Divide-and-Conquer: Polynomial Multiplication

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Data Structures and Algorithms
Algorithmic Toolbox

Outline

Uses of multiplying polynomials

- Error-correcting codes
- Large-integer multiplication
- Generating functions
- Convolution in signal processing

Multiplying Polynomials

Example

Multiplying Polynomials

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$$A(x) = 3x^2 + 2x + 5$$

Multiplying Polynomials

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$$B(x) = 5x^2 + x + 2$$

Multiplying Polynomials

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$$A(x) = 3x^2 + 2x + 5$$
$$B(x) = 5x^2 + x + 2$$
$$A(x)B(x) = 15x^4 + 13x^3 + 33x^2 + 9x + 10$$

Multiplying polynomials

Input: Two $n-1$ degree polynomials:

$$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

$$b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$$

Output:

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Output: The product polynomial:

$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1 x + c_0$$

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Multiplying Polynomials

Example

Input: $n = 3, A = (3, 2, 5), B = (5, 1, 2)$

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Multiplying Polynomials

Example

Input: $n = 3, A = (3, 2, 5), B = (5, 1, 2)$

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Output: $C = (15, 13, 33, 9, 10)$

Outline

1. Problem Overview
2. Naïve Algorithm
3. Naïve Divide and Conquer Algorithm
4. Faster Divide and Conquer

`MultPoly`(A, B, n)

```
pair ← Array[n][n]
for i from 0 to n − 1:
    for j from 0 to n − 1:
        pair[i][j] ← A[i] ∗ B[j]
```

MultPoly(A, B, n)

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for i from 0 to n − 1:
  for j from 0 to n − 1:
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product ← Array[2n − 1]
for i from 0 to 2n − 1:
  product[i] ← 0
```

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product ← Array[2n − 1]
for i from 0 to 2n − 1:
  product[i] ← 0
for i from 0 to n − 1:
  for j from 0 to n − 1:
    product[i + j] ← product[i + j] + pair[i][j]
```

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  for j from 0 to n − 1:
    product[i + j] ← product[i + j] + pair[i][j]
return product
```

Multiplying Polynomials

Naïve Solution: $O(n^2)$

- Multiply all $a_i b_j$ pairs (n^2 multiplications)

Multiplying Polynomials

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- Multiply all $a_i b_j$ pairs (n^2 multiplications)
- Sum needed pairs (n^2 additions)

Outline

Multiplying Polynomials

- Let $A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$ where
  $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + \ldots + a_{\frac{n}{2}}$
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- Let $B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$ where
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  $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \ldots + b_0$
- $AB = (D_1x^{\frac{n}{2}} + D_0)(E_1x^{\frac{n}{2}} + E_0)$
  $= (D_1E_1)x^n + (D_1E_0 + D_0E_1)x^{\frac{n}{2}} + D_0E_0$

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- Calculate D_1E_1, D_1E_0, D_0E_1, and D_0E_0

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Recurrence: $T(n) = 4T(\frac{n}{2}) + kn$.

Polynomial Mult: Divide & Conquer

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

Polynomial Mult: Divide & Conquer

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$$\begin{aligned} AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 + \\ & 6x^2 + 11x + 4 \\ = {} & 4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4 \end{aligned}$$

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$R[0..n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l)$

$R[n..2n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

D_0E_1 = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

Function Mult2(A, B, n, a_l, b_l)

$R =$ `array`$[0..2n-2]$
`if` $n = 1$:
  $R[0] = A[a_l] * B[b_l]$; `return` R
$R[0\,..\,n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l)$
$R\,[n\,..\,2n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$
D_0E_1 = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$
D_1E_0 = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$
$R[\frac{n}{2} \ldots n + \frac{n}{2} - 2]$ += $D_1E0 + D_0E_1$

Function Mult2(A, B, n, a_l, b_l)

$R =$ `array`$[0..2n-2]$

`if` $n = 1$:

  $R[0] = A[a_l] * B[b_l]$; `return` R

$R[0\,..\,n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l)$

$R\,[n\,..\,2n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

D_0E_1 = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

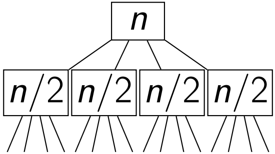
D_1E_0 = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

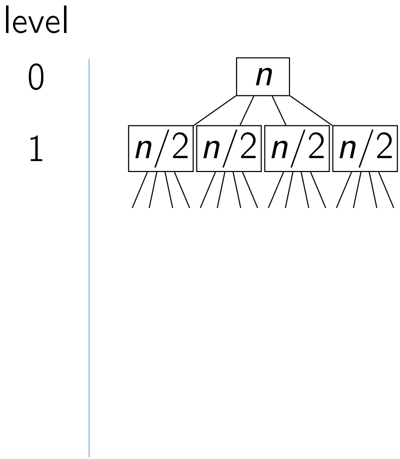
$R[\frac{n}{2} \ldots n + \frac{n}{2} - 2]$ += $D_1E0 + D_0E_1$

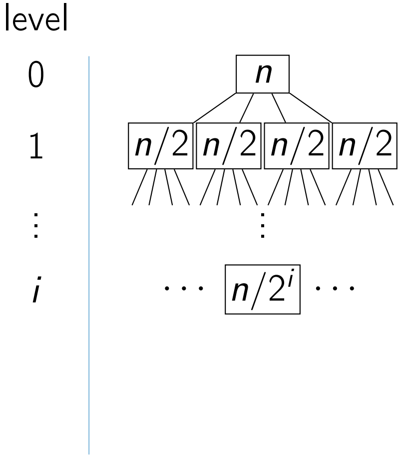
`return` R

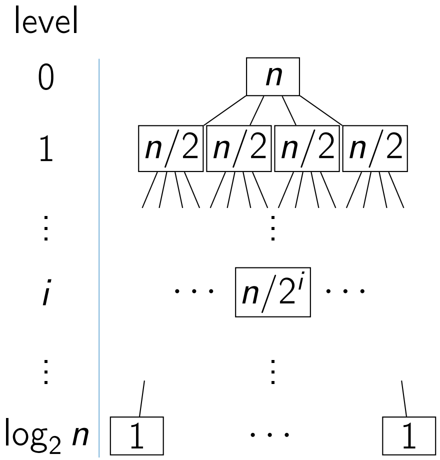
$\boxed{n}$

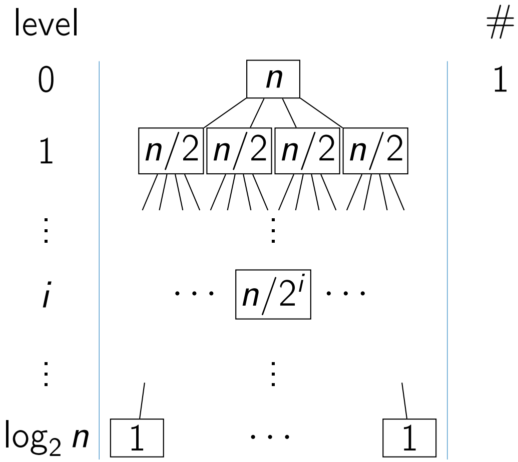
level

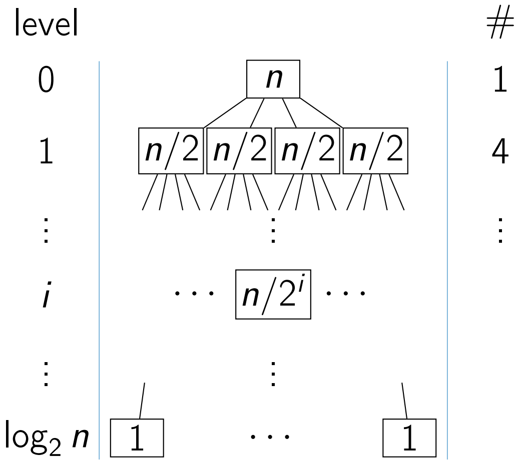


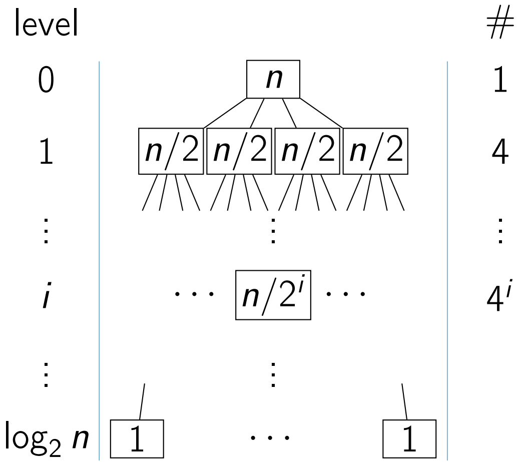

level
0
1
n
n/2
n/2
n/2
n/2


level
0
1
i
n
n/2
n/2
n/2
n/2
n/2^i

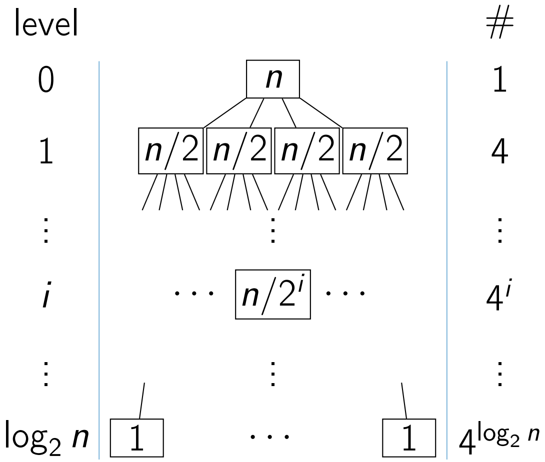

level
0
1
$\vdots$
i
$\vdots$
$\log_2 n$
n
$n/2$
$n/2$
$n/2$
$n/2$
$n/2^i$
1
1

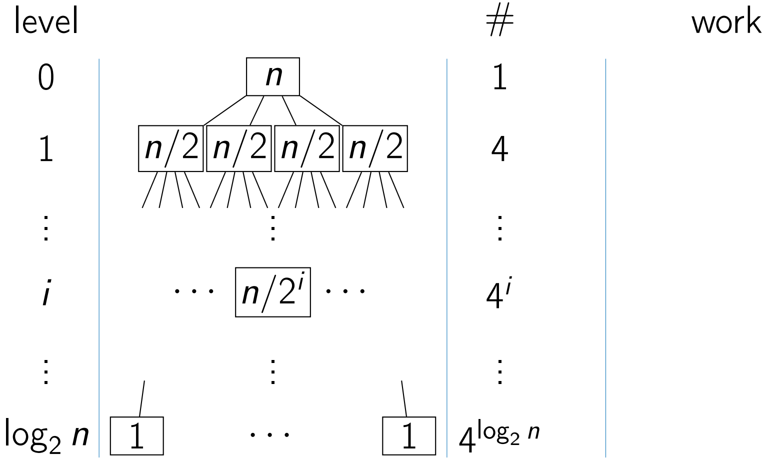

level
#
0
1
1
i
log2 n
n
n/2
n/2
n/2
n/2
n/2^i
1
1

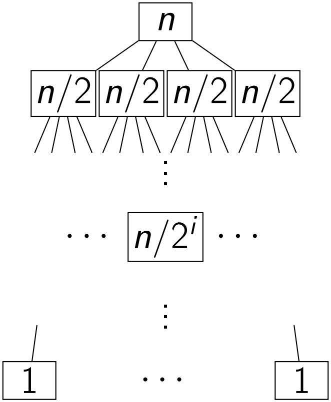

level
#
0
n
1
1
n/2
n/2
n/2
n/2
4
i
n/2^i
log_2 n
1
1



level
#
0
n
1
1
$n/2$
$n/2$
$n/2$
$n/2$
4
i
$n/2^i$
4^i
$\log_2 n$
1
1


level
#
0
n
1
1
$n/2$
$n/2$
$n/2$
$n/2$
4
i
$n/2^i$
4^i
$\log_2 n$
1
1
$4^{\log_2 n}$

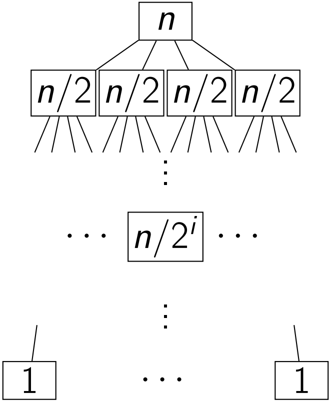

level
#
work
0
n
1
1
n/2
n/2
n/2
n/2
4
i
n/2^i
4^i
log_2 n
1
1
4^{log_2 n}



level	#	work
0	1	kn
1	4	
$\vdots$	$\vdots$	
i	4^i	
$\vdots$	$\vdots$	
$\log_2 n$	$4^{\log_2 n}$	

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$	
i	$\cdots\ \boxed{n/2^i}\ \cdots$	4^i	
$\vdots$	$\vdots$	$\vdots$	
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$4^{\log_2 n}$	

level	#	work
0	1	kn
1	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$
i	4^i	$4^i k \frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	
$\log_2 n$	$4^{\log_2 n}$	



level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots\ \boxed{n/2^i}\ \cdots$	4^i	$4^i k \frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots\ \boxed{n/2^i}\ \cdots$	4^i	$4^i k \frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

Total: $\sum_{i=0}^{\log_2 n} 4^i k \frac{n}{2^i} = \Theta(n^2)$

Outline

1. Problem Overview
2. Naïve Algorithm
3. Naïve Divide and Conquer Algorithm
4. Faster Divide and Conquer

Karatsuba approach

Karatsuba approach

$A(x) = a_1 x + a_0$

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

Karatsuba approach

$$\begin{aligned} A(x) &= a_1 x + a_0 \\ B(x) &= b_1 x + b_0 \\ C(x) &= a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0 \end{aligned}$$

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Karatsuba approach

$$A(x) = a_1 x + a_0$$
$$B(x) = b_1 x + b_0$$
$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

$$C(x) = a_1 b_1 x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0)x + a_0 b_0$$

Karatsuba approach

$$A(x) = a_1 x + a_0$$
$$B(x) = b_1 x + b_0$$
$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

$$C(x) = a_1 b_1 x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0)x + a_0 b_0$$

Needs 3 multiplications

Karatsuba approach

$$
\begin{aligned}
A(x) &= a_1 x + a_0 \\
B(x) &= b_1 x + b_0 \\
C(x) &= a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0
\end{aligned}
$$

Needs 4 multiplications

Rewrite as:

$$
\begin{aligned}
C(x) = {} & a_1 b_1 x^2 + \\
& ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0)x + \\
& a_0 b_0
\end{aligned}
$$

Needs 3 multiplications

Karatsuba approach

$$A(x) = a_1 x + a_0$$
$$B(x) = b_1 x + b_0$$
$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

$$\begin{aligned} C(x) = {} & a_1 b_1 x^2 + \\ & ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0)x + \\ & a_0 b_0 \end{aligned}$$

Needs 3 multiplications

Karatsuba Example

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

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$$E_1(x) = x + 2$$

Karatsuba Example

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$$B(x) = x^3 + 2x^2 + 3x + 4$$

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$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

Karatsuba Example

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$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

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$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) =$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$
$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$
$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$
$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6)\\ &= 24x^2 + 52x + 24\end{aligned}$$

$$AB = (4x^2 + 11x + 6)x^4 +$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (\\ & \qquad\qquad\qquad\qquad)x^2 +\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

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$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

$$\begin{aligned}AB = &(4x^2 + 11x + 6)x^4 + \\ &(24x^2 + 52x + 24 \\ &\qquad\qquad\qquad\qquad)x^2 +\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ &)x^2 +\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = {\color{red}6x^2 + 11x + 4}$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6)\\ &= 24x^2 + 52x + 24\end{aligned}$$

$$\begin{aligned}AB = {}&(4x^2 + 11x + 6)x^4 +\\ &(24x^2 + 52x + 24 - (4x^2 + 11x + 6)\\ &\qquad - ({\color{red}6x^2 + 11x + 4}))x^2 +\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6)\\ &= 24x^2 + 52x + 24\end{aligned}$$

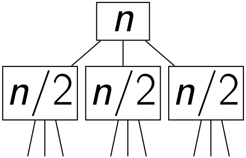
$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ & \qquad - (6x^2 + 11x + 4))x^2 + \\ & 6x^2 + 11x + 4\end{aligned}$$

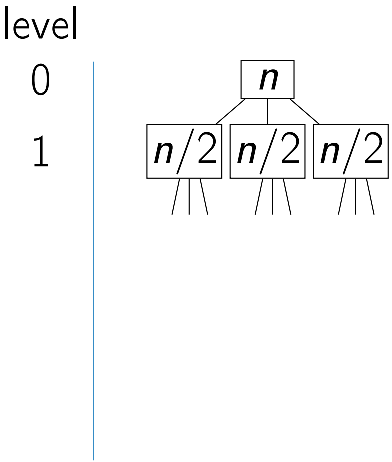
Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$
$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$
$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$
$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$
$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ & \qquad - (6x^2 + 11x + 4))x^2 + \\ & 6x^2 + 11x + 4 \\ = {} & 4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4\end{aligned}$$

$\boxed{n}$

level




level
0
1
n
n/2
n/2
n/2

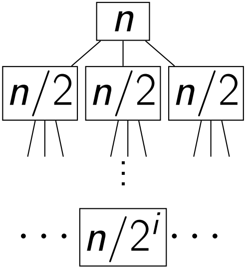
level

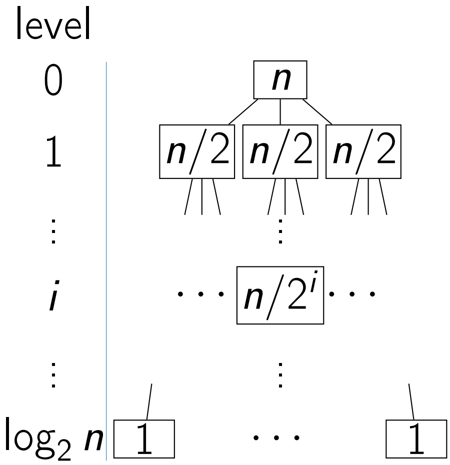
0

1

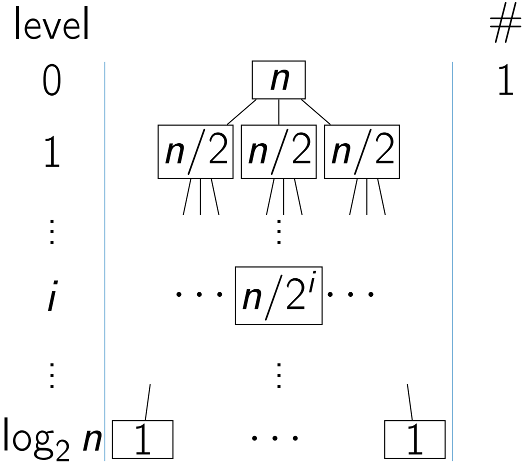
⋮

i

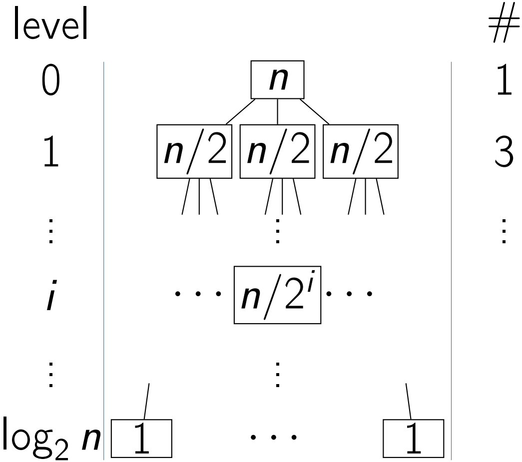


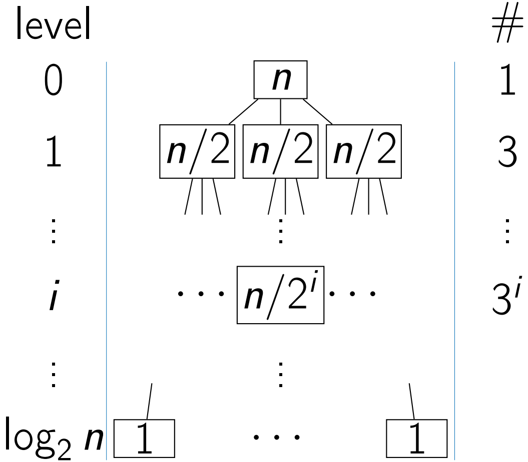


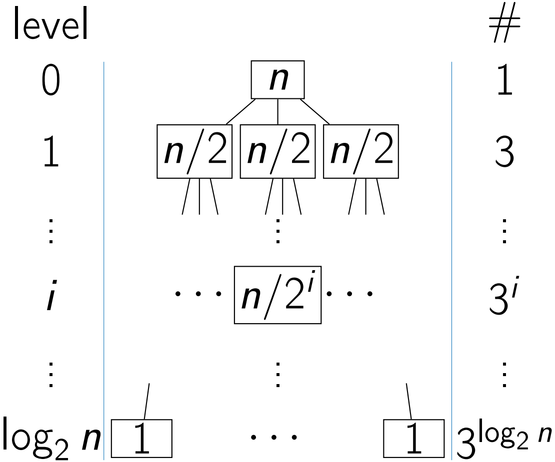
level
0
1
i
log2 n
n
n/2
n/2
n/2
n/2^i
1
1



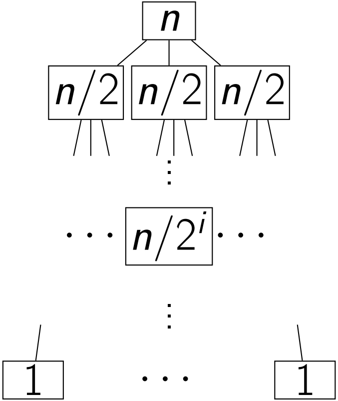
level
#
0
1
n
1
n/2
n/2
n/2
i
n/2^i
log_2 n
1
1


level
#
0
n
1
1
$n/2$
$n/2$
$n/2$
3
i
$\cdots n/2^i \cdots$
$\log_2 n$
1
$\cdots$
1


level
#
0
n
1
1
n/2
n/2
n/2
3
i
n/2^i
3^i
log_2 n
1
1


level
#
0
n
1
1
$n/2$
$n/2$
$n/2$
3
i
$n/2^i$
3^i
$\log_2 n$
1
1
$3^{\log_2 n}$

level		#	work
0	$\boxed{n}$	1	
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	3	
$\vdots$	$\vdots$	$\vdots$	
i	$\cdots \boxed{n/2^i} \cdots$	3^i	
$\vdots$	$\vdots$	$\vdots$	
$\log_2 n$	$\boxed{1} \quad \cdots \quad \boxed{1}$	$3^{\log_2 n}$	



level	#	work
0	1	kn
1	3	
$\vdots$	$\vdots$	
i	3^i	
$\vdots$	$\vdots$	
$\log_2 n$	$3^{\log_2 n}$	

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$	
i	$\cdots \boxed{n/2^i} \cdots$	3^i	
$\vdots$	$\vdots$	$\vdots$	
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$3^{\log_2 n}$	

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots\ \boxed{n/2^i}\ \cdots$	3^i	$3^i k \frac{n}{2^i} = k(\frac{3}{2})^i n$
$\vdots$	$\vdots$	$\vdots$	
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$3^{\log_2 n}$	

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots \boxed{n/2^i} \cdots$	3^i	$3^i k \frac{n}{2^i} = k(\frac{3}{2})^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

level		#	work
0	n	1	kn
1	$n/2 \quad n/2 \quad n/2$	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots\ n/2^i\ \cdots$	3^i	$3^i k\frac{n}{2^i} = k(\frac{3}{2})^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$1 \quad \cdots \quad 1$	$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

$$\text{Total: } \sum_{i=0}^{\log_2 n} 3^i k\frac{n}{2^i} = \Theta(n^{\log_2 3})$$

$$= \Theta(n^{1.58})$$