Divide-and-Conquer: Polynomial Multiplication

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Data Structures and Algorithms Algorithmic Toolbox

Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer

Uses of multiplying polynomials

- Error-correcting codes
- Large-integer multiplication
- Generating functions
- Convolution in signal processing

$$A(x) = 3x^2 + 2x + 5$$

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$$B(x) = 5x^2 + x + 2$$

$$A(x) = 3x^{2} + 2x + 5$$

$$B(x) = 5x^{2} + x + 2$$

$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Input: Two n-1 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output:

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$
 $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output: The product polynomial:
$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$

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$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$
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$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$
 where: $c_{2n-2} = a_{n-1}b_{n-1}$

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Output: The product polynomial:
$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$
 where:

$$c_{2n-2} = a_{n-1}b_{n-1}$$

 $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output: The product polynomial: $c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$ where: $c_{2n-2} = a_{n-1}b_{n-1}$ $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$... $c_2 = a_2b_0 + a_1b_1 + a_0b_2$

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output: The product polynomial: $c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$

where:
$$c_{2n-2}=a_{n-1}b_{n-1}\ c_{2n-3}=a_{n-1}b_{n-2}+a_{n-2}b_{n-1}$$

 $c_2 = a_2b_0 + a_1b_1 + a_0b_2$ $c_1 = a_1b_0 + a_0b_1$

Input: Two n-1 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$ Output: The product polynomial: $c_{2n-2}x^{2n-2}+c_{2n-3}x^{2n-3}+\cdots+c_1x+c_0$ where: $c_{2n-2} = a_{n-1}b_{n-1}$ $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$

$$c_{2n-2} = a_{n-1}b_{n-1}$$
 $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b$
...
 $c_2 = a_2b_0 + a_1b_1 + a_0b_2$
 $c_1 = a_1b_0 + a_0b_1$
 $c_0 = a_0b_0$

Example

Input: n = 3, A = (3, 2, 5), B = (5, 1, 2)

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$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Example

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$$n = 3, A = (3, 2, 5), B = (5, 1, 2)$$

$$A(x) = 3x^{2} + 2x + 5$$

$$B(x) = 5x^{2} + x + 2$$

$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Output: C = (15, 13, 33, 9, 10)

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MultPoly(A, B, n)

```
\begin{array}{l} \textit{pair} \leftarrow \textit{Array}[n][n] \\ \textit{for } \textit{i} \;\; \textit{from 0 to } n-1 \text{:} \\ \textit{for } \textit{j} \;\; \textit{from 0 to } n-1 \text{:} \\ \textit{pair}[\textit{i}][\textit{j}] \leftarrow \textit{A}[\textit{i}] * \textit{B}[\textit{j}] \end{array}
```

MultPoly(A, B, n)

```
pair \leftarrow Array[n][n]
for i from 0 to n-1:
for j from 0 to n-1:
pair[i][j] \leftarrow A[i] * B[j]
product \leftarrow Array[2n-1]
for i from 0 to 2n-1:
product[i] \leftarrow 0
```

```
MultPoly(A, B, n)

pair \leftarrow Array[n][n]

for i from 0 to n-1:

for j from 0 to n-1:

pair[i][j] \leftarrow A[i] * B[<math>j]
```

product \leftarrow Array[2n-1] for i from 0 to 2n-1:

for i from 0 to n-1:

for i from 0 to n-1:

 $product[i + j] \leftarrow product[i + j] + pair[i][j]$

 $product[i] \leftarrow 0$

```
MultPoly(A, B, n)
pair \leftarrow Array[n][n]
for i from 0 to n-1:
```

 $pair[i][j] \leftarrow A[i] * B[j]$ $product \leftarrow Array[2n-1]$ for i from 0 to 2n-1: $product[i] \leftarrow 0$ for i from 0 to n-1: for i from 0 to n-1:

 $product[i + j] \leftarrow product[i + j] + pair[i][j]$

for i from 0 to n-1:

return *product*

Naïve Solution: $O(n^2)$

■ Multiply all $a_i b_j$ pairs (n^2 multiplications)

Naïve Solution: $O(n^2)$

- Multiply all $a_i b_j$ pairs (n^2 multiplications)
- Sum needed pairs $(n^2 \text{ additions})$

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Let
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_n$

$$D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + \dots + a_{\frac{n}{2}}$$

$$D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + a_0$$

Let
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$ $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$

Let
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + \dots + b_{\frac{n}{2}}$ $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + b_0$

Let
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$ $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$

Let
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + ... + b_{\frac{n}{2}}$ $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + b_0$

$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

= $(D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$

Let
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 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$ $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$

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$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
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$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

$$= (D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$$

lacktriangle Calculate D_1E_1, D_1E_0, D_0E_1 , and D_0E_0

Let
$$A(x) = D_1(x)x^{\frac{n}{2}} + D_0(x)$$
 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$ $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$

Let
$$B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$$
 where $E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + ... + b_{\frac{n}{2}}$ $E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + b_0$

$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

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Let
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 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$

$$D_1(x) = a_{n-1}x^2 + a_{n-2}x^2 + \dots + a_{\frac{n}{2}}$$

$$D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + a_0$$
• Let $B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$ where

$$E_{1}(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + \dots + b_{\frac{n}{2}}$$

$$E_{0}(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + b_{0}$$

■
$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

= $(D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$
■ Calculate $D_1 E_1$, $D_1 E_0$, $D_0 E_1$, and $D_0 E_0$

Recurrence: $T(n) = 4T(\frac{n}{2}) + kn$.

Polynomial Mult: Divide & Conquer $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$A(x) = 4x^{3} + 3x^{2} + 2x + 4$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

Polynomial Mult: Divide & Conquer $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$A(x) = \frac{4x^3 + 3x^2 + 2x + 1}{B(x)}$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$

Polynomial Mult: Divide & Conquer $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_0(x) = 2x + 1$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$

 $E_1(x) = x + 2$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $E_1(x) = x + 2$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$ $D_0(x) = 2x + 1$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $E_1(x) = x + 2$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$ $D_0(x) = 2x + 1$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x +$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$= 4x + 3$$
$$= x + 2$$

$$D_0(x)=2x+1$$

$$D_0(x) = 2x + 1$$

 $E_0(x) = 3x + 4$

$$= 2x + 1$$

= $3x + 4$

 $D_1 E_0 = 12x^2 + 25x + 12$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$= 4x + 3$$
$$= x + 2$$

$$E_1(x) = \frac{x+2}{D_1 E_1} = 4x^2 + 11x + 6$$

$$+11x + 6$$

$$E_0(x) = 3x + 4$$

 $D_1E_0 = 12x^2 + 4$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_0 E_1 = 2x^2 + 5x + 2$$

$$D_1 E_0 = 12x^2 + 25x + 12$$

 $D_0(x) = 2x + 1$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $D_0 E_1 = 2x^2 + 5x + 2$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$



$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$D_1 E_0 = 12x^2 + 25x +$$

$$D_0 E_0 = 6x^2 + 11x + 4$$



 $D_0(x) = 2x + 1$

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) - 4x + 3$$

$$D_1(x) = 4x + 3$$

 $D_0 E_1 = 2x^2 + 5x + 2$

AB =

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

$$c+2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_1 E_0 = 12x^2 + 25x + 12$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$D_0 E_0 =$$

 $D_0(x) = 2x + 1$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $D_0 E_1 = 2x^2 + 5x + 2$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$E_0 = 0$$

 $D_0(x) = 2x + 1$

$$D_1 E_0 = 12x^2 + 25x + 12$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $F_1(x) = x + 2$

$$E_1(x) = 1x + 3$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_1E_1 = 4x^2 + 11x + 0$$
$$D_0E_1 = 2x^2 + 5x + 2$$

$$D_0E_1 = 2x^2 + 5x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 + 6x^4$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$D_1 E_0 = 12x^2 + 25x + 12$$

 $)x^{2} +$



$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

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 $E_1(x) = x + 2$

 $(12x^2 + 25x + 12)$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_0E_1 = 4x^2 + 11x + 0$$

$$D_0E_1 = 2x^2 + 5x + 2$$

$$AB = (4x^2 + 11x + 6)x^4 + 1$$

$$D_0 E_0 =$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

 $)x^{2} +$

 $D_0(x) = 2x + 1$

$$25x + 1x + 1$$

$$D_1 E_0 = 12x^2 + 25x + 12$$
$$D_0 E_0 = 6x^2 + 11x + 4$$



$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $E_2(x) = x + 2$

$$E_1(x) = 4x + 3$$
$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$11x + 6$$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_1E_0 = 12x^2 + 25x + 12$
 $D_0E_1 = 2x^2 + 5x + 2$ $D_0E_0 = 6x^2 + 11x + 4$

$$D_0 E_1 = \frac{2x^2 + 5x + 2}{AB} = (4x^2 + 11x + 6)x^4 +$$

$$(4x^2 + 11x + 6)x^4 +$$

 $(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$

 $D_0(x) = 2x + 1$

$$= 6x^2 + 11x$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $6x^2 + 11x + 4$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$D_1 E_1 = 4x^2 + 11x + 6$$
$$D_0 E_1 = 2x^2 + 5x + 2$$

$$D_0E_1 = 2x^2 + 5x + 2$$

$$AB = (4x^2 + 11x + 6)x^4 + 6x^4 + 6x^4$$

$$(4x^2 + 11x + 6)x^4 +$$

 $(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$

 $D_0(x) = 2x + 1$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$D_1 E_0 = 12x^2 + 25x + 12$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_2(x) = 2x + 1$$

$$A(x) = 4x^{3} + 3x^{2} + 2x + 1$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_{1}(x) = 4x + 3$$

$$D_{0}(x) = 2x + 1$$

 $E_1(x) = x + 2$

 $D_1 E_1 = 4x^2 + 11x + 6$

 $D_0E_1 = 2x^2 + 5x + 2$

 $AB = (4x^2 + 11x + 6)x^4 +$

 $6x^2 + 11x + 4$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

 $(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$

 $=4x^{6} + 11x^{5} + 20x^{4} + 30x^{3} + 20x^{2} + 11x + 4$

 $E_0(x) = 3x + 4$

 $D_1 E_0 = 12x^2 + 25x + 12$

 $D_0 E_0 = 6x^2 + 11x + 4$

Function Mult2(A, B, n, a_l, b_l)

Function Mult2 (A, B, n, a_l, b_l)

$$R = \text{array}[0..2n - 2]$$

if $n = 1$:
 $R[0] = A[a_l] * B[b_l]$; return R

Function Mult2 (A, B, n, a_l, b_l)

$$R = array[0..2n - 2]$$

if $n = 1$:
 $R[0] = A[a_I] * B[b_I]$; return R
 $R[0..n - 2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$

 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l +$

if
$$n = 1$$
:
 $R[0] = A[a_l] * B[b_l]$; return R
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_l, b_l)$

 $\frac{n}{2}$

if
$$n = 1$$
:
 $R[0] = A[a_i] * B[b_i]$

$$R[0] = A[a_I] * B[b_I]$$
; return R
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$
 $R[n-2n-2] = Mult2(A, B, \frac{n}{2}, a_I + \frac{1}{2})$

$$R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$$

if
$$n = 1$$
:
 $R[0] = A[a_I] * B[b_I]$; return R
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$

$$R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$$

$$\frac{n}{2}$$
)
 $D_0 E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$
 $D_1 E_0 = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$
 $R[\frac{n}{2} \dots n + \frac{n}{2} - 2] + D_1 E_0 + D_0 E_1$

if
$$n = 1$$
:
 $R[0] = A[a_I] * B[b_I]$; return R
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$

$$R[n..2n-2] = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$$

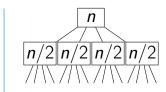
$$D_0 E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$$

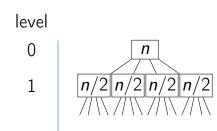
$$R[n...2n-2]$$
 - Mult2 $(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$
 D_0E_1 = Mult2 $(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$
 D_1E_0 = Mult2 $(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$
 $R[\frac{n}{2}...n + \frac{n}{2} - 2]$ += $D_1E_0 + D_0E_1$

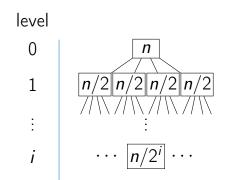
return R

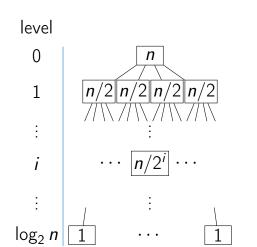


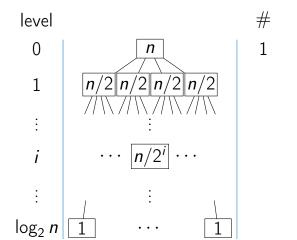
level

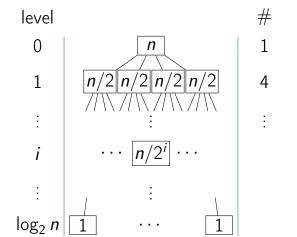


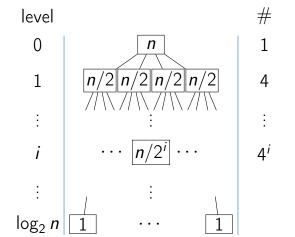


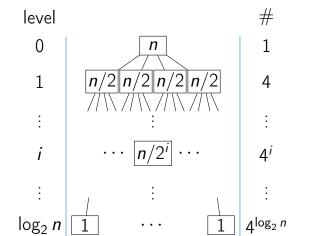


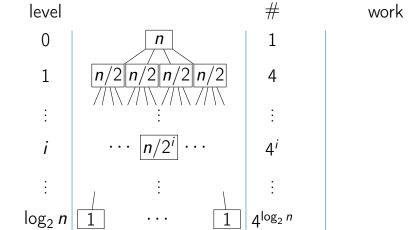


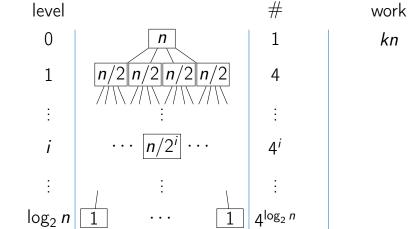


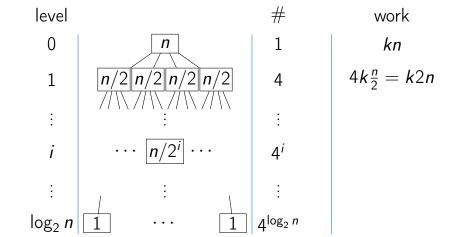


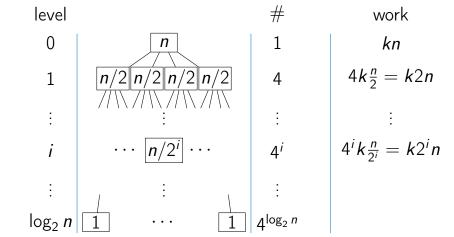


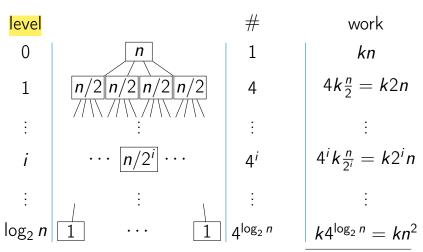








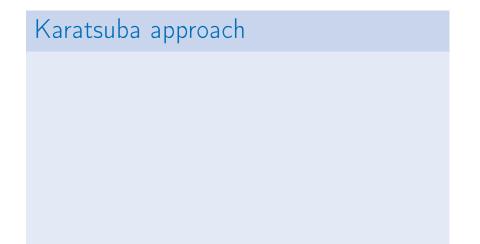




Total: $\sum_{i=0}^{\log_2 n} 4^i k \frac{n}{2^i} = \Theta(n^2)$

Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer



$$A(x) = a_1x + a_0$$

$$A(x) = a_1x + a_0$$
$$B(x) = b_1x + b_0$$

 $C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$x + L$$

Needs 4 multiplications

$$C(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1) x + a_0 b_0$$

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

$$b_1 x + b_1 x^2$$

 $C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$ Needs 4 multiplications

Needs 4 multip

Rewrite as:

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

 $C(x) = a_1 b_1 x^2 + b_0$

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

Needs 4 multiplications

Rewrite as:
$$C(x) = a_1b_1x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x + a_0b_0$$

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

 $C(x) = a_1 b_1 x^2 + b$

 $C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$ Needs 4 multiplications

Rewrite as:
$$C(x) = ab x^2$$

 $C(x) = \frac{a_1b_1x^2}{a_1b_1x^2} + \frac{a_1b_1x$

$$y$$
rite as:
 $(x) = \frac{1}{a_1} \frac{b_1}{b_1} x^2$

 a_0b_0

Needs 3 multiplications

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

 a_0b_0

Needs 3 multiplications

Rewrite as:
$$C(x) = a_1 b_1 x^2 +$$

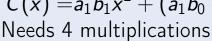
 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$

$$A(x) = a_1x + a_0$$







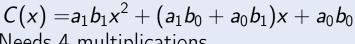


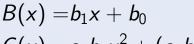
Rewrite as:

 $C(x) = a_1b_1x^2 +$

 a_0b_0

Needs 3 multiplications





$$x + a_0$$



 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$





$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$A(x) = 4x^{3} + 3x^{2} + 2x + 4$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_{1}(x) = 4x + 3$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $D_0(x) = 2x + 1$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_0$$

$$D_0($$

$$D_0(x)=2x+1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D(x) = x + 2x + 3x + 4$$

 $D_1(x) = 4x + 3$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$\begin{aligned} (x) &= 4x + 3 \\ (x) &= x + 2 \end{aligned}$$

$$x) = 4x + 3$$
$$x) = x + 2$$

$$(x) = 4x + 3$$
$$(x) = x + 2$$

$$D_0(x) = 2x + 1$$

 $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$F_1(x) = x + 2$$

$$D_1(x) = 4x + 5$$

$$E_1(x) = x + 2$$

$$D_1E_1 = 4x^2 + 11x + 6$$

$$D_0(x) = 2x + 1$$
$$E_0(x) = 3x + 4$$

$$E_0(x)=3x+4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $F_1(x) = x + 2$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$
 $D_1E_1 = 4x^2 + 11x + 6$

$$D_0(x) = 2x + 1$$

$$E_0(x) = 3x + 4$$

$$3x - 6x^2$$

$$E_0(x) = 3x + 4$$

 $D_0 E_0 = 6x^2 + 11x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

 $(D_1 + D_0)(E_1 + E_0) =$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $D_0 E_0 = 6x^2 + 11x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$D_0(x) = 2x + 1$$

$$E_0(x) = 3x + 4$$

$$E_0(x) = 3x + 4$$

$$D_0 E_0 = 6x^2 + 11x + 4$$



$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$E_0(x) = \frac{3x + 4}{D_0 E_0} = 6x^2 + 11x + 4$$

 $D_0(x) = 2x + 1$

$$\delta x^2$$

$$x^{2} + 11x$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$



$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$E_0(x) = 3x + 4$$

 $D_0(x) = 2x + 1$

$$E_0(x) = 3x - 6x^2$$

$$D_0 E_0 = 6x^2$$

 $= 24x^2 + 52x + 24$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$x^2 + 11x -$$

$$5x^2 + 11x +$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$1x + 6$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$= (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_x(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$D_0 E_0 = 6x^2$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$= (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $)x^{2} +$

- $D_0 E_0 = 6x^2 + 11x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 +$

 $(24x^2 + 52x + 24)$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $)x^{2} +$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

$$D_1 E_1 = 4x^2 + 11x + 6$$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$11x + 6$$

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

= $24x^2 + 52x + 24$

$$x + 24$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $D_0 E_0 = 6x^2 + 11x + 4$

 $)x^{2} +$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$11x + 6$$

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$(6x + 4)(4x + 6)$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

$$+4)(4x + 6)$$

 $+52x + 24$

$$-52x + 24$$

$$52x + 24$$

 $-(6x^2+11x+4))x^2+$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

$$D_1E_1 = 4x^2 + 11x + 6$$

 $AB = (4x^2 + 11x + 6)x^4 +$

 $6x^2 + 11x + 4$

$$11x + 6$$

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

$$= (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

$$+52x + 24$$

 $-(6x^2+11x+4))x^2+$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$x + 24$$

 $D_0 E_0 = 6x^2 + 11x + 4$

Karatsuba Example
$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_1E_1 = 4x^2 + 11x + 6$$

$$D_0(x) = 2x + 1$$

$$E_0(x) = 3x + 4$$

$$D_0E_0 = 6x^2 + 11x + 4$$

$$E_1(x) = x + 2$$
 $E_0(x) = 3$
 $D_1E_1 = 4x^2 + 11x + 6$ $D_0E_0 = 6$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

 $6x^2 + 11x + 4$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4x^2 + 4x^2$$

$$E_0) = (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

$$9x + 4)$$

 $9x + 5$

 $=4x^{6} + 11x^{5} + 20x^{4} + 30x^{3} + 20x^{2} + 11x + 4$

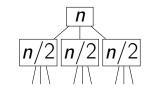
$$(4x^2 + 11x + 6)x^4 +$$

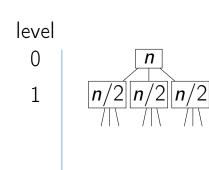
 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

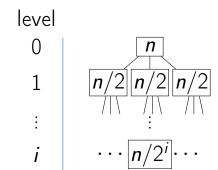
 $-(6x^2+11x+4))x^2+$

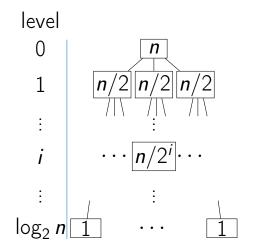


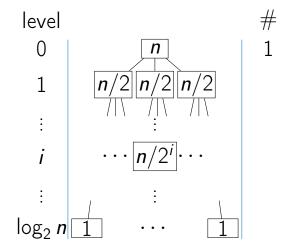
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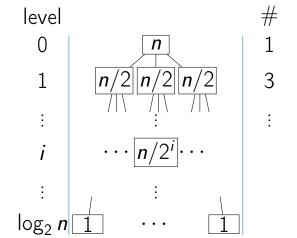


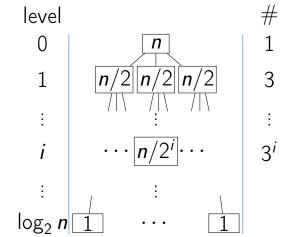


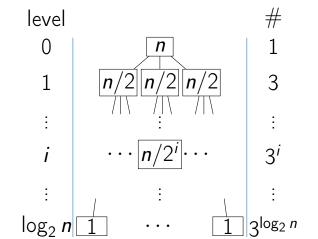


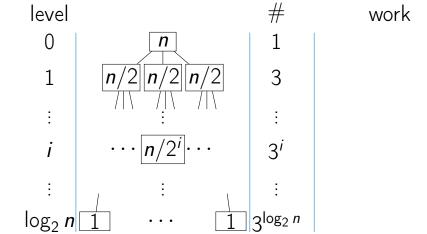


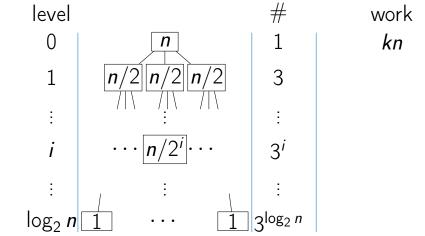












level

 $\log_2 n$ 1

#

work

 $3^{\log_2 n} k 3^{\log_2 n} = k n^{\log_2 3}$

 $=\Theta(n^{1.58})$