Multiple View Geometry exercise 5

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1 Maximum Likelihood Estimation for Structure from Motion Problems

Ex1

Setup: The point $X_j \in \mathbb{P}^3$ is mapped to the point (x_{ij}^1, x_{ij}^2) by projecting it with camera P_i and adding noise $\epsilon_{ij} \sim \mathcal{N}(0, \sigma I)$.

$$\begin{bmatrix} x_{ij}^1 \\ x_{ij}^2 \end{bmatrix} = \begin{bmatrix} P_i^1 X_j / P_i^3 X_j \\ P_i^2 X_j / P_i^3 X_j \end{bmatrix} + \begin{bmatrix} \epsilon_{ij}^1 \\ \epsilon_{ij}^2 \end{bmatrix}$$

We want to find the cameras and real world points that maximize the likelihood of the noise:

$$\max_{P_1, \dots P_n, X_1, \dots, X_m} \Pr(\epsilon)$$

Which is the same as minimizing the negative-log, since log is a monotonic function:

$$\min - \log\left(\Pr(\epsilon)\right)$$

The noise is independent between points and cameras, so:

$$\Pr(\epsilon) = \prod_{i,j} \Pr(\epsilon_{ij})$$

And our optimization becomes:

$$\min - \sum_{i,j} \log\left(\Pr(\epsilon_{ij})\right)$$

The probability of getting noise ϵ_{ij}:

$$\Pr(\epsilon_{ij}) = \frac{1}{2\pi\sigma} e^{-\frac{1}{2\sigma^2}\|\epsilon_{ij}\|^2}$$

Taking the log of this probability:

$$\log \Pr(\epsilon_{ij}) = \log\left(\frac{1}{2\pi\sigma}\right) - \frac{1}{2\sigma^2}\|\epsilon_{ij}\|^2$$

Assigning this back to the optimization:

$$\min_{P_1,\dots P_n,X_1,\dots,X_m} \sum_{ij} \left(\frac{1}{2\sigma^2} \|\epsilon_{ij}\|^2 - \log\left(\frac{1}{2\pi\sigma}\right)\right)$$

Since σ does not depend directly on the choice of the optimization variables we can remove the terms that contain it and we are left with:

$$\min_{P_1,\dots P_n,X_1,\dots,X_m} \sum_{ij} \|\epsilon_{ij}\|^2$$

The noise term is the difference between the observed points and the points that are the result of projecting the real world points with the camera, and so we have the final formulation of the problem:

$$\min_{P_1,\dots P_n,X_1,\dots,X_m} \sum_{i=1}^{n}\sum_{j=1}^{m} \left\| \left(x_{ij}^1 - \frac{P_i^1 X_j}{P_i^3 X_j}, x_{ij}^2 - \frac{P_i^2 X_j}{P_i^3 X_j} \right) \right\|^2$$

2 Calibrated Structure from Motion and Local Optimization

Ex2

Assume we have n residuals we want to minimize $r \in \mathbb{R}^n$ (here they're the distances of the observed points from the projected points). We will denote the list of m variables we optimize as $v \in \mathbb{R}^m$ (In our case- the cameras and real world points). Each residual is a function $r_i : \mathbb{R}^m \to \mathbb{R}$.
The objective:

$$f(v) = \|r(v)\|^2 = \sum_i r_i(v)^2$$

Our goal is to minimize the residuals:

$$\min_v f(v)$$

If we start at some v_0, the direction of the steepest descent is the negative of the gradient.
The partial derivative of $f(v)$ with respect to the j'th entry of v:

$$\frac{\partial f}{\partial v_j} = 2\sum_i r_i(v) \frac{\partial r_i}{\partial v_j} = 2\sum_i r_i(v) J(v)_{ij} = 2J(v)_j^T r(v)$$

And $J(v)_j^T$ is the j'th column of $J(v)$, the Jacobian matrix:

$$J(v) = \begin{bmatrix} \frac{\partial r_1}{\partial v_1} & \cdots & \frac{\partial r_1}{\partial v_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_n}{\partial v_1} & \cdots & \frac{\partial r_n}{\partial v_m} \end{bmatrix}$$

So the gradient of f:

$$\nabla f(v) = \begin{bmatrix} \frac{\partial f}{\partial v_1} \\ \vdots \\ \frac{\partial f}{\partial v_m} \end{bmatrix} = \begin{bmatrix} 2J(v)_1^T r(v) \\ \vdots \\ 2J(v)_m^T r(v) \end{bmatrix} = 2J(v)^T r(v)$$

Therefor, when starting at point v_0 the direction of the steepest descent is:

$$-\nabla f(v_0) = -2J(v_0)^T r(v_0)$$

Ex3

A direction d is called a descent direction for f at v_0 if:

$$\nabla f(v_0)^T d < 0$$

We saw in the previous section that $d = -\nabla f(v)$:

$$-\nabla f(v_0)^T \nabla f(v) = -\|\nabla f(v)\|^2$$

If v_0 is not a local minimum $\|\nabla f(v)\|^2 \neq 0$ and we have a strict inequality and a descent direction.
Let $M \succ 0$. The direction:

$$d = -M\nabla f(v)$$

Is a descent direction:

$$\nabla f(v)^T d = -\nabla f(v) M \nabla f(v) < 0$$

Since $xMx > 0$ for any $x \neq 0$. If $\nabla f(v) \neq 0$, meaning v is not a local minimum we have a descent direction.
In Levenberg-Marquardt the direction is:

$$\begin{aligned} \delta v &= -\left(J(v_0)^T J(v_0) + \lambda I\right)^{-1} J(v_0)^T r(v_0) \\ &= -\left(J(v_0)^T J(v_0) + \lambda I\right)^{-1} \nabla f(v_0) \end{aligned}$$

Since the inverse of a positive definite matrix is also positive definite, we only need to show that:

$$M = J(v_0)^T J(v_0) + \lambda I \succ 0$$

Let $x \in \mathbb{R}^m \neq 0$ and assume $\lambda > 0$.

$$\begin{aligned} xMx =& x^T J(v_0)^T J(v_0) x + x^T \lambda I x \\ =& \|J(v_0)x\|^2 + \lambda \|x\|^2 \\ >& 0 \end{aligned}$$

CompEx1

We tried using three values for λ. While the three of them improved the residuals comparing to the original solution, the most dramatic improvement was with the smallest value we tried, $\lambda = 0.01$ and the least improvement was when using the biggest value $\lambda = 100$.

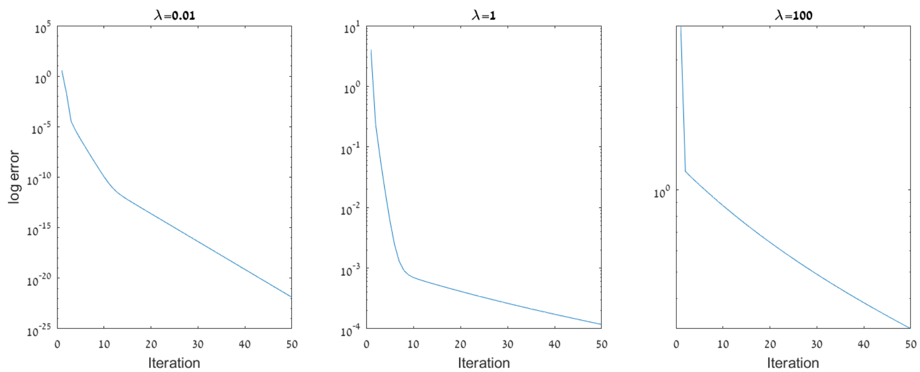

Figure 1: The log of the error of the reprojection points as the optimization progresses.

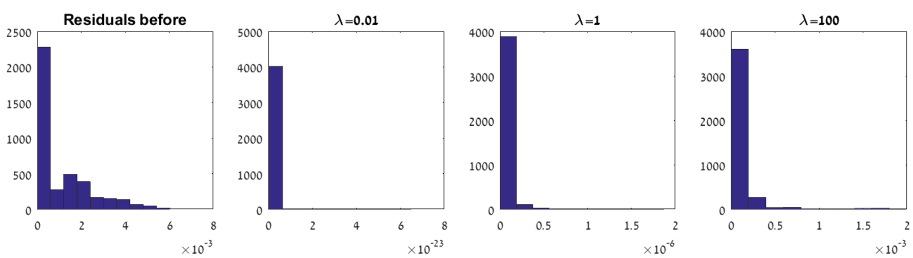

Figure 2: Histogram of the residuals before and after optimization.

compEx2

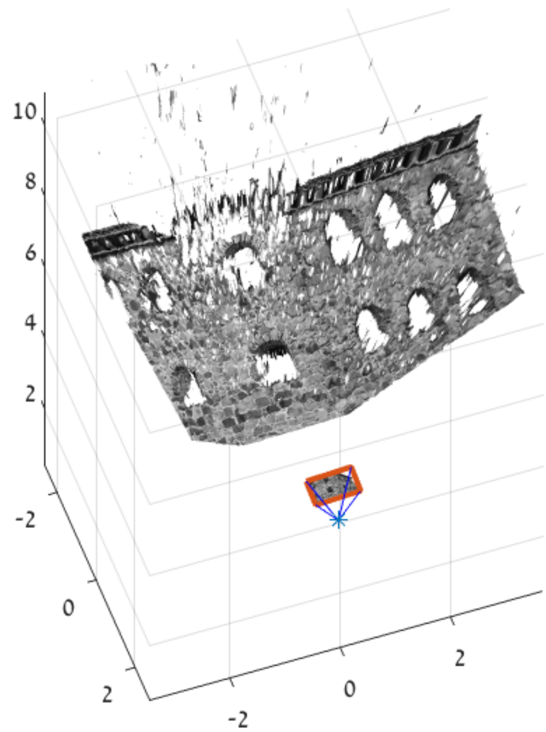

Figure 3: 3D dense reconstruction