Divide-and-Conquer: Quick Sort

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Algorithmic Toolbox
Data Structures and Algorithms

Outline

Quick Sort

- comparison based algorithm
- running time: $O(n \log n)$ (on average)
- efficient in practice

Example: quick sort

6	4	8	2	9	3	9	4	7	6	1

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partition with respect to $x = A[1]$

in particular, x is in its final position

1	4	2	3	4	6	6	9	7	8	9

≤ 6 $\qquad$ > 6

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1	4	2	3	4	6	6	9	7	8	9

sort the two parts recursively

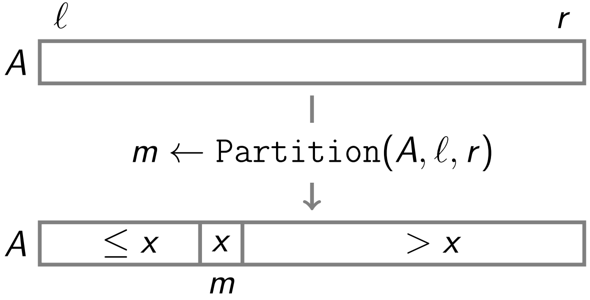
1	2	3	4	4	6	6	7	8	9	9

Outline

```
QuickSort(A, ℓ, r)

if ℓ ≥ r:
  return
m ← Partition(A, ℓ, r)
{A[m] is in the final position}
QuickSort(A, ℓ, m − 1)
QuickSort(A, m + 1, r)
```


ℓ
r
A


ℓ
r
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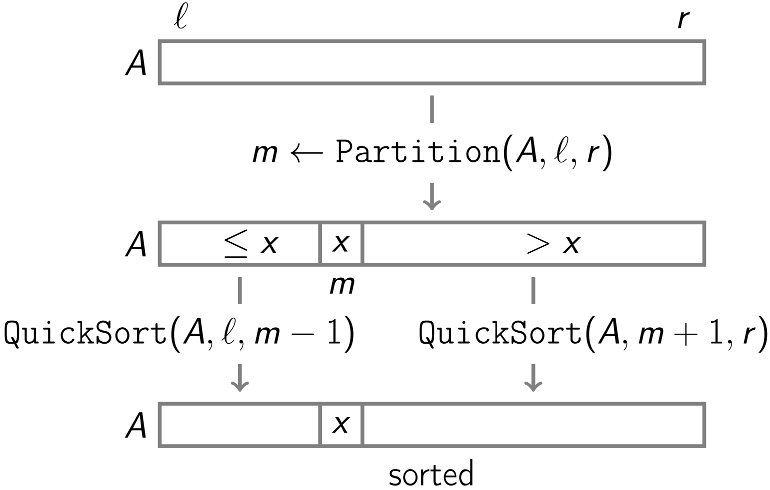
$m \leftarrow \texttt{Partition}(A, \ell, r)$
A
$\leq x$
x
$> x$
m



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$m \leftarrow \texttt{Partition}(A, \ell, r)$
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x
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$\texttt{QuickSort}(A, \ell, m-1)$
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A
x
sorted

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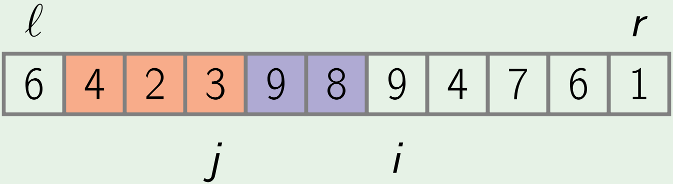

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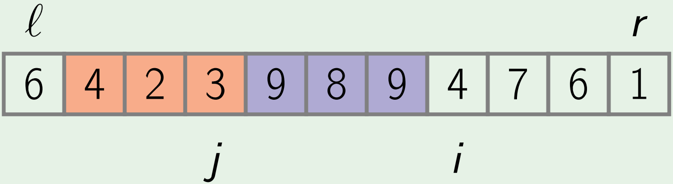

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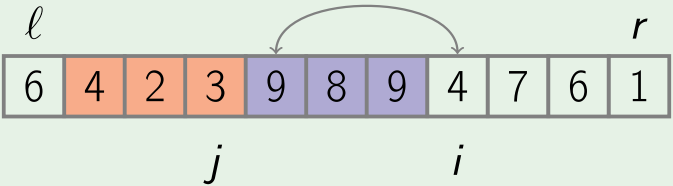

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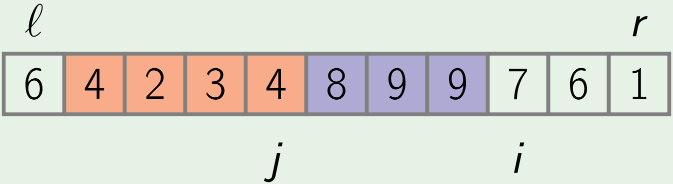

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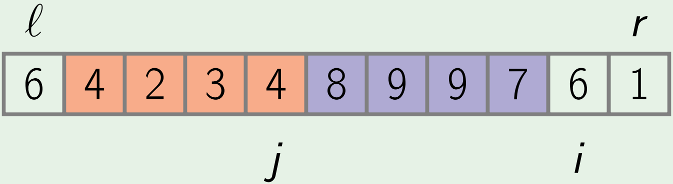

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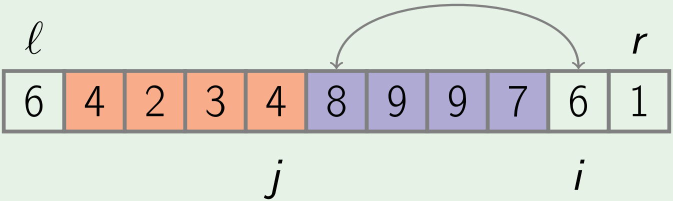

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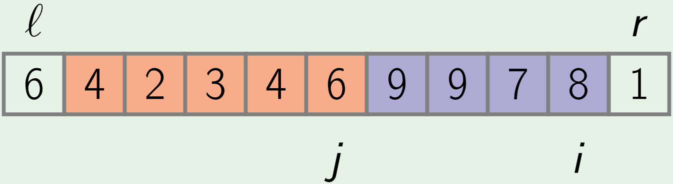

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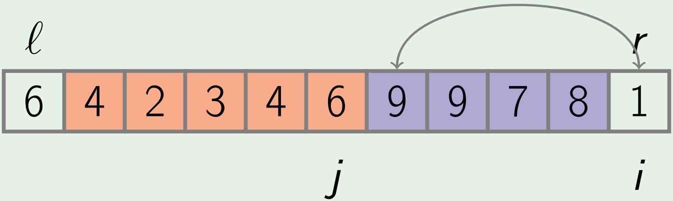

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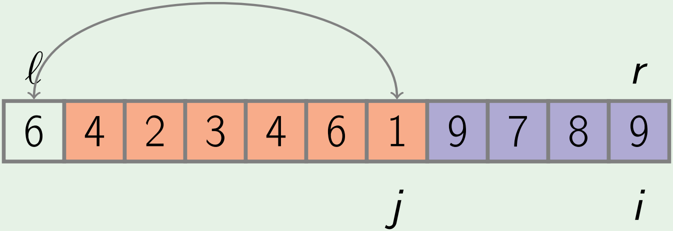

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						j				i

$\texttt{Partition}(A, \ell, r)$

```
x ← A[ℓ]     {pivot}
j ← ℓ
for i from ℓ+1 to r:
   if A[i] ≤ x:
      j ← j+1
      swap A[j] and A[i]
   {A[ℓ+1...j] ≤ x, A[j+1...i] > x}
swap A[ℓ] and A[j]
return j
```

Outline

Unbalanced Partitions

- $T(n) = n + T(n-1)$:

  $$T(n) = n+(n-1)+(n-2)+\cdots = \Theta(n^2)$$

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- $T(n) = n + T(n-5) + T(4)$:

  $$T(n) \geq n+(n-5)+(n-10)+\cdots = \Theta(n^2)$$

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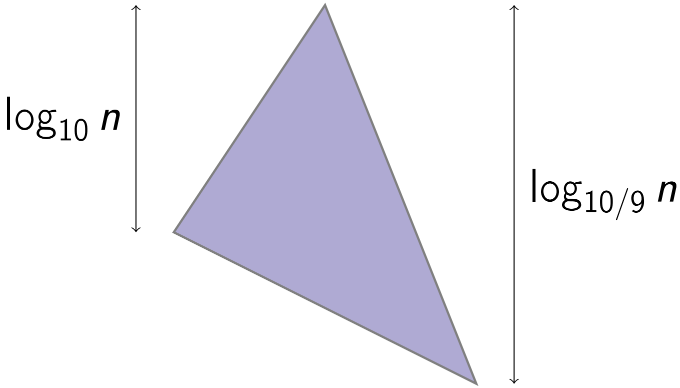
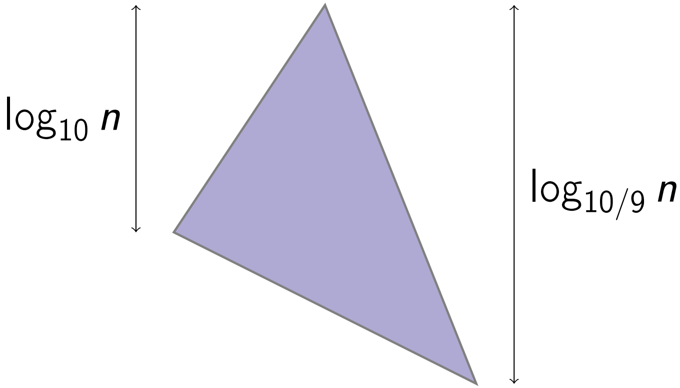


Balanced Partitions

$$T(n) = T(n/10) + T(9n/10) + O(n)$$



$$T(n) = O(n \log n)$$

Random Pivot

```
RandomizedQuickSort(A, ℓ, r)

if ℓ ≥ r:
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k ← random number between ℓ and r
swap A[ℓ] and A[k]
m ← Partition(A, ℓ, r)
{A[m] is in the final position}
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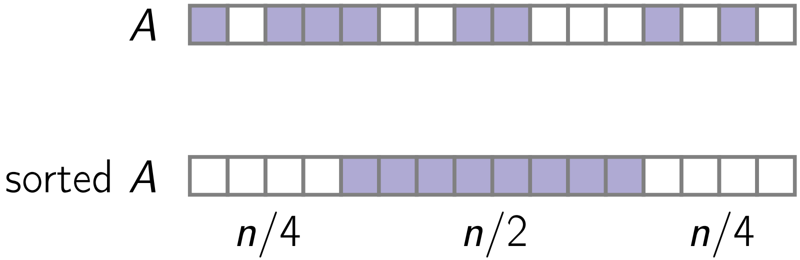
RandomizedQuickSort(A, m + 1, r)
```

Why Random?

half of the elements of A guarantees a balanced partition:



Theorem

Assume that all the elements of $A[1 \dots n]$ are pairwise different. Then the average running time of `RandomizedQuickSort`(A) is $O(n \log n)$ while the worst case running time is $O(n^2)$.

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Remark

Averaging is over random numbers used by the algorithm, but not over the inputs.

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Proof Ideas: Comparisons

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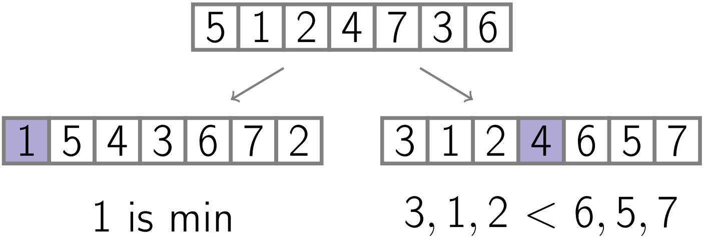
- balanced partition are better since they reduce the number of comparisons needed:



Proof Ideas: Probability

A	5	1	8	9	2	4	7	3	6
A'	1	2	3	4	5	6	7	8	9

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$$\text{Prob}\left(3 \text{ and } 4 \text{ are compared}\right) =$$

Proof Ideas: Probability

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A'	1	2	3	4	5	6	7	8	9

$$\text{Prob}\left(1 \text{ and } 9 \text{ are compared}\right) = \frac{2}{9}$$

$$\text{Prob}\left(3 \text{ and } 4 \text{ are compared}\right) = 1$$

Proof

- let, for $i < j$,

$$\chi_{ij} = \begin{cases} 1 & A'[i] \text{ and } A'[j] \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$$

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- for all $i < j$, $A'[i]$ and $A'[j]$ are either compared exactly once or not compared at all (as we compare with a pivot)
- this, in particular, implies that the worst case running time is $O(n^2)$

Proof (continued)

- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \ldots j]$ is $A'[i]$ or $A'[j]$

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- then $\text{Prob}(\chi_{ij}) = \frac{2}{j-i+1}$ and $\text{E}(\chi_{ij}) = \frac{2}{j-i+1}$

Proof (continued)

Then (the expected value of) the running time is

$$
\begin{aligned}
\mathrm{E} \sum_{i=1}^{n} \sum_{j=i+1}^{n} \chi_{ij} &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathrm{E}(\chi_{ij}) \\
&= \sum_{i<j} \frac{2}{j-i+1} \\
&\leq 2n \cdot \left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right) \\
&= \Theta(n \log n)
\end{aligned}
$$

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- the array is always split into two parts of size 0 and $n-1$
- $T(n) = n + T(n-1) + T(0)$ and hence $T(n) = \Theta(n^2)$!

To handle equal elements, we replace the line

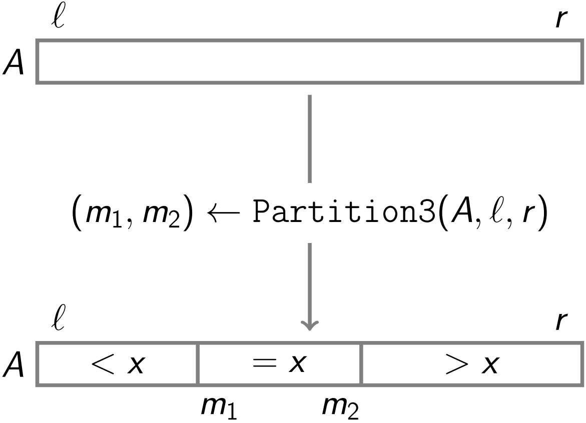
$$m \leftarrow \texttt{Partition}(A, \ell, r)$$

with the line

$$(m_1, m_2) \leftarrow \texttt{Partition3}(A, \ell, r)$$

such that

- for all $\ell \leq k \leq m_1 - 1$, $A[k] < x$
- for all $m_1 \leq k \leq m_2$, $A[k] = x$
- for all $m_2 + 1 \leq k \leq r$, $A[k] > x$


ℓ
r
A
$(m_1, m_2) \leftarrow \texttt{Partition3}(A, \ell, r)$
ℓ
r
A
$< x$
$= x$
$> x$
m_1
m_2

RandomizedQuickSort(A, ℓ, r)

```
if ℓ ≥ r:
  return
k ← random number between ℓ and r
swap A[ℓ] and A[k]
(m₁, m₂) ← Partition3(A, ℓ, r)
{A[m₁ ... m₂] is in final position}
RandomizedQuickSort(A, ℓ, m₁ − 1)
RandomizedQuickSort(A, m₂ + 1, r)
```

Outline

Tail Recursion Elimination

$\texttt{QuickSort}(A, \ell, r)$

```
while ℓ < r:
   m ← Partition(A, ℓ, r)
   QuickSort(A, ℓ, m − 1)
   ℓ ← m + 1
```

$\texttt{QuickSort}(A, \ell, r)$

```
while ℓ < r:
    m ← Partition(A, ℓ, r)
    if (m − ℓ) < (r − m):
        QuickSort(A, ℓ, m − 1)
        ℓ ← m + 1
    else:
        QuickSort(A, m + 1, r)
        r ← m − 1
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QuickSort(A, ℓ, r)

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Worst-case space requirement: $O(\log n)$

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- if the recursion depth exceeds a certain threshold $c \log n$ the algorithm switches to heap sort
- the running time is $O(n \log n)$ in the worst case

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- Efficient in practice