

# Week 3 research projects: Probability - Python

## 1 The construction of numbers

With this subject you are asked to present the construction(s) of the classical mathematical sets. Starting from the Peano axiomatisation of  $\mathbb{N}$  up to the field of complex numbers  $\mathbb{C}$ .

On the way you will consider investigating algebraic numbers, computable numbers and transcendental numbers.

## 2 Statistics of random functions

Let  $S_n$  be the set of bijections from  $\{1, 2, \dots, n\}$  to itself. An element of  $S_n$  is called a *permutation*. Each permutation  $s$  can be decomposed into *cycles* (that is an ordered list  $(i_0, i_1, \dots, i_k)$  of elements of  $\{1, 2, \dots, n\}$  so that  $s(i_0) = i_1, s(i_1) = i_2, \dots, s(i_k) = i_0$ ). With this subject you are asked to study the distribution (in the probabilistic sense) of these cycles when  $n$  is large. This subject is a mix of experimentations with Python and probability.

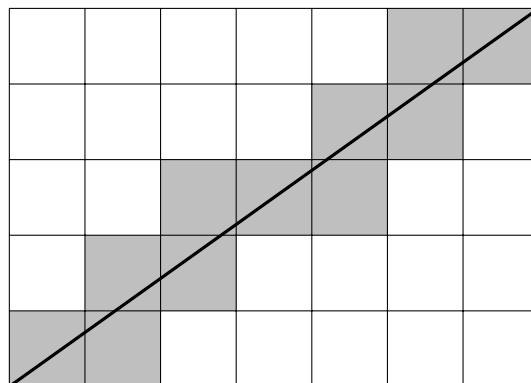
## 3 Gap distributions

Given  $n$  real numbers in  $[0, 1]$ , they cut the interval into  $n + 1$  pieces. We can then consider the lengths of these pieces  $y_0, y_1, \dots, y_n$  and their empirical distribution. With this subject you are asked to study this distribution when  $x_i$  come from random samples of the uniform distribution  $\mathcal{U}(0, 1)$ .

Next, you might consider other cases such as  $x_n$  being rational numbers with bounded denominators,  $x_n = \alpha n \bmod 1$  where  $\alpha$  is a real number, or  $x_n = \sqrt{n} \bmod 1$ .

## 4 How do we draw a line on a screen?

Imagine the pixels of a screen computer as a grid where each square is either black or white. The simplest way to draw a line is just to lighten the pixels that cross the trajectory of a given line as in the picture below.



This subject consists in two parts:

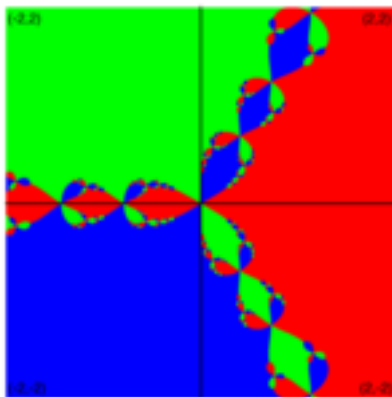
- given a line how to compute the set of pixels to lighten,
- conversely, given a set of pixels check whether there is a line that pass through them.

## 5 Prime numbers

A number is prime if it has exactly two divisors. The aim of this project is to study various algorithms related to prime numbers (making the list of primes, testing if a given number is prime, randomized tests, why primes are so important today in cryptography, etc).

## 6 The Newton method

The Newton method is a powerful tool to approximate the roots of a real or complex polynomial. However the convergence of this method is only ensured in a small neighborhood of each root.



With this subject you are asked to implement the Newton method for polynomials and experiment its convergence. You will also have to study theoretical results about the convergence (Newton-Raphson theorem, Smale theorem).

## 7 Limit cycles of ODE in the plane

Given an ordinary differential equation (ODE) in the plane, we are interested in their (isolated) periodic trajectories. In the case where the coefficient of the vector field are polynomials it is known that there are finitely many of them but without effective bound. With this subject you are asked to present ODE in the plane, the definition of periodic orbits and stability. You also have to experiment these with `numpy` and `scipy` in Python.

## 8 Sum of squares

Which positive integers are the sum of two squares  $a^2 + b^2$ ? of three squares? The aim of this subject is to start with some experiments in Python making the list of the possible way of writing each number  $n$  as a sum of two squares. Then conjecture should be formulated and tentatively proven.

## 9 The $n$ -body problem

Newton's equations describe the displacement of mass bodies in space. In this subject you are asked to understand the general solution of the 2-body problem and experiment with Python the 3-body problem in the 2-dimensional space.

## 10 Monte Carlo methods

Monte Carlo method is a generic name for algorithms that use random samples to compute certain quantity. Important examples include

- computing  $\pi$ ,
- integration of functions in high dimension,
- primality testing.

With this subject you are asked to understand and implement some algorithms involving Monte Carlo method and also to study the (probabilistic) speed of convergence of these methods.

## 11 The number pi

The number  $\pi$  is a number known since more than 2000 years. With this subject you are asked to construct some algorithms to compute this number as well as understanding some of its properties.

## 12 The quadratic family

Let  $f_c(z) = z^2 + c$  be a polynomial where  $c \in \mathbb{C}$ . Starting from an initial value  $z_0 \in \mathbb{C}$  one can consider the sequence  $z_1 = f(z_0)$ ,  $z_2 = f(z_1) = f(f(z_0))$ ,  $\dots$ . Depending on the initial condition this sequence might either diverge at  $\infty$ , converge to an attractive fixed point of  $f$ , or have some complicated orbits.



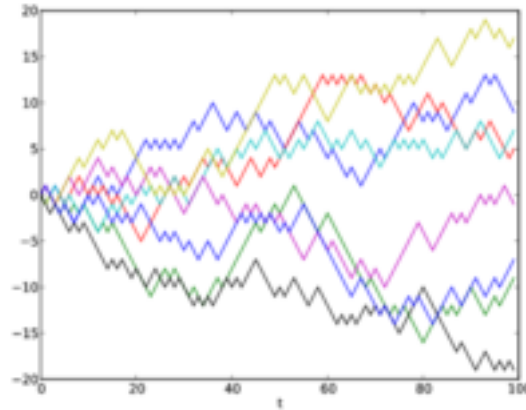
With this subject you will be asked to experiment in Python these sequences and learn about the mysterious fractal structure of this problem.

## 13 Random number generation

Can an algorithm generate a random number? What is a random number?

## 14 Random walks

Let  $S_0 = 0$ . Then for each  $n \geq 1$  we toss a coin and depending on the issue we either set  $S_{n+1} = S_n + 1$  or  $S_{n+1} = S_n - 1$ . The random variable  $S_n$  is called a random walk (in  $\mathbb{Z}$ ).

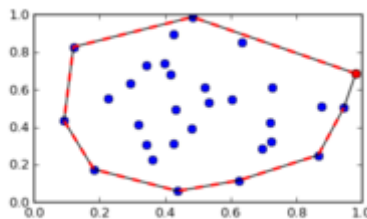


With this subject you are asked to program and experiment random walks. For example

1. Does the walk returns to 0? how frequently?
2. How far the walk goes up to time  $n$ ? In other words, can we estimate  $\max(|S_0|, |S_1|, \dots, |S_n|)$ ?
3. What happens if we change the rules, for example changing the probability of head and tail? changing the size of the steps?
4. What happens in higher dimension?

## 15 2d convex hulls

In this subject you are first asked to implement an algorithm in Python to compute the convex hull of a set of points in 2 dimensions.



Then you will turn to the following experimentations: given a random sample of  $n$  points in  $\mathbb{R}^2$  what are the statistics of the convex hull of these points? Namely: how many points are there on the boundary? what is the volume of the convex hull? the perimeter? Are these statistics independent?