \in \operatorname{dom}(\Re)$ requires that $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ be pointwise stabilizable. Hence, Lemma 5 implies that, under the (nonrestrictive) condition $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ is stabilizable for each $\mathbf{x} \in \Omega$, the SDRE in Eq. (28) has a unique pointwise nonnegative-definite solution $\mathbf{P}(\mathbf{x}) = \mathbf{P}^T(\mathbf{x}) \geq \mathbf{0}$ for each $\mathbf{x} \in \Omega$, which is indeed the corresponding nonnegative-definite solution to the standard SDRE [Eq. (15)]. Additionally, if $\{\mathbf{C}(\mathbf{x}), \mathbf{A}(\mathbf{x})\}$ is pointwise observable, then this solution is positive definite. The following result, from Theorem V.1 of Lu and Doyle [119], further shows that, if such a solution exists, then there exists a continuous one.

Lemma 6: Suppose the SDRE [Eqs. (15) and (28)] has a positive-definite solution $\mathbf{P}(\mathbf{x}) = \mathbf{P}^T(\mathbf{x}) > \mathbf{0}$ for each $\mathbf{x} \in \Omega$. Then, there exists a C^0 positive-definite matrix-valued function $\mathbf{P}\colon \Omega \to \mathbb{R}^{n\times n}$, such that $\mathbf{P}(\mathbf{x}) = \mathbf{P}^T(\mathbf{x}) > \mathbf{0}$ is a solution to the SDRE [Eqs. (15) and (28)] $\forall\ \mathbf{x} \in \Omega$.

The following lemma now shows that, if $V(\mathbf{x})$ with $V(\mathbf{0}) = 0$ satisfies $\partial V^T(\mathbf{x})/\partial\mathbf{x} = \mathbf{P}(\mathbf{x})\mathbf{x}$ for some positive (or nonnegative) definite matrix-valued function $\mathbf{P}\colon \Omega \to \mathbb{R}^{n\times n}$, then $V(\mathbf{x})$ is also a positive (or nonnegative) definite function on Ω (for the proof, see Technical Memorandum version of Lu and Doyle [120]).

Lemma 7: Let $V\colon \Omega \to \mathbb{R}$ with $V(\mathbf{0}) = 0$ be such that $\partial V^T(\mathbf{x})/\partial\mathbf{x} = \mathbf{P}(\mathbf{x})\mathbf{x}$ for some $\mathbf{P} = \mathbf{P}^T\colon \Omega \to \mathbb{R}^{n\times n}$. If $\mathbf{P}(\mathbf{x}) > \mathbf{0}$ $\forall\ \mathbf{x} \in \Omega$, then $V(\mathbf{x}) > 0\ \forall\ \mathbf{x} \in \Omega\backslash\{0\}$; if $\mathbf{P}(\mathbf{x}) \geq \mathbf{0}\ \forall\ \mathbf{x} \in \Omega$, then $V(\mathbf{x}) \geq 0\ \forall\ \mathbf{x} \in \Omega$.

Even though the representation in Eq. (25) of $\partial V(\mathbf{x})/\partial\mathbf{x}$ is not unique, the following theorem from Huang and Lu [108] essentially follows using the preceding results.

Theorem 1: If V is a positive-definite solution of the HJB equation [Eq. (24)], then there exists at most one C^0 positive-definite matrix-valued function $\mathbf{P}(\mathbf{x})$ such that Eq. (25) is satisfied.

Proof: The proof follows from Lemma 5, Lemma 6 and Lemma 7. □

From Sec. V.A, recall that the HJB PDE associated with the ITHNOC problem does not in general have a smooth solution V existing for all times $t \geq 0$. Consequently, the HJB PDE must be solved in a viscosity sense, which is impractical for finding any use in control applications. On the other hand, using the preceding results, the existence of nonnegative-definite solutions characterized by SDREs can be guaranteed under the very mild condition of pointwise stabilizability. Although the preceding derivation of the SDRE control law takes place in the region Ω where $V(\mathbf{x})$ is smooth, it can clearly be applied independently of this assumption.

C. Existence of an Optimum Positive-Definite State-Dependent Riccati Equation Solution

The existence and uniqueness of a nonnegative-definite solution to the SDRE in Eq. (28), hence the standard SDRE [Eq. (15)], have been justified in Sec. V.B. In this case, the vector Riccati equation [Eq. (26)] simplifies to

$$\begin{aligned}\mathbf{x}^T[&\mathbf{P}(\mathbf{x})\mathbf{A}(\mathbf{x}) + \mathbf{A}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) - \mathbf{P}(\mathbf{x})\mathbf{B}(\mathbf{x})\mathbf{R}^{-1}(\mathbf{x})\mathbf{B}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) \\ &+ \mathbf{Q}(\mathbf{x})]\mathbf{x} = 0\end{aligned} \tag{31}$$

However, in Sec. III.F, it is shown in Lemma 3 that the state-dependent parameterization is not unique, and there are an infinite number of representations given by Proposition 2. Huang and Lu [108] have shown that, with a certain type of value function, there will always exist a SDC parameterization such that the SDRE feedback produces the optimal feedback control law. Let us now formally state this property.

Theorem 2: Under Condition 1 and Condition 2, if the value function $V(\mathbf{x})$ has gradient of the form in Eq. (25) with $\mathbf{P}(\mathbf{x}) = \mathbf{P}^T(\mathbf{x})$ for some positive definite $\mathbf{P}\colon \Omega \to \mathbb{R}^{n\times n}$, then there always exists a parameterization [Eq. (11)] such that $\mathbf{P}(\mathbf{x})$ is the stabilizing solution of the SDRE [Eq. (15)] that gives the optimal feedback controller [Eq. (16)] $\forall\ \mathbf{x} \in \Omega$.

Proof: (For details, see [108].) Suppose $\mathbf{A}(\mathbf{x})$ satisfies Eq. (11). Then, by Lemma 3, all the possible state matrices $\mathbf{A}_0(\mathbf{x})$ with $\mathbf{f}(\mathbf{x}) = \mathbf{A}_0(\mathbf{x})\mathbf{x}$ can be parameterized as $\mathbf{A}_0(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x})$ with $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{0}$. In this case, the HJB PDE [Eq. (24)] becomes $\mathbf{x}^T\mathbf{S}(\mathbf{x})\mathbf{x} = 0 \;\; \forall\; \mathbf{x} \in \Omega$, where $\mathbf{S}(\mathbf{x})$ is equal to the bracketed expression in Eq. (31), with $\mathbf{A}(\mathbf{x})$ replaced by $\mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x})$, as in Eq. (19). Hence, the SDRE [Eq. (15)] is equivalent to the HJB equation [Eq. (24)] if and only if $\mathbf{E}(\mathbf{x})$ can be found such that $\mathbf{S}(\mathbf{x}) + \mathbf{P}(\mathbf{x})\mathbf{E}(\mathbf{x}) + \mathbf{E}^T(\mathbf{x})\mathbf{P}(\mathbf{x}) = \mathbf{0} \;\; \forall\; \mathbf{x} \in \Omega$. Since $\mathbf{S}(\mathbf{x})$ is symmetric, $\mathbf{E}(\mathbf{x})$ can be parameterized as $\mathbf{E}(\mathbf{x}) = -\frac{1}{2}\mathbf{P}^{-1}(\mathbf{x})[\mathbf{S}(\mathbf{x}) - \mathbf{T}(\mathbf{x})]$ for some skew-symmetric matrix $\mathbf{T}(\mathbf{x})$; that is, $\mathbf{T}(\mathbf{x}) = -\mathbf{T}^T(\mathbf{x})$. Since $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{0}$, $\mathbf{T}(\mathbf{x})$ must satisfy $[\mathbf{S}(\mathbf{x}) - \mathbf{T}(\mathbf{x})]\mathbf{x} = \mathbf{0} \;\; \forall\; \mathbf{x} \in \Omega \subseteq \mathbb{R}^n$. However, it is always possible to construct such $\mathbf{T}(\mathbf{x})$ in ways, for example, as shown in [108]. Therefore, $\mathbf{E}(\mathbf{x})$ defined above can be obtained for all $\mathbf{x} \in \Omega$. □

Theorem 2 confirms that the global optimal controller can always be formed from the positive-definite solution to the SDRE [Eq. (15)] if the gradient of the value function $V(\mathbf{x})$ has the form $\mathbf{P}(\mathbf{x})\mathbf{x}$ with $\mathbf{P}(\mathbf{x}) = \mathbf{P}^T(\mathbf{x})$ and the right $\mathbf{A}(\mathbf{x})$ is chosen. Although there are multiple solutions to an ARE, there is at most one solution that gives the optimal performance for both the original system (that is, the HJB equation) and the SDRE system. From Lemma 5, this corresponds to the positive-definite solution of the SDRE [Eq. (15)], which gives a pointwise stabilizing state-feedback solution. Under the conditions $\mathbf{H}(\mathbf{x}) \in \mathrm{dom}(\Re)$ and $\cap_{i=0}^{n-1} \ker(\mathbf{C}(\mathbf{x})\mathbf{A}^i(\mathbf{x})) = \emptyset$ for each $\mathbf{x} \in \Omega$, $\mathbf{P}(\mathbf{x})$ is uniquely determined by $\mathbf{H}(\mathbf{x})$, with $\mathbf{P}(\mathbf{x}) = \Re(\mathbf{H}(\mathbf{x})) > \mathbf{0}$. Therefore, since $\mathbf{P}(\mathbf{x})$ in Eq. (25) is positive definite, then with a right choice of $\mathbf{A}(\mathbf{x})$ in Eq. (12) for representing Eq. (1), the unique positive-definite solution of Eq. (15), which is thus $\mathbf{P}(\mathbf{x})$, recovers the optimal.

While there always exists some choice of SDC parameterization in which the SDRE recovers the global optimal for the original ITHNOC problem, finding the right representation in SDC form using the above result is very difficult since the value function $V(\mathbf{x})$ is assumed to be known a priori. The following example from [108] illustrates that the optimal choice of $\mathbf{A}(\mathbf{x})$ may not always be the seemingly likely one.

Example 3: Consider the optimal regulation problem for the nonlinear system $\dot{x}_1 = -x_1 + x_1x_2^2$ and $\dot{x}_2 = -x_2 + x_1u$ and the cost functional in Eq. (2) with $\mathbf{Q} = 4\mathbf{I}_{2\times2}$ and $r = 1$. The solution of the HJB equation [Eq. (24)] for this problem is given by the value function $V(\mathbf{x}) = x_1^2 + x_2^2$. The optimal feedback is constructed using Eq. (22), giving $u_*(\mathbf{x}) = -2x_1x_2$. Now, consider the following two SDC matrices:

$$\mathbf{A}_1(\mathbf{x}) = \begin{bmatrix} -1 + x_2^2 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \mathbf{A}_2(\mathbf{x}) = \begin{bmatrix} -1 & x_1x_2 \\ 0 & -1 \end{bmatrix}$$

If $\mathbf{A}_1(\mathbf{x})$ is chosen, then the pair $\{\mathbf{A}_1(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ is not controllable for any $\mathbf{x}$ since $\det(\mathbf{M}_C) = 0 \;\forall\; \mathbf{x}$, and the SDRE [Eq. (15)] fails to give a positive-definite solution at any point. If $\mathbf{A}_2(\mathbf{x})$ is chosen, then the SDRE defines a pointwise stabilizing controller. However, it is not optimal. Surprisingly, the optimal choice $\mathbf{A}_*(\mathbf{x})$ for this system is obtained by parameterizing either $\mathbf{A}_1(\mathbf{x})$ or $\mathbf{A}_2(\mathbf{x})$ in the form

$$\mathbf{A}_*(\mathbf{x}) = \mathbf{A}_1(\mathbf{x}) + \mathbf{E}_1(\mathbf{x}) = \mathbf{A}_2(\mathbf{x}) + \mathbf{E}_2(\mathbf{x})$$

using Lemma 3, with

$$\mathbf{E}_1(\mathbf{x}) = \begin{bmatrix} -x_2^2 & x_1x_2 \\ -x_1x_2 & x_1^2 \end{bmatrix}, \qquad \mathbf{E}_2(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ -x_1x_2 & x_1^2 \end{bmatrix}$$

$$\mathbf{A}_*(\mathbf{x}) = \begin{bmatrix} -1 & x_1x_2 \\ -x_1x_2 & -1 + x_1^2 \end{bmatrix}$$

where $\mathbf{E}_1(\mathbf{x})\mathbf{x} = \mathbf{E}_2(\mathbf{x})\mathbf{x} = \mathbf{0}$. Note that $\mathbf{A}_*(\mathbf{x})$ is hard to guess from the dynamics of the plant unless the optimal cost is known, and it cannot even be obtained by constructing $\mathbf{A}(\mathbf{x}, \alpha)$ from the given two distinct SDC matrices. Note also that, using $\mathbf{A}_*(\mathbf{x})$, the SDRE [Eq. (15)] yields a constant (state-independent) Riccati matrix solution $\mathbf{P} = \mathbf{P}^T = 2\mathbf{I}_{2\times2} > \mathbf{0}$, such that $\mathbf{P}\mathbf{x}$ is indeed equal to the gradient of $V(\mathbf{x})$, $\partial V(\mathbf{x})/\partial\mathbf{x}$.

Remark 10: For an arbitrary positive-definite value function $V\colon\, \Omega \to \mathbb{R}^+$, there may not exist any positive definite $\mathbf{P}(\mathbf{x})$ such that Eq. (25) is satisfied. In this case, no parameterization [Eq. (11)] will exist such that $\mathbf{P}(\mathbf{x}) > \mathbf{0}$ is the solution of the SDRE [Eq. (15)], which gives the optimal feedback controller $\forall\; \mathbf{x} \in \Omega$, as shown in the following example.

Example 2 revisited: Let us reconsider Example 2, which was used in Sec. V.B to illustrate the nonsymmetric-SDRE method. Cloutier et al. [107], based on Lemma 3, showed that there exists no SDC parameterization of this problem that will yield a closed-loop dynamics matrix $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$ that is Hurwitz everywhere, meaning that, for this particular example, the optimal control law cannot be globally recovered by the SDRE method, or any other extended linearization control method for that matter. Even though the feedback law can be recovered by using the unsymmetrical solution $\mathbf{P}_3(\mathbf{x})$ of the SDRE in Eq. (28), which simultaneously satisfies the symmetry condition [Eq. (29)], this approach is not an extended linearization control method as noted in Remark 9. Moreover, although $\mathbf{P}_3(\mathbf{x})\mathbf{x}$ yields the gradient of $V(\mathbf{x})$, note that $\mathbf{P}_3(\mathbf{x}) \neq \mathbf{P}_3^T(\mathbf{x})$. Thus, from Theorem 2, no parameterization [Eq. (11)] for this particular problem exists such that $\mathbf{P}_3(\mathbf{x})$ is the solution of the SDRE [Eq. (15)], which gives the optimal feedback controller $\forall\; \mathbf{x} \in \Omega$. It is also worth mentioning that various methods applied to this academic problem in order to achieve optimal performance use a combination of different methods [31] (for example, receding-horizon control together with either SDRE or feedback linearization), whereas Jacobian linearization, in this particular case, acquires this objective single-handedly.

Remark 11: Although Huang and Lu [108] showed that, under certain conditions, there will always exist an optimum SDC matrix $\mathbf{A}_*(\mathbf{x})$, Cloutier et al. [107] derived the necessary and sufficient conditions on $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ for the existence of any feedback gain matrix $\mathbf{K}(\mathbf{x})$ that results in Eq. (14) being pointwise Hurwitz, such that the SDRE feedback produces the optimal feedback control law. Additionally, Shamma and Cloutier [109] showed that, if there exists any stabilizing state-feedback control law $\mathbf{u}(\mathbf{x}) = -\mathbf{k}(\mathbf{x}) = -\mathbf{K}(\mathbf{x})\mathbf{x}$ given by Eqs. (3) and (13) that admits a certain type of Lyapunov function (with star-convex level sets), then there always exists a SDC parameterization of the open-loop state dynamics [Eq. (1)] such that the closed-loop dynamics (SDC) matrix [Eq. (14)], represented in the form $\mathbf{A}(\mathbf{x}) + \mathbf{E}(\mathbf{x}) - \mathbf{B}(\mathbf{x})\mathbf{K}(\mathbf{x})$, is pointwise Hurwitz, where $\mathbf{E}(\mathbf{x})\mathbf{x} = \mathbf{0}$. Under the same assumptions, a stabilizing state-dependent feedback gain $\mathbf{K}(\mathbf{x})$ can be found by solving a Riccati equation. Although these results do not suggest how to derive an appropriate SDC parameterization, the implications are that, if a system is stabilizable (under an appropriate Lyapunov function), then it is stabilizable via the SDRE method.

VI. Optimality Analyses

This section reviews the main optimality and suboptimality properties of SDRE nonlinear regulators associated with ITHNOC problems. Large-scale asymptotic optimality properties of SDRE feedback controls are reviewed in Sec. VI.A, whereas global optimality results are reviewed in Sec. VI.B. Throughout the analyses, Condition 1 through Condition 4 (set out in Sec. IV) are assumed to hold, so that $\mathbf{P}(\mathbf{x}) \geq \mathbf{0}$ exists for all $\mathbf{x}$.

A. Large-Scale Asymptotic Optimality

Since, by Lemma 1, $\mathbf{A}(\mathbf{x}) \to \partial\mathbf{f}(\mathbf{0})/\partial\mathbf{x}$ as $\mathbf{x} \to \mathbf{0}$, $\mathbf{P}(\mathbf{x})$ tends to the solution of the ARE [Eq. (9)] for the linearized problem [Eq. (8)] at the origin. Hence, in a sufficiently small region Ω_0 around the origin, the feedback from the SDRE control law is arbitrarily close to the optimal feedback. This approximation is asymptotically optimal, in that it converges to the optimal control close to the origin as $\mathbf{x} \to \mathbf{0}$. In [3,6], optimality of the SDRE method has been addressed in the region Ω of state space strictly larger than Ω_0 by considering the necessary conditions for optimality of the nonlinear regulator [Eqs. (1) and (2)]. In this section, the main results are presented.

From Pontryagin's minimum principle, the necessary conditions for optimality are given by Eqs. (7) and (20), where H is the Hamiltonian function defined in Eq. (5), with $\mathbf{p} = \partial V^T/\partial \mathbf{x}$ from Eq. (21). Hence, from Eqs. (7) and (16),

$$\frac{\partial H^T}{\partial \mathbf{u}} = \mathbf{B}^T(\mathbf{x})[\mathbf{p} - \mathbf{P}(\mathbf{x})\mathbf{x}] \tag{32}$$

Since the costate vector satisfies $\mathbf{p} = \mathbf{P}(\mathbf{x})\mathbf{x}$ from Eq. (25), substituting for $\mathbf{p}$ above thus gives $\partial H^T/\partial \mathbf{u} = \mathbf{0}$. Therefore, the SDRE feedback solution always satisfies the necessary condition [Eq. (7)] of the ITHNOC problem [Eqs. (1) and (2)]. Now, taking the partial derivatives of the Hamiltonian function [Eq. (5)] with respect to $\mathbf{x}$, the necessary condition on the costate $\dot{\mathbf{p}} = -\partial H^T/\partial \mathbf{x}$ becomes

$$\dot{\mathbf{p}} = -\frac{\partial \mathbf{f}^T(\mathbf{x})}{\partial \mathbf{x}}\mathbf{p} - \mathbf{u}^T\frac{\partial \mathbf{B}^T(\mathbf{x})}{\partial \mathbf{x}}\mathbf{p} - \mathbf{Q}(\mathbf{x})\mathbf{x} - \frac{1}{2}\mathbf{x}^T\frac{\partial \mathbf{Q}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{x} - \frac{1}{2}\mathbf{u}^T\frac{\partial \mathbf{R}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{u} \tag{33}$$

Differentiating $\mathbf{p} = \mathbf{P}(\mathbf{x})\mathbf{x}$ with respect to time gives $\dot{\mathbf{p}} = \dot{\mathbf{P}}(\mathbf{x})\mathbf{x} + \mathbf{P}(\mathbf{x})\dot{\mathbf{x}}$. Substituting this into Eq. (33) with Eqs. (11), (16), and (17), rearranging terms, using SDRE [Eq. (15)], and dropping the argument $\mathbf{x}$ for notational simplicity gives

$$\dot{\mathbf{P}}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\frac{\partial \mathbf{Q}}{\partial \mathbf{x}}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial \mathbf{x}}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}\mathbf{x} + \mathbf{x}^T\frac{\partial \mathbf{A}^T}{\partial \mathbf{x}}\mathbf{P}\mathbf{x} - \mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{B}^T}{\partial \mathbf{x}}\mathbf{P}\mathbf{x} = \mathbf{0} \tag{34}$$

This equation is related to requiring the solution to follow the optimal costate vector trajectory, and it is called the SDRE necessary condition for optimality [3,6], which must be satisfied for a strong local minimum with respect to Eq. (2). Therefore, whenever this condition is satisfied, the SDRE closed-loop solution satisfies all of the necessary conditions of the minimum principle, since $\partial H^T/\partial \mathbf{u} = \mathbf{0}$ and $\dot{\mathbf{x}} = \partial H^T/\partial \mathbf{p}$ are also satisfied.

In the case of infinite-time LQR, the left-hand side of Eq. (34) collapses to zero. In general, however, the SDRE necessary condition for optimality is not satisfied for an arbitrary SDC matrix $\mathbf{A}(\mathbf{x})$ in the multivariable case. Nevertheless, a suboptimality property of SDRE solutions can be revealed for any fixed $\mathbf{A}(\mathbf{x})$, which additionally requires the following boundedness conditions.

Hypothesis 2: Suppose $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{P}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$, and $\mathbf{R}(\mathbf{x})$ are C^1 functions and that these SDC matrices and their Jacobians are bounded in an arbitrarily large open ball B_r in Ω centered at the origin with radius $r < \infty$.

Theorem 3: In the general multivariable case ($n > 1$), the SDRE nonlinear feedback solution and its associated state and costate trajectories satisfy the necessary condition $\partial H^T/\partial \mathbf{u}_* = \mathbf{0}$ in Eq. (7) for optimality of the ITHNOC problem [Eqs. (1) and (2)]. Additionally, if Hypothesis 2 holds, under asymptotic stability, as the state $\mathbf{x}$ is driven to zero, the necessary condition on the costate $\dot{\mathbf{p}} = -\partial H_*^T/\partial \mathbf{x}$ in Eq. (20) is asymptotically satisfied at a quadratic rate.

Proof: The first assertion immediately follows from Eq. (32) with Eq. (25). The latter part follows directly from the SDRE necessary condition for optimality [Eq. (34)] as follows. Expanding $\dot{\mathbf{P}}$ yields

$$\dot{\mathbf{P}}(\mathbf{x})\mathbf{x} = \sum_{i=1}^{n} \partial \mathbf{P}(\mathbf{x})/\partial x_i \dot{x}_i \mathbf{x}$$

and so

$$\dot{\mathbf{P}}(\mathbf{x})\mathbf{x} = \sum_{i=1}^{n} \partial \mathbf{P}(\mathbf{x})/\partial x_i [(a_{\mathrm{CL}})_i \mathbf{x}]\mathbf{x}$$

where $(a_{\mathrm{CL}})_i$ is the ith row of the closed-loop coefficient matrix $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$ defined by Eq. (14). Rewriting this as $\mathbf{x}^T\mathbf{N}_i\mathbf{x}$, where the elements of $\mathbf{N}_i$ are functions of the elements of $\partial \mathbf{P}(\mathbf{x})/\partial x_i$ and $(a_{\mathrm{CL}})_i$, $i = 1, \ldots, n$, and substituting this result into Eq. (34) yields $\mathbf{x}^T\mathbf{M}_i\mathbf{x} = \mathbf{0}$, where

$$\mathbf{M}_i \triangleq \mathbf{N}_i + \frac{1}{2}\frac{\partial \mathbf{Q}}{\partial \mathbf{x}} + \frac{1}{2}\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{R}}{\partial \mathbf{x}}\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P} + \frac{\partial \mathbf{A}^T}{\partial \mathbf{x}}\mathbf{P}\mathbf{x} - \mathbf{x}^T\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\frac{\partial \mathbf{B}^T}{\partial \mathbf{x}}\mathbf{P}$$

for which the elements are functions of the elements of $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{P}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$, and $\mathbf{R}(\mathbf{x})$ as well as their Jacobians. From boundedness assumptions on the functions in Hypothesis 2, under asymptotic stability, there exists a constant positive-definite matrix such that

$$\max_i |\mathbf{x}^T\mathbf{M}_i\mathbf{x}| \leq \mathbf{x}^T\mathbf{U}\mathbf{x}$$

$\forall\ \mathbf{x} \in \Omega$. Thus, the ∞ norm of the left-hand side of Eq. (34) is bounded above by a quadratic function of $\mathbf{x}$, and so the result follows. □

Theorem 3 presents a desirable suboptimality property of SDRE nonlinear regulation for the general multivariable case. The theorem also gives rise to a phenomenon observed in numerous applications of the SDRE method; namely, as the states are driven to zero, the SDRE control trajectories converge, at a quadratic rate, to the optimal control trajectories, with the latter being obtained by iterative numerical methods.

B. Global Optimality

The following theorem reveals when the SDRE [Eq. (15)] gives the global optimal solution and the optimal cost for a given SDC parameterization.

Theorem 4: If the vector-valued function [Eq. (25)] satisfies the symmetry condition [Eq. (23)], then Eq. (16) is the optimal state feedback for the ITHNOC problem [Eqs. (1) and (2)], and the value function is defined by

$$V(\mathbf{x}) = \mathbf{x}^T\int_0^1 t\mathbf{P}(t\mathbf{x})\,\mathrm{d}t\mathbf{x}$$

$\mathbf{x} \geq 0$. In addition, if $\mathbf{P}$: $\Omega \to \mathbb{R}^{n\times n}$ is a positive-definite matrix-valued function, then $V(\mathbf{x})$ is also positive definite on Ω.

Proof: The result follows directly from Lemma 4 and Lemma 7.□

Example 4: Consider a Van der Pol-type system governed by the equations $\dot{x}_1 = x_2$ and $\dot{x}_2 = (1 - \varepsilon x_1^2)x_2 - x_1 + u$ [2,110]. The free (unforced) system has strong limiting trajectories, which are well known. The control signal u is selected to minimize Eq. (2) with $\mathbf{Q} = \mathbf{I}_{2\times 2}$ and $r = 1$. For illustrative purposes, let us select the SDC matrix $\mathbf{A}(\mathbf{x}) = [a_{ij}(\mathbf{x})]$ with $a_{11} = 0$, $a_{12} = 1$, $a_{21} = -1$, and $a_{22} = 1 - \varepsilon x_1^2$. Noting that $\mathbf{Q} > \mathbf{0}$, the positive-definite solution of Eq. (15) for $\mathbf{P}(\mathbf{x}) = [P_{ij}(\mathbf{x})]$ yields $P_{11}(x_1) = a_{22} + \beta\gamma$, $P_{12} = \beta - 1$, and $P_{22}(x_1) = a_{22} + \gamma$, where $\beta \triangleq \sqrt{2}$, $\gamma \triangleq (a_{22}^2 + 2\sqrt{2} - 1)^{1/2}$. The resultant SDRE control law is then obtained from Eq. (16), in closed form, giving

$$u_{\mathrm{SDRE}} = -[P_{12}x_1 + P_{22}(x_1)x_2] \tag{35}$$

with the closed-loop system equations $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\beta x_1 - \gamma x_2$. Using the Lyapunov function $V(\mathbf{x}) = \frac{1}{2}(\beta x_1^2 + x_2^2)$ gives $\dot{V}(\mathbf{x}) = -\gamma x_2^2$, which is sufficient to deduce global asymptotic stability of the SDRE-controlled system $\forall\ \mathbf{x} \in \mathbb{R}^2$.

The Van der Pol-type system considered in this example has been treated in several suboptimal studies in the literature, where comparisons of different techniques have been performed for small initial conditions [121–124]. Therefore, it offers an ideal example to compare the results using the SDRE method to results from previously suggested suboptimal methods.

It is desired to examine the effects of initial conditions on these methods by progressively moving away into large regions around the origin. Four initial states are therefore investigated: 1) $[1\quad 1]^T$ and 2) $[3\quad 3]^T$, corresponding to the problems in [121,124]; 3) $[4.5\quad 4.5]^T$, representing an intermediate but more stringent requirement for the controllers [123]; and 4) $[10\quad 10]^T$, a very

Table 1 Comparison of different methods for the Van der Pol-type control system in Example 4

Method	Control law	Normalized cost (J/J_*) $\mathbf{x}_0 = [1 \quad 1]^T$	$\mathbf{x}_0 = [3 \quad 3]^T$	$\mathbf{x}_0 = [4.5 \quad 4.5]^T$	$\mathbf{x}_0 = [10 \quad 10]^T$
Perturbation [125]	$-(0.414x_1 + 2.685x_2 - 1.086x_1^2x_2 - 0.583x_1x_2 - 0.072x_2^3)$	1.09	∞ (unstable)	∞ (unstable)	∞ (unstable)
HJB series approximation [121]	$-(0.414x_1 + 2.685x_2 - 1.586x_1^2x_2 + 0.194x_1^4x_2)$	1.03	1.36	5.80	∞ (unstable)
Parameter optimization [126]	$-(1.422x_1 + 3.080x_2 - 0.219x_2^3)$	1.22	1.23	1.22	1.43
Linear approximation by arbitrary hyperplane [123]	$-(a_{20} + k_{12}x_1 + k_{22}x_2)$, $k_{12} = a_{21} + (a_{21}^2 + 1)^{1/2}$, $k_{22} = a_{22} + (a_{22}^2 + 2k_{12} + 1)^{1/2}$, $a_{20} = \frac{1}{4}x_1^2x_2$, $a_{21} = -(\frac{1}{2}x_1x_2 + 1)$, $a_{22} = -(\frac{1}{3}x_1^2 - 1)$	1.02	1.07	1.09	1.11
ASREs [127]	Iterative sequence	1.02	1.02	1.03	1.04
LQR (linearization)	$-(0.414x_1 + 2.685x_2)$	1.05	1.22	1.10	1.01
SDRE	$-(0.414x_1 + \{(1 - x_1^2) + [1.828 + (1 - x_1^2)^2]^{1/2}\}x_2)$	1.04	1.10	1.05	1.00
Optimal [127]	Iterative sequence	1	1	1	1

stringent starting point in order to demonstrate the performance of the SDRE controller compared with other suboptimal methods. Let us therefore illustrate the degree of correspondence between the optimal trajectories and the suboptimal trajectories obtained using various techniques and compare them with SDRE solutions.

Table 1 presents a list of the several suboptimal control laws that have been derived for this particular system by using various well-established methods, which are performed by setting the parameter $\varepsilon = 1$, in accordance with previous studies. The optimal control for this system has also been deduced numerically, using the technique set out in [127], which provides explicit solutions based on Pontryagin's minimum principle. Normalized costs of the suboptimal control laws corresponding to each initial condition, which are obtained after steady state is reached, are also given in Table 1 as a ratio to the optimal cost for better comparison. The parameter optimization solution calculated by Durbeck [126] was for initial conditions $[1.75 \quad -2.0]^T$. To see how well this control performed for a variety of initial conditions, it was tested using initial conditions different from those for which the control law was originally computed [121].

Table 1 shows an excellent correlation between the optimal and suboptimal costs, which correspond to various nonlinear control laws for each of the starting points. All the methods provide satisfactory performance for small initial conditions but, with the exception of LQR, approximating sequence of Riccati equations (ASREs), and SDRE control laws, all investigated methods lead to undesirable costs for large initial conditions. In particular, for progressively larger initial conditions, the SDRE controller begins to outperform all other previous designs in this area, including the newly added ASRE method. The strength of the method in this particular situation becomes evident by analyzing the optimality of the system with the control law given by u_{SDRE} for any $\varepsilon \in \mathbb{R}$. From Theorem 4, if the vector-valued function [Eq. (25)] satisfies the symmetry (or curl) condition [Eq. (23)], then Eq. (16) gives the optimal state feedback for the ITHNOC problem [Eqs. (1) and (2)]. Since

$$\mathbf{p}(\mathbf{x}) = \mathbf{P}(\mathbf{x})\mathbf{x} = \begin{bmatrix} P_{11}(x_1)x_1 + P_{12}x_2 \\ P_{12}x_1 + P_{22}(x_1)x_2 \end{bmatrix}$$

applying Eq. (23) to analyze the optimality of the SDRE control law gives

$$\frac{\partial p_1(\mathbf{x})}{\partial x_2} = P_{12} \quad \text{and} \quad \frac{\partial p_2(\mathbf{x})}{\partial x_1} = P_{12} - \frac{2\varepsilon x_1 x_2 P_{22}(x_1)}{\gamma}$$

Hence, the SDRE controller u_{SDRE} is optimal when 1) $x_1 = 0 \ \forall \ x_2$, $\varepsilon \in \mathbb{R}$; 2) $x_2 = 0 \ \forall \ x_1$, $\varepsilon \in \mathbb{R}$; or 3) $\varepsilon = 0 \ \forall \ \mathbf{x} \in \mathbb{R}^2$, the latter corresponding to the trivial case associated with a LQ problem. Furthermore, with fast regulation of either of the states to the origin, note that $\partial p_1(\mathbf{x})/\partial x_2 \approx \partial p_2(\mathbf{x})/\partial x_1$, and so u_{SDRE} generates near-optimal feedback. This supports the close agreement of the optimal and SDRE costs (even for larger initial starting points), and it verifies the large-scale asymptotic optimality of the SDRE methodology for this particular problem, an attribute that also holds in general.

Therefore, provided that $\partial[\mathbf{P}(\mathbf{x})\mathbf{x}]/\partial\mathbf{x}$ is a symmetric matrix, the SDRE control [Eq. (16)] is in fact optimal with respect to Eq. (2). While this condition is true in the scalar case for $n = 1$, it does not hold in general for higher-dimensional systems. The next result uses the former fact to highlight a unique property of the SDRE method for nonlinear regulation.

Theorem 5: For scalar systems ($x \in \mathbb{R}$), the SDRE feedback controller of the ITHNOC problem [Eqs. (1) and (2)] is always (globally) optimal on $\mathbb{R}$.

Proof: In the case of scalar x, the symmetry condition [Eq. (23) or Eq. (29)] is always satisfied. Hence, the solution is optimal. Alternatively, optimality can be shown using the SDRE necessary condition for optimality [Eq. (34)]. However, this requires detailed algebraic manipulations, which are omitted for brevity. □

Corollary 1: For scalar systems, the globally optimal nonnegative-definite SDRE solution and SDRE feedback control law for the ITHNOC problem [Eqs. (1) and (2)] are, respectively, given by

$$p(x) = \frac{r(x)}{b^2(x)}\left\{\frac{f(x)}{x} + \left[\frac{f^2(x)}{x^2} + \frac{1}{r(x)}b^2(x)q(x)\right]^{1/2}\right\} \tag{36}$$

$$u(x) = -\frac{1}{b(x)}\left\{f(x) + \operatorname{sgn}(x)\left[f^2(x) + \frac{1}{r(x)}b^2(x)x^2q(x)\right]^{1/2}\right\} \tag{37}$$

Proof: Using lowercase notation, by Lemma 2, there exists only one SDC parameterization in the scalar case, with $a(x) = f(x)/x$. Hence, the SDRE [Eq. (15)] is given by $2pf(x)/x - p^2b^2(x)/r(x) + q(x) = 0$, which has the nonnegative-definite solution in Eq. (36). Substituting this into Eq. (16) gives the result in Eq. (37). □

Therefore, in the scalar case, even when the performance index [Eq. (2)] is nonquadratic in $\mathbf{x}$ (that is, $\mathbf{Q}$ and $\mathbf{R}$ are state dependent), the SDRE method produces the optimal solution of the ITHNOC problem [Eqs. (1) and (2)] in feedback form, which is given by Eq. (37).

Remark 12: In Corollary 1, note that while x denominator terms are present in Eq. (36), there are no such terms present in the SDRE controller expression [Eq. (37)]. Thus, if the SDRE can be solved analytically, it is not a requirement that

$$\lim_{x\to 0} f(x)/x$$

be finite. This is also true for multivariable problems in general; that is, if the SDRE can be solved analytically,

$$\lim_{\|\mathbf{x}\|\to 0} \mathbf{A}(\mathbf{x})$$

does not have to be finite as long as

$$\lim_{\|\mathbf{x}\|\to 0} \mathbf{A}(\mathbf{x})\mathbf{x}$$

is finite. This is due to the fact that, in the calculation of the control [Eq. (16)], the matrix solution to the SDRE [Eq. (15)] is multiplied by $\mathbf{x}$, which cancels the $\mathbf{x}$ denominator terms in $\mathbf{P}(\mathbf{x})$. However, if $\mathbf{A}(\mathbf{x})$ is unbounded for some zero-valued state component, then the SDRE must be solved analytically. If the SDRE is solved numerically, indeed, ill-conditioning may occur as the state approaches zero.

Example 5: Consider the scalar problem of minimizing Eq. (2) with respect to x and u subject to the constraint $\dot{x} = x - x^3 + u$. For this problem, $f(x) = x - x^3$, $a(x) = 1 - x^2$, $b = 1$, $q = 1$, and $r = 1$. Therefore, using the SDRE method, the desired optimal feedback control law is obtained by substituting these expressions into Eq. (37), giving

$$u_{\mathrm{SDRE}} = -(x - x^3) - x\sqrt{x^4 - 2x^2 + 2} \tag{38}$$

which is the corresponding optimal control law u_* Freeman and Kokotović [128] also derived for this problem.

Remark 13: The simple scalar problem of Example 5 was used in [128] to illustrate one of the potential pitfalls of feedback linearization. One such stabilizing controller was given by $u_{\mathrm{FL}} = x^3 - 2x$, which cancels the nonlinearity $-x^3$ and results in the closed-loop dynamics of $\dot{x}(t) = -x(t)$. Although this controller guarantees global exponential stability of the open-loop unstable equilibrium at $x = 0$, for large initial conditions, the cancellation term x^3 reduces the speed of convergence and produces huge control activity, which can cause instability in the presence of actuator saturation or uncertainties if such a controller were to be implemented in practice. It should be noted that the SDRE solution does not cancel out the beneficial nonlinearity $-x^3$ since it is accounted for in x times the radical in Eq. (38). Although u_{FL} produces a large control effort to blindly cancel the nonlinearity $-x^3$, u_{SDRE} uses the fact that the nonlinearity is stabilizing for large x, and thus commands very little control activity in that region [4]. This is also true in the case of multivariable problems, which can be illustrated using Example 4, where the nonlinearity $-x_1^2 x_2$ has been accounted for in x_2 times the radical in u_{SDRE} in Eq. (35).

VII. Stability Analyses

It is important to be able to state some type of stability claim for the methodology before its implementation in practice. This is the main focus of this section. A theoretical study of large-scale asymptotic stability properties of SDRE feedback controls is carried out in Sec. VII.A. Global asymptotic stability results are presented in Sec. VII.B. Finally, a discussion on stability tests for estimating the region (or domain) of attraction (hereafter referred to as the ROA) for asymptotic stability of SDRE control of general multivariable nonlinear systems is pursued in Sec. VII.C.

A. Large-Scale Asymptotic Stability

Because of the nature of the LQR formulation, under very mild conditions of local stabilizability and detectability at the origin set by Hypothesis 1, $\mathbf{A}_{\mathrm{CL}}(\mathbf{0})$ is Hurwitz; that is, all eigenvalues of the closed-loop dynamics matrix [Eq. (14)] have negative real parts at $\mathbf{x} = \mathbf{0}$. Hence, the SDRE-controlled system is locally asymptotically stable, however in a sufficiently small region Ω_0 around the origin. On the other hand, large-scale asymptotic stability of SDRE-controlled systems for ITHNOC problems has been addressed in the literature in [3,6]. The proof, however, relies on the assumption that $\mathbf{P}(\mathbf{x})$ is smooth, thus requiring additional conditions regarding smoothness of the SDC matrices $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$, and $\mathbf{R}(\mathbf{x})$. The fact that stable responses can be obtained in various examples and applications, in spite of the nonsmoothness of SDC matrices, suggests the use of an alternative approach to proving large-scale asymptotic stability. Accordingly, Langson and Alleyne [16] showed that, if $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are C^0 matrix-valued functions of $\mathbf{x}$, then under Condition 4, the origin of the SDRE-controlled system is asymptotically stable on a region Ω of state space strictly larger than Ω_0. Therefore, this is the result presented in the sequel.

Theorem 6: Consider the nonlinear system [Eq. (1)] with SDRE feedback [Eqs. (15) and (16)], and suppose that Condition 1 through Condition 4 hold. Then, the origin of the SDRE closed-loop system [Eq. (17)] is asymptotically stable in the large region $\Omega \subseteq \mathbb{R}^n$.

Proof: The solution of the SDRE [Eq. (15)] $\mathbf{P}(\mathbf{x})$ depends continuously on the entries of the SDC matrices. Therefore, from Riccati equation theory, the closed-loop SDC matrix $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$ in Eq. (14), with $\mathbf{K}(\mathbf{x})$ given by Eq. (18), is a pointwise Hurwitz C^0 function of $\mathbf{x}$ on $\Omega \subseteq \mathbb{R}^n$. Given $\varepsilon > 0$, the set $B_\varepsilon = \{\mathbf{x} \in \Omega\colon \|\mathbf{x}\| < \varepsilon\}$ is compact. Since $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$ is Hurwitz for each $\mathbf{x} \in B_\varepsilon$, $\exists M(\mathbf{x}) > 0$ and $\exists \mu(\mathbf{x}) > 0$, such that $\|e^{\mathbf{A}_{\mathrm{CL}}(\mathbf{x})t}\| \le M(\mathbf{x}) e^{-\mu(\mathbf{x})t}$ $\forall\ t \in \mathbb{R}^+$. Since $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$ is C^0 in $\mathbf{x}$, M and μ may be chosen to be C^0 functions of $\mathbf{x}$. M attains a maxima $M_* > 0$, and μ attains a minima $\mu_* > 0$ on the compact set B_ε. Choosing $\delta(\varepsilon) < \varepsilon / M_*$, let $\mathbf{x}_0 \in B_{\delta(\varepsilon)} = \{\mathbf{x} \in \Omega\colon \|\mathbf{x}\| < \delta(\varepsilon)\}$. The matrix-valued function $\mathbf{A}_{\mathrm{CL}}$ is uniformly continuous on B_ε. Therefore, given $\varepsilon' > 0$, $\exists \delta'(\varepsilon')$ such that $\forall\ \mathbf{x}, \mathbf{y} \in B_\varepsilon$ with $\|\mathbf{x} - \mathbf{y}\| < \delta'(\varepsilon')$, $\|\mathbf{A}_{\mathrm{CL}}(\mathbf{x}) - \mathbf{A}_{\mathrm{CL}}(\mathbf{y})\| < \varepsilon'$. Let $B_{\delta'(\varepsilon')}(\mathbf{x}) \subset \Omega$ be the open ball centered at $\mathbf{x} \in \Omega$ with radius $\delta'(\varepsilon')$. The collection $\vartheta = \{B_{\delta'(\varepsilon')}(\mathbf{x})\}_{\mathbf{x} \in B_\varepsilon}$ is an open cover of B_ε. Choose a finite cover $\vartheta_f = \{B_{\delta'(\varepsilon')}(\mathbf{x}_i)\}_{i=1}^N$ (each $\mathbf{x}_i$ is the center of some ball from the collection ϑ) from ϑ that contains B_ε in its union. If $\mathbf{x} \in B_{\delta'(\varepsilon')}(\mathbf{x}_i)$ for some i, $1 \le i \le N$, then $\dot{\mathbf{x}} = \mathbf{A}_{\mathrm{CL}}(\mathbf{x}_i)\mathbf{x} + (\mathbf{A}_{\mathrm{CL}}(\mathbf{x}) - \mathbf{A}_{\mathrm{CL}}(\mathbf{x}_i))\mathbf{x}$. Since $\mathbf{A}_{\mathrm{CL}}(\mathbf{x}_i)$ is a constant matrix, using the solution of the preceding differential equation for $1 \le i \le N$ thus gives

$$\|\mathbf{x}(t)\| \le M_* e^{-\mu_* t} \|\mathbf{x}_0\| + \int_0^t M_* e^{-\mu_*(t-s)} \varepsilon' \|\mathbf{x}(s)\|\, \mathrm{d}s$$

This holds in every ball $B_{\delta'(\varepsilon')}(\mathbf{x}_i)$, $1 \le i \le N$, and therefore holds throughout B_ε. From the Gronwall–Bellman inequality, it follows that $\|\mathbf{x}(t)\| \le M_* e^{-\mu_* t} \|\mathbf{x}_0\| < \varepsilon$, which proves asymptotic stability. □

B. Global Asymptotic Stability

Global asymptotic stability of the closed-loop system implies that it is possible to regulate the states to the origin, regardless of the initial conditions. This is obviously a very desirable property; however, it is usually difficult to achieve and/or prove. Using SDRE control [Eqs. (16–18)], the closed-loop solution becomes $\dot{x} = \mathbf{A}_{\mathrm{CL}}(\mathbf{x})\mathbf{x}$, where $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$ is the closed-loop SDC matrix given by Eq. (14). From Riccati equation theory (Sec. V.B), $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$ is guaranteed to be stable pointwise, that is, at every point $\mathbf{x}$. Since the characterization of the resulting SDRE controller has a similar structure to the LQR problem, in order for the SDRE [Eq. (15)] to have a nonnegative-definite solution for all $\mathbf{x}$, it is therefore sufficient that $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x}), \mathbf{C}(\mathbf{x})\}$ be pointwise stabilizable and detectable in the linear sense in the domain of interest $\Omega \subseteq \mathbb{R}^n$ for all $\mathbf{x} \in \Omega$. This will also guarantee that the SDRE-controlled system is large-scale asymptotically stable, as discussed in Sec. VII.A. Hence, under Condition 3 and Condition 4, the SDRE control law is pointwise stabilizing, such that $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$ is pointwise Hurwitz $\forall\ \mathbf{x} \in \Omega$. However, this latter property is not sufficient to deduce the global stability of a nonlinear system. In fact, even if all eigenvalues of $\mathbf{A}_{\mathrm{CL}}(\mathbf{x})$ have negative real parts $\forall\ \mathbf{x} \in \mathbb{R}^n$, global stability of a nonlinear system still cannot be guaranteed, as illustrated next using Example 1.

Example 1 Revisited: Let us reconsider Example 1 with SDC matrix $\mathbf{A}_1(\mathbf{x})$, which has (stable) eigenvalues $\lambda_1 = \lambda_2 = -1\ \forall\ \mathbf{x}$. With initial states $x_1(0) = x_{10}$ and $x_2(0) = x_{20}$, the solution for the evolution of the states as an explicit function of time for $t \in [0, T_e)$ becomes

$$x_1(t) = x_{10} e^{-t} \quad \text{and} \quad x_2(t) = \frac{2x_{20} e^{-t}}{x_{10}x_{20} e^{-2t} + (2 - x_{10}x_{20})}$$

which has a finite escape time at $t = T_e$ with $T_e = \frac{1}{2} \ln[x_{10}x_{20}/(x_{10}x_{20} - 2)]$, since $x_2(t)$ grows without bound as $t \to T_e$, and $x_2(T_e) = \infty$. Hence, although $\mathbf{A}_1(\mathbf{x})$ is pointwise Hurwitz $\forall\ \mathbf{x}$, the system is not globally asymptotically stable.

Global stability results are now presented for three cases. The first case considers general multivariable problems $(n > 1)$ with $1 \le m \le n$. In the second case, the system is assumed multivariable

($n > 1$), having a full-rank constant $\mathbf{B}$ matrix (where the number of inputs are equal to the number of states; that is, $m = n$) and constant $\mathbf{R} > \mathbf{0}$. The last case concerns scalar systems, where $\mathbf{x} \in \mathbb{R}^n$ with $n = 1$.

Theorem 7: Consider the nonlinear system [Eq. (1)] with SDRE feedback [Eqs. (15) and (16)] and suppose that Condition 1 through Condition 4 hold. If $\|e^{\mathbf{A}_{\mathrm{CL}}(\mathbf{x})t}\| \leq M$ for some real $M > 0$ and $\forall\ \mathbf{x} \in \Omega \subseteq \mathbb{R}^n$, $\forall\ t \in \mathbb{R}^+$, then the SDRE closed-loop system $\dot{\mathbf{x}} = \mathbf{A}_{\mathrm{CL}}(\mathbf{x})\mathbf{x}$ given by Eq. (17) is globally asymptotically stable in Ω.

Proof: In this case, $\delta(\varepsilon)$ in the proof of Theorem 6 (in Sec. VII.A) can be chosen to satisfy

$$\lim_{\varepsilon \to \infty} \delta(\varepsilon) = \infty$$

By choosing ε sufficiently large, any initial condition $\mathbf{x}_0 \in \Omega \subseteq \mathbb{R}^n$ can be included in the set $B_{\delta(\varepsilon)}$. The solution $\mathbf{x}(t)$ starting at $\mathbf{x}_0 \in \Omega$ satisfies $\|\mathbf{x}(t)\| \leq M e^{-\mu_* t} \|\mathbf{x}_0\|$, where μ_*, as in the proof of Theorem 6, is a minima of $\mu(\mathbf{x})$ on the compact ball centered at $\mathbf{x} = \mathbf{0}$ with radius M. □

Remark 14: From Theorem 6, if $\mathbf{A}(\mathbf{x})$ is pointwise Hurwitz and continuous (C^0) $\forall\ \mathbf{x} \in \Omega$, then the origin is an asymptotically stable equilibrium point in Ω. Additionally, by Theorem 7, if there exists an upper bound $M > 0$ to the solution $\|e^{\mathbf{A}_{\mathrm{CL}}(\mathbf{x})t}\| \leq M$ for all t, then the origin is globally asymptotically stable. Unfortunately, the global upper bound expressed in the theorem is very conservative and is not easy to determine or enforce by using feedback control a priori for global asymptotic stabilization.

Theorem 8: In addition to satisfying Condition 1 through Condition 4, suppose that $\mathbf{R} > \mathbf{0}$ is constant. Then, the SDRE closed-loop solution is globally asymptotically stable if $\mathbf{B} \in \mathbb{R}^{n \times m}$ is constant and full rank ($m = n$).

Proof: (For details, see Theorem 7.3.1 in [15].) In Theorem 4 of Hammett et al. [58], it is shown that systems of the form in Eq. (1) with full-rank $\mathbf{B}$ matrices are nonlinearly controllable for all $\mathbf{x}$ in addition to having $\{\mathbf{A}(\mathbf{x}), \mathbf{B}\}$ controllable in the linear sense for all $\mathbf{x}$, regardless of the choice of $\mathbf{A}(\mathbf{x})$. Thus, the reachable set from each $\mathbf{x}$ has dimension n, so that the uncontrollable subspace for such systems consists of only the zero vector. Then, using the equivalence between factored and nonlinear controllabilities in such cases, the proof follows from standard Lyapunov analysis via the direct method in a transformed set of coordinates. □

Theorem 9: In the scalar case ($x \in \mathbb{R}$), the (optimal) SDRE closed-loop solution obtained by applying the optimal SDRE feedback [Eq. (37)] is globally asymptotically stable.

Proof: In the case of scalar x, global asymptotic stability under the globally optimal SDRE feedback control [Eq. (37)] on $\mathbb{R}$ (by Theorem 5) is easily deduced using the Lyapunov function

$$V(x) = \int_0^x p(\tau)\tau \, \mathrm{d}\tau \quad \text{or} \quad V(x) = 0.5x^2$$

□

In terms of the stability features of SDRE-controlled systems, Theorem 7 and Theorem 8 on multivariable problems and Theorem 9 on scalar systems provide the best results. While extension to general multivariable problems is possible, global asymptotic stability of the continuous-time state-feedback SDRE regulator can only be guaranteed under several, sometimes restrictive, conditions. Accordingly, Qu et al. [14] proposed a method for establishing global asymptotic stability of the state-feedback SDRE regulator, based on satisfying the SDRE necessary condition for optimality [Eq. (34)] and a restricted class of weighting matrix functions. By constructing a SDC matrix $\mathbf{A}(\mathbf{x}, \boldsymbol{\alpha})$, conditions on $\boldsymbol{\alpha}(\mathbf{x})$, $\mathbf{Q}(\mathbf{x})$, and $\mathbf{R}(\mathbf{x})$ are developed to ensure global asymptotic stability. Unfortunately, these conditions, being imposed on the performance (design) parameters, are too restrictive. However, the results do show that global asymptotic stability can only be guaranteed by choosing state-dependent weighting matrices.

As an alternative to global asymptotic stability, it is desirable to be able to estimate the ROA for asymptotic stability. This is the region in state space that encloses all initial conditions such that, when the system is steered from them, the origin will be reached asymptotically. The subsequent discussion addresses this issue.

C. Stability Tests for Estimating the Region of Attraction

Obtaining a good estimate of the ROA for SDRE control of general multivariable nonlinear systems is a challenging task, since the closed-loop dynamics matrix [Eq. (14)] is usually not available in closed form due to the difficulty in explicitly solving the SDRE [Eq. (15)], especially for systems with state dimensions greater than two. Apart from brute-force time-domain simulations of the closed-loop system from a variety of initial conditions [6,86], there have been few results in the literature on the estimation of the ROA for SDRE-regulated systems, and the ones presented have rather conservative results.

The first set of techniques involves upper bounding the state trajectory for estimating the ROA [16,129]. Langson and Alleyne [16] used their proof of Theorem 6 on globally asymptotically stabilizing controls for nth-order nonlinear systems satisfying those growth conditions for additionally providing a basis on which a ROA for local asymptotic stability of SDRE-controlled systems can be estimated. This estimate, however, turned out to be very conservative. Accordingly, Erdem and Alleyne [129] provided an alternative approach based on defining an overvaluing comparison system for the original one by using vector norms. Although the resulting estimates are prone to be conservative, the method does not require that the closed-loop system equations be known explicitly and eliminates the need for time-domain simulations, but it requires a Hurwitz overvaluing matrix in the stability region, with knowledge of maximum and minimum values of the feedback gains over a chosen domain Ω in state space.

The second set of techniques for estimating the ROA relies on the choice of a Lyapunov function, based on the linearization of the closed-loop system [130,131]. Seiler [130] presented an approach that transformed stability region estimation into a search problem for several functions satisfying inequality constraints and used sum of squares optimization to convert this search into a semidefinite programming problem, which can be used to estimate the ROA. Unfortunately, this method relies on the assumption that the SDC matrices are polynomial functions of $\mathbf{x}$. The procedure proposed by Bracci et al. [131] calculates a Lyapunov function V for the linearized system in the neighborhood of the origin, and then it applies it to the complete nonlinear system. The largest level set of V that is completely inside the region where $\dot{V} < 0$ then defines a lower bound of the ROA for the system. For these Lyapunov-based methods, if several Lyapunov functions can be defined for the linearized system, an estimate of the ROA can be obtained by the union of the different estimates. However, this has limitations for SDRE-controlled systems, because a closed-form SDRE solution is not generally known. This necessitates the generation of a grid in a region of the state space so that the procedure is carried out on these grid points, resulting in a computationally intensive procedure, especially for high-order systems.

Contrary to these two sets of techniques, McCaffrey and Banks [132] proposed a stability test for determining the size of the region on which large-scale asymptotic stability holds for the SDRE algorithm. The analysis is based on the geometrical construction of a viscosity-type Lyapunov function from a stable Lagrangian submanifold M, and it uses the value function that solves the ITHNOC problem [Eqs. (1) and (2)] as a Lyapunov function. The resulting test involves evaluating an inequality along trajectories of the Hamiltonian dynamical system [Eq. (20)], without the need to find the value function. By considering regions of state space over which the stable manifold is multisheeted rather than just single-sheeted, the resulting estimate of the ROA for the SDRE feedback is far closer to the true ROA than conservative estimates arising from the smoothness assumptions of the existing literature. Another advantage of this approach is that the proposed inequality gives some measure of the distance between the SDRE feedback and the optimal control, that is, a measure of the degree of suboptimality. It also shows that sufficient conditions for optimality are asymptotically satisfied by SDRE feedback.

As an alternative to asymptotic stability region estimates, Chang and Chung [133] offered a new perspective on stability tests of SDRE-controlled systems by estimating exponential stability regions using contraction theory.

Despite all the aforementioned methods proposed in the literature, the safe path to estimating the ROA is simply to integrate the closed-loop state-space dynamics given by the SDRE feedback forward in time from a variety of initial conditions and, through extensive simulations, identify which trajectories converge to the equilibrium. This also involves fewer equations (n instead of $2n$ equations) in the integration compared with the method proposed in [132]. Although it requires iteration through numerous sets of initial conditions in order to identify the boundary of the ROA, it does accomplish this task very accurately, and it is still considered as the best approach to use in practice.

VIII. State-Dependent Riccati Equation Controller Design and Implementation

This section addresses systematic design (as opposed to heuristic tuning) of SDRE feedback controllers, and it discusses their implementation. Systematic selection of SDC parameterization and state-dependent weighting matrices are addressed in Secs. VIII.A and VIII.B, respectively, assessing the design flexibility (that is, the additional degrees of freedom) provided by the nonuniqueness of the SDC parameterization (see Sec. III.F). Implementation prospects are then considered in Sec. VIII.C.

A. Selection of the State-Dependent Coefficient Parameterization

The nonuniqueness of the SDC parameterization [Eq. (11)] for multivariable systems creates additional degrees of freedom that are not available in other nonlinear design methods and can be used to provide great design flexibility. Recall from Remark 7 that a complete characterization of the possible factorizations of $\mathbf{f}(\mathbf{x})$ that span the space of all valid parameterizations can be obtained in the form $\mathbf{A}(\mathbf{x},\boldsymbol{\alpha})\mathbf{x}$ by constructing a SDC matrix $\mathbf{A}(\mathbf{x},\boldsymbol{\alpha})$, which is the parametric representation of a hypersurface, where $\boldsymbol{\alpha}$ is a vector of dimension k. The additional degrees of freedom available through $\boldsymbol{\alpha}$ provide design flexibility that can be used to enhance controller performance or affect tradeoffs between optimality, stability, robustness, and disturbance rejection. The control designer can exploit this flexibility of choice for design and analysis purposes. On the estimation/filtering side, the additional degrees of freedom can be used to enhance observer/filter performance, avoid singularities, or avoid loss of observability (for illustrative examples, readers may refer to [25,28]). Such capabilities are not available in traditional nonlinear estimation and filtering techniques. Therefore, some consideration must be given concerning the choice of the SDC matrices. The issues on how the additional degrees of freedom provided by the nonuniqueness of the SDC parameterization can be used to enhance performance and how to select appropriate SDC matrices, as well as state-dependent weighting matrices, in a systematic fashion for carrying out an effective SDRE design have been addressed in [8]. These are now discussed.

For controller design, the primary condition in selecting the right $\mathbf{A}(\mathbf{x})$ is that the pair $\{\mathbf{A}(\mathbf{x}),\mathbf{B}(\mathbf{x})\}$ is a pointwise stabilizable SDC parameterization of the nonlinear system [Eq. (1)] $\forall\ \mathbf{x}\in\Omega$. In addition to satisfying this requirement given in Condition 4, in selecting the state-dependent factorization, any term containing more than one state variable in $\mathbf{f}(\mathbf{x})$ must be parameterized and apportioned among the corresponding elements of the $\mathbf{A}(\mathbf{x})$ matrix [7,8,42]. For example, if $\dot{x}_3 = x_1x_2$, two possible factorizations for $\mathbf{A}(\mathbf{x}) = [a_{ij}]$ are $a_{31}=0$, $a_{32}=x_1$ and $a_{31}=x_2$, $a_{32}=0$. Neither one of these parameterizations reflect, in the $\mathbf{A}(\mathbf{x})$ matrix, the fact that $\dot{x}_3$ depends on both x_1 and x_2. While both these parameterizations may work, better responses and a larger stability region can be obtained using the parameterizations $a_{31}=\alpha_1x_2$ and $a_{32}=(1-\alpha_1)x_1$. As noted in Remark 2, the free design parameter $\alpha_1\in[0,1]$ forms a convex combination. Additionally, both of the previous factorizations can be tested by setting $\alpha_1=0$ and $\alpha_1=1$, respectively. It is also desirable to shift state-dependent factors that exclude the origin, even though they are embedded in a term that goes to zero as the state goes to zero, as indicated in [7,8]. For example, consider $\dot{x}_2 = x_3\cos x_1$. Obviously, this term goes to zero as x_3 goes to zero, but it is desirable to have a nonzero entry in the (2,1) element of the $\mathbf{A}(\mathbf{x},\boldsymbol{\alpha})$ matrix that reflects the fact that $\dot{x}_2$ depends on x_1. This is accomplished by shifting the term so that it goes through the origin. For the given example, adding and subtracting 1 gives $\cos x_1 = [\cos x_1 - 1] + 1$. The function $\cos x_1 - 1$ goes through the origin and can therefore be factored as $\cos x_1 - 1 = [(\cos x_1 - 1)/x_1]x_1$. Note that the expression in brackets is well behaved at the origin when $x_1 = 0$. Then, writing $\dot{x}_2 = [(\cos x_1 - 1)/x_1]x_1x_3 + x_3$ allows the system to be parameterized as $a_{21} = \alpha_2[(\cos x_1 - 1)/x_1]x_3$ and $a_{23} = (1-\alpha_2)(\cos x_1 - 1) + 1$, which yields the desired nonzero entry in a_{21}. Similarly, considering the example $\dot{x}_2 = e^{x_1}x_3$, although this term can be factored as $a_{21}=0$ and $a_{23}=e^{x_1}$, this factorization does not reflect the fact that $\dot{x}_2$ depends on x_1 within the pointwise LQR structure since $a_{21}=0$. Shifting e^{x_1} and writing

$$\dot{x}_2 = [(e^{x_1}-1)+1]x_3 = [(e^{x_1}-1)/x_1]x_1x_3 + x_3$$

allows the system to be parameterized as $a_{21} = \alpha_3[(e^{x_1}-1)/x_1]x_3$ and $a_{23} = (1-\alpha_3)(e^{x_1}-1)+1$, which yields the desired nonzero entry in a_{21}. Such an approach provides a complete characterization of the possible factorizations of $\mathbf{f}(\mathbf{x})$ in the form $\mathbf{A}(\mathbf{x},\boldsymbol{\alpha})\mathbf{x}$, where $\boldsymbol{\alpha} = [\alpha_1,\ldots,\alpha_k]$, leading to a solution of the SDRE [Eq. (15)] of the form $\mathbf{P}(\mathbf{x},\boldsymbol{\alpha})$. For optimality, $\mathbf{P}(\mathbf{x},\boldsymbol{\alpha})$ must satisfy the SDRE necessary condition for optimality [Eq. (34)], which must hold for a strong local minimum. Then, to obtain the optimal feedback solution of the nonlinear regulator, $\boldsymbol{\alpha}$ may be required to vary as an explicit function of time, an explicit function of the state vector $\mathbf{x}$, or both (see [3,15]). The procedure thus involves solving Eq. (34) for optimal $\boldsymbol{\alpha}$ based on $\boldsymbol{\alpha}(t)$, $\boldsymbol{\alpha}(\mathbf{x})$, or $\boldsymbol{\alpha}(\mathbf{x},t)$. This, however, results in a cumbersome procedure that requires solving either a TPBVP or a PDE, neither of which have any hope for real-time implementation. Fortunately, the real advantage of the SDRE technique is that there is no need to do so, because with $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ a priori specified, satisfactory performance relative to the optimal value of the performance index is generally obtained using a constant value of $\boldsymbol{\alpha}$. Hence, $\boldsymbol{\alpha}$ is normally used as a constant tuning parameter to aid in the achievement of the desired response specifications. The best SDC parameterization with regard to stability and performance using a constant value of $\boldsymbol{\alpha}$ can be obtained by attempting to maximize the pointwise controllable spaces of the possible factorizations by evaluating the state-dependent controllability matrix (see Remark 4)

$$\mathbf{M}_C(\mathbf{x},\boldsymbol{\alpha}) = [\,\mathbf{B}(\mathbf{x}) \quad \mathbf{A}(\mathbf{x},\boldsymbol{\alpha})\mathbf{B}(\mathbf{x}) \quad \cdots \quad \mathbf{A}^{n-1}(\mathbf{x},\boldsymbol{\alpha})\mathbf{B}(\mathbf{x})\,]$$

for different values of $\boldsymbol{\alpha}$. This provides a very effective approach and is clearly the logical choice, since pointwise control effort can be directly linked to these issues (for illustrative examples and applications, see [8]).

B. Selection of State-Dependent Weighting Matrices

For the SDRE to have a solution, the pointwise detectability requirement in Condition 4 must also be satisfied (see Remark 4). A primary advantage offered by SDRE methodology to the control designer is the opportunity to make tradeoffs between control effort and state errors by tuning the penalty matrices $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$, a capability that is not readily available in other nonlinear control methods. Consequently, much of the physical intuition in controller design associated with linear H_2 methods is available to the designer by adjusting physically significant parameters, depending on the behavior of the controlled system. Moreover, $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ are not only allowed to be constant but can also vary as functions of the states so as to obtain the desired system response. Thus, different modes of behavior can be imposed in different regions of state space. For instance, the weighting matrices can be chosen such that, as $\mathbf{x}$ increases, $\mathbf{Q}(\mathbf{x})$ increases and $\mathbf{R}(\mathbf{x})$ decreases, thus saving control effort near the origin and making sure that the system is driven to its equilibrium far from the origin. This flexibility increases the ability in classical LQR design of making tradeoffs between input and state error. In addition, global minimizations will rely on the convexity of the Hamiltonian function [Eq. (5)] with respect to both $\mathbf{x}$ and $\mathbf{u}$. Therefore, it may also be desirable to select $\mathbf{Q}(\mathbf{x})$ and $\mathbf{R}(\mathbf{x})$ such that the performance index [Eq. (2)] is globally convex with respect to these variables [8,15]. The convexity of Eq. (2) with respect to the control $\mathbf{u}$ is always guaranteed since, by Condition 3, $\partial^2 J/\partial \mathbf{u}^2 = \mathbf{R}(\mathbf{x}) > \mathbf{0}\ \forall\ \mathbf{x} \in \Omega$. In addition, the state weighting matrix function $\mathbf{Q}(\mathbf{x})$ may be chosen so as to ensure $\partial^2[\mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x}]/\partial \mathbf{x}^2 > \mathbf{0}\ \forall\ \mathbf{x} \in \Omega$. The following proposition from [15] provides a sufficient condition for convexity of the state component of Eq. (2), and it justifies the logical choice for state-dependent penalty weightings. It also ensures that $\mathbf{Q}(\mathbf{x})$ is positive definite so that the detectability requirement in Condition 3 is always satisfied. Besides, global asymptotic stability can only be guaranteed by choosing state-dependent weighting matrices (see Sec. VII.B).

Proposition 3: Consider the scalar function $l(\mathbf{x}) = \mathbf{x}^T\mathbf{Q}(\mathbf{x})\mathbf{x}$. Let $\mathbf{Q}(\mathbf{x}) = \mathbf{Q}_0 + \mathbf{Q}_1(\mathbf{x})$, where $\mathbf{Q}_0 = \mathrm{diag}\{c_{i0}, \ldots, c_{n0}\}$ is any constant symmetric positive-definite matrix with $c_{i0} > 0$, $i = 1, 2, \ldots, n$, and $\mathbf{Q}_1(\mathbf{x}) = \mathrm{diag}\{q_1(x_1), \ldots, q_n(x_n)\}$ is such that each q_i takes the form $q_i(x_i) = c_{i2}x_i^2 + c_{i4}x_i^4 + \cdots + c_{is_i}x_i^{s_i}$ with $c_{ij} \geq 0$, $j = 2, 4, \ldots, s_i$. Then, $l(\mathbf{x})$ is globally convex with respect to $\mathbf{x}$.

Thus, if $\mathbf{Q}(\mathbf{x})$ is chosen according to Proposition 3, each state x_i has a term in the cost functional with $c_{ij} \geq 0$. Such a state weighting provides a steeper penalty for nonzero state deviations far from the origin than purely constant state weightings do, as in LQR design, and thus can be expected to increase control gains far from the origin. Hence, by choosing state weighting matrix functions so as to satisfy the requirements of Proposition 3, the desirable properties of convex functionals are obtained by simultaneously imposing different modes of behavior in different regions of state space. Of course, it is always possible to relax these requirements by choosing $\mathbf{Q}(\mathbf{x})$ to satisfy $\partial^2 l(\mathbf{x})/\partial \mathbf{x}^2 \geq \mathbf{0}$ so that convexity is only preserved on a reduced part of the state space. As such, the SDRE method allows significant design flexibility through the option of penalizing various combinations of states that can be exploited to meet the desired performance characteristics.

The final step in the design process involves penalizing each state and control by either heuristic tuning or systematic selection of the corresponding design (penalty) parameters in the weighting matrices $\mathbf{Q}(\mathbf{x}) = \mathrm{diag}\{q_i(x_i)\}$ and $\mathbf{R}(\mathbf{x}) = \mathrm{diag}\{r_j(x_j)\}$. By assuming equal weight on each control, so that $\mathbf{R} = \mathbf{I}_{m\times m}$, the penalties can be absorbed into $\mathbf{Q}(\mathbf{x})$. Since pointwise LQR design guarantees a pointwise Hurwitz closed-loop system as the penalties q_i are made arbitrarily large, it is important to limit the bandwidth so as to ensure robustness against high-frequency unmodeled dynamics, which would otherwise cause the actual system to go unstable. The design parameters q_i can be selected using design charts [134] created from a frequency-domain analysis of pointwise LQR designs. To determine performance, stability, and robustness in the frequency domain, singular values of several matrix functions are used. These are the loop gain crossover frequency (that is, the bandwidth) of the system, the loop-transfer function matrix, the return-difference function matrix, and the stability-robustness function matrix (SRFM). The design procedure for systematic selection of q_i pointwise then consists of three design charts, which relate the magnitude of q_i to bandwidth, the bandwidth to performance (rise time and settling time), and the magnitude of q_i to closed-loop sensitivity (singular values of SRFM). The goals are to maximize the singular value measures in order minimize closed-loop peak resonance and to achieve fast speed of response without making the bandwidth too high. The time- and frequency-domain performance metrics versus bandwidth plots allow designers to identify what is actually achievable from the system by observing the diminishing return in performance (smaller time constants) with increased bandwidth. Use of these design charts to maximize performance and minimize sensitivity provides a systematic and automatable selection process for numerically selecting design (penalty) parameters q_i that meet the design objectives for effective design of SDRE state-feedback controllers.

C. Implementation Prospects

The SDRE approach, as originally proposed by Pearson [2], necessitates repeated online computation of a SDRE, with the control applied at a particular point in state space determined by solving the infinite-time horizon LQR problem using the linear model for the particular point. This involves finding the steady-state solution to the state-dependent Riccati differential equation, which requires the solution of an ARE at each sampling instant. This, however, demands advanced numerical algorithms and significantly more computational resources than conventional control algorithms, particularly in real-time applications. Several authors have therefore addressed the problem of economically implementing the SDRE control law by using a truncated series expansion about a nominal value of the present state [9,124,135–138]. The terms of this series are calculated recursively offline. In so doing, the problem of repeatedly solving the SDRE is circumvented, which in turn significantly reduces the required online computations, and the control law becomes an explicit function of the state. Under the assumptions that the SDC matrices are continuous, the Taylor series converges locally to the solution of the SDRE, and the control law obtained by truncating after any number of terms in this series preserves the stability of the closed-loop system in a sufficiently small region around the origin [9]. However, there are severe limitations in using the Taylor-series approach nonlocally for approximating the solution to the SDRE outside of a small region around the origin. In fact, using more terms in the series does not necessarily result in greater accuracy, but instead may produce an approximate feedback control that is far from accurate, or it may even cause instability (see, for example, [137]). In the end, this method does not serve the real-time implementation prospect of SDRE controllers. Alternatively, the θ-D approximation [138], based on approximate (truncated) power-series solutions to the HJB equation, gives an approximate closed-form solution to the SDRE [Eq. (15)], which guarantees convergence and closed-loop stability.

To keep the payoff low for real-time implementation, there is also the option to produce SDRE point designs and use gain-scheduling. The SDRE gain-scheduling approach, alluded to by Cloutier et al. [3] and Mracek and Cloutier [6], and investigated by Banks et al. [137], is very similar to the linear parameter-varying (LPV) framework that has been used for gain-scheduling [103]. The difference is that, in the LPV structure, the coefficient matrices are functions of a parameter, while in the SDC structure, the coefficient matrices are explicit functions of the state. This approach produces relatively accurate SDRE feedback controls, which are computationally low cost compared with series approximations. It involves varying the state over the domain of interest and solving offline for the SDRE feedback control $\mathbf{u}(\mathbf{x})$, the SDRE matrix solution $\mathbf{P}(\mathbf{x})$, or even the SDRE feedback gain matrix $\mathbf{K}(\mathbf{x})$. Any of these functions can then be stored in a grid, so that the control is approximated in real time by interpolating over the stored solutions as the state varies. However, online computation of SDRE feedback controls makes the technique ideal for adaptation of feedback gains due to parameter variations, modeling uncertainties, etc. Fortunately, contemporary technology with increased microprocessor execution speed and storage provides evidence that the need to revert to the SDRE gain-scheduling option is no longer required for most problems of practical interest [8].

Although its potential is well recognized, due to the myth of extreme computational complexity associated with solving an ARE online at each sampling time, the industry acceptance of the SDRE technique has been slow. However, the computational burden associated with real-time SDRE control is not as high as anticipated and, with the development of faster embedded processors and subsystems, this has ceased to be an issue, as evidenced by several application papers on real-time implementation, testing the feasibility of computing the solution to the SDRE [Eq. (15)] online at each sampling time (for a detailed discussion on this subject, as well as algorithms for solving the SDRE pointwise in real time, readers may refer to [8] and the references therein). Advances in technology now deliver computational speeds far in excess of that required for implementing high-performance weapon flight control systems using commercial off-the-shelf processors [139]. Today, implementation of SDRE controllers containing more than 10 states can easily be realized in real time at sampling rates of several hundred hertz [8], thus demonstrating the feasibility of implementing the SDRE approach in real time for realistic high-order multivariable systems for most problems of practical interest. Since the computational complexity of an ARE is only of polynomial growth rate with the state dimension, the SDRE algorithm indeed provides the possibility of dealing with even higher-dimensional systems.

IX. Summary and Recommendations for Further Research

Although empirical experience in the abundant number of successful practical applications and design problems over the past two decades has demonstrated the effectiveness and capabilities of the SDRE method in very diverse fields of study, as discussed in Sec. II.C, a number of issues remains regarding adaptive control, robustness, optimality, and the theoretical issue of global asymptotic stability, which are now discussed for further investigation.

A. Adaptive Control

In dealing with internal parameter variations, the ability of SDRE design methods to adapt controller gains in real time based on the SDC pair $\{\mathbf{A}(\mathbf{x}), \mathbf{B}(\mathbf{x})\}$ can be viewed as an equivalent approach to the adaptive control design paradigm. The SDC pair captures new knowledge of the plant dynamics from the dynamic state vector signatures in real time without explicitly relying on an onboard parameter estimator. This is virtually achieved by having the state vector $\mathbf{x}(t)$ driving the SDC pair as the provider of new knowledge, which is supplied to the controller tuning process. This perspective of perceiving SDRE control methods as one branch of adaptive control without relying on any estimation theory facilitates practitioners in using this design methodology for problems requiring adaptive control [52,94], and it should encourage theorists to further solidify this thinking.

B. Robustness

Geometric methods provide a variety of tools for the design of feedback controllers for nonlinear systems. The importance of developing methods that are near optimal and do not cancel beneficial nonlinearities (for instance, with regard to stability) has been emphasized in several works, including [87,128]. Typically, the cancellation of such nonlinearities results in a significant increase in control effort. Consequently, such designs can be far from optimal and, even worse, severely suffer from robustness problems. In contrast, the SDRE method avoids such cancellations due to the suboptimality property of the method. Extensive simulation results also show that SDRE controllers exhibit robustness against parametric uncertainties/variations and unmodeled dynamics, and they attenuate disturbances. However, robustness characteristics of SDRE controllers have been carried out for individual problems rather than explicitly investigated as a general topic, and a formal robustness analysis does not yet exist. Of course, in its more general form, the SDRE technique allows the inclusion of disturbance terms using the SDRE nonlinear H_∞ control formulation [3]. If desired, the designer can augment this system by introducing integral states and dynamic compensators to improve the tracking and disturbance rejection characteristics of the SDRE controller.

C. Optimality

The solution $\mathbf{P}(\mathbf{x})$ of the SDRE is dependent on the chosen SDC matrix $\mathbf{A}(\mathbf{x})$, which is not unique for systems of orders greater than one. This being the case, it is apparent that different choices of parameterizing the nonlinear dynamics lead to different control laws, and hence different performance. However, under a certain (restrictive) constraint on the value function, there always exists an optimum factorization $\mathbf{A}_*(\mathbf{x})$ in which the SDRE yields the unique $\mathbf{P}(\mathbf{x})$ that will recover the optimal control, in the sense that the SDRE control actually achieves the minimum performance value. Even so, a method of determining this factorization is not yet known, and finding the $\mathbf{A}_*(\mathbf{x})$ that will recover the optimal control is usually not straightforward. In addition, although the existence of pointwise stabilizing nonnegative-definite solutions to the SDRE is guaranteed under very mild conditions of pointwise stabilizability, this alone is not sufficient to guarantee global optimality, and an additional symmetry (or curl type) condition is required for the latter. Indeed, a computationally efficient approach for solving the nonsymmetric-SDRE equations online is yet to be established for the optimal choice. These limitations are acknowledged, although several investigations indicate that the suboptimality is of minor consequence when comparing the benefits afforded by the standard SDRE method, such as design flexibility. The degree of suboptimality depends on the chosen SDC parameterization. Even so, under asymptotic stability, optimality is always approached at a quadratic rate, independent of the parameterization.

D. Global Asymptotic Stability

As with all suboptimum control methods, stability is an issue in extended linearization control methods. These methods theoretically only guarantee large-scale asymptotic stability, as shown in this Survey Paper, but work extremely well in practice. Much criticism has been leveled against the SDRE method because of the common misconception that it does not provide assurance of global asymptotic stability. General conditions do, however, exist on how to systematically achieve global asymptotic stability [32], albeit based on a restricted choice of state-dependent weighting matrices. Even so, typically, most SDRE controllers are simply implemented by choosing an $\mathbf{A}(\mathbf{x})$ that satisfies Condition 4 and constructing a suboptimal and locally stabilizing controller in the region of interest, without having to sacrifice controller performance for global asymptotic stability. Thus far, this approach has shown great promise, as evidenced by the growing number of application papers dealing with SDRE control. A systematic procedure for selecting the best parameterization with regard to stability and performance has also been proposed, although a general rule for achieving global asymptotic stability without overly sacrificing performance is yet to be established.

X. Conclusions

State-dependent Riccati equation (SDRE) design does not yet provide assurance of global asymptotic stability unless certain restrictions are imposed on the performance (design) parameters. On the contrary, stability-based nonlinear synthesis methods, when applicable, do provide such a proof but very often ignore the consequences on the control signals, and hence on the performance criterion. They may result in the requirement of large control signals that may cause instability in the presence of actuator saturation or uncertainties if implemented on an actual plant, which the suboptimum control methods seek to avoid. Furthermore, in reality, even the proven stability theory no longer holds due to state of practice (modeling assumptions, safety requirements, and physical limits, to name just a few) and, in applications (especially in aerospace), performance in the region of attraction becomes more important. In these cases, the suboptimum methods have great promise. Moreover, empirical experience in numerous applications and design problems shows that the domain of attraction of extended linearization control methods is almost always as large as the domain of interest. Nevertheless, given the lack of an a priori guarantee of global asymptotic stability and given the wealth of well-understood and theoretically supported nonlinear synthesis methods, stability-based design is usually the method of choice when the only concern is theoretical stability of the system. However, the situation changes significantly when, in addition to stability, the goal involves performance, measured by certain optimality criteria such as Eq. (2), which is the sole concern in most applications of practical interest. In this case, performance of stability-based nonlinear designs is highly problem dependent, ranging, for any given method, from near optimal to very poor. The greatest advantage offered by SDRE control is the opportunity to make tradeoffs between control effort and state errors by systematic selection, or even heuristic tuning of the corresponding weighting matrices as functions of the state. The SDRE paradigm eliminates the gain-scheduling requirement and essentially provides a unified control methodology that, in addition to stability, has the ability to address various performance characteristics (such as nonlinear control, robustness, optimality, and adaptation to accommodate system/parameter variations, to name a few) to a satisfying extent for a wide variety of nonlinear systems and a vast number of applications in very diverse fields of study across all industrial sectors.

During the inaugural Hendrik W. Bode Prize Lecture at the IEEE Conference on Decision and Control in Tampa, Florida, in December 1989, Gunter Stein, a unique authority in the control community, illustrated and emphasized that the trend to uphold mathematical rigor as the only virtue to strive for in control is incompatible with the trend in applications [140]. Stein's influential message that there will always be a leap of faith going from the paper design (with its model-based global asymptotic stability guarantee, for instance) to implementation on the actual physical plant is an inevitable fact of reality. There is no doubt that the theoretical progress that has been made in nonlinear control over the last several decades has reduced the leap of faith in practice; albeit, the leap of faith will never be eliminated. There has also been substantial progress in developing a coherent and rigorous theory for SDRE control of nonlinear systems, with theoretical justifications for design choices and properties. There is also theoretical evidence to support that, if a system can be stabilized over a domain of interest using a nonlinear technique, then it can be stabilized over the same domain using SDRE control. Since its debut in the literature half a century ago, SDRE control has evolved into a systematic design methodology, fulfilling performance expectations of practicing engineers and outperforming most other state-of-the-art nonlinear design methods. Future research advances concerning the SDRE method will only contribute to further reducing the leap of faith.

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