Greedy Algorithms: Fractional Knapsack

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Algorithmic Toolbox
Data Structures and Algorithms

Outline

1. Long Hike
2. Fractional Knapsack
3. Pseudocode and Running Time

Long Hike

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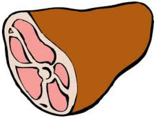

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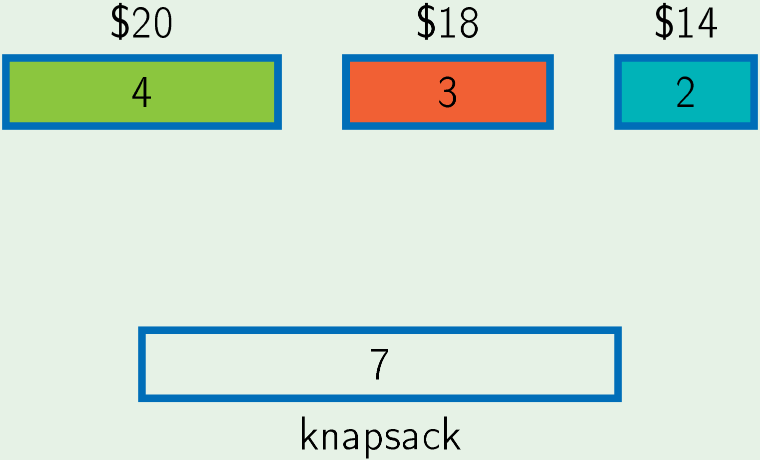
Fractional knapsack

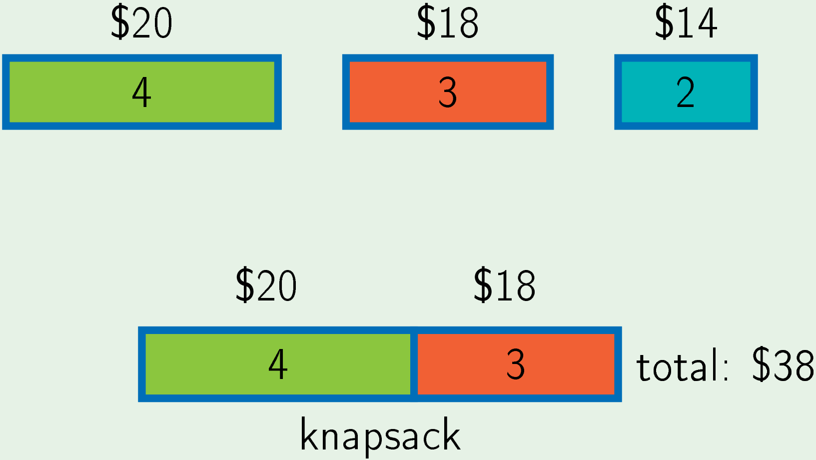
Input: Weights $w_1, \ldots, w_n$ and values $v_1, \ldots, v_n$ of n items; capacity W.

Output: The maximum total value of fractions of items that fit into a bag of capacity W.

Example



Example



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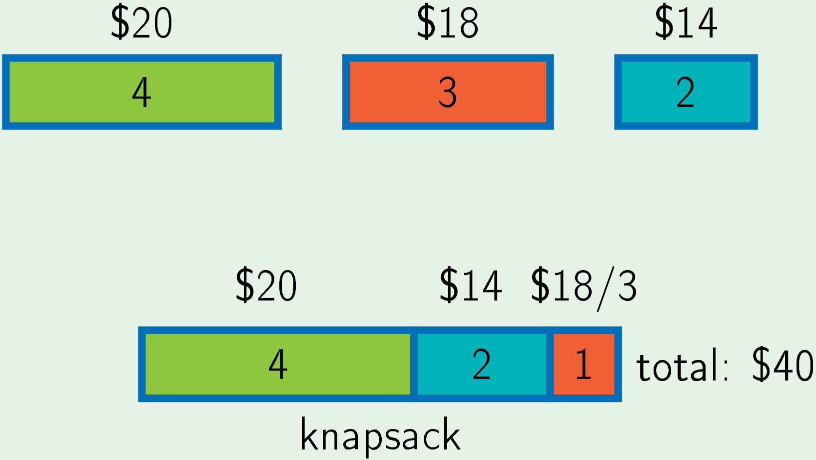

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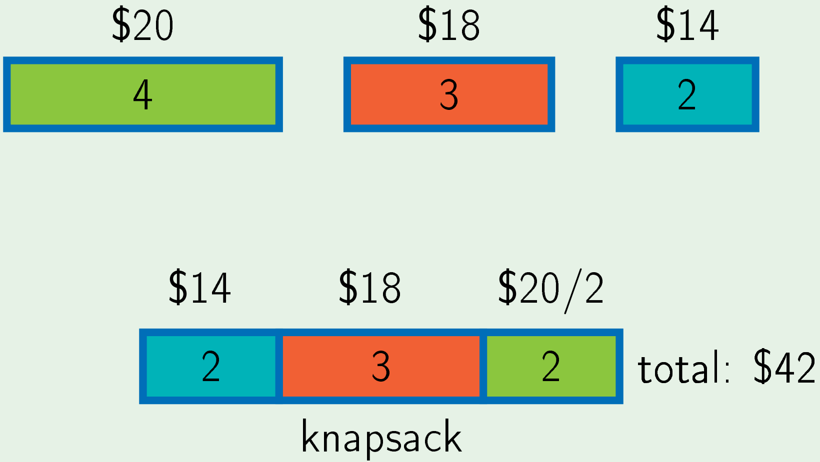

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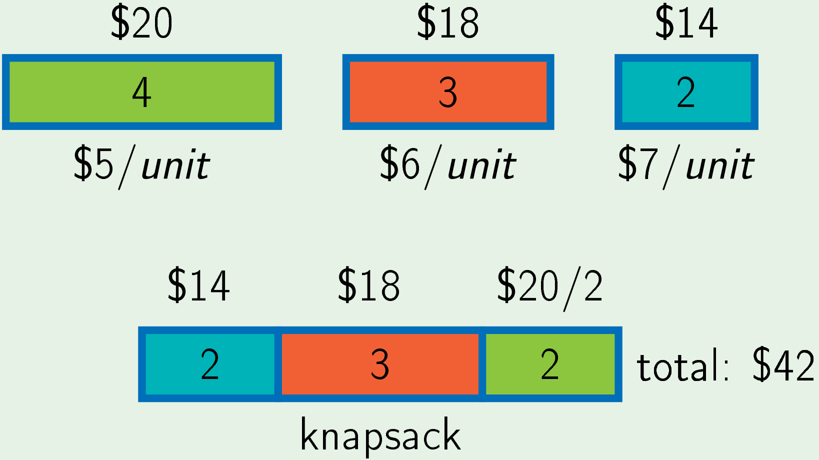


Safe move

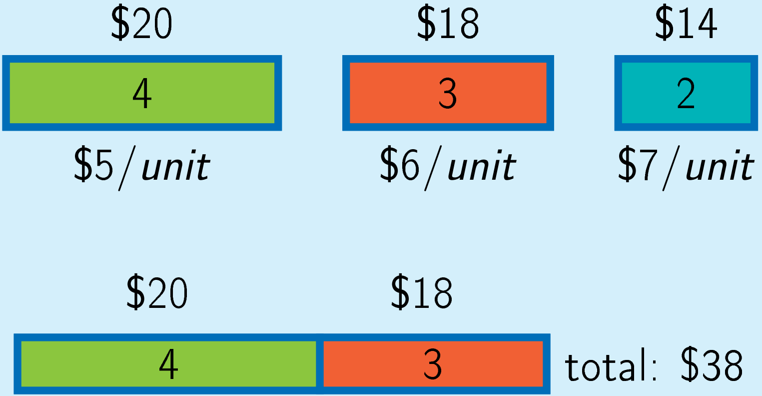
Lemma

There exists an optimal solution that uses as much as possible of an item with the maximal value per unit of weight.

Proof

\$20	\$18	\$14
4	3	2
\$5/*unit*	\$6/*unit*	\$7/*unit*

Proof



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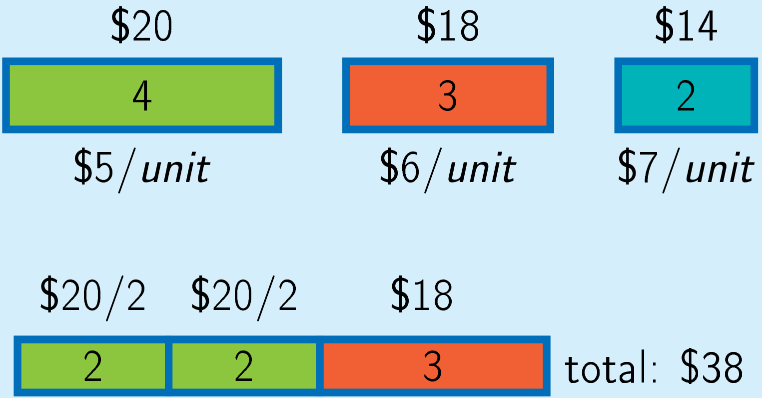

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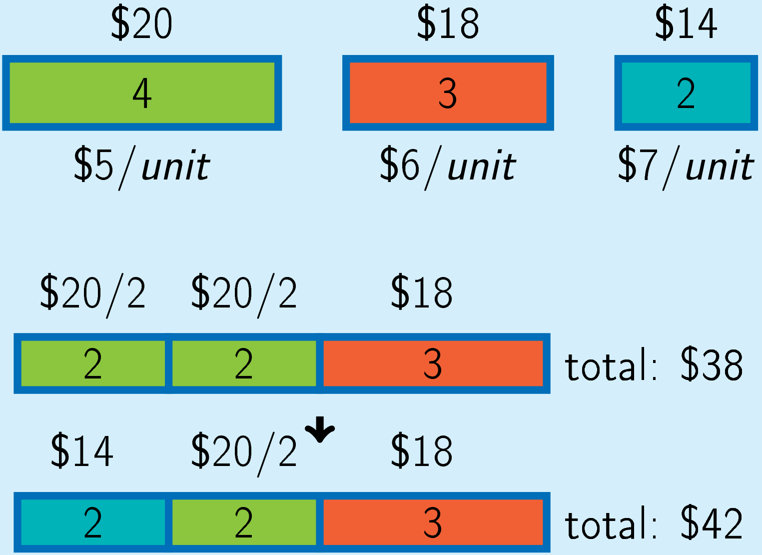

$20
4
$5/unit
$18
3
$6/unit
$14
2
$7/unit
$20/2
$20/2
$18
2
2
3
total: $38
$14
$20/2
$18
2
2
3
total: $42

Greedy Algorithm

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Knapsack($W, w_1, v_1, \ldots, w_n, v_n$)

```
A ← [0, 0, ..., 0], V ← 0
repeat n times:
    if W = 0:
        return (V, A)
    select i with w_i > 0 and max v_i/w_i
    a ← min(w_i, W)
    V ← V + a·v_i/w_i
    w_i ← w_i − a, A[i] ← A[i] + a, W ← W − a
return (V, A)
```

$A \leftarrow [0, 0, \ldots, 0], V \leftarrow 0$

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  select i with $w_i > 0$ and max $\frac{v_i}{w_i}$

  $a \leftarrow \min(w_i, W)$

  $V \leftarrow V + a\frac{v_i}{w_i}$

  $w_i \leftarrow w_i - a, A[i] \leftarrow A[i] + a, W \leftarrow W - a$

return (V, A)

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The running time of `Knapsack` is $O(n^2)$.

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Proof

- Select best item on each step is $O(n)$
- Main loop is executed n times
- Overall, $O(n^2)$ $\square$

Optimization

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- First, sort items by decreasing $\frac{v}{w}$

Assume $\frac{v_1}{w_1} \geq \frac{v_2}{w_2} \geq \cdots \geq \frac{v_n}{w_n}$

$\texttt{Knapsack}(W, w_1, v_1, \ldots, w_n, v_n)$

$A \leftarrow [0, 0, \ldots, 0]$, $V \leftarrow 0$
for i from 1 to n:
  if $W = 0$:
    return (V, A)
  $a \leftarrow \min(w_i, W)$
  $V \leftarrow V + a\frac{v_i}{w_i}$
  $w_i \leftarrow w_i - a$, $A[i] \leftarrow A[i] + a$, $W \leftarrow W - a$
return (V, A)

Asymptotics

- Now each iteration is $O(1)$
- `Knapsack` after sorting is $O(n)$
- Sort + `Knapsack` is $O(n \log n)$