Multiple View Geometry exercise 3

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1 The fundamental matrix

Ex1

$$P_1 = \begin{bmatrix} I & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The fundamental matrix can be computed as:

$$F = [t]_\times A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix}$$

The point $x = (1, 1)^T$ creates the epipolar line:

$$l = Fx = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}$$

For a point to be on the line it needs to satisfy $l^T p = 0$.

- $p_1 = (2, 0)^T$
  $2 \cdot 2 + 0 \cdot 0 + 1 \cdot (-4) = 0$
- $p_2 = (2, 1)^T$
  $2 \cdot 2 + 0 \cdot 1 + 1 \cdot (-4) = 0$
- $p_3 = (4, 2)^T$
  $2 \cdot 4 + 0 \cdot 0 + 2 \cdot (-4) = 4$

So only p_1, p_2 are on the line.

Ex2

$$P_1 = \begin{bmatrix} I & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The camera centers:

$$C_1 = (0,0,0,1)^T, C_2 = (-1,-1,0,1)^T$$

$$e_2 = P_2C_1 = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$e_1 = P_1C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$$

The fundamental matrix:

$$F = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix}$$

Let's confirm that the epipoles are the left and right null spaces of F:

$$Fe_1 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$e_2^T F = \begin{bmatrix} 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Optional Ex2

$$P_1 = \begin{bmatrix} I & 0 \end{bmatrix}, P_2 = \begin{bmatrix} A & \mathbf{t} \end{bmatrix}$$

The first camera center:

$$C_1 = (0,0,0,1)^T$$

$$e_2 = P_2C_1 = \mathbf{t}$$

The second camera center denoted $C_2 = (c_1, c_2, c_3, 1)$ satisfies: $P_2C_2 = \mathbf{0}$

$$\begin{bmatrix} A & \mathbf{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \mathbf{0}$$

When organizing we get:

$$C_2 = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ 1 \end{bmatrix} = \begin{bmatrix} -A^{-1}\mathbf{t} \\ 1 \end{bmatrix}$$

And the first epipole:

$$e_1 = P_1 C_2 = -A^{-1}\mathbf{t}$$

The fundamental matrix:

$$F = [\mathbf{t}]_{\times} A$$

Now let's verify these are the null spaces of F:

$$Fe_1 = -[\mathbf{t}]_{\times} A A^{-1}\mathbf{t} = -[\mathbf{t}]_{\times}\mathbf{t} = \mathbf{0}$$

$$e_2^T F = \mathbf{t}^T [\mathbf{t}]_{\times} A = \mathbf{0}$$

The fundamental matrix has a non trivial null space so it has to have a zero determinant.

Ex3

If:

$$\tilde{x_1} = N_1 x_1$$

$$\tilde{x_2} = N_2 x_2$$

And:

$$\tilde{x_2}^T \tilde{F} \tilde{x_1} = 0$$

Then:

$$x_2^T \underbrace{N_2^T \tilde{F} N_1}_{F} x_1 = 0$$

Computer Ex1

After calculating the normalization matrices and M, we used svd to find F. The minimal singular value and $\|Mv\|$ are 0.0497. The normalized matrix:

$$\tilde{F} = \begin{bmatrix} 0.15 & 13.32 & -35.05 \\ -16.07 & -0.79 & 184.56 \\ 40.98 & -186.55 & 1 \end{bmatrix}$$

The un-normalized matrix:

$$F = \begin{bmatrix} 0 & 0 & 0.0058 \\ 0 & 0 & -0.0267 \\ -0.0072 & 0.0263 & 1 \end{bmatrix}$$

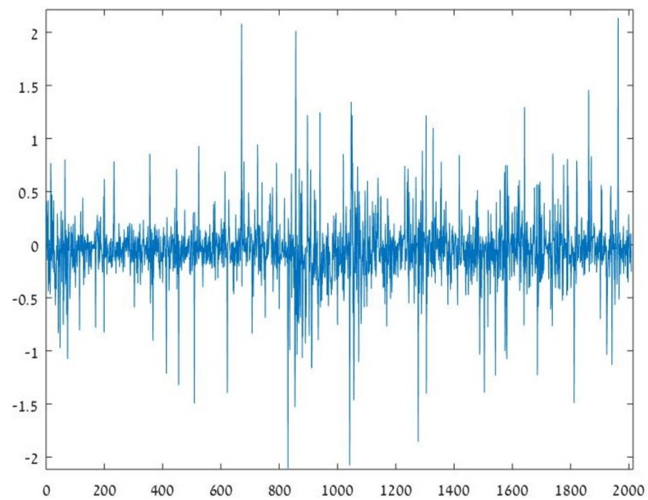

Figure 1: $x_2^T F x_1$ For all the points. The results are close to zero, which indicates an approximately good solution.

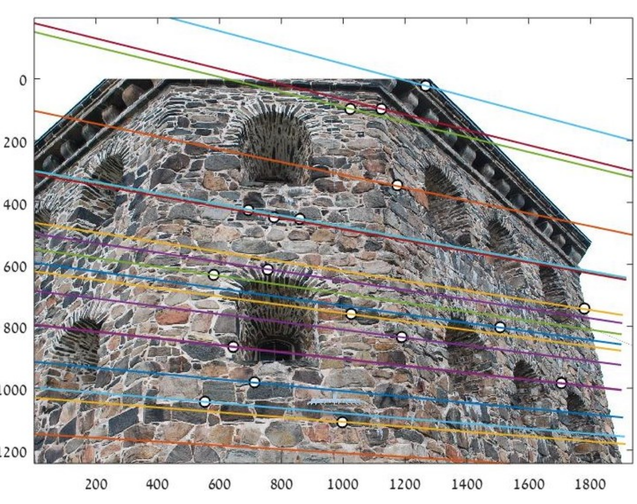

Figure 2: 20 Points sampled from the second image together with the epipolar lines. They seem to intersect.

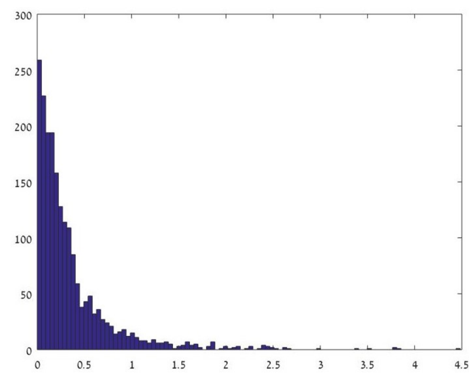

Figure 3: Histogram of the distances of the calculated epipolar lines from the points in the second image. Mean=0.3612, indicating a good solution.

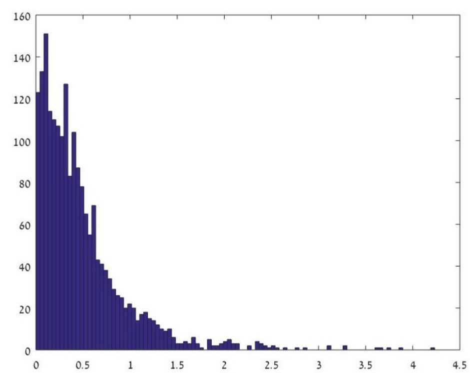

Figure 4: The distances between the epipolar lines and the points in the second image. Mean=0.4878, higher than the mean when normalizing.

Computer ex1 optional

When not normalizing the points before calculating the fundamental matrix the distances between the epipolar lines and the points in the second image increase. However, the difference does not seem too drastic.
The fundamental matrix is:

$$F = \begin{bmatrix} 0 & 0 & 0.0058 \\ 0 & 0 & -0.0266 \\ -0.0072 & 0.0262 & 1 \end{bmatrix}$$

Ex4

Given the matrix:

$$F = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

And the camera matrices:

$$P_1 = \begin{bmatrix} I & 0 \end{bmatrix}, P_2 = \begin{bmatrix} [e_2]_\times F & e_2 \end{bmatrix}$$

First, we can calculate e_2, the null space of F^T:

$$e_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Which gives us the second camera matrix:

$$[e_2]_\times F = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 2 \\ -1 & 0 & 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

For a point X, the following should be satisfied:

$$(P_2X)^T F(P_1X) = 0$$

If we reorder:

$$X^T P_2^T F P_1 X = X^T \begin{bmatrix} 0 & -2 & -2 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} X = -2x_1x_2 - 2x_1x_3 + 2x_1x_2 + 2x_1x_3 = 0$$

So we have confirmed a more **general** claim, that any point will satisfy the equation of the fundamental matrix.
The camera center C_2 is the null space of P_2:

$$C_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Computer Ex2

We will assume $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$ and find P_2. First we will calculate e_2 and use the previous question.

$$e_2 = (0.9763, 0.2163, 0.0001)^T$$

$$P_2 = \begin{bmatrix} -0.0016 & 0.0057 & 0.2163 & 0.9763 \\ 0.0070 & -0.0257 & -0.9763 & 0.2163 \\ 0 & 0 & -0.0273 & 0.0001 \end{bmatrix}$$

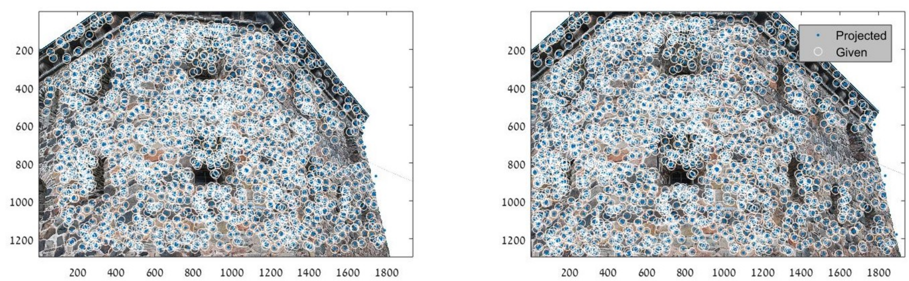

Figure 5: The projected points from triangulation and the given image points for the first camera (left) and the second camera (right).

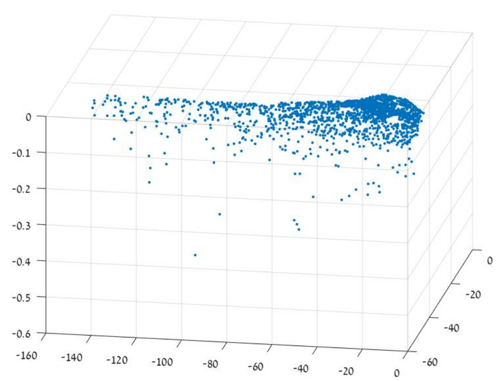

Figure 6: The 3D points computed using DLT with normalization.

The 3D points look distorted. We have created a Projective reconstruction rather than a Euclidean one, because we don't know the calibration matrices. If we had a calibrated problem, we could use the essential matrix and get a more plausible solution, with the right angles.

2 The essential matrix

Ex5

Suppose $[t]_\times = USV^T$. Then:

$$[t]_\times^T [t]_\times = VS^T U^T USV^T = VS^2 V^T$$

And:

$$[t]_\times^T [t]_\times V = VS^2$$
$$[t]_\times^T [t]_\times V[:, i] = V[:, i] S^2[i, i]$$

Which means that V's columns are the eigenvectors of $[t]_\times$, and the corresponding eigenvalue for the i'th eigenvector is $S^2[i, i] = \sigma_i^2$.
For some fixed i, denote: $V[:, i] = w, S^2[i, i] = \lambda$.

$$\begin{aligned} \lambda w =& [t]_\times^T [t]_\times w \\ =& -[t]_\times [t]_\times w \\ =& -t \times ([t]_\times w) \\ =& -t \times (t \times w) \end{aligned}$$

The first inequality by definition of eigenvectors and eigenvalues. The second is because $[t]_\times^T = -[t]_\times$. The third and fourth are by definition of cross product matrix.
Given the equation:

$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$$

We can set $u = -t$ and $v = t$:

$$-t \times (t \times w) = (-t \cdot w)t - (-t \cdot t)w$$

Knowing that $[t]_\times^T [t]_\times w = -t \times (t \times w)$ and reordering:

$$[t]_\times^T [t]_\times w = \|t\|^2 w - (t \cdot w)t$$

<u>If $w = t$</u>:

$$[t]_\times^T [t]_\times t = \|t\|^2 t - \|t\| t = 0 = 0 \cdot t$$

Since we assume $t \neq 0$, this means that t is an eigenvector of $[t]_\times^T [t]_\times t$ with a zero eigenvalue.
<u>If $w \cdot t = 0$</u>:

$$[t]_\times^T [t]_\times w = \|t\|^2 w$$

Meaning a vector w that is perpendicular to t is an eigenvector of $[t]_\times^T [t]_\times$ with an eigenvalue $\|t\|^2$.
These are all the eigenvectors:

$$[t]_\times^T [t]_\times w = \lambda w = \|t\|^2 w - (t \cdot w)t$$

$$(\|t\|^2 - \lambda)w = (t \cdot w)t$$

Either $t = w$ and $\lambda = 0$ or $\lambda = \|t\|^2$ and $t \cdot w = 0$. (Because $t, w \neq 0$)
The singular values of a matrix A can be computed by taking the square root of the eigenvalues of $A^T A$. We have shown that the eigenvalues of $[t]_\times^T [t]_\times$ are 0 with multiplicity of 1, and $\|t\|^2$ with multiplicity of $d - 1 = 2$. This means that the singular values of $[t]_\times$ are $\sigma_1 = \sigma_2 = \|t\|, \sigma_3 = 0$.
If:

$$E = [t]_x R = USV^T R$$

(All the matrices are in $\mathbb{R}^{3\times3}$, and U, V, R are orthogonal so $V^T R$ is also orthogonal) then:

$$E^T E = (USV^T R)^T USV^T R = R^T V S U^T U S V^T R = \underbrace{R^T V}_{\tilde{V}} \underbrace{S^2}_{\tilde{S}^2} \underbrace{V^T R}_{\tilde{V}^T}$$

$$EE^T = TUSV^T R(USV^T R)^T = \underbrace{U}_{\tilde{U}} \underbrace{S^2}_{\tilde{S}^2} \underbrace{U^T}_{\tilde{U}^T}$$

Altogether we have shown that $E = US\tilde{V}^T$ where $\tilde{V} = RV$.
In particular, the singular values of E are the same as of $[t]_\times$.

Computer Ex3

The minimal singular value of the calibrated matrix M was 0.0005
The essential matrix:

$$E = \begin{bmatrix} -9 & -1006 & 377 \\ 1253 & 78 & -2448 \\ -473 & 2550 & 1 \end{bmatrix}$$

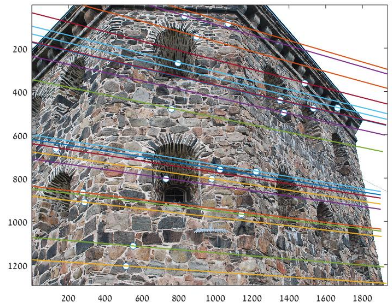

Figure 7: The second image with 20 sampled points and the corresponding epipolar lines that should intersect them.

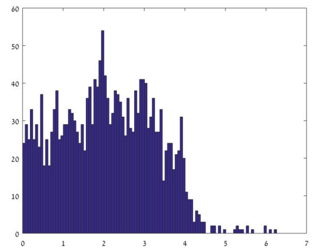

Figure 8: The distances between epipolar lines and points in the second image. The epipolar lines are computed with the Essential matrix. Mean=2.0838.

Ex6

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} V = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E = U\text{diag}([1,1,0])V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\det(UV^T) = \det\left(\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 \end{bmatrix}\right) = 1$$

The points: $x_1 = (0,0,1)^T$ and $x_2 = (1,1,1)^T$ are plausible:

$$x_2^T E x_1 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

If $(0,0,1)^T$ is the projection of $\mathbf{X}$ with the camera $P = [I\ 0]$:

$$\lambda \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Meaning $x_1 = x_2 = 0$, and $x_3 = \lambda$. If we divide by λ:

$$\mathbf{X} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ x_4/\lambda \end{bmatrix}$$

Now we'll denote $s = x_4/\lambda$ to get:

$$\mathbf{X}(s) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix} \sim \begin{bmatrix} 0 \\ 0 \\ 1/s \\ 1 \end{bmatrix}$$

The point $\mathbf{X}$ is projected to the second camera by:

$$\lambda \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = P_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ s \end{bmatrix}$$

Denote the i'th column of P_2: P^i:

$$P^3 + sP^4 = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix}$$

The four possible cameras are:

1. $P = [UWV^T \; u_3]$

$$P^3 + sP^4 = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ s \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix}$$

   $s = -1/\sqrt{2}$

2. $P = [UWV^T \; -u_3]$

$$P^3 + sP^4 = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ -s \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix}$$

   $s = 1/\sqrt{2}$

3. $P = [UW^TV^T \; u_3]$

$$P^3 + sP^4 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ s \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix}$$

   $s = 1/\sqrt{2}$

4. $P = [UW^TV^T \; u_3]$

$$P^3 + sP^4 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ -s \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ \lambda \end{bmatrix}$$

   $s = -1/\sqrt{2}$

The point $\mathbf{X}$ is in front of the second camera for the third and fourth options, where $\lambda > 0$, whereas it is in front of the first camera for the first and third options, where $s > 0$. Therefor it is in front of the two cameras only for the third option.

Computer Ex4

The points are in front of both cameras only when computing by the second camera: $P_2 = \begin{bmatrix} UWV^T & -u_3 \end{bmatrix}$

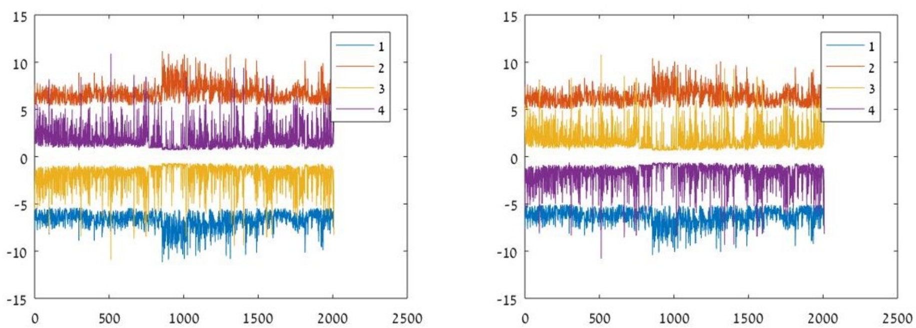

Figure 9: The third coordinate of the projected triangulated points to the first camera (left) and the second camera (right).

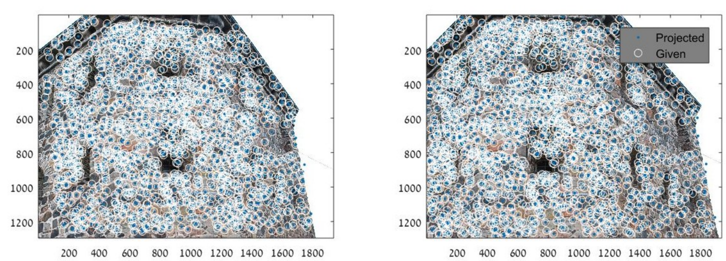

Figure 10: The projected points computed with the essential matrix..

Finally, as we expected, when using the calibrated points and the essential matrix we get a better looking 3D reconstruction - not distorted. (euclidean reconstruction, as opposed to projective reconstruction).

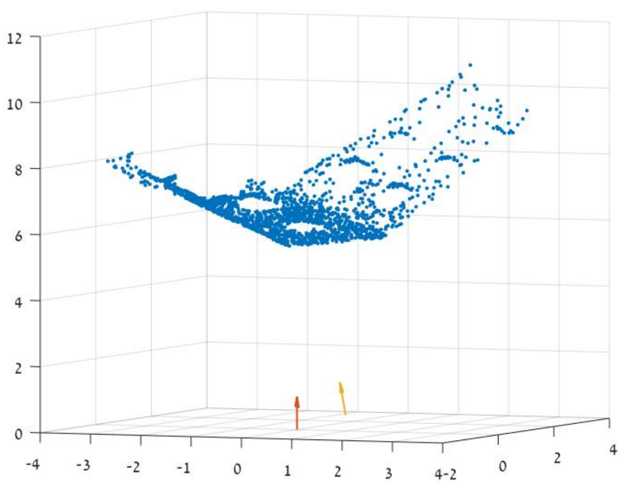

Figure 11: The 3D points triangulated using the second camera and the essential matrix.