Multiple View Geometry exercise 4

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1 Plane Fitting

Ex1

A plane $ax + by + cz + d = 0$ has four parameters but is determined up to scale, so it has three degrees of freedom.
We need three correct points to compute the plane. The probability of a point being correct is 0.9. The probability that when sampling three points, all of them are correct is $0.9^3 = 0.729$, and the probability that at least one point is an outlier is 0.271. If we repeat the experiment k times, the probability that all trials are not successful is 0.271^k. We want this probability to be smaller than 0.02.

$$0.271^k < 0.02$$

$$k > \log_{0.271} 0.02 = 2.9963$$

Therefor when taking $k \geq 3$, we have $\Pr(Error) \leq 0.271^3 = 0.0199 < 0.02$.

Ex2

We'll use total least squares to fit a plane to 3D points $(x_i, y_i, z_i),\ i = 1, \ldots, m$. The objective:

$$\begin{aligned} \min_{a,b,c,d} \quad & \sum_{i=1}^{m} (ax_i + by_i + cz_i + d)^2 \\ \text{s.t.} \quad & a^2 + b^2 + c^2 = 1 \end{aligned}$$

Taking the derivative of the objective function with respect to d:

$$2 \sum_{i=1}^{m} (ax_i + by_i + cz_i + d) = 0$$

$$
\begin{aligned}
d &= -\frac{1}{m}\sum_{i=1}^{m}(ax_i + by_i + cz_i) \\
&= -\left(\frac{1}{m}\sum_{i=1}^{m} ax_i + \frac{1}{m}\sum_{i=1}^{m} by_i + \frac{1}{m}\sum_{i=1}^{m} cz_i\right) \\
&= -\left(a\overline{x} + b\overline{y} + c\overline{z}\right)
\end{aligned}
$$

Setting in the optimization objective function, it turns to:

$$
\sum_{i=1}^{m}(ax_i + by_i + cz_i - (a\overline{x} + b\overline{y} + c\overline{z}))^2 = \sum_{i=1}^{m}(a(x_i - \overline{x}) + b(y_i - \overline{y}) + c(z_i - \overline{z}))^2
$$

Next we'll denote the zero centered coordinates with a tilde and get the new optimization problem:

$$
\begin{aligned}
\min_{a,b,c} \quad & \sum_{i=1}^{m}(a\tilde{x}_i + b\tilde{y}_i + c\tilde{z}_i)^2 \\
\text{s.t.} \quad & 1 - (a^2 + b^2 + c^2) = 0
\end{aligned}
$$

Denote:

$$
t = \begin{bmatrix} a \\ b \\ c \end{bmatrix}
$$

And:

$$
M = \begin{bmatrix}
\sum_i \tilde{x}_i{}^2 & \sum_i \tilde{x}_i\tilde{y}_i & \sum_i \tilde{x}_i\tilde{z}_i \\
\sum_i \tilde{y}_i\tilde{x}_i & \sum_i \tilde{y}_i{}^2 & \sum_i \tilde{y}_i\tilde{z}_i \\
\sum_i \tilde{z}_i\tilde{x}_i & \sum_i \tilde{z}_i\tilde{y}_i & \sum_i \tilde{z}_i{}^2
\end{bmatrix}
$$

Then:

$$
f(t) = t^T M t
$$

We can simplify by forming three vectors:

$$
\mathbf{x} = (\tilde{x}_1, \dots, \tilde{x}_m)^T, \mathbf{y} = (\tilde{y}_1, \dots, \tilde{y}_m)^T, \mathbf{z} = (\tilde{z}_1, \dots, \tilde{z}_m)^T
$$

And rewriting in vector form:

$$
f(t) = t^T M t = \begin{bmatrix} a & b & c \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{x}^T \\ \mathbf{y}^T \\ \mathbf{z}^T \end{bmatrix} \begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix}}_{M} \begin{bmatrix} a \\ b \\ c \end{bmatrix}
$$

Finally, the optimization problem we will solve is:

$$
\begin{aligned}
\min_{t} \quad & t^T M t \\
\text{s.t.} \quad & 1 - \|t\|^2 = 0
\end{aligned}
$$

This can be solved easily with Lagrange multipliers.

$$\nabla f(t) + \lambda g(t) = 0$$

$$Mt - \lambda t = 0$$

So a solution t must be an eigenvector of the matrix M. In order to minimize the objective, we need to choose the vector with the minimal eigenvalue.

Computer Ex1

Total least squares on all of the points
The plane:

$$(a,b,c,d)^T = \begin{bmatrix} -0.2550 \\ -0.3785 \\ 0.8898 \\ -14.6271 \end{bmatrix}$$

RMSE=0.5168.
RANSAC
The plane we found:

$$(a,b,c,d)^T = \begin{bmatrix} 0.1760 \\ 0.4130 \\ -0.8936 \\ 14.4592 \end{bmatrix}$$

RMSE=0.5529.
Over 92% of the points agree with this plane, and the distances are very close to zero.

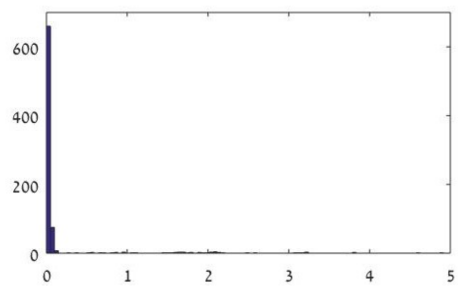

Figure 1: Histogram of the distances of all the points from the plane computed via RANSAC

Total least squares on the inliers found using RANSAC
This gave the best results.
The plane:

$$(a,b,c,d)^T = \begin{bmatrix} -0.1630 \\ -0.4283 \\ 0.8888 \\ -14.3512 \end{bmatrix}$$

EMSE=0.0254.

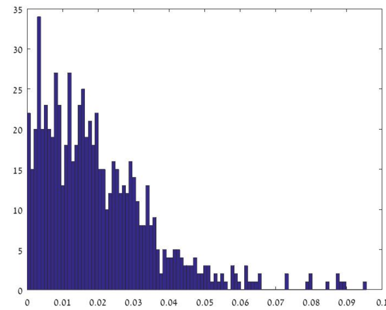

Figure 2: Histogram of the distances of inlier points found with RANSAC to the plane found with total least squares.

When projecting the inlier points to the original images we can see that they lie on the front wall, which makes sense.

Figure 3: Inlier points from RANSAC projected to the images using the given camera matrices.

When mapping the points from one image to the other using a homogrpahy, the points that are located on the front wall seem to be mapped correctly, whereas points from the farther wall are misplaced. This makes sense because we calculated the homogrpahy based on the points from the front wall. We can also see a clear outlier - a point from the wheel of a biker passing by that should not be mapped to any other point in the second image.

Figure 4: 10 points projected using a homography

2 Robust Homography Estimation & Stitching

Ex3

Two cameras $P_1' = \begin{bmatrix} A_1 & t_1 \end{bmatrix}$ and $P_2' = \begin{bmatrix} A_2 & t_2 \end{bmatrix}$ have the same center. We can apply a projective transformation to go to the first camera's coordinate system and have the following two camera matrices:

$$H = \begin{bmatrix} A_1^{-1} & -A_1^{-1}t_1 \\ 0 & 1 \end{bmatrix}$$

$P_1 = P_1'H, P_2 = P_2'H$.

$$P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} A_2A_1^{-1} & -A_2A_1^{-1}t_1 + t_2 \end{bmatrix} = \begin{bmatrix} A & t \end{bmatrix}$$

The camera center of the first camera is $C = (0, 0, 0, 1)$ and it is the same as the center of the second camera. This means that $t = \mathbf{0}$.

$$P_1 = \begin{bmatrix} I & 0 \end{bmatrix}, P_2 = \begin{bmatrix} A & 0 \end{bmatrix}$$

Given a point $\mathbf{X} = (x_1, x_2, x_3, 1)^T$ it is projected to the two images:

$$\mathbf{x}_1 = P_1\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\mathbf{x}_2 = P_2\mathbf{X} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{A_2 A_1^{-1}}_{H} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Since A_1, A_2 are invertible we have an invertible 3×3 transformation - H, Homography that relates the 2D projected points to the two images.

Ex4

The Homography matrix has $3 \cdot 3 = 9$ entries, but it is estimated up to scale, so it has 8 DOF. Every matching pair of points gives two equations, so we need four such pairs in order to solve for H.
If 10% of the matches are incorrect, The probability that all four are good is $0.9^4 = 0.6561$. In order for the probability of having at least one pair incorrect to be less than 0.02:

$$0.3439^k < 0.02$$

$$k > \log_{0.3439} 0.02 = 3.665$$

So we need k to be at least four.

Computer Ex2

The toolbox found 947 features in the first image and 865 in the second one, and 204 matches.
The homogrpahy we calculated with agreement fraction of 0.7304:

$$H = \begin{bmatrix} 0.6216 & 0.7940 & -70.4433 \\ -0.7873 & 0.5884 & 170.7666 \\ 0 & 0 & 1 \end{bmatrix}$$

And the concatenated image:

Figure 5: The two images concatenated