Multiple View Geometry

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1 Points in Homogeneous Coordinates

$$x_1 = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \sim \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

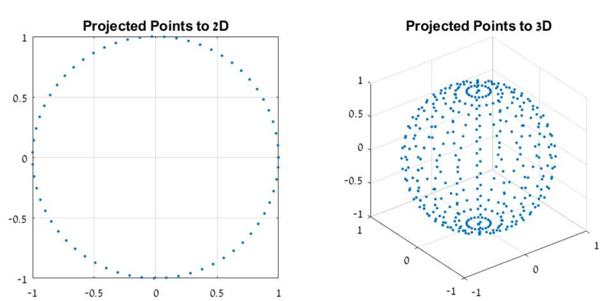
For $\lambda \neq 0$:

$$x_3 = \begin{bmatrix} 4\lambda \\ -2\lambda \\ 2\lambda \end{bmatrix} \sim \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$x_4 = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 4/0 \\ -2/0 \end{bmatrix} \rightarrow \begin{bmatrix} \infty \\ \infty \end{bmatrix}$$

The point $x_4 \in \mathbb{P}^2$ has no corresponding point in $\mathbb{R}^2$. It is called an ideal point, or point at infinity.

1.1 Computer Exercise 1



2 Lines

Given the two lines:

$$l_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, l_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

The homogeneous coordinate of the lines' intersection is their cross-product:

$$l_1 \times l_2 = \begin{bmatrix} 1 \cdot 1 - 1 \cdot 2 \\ 1 \cdot 3 - 1 \cdot 1 \\ 1 \cdot 2 - 1 \cdot 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Similarly, we can try to calculate for the two next lines:

$$l_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, l_4 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

We can already see that they are parallel, but yet When calculating the cross product:

$$l_3 \times l_4 = \begin{bmatrix} 2 \cdot 1 - 3 \cdot 2 \\ 3 \cdot 1 - 1 \cdot 1 \\ 1 \cdot 2 - 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \sim \infty$$

We get that the intersection is a point at infinity, which is where two parallel lines intersect on the projective plane.
Given the two points in $\mathbb{R}^2$:

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Using the duality of points and lines in $\mathbb{P}^2$, instead of computing the line through two points we can use the point that intersects the lines l_1, l_2. By adding the homogeneous coordinate to the points we get the same calculation. So the line is $(1, -2, 1)^T$.
The null space of:

$$M = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Is given by:

$$M = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which induces the following two equations:

$$3x + 2y + z = 0$$

$$x + y + z = 0$$

If $z \neq 0$:

$$3\frac{x}{z} + 2\frac{y}{z} + 1 = 0$$
$$\frac{x}{z} + \frac{y}{z} + 1 = 0$$

In that case $(x/z\ y/z\ 1)^T$ satisfies the equations of l_1, l_2, i.e it is their intersection.
If $z = 0$:

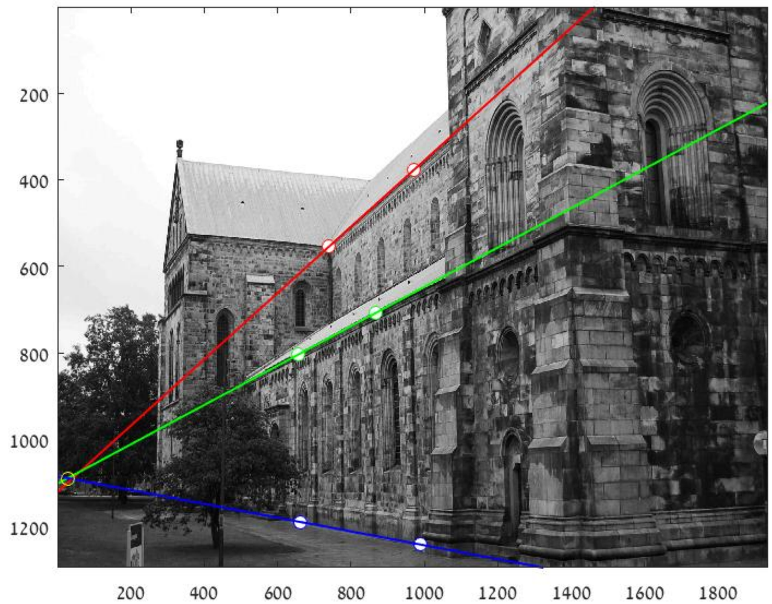
$$3x + y = 0$$

$$x + y = 0$$

The only point that satisfies these equations is $(0, 0, 0)^T$ so there aren't points in $\mathcal{N}(M)$ which are not the intersection.

2.1 Computer Exercise 2

We plotted the three lines, each line goes through two points. The intersection of the second and third lines is marked by a yellow circle.



The three lines we got were:

$$l1 = (-6.8 \times 10^{-4}, -8.9 \times 10^{-4}, 1)^T$$

$$l2 = (-4.1 \times 10^{-4}, -9.0 \times 10^{-4}, 1)^T$$

$$l3 = (1.4 \times 10^{-4}, -9.2 \times 10^{-4}, 1)^T$$

The intersection between the second and third lines is calculated by the cross product of the lines, divided by the third entry to get the projection to $z = 1$.

$$p = (23, 1089)^T$$

The distance to the first line:

$$dist = \frac{|ax_1 + bx_2 + c|}{\sqrt{a^2 + b^2}} = 8.27$$

3 Projective Transformations

The projective transformations matrix:

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

H moves the two points:

$$y_1 = Hx_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$y_2 = Hx_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The line through x_1 and x_2:

$$l_1 = x_1 \times x_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

And the line through y_1 and y_2:

$$l_2 = y_1 \times y_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

The inverse of the transformation:

$$H^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

Finally:

$$H^{-T} l_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = l_2$$

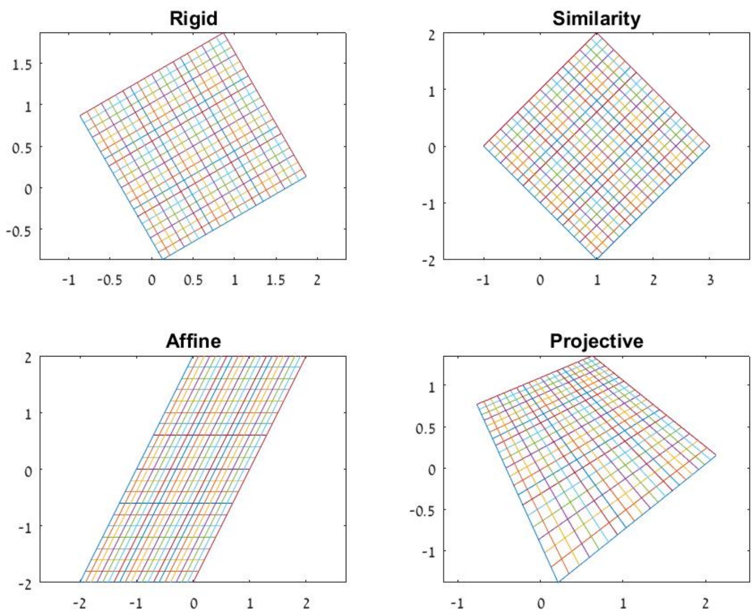
In order to show that lines are mapped to lines, we take a line l_1 and points on it, that satisfy $l_1^T x = 0$. Since H is invertible, $H^{-1}H = I$:

$$l_1^T x = \underbrace{l_1^T H^{-1}}_{l_2} \underbrace{Hx}_{y} = 0$$

So applying the transformation on points that satisfy a line equation results in new points that all satisfy a single line equation: i.e lines are mapped to lines.

3.1 Computer exercise 3

The projective transformation preserves straight lines. The affine is a special case which also preserves parallel lines. The similarity is a special case of affine where the angles are preserved, and rigid is a special case where length is preserved.
Therefor we can classify the transformations:



4 The Pinhole Model

Denote the third row of matrix P as P_3.

$$PX_1/P_3X_1 = \begin{bmatrix} 0.25 \\ 0.5 \\ 1 \end{bmatrix} \rightarrow x_1 = \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}$$

$$PX_2/P_3X_2 = \begin{bmatrix} 0.5 \\ 0.5 \\ 1 \end{bmatrix} \rightarrow x_2 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

However:

$$PX_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

When applying P on X_3 we get a point at infinity.
The center of the camera is $(0,0)^T$, and the principle axis is in the z direction.

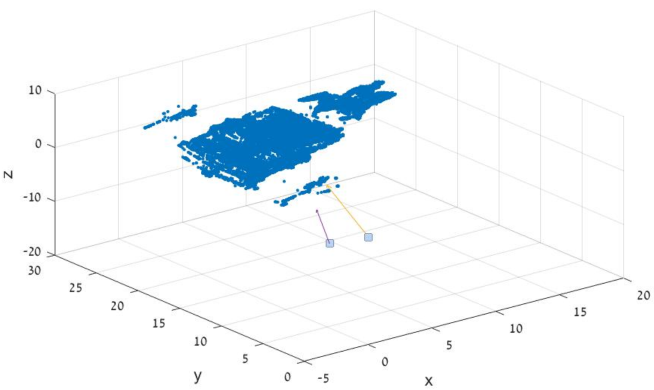
4.1 Computer Exercise 4

In order to find the camera centers, we will compute the null space of the camera matrix - the points that are mapped to 0.
The center of camera 1 is $(0,0,0)^T$ and of camera 2 $(6.63, 14.85, -15.07)^T$.
The principle axis is the line that originates in the camera center, and goes in the direction of the normal to the image plane, where the image plane is the third row of the camera matrix.
Therefor we can calculate:

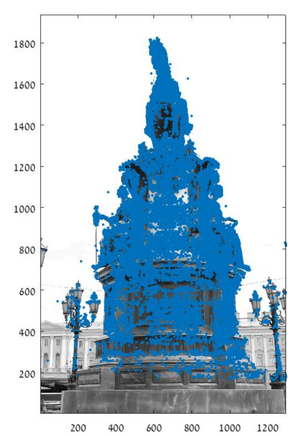
$$pa_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0.31 \\ 0.95 \\ 0.08 \end{bmatrix}$$

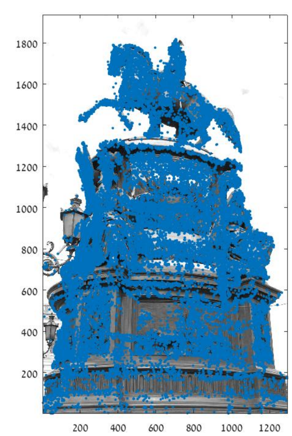
$$pa_2 = \begin{bmatrix} 6.63 \\ 14.85 \\ -15.07 \end{bmatrix} + t \begin{bmatrix} 0.03 \\ 0.34 \\ 0.94 \end{bmatrix}$$

When plotting the cameras positions and principle axis with the points of the statue:



When projecting the points to the images we get:





4.2 Bonus 1

Camera matrix P_1 projects a point $U = (x, y, z, s)^T$:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Regardless of the value of s. These points are co-planar. There is no way to recover s using the projected points with P_1.

Given the following plane, with $\pi \in \mathbb{R}^3$:

$$\Pi = \begin{bmatrix} \pi \\ 1 \end{bmatrix}$$

We can find s s.t the point $U = (x, y, z, s)^T$ belong to Π, i.e $\Pi^T U = 0$.

$$\begin{bmatrix} \pi_1 & \pi_2 & \pi_3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \end{bmatrix} = 0$$

$$s = -\pi^T \mathbf{x}$$

The points on Π are of the form:

$$U = \begin{bmatrix} x \\ y \\ z \\ -\pi^T \mathbf{x} \end{bmatrix}$$

If:

$$\mathbf{x} \sim P_1 U$$
$$\mathbf{y} \sim P_2 U$$
$$\Pi^T U = 0$$

The homography $H = (R - t\pi^T)$ maps $\mathbf{x}$ to $\mathbf{y}$.

$$\mathbf{y} \sim P_2 U = \begin{bmatrix} R & t \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ -\pi^T \mathbf{x} \end{bmatrix} = R\mathbf{x} - t\pi^T \mathbf{x} = (R - t\pi^T)\mathbf{x} = H\mathbf{x}$$

4.3 Bonus 2 - computer exercise 5

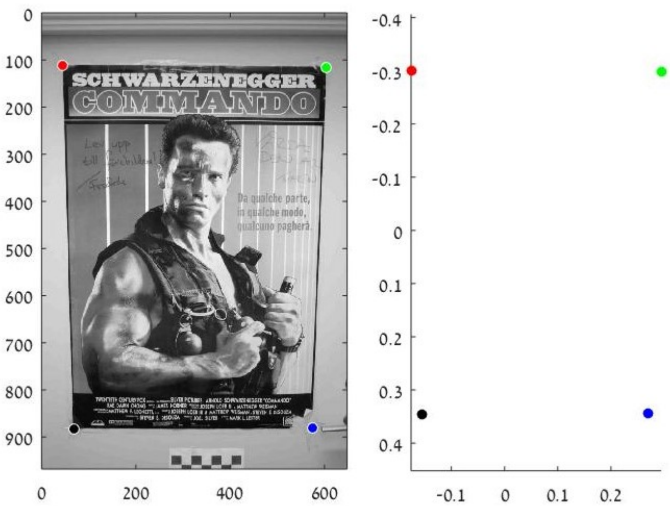


Figure 1: The four corners in the image space and in the "real" space

The origin of the photo in image space is at the upper-left corner, while in real space it is in the middle.

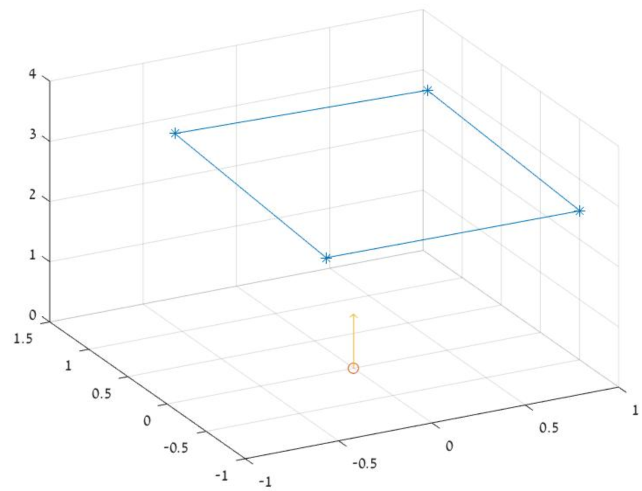


Figure 2: The corners of the poster in 3D with the camera center and principle axis.

Given the rotation matrix and the camera center we can calculate the translation vector, because the center is the null space of [R t]:

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & t_1 \\ 0 & 1 & 0 & t_2 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & t_3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

So the camera matrix is:

$$P = \begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & -\sqrt{3} \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

when moving the camera 2 units and changing its rotation by the given $[Rt]$ we have the following camera center and principle axis:

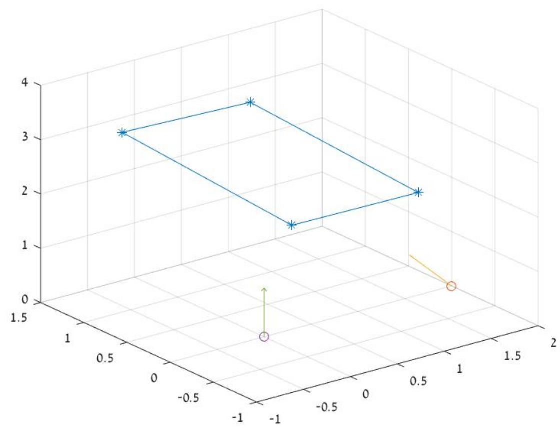


Figure 3: The new camera center in 3D

After applying H to the corners we can see as expected that they seem as if we are looking at them from the right.

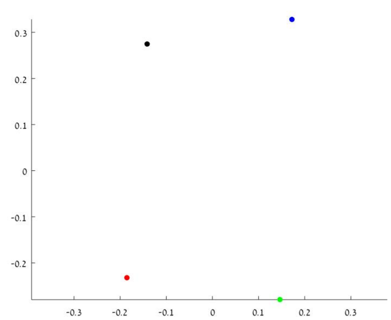


Figure 4: The transformed corners

We have the exact same result when applying $[Rt]$ on the 3D points $U \in \mathbb{P}^3$ and when applying H on the 2D points $\in \mathbb{P}^2$.
And now finally, the transformed image:

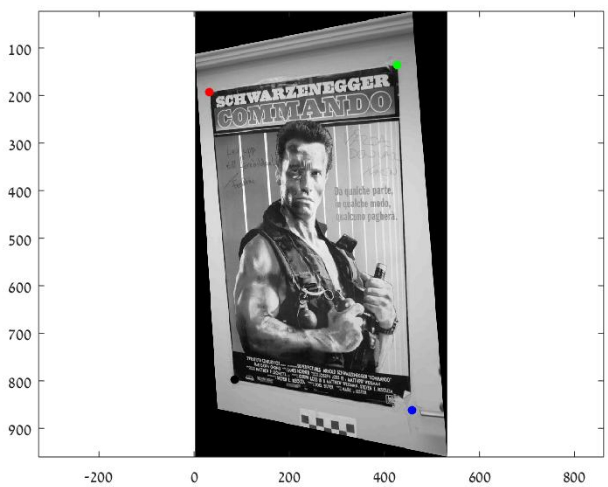


Figure 5: After applying KHK^{-1} on the image