Exercises week 1: arithmetic functions

For this first sheet of exercises you are asked to create a Jupyter notebook in which you will program several arithmetic functions.

Exercise 1

Write a function `prod(l)` that returns the product of the elements in the list `l`. In case the list is empty, the function should return `1`.

For example

```
>>> prod([1, 3, 2, 3])
18
>>> prod([])
1
```

Exercise 2

Write a function `gcd(x, y)` that computes the greatest common divisor of two integers using Euclide algorithm.

Plot the graphic of the function $n \mapsto \#\{(p,q) : 1 \leq p \leq n,\ 1 \leq q \leq n,\ \gcd(p,q) = 1\}$.

Could you guess the asymptotic?

Recall that the *digits* of an integer n in base b is the sequence of numbers $(n_0, n_1, \ldots, n_d)$ in $\{0, 1, \ldots, b-1\}$ so that

$$n = \sum_{i=0}^{d} n_i b^i.$$

For example, the digits of 12 in base 10 are $(2, 1)$ and in base 2 are $(0, 1, 1)$.

Exercise 3

Write a function `digits(n, b)` that return the list of digits of the number `n` in base `b`.

Let $n = 123576537645123412$ written in base 10. What are its digits in base 2? In base 3?

What is the sum of the digits of 2^{100} written in base 3?

Exercise 4

Write a function `prime_range(n)` that returns the list of prime numbers less than `n`. (*hint: use the sieve of Eratosthenes*)

Use your function `prime_range` to answer the following questions

- How many primes are less than 1243? less than 254321?
- How many primes are there between 43123 and 122505?
- Make a graphic of the function $n \mapsto \#\{p : p \leq n \text{ and } p \text{ is prime}\}$. Could you guess the asymptotic of this function?

We recall that the set of integers is a unique factorization domain (UFD): any non-zero integer can be uniquely written as a product $n = s\ p_0^{k_0}\ p_1^{k_1}\ \ldots\ p_m^{k_m}$, s is the sign (ie either $+1$ or -1), p_i are primes and k_i are positive integers.

Exercise 5

Write a function `factor(n)` that returns the factorisation of a positive number `n` as a list of pairs `[[p0, k0], [p1, k1], ..., [pm, km]]` where the `pi` and `ki` are respectively the prime numbers and exponents.

For example

```
>>> factor(12)
[[2, 2], [3, 1]]
>>> factor(19)
[[19, 1]]
```

What are the factorisation of

- 533850245821893?
- $2^{97} + 1$?
- $2^{97} - 1$?

Exercise 6

A positive number n is k-th power free if there is no integer m greater than 1 so that m^k divides n.

- Write a function `is_kth_power_free(n)` that tests whether the number n is k-th power free.
- Make the list of squarefree (ie 2-nd power free) numbers less than 100.
- Program the function `moebius_mu(n)` that is the Möbius μ function

$$n \mapsto \begin{cases} 0 & \text{if } n \text{ is not squarefree} \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct prime numbers} \end{cases}$$

- Find the smallest integer $n \geq 2$ so that $\mu(n) = 1$, $\mu(n+1) = -1$ and $\mu(n) = 1$.

Exercise 7

Write a function `sigma(n, k)` that return the sum of the k-th power of the divisors of n. For example $\sigma(n, 0)$ is the number of divisors of n and $\sigma(n, 1)$ is the sum of divisors. (*hint: you should use the function `factor`*)

Could you write a more efficient function `num_divisors(n)` that is equivalent to `sigma(n, 0)`?

What is the number less than 1000 that has the largest number of divisors?

What is the number less than 1000 that has the largest sum of divisors?