

An Introduction to Quantum Field Theory (Peskin and Schroeder)

参考本：

- [1] Feynman ルールの証明はせずに摂動計算の練習
- [2] 非可換ゲージ理論について分かりやすく書いてある
- [3] 標準模型の Feynman rule 一覧. Higgs field ϕ の定義が異なるので注意が必要

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- [3] Jorge C Romao and Joao P Silva. “A resource for signs and Feynman diagrams of the Standard Model”. In: *International Journal of Modern Physics A* 27.26 (2012), p. 1230025.
- [4] Abhijit Sen and Zurab K Silagadze. “Two-photon decay of P-wave positronium: a tutorial”. In: *Canadian Journal of Physics* 97.7 (2019), pp. 693–700.
- [5] David J Gross and Frank Wilczek. “Asymptotically free gauge theories. II”. In: *Physical Review D* 9.4 (1974), p. 980.

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Part I

Feynman Diagrams and Quantum Electrodynamics

Fourier 変換

場の Fourier 変換は xxi のように

$$\phi(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \phi(k), \quad \phi(k) = \int d^4 x e^{ik \cdot x} \phi(x)$$

と定める. Fermion の場合は

$$\psi(p) = \int d^4 x e^{ip \cdot x} \psi(x), \quad \bar{\psi}(p) = \int d^4 x e^{-ip \cdot x} \bar{\psi}(x)$$

とする^{*1}. Propagator は

$$\begin{aligned} \overline{\psi(p)} \bar{\psi}(q) &= \int d^4 x e^{ip \cdot x} \int d^4 y e^{-iq \cdot x} \overline{\psi(x)} \bar{\psi}(y) \\ &= \int d^4 x e^{ip \cdot x} \int d^4 y e^{-iq \cdot y} \int \frac{d^4 k}{(2\pi)^4} \frac{i \not{k}}{k^2} e^{-ik \cdot (x-y)} \\ &= \frac{i \not{p}}{p^2} (2\pi)^4 \delta^{(4)}(p - q). \end{aligned}$$

(4.47) より e^{-ipx} は位置 x に運動量 p が入るものとする (e.g. (4.47), p. 507).

^{*1} $\bar{\psi}(p)$ は $\psi(p)$ の Hermite 共役に対し右から γ^0 をかけたもの

Chapter 2

The Klein-Gordon Field

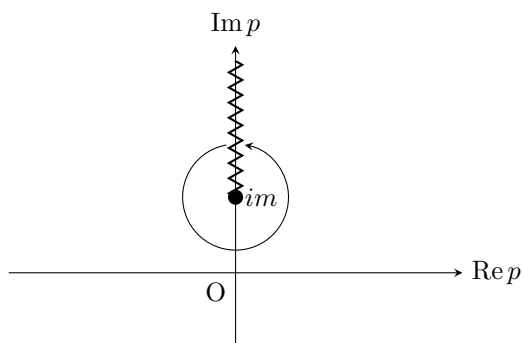
2.4 The Klein-Gordon Field in spacetime

(2.52)

積分

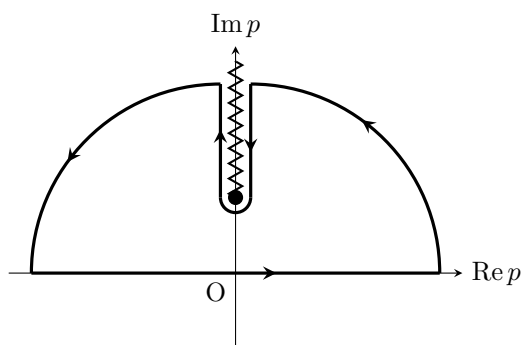
$$\int_{-\infty}^{\infty} dp \frac{pe^{ipr}}{\sqrt{p^2 + m^2}}$$

を計算する。複素平面では、 $1/\sqrt{p^2 + m^2}$ は $p = \pm im$ に極をもち、 $[\pm im, \pm\infty)$ を載線 (branch cut) に取れる。

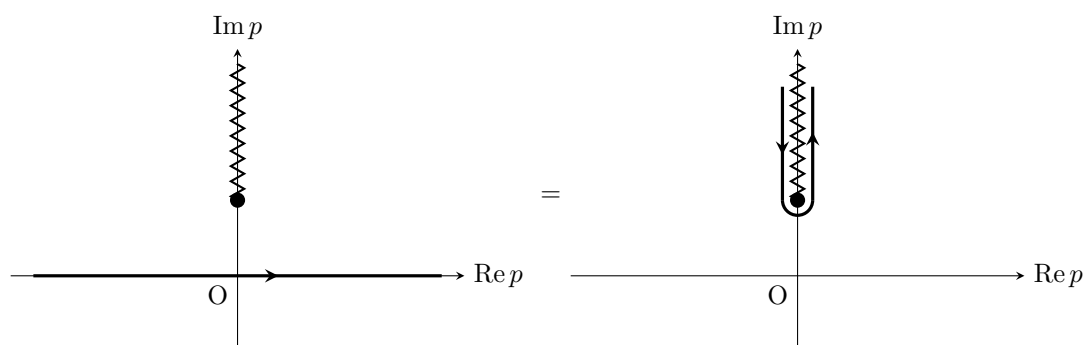


branch cut の左右で $p^2 + m^2$ の偏角は 2π 異なるので、 $\sqrt{p^2 + m^2}$ の偏角は π 異なる。すなわち、左右で被積分関数の符号は入れ替わる。

次の経路で、積分値は 0.



大きい円弧に沿った積分は 0 なので、結局、



左側と右側の積分は逆向きで、被積分函数の符号が逆なので、積分値は等しい。従って、 $p = i\rho$ とすれば、

$$\int_{-\infty}^{\infty} = 2 \int_{im}^{i\infty} dp = 2i \int_m^{\infty} d\rho$$

となる。

Chapter 3

The Dirac Field

Problems

Problem 3.4: The Quantized Majorana Field

Majorana フェルミオンのモード展開は次式で与えられる：

$$\begin{aligned}
 \chi &= \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{(p\sigma)}{2E_{\mathbf{p}}}} \sum_s (a_{\mathbf{p}}^s \xi^s e^{-ipx} - i\sigma^2 a_{\mathbf{p}}^{s\dagger} \xi^s e^{ipx}), \\
 \chi^\dagger &= \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\mathbf{p}}}} \sum_s (a_{\mathbf{p}}^{s\dagger} \xi^{s\dagger} e^{ipx} + i a_{\mathbf{p}}^s \xi^{s\dagger} \sigma^2 e^{-ipx}) \sqrt{(p\sigma)}, \\
 \chi^* &= \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{(p\sigma^*)}{2E_{\mathbf{p}}}} \sum_s (a_{\mathbf{p}}^{s\dagger} \xi^s e^{ipx} - i\sigma^2 a_{\mathbf{p}}^s \xi^s e^{-ipx}), \\
 \chi^\top &= \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{1}{2E_{\mathbf{p}}}} \sum_s (a_{\mathbf{p}}^s \xi^{s\dagger} e^{-ipx} + i a_{\mathbf{p}}^{s\dagger} \xi^{s\dagger} \sigma^2 e^{ipx}) \sqrt{(p\sigma^\top)}, \\
 (\boldsymbol{\sigma} \cdot \boldsymbol{\nabla})\chi &= \int \frac{d^3p}{(2\pi)^3} i(\mathbf{p} \cdot \boldsymbol{\sigma}) \sqrt{\frac{(p\sigma)}{2E_{\mathbf{p}}}} \sum_s (a_{\mathbf{p}}^s \xi^s e^{-ipx} + i\sigma^2 a_{\mathbf{p}}^{s\dagger} \xi^s e^{ipx}).
 \end{aligned}$$

ハミルトニアンは

$$H_{\text{Majorana}} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\chi}} \dot{\chi} - \mathcal{L} \right) = \int d^3x \left[i\chi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \chi + \frac{im}{2} (\chi^\dagger \sigma^2 \chi^* - \chi^\top \sigma^2 \chi) \right]. \quad [3.0.1]$$

[3.0.1] 第1項 ($e^{\pm p_0 t}$ は省略) は

$$\begin{aligned}
 &\int d^3x i\chi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \chi \\
 &= \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{-1}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \sum_{r,s} (a_{\mathbf{p}}^{r\dagger} \xi^{r\dagger} e^{ipx} + i a_{\mathbf{p}}^r \xi^{r\dagger} \sigma^2 e^{-ipx}) \\
 &\quad \times \sqrt{(p\sigma)} (\mathbf{q} \cdot \boldsymbol{\sigma}) \sqrt{(q\sigma)} (a_{\mathbf{q}}^s \xi^s e^{-iqx} + i\sigma^2 a_{\mathbf{q}}^{s\dagger} \xi^s e^{iqx}) \\
 &= \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{-1}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \sum_{r,s} (a_{\mathbf{p}}^{r\dagger} \xi^{r\dagger} e^{ipx}) \sqrt{(p\sigma)} (\mathbf{q} \cdot \boldsymbol{\sigma}) \sqrt{(q\sigma)} (a_{\mathbf{q}}^s \xi^s e^{-iqx}) \\
 &\quad + \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{-1}{\sqrt{2E_{\mathbf{p}} 2E_{\mathbf{q}}}} \sum_{r,s} (a_{\mathbf{p}}^{r\dagger} \xi^{r\dagger} e^{ipx}) \sqrt{(p\sigma)} (\mathbf{q} \cdot \boldsymbol{\sigma}) \sqrt{(q\sigma)} (i\sigma^2 a_{\mathbf{q}}^{s\dagger} \xi^s e^{iqx})
 \end{aligned}$$

$$\begin{aligned}
 & + \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{-1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (i a_p^r \xi^{r\dagger} \sigma^2 e^{-ipx}) \sqrt{(p\sigma)}(\mathbf{q} \cdot \boldsymbol{\sigma}) \sqrt{(q\sigma)} (a_q^s \xi^s e^{-iqx}) \\
 & + \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{-1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (i a_p^r \xi^{r\dagger} \sigma^2 e^{-ipx}) \sqrt{(p\sigma)}(\mathbf{q} \cdot \boldsymbol{\sigma}) \sqrt{(q\sigma)} (i \sigma^2 a_q^{s\dagger} \xi^s e^{iqx}) \\
 = & \int \frac{d^3p}{(2\pi)^3} \frac{-1}{2E_p} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} \sqrt{(p\sigma)}(\mathbf{p} \cdot \boldsymbol{\sigma}) \sqrt{(p\sigma)} \xi^s \\
 & + \int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_p} \sum_{r,s} a_p^{r\dagger} a_{-p}^{s\dagger} \times \xi^{r\dagger} \sqrt{(p\sigma)}(\mathbf{p} \cdot \boldsymbol{\sigma}) \sqrt{(p\bar{\sigma})} \sigma^2 \xi^s \\
 & + \int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_p} \sum_{r,s} a_p^r a_{-p}^s \times \xi^{r\dagger} \sigma^2 \sqrt{(p\sigma)}(\mathbf{p} \cdot \boldsymbol{\sigma}) \sqrt{(p\bar{\sigma})} \xi^s \\
 & + \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_p^r a_p^{s\dagger} \times \xi^{r\dagger} \sigma^2 \sqrt{(p\sigma)}(\mathbf{p} \cdot \boldsymbol{\sigma}) \sqrt{(p\sigma)} \sigma^2 \xi^s
 \end{aligned}$$

である。 $\sqrt{(p\sigma)}$ などを明示的に書くと

$$\sqrt{(p\sigma)} = \frac{E_p + m - \mathbf{p} \cdot \boldsymbol{\sigma}}{\sqrt{2(E_p + m)}}, \quad \sqrt{(p\bar{\sigma})} = \frac{E_p + m + \mathbf{p} \cdot \boldsymbol{\sigma}}{\sqrt{2(E_p + m)}}$$

となるので、

$$\begin{aligned}
 \sqrt{(p\sigma)}(\mathbf{p} \cdot \boldsymbol{\sigma}) \sqrt{(p\sigma)} &= E_p(\mathbf{p} \cdot \boldsymbol{\sigma}) - |\mathbf{p}|^2, \\
 \sqrt{(p\sigma)}(\mathbf{p} \cdot \boldsymbol{\sigma}) \sqrt{(p\bar{\sigma})} \sigma^2 &= m(\mathbf{p} \cdot \boldsymbol{\sigma}) \sigma^2, \\
 \sigma^2 \sqrt{(p\sigma)}(\mathbf{p} \cdot \boldsymbol{\sigma}) \sqrt{(p\bar{\sigma})} &= m \sigma^2(\mathbf{p} \cdot \boldsymbol{\sigma}), \\
 \sigma^2 \sqrt{(p\sigma)}(\mathbf{p} \cdot \boldsymbol{\sigma}) \sqrt{(p\sigma)} \sigma^2 &= -E_p(\mathbf{p} \cdot \boldsymbol{\sigma}^*) - |\mathbf{p}|^2.
 \end{aligned}$$

従って、第1項を引き続き計算して

$$= \int \frac{d^3p}{(2\pi)^3} \frac{-1}{2E_p} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} [E_p(\mathbf{p} \cdot \boldsymbol{\sigma}) - |\mathbf{p}|^2] \xi^s \quad [3.0.2]$$

$$+ \int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_p} \sum_{r,s} a_p^{r\dagger} a_{-p}^{s\dagger} \times \xi^{r\dagger} [m(\mathbf{p} \cdot \boldsymbol{\sigma}) \sigma^2] \xi^s \quad [3.0.3]$$

$$+ \int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_p} \sum_{r,s} a_p^r a_{-p}^s \times \xi^{r\dagger} [m \sigma^2(\mathbf{p} \cdot \boldsymbol{\sigma})] \xi^s \quad [3.0.4]$$

$$+ \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_p^r a_p^{s\dagger} \times \xi^{r\dagger} [-E_p(\mathbf{p} \cdot \boldsymbol{\sigma}^*) - |\mathbf{p}|^2] \xi^s \quad [3.0.5]$$

となる。

[3.0.1] の第2項、

$$\begin{aligned}
 & \frac{im}{2} \int d^3x \chi^\dagger \sigma^2 \chi^* \\
 = & \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (a_p^{r\dagger} \xi^{r\dagger} e^{ipx} + i a_p^r \xi^{r\dagger} \sigma^2 e^{-ipx}) \\
 & \times \sqrt{(p\sigma)} \sigma^2 \sqrt{(q\sigma^*)} (a_q^{s\dagger} \xi^s e^{iqx} - i \sigma^2 a_q^s \xi^s e^{-iqx})
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (a_{\mathbf{p}}^{r\dagger} \xi^{r\dagger} e^{ipx} + i a_{\mathbf{p}}^r \xi^{r\dagger} \sigma^2 e^{-ipx}) \\
 &\quad \times \sqrt{(p\sigma)} \sqrt{(q\bar{\sigma})} \sigma^2 (a_{\mathbf{q}}^{s\dagger} \xi^s e^{iqx} - i \sigma^2 a_{\mathbf{q}}^s \xi^s e^{-iqx}) \\
 &= \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (a_{\mathbf{p}}^{r\dagger} \xi^{r\dagger} e^{ipx}) \sqrt{(p\sigma)} \sqrt{(q\bar{\sigma})} \sigma^2 (a_{\mathbf{q}}^{s\dagger} \xi^s e^{iqx}) \\
 &\quad + \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (a_{\mathbf{p}}^{r\dagger} \xi^{r\dagger} e^{ipx}) \sqrt{(p\sigma)} \sqrt{(q\bar{\sigma})} \sigma^2 (-i \sigma^2 a_{\mathbf{q}}^s \xi^s e^{-iqx}) \\
 &\quad + \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (i a_{\mathbf{p}}^r \xi^{r\dagger} \sigma^2 e^{-ipx}) \sqrt{(p\sigma)} \sqrt{(q\bar{\sigma})} \sigma^2 (a_{\mathbf{q}}^{s\dagger} \xi^s e^{iqx}) \\
 &\quad + \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (i a_{\mathbf{p}}^r \xi^{r\dagger} \sigma^2 e^{-ipx}) \sqrt{(p\sigma)} \sqrt{(q\bar{\sigma})} \sigma^2 (-i \sigma^2 a_{\mathbf{q}}^s \xi^s e^{-iqx}) \\
 &= \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_{\mathbf{p}}^{r\dagger} a_{-\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} \sqrt{(p\sigma)} \sqrt{(p\sigma)} \sigma^2 \xi^s \\
 &\quad + \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{-i}{2E_p} \sum_{r,s} a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s \times \xi^{r\dagger} \sqrt{(p\sigma)} \sqrt{(p\bar{\sigma})} \xi^s \\
 &\quad + \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_p} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} \sigma^2 \sqrt{(p\sigma)} \sqrt{(p\bar{\sigma})} \sigma^2 \xi^s \\
 &\quad + \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_{\mathbf{p}}^r a_{-\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} \sigma^2 \sqrt{(p\sigma)} \sqrt{(p\sigma)} \xi^s
 \end{aligned}$$

に $\sqrt{(p\sigma)} \sqrt{(p\bar{\sigma})} = m$ などを代入して,

$$= \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_{\mathbf{p}}^{r\dagger} a_{-\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} [(p\sigma)\sigma^2] \xi^s \quad [3.0.6]$$

$$+ \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{-im}{2E_p} \sum_{r,s} a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s \times \xi^{r\dagger} \xi^s \quad [3.0.7]$$

$$+ \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{im}{2E_p} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} \xi^s \quad [3.0.8]$$

$$+ \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_{\mathbf{p}}^r a_{-\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} \sigma^2 (p\sigma) \xi^s \quad [3.0.9]$$

を得る.

[3.0.1] の第 3 項.

$$\begin{aligned}
 &- \frac{im}{2} \int d^3x \chi^\top \sigma^2 \chi \\
 &= - \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (a_{\mathbf{p}}^r \xi^{r\dagger} e^{-ipx} + i a_{\mathbf{p}}^{r\dagger} \xi^{r\dagger} \sigma^2 e^{ipx}) \\
 &\quad \times \sqrt{(p\sigma^\top)} \sigma^2 \sqrt{(q\sigma)} (a_{\mathbf{q}}^s \xi^s e^{-iqx} - i \sigma^2 a_{\mathbf{q}}^{s\dagger} \xi^s e^{iqx}) \\
 &= - \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (a_{\mathbf{p}}^r \xi^{r\dagger} e^{-ipx} + i a_{\mathbf{p}}^{r\dagger} \xi^{r\dagger} \sigma^2 e^{ipx}) \\
 &\quad \times \sigma^2 \sqrt{(p\bar{\sigma})} \sqrt{(q\sigma)} (a_{\mathbf{q}}^s \xi^s e^{-iqx} - i \sigma^2 a_{\mathbf{q}}^{s\dagger} \xi^s e^{iqx})
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (a_p^r \xi^{r\dagger} e^{-ipx}) \sigma^2 \sqrt{(p\bar{\sigma})} \sqrt{(q\sigma)} (a_q^s \xi^s e^{-iqx}) \\
 &\quad - \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (a_p^r \xi^{r\dagger} e^{-ipx}) \sigma^2 \sqrt{(p\bar{\sigma})} \sqrt{(q\sigma)} (-i\sigma^2 a_q^{s\dagger} \xi^s e^{iqx}) \\
 &\quad - \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (i a_p^{r\dagger} \xi^{r\dagger} \sigma^2 e^{ipx}) \sigma^2 \sqrt{(p\bar{\sigma})} \sqrt{(q\sigma)} (a_q^s \xi^s e^{-iqx}) \\
 &\quad - \frac{im}{2} \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2E_p 2E_q}} \sum_{r,s} (i a_p^{r\dagger} \xi^{r\dagger} \sigma^2 e^{ipx}) \sigma^2 \sqrt{(p\bar{\sigma})} \sqrt{(q\sigma)} (-i\sigma^2 a_q^{s\dagger} \xi^s e^{iqx}) \\
 &= -\frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_p^r a_{-p}^s \times \xi^{r\dagger} \sigma^2 \sqrt{(p\bar{\sigma})} \sqrt{(p\sigma)} \xi^s \\
 &\quad - \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{-i}{2E_p} \sum_{r,s} a_p^r a_p^{s\dagger} \times \xi^{r\dagger} \sigma^2 \sqrt{(p\bar{\sigma})} \sqrt{(p\sigma)} \sigma^2 \xi^s \\
 &\quad - \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_p} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} \sigma^2 \sigma^2 \sqrt{(p\bar{\sigma})} \sqrt{(p\sigma)} \xi^s \\
 &\quad - \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_p^{r\dagger} a_{-p}^{s\dagger} \times \xi^{r\dagger} \sigma^2 \sigma^2 \sqrt{(p\bar{\sigma})} \sqrt{(p\sigma)} \sigma^2 \xi^s
 \end{aligned}$$

となるので,

$$= -\frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_p^r a_{-p}^s \times \xi^{r\dagger} [\sigma^2(p\bar{\sigma})] \xi^s \quad [3.0.10]$$

$$- \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{-im}{2E_p} \sum_{r,s} a_p^r a_p^{s\dagger} \times \xi^{r\dagger} \xi^s \quad [3.0.11]$$

$$- \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{im}{2E_p} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} \xi^s \quad [3.0.12]$$

$$- \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_p^{r\dagger} a_{-p}^{s\dagger} \times \xi^{r\dagger} [(p\bar{\sigma})\sigma^2] \xi^s \quad [3.0.13]$$

を得る.

$a_p^{r\dagger} a_p^s$ の項は次のようになる :

$$\begin{aligned}
 &[3.0.2] + [3.0.7] + [3.0.12] \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{-1}{2E_p} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} [E_p(\mathbf{p} \cdot \boldsymbol{\sigma}) - |\mathbf{p}|^2] \xi^s \\
 &\quad + \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{-im}{2E_p} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} \xi^s \\
 &\quad - \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{im}{2E_p} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} \xi^s \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} [-E_p(\mathbf{p} \cdot \boldsymbol{\sigma}) + |\mathbf{p}|^2] \xi^s + \int \frac{d^3p}{(2\pi)^3} \frac{m^2}{2E_p} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} \xi^s \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{|\mathbf{p}|^2 + m^2}{2E_p} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} \xi^s - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{r,s} a_p^{r\dagger} a_p^s \times \xi^{r\dagger} (\mathbf{p} \cdot \boldsymbol{\sigma}) \xi^s
 \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_{r,s} a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s \delta^{rs} - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{r,s} a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s \times \xi^{r\dagger} (\mathbf{p} \cdot \boldsymbol{\sigma}) \xi^s \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_s a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s
 \end{aligned} \tag{3.0.14}$$

(最後の式変形では被積分関数が奇関数であることを使った). $a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger}$ の項は次のようになる :

$$\begin{aligned}
 &[3.0.5] + [3.0.8] + [3.0.11] \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} [-E_{\mathbf{p}}(\mathbf{p} \cdot \boldsymbol{\sigma}^*) - |\mathbf{p}|^2] \xi^s \\
 &\quad + \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{im}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} \xi^s \\
 &\quad - \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{-im}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} \xi^s \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} [-E_{\mathbf{p}}(\mathbf{p} \cdot \boldsymbol{\sigma}^*) - |\mathbf{p}|^2] \xi^s - \int \frac{d^3p}{(2\pi)^3} \frac{m^2}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} \xi^s \\
 &= - \int \frac{d^3p}{(2\pi)^3} \frac{|\mathbf{p}|^2 + m^2}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} \xi^s - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} (\mathbf{p} \cdot \boldsymbol{\sigma}^*) \xi^s \\
 &= - \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \delta^{rs} - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \sum_{r,s} a_{\mathbf{p}}^r a_{\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} (\mathbf{p} \cdot \boldsymbol{\sigma}^*) \xi^s \\
 &= - \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_s a_{\mathbf{p}}^s a_{\mathbf{p}}^{s\dagger}
 \end{aligned} \tag{3.0.15}$$

(最後の式変形では被積分関数が奇関数であることを使った). $a_{\mathbf{p}}^{r\dagger} a_{-\mathbf{p}}^{s\dagger}$ の項は次のようになる :

$$\begin{aligned}
 &[3.0.3] + [3.0.6] + [3.0.13] \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^{r\dagger} a_{-\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} [m(\mathbf{p} \cdot \boldsymbol{\sigma}) \sigma^2] \xi^s \\
 &\quad + \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^{r\dagger} a_{-\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} [(p\sigma) \sigma^2] \xi^s \\
 &\quad - \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^{r\dagger} a_{-\mathbf{p}}^{s\dagger} \times \xi^{r\dagger} [(p\bar{\sigma}) \sigma^2] \xi^s \\
 &= 0.
 \end{aligned} \tag{3.0.16}$$

$a_{\mathbf{p}}^r a_{-\mathbf{p}}^s$ の項は次のようになる :

$$\begin{aligned}
 &[3.0.4] + [3.0.9] + [3.0.10] \\
 &= \int \frac{d^3p}{(2\pi)^3} \frac{i}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^r a_{-\mathbf{p}}^s \times \xi^{r\dagger} [m\sigma^2(\mathbf{p} \cdot \boldsymbol{\sigma})] \xi^s \\
 &\quad + \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^r a_{-\mathbf{p}}^s \times \xi^{r\dagger} \sigma^2(p\sigma) \xi^s \\
 &\quad - \frac{im}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} a_{\mathbf{p}}^r a_{-\mathbf{p}}^s \times \xi^{r\dagger} [\sigma^2(p\bar{\sigma})] \xi^s
 \end{aligned}$$

$$= 0.$$

[3.0.17]

[3.0.14][3.0.15][3.0.16][3.0.17] から,

$$\begin{aligned} H_{\text{Majorana}} &= \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_s a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - \int \frac{d^3p}{(2\pi)^3} \frac{E_{\mathbf{p}}}{2} \sum_s a_{\mathbf{p}}^s a_{\mathbf{p}}^{s\dagger} \\ &= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_s a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s \end{aligned}$$

となる (最後の計算では, 発散する定数を無視した). これは Dirac 場のハミルトニアン

$$H_{\text{Dirac}} = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_s (a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s)$$

の半分である.

Chapter 4

Interacting Fields and Feynman Diagrams

Problems

Problem 4.3: Linear sigma model

(d)

ポテンシャルは

$$V = -\frac{1}{2}\mu^2 \Phi \cdot \Phi + \frac{\lambda}{4}(\Phi \cdot \Phi)^2 - a\Phi^N$$

で与えられる. V が $\Phi^i = 0$ で極小となる v を求める.

$$\frac{\partial V}{\partial \Phi^i} = (-\mu^2 + \lambda \Phi \cdot \Phi) \Phi^i - a\delta^{iN}$$

に $\Phi^i = 0$ ($1 \leq i \leq N-1$), $\Phi^N = v$ を代入して,

$$(-\mu^2 + \lambda v^2)v\delta^{iN} - a\delta^{iN} = 0.$$

a は十分小さいので,

$$v = \frac{\mu}{\sqrt{\lambda}} + \frac{a}{2\mu^2}$$

であり,

$$\Phi^N = \frac{\mu}{\sqrt{\lambda}} + \sigma + \frac{a}{2\mu^2}.$$

V の表式は

$$\begin{aligned} V &= -\frac{1}{2}\mu^2 \Phi \cdot \Phi + \frac{\lambda}{4}(\Phi \cdot \Phi)^2 - a\Phi^N \\ &= -\frac{\mu^2}{2} \left\{ \pi \cdot \pi + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma + \frac{a}{2\mu^2} \right)^2 \right\} + \frac{\lambda}{4} \left\{ \pi \cdot \pi + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma + \frac{a}{2\mu^2} \right)^2 \right\}^2 - a \left(\frac{\mu}{\sqrt{\lambda}} + \sigma + \frac{a}{2\mu^2} \right) \\ &\simeq -\frac{\mu^2}{2} \left\{ \pi \cdot \pi + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right)^2 + 2 \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right) \frac{a}{2\mu^2} \right\} \\ &\quad + \frac{\lambda}{4} \left[\left\{ \pi \cdot \pi + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right)^2 \right\}^2 + 2 \left\{ \pi \cdot \pi + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right)^2 \right\} 2 \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right) \frac{a}{2\mu^2} \right] \\ &\quad - a \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right). \end{aligned}$$

a を含まない項を先に計算する (これは (b) で計算した) :

$$\begin{aligned}
 V_0 &= -\frac{\mu^2}{2} \left\{ \pi \cdot \pi + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right)^2 \right\} + \frac{\lambda}{4} \left\{ \pi \cdot \pi + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right)^2 \right\}^2 \\
 &= -\frac{\mu^2}{2} (\pi \cdot \pi) - \frac{\mu^2}{2} \left(\frac{\mu^2}{\lambda} + 2 \frac{\mu}{\sqrt{\lambda}} \sigma + \sigma^2 \right) + \frac{\lambda}{4} \left\{ (\pi \cdot \pi) + \left(\frac{\mu^2}{\lambda} + 2 \frac{\mu}{\sqrt{\lambda}} \sigma + \sigma^2 \right) \right\}^2 \\
 &= -\frac{\mu^2}{2} (\pi \cdot \pi) - \frac{\mu^2}{2} \left(\frac{\mu^2}{\lambda} + 2 \frac{\mu}{\sqrt{\lambda}} \sigma + \sigma^2 \right) \\
 &\quad + \frac{\lambda}{4} \left\{ (\pi \cdot \pi) + 2(\pi \cdot \pi) \left(\frac{\mu^2}{\lambda} + 2 \frac{\mu}{\sqrt{\lambda}} \sigma + \sigma^2 \right) + \left(\frac{\mu^2}{\lambda} + 2 \frac{\mu}{\sqrt{\lambda}} \sigma + \sigma^2 \right)^2 \right\} \\
 &= -\frac{\mu^2}{2} (\pi \cdot \pi) - \frac{\mu^4}{2\lambda} - \frac{\mu^3}{\sqrt{\lambda}} \sigma - \frac{\mu^2}{2} \sigma^2 \\
 &\quad + \frac{\lambda}{4} (\pi \cdot \pi)^2 + \frac{\mu^2}{2} (\pi \cdot \pi) + \sqrt{\lambda} \mu (\pi \cdot \pi) \sigma + \frac{\lambda}{2} (\pi \cdot \pi) \sigma^2 \\
 &\quad + \frac{\mu^4}{4\lambda} + \mu^2 \sigma^2 + \frac{\lambda}{4} \sigma^4 + \frac{\mu^3}{\sqrt{\lambda}} \sigma + \frac{1}{2} \mu^2 \sigma^2 + \sqrt{\lambda} \mu \sigma^3 \\
 &= -\frac{\mu^4}{4\lambda} + \sqrt{\lambda} \mu \sigma^3 + \mu^2 \sigma^2 + \frac{\lambda}{4} \sigma^4 + \frac{\lambda}{4} (\pi \cdot \pi)^2 + \sqrt{\lambda} \mu (\pi \cdot \pi) \sigma + \frac{\lambda}{2} (\pi \cdot \pi) \sigma^2.
 \end{aligned}$$

次に, a を含む項を計算する :

$$\begin{aligned}
 V_a &= -\mu^2 \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right) \frac{a}{2\mu^2} + \lambda \left\{ \pi \cdot \pi + \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right)^2 \right\} \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right) \frac{a}{2\mu^2} - a \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right) \\
 &= a \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right) \left\{ -\frac{3}{2} + \frac{\lambda}{2\mu^2} \left(\pi \cdot \pi + \frac{\mu^2}{\lambda} + 2 \frac{\mu}{\sqrt{\lambda}} \sigma + \sigma^2 \right) \right\} \\
 &= a \left(\frac{\mu}{\sqrt{\lambda}} + \sigma \right) \frac{\lambda}{2\mu^2} \left(\pi \cdot \pi - 2 \frac{\mu^2}{\lambda} + 2 \frac{\mu}{\sqrt{\lambda}} \sigma + \sigma^2 \right).
 \end{aligned}$$

以上から

$$\begin{aligned}
 V &= V_0 + V_a \\
 &= \frac{1}{2} \left(2\mu^2 + \frac{3a\sqrt{\lambda}}{\mu} \right) \sigma^2 + \frac{1}{2} \frac{a\sqrt{\lambda}}{\mu} (\pi \cdot \pi) \\
 &\quad + \left(\sqrt{\lambda} \mu + \frac{a\lambda}{2\mu^2} \right) (\pi \cdot \pi) \sigma + \left(\sqrt{\lambda} \mu + \frac{a\lambda}{2\mu^2} \right) \sigma^3 + \frac{\lambda}{4} (\pi \cdot \pi)^2 + \frac{\lambda}{2} (\pi \cdot \pi) \sigma^2 + \frac{\lambda}{4} \sigma^4 \\
 &\quad + \text{const.}
 \end{aligned}$$

質量は

$$m_\sigma^2 = 2\mu^2 + \frac{3a\sqrt{\lambda}}{\mu}, \quad m_\pi^2 = \frac{a\sqrt{\lambda}}{\mu}.$$

propagator は

$$\begin{aligned}
 \text{=====} &= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_\sigma^2 + i\epsilon} e^{-ip(x-y)}, \\
 \text{-----} &= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m_\pi^2 + i\epsilon} e^{-ip(x-y)} \delta^{ij}.
 \end{aligned}$$

vertex factor は

$$\begin{aligned}
 & \text{Diagram 1: A vertex with two external lines (labeled } i \text{ and } j \text{) and one internal double line.} & = -2i \left(\sqrt{\lambda} \mu + \frac{a\lambda}{2\mu^2} \right) \delta^{ij} \\
 & \text{Diagram 2: A vertex with two external lines (labeled } k \text{ and } l \text{) and one internal double line.} & = -6i \left(\sqrt{\lambda} \mu + \frac{a\lambda}{2\mu^2} \right) \\
 & \text{Diagram 3: A four-point vertex with external lines } i, j, k, l \text{ meeting at a central dot.} & = -2i\lambda (\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}) \\
 & \text{Diagram 4: A four-point vertex with external lines } i, j \text{ and two internal double lines meeting at a central dot.} & = -2i\lambda \delta^{ij} \\
 & \text{Diagram 5: A four-point vertex with external lines } i, j \text{ and two internal double lines meeting at a central dot.} & = -6i\lambda
 \end{aligned}$$

で与えられる.

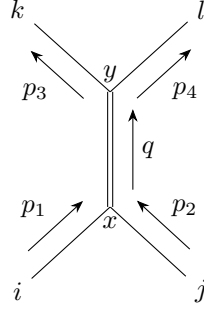
T 行列要素

$$T = \left\langle p_3^k p_4^l \left| T \exp \left(- \int d^4x \mathcal{H}_{\text{int}} \right) \right| p_1^i p_2^j \right\rangle$$

を計算する.

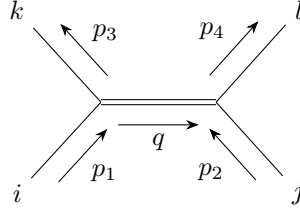
まず, 2 次の展開を考える:

$$\sum_{m,n} \left\langle 0 \left| a_{\mathbf{p}_3}^k a_{\mathbf{p}_4}^l \frac{(-i)^2}{2!} (\sqrt{\lambda} \mu)^2 \int d^4x d^4y N \{ \pi^m(y) \pi^m(y) \sigma(y) \pi^n(x) \pi^n(x) \sigma(x) \} a_{\mathbf{p}_1}^{i\dagger} a_{\mathbf{p}_2}^{j\dagger} \right| 0 \right\rangle.$$



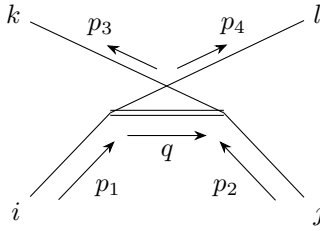
上図に対応する項は 4 通り存在し, x と y の交換を考慮に入れて,

$$\begin{aligned}
 & -4\lambda\mu^2 \int d^4x d^4y e^{i(p_3+p_4)y} \int \frac{d^4q}{(2\pi^4)} \frac{ie^{iq(y-x)}}{q^2 - m_\sigma^2} e^{-i(p_1+p_2)x} \delta^{ij} \delta^{kl} \\
 &= -4\lambda\mu^2 \int \frac{d^4q}{(2\pi^4)} \frac{1}{q^2 - m_\sigma^2} \int d^4y e^{i(p_3+p_4-q)y} \int d^4x e^{-i(p_1+p_2-q)x} \delta^{ij} \delta^{kl} \\
 &= -(2\pi)^4 4i\lambda\mu^2 \int \frac{d^4q}{q^2 - m_\sigma^2} \delta^{(4)}(p_3 + p_4 - q) \delta^{(4)}(p_1 + p_2 - q) \delta^{ij} \delta^{kl} \\
 &= \frac{-4\lambda\mu^2}{(p_1 + p_2)^2 - m_\sigma^2} \delta^{ij} \delta^{kl} i(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4).
 \end{aligned}$$



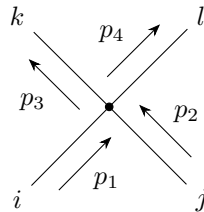
この場合は

$$\frac{-4\lambda\mu^2}{(p_1 - p_3)^2 - m_\sigma^2} \delta^{ik} \delta^{jl} i(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4).$$



この場合は

$$\frac{-4\lambda\mu^2}{(p_1 - p_4)^2 - m_\sigma^2} \delta^{il} \delta^{jk} i(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4).$$



この場合は

$$-2i\lambda (\delta^{ij}\delta^{kl} + \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl}) i(2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4).$$

ΔV によって $m_\sigma^2 \neq 2\mu^2$ となったので, $p_i \rightarrow 0$ の極限でも, これらの和は 0 とならない.

Chapter 5

Elementary Processes of Quantum Electrodynamics

5.2 $e^+e^- \rightarrow \mu^+\mu^-$: Helicity Structure

(5.28)

Dirac 方程式の解は (A.19) で与えられる：

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix}, \quad v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \eta^s \\ -\sqrt{p \cdot \bar{\sigma}} \eta^s \end{pmatrix}.$$

高エネルギー極限では (A.20) のように,

$$u^s(p) \approx \sqrt{2E} \begin{pmatrix} \frac{1}{2}(1 - \hat{p} \cdot \boldsymbol{\sigma}) \xi^s \\ \frac{1}{2}(1 + \hat{p} \cdot \boldsymbol{\sigma}) \xi^s \end{pmatrix}, \quad v^s(p) \approx \sqrt{2E} \begin{pmatrix} \frac{1}{2}(1 - \hat{p} \cdot \boldsymbol{\sigma}) \eta^s \\ -\frac{1}{2}(1 + \hat{p} \cdot \boldsymbol{\sigma}) \eta^s \end{pmatrix}.$$

電子の spinor は $\xi = {}^\top(1, 0)$ が z ($\sigma^3 \xi = +\xi$). 陽電子の spinor は $\eta = {}^\top(0, 1)$ が z (電子と逆) (p. 61)

2 成分の spinor ξ が $(\hat{p} \cdot \boldsymbol{\sigma})\xi = +\xi$ を満たすとき helicity を右と定義する. 陽電子の場合は spinor と粒子の spin が逆なので, helicity も逆になる (p. 142, 144)

電子は z 方向の spin 上向きなので, spinor は $\xi = {}^\top(1, 0)$. $\hat{p} = (0, 0, 1)$ の向きに進むので, helicity は右. $\hat{p} \cdot \boldsymbol{\sigma} = \sigma^3$ なので,

$$u \approx \sqrt{2E} \begin{pmatrix} \frac{1}{2}(1 - \hat{p} \cdot \boldsymbol{\sigma}) \xi \\ \frac{1}{2}(1 + \hat{p} \cdot \boldsymbol{\sigma}) \xi \end{pmatrix} = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

陽電子は z 方向の粒子 spin が上向きなので, spinor は $\eta = {}^\top(0, 1)$. $\hat{p} = (0, 0, -1)$ の向きに進むので, 粒

子 helicity は左. $\hat{p} \cdot \boldsymbol{\sigma} = -\sigma^3$ なので,

$$v \approx \sqrt{2E} \begin{pmatrix} \frac{1}{2}(1 - \hat{p} \cdot \boldsymbol{\sigma})\eta \\ -\frac{1}{2}(1 + \hat{p} \cdot \boldsymbol{\sigma})\eta \end{pmatrix} = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}.$$

5.5 Compton Scattering

(5.99)

入射電子は $-z$ の向きに進み, helicity は右とする. z 方向の spin 下向きなので

$$\hat{p} = (0, 0, -1), \quad \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u(p) = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(5.97) が非零となるのは散乱電子 $u^\dagger(p')$ の第 3, 4 成分が非零, すなわち helicity が右の場合. さらに, 電子は $+z$ 側に散乱される (Figure 5.6) ので, $\xi^\dagger = (1, 0)$ である.

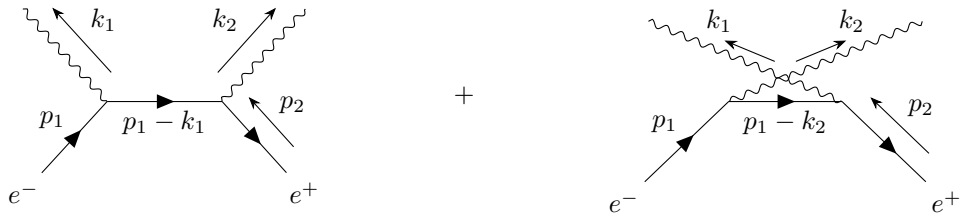
Problems

Problem 5.4: Positronium lifetime

[4] を簡略化した.

対消滅の不変振幅 (0 次)

$e^-e^+ \rightarrow 2\gamma$ の過程を考える.



対消滅の不変振幅を運動量 p の 0 次までの精度で求める.

$$p_1^\mu = (E, 0), \quad p_2^\mu = (E, 0), \quad k_1^\mu = (E, \mathbf{k}), \quad k_2^\mu = (E, -\mathbf{k}), \quad |\mathbf{k}| = E.$$

光子の偏極は次のようにおく:

$$\epsilon_\pm^\mu(k_1) = \epsilon_{1\pm}^\mu = (0, \boldsymbol{\epsilon}_1), \quad \boldsymbol{\epsilon}_1 \cdot \mathbf{k}_1 = 0, \quad \epsilon_\pm^\mu(k_2) = \epsilon_{2\pm}^\mu = (0, \boldsymbol{\epsilon}_2), \quad \boldsymbol{\epsilon}_2 \cdot \mathbf{k} = 0.$$

スピノルは次のように近似される:

$$u(p_1) = \begin{pmatrix} \sqrt{\sigma \cdot p_1} \xi \\ \sqrt{\bar{\sigma} \cdot p_1} \xi \end{pmatrix} \approx \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}, \quad v(p_2) = \begin{pmatrix} \sqrt{\sigma \cdot p_2} \eta \\ -\sqrt{\bar{\sigma} \cdot p_2} \eta \end{pmatrix} \approx \sqrt{m} \begin{pmatrix} \eta \\ -\eta \end{pmatrix}. \quad [5.5.1]$$

さらに,

$$p_1 \cdot k_1 = E^2 \approx m^2, \quad p_1 \cdot k_2 = E^2 \approx m^2, \quad (p_1 \cdot k_1)(p_1 \cdot k_2) \approx m^4.$$

Dirac 方程式から次を得る：

$$\begin{aligned} (\not{p}_1 + m)\not{\epsilon}_1^* u(p_1) &= 2(p_1 \cdot \epsilon_1^*)u(p_1) - \not{\epsilon}_1^*(\not{p}_1 - m)u(p_1) = 0, \\ (\not{p}_1 + m)\not{\epsilon}_2^* u(p_1) &= 2(p_1 \cdot \epsilon_2^*)u(p_1) - \not{\epsilon}_2^*(\not{p}_1 - m)u(p_1) = 0. \end{aligned}$$

従って, 不変振幅は

$$\begin{aligned} i\mathcal{M} &= -ie^2 \bar{v}(p_2) \left[\not{\epsilon}_2^* \frac{\not{p}_1 - \not{k}_1 + m}{(p_1 - k_1)^2 - m^2} \not{\epsilon}_1^* + \not{\epsilon}_1^* \frac{\not{p}_1 - \not{k}_2 + m}{(p_1 - k_2)^2 - m^2} \not{\epsilon}_2^* \right] u(p_1) \\ &= -ie^2 \bar{v}(p_2) \left[\not{\epsilon}_2^* \frac{-\not{p}_1 + \not{k}_1 - m}{2p_1 \cdot k_1} \not{\epsilon}_1^* + \not{\epsilon}_1^* \frac{-\not{p}_1 + \not{k}_2 - m}{2p_1 \cdot k_2} \not{\epsilon}_2^* \right] u(p_1) \\ &= -i \frac{e^2}{2} \bar{v}(p_2) \left[\frac{\not{\epsilon}_2^* \not{k}_1 \not{\epsilon}_1^*}{m^2} + \frac{\not{\epsilon}_1^* \not{k}_2 \not{\epsilon}_2^*}{m^2} \right] u(p_1) \\ &= -i \frac{e^2}{2m^2} \bar{v}(p_2) [\not{\epsilon}_2^* \not{k}_1 \not{\epsilon}_1^* + \not{\epsilon}_1^* \not{k}_2 \not{\epsilon}_2^*] u(p_1) \end{aligned} \tag{5.5.2}$$

となる. $[\dots]$ を計算するが, まず次の式を示しておく：

$$\begin{aligned} \not{\epsilon} \not{k} \not{\epsilon} &= a_\mu b_\nu c_\rho \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\rho \\ \bar{\sigma}^\rho & 0 \end{pmatrix} \\ &= a_\mu b_\nu c_\rho \begin{pmatrix} 0 & \sigma^\mu \bar{\sigma}^\nu \sigma^\rho \\ \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (a \cdot \sigma)(b \cdot \bar{\sigma})(c \cdot \sigma) \\ (a \cdot \bar{\sigma})(b \cdot \sigma)(c \cdot \bar{\sigma}) & 0 \end{pmatrix}. \end{aligned}$$

[5.5.2] の $[\dots]$ のうち 3 つのガンマ行列を含む項は

$$\begin{aligned} [\not{\epsilon}_2^* \not{k}_1 \not{\epsilon}_1^* + \not{\epsilon}_1^* \not{k}_2 \not{\epsilon}_2^*] &= \begin{pmatrix} 0 & (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma) \\ (\epsilon_2^* \cdot \bar{\sigma})(k_1 \cdot \sigma)(\epsilon_1^* \cdot \bar{\sigma}) & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) \\ (\epsilon_1^* \cdot \bar{\sigma})(k_2 \cdot \sigma)(\epsilon_2^* \cdot \bar{\sigma}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma) \\ (\epsilon_2^* \cdot \sigma)(k_1 \cdot \sigma)(\epsilon_1^* \cdot \bar{\sigma}) & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) \\ (\epsilon_1^* \cdot \sigma)(k_2 \cdot \sigma)(\epsilon_2^* \cdot \bar{\sigma}) & 0 \end{pmatrix} \\ &=: \begin{pmatrix} 0 & m^2 \bar{B}_+ \\ m^2 B_+ & 0 \end{pmatrix}. \end{aligned}$$

以上から

$$i\mathcal{M} = -i \frac{e^2}{2m^4} \bar{v}(p_2) \begin{pmatrix} 0 & -2m^2 A + m^2 \bar{B}_+ - (\mathbf{p} \cdot \mathbf{k}) \bar{B}_- \\ 2m^2 A + m^2 B_+ - (\mathbf{p} \cdot \mathbf{k}) B_- & 0 \end{pmatrix} u(p_1) \tag{5.5.3}$$

となる. 各項の定義は

$$\begin{aligned} B_+ &= (\epsilon_1^* \cdot \sigma)(k_2 \cdot \sigma)(\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma)(k_1 \cdot \sigma)(\epsilon_1^* \cdot \sigma) \\ \bar{B}_+ &= (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma). \end{aligned} \tag{5.5.4}$$

$(\boldsymbol{\sigma} \cdot \boldsymbol{a})(\boldsymbol{\sigma} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} + i\boldsymbol{\sigma} \cdot (\boldsymbol{a} \times \boldsymbol{b})$ から

$$\begin{aligned}
 (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(\boldsymbol{k} \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) &= [\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{k} + i\boldsymbol{\sigma} \cdot (\boldsymbol{\epsilon}_1^* \times \boldsymbol{k})](\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) \\
 &= i\boldsymbol{\sigma} \cdot (\boldsymbol{\epsilon}_1^* \times \boldsymbol{k})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) \\
 &= i(\boldsymbol{\epsilon}_1^* \times \boldsymbol{k}) \cdot \boldsymbol{\epsilon}_2^* - \boldsymbol{\sigma} \cdot [(\boldsymbol{\epsilon}_1^* \times \boldsymbol{k}) \times \boldsymbol{\epsilon}_2^*] \\
 &= i(\boldsymbol{\epsilon}_2^* \times \boldsymbol{\epsilon}_1^*) \cdot \boldsymbol{k} - \boldsymbol{\sigma} \cdot [(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*)\boldsymbol{k} - (\boldsymbol{k} \cdot \boldsymbol{\epsilon}_2^*)\boldsymbol{\epsilon}_1^*] \\
 &= -i(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \boldsymbol{k} - (\boldsymbol{\sigma} \cdot \boldsymbol{k})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*) \\
 (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(\boldsymbol{k} \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) &= i(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \boldsymbol{k} - (\boldsymbol{\sigma} \cdot \boldsymbol{k})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*)
 \end{aligned}$$

となる。従って、

$$\begin{aligned}
 \bar{B}_+ - B_+ &= (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(\boldsymbol{k}_2 \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(\boldsymbol{k}_1 \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
 &\quad - (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(\boldsymbol{k}_2 \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) - (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(\boldsymbol{k}_1 \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
 &= (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(\boldsymbol{k}_2 \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}))(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(\boldsymbol{k}_1 \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}))(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
 &= -2(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma})(\boldsymbol{k} \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + 2(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma})(\boldsymbol{k} \cdot \boldsymbol{\sigma})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) \\
 &= 4i(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \boldsymbol{k},
 \end{aligned} \tag{5.5.5}$$

[5.5.3] に [5.5.1] を代入して、[5.5.5] を使えば、

$$\begin{aligned}
 i\mathcal{M}(e_s^- e_r^+ \rightarrow 2\gamma) &= -i\frac{e^2}{2m^2} \bar{v}(p_2) \begin{pmatrix} 0 & \bar{B}_+ \\ B_+ & 0 \end{pmatrix} u(p_1) \\
 &= -i\frac{e^2}{2m} (\eta^{r\dagger} \quad -\eta^{r\dagger}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \bar{B}_+ \\ B_+ & 0 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \\
 &= i\frac{e^2}{2m} \eta^{r\dagger} (\bar{B}_+ - B_+) \xi^s \\
 &= -\frac{2e^2}{m} (\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \boldsymbol{k} (\eta^{r\dagger} \xi^s)
 \end{aligned} \tag{5.5.6}$$

となる。

ここで、各スピノルは (3.135)(3.136) から

$$\xi^\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta^\uparrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \eta^\downarrow = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \tag{5.5.7}$$

で与えられる。ラベルは全て電子・陽電子のスピンを表す。

S ポジトロニウムの構成

束縛状態の式 (5.43)

$$|B\rangle = \sqrt{2M} \int \frac{d^3k}{(2\pi)^3} \tilde{\psi}(\boldsymbol{k}) \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |\boldsymbol{k} \uparrow, -\boldsymbol{k} \uparrow\rangle$$

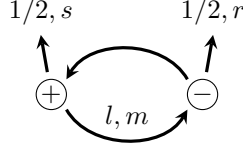
を角運動量を考慮した形に拡張する。すなわち、スピン S 、軌道角運動量 l 、全角運動量 J 、全角運動量の射影 M の状態 $|^{2S+1}l_J; M\rangle$ を構成しよう^{*1}。これには、スピンの射影が S_z で軌道角運動量の射影が $M - S_z =: m$

^{*1} スピンは粒子の内在的な量なので、電子のスピン、陽電子のスピンを足す。それに対し軌道角運動量は粒子の相対運動に起因するので、電子と陽電子について和を取ることは行わない

の状態を Clebsh-Gordan 係数 $\langle lS; mS_z | lS; JM \rangle$ によって足せば良い：

$$|^{2S+1}l_J; M\rangle = \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \sum_{S_z=-S}^S \langle lS; mS_z | lS; JM \rangle \tilde{\psi}_{lm}(\mathbf{p}) |S, S_z\rangle_{\mathbf{p}} \quad [5.5.8]$$

(ただし、慣例に従い $l=0$ は S , $l=1$ は P などと表記する). なお、電子の運動量は \mathbf{p} , 陽電子の運動量は $-\mathbf{p}$ とする. 換算質量は $m/2$ なので, 相対運動量は \mathbf{p} である.



(電子と陽電子の合計) スピン S , 射影 S_z , 相対運動量 \mathbf{p} の状態 $|S, S_z\rangle_{\mathbf{p}}$ は, スピン射影 s で運動量 \mathbf{p} の電子と, スピン射影 $r := S_z - s$ で運動量 \mathbf{p} の陽電子の線形結合で表せる：

$$\begin{aligned} |S, S_z\rangle_{\mathbf{p}} &= \sum_s \left\langle \frac{1}{2} \frac{1}{2}; sr \left| \frac{1}{2} \frac{1}{2}; SS_z \right\rangle \left| \frac{1}{2}, s \right\rangle_{\mathbf{p}} \left| \frac{1}{2}, r \right\rangle_{-\mathbf{p}} \right. \\ &= \sum_s \left\langle \frac{1}{2} \frac{1}{2}; sr \left| \frac{1}{2} \frac{1}{2}; SS_z \right\rangle \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} a_{\mathbf{p}}^{s\dagger} b_{-\mathbf{p}}^{r\dagger} |0\rangle. \end{aligned}$$

まず, S 状態 ($l=0$) のポジトロニウムについて考える ($m=0$ なので $M=S_z$). $|^1S_0; 0\rangle$ は $J=M=0$, $S=S_z=0$ なのでスピンは singlet：

$$|^1S_0; 0\rangle = 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\mathbf{p}) \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} \frac{|\mathbf{p} \uparrow, -\mathbf{p} \downarrow\rangle - |\mathbf{p} \downarrow, -\mathbf{p} \uparrow\rangle}{\sqrt{2}}. \quad [5.5.9]$$

$|^3S_1; 0\rangle$ は $J=1$, $M=0$, $S=1$, $S_z=0$ なのでスピンは triplet：

$$|^3S_1; 0\rangle = 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\mathbf{p}) \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} \frac{|\mathbf{p} \uparrow, -\mathbf{p} \downarrow\rangle + |\mathbf{p} \downarrow, -\mathbf{p} \uparrow\rangle}{\sqrt{2}}.$$

$|^3S_1; 1\rangle$ は $J=1$, $M=1$, $S=1$, $S_z=1$ なのでスピンは triplet：

$$|^3S_1; 1\rangle = 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\mathbf{p}) \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |\mathbf{p} \uparrow, -\mathbf{p} \uparrow\rangle.$$

$|^3S_1; -1\rangle$ は $J=1$, $M=-1$, $S=1$, $S_z=-1$ なのでスピンは triplet：

$$|^3S_1; -1\rangle = 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\mathbf{p}) \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |\mathbf{p} \downarrow, -\mathbf{p} \downarrow\rangle.$$

S ポジトロニウムの崩壊

不変振幅 \mathcal{M} の定義 (4.73) に注意して, [5.5.6][5.5.7][5.5.9] から

$$\begin{aligned}
 i\mathcal{M}(1^1S_0 \rightarrow 2\gamma) &= 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\mathbf{p}) \frac{1}{2m} \frac{i\mathcal{M}(e_{\uparrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) - i\mathcal{M}(e_{\downarrow}^- e_{\uparrow}^+ \rightarrow 2\gamma)}{\sqrt{2}} \\
 &= 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\mathbf{p}) \frac{1}{2m} \left(-\frac{2e^2}{m} \right) (\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \mathbf{k} \frac{\eta^{\uparrow\downarrow} \xi^{\uparrow} - \eta^{\uparrow\uparrow} \xi^{\downarrow}}{\sqrt{2}} \\
 &= 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\mathbf{p}) \frac{1}{2m} \left(-\frac{2e^2}{m} \right) (\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \mathbf{k} \frac{-2}{\sqrt{2}} \\
 &= 2\sqrt{m} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\mathbf{p}) \frac{1}{2m} \left(-\frac{2e^2}{m} \right) (\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \mathbf{k} \frac{-2}{\sqrt{2}} \\
 &= 2\sqrt{2} \frac{e^2}{m\sqrt{m}} (\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \mathbf{k} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{00}(\mathbf{p}) \\
 &= 2\sqrt{2} \frac{e^2}{m\sqrt{m}} (\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \mathbf{k} \psi_{00}(0).
 \end{aligned} \tag{5.5.10}$$

(A.26) から

$$\sum_{\text{polarization}} \epsilon^{\mu*} \epsilon^{\nu} \rightarrow -g^{\mu\nu}$$

なので,

$$\begin{aligned}
 |(\boldsymbol{\epsilon}_1^* \times \boldsymbol{\epsilon}_2^*) \cdot \mathbf{k}|^2 &= \sum_{\text{pol } 1} \sum_{\text{pol } 2} \sum_{ijklmn} \epsilon^{ijk} \epsilon_1^{i*} \epsilon_2^{j*} k^k \epsilon^{lmn} \epsilon_1^l \epsilon_2^m k^n \\
 &\rightarrow \sum_{ijklmn} \epsilon^{ijk} \epsilon^{lmn} g^{il} g^{jm} k^k k^n \\
 &= \sum_{klmn} \epsilon^{lmk} \epsilon^{lmn} k^k k^n \\
 &= \sum_{lmn} \epsilon^{lmn} \epsilon^{lmn} k^n k^n \\
 &= 2|\mathbf{k}|^2 = 2E^2 \approx 2m^2
 \end{aligned} \tag{5.5.11}$$

を得る.

$n = 1$ の場合*2を考える.

$$\psi_{100} = R_{10}(r) Y_{00}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} \frac{2}{a_0^{3/2}} \exp\left(-\frac{r}{2a_0}\right)$$

なので,

$$|\psi_{100}(0)|^2 = \frac{m^3 \alpha^3}{8\pi}. \tag{5.5.12}$$

[5.5.10][5.5.11][5.5.12] から

$$\sum_{\text{polarization}} |\mathcal{M}(1^1S_0 \rightarrow 2\gamma)|^2 = 8 \frac{e^4}{m^3} 2m^2 \frac{m^3 \alpha^3}{8\pi} = 32\pi m^2 \alpha^5. \tag{5.5.13}$$

*2 換算質量は $m/2$ なので, Bohr 半径 $a_0 = 2/\alpha m$

(4.86) から

$$\begin{aligned}
 \Gamma(1^1S_0 \rightarrow 2\gamma) &= \frac{1}{2} \frac{1}{4m} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{1}{4E_1E_2} |\mathcal{M}(1^1S_0 \rightarrow 2\gamma)|^2 (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \\
 &= \frac{1}{8m} \int \frac{d^3k_1}{(2\pi)^2} \frac{1}{4|\mathbf{k}_1|^2} 32\pi m^2 \alpha^5 \delta(2E_1 - 2m) \\
 &= \pi m \alpha^5 4\pi \int \frac{d|\mathbf{k}_1|}{(2\pi)^2} \frac{1}{2} \delta(|\mathbf{k}_1| - m) \\
 &= \frac{m\alpha^5}{2}
 \end{aligned}$$

(出てくる光子は区別できないので, $1/2$ 倍する).

[5.5.10] と同様に考えれば,

$$\eta^{\uparrow\uparrow}\xi^\uparrow = 0, \quad \frac{\eta^{\uparrow\downarrow}\xi^\downarrow + \eta^{\downarrow\uparrow}\xi^\uparrow}{\sqrt{2}} = 0, \quad \eta^{\downarrow\downarrow}\xi^\downarrow = 0$$

なので, $\mathcal{M}(3S_1 \rightarrow 2\gamma) = 0$ であることが分かる. すなわち, スピン 1 の 1^3S は 2 光子に崩壊しない.

対消滅の不変振幅 (1 次)

対消滅の不変振幅を運動量 p の 1 次までの精度で求める.

$$p_1^\mu = (E, \mathbf{p}), \quad p_2^\mu = (E, -\mathbf{p}), \quad k_1^\mu = (E, \mathbf{k}), \quad k_2^\mu = (E, -\mathbf{k}).$$

光子の偏極は次のようにおく:

$$\epsilon_\pm^\mu(k_1) = \epsilon_{1\pm}^\mu = (0, \boldsymbol{\epsilon}_1), \quad \boldsymbol{\epsilon}_1 \cdot \mathbf{k}_1 = 0, \quad \epsilon_\pm^\mu(k_2) = \epsilon_{2\pm}^\mu = (0, \boldsymbol{\epsilon}_2), \quad \boldsymbol{\epsilon}_2 \cdot \mathbf{k}_2 = 0.$$

スピノルは次のように近似される:

$$u(p_1) = \begin{pmatrix} \sqrt{\sigma \cdot p_1} \xi \\ \sqrt{\sigma \cdot p_1} \xi \end{pmatrix} \approx \sqrt{m} \begin{pmatrix} \left(1 - \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m}\right) \xi \\ \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m}\right) \xi \end{pmatrix}, \quad v(p_2) = \begin{pmatrix} \sqrt{\sigma \cdot p_2} \eta \\ -\sqrt{\sigma \cdot p_2} \eta \end{pmatrix} \approx \sqrt{m} \begin{pmatrix} \left(1 + \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m}\right) \eta \\ -\left(1 - \frac{\boldsymbol{\sigma} \cdot \mathbf{p}}{2m}\right) \eta \end{pmatrix}. \quad [5.5.14]$$

さらに,

$$\begin{aligned}
 p_1 \cdot k_1 &= E^2 - \mathbf{p} \cdot \mathbf{k} \approx m^2 - \mathbf{p} \cdot \mathbf{k}, \\
 p_1 \cdot k_2 &= E^2 + \mathbf{p} \cdot \mathbf{k} \approx m^2 + \mathbf{p} \cdot \mathbf{k}, \\
 (p_1 \cdot k_1)(p_1 \cdot k_2) &\approx m^4.
 \end{aligned}$$

Dirac 方程式から次を得る:

$$\begin{aligned}
 (\not{p}_1 + m) \not{\epsilon}_1^* u(p_1) &= 2(p_1 \cdot \epsilon_1^*) u(p_1) - \not{\epsilon}_1^* (\not{p}_1 - m) u(p_1) = 2(p_1 \cdot \epsilon_1^*) u(p_1), \\
 (\not{p}_1 + m) \not{\epsilon}_2^* u(p_1) &= 2(p_1 \cdot \epsilon_2^*) u(p_1) - \not{\epsilon}_2^* (\not{p}_1 - m) u(p_1) = 2(p_1 \cdot \epsilon_2^*) u(p_1).
 \end{aligned}$$

従って、不変振幅は

$$\begin{aligned}
 i\mathcal{M} &= -ie^2 \bar{v}(p_2) \left[\not{\epsilon}_2^* \frac{\not{p}_1 - \not{k}_1 + m}{(p_1 - k_1)^2 - m^2} \not{\epsilon}_1^* + \not{\epsilon}_1^* \frac{\not{p}_1 - \not{k}_2 + m}{(p_1 - k_2)^2 - m^2} \not{\epsilon}_2^* \right] u(p_1) \\
 &= -ie^2 \bar{v}(p_2) \left[\not{\epsilon}_2^* \frac{-\not{p}_1 + \not{k}_1 - m}{2p_1 \cdot k_1} \not{\epsilon}_1^* + \not{\epsilon}_1^* \frac{-\not{p}_1 + \not{k}_2 - m}{2p_1 \cdot k_2} \not{\epsilon}_2^* \right] u(p_1) \\
 &= -i \frac{e^2}{2} \bar{v}(p_2) \left[\frac{\not{\epsilon}_2^* \not{k}_1 \not{\epsilon}_1^* - 2(p_1 \cdot \epsilon_1^*) \not{\epsilon}_2^*}{m^2 - \mathbf{p} \cdot \mathbf{k}} + \frac{\not{\epsilon}_1^* \not{k}_2 \not{\epsilon}_2^* - 2(p_1 \cdot \epsilon_2^*) \not{\epsilon}_1^*}{m^2 + \mathbf{p} \cdot \mathbf{k}} \right] u(p_1) \\
 &= -i \frac{e^2}{2m^4} \bar{v}(p_2) \left[(m^2 + \mathbf{p} \cdot \mathbf{k}) (\not{\epsilon}_2^* \not{k}_1 \not{\epsilon}_1^* - 2(p_1 \cdot \epsilon_1^*) \not{\epsilon}_2^*) + (m^2 - \mathbf{p} \cdot \mathbf{k}) (\not{\epsilon}_1^* \not{k}_2 \not{\epsilon}_2^* - 2(p_1 \cdot \epsilon_2^*) \not{\epsilon}_1^*) \right] u(p_1)
 \end{aligned} \tag{5.5.15}$$

となる. $[\dots]$ を計算するが, まず次の式を示しておく:

$$\begin{aligned}
 \not{a} \not{b} \not{c} &= a_\mu b_\nu c_\rho \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu \\ \bar{\sigma}^\nu & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\rho \\ \bar{\sigma}^\rho & 0 \end{pmatrix} \\
 &= a_\mu b_\nu c_\rho \begin{pmatrix} 0 & \sigma^\mu \bar{\sigma}^\nu \sigma^\rho \\ \bar{\sigma}^\mu \sigma^\nu \bar{\sigma}^\rho & 0 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & (a \cdot \sigma)(b \cdot \bar{\sigma})(c \cdot \sigma) \\ (a \cdot \bar{\sigma})(b \cdot \sigma)(c \cdot \bar{\sigma}) & 0 \end{pmatrix}.
 \end{aligned}$$

[5.5.15] の $[\dots]$ のうち m^2 と 1 つのガンマ行列を含む項は

$$\begin{aligned}
 &-2m^2(p_1 \cdot \epsilon_1^*) \not{\epsilon}_2^* - 2m^2(p_1 \cdot \epsilon_2^*) \not{\epsilon}_1^* \\
 &= -2m^2(p_1 \cdot \epsilon_1^*) \begin{pmatrix} 0 & \epsilon_2^* \cdot \sigma \\ \epsilon_2^* \cdot \bar{\sigma} & 0 \end{pmatrix} - 2m^2(p_1 \cdot \epsilon_2^*) \begin{pmatrix} 0 & \epsilon_1^* \cdot \sigma \\ \epsilon_1^* \cdot \bar{\sigma} & 0 \end{pmatrix} \\
 &= -2m^2(\mathbf{p} \cdot \epsilon_1^*) \begin{pmatrix} 0 & \epsilon_2^* \cdot \sigma \\ -\epsilon_2^* \cdot \sigma & 0 \end{pmatrix} - 2m^2(\mathbf{p} \cdot \epsilon_2^*) \begin{pmatrix} 0 & \epsilon_1^* \cdot \sigma \\ -\epsilon_1^* \cdot \sigma & 0 \end{pmatrix} \\
 &= -2m^2 \begin{pmatrix} 0 & (\mathbf{p} \cdot \epsilon_1^*)(\epsilon_2^* \cdot \sigma) + (\mathbf{p} \cdot \epsilon_2^*)(\epsilon_1^* \cdot \sigma) \\ -(\mathbf{p} \cdot \epsilon_1^*)(\epsilon_2^* \cdot \sigma) - (\mathbf{p} \cdot \epsilon_2^*)(\epsilon_1^* \cdot \sigma) & 0 \end{pmatrix} \\
 &=: \begin{pmatrix} 0 & -2m^2 A \\ 2m^2 A & 0 \end{pmatrix}.
 \end{aligned}$$

[5.5.15] の $[\dots]$ のうち m^2 と 3 つのガンマ行列を含む項は

$$\begin{aligned}
 &m^2 [\not{\epsilon}_2^* \not{k}_1 \not{\epsilon}_1^* + \not{\epsilon}_1^* \not{k}_2 \not{\epsilon}_2^*] \\
 &= m^2 \begin{pmatrix} 0 & (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma) \\ (\epsilon_2^* \cdot \bar{\sigma})(k_1 \cdot \sigma)(\epsilon_1^* \cdot \bar{\sigma}) & 0 \end{pmatrix} \\
 &\quad + m^2 \begin{pmatrix} 0 & (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) \\ (\epsilon_1^* \cdot \bar{\sigma})(k_2 \cdot \sigma)(\epsilon_2^* \cdot \bar{\sigma}) & 0 \end{pmatrix} \\
 &= m^2 \begin{pmatrix} 0 & (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma) \\ (\epsilon_2^* \cdot \sigma)(k_1 \cdot \sigma)(\epsilon_1^* \cdot \sigma) & 0 \end{pmatrix} \\
 &\quad + m^2 \begin{pmatrix} 0 & (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) \\ (\epsilon_1^* \cdot \sigma)(k_2 \cdot \sigma)(\epsilon_2^* \cdot \sigma) & 0 \end{pmatrix} \\
 &=: \begin{pmatrix} 0 & m^2 \bar{B}_+ \\ m^2 B_+ & 0 \end{pmatrix}.
 \end{aligned}$$

[5.5.15] の $[\dots]$ のうち $\mathbf{p} \cdot \mathbf{k}$ と 3 つのガンマ行列を含む項は

$$\mathbf{p} \cdot \mathbf{k} [\not{\epsilon}_2^* \not{k}_1 \not{\epsilon}_1^* - \not{\epsilon}_1^* \not{k}_2 \not{\epsilon}_2^*]$$

$$\begin{aligned}
 &= \mathbf{p} \cdot \mathbf{k} \begin{pmatrix} 0 & (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma) \\ (\epsilon_2^* \cdot \bar{\sigma})(k_1 \cdot \sigma)(\epsilon_1^* \cdot \bar{\sigma}) & 0 \end{pmatrix} \\
 &- \mathbf{p} \cdot \mathbf{k} \begin{pmatrix} 0 & (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) \\ (\epsilon_1^* \cdot \bar{\sigma})(k_2 \cdot \sigma)(\epsilon_2^* \cdot \bar{\sigma}) & 0 \end{pmatrix} \\
 &= \mathbf{p} \cdot \mathbf{k} \begin{pmatrix} 0 & (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma) \\ (\epsilon_2^* \cdot \sigma)(k_1 \cdot \sigma)(\epsilon_1^* \cdot \sigma) & 0 \end{pmatrix} \\
 &- \mathbf{p} \cdot \mathbf{k} \begin{pmatrix} 0 & (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) \\ (\epsilon_1^* \cdot \sigma)(k_2 \cdot \sigma)(\epsilon_2^* \cdot \sigma) & 0 \end{pmatrix} \\
 &=: \begin{pmatrix} 0 & -(\mathbf{p} \cdot \mathbf{k})\bar{B}_- \\ -(\mathbf{p} \cdot \mathbf{k})B_- & 0 \end{pmatrix}.
 \end{aligned}$$

[5.5.15] の残りの項は運動量の 2 乗なので無視する：

$$\begin{aligned}
 &-2(\mathbf{p} \cdot \mathbf{k})(p_1 \cdot \epsilon_1^*)\not{\epsilon}_2^* + 2(\mathbf{p} \cdot \mathbf{k})(p_1 \cdot \epsilon_2^*)\not{\epsilon}_1^* \\
 &= -2(\mathbf{p} \cdot \mathbf{k})(p_1 \cdot \epsilon_1^*) \begin{pmatrix} 0 & \epsilon_2^* \cdot \sigma \\ \epsilon_2^* \cdot \bar{\sigma} & 0 \end{pmatrix} + 2(\mathbf{p} \cdot \mathbf{k})(p_1 \cdot \epsilon_2^*) \begin{pmatrix} 0 & \epsilon_1^* \cdot \sigma \\ \epsilon_1^* \cdot \bar{\sigma} & 0 \end{pmatrix} \\
 &= -2(\mathbf{p} \cdot \mathbf{k})(\mathbf{p} \cdot \epsilon_1^*) \begin{pmatrix} 0 & \epsilon_2^* \cdot \sigma \\ -\epsilon_2^* \cdot \sigma & 0 \end{pmatrix} + 2(\mathbf{p} \cdot \mathbf{k})(\mathbf{p} \cdot \epsilon_2^*) \begin{pmatrix} 0 & \epsilon_1^* \cdot \sigma \\ -\epsilon_1^* \cdot \sigma & 0 \end{pmatrix} \\
 &= -2(\mathbf{p} \cdot \mathbf{k}) \begin{pmatrix} 0 & (\mathbf{p} \cdot \epsilon_1^*)(\epsilon_2^* \cdot \sigma) - (\mathbf{p} \cdot \epsilon_2^*)(\epsilon_1^* \cdot \sigma) \\ -(\mathbf{p} \cdot \epsilon_1^*)(\epsilon_2^* \cdot \sigma) + (\mathbf{p} \cdot \epsilon_2^*)(\epsilon_1^* \cdot \sigma) & 0 \end{pmatrix} \\
 &\approx 0.
 \end{aligned}$$

以上から

$$i\mathcal{M} = -i\frac{e^2}{2m^4}\bar{v}(p_2) \begin{pmatrix} 0 & -2m^2A + m^2\bar{B}_+ - (\mathbf{p} \cdot \mathbf{k})\bar{B}_- \\ 2m^2A + m^2B_+ - (\mathbf{p} \cdot \mathbf{k})B_- & 0 \end{pmatrix} u(p_1) \quad [5.5.16]$$

となる。各項の定義は

$$\begin{aligned}
 A &= (\mathbf{p} \cdot \epsilon_1^*)(\epsilon_2^* \cdot \sigma) + (\mathbf{p} \cdot \epsilon_2^*)(\epsilon_1^* \cdot \sigma) \\
 B_{\pm} &= (\epsilon_1^* \cdot \sigma)(k_2 \cdot \sigma)(\epsilon_2^* \cdot \sigma) \pm (\epsilon_2^* \cdot \sigma)(k_1 \cdot \sigma)(\epsilon_1^* \cdot \sigma) \\
 \bar{B}_{\pm} &= (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) \pm (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma).
 \end{aligned} \quad [5.5.17]$$

$(\sigma \cdot \mathbf{a})(\sigma \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\sigma \cdot (\mathbf{a} \times \mathbf{b})$ から

$$\begin{aligned}
 (\epsilon_1^* \cdot \sigma)(\mathbf{k} \cdot \sigma)(\epsilon_2^* \cdot \sigma) &= [\epsilon_1^* \cdot \mathbf{k} + i\sigma \cdot (\epsilon_1^* \times \mathbf{k})](\epsilon_2^* \cdot \sigma) \\
 &= i\sigma \cdot (\epsilon_1^* \times \mathbf{k})(\epsilon_2^* \cdot \sigma) \\
 &= i(\epsilon_1^* \times \mathbf{k}) \cdot \epsilon_2^* - \sigma \cdot [(\epsilon_1^* \times \mathbf{k}) \times \epsilon_2^*] \\
 &= i(\epsilon_2^* \times \epsilon_1^*) \cdot \mathbf{k} - \sigma \cdot [(\epsilon_1^* \cdot \epsilon_2^*)\mathbf{k} - (\mathbf{k} \cdot \epsilon_2^*)\epsilon_1^*] \\
 &= -i(\epsilon_1^* \times \epsilon_2^*) \cdot \mathbf{k} - (\sigma \cdot \mathbf{k})(\epsilon_1^* \cdot \epsilon_2^*) \\
 (\epsilon_2^* \cdot \sigma)(\mathbf{k} \cdot \sigma)(\epsilon_1^* \cdot \sigma) &= i(\epsilon_1^* \times \epsilon_2^*) \cdot \mathbf{k} - (\sigma \cdot \mathbf{k})(\epsilon_1^* \cdot \epsilon_2^*) \\
 (\epsilon_1^* \cdot \sigma)(\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma)(\epsilon_1^* \cdot \sigma) &= 2\epsilon_1^* \cdot \epsilon_2^*
 \end{aligned}$$

となる。従って、

$$\begin{aligned}
 \bar{B}_+ - B_+ &= (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma) \\
 &\quad - (\epsilon_1^* \cdot \sigma)(k_2 \cdot \sigma)(\epsilon_2^* \cdot \sigma) - (\epsilon_2^* \cdot \sigma)(k_1 \cdot \sigma)(\epsilon_1^* \cdot \sigma) \\
 &= (\epsilon_1^* \cdot \sigma)(k_2 \cdot (\bar{\sigma} - \sigma))(\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma)(k_1 \cdot (\bar{\sigma} - \sigma))(\epsilon_1^* \cdot \sigma) \\
 &= -2(\epsilon_1^* \cdot \sigma)(\mathbf{k} \cdot \sigma)(\epsilon_2^* \cdot \sigma) + 2(\epsilon_2^* \cdot \sigma)(\mathbf{k} \cdot \sigma)(\epsilon_1^* \cdot \sigma) \\
 &= 4i(\epsilon_1^* \times \epsilon_2^*) \cdot \mathbf{k}, \\
 B_- - \bar{B}_- &= (\epsilon_1^* \cdot \sigma)(k_2 \cdot \sigma)(\epsilon_2^* \cdot \sigma) - (\epsilon_2^* \cdot \sigma)(k_1 \cdot \sigma)(\epsilon_1^* \cdot \sigma) \\
 &\quad - (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma) \\
 &= (\epsilon_1^* \cdot \sigma)(k_2 \cdot (\sigma - \bar{\sigma}))(\epsilon_2^* \cdot \sigma) - (\epsilon_2^* \cdot \sigma)(k_1 \cdot (\sigma - \bar{\sigma}))(\epsilon_1^* \cdot \sigma) \\
 &= 2(\epsilon_1^* \cdot \sigma)(\mathbf{k} \cdot \sigma)(\epsilon_2^* \cdot \sigma) + 2(\epsilon_2^* \cdot \sigma)(\mathbf{k} \cdot \sigma)(\epsilon_1^* \cdot \sigma) \\
 &= -4(\sigma \cdot \mathbf{k})(\epsilon_1^* \cdot \epsilon_2^*), \\
 B_+ + \bar{B}_+ &= (\epsilon_1^* \cdot \sigma)(k_2 \cdot \sigma)(\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma)(k_1 \cdot \sigma)(\epsilon_1^* \cdot \sigma) \\
 &\quad + (\epsilon_1^* \cdot \sigma)(k_2 \cdot \bar{\sigma})(\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma)(k_1 \cdot \bar{\sigma})(\epsilon_1^* \cdot \sigma) \\
 &= 2E(\epsilon_1^* \cdot \sigma)(\epsilon_2^* \cdot \sigma) + 2E(\epsilon_2^* \cdot \sigma)(\epsilon_1^* \cdot \sigma) \\
 &\approx 2m(\epsilon_1^* \cdot \sigma)(\epsilon_2^* \cdot \sigma) + 2m(\epsilon_2^* \cdot \sigma)(\epsilon_1^* \cdot \sigma) \\
 &= 4m\epsilon_1^* \cdot \epsilon_2^*.
 \end{aligned} \tag{5.5.18}$$

[5.5.16] に [5.5.14] を代入して、[5.5.18] を使えば、

$$\begin{aligned}
 &i\mathcal{M}(e_s^- e_r^+ \rightarrow 2\gamma) \\
 &= -i\frac{e^2}{2m^4} \bar{v}(p_2) \begin{pmatrix} 0 & -2m^2 A + m^2 \bar{B}_+ - (\mathbf{p} \cdot \mathbf{k}) \bar{B}_- \\ 2m^2 A + m^2 B_+ - (\mathbf{p} \cdot \mathbf{k}) B_- & 0 \end{pmatrix} u(p_1) \\
 &= -i\frac{e^2}{2m^3} \left(\eta^{r\dagger} \left(1 + \frac{\sigma \cdot \mathbf{p}}{2m} \right) \quad -\eta^{r\dagger} \left(1 - \frac{\sigma \cdot \mathbf{p}}{2m} \right) \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
 &\quad \times \begin{pmatrix} 0 & -2m^2 A + m^2 \bar{B}_+ - (\mathbf{p} \cdot \mathbf{k}) \bar{B}_- \\ 2m^2 A + m^2 B_+ - (\mathbf{p} \cdot \mathbf{k}) B_- & 0 \end{pmatrix} \begin{pmatrix} \left(1 - \frac{\sigma \cdot \mathbf{p}}{2m} \right) \xi^s \\ \left(1 + \frac{\sigma \cdot \mathbf{p}}{2m} \right) \xi^s \end{pmatrix} \\
 &= i\frac{e^2}{2m^3} \eta^{r\dagger} \left(1 - \frac{\sigma \cdot \mathbf{p}}{2m} \right) [-2m^2 A + m^2 \bar{B}_+ - (\mathbf{p} \cdot \mathbf{k}) \bar{B}_-] \left(1 + \frac{\sigma \cdot \mathbf{p}}{2m} \right) \xi^s \\
 &\quad - i\frac{e^2}{2m^3} \eta^{r\dagger} \left(1 + \frac{\sigma \cdot \mathbf{p}}{2m} \right) [2m^2 A + m^2 B_+ - (\mathbf{p} \cdot \mathbf{k}) B_-] \left(1 - \frac{\sigma \cdot \mathbf{p}}{2m} \right) \xi^s \\
 &\approx i\frac{e^2}{2m^3} \eta^{r\dagger} \left\{ -2m^2 A + m^2 \bar{B}_+ - (\mathbf{p} \cdot \mathbf{k}) \bar{B}_- + \frac{m}{2} [\bar{B}_+, \sigma \cdot \mathbf{p}] \right\} \xi^s \\
 &\quad - i\frac{e^2}{2m^3} \eta^{r\dagger} \left\{ 2m^2 A + m^2 B_+ - (\mathbf{p} \cdot \mathbf{k}) B_- - \frac{m}{2} [B_+, \sigma \cdot \mathbf{p}] \right\} \xi^s \\
 &= i\frac{e^2}{2m^3} \eta^{r\dagger} \left\{ -4m^2 A + m^2 (\bar{B}_+ - B_+) + (\mathbf{p} \cdot \mathbf{k}) (B_- - \bar{B}_-) + \frac{m}{2} [B_+ + \bar{B}_+, \sigma \cdot \mathbf{p}] \right\} \xi^s \\
 &= -i\frac{2e^2}{m} \eta^{r\dagger} \left[(\mathbf{p} \cdot \epsilon_1^*)(\epsilon_2^* \cdot \sigma) + (\mathbf{p} \cdot \epsilon_2^*)(\epsilon_1^* \cdot \sigma) - i(\epsilon_1^* \times \epsilon_2^*) \cdot \mathbf{k} + \frac{1}{m^2} (\mathbf{p} \cdot \mathbf{k})(\sigma \cdot \mathbf{k})(\epsilon_1^* \cdot \epsilon_2^*) \right] \xi^s
 \end{aligned}$$

となる。

ここで、

$$i\mathcal{M}^{(1)j} = -i\frac{2e^2}{m} \eta^{r\dagger} \left[\epsilon_1^{j*} (\epsilon_2^* \cdot \sigma) + \epsilon_2^{j*} (\epsilon_1^* \cdot \sigma) + \frac{k^j}{m^2} (\sigma \cdot \mathbf{k})(\epsilon_1^* \cdot \epsilon_2^*) \right] \xi^s \tag{5.5.19}$$

とおけば,

$$i\mathcal{M}(e_s^- e_r^+ \rightarrow 2\gamma) = i\mathcal{M}^{(0)} + \sum_j p^j i\mathcal{M}^{(1)j} \quad [5.5.20]$$

とかける.

3P_0 ポジトロニウムの構成

$|^3P_0; 0\rangle$ は [5.5.8] で $S = 1, l = 1, J = 0, M = 0$ なので

$$\begin{aligned} |^3P_0; 0\rangle &= \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \sum_{m=-1}^1 \langle 1, 1; m, -m | 1, 1; 00 \rangle \tilde{\psi}_{1,m} \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |1, -m\rangle \\ &= \frac{1}{\sqrt{3m}} \int \frac{d^3p}{(2\pi)^3} \left[\tilde{\psi}_{11} |\downarrow\downarrow\rangle - \tilde{\psi}_{10} \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} + \tilde{\psi}_{1,-1} |\uparrow\uparrow\rangle \right] \end{aligned}$$

となる. ここで

$$\tilde{\psi}^1 = \frac{\tilde{\psi}_{1,-1} - \tilde{\psi}_{1,1}}{\sqrt{2}}, \quad \tilde{\psi}^2 = i \frac{\tilde{\psi}_{1,-1} + \tilde{\psi}_{1,1}}{\sqrt{2}}, \quad \tilde{\psi}^3 = \tilde{\psi}_{1,0} \quad [5.5.21]$$

とおけば,

$$|^3P_0; 0\rangle = \frac{1}{\sqrt{6m}} \int \frac{d^3p}{(2\pi)^3} \left[\tilde{\psi}^1 (|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) - i\tilde{\psi}^2 (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) - \tilde{\psi}^3 (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \right]$$

となる.

2^3P_0 ポジトロニウムの崩壊

$n = 2$ の場合を考える.

$$\begin{aligned} \psi^1(\mathbf{x}) &= \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0^{5/2}} x \exp\left(-\frac{r}{2a_0}\right) \\ \psi^2(\mathbf{x}) &= \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0^{5/2}} y \exp\left(-\frac{r}{2a_0}\right) \\ \psi^3(\mathbf{x}) &= \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0^{5/2}} z \exp\left(-\frac{r}{2a_0}\right) \end{aligned}$$

なので,

$$\frac{\partial \psi^i}{\partial x^j}(0) = \frac{\delta^{ij}}{4\sqrt{2\pi} a_0^{5/2}}.$$

ポジトロニウムの崩壊の不変振幅は

$$\begin{aligned} i\mathcal{M}(2^3P_0 \rightarrow 2\gamma) &= \frac{1}{\sqrt{6m}} \int \frac{d^3p}{(2\pi)^3} \left[\tilde{\psi}^1(\mathbf{p}) \left\{ i\mathcal{M}(e_{\uparrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) - i\mathcal{M}(e_{\downarrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) \right\} \right. \\ &\quad - i\tilde{\psi}^2(\mathbf{p}) \left\{ i\mathcal{M}(e_{\uparrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) + i\mathcal{M}(e_{\downarrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) \right\} \\ &\quad \left. - \tilde{\psi}^3(\mathbf{p}) \left\{ i\mathcal{M}(e_{\uparrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) + i\mathcal{M}(e_{\downarrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) \right\} \right] \end{aligned}$$

で与えられる. [5.5.20] のように, 不変振幅は $i\mathcal{M} = i\mathcal{M}^{(0)} + \sum_j ip^j \mathcal{M}^{(1)j}$ と表すことができたが,

$$\int \frac{d^3p}{(2\pi)^3} \tilde{\psi}^i i\mathcal{M}^{(0)} = \psi^i(0) i\mathcal{M}^{(0)} = 0$$

である。さらに,

$$\int \frac{d^3p}{(2\pi)^3} \tilde{\psi}^i \sum_j i p^j \mathcal{M}^{(1)j} = i \frac{1}{4\sqrt{2\pi}a_0^{5/2}} i \mathcal{M}^{(1)i}$$

なので

$$\int \frac{d^3p}{(2\pi)^3} \tilde{\psi}^i i \mathcal{M} = i \frac{1}{4\sqrt{2\pi}a_0^{5/2}} i \mathcal{M}^{(1)i}$$

となる。従って

$$\begin{aligned} i \mathcal{M}(2^3P_0 \rightarrow 2\gamma) &= \frac{1}{\sqrt{6m}} \frac{i}{4\sqrt{2\pi}a_0^{5/2}} \left[\left\{ i \mathcal{M}^{(1)1}(e_{\uparrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) - i \mathcal{M}^{(1)1}(e_{\downarrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) \right\} \right. \\ &\quad \left. - i \left\{ i \mathcal{M}^{(1)2}(e_{\uparrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) + i \mathcal{M}^{(1)2}(e_{\downarrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) \right\} \right. \\ &\quad \left. - \left\{ i \mathcal{M}^{(1)3}(e_{\uparrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) + i \mathcal{M}^{(1)3}(e_{\downarrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) \right\} \right] \end{aligned}$$

となる。[5.5.7] から

$$\xi^{\uparrow} \eta^{\uparrow\uparrow} - \xi^{\downarrow} \eta^{\downarrow\uparrow} = \sigma^1, \quad -i(\xi^{\uparrow} \eta^{\uparrow\uparrow} + \xi^{\downarrow} \eta^{\downarrow\uparrow}) = \sigma^2, \quad -(\xi^{\uparrow} \eta^{\downarrow\uparrow} + \xi^{\downarrow} \eta^{\uparrow\uparrow}) = \sigma^3$$

である。[5.5.19] は

$$\begin{aligned} i \mathcal{M}^{(1)j} &= -i \frac{2e^2}{m} \eta^{r\dagger} \left[\epsilon_1^{j*} (\epsilon_2^* \cdot \boldsymbol{\sigma}) + \epsilon_2^{j*} (\epsilon_1^* \cdot \boldsymbol{\sigma}) + \frac{k^j}{m^2} (\boldsymbol{\sigma} \cdot \mathbf{k}) (\epsilon_1^* \cdot \epsilon_2^*) \right] \xi^s \\ &= -i \frac{2e^2}{m} \text{Tr} \left[\xi^s \eta^{r\dagger} \left\{ \epsilon_1^{j*} (\epsilon_2^* \cdot \boldsymbol{\sigma}) + \epsilon_2^{j*} (\epsilon_1^* \cdot \boldsymbol{\sigma}) + \frac{k^j}{m^2} (\boldsymbol{\sigma} \cdot \mathbf{k}) (\epsilon_1^* \cdot \epsilon_2^*) \right\} \right] \end{aligned}$$

と表せるので,

$$\begin{aligned} i \mathcal{M}(2^3P_0 \rightarrow 2\gamma) &= \frac{1}{\sqrt{6m}} \frac{1}{4\sqrt{2\pi}a_0^{5/2}} \frac{2e^2}{m} \\ &\quad \times \text{Tr} \left[(\epsilon_1^* \cdot \boldsymbol{\sigma}) (\epsilon_2^* \cdot \boldsymbol{\sigma}) + (\epsilon_2^* \cdot \boldsymbol{\sigma}) (\epsilon_1^* \cdot \boldsymbol{\sigma}) + \frac{1}{m^2} (\boldsymbol{\sigma} \cdot \mathbf{k}) (\boldsymbol{\sigma} \cdot \mathbf{k}) (\epsilon_1^* \cdot \epsilon_2^*) \right]. \end{aligned}$$

$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i \boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b})$ から

$$\begin{aligned} i \mathcal{M}(2^3P_0 \rightarrow 2\gamma) &= \frac{e^2}{4\sqrt{3\pi m} a_0^{5/2}} \text{Tr} \left[2(\epsilon_1^* \cdot \epsilon_2^*) + \frac{|\mathbf{k}|^2}{m^2} (\epsilon_1^* \cdot \epsilon_2^*) \right] \\ &= \frac{e^2}{2\sqrt{3\pi m} a_0^{5/2}} \frac{2m^2 + |\mathbf{k}|^2}{m^2} (\epsilon_1^* \cdot \epsilon_2^*). \end{aligned}$$

光子の偏極ベクトルの完全性

$$\sum_{\text{polarization}} \epsilon^{i*}(k) \epsilon^j(k) = \delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}$$

から

$$\sum_{\text{polarization}} |\epsilon_1^* \cdot \epsilon_2^*|^2 = \sum_{\text{pol } 1} \sum_{\text{pol } 2} \sum_{i=1}^3 \sum_{j=1}^3 \epsilon_1^{i*} \epsilon_1^j \epsilon_2^{i*} \epsilon_2^j = \sum_{i=1}^3 \sum_{j=1}^3 \left(\delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2} \right) \left(\delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2} \right) = 2$$

なので

$$\sum_{\text{polarization}} |\mathcal{M}(2^3P_0 \rightarrow 2\gamma)|^2 = \frac{e^4}{12\pi m^3 a_0^5} \left(\frac{2m^2 + |\mathbf{k}|^2}{m^2} \right)^2 \sum_{\text{polarization}} |\epsilon_1^* \cdot \epsilon_2^*|^2$$

$$\begin{aligned}
 &= \frac{e^4}{6\pi m^3 a_0^5} \left(\frac{2m^2 + |\mathbf{k}|^2}{m^2} \right)^2 \\
 &= \frac{\pi}{12} m^2 \alpha^7 \left(\frac{2m^2 + |\mathbf{k}|^2}{m^2} \right)^2.
 \end{aligned}$$

(4.86) から

$$\begin{aligned}
 \Gamma(2^3P_0 \rightarrow 2\gamma) &= \frac{1}{2} \frac{1}{4m} \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{1}{4E_1 E_2} |\mathcal{M}(2^3P_0 \rightarrow 2\gamma)|^2 (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \\
 &= \frac{1}{8m} \int \frac{d^3k_1}{(2\pi)^2} \frac{1}{4|\mathbf{k}|^2} |\mathcal{M}(2^3P_2 \rightarrow 2\gamma)|^2 \delta(2E_1 - 2m) \\
 &= \frac{1}{256\pi^2 m} \int d|\mathbf{k}| \delta(|\mathbf{k}| - m) \int d\Omega |\mathcal{M}(2^3P_2 \rightarrow 2\gamma)|^2 \\
 &= \frac{1}{256\pi^2 m} \int d|\mathbf{k}| \delta(|\mathbf{k}| - m) \times 4\pi \frac{\pi}{12} m^2 \alpha^7 \left(\frac{2m^2 + |\mathbf{k}|^2}{m^2} \right)^2 \\
 &= \frac{3}{256} m \alpha^7.
 \end{aligned}$$

2^3P_2 ポジトロニウムの崩壊

$n = 2$ の場合を考える.

$|2^3P_2; 0\rangle$ は $S = 1, l = 1, J = 2, M = 0$ なので

$$\begin{aligned}
 |2^3P_2; 0\rangle &= \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \sum_{m=-1}^1 \langle 1, 1; m, -m | 1, 1; 20 \rangle \tilde{\psi}_{1,m} \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |1, -m\rangle \\
 &= \frac{1}{\sqrt{6m}} \int \frac{d^3p}{(2\pi)^3} \left[\tilde{\psi}_{11} |\downarrow\downarrow\rangle + 2\tilde{\psi}_{10} \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} + \tilde{\psi}_{1,-1} |\uparrow\uparrow\rangle \right] \\
 &= \frac{1}{2\sqrt{3m}} \int \frac{d^3p}{(2\pi)^3} \left[\tilde{\psi}^1(|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle) - i\tilde{\psi}^2(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle) + 2\tilde{\psi}^3(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \right].
 \end{aligned}$$

$\boldsymbol{\sigma}' = (\sigma^1, \sigma^2, -2\sigma^3)$ とすれば

$$\text{Tr}[(\boldsymbol{\sigma}' \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b})] = 2a^1 b^1 + 2a^2 b^2 - 4a^3 b^3.$$

$M = 0$ の不変振幅は

$$\begin{aligned}
 i\mathcal{M}(2^3P_2; 0 \rightarrow 2\gamma) &= \frac{1}{2\sqrt{3m}} \frac{i}{4\sqrt{2\pi} a_0^{5/2}} \left[\left\{ i\mathcal{M}^{(1)1}(e_{\uparrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) - i\mathcal{M}^{(1)1}(e_{\downarrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) \right\} \right. \\
 &\quad \left. - i \left\{ i\mathcal{M}^{(1)2}(e_{\uparrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) + i\mathcal{M}^{(1)2}(e_{\downarrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) \right\} \right. \\
 &\quad \left. + 2 \left\{ i\mathcal{M}^{(1)3}(e_{\uparrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) + i\mathcal{M}^{(1)3}(e_{\downarrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) \right\} \right] \\
 &= \frac{1}{2\sqrt{3m}} \frac{1}{4\sqrt{2\pi} a_0^{5/2}} \frac{2e^2}{m} \\
 &\quad \times \text{Tr} \left[(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}') (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}') (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) + \frac{1}{m^2} (\boldsymbol{\sigma}' \cdot \mathbf{k}) (\boldsymbol{\sigma} \cdot \mathbf{k}) (\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*) \right] \\
 &= \frac{e^2}{2\sqrt{\pi m m a_0^{5/2}}} \left[2h_0^{ij} \epsilon_1^{i*} \epsilon_2^{j*} + \frac{(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*)}{m^2} h_0^{ij} k^i k^j \right], \quad h_0^{ij} = \frac{1}{\sqrt{6}} \text{diag}(1, 1, -2).
 \end{aligned}$$

$|2^3P_2; 1\rangle$ は $S = 1, l = 1, J = 2, M = 1$ なので

$$\begin{aligned} |2^3P_2; 1\rangle &= \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \sum_{m=0}^1 \langle 1, 1; m, 1-m | 1, 1; 21 \rangle \tilde{\psi}_{1,m} \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |1, 1-m\rangle \\ &= \frac{1}{\sqrt{2m}} \int \frac{d^3p}{(2\pi)^3} \left[\tilde{\psi}_{10} |\uparrow\uparrow\rangle + \tilde{\psi}_{11} \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \right] \\ &= \frac{1}{2\sqrt{2m}} \int \frac{d^3p}{(2\pi)^3} \left[-\tilde{\psi}^1(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) - i\tilde{\psi}^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) + 2\tilde{\psi}^3|\uparrow\uparrow\rangle \right]. \end{aligned}$$

$\boldsymbol{\sigma}' = (\sigma^3, i\sigma^3, 2\sigma^+)$ とすれば

$$\text{Tr}[(\boldsymbol{\sigma}' \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b})] = 2a^1b^3 + 2ia^2b^3 + 2a^3b^1 + 2ia^3b^2.$$

$M = 1$ の不変振幅は

$$\begin{aligned} i\mathcal{M}(2^3P_2; 1 \rightarrow 2\gamma) &= \frac{1}{2\sqrt{2m}} \frac{i}{4\sqrt{2\pi a_0^{5/2}}} \left[-\left\{ i\mathcal{M}^{(1)1}(e_{\uparrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) + i\mathcal{M}^{(1)1}(e_{\downarrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) \right\} \right. \\ &\quad \left. - i\left\{ i\mathcal{M}^{(1)2}(e_{\uparrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) + i\mathcal{M}^{(1)2}(e_{\downarrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) \right\} \right. \\ &\quad \left. + 2\left\{ i\mathcal{M}^{(1)3}(e_{\uparrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) \right\} \right] \\ &= \frac{1}{2\sqrt{2m}} \frac{1}{4\sqrt{2\pi a_0^{5/2}}} \frac{2e^2}{m} \\ &\quad \times \text{Tr} \left[(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}')(\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}_2^* \cdot \boldsymbol{\sigma}')(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\sigma}) + \frac{1}{m^2}(\boldsymbol{\sigma}' \cdot \mathbf{k})(\boldsymbol{\sigma} \cdot \mathbf{k})(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*) \right] \\ &= \frac{e^2}{2\sqrt{\pi m m a_0^{5/2}}} \left[2h_1^{ij} \epsilon_1^{i*} \epsilon_2^{j*} + \frac{(\boldsymbol{\epsilon}_1^* \cdot \boldsymbol{\epsilon}_2^*)}{m^2} h_1^{ij} k^i k^j \right], \quad h_1^{ij} = \frac{1}{2} \begin{pmatrix} & & 1 \\ & i & \\ 1 & & \end{pmatrix}. \end{aligned}$$

$|2^3P_2; -1\rangle$ は $S = 1, l = 1, J = 2, M = -1$ なので

$$\begin{aligned} |2^3P_2; -1\rangle &= \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \sum_{m=-1}^0 \langle 1, 1; m, -1-m | 1, 1; 2, -1 \rangle \tilde{\psi}_{1,m} \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |1, -1-m\rangle \\ &= \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \sum_{m=-1}^0 \langle 1, 1; -m, 1+m | 1, 1; 2, 1 \rangle \tilde{\psi}_{1,m} \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |1, -1-m\rangle \\ &= \frac{1}{\sqrt{2m}} \int \frac{d^3p}{(2\pi)^3} \left[\tilde{\psi}_{1,-1} \frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} + \tilde{\psi}_{10} |\downarrow\downarrow\rangle \right] \\ &= \frac{1}{2\sqrt{2m}} \int \frac{d^3p}{(2\pi)^3} \left[\tilde{\psi}^1(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) - i\tilde{\psi}^2(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) + 2\tilde{\psi}^3|\downarrow\downarrow\rangle \right]. \end{aligned}$$

$\boldsymbol{\sigma}' = (-\sigma^3, i\sigma^3, -2\sigma^-)$ とすれば

$$\text{Tr}[(\boldsymbol{\sigma}' \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b})] = -2a^1b^3 + 2ia^2b^3 - 2a^3b^1 + 2ia^3b^2.$$

$M = -1$ の不変振幅は

$$\begin{aligned} i\mathcal{M}(2^3P_2; 1 \rightarrow 2\gamma) &= \frac{1}{2\sqrt{2m}} \frac{i}{4\sqrt{2\pi a_0^{5/2}}} \left[\left\{ i\mathcal{M}^{(1)1}(e_{\uparrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) + i\mathcal{M}^{(1)1}(e_{\downarrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) \right\} \right. \\ &\quad \left. - i\left\{ i\mathcal{M}^{(1)2}(e_{\uparrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) + i\mathcal{M}^{(1)2}(e_{\downarrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) \right\} \right] \end{aligned}$$

$$\begin{aligned}
 & + 2 \left\{ i\mathcal{M}^{(1)3}(e_{\downarrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) \right\} \Big] \\
 &= \frac{1}{2\sqrt{2m}} \frac{1}{4\sqrt{2\pi}a_0^{5/2}} \frac{2e^2}{m} \\
 & \quad \times \text{Tr} \left[(\epsilon_1^* \cdot \sigma')(\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma')(\epsilon_1^* \cdot \sigma) + \frac{1}{m^2}(\sigma' \cdot \mathbf{k})(\sigma \cdot \mathbf{k})(\epsilon_1^* \cdot \epsilon_2^*) \right] \\
 &= \frac{e^2}{2\sqrt{\pi m m a_0^{5/2}}} \left[2h_{-1}^{ij} \epsilon_1^{i*} \epsilon_2^{j*} + \frac{(\epsilon_1^* \cdot \epsilon_2^*)}{m^2} h_{-1}^{ij} k^i k^j \right], \quad h_{-1}^{ij} = \frac{1}{2} \begin{pmatrix} & & -1 \\ -1 & i & \end{pmatrix}.
 \end{aligned}$$

$|2^3P_2; 2\rangle$ は $S=1, l=1, J=2, M=2$ なので ($m=S_z=1$)

$$\begin{aligned}
 |2^3P_2; 2\rangle &= \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \langle 1, 1; 1, 1 | 1, 1; 22 \rangle \tilde{\psi}_{1,1} \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |1, 1\rangle \\
 &= \frac{1}{\sqrt{m}} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{11} |\uparrow\uparrow\rangle \\
 &= \frac{1}{\sqrt{2m}} \int \frac{d^3p}{(2\pi)^3} \left[-\tilde{\psi}^1 |\uparrow\uparrow\rangle - i\tilde{\psi}^2 |\uparrow\uparrow\rangle \right].
 \end{aligned}$$

$\sigma' = (-\sigma^+, -i\sigma^+, 0)$ とすれば

$$\text{Tr}[(\sigma' \cdot \mathbf{a})(\sigma \cdot \mathbf{b})] = -a^1 b^2 - ia^1 b^2 - ia^2 b^1 + a^2 b^2.$$

$M=2$ の不変振幅は

$$\begin{aligned}
 i\mathcal{M}(2^3P_2; 2 \rightarrow 2\gamma) &= \frac{1}{\sqrt{2m}} \frac{i}{4\sqrt{2\pi}a_0^{5/2}} \left[-\left\{ i\mathcal{M}^{(1)1}(e_{\uparrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) \right\} - i \left\{ i\mathcal{M}^{(1)2}(e_{\uparrow}^- e_{\uparrow}^+ \rightarrow 2\gamma) \right\} \right] \\
 &= \frac{1}{\sqrt{2m}} \frac{1}{4\sqrt{2\pi}a_0^{5/2}} \frac{2e^2}{m} \\
 & \quad \times \text{Tr} \left[(\epsilon_1^* \cdot \sigma')(\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma')(\epsilon_1^* \cdot \sigma) + \frac{1}{m^2}(\sigma' \cdot \mathbf{k})(\sigma \cdot \mathbf{k})(\epsilon_1^* \cdot \epsilon_2^*) \right] \\
 &= \frac{e^2}{2\sqrt{\pi m m a_0^{5/2}}} \left[2h_2^{ij} \epsilon_1^{i*} \epsilon_2^{j*} + \frac{(\epsilon_1^* \cdot \epsilon_2^*)}{m^2} h_2^{ij} k^i k^j \right], \quad h_2^{ij} = \frac{1}{2} \begin{pmatrix} -1 & -i \\ -i & 1 \end{pmatrix}.
 \end{aligned}$$

$|2^3P_2; -2\rangle$ は $S=1, l=1, J=2, M=-2$ なので ($m=S_z=-1$)

$$\begin{aligned}
 |2^3P_2; -2\rangle &= \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \langle 1, 1; -1, -1 | 1, 1; 2, -2 \rangle \tilde{\psi}_{1,-1} \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |1, -1\rangle \\
 &= \sqrt{2M} \int \frac{d^3p}{(2\pi)^3} \langle 1, 1; 1, 1 | 1, 1; 22 \rangle \tilde{\psi}_{1,-1} \frac{1}{\sqrt{2m}} \frac{1}{\sqrt{2m}} |1, -1\rangle \\
 &= \frac{1}{\sqrt{m}} \int \frac{d^3p}{(2\pi)^3} \tilde{\psi}_{1,-1} |\downarrow\downarrow\rangle \\
 &= \frac{1}{\sqrt{2m}} \int \frac{d^3p}{(2\pi)^3} \left[\tilde{\psi}^1 |\downarrow\downarrow\rangle - i\tilde{\psi}^2 |\downarrow\downarrow\rangle \right].
 \end{aligned}$$

$\sigma' = (-\sigma^-, i\sigma^-, 0)$ とすれば

$$\text{Tr}[(\sigma' \cdot \mathbf{a})(\sigma \cdot \mathbf{b})] = -a^1 b^2 + ia^1 b^2 + ia^2 b^1 + a^2 b^2.$$

$M=-2$ の不変振幅は

$$i\mathcal{M}(2^3P_2; -2 \rightarrow 2\gamma) = \frac{1}{\sqrt{2m}} \frac{i}{4\sqrt{2\pi}a_0^{5/2}} \left[\left\{ i\mathcal{M}^{(1)1}(e_{\downarrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) \right\} - i \left\{ i\mathcal{M}^{(1)2}(e_{\downarrow}^- e_{\downarrow}^+ \rightarrow 2\gamma) \right\} \right]$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2m}} \frac{1}{4\sqrt{2\pi a_0^{5/2}}} \frac{2e^2}{m} \\
 &\quad \times \text{Tr} \left[(\epsilon_1^* \cdot \sigma') (\epsilon_2^* \cdot \sigma) + (\epsilon_2^* \cdot \sigma') (\epsilon_1^* \cdot \sigma) + \frac{1}{m^2} (\sigma' \cdot \mathbf{k}) (\sigma \cdot \mathbf{k}) (\epsilon_1^* \cdot \epsilon_2^*) \right] \\
 &= \frac{e^2}{2\sqrt{\pi m m a_0^{5/2}}} \left[2h_{-2}^{ij} \epsilon_1^{i*} \epsilon_2^{j*} + \frac{(\epsilon_1^* \cdot \epsilon_2^*)}{m^2} h_{-2}^{ij} k^i k^j \right], \quad h_{-2}^{ij} = \frac{1}{2} \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}.
 \end{aligned}$$

以上から,

$$i\mathcal{M}(2^3P_2; M \rightarrow 2\gamma) = \frac{e^2}{2\sqrt{\pi m m a_0^{5/2}}} \left[2h_M^{ij} \epsilon_1^{i*} \epsilon_2^{j*} + \frac{(\epsilon_1^* \cdot \epsilon_2^*)}{m^2} h_M^{ij} k^i k^j \right].$$

h_M はトレースが 0 の対称行列

$$h_0 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}, \quad h_{\pm 1} = \frac{1}{2} \begin{pmatrix} & \pm 1 & \\ & i & \\ \pm 1 & & \end{pmatrix}, \quad h_{\pm 2} = \frac{1}{2} \begin{pmatrix} -1 & \mp i & \\ \mp i & 1 & \end{pmatrix}$$

で与えられ,

$$\text{Tr} \left(h_M h_{M'}^\dagger \right) = \sum_{ij} h_M^{ij} h_{M'}^{ij*} = \delta_{MM'} \quad [5.5.22]$$

を満たす.

光子の偏極と全角運動量の射影について和を取って,

$$\begin{aligned}
 &\sum_{\text{polarization}} \sum_M |\mathcal{M}(2^3P_2; M \rightarrow 2\gamma)|^2 \\
 &= \frac{e^4}{4\pi m^3 a_0^5} \sum_{\text{polarization}} \sum_M \left| 2h_M^{ij} \epsilon_1^{i*} \epsilon_2^{j*} + \frac{(\epsilon_1^* \cdot \epsilon_2^*)}{m^2} h_M^{ij} k^i k^j \right|^2 \\
 &= \frac{e^4}{4\pi m^3 a_0^5} \sum_{\text{polarization}} \sum_M \left[2h_M^{ij} \epsilon_1^{i*} \epsilon_2^{j*} + \frac{(\epsilon_1^* \cdot \epsilon_2^*)}{m^2} h_M^{ij} k^i k^j \right] \left[2h_M^{kl*} \epsilon_1^k \epsilon_2^l + \frac{(\epsilon_1 \cdot \epsilon_2)}{m^2} h_M^{kl*} k^k k^l \right] \\
 &= \frac{e^4}{4\pi m^3 a_0^5} \sum_{\text{pol}} \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \\
 &\quad \times \left[4\epsilon_1^{i*} \epsilon_1^k \epsilon_2^{j*} \epsilon_2^l + \frac{2k^k k^l}{m^2} (\epsilon_1 \cdot \epsilon_2) \epsilon_1^{i*} \epsilon_2^{j*} + \frac{2k^i k^j}{m^2} (\epsilon_1^* \cdot \epsilon_2^*) \epsilon_1^k \epsilon_2^l + \frac{k^i k^j k^k k^l}{m^4} |\epsilon_1^* \cdot \epsilon_2^*|^2 \right] \\
 &= \frac{e^4}{4\pi m^3 a_0^5} \sum_{\text{pol}} \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \\
 &\quad \times \left[4\epsilon_1^{i*} \epsilon_1^k \epsilon_2^{j*} \epsilon_2^l + \frac{2k^k k^l}{m^2} \sum_m \epsilon_1^{i*} \epsilon_1^m \epsilon_2^{j*} \epsilon_2^m + \frac{2k^i k^j}{m^2} \sum_m \epsilon_1^{m*} \epsilon_1^k \epsilon_2^{m*} \epsilon_2^l + \frac{k^i k^j k^k k^l}{m^4} |\epsilon_1^* \cdot \epsilon_2^*|^2 \right].
 \end{aligned}$$

ϵ^μ の完全性から

$$|\mathcal{M}(2^3P_2 \rightarrow 2\gamma)|^2 = \frac{e^4}{4\pi m^3 a_0^5} 4 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \left(\delta^{ik} - \frac{k^i k^k}{|\mathbf{k}|^2} \right) \left(\delta^{jl} - \frac{k^j k^l}{|\mathbf{k}|^2} \right) \quad [5.5.23]$$

$$+ \frac{e^4}{4\pi m^3 a_0^5} 2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \frac{k^k k^l}{m^2} \sum_m \left(\delta^{im} - \frac{k^i k^m}{|\mathbf{k}|^2} \right) \left(\delta^{jm} - \frac{k^j k^m}{|\mathbf{k}|^2} \right) \quad [5.5.24]$$

$$+ \frac{e^4}{4\pi m^3 a_0^5} 2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \frac{k^i k^j}{m^2} \sum_m \left(\delta^{mk} - \frac{k^m k^k}{|\mathbf{k}|^2} \right) \left(\delta^{ml} - \frac{k^m k^l}{|\mathbf{k}|^2} \right) \quad [5.5.25]$$

$$+ \frac{e^4}{4\pi m^3 a_0^5} 2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \frac{k^i k^j k^k k^l}{m^4}. \quad [5.5.26]$$

(4.86) から

$$\begin{aligned} \Gamma(2^3P_2 \rightarrow 2\gamma) &= \frac{1}{2} \frac{1}{4m} \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} \frac{1}{4E_1 E_2} |\mathcal{M}(2^3P_2 \rightarrow 2\gamma)|^2 (2\pi)^4 \delta^{(4)}(k_1 + k_2 - p_1 - p_2) \\ &= \frac{1}{8m} \int \frac{d^3 k_1}{(2\pi)^2} \frac{1}{4|\mathbf{k}|^2} |\mathcal{M}(2^3P_2 \rightarrow 2\gamma)|^2 \delta(2E_1 - 2m) \\ &= \frac{1}{256\pi^2 m} \int d|\mathbf{k}| \delta(|\mathbf{k}| - m) \int d\Omega |\mathcal{M}(2^3P_2 \rightarrow 2\gamma)|^2. \end{aligned}$$

$|\mathcal{M}|^2$ の中に現れる積分を計算する。まず,

$$\int d\Omega \frac{k^i k^j}{|\mathbf{k}|^2}$$

は $i \neq j$ ならば 0 なので, δ^{ij} に比例する:

$$\int d\Omega \frac{k^i k^j}{|\mathbf{k}|^2} = A \delta^{ij}.$$

$i = j = 1, 2, 3$ について和を取れば $4\pi = 3A$ となるので,

$$\int d\Omega \frac{k^i k^j}{|\mathbf{k}|^2} = \frac{4\pi}{3} \delta^{ij}.$$

次に,

$$\int d\Omega \frac{k^i k^j k^k k^l}{|\mathbf{k}|^4} = B(\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk})$$

とおく. $i = j = 1, 2, 3$ について和を取れば

$$\int d\Omega \frac{k^k k^l}{|\mathbf{k}|^2} = \frac{4\pi}{3} \delta^{kl} = B(3\delta^{kl} + \delta^{kl} + \delta^{kl}) = 5B\delta^{kl}$$

なので

$$\int d\Omega \frac{k^i k^j k^k k^l}{|\mathbf{k}|^4} = \frac{4\pi}{15} (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}).$$

[5.5.22] に注意して [5.5.23] を積分すれば

$$\begin{aligned} & 4 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \int d|\mathbf{k}| \delta(|\mathbf{k}| - m) \int d\Omega \left(\delta^{ik} - \frac{k^i k^k}{|\mathbf{k}|^2} \right) \left(\delta^{jl} - \frac{k^j k^l}{|\mathbf{k}|^2} \right) \\ &= 4 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \left[4\pi \delta^{ik} \delta^{jl} - \frac{4\pi}{3} \delta^{ik} \delta^{jl} - \frac{4\pi}{3} \delta^{ik} \delta^{jl} + \frac{4\pi}{15} (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) \right] \\ &= 4 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \left[\frac{4\pi}{15} (\delta^{ij} \delta^{kl} + \delta^{il} \delta^{jk}) + \frac{8\pi}{5} \delta^{ik} \delta^{jl} \right] \\ &= 4 \sum_M \sum_{ij} \left[\frac{4\pi}{15} h_M^{ij} h_M^{ji*} + \frac{8\pi}{5} h_M^{ij} h_M^{ij*} \right] \end{aligned}$$

$$\begin{aligned}
 &= 4 \sum_M \frac{12\pi}{15} \delta_{MM} \\
 &= \frac{48\pi}{5}.
 \end{aligned}$$

[5.5.24][5.5.25] を積分すれば

$$\begin{aligned}
 &2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \int d|\mathbf{k}| \delta(|\mathbf{k}| - m) \int d\Omega \frac{k^k k^l}{m^2} \sum_m \left(\delta^{im} - \frac{k^i k^m}{|\mathbf{k}|^2} \right) \left(\delta^{jm} - \frac{k^j k^m}{|\mathbf{k}|^2} \right) \\
 &= 2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \int d|\mathbf{k}| \delta(|\mathbf{k}| - m) \int d\Omega \frac{k^k k^l}{m^2} \left(\delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2} \right) \\
 &= 2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \int d|\mathbf{k}| \delta(|\mathbf{k}| - m) \int d\Omega \left(\frac{k^k k^l}{|\mathbf{k}|^2} \delta^{ij} - \frac{k^i k^j k^k k^l}{|\mathbf{k}|^4} \right) \\
 &= 2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \left[\frac{4\pi}{3} \delta^{ij} \delta^{kl} - \frac{4\pi}{15} (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) \right] \\
 &= 2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \left[\frac{16\pi}{15} \delta^{ij} \delta^{kl} - \frac{4\pi}{15} (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) \right] \\
 &= 2 \sum_M \sum_{ij} \left[-\frac{4\pi}{15} h_M^{ij} h_M^{ij*} - \frac{4\pi}{15} h_M^{ij} h_M^{ji*} \right] \\
 &= -\frac{16\pi}{15} \sum_M \delta_{MM} \\
 &= -\frac{16\pi}{5}.
 \end{aligned}$$

[5.5.26] を積分すれば

$$\begin{aligned}
 &2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \int d|\mathbf{k}| \delta(|\mathbf{k}| - m) \int d\Omega \frac{k^i k^j k^k k^l}{m^4} \\
 &= 2 \sum_M \sum_{ijkl} h_M^{ij} h_M^{kl*} \frac{4\pi}{15} (\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}) \\
 &= 2 \sum_M \sum_{ijkl} \frac{4\pi}{15} (h_M^{ij} h_M^{ij*} + h_M^{ij} h_M^{ji*}) \\
 &= 2 \sum_M \sum_{ijkl} \frac{4\pi}{15} (h_M^{ij} h_M^{ij*} + h_M^{ij} h_M^{ji*}) \\
 &= \frac{16\pi}{15} \sum_M \delta_{MM} \\
 &= \frac{16\pi}{5}.
 \end{aligned}$$

以上から

$$\Gamma(2^3P_2 \rightarrow 2\gamma) = \frac{1}{256\pi^2 m} \frac{e^4}{4\pi m^3 a_0^5} \left(\frac{48\pi}{5} - \frac{16\pi}{5} - \frac{16\pi}{5} + \frac{16\pi}{5} \right) = \frac{m\alpha^7}{320}.$$

Problem 5.6

使う式を列挙しておく．Fierz 恒等式：

$$\bar{u}_L(p_1)\gamma^\mu u_L(p_2)[\gamma^\mu]_{ab} = \bar{u}_R(p_2)\gamma^\mu u_R(p_1) = 2[u_L(p_2)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(p_2)]_{ab}.$$

s, t の定義と性質：

$$\begin{aligned} s(p_1, p_2) &= \bar{u}_R(p_1)u_L(p_2), & t(p_1, p_2) &= \bar{u}_L(p_1)u_R(p_2), \\ t(p_1, p_2) &= (s(p_2, p_1))^*, & s(p_1, p_2) &= s(p_2, p_1), & |s(p_1, p_2)|^2 &= 2p_1 \cdot p_2. \end{aligned}$$

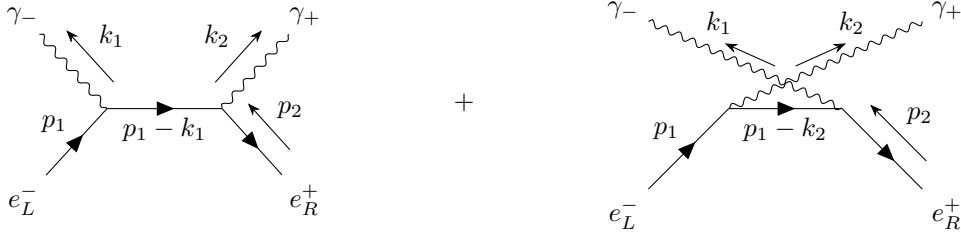
射影に関する性質

$$u_L(p)\bar{u}_L(p) + u_R(p)\bar{u}_R(p) = \not{p}$$

及び^{*3}

$$\bar{u}_L u_L = \bar{u}_R u_R = 0.$$

$e_L^- e_R^+ \rightarrow \gamma_- \gamma_+$ の過程を考える．



(a) で定義した偏極ベクトルを使い，不変振幅は

$$\begin{aligned} i\mathcal{M} &= -ie^2 \epsilon_{-\mu}^*(k_1) \epsilon_{+\nu}^*(k_2) \bar{u}_L(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{(p_1 - k_1)^2} \gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2}{(p_1 - k_2)^2} \gamma^\nu \right] u_L(p_1) \\ &= -ie^2 \frac{\bar{u}_L(p_2) \gamma_\mu u_L(k_1) u_R(p_1) \gamma_\nu u_R(k_2)}{4\sqrt{(p_2 \cdot k_1)(p_1 \cdot k_2)}} \bar{u}_L(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{t} \gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2}{u} \gamma^\nu \right] u_L(p_1) \end{aligned} \quad [5.5.27]$$

となる^{*4}．

^{*3} 全ての粒子が質量 0 なので，左巻きスピノルは上半分；右巻きスピノルは下半分の成分のみ持つ． $\bar{u}_L = u_L^\dagger \gamma_0$ は下半分のみ

^{*4} $\epsilon(k_1)$ の定義に p_2 ， $\epsilon(k_2)$ の定義に p_1 を使った

[5.5.27] の第 1 項は,

$$\begin{aligned}
 & \frac{\bar{u}_L(p_2)\gamma_\mu u_L(k_1)u_R(p_1)\gamma_\nu u_R(k_2)}{4\sqrt{(p_2 \cdot k_1)(p_1 \cdot k_2)}} \bar{u}_L(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{t} \gamma^\mu \right] u_L(p_1) \\
 &= -\frac{2}{ut} \bar{u}_L(p_2) [u_L(p_1)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(p_1)] \\
 &\quad \times [u_L(p_1)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(p_1)] \\
 &\quad \times [u_L(k_1)\bar{u}_L(p_2) + u_R(p_2)\bar{u}_R(k_1)] u_L(p_1) \\
 &\quad + \frac{2}{ut} \bar{u}_L(p_2) [u_L(p_1)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(p_1)] \\
 &\quad \times [u_L(k_1)\bar{u}_L(k_1) + u_R(k_1)\bar{u}_R(k_1)] \\
 &\quad \times [u_L(k_1)\bar{u}_L(p_2) + u_R(p_2)\bar{u}_R(k_1)] u_L(p_1) \\
 &= -\frac{2}{ut} \bar{u}_L(p_2) u_R(k_2) \bar{u}_R(p_1) u_L(p_1) \bar{u}_L(p_1) u_R(p_2) \bar{u}_R(k_1) u_L(p_1) \\
 &\quad + \frac{2}{ut} \bar{u}_L(p_2) u_R(k_2) \bar{u}_R(p_1) u_L(k_1) \bar{u}_L(k_1) u_R(p_2) \bar{u}_R(k_1) u_L(p_1) \\
 &= -\frac{2}{ut} t(p_2, k_2) s(p_1, p_1) t(p_1, p_2) s(k_1, p_1) + \frac{2}{ut} t(p_2, k_2) s(p_1, k_1) t(k_1, p_2) s(k_1, p_1) \\
 &= \frac{2}{ut} t(p_2, k_2) s(p_1, k_1) t(k_1, p_2) s(k_1, p_1).
 \end{aligned} \tag{5.5.28}$$

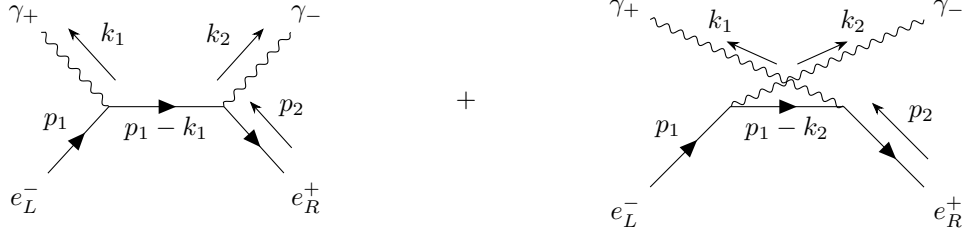
同様に, [5.5.27] の第 2 項は,

$$\begin{aligned}
 & \frac{\bar{u}_L(p_2)\gamma_\mu u_L(k_1)u_R(p_1)\gamma_\nu u_R(k_2)}{4\sqrt{(p_2 \cdot k_1)(p_1 \cdot k_2)}} \bar{u}_L(p_2) \left[\gamma^\mu \frac{\not{p}_1 - \not{k}_2}{u} \gamma^\nu \right] u_L(p_1) \\
 &= -\frac{2}{u^2} \bar{u}_L(p_2) [u_L(k_1)\bar{u}_L(p_2) + u_R(p_2)\bar{u}_R(k_1)] \\
 &\quad \times [u_L(p_1)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(p_1)] \\
 &\quad \times [u_L(p_1)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(p_1)] u_L(p_1) \\
 &\quad + \frac{2}{u^2} \bar{u}_L(p_2) [u_L(k_1)\bar{u}_L(p_2) + u_R(p_2)\bar{u}_R(k_1)] \\
 &\quad \times [u_L(k_2)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(k_2)] \\
 &\quad \times [u_L(p_1)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(p_1)] u_L(p_1) \\
 &= -\frac{2}{u^2} \bar{u}_L(p_2) u_R(p_2) \bar{u}_R(k_1) u_L(p_1) \bar{u}_L(p_1) u_R(k_2) \bar{u}_R(p_1) u_L(p_1) \\
 &\quad + \frac{2}{u^2} \bar{u}_L(p_2) u_R(p_2) \bar{u}_R(k_1) u_L(k_2) \bar{u}_L(k_2) u_R(k_2) \bar{u}_R(p_1) u_L(p_1) \\
 &= -\frac{2}{u^2} t(p_2, p_2) s(k_1, p_1) t(p_1, k_2) s(p_1, p_1) + \frac{2}{u^2} t(p_2, p_2) s(k_1, k_2) t(k_2, k_2) s(p_1, p_1) \\
 &= 0.
 \end{aligned} \tag{5.5.29}$$

[5.5.27][5.5.28][5.5.29] から

$$i\mathcal{M}(e_L^- e_R^+ \rightarrow \gamma_- \gamma_+) = -ie^2 \frac{2}{ut} t(p_2, k_2) s(p_1, k_1) t(k_1, p_2) s(k_1, p_1). \tag{5.5.30}$$

$e_L^- e_R^+ \rightarrow \gamma_+ \gamma_-$ の過程を考える.



不変振幅は

$$\begin{aligned}
 i\mathcal{M} &= -ie^2 \epsilon_{+\mu}^*(k_1) \epsilon_{-\nu}^*(k_2) \bar{u}_L(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{(p_1 - k_1)^2} \gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2}{(p_1 - k_2)^2} \gamma^\nu \right] u_L(p_1) \\
 &= -ie^2 \frac{\bar{u}_R(p_1) \gamma_\mu u_R(k_1) u_L(p_2) \gamma_\nu u_L(k_2)}{4\sqrt{(p_1 \cdot k_1)(p_2 \cdot k_2)}} \bar{u}_L(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{t} \gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2}{u} \gamma^\nu \right] u_L(p_1)
 \end{aligned} \tag{5.5.31}$$

となる*5.

[5.5.31] の第 1 項は,

$$\begin{aligned}
 &\frac{\bar{u}_R(p_1) \gamma_\mu u_R(k_1) u_L(p_2) \gamma_\nu u_L(k_2)}{4\sqrt{(p_1 \cdot k_1)(p_2 \cdot k_2)}} \bar{u}_L(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{t} \gamma^\mu \right] u_L(p_1) \\
 &= -\frac{2}{t^2} \bar{u}_L(p_2) [u_L(k_2) \bar{u}_L(p_2) + u_R(p_2) \bar{u}_R(k_2)] \\
 &\quad \times [u_L(p_1) \bar{u}_L(p_1) + u_R(p_1) \bar{u}_R(p_1)] \\
 &\quad \times [u_L(p_1) \bar{u}_L(k_1) + u_R(k_1) \bar{u}_R(p_1)] u_L(p_1) \\
 &+ \frac{2}{t^2} \bar{u}_L(p_2) [u_L(k_2) \bar{u}_L(p_2) + u_R(p_2) \bar{u}_R(k_2)] \\
 &\quad \times [u_L(k_1) \bar{u}_L(k_1) + u_R(k_1) \bar{u}_R(k_1)] \\
 &\quad \times [u_L(p_1) \bar{u}_L(k_1) + u_R(k_1) \bar{u}_R(p_1)] u_L(p_1) \\
 &= -\frac{2}{t^2} \bar{u}_L(p_2) u_R(p_2) \bar{u}_R(k_2) u_L(p_1) \bar{u}_L(p_1) u_R(k_1) \bar{u}_R(p_1) u_L(p_1) \\
 &\quad + \frac{2}{t^2} \bar{u}_L(p_2) u_R(p_2) \bar{u}_R(k_2) u_L(k_1) \bar{u}_L(k_1) u_R(k_1) \bar{u}_R(p_1) u_L(p_1) \\
 &= -\frac{2}{t^2} t(p_2, p_2) s(k_2, p_1) t(p_1, k_1) s(p_1, p_1) + \frac{2}{ut} t(p_2, p_2) s(k_2, k_1) t(k_1, k_1) s(p_1, p_1) \\
 &= 0.
 \end{aligned} \tag{5.5.32}$$

*5 $\epsilon(k_1)$ の定義に p_1 , $\epsilon(k_2)$ の定義に p_2 を使った

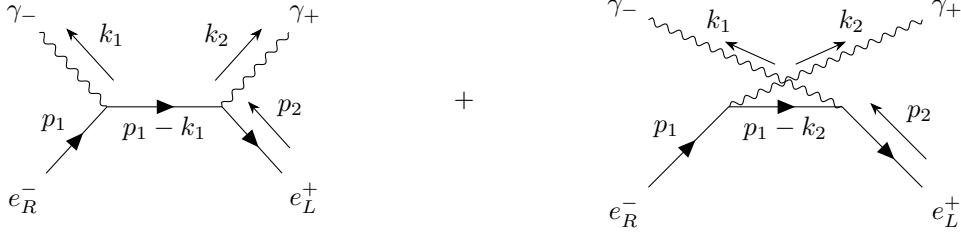
同様に, [5.5.31] の第 2 項は,

$$\begin{aligned}
 & \frac{\bar{u}_R(p_1)\gamma_\mu u_R(k_1)u_L(p_2)\gamma_\nu u_L(k_2)}{4\sqrt{(p_1 \cdot k_1)(p_2 \cdot k_2)}} \bar{u}_L(p_2) \left[\gamma^\mu \frac{\not{p}_1 - \not{k}_2}{u} \gamma^\nu \right] u_L(p_1) \\
 &= -\frac{2}{ut} \bar{u}_L(p_2) [u_L(p_1)\bar{u}_L(k_1) + u_R(k_1)\bar{u}_R(p_1)] \\
 &\quad \times [u_L(p_1)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(p_1)] \\
 &\quad \times [u_L(k_2)\bar{u}_L(p_2) + u_R(p_2)\bar{u}_R(k_2)] u_L(p_1) \\
 &+ \frac{2}{ut} \bar{u}_L(p_2) [u_L(p_1)\bar{u}_L(k_1) + u_R(k_1)\bar{u}_R(p_1)] \\
 &\quad \times [u_L(k_2)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(k_2)] \\
 &\quad \times [u_L(k_2)\bar{u}_L(p_2) + u_R(p_2)\bar{u}_R(k_2)] u_L(p_1) \\
 &= -\frac{2}{ut} \bar{u}_L(p_2) u_R(k_1) \bar{u}_R(p_1) u_L(p_1) \bar{u}_L(p_1) u_R(p_2) \bar{u}_R(k_2) u_L(p_1) \\
 &\quad + \frac{2}{ut} \bar{u}_L(p_2) u_R(k_1) \bar{u}_R(p_1) u_L(k_2) \bar{u}_L(k_2) u_R(p_2) \bar{u}_R(k_2) u_L(p_1) \\
 &= -\frac{2}{ut} t(p_2, k_1) s(p_1, p_1) t(p_1, p_2) s(k_2, p_1) + \frac{2}{ut} t(p_2, k_1) s(p_1, k_2) t(k_2, p_2) s(k_2, p_1) \\
 &= \frac{2}{ut} t(p_2, k_1) s(p_1, k_2) t(k_2, p_2) s(k_2, p_1).
 \end{aligned} \tag{5.5.33}$$

[5.5.31][5.5.32][5.5.33] から

$$i\mathcal{M}(e_L^- e_R^+ \rightarrow \gamma_- \gamma_+) = -ie^2 \frac{2}{ut} t(p_2, k_1) s(p_1, k_2) t(k_2, p_2) s(k_2, p_1). \tag{5.5.34}$$

$e_R^- e_L^+ \rightarrow \gamma_- \gamma_+$ の過程を考える.



不変振幅は

$$\begin{aligned}
 i\mathcal{M} &= -ie^2 \epsilon_{-\mu}^*(k_1) \epsilon_{+\nu}^*(k_2) \bar{u}_R(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{(p_1 - k_1)^2} \gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2}{(p_1 - k_2)^2} \gamma^\nu \right] u_R(p_1) \\
 &= -ie^2 \frac{\bar{u}_L(p_1)\gamma_\mu u_L(k_1)u_R(p_2)\gamma_\nu u_R(k_2)}{4\sqrt{(p_1 \cdot k_1)(p_2 \cdot k_2)}} \bar{u}_R(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{t} \gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2}{u} \gamma^\nu \right] u_R(p_1)
 \end{aligned} \tag{5.5.35}$$

となる*6.

*6 $\epsilon(k_1)$ の定義に p_1 , $\epsilon(k_2)$ の定義に p_2 を使った

[5.5.35] の第 1 項は,

$$\begin{aligned}
 & \frac{\bar{u}_L(p_1)\gamma_\mu u_L(k_1)u_R(p_2)\gamma_\nu u_R(k_2)}{4\sqrt{(p_1 \cdot k_1)(p_2 \cdot k_2)}} \bar{u}_R(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{t} \gamma^\mu \right] u_R(p_1) \\
 &= -\frac{2}{t^2} \bar{u}_R(p_2) [u_L(p_2)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(p_2)] \\
 &\quad \times [u_L(p_1)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(p_1)] \\
 &\quad \times [u_L(k_1)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(k_1)] u_R(p_1) \\
 &+ \frac{2}{t^2} \bar{u}_R(p_2) [u_L(p_2)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(p_2)] \\
 &\quad \times [u_L(k_1)\bar{u}_L(k_1) + u_R(k_1)\bar{u}_R(k_1)] \\
 &\quad \times [u_L(k_1)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(k_1)] u_R(p_1) \\
 &= -\frac{2}{t^2} \bar{u}_R(p_2) u_L(p_2) \bar{u}_L(k_2) u_R(p_1) \bar{u}_R(p_1) u_L(k_1) \bar{u}_L(p_1) u_R(p_1) \\
 &\quad + \frac{2}{t^2} \bar{u}_R(p_2) u_L(p_2) \bar{u}_L(k_2) u_R(k_1) \bar{u}_R(k_1) u_L(k_1) \bar{u}_L(p_1) u_R(p_1) \\
 &= -\frac{2}{t^2} s(p_2, p_2) t(k_2, p_1) s(p_1, k_1) t(p_1, p_1) + \frac{2}{ut} s(p_2, p_2) t(k_2, k_1) s(k_1, k_1) t(p_1, p_1) \\
 &= 0.
 \end{aligned} \tag{5.5.36}$$

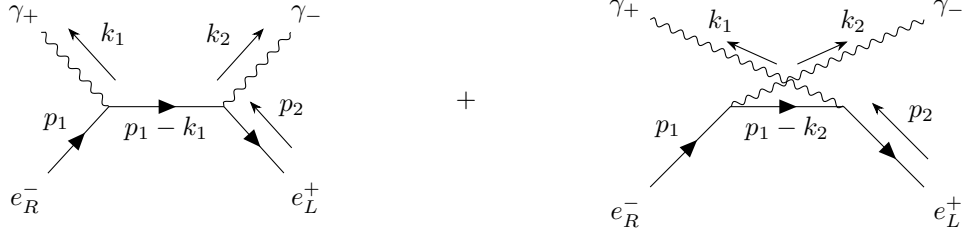
同様に, [5.5.35] の第 2 項は,

$$\begin{aligned}
 & \frac{\bar{u}_L(p_1)\gamma_\mu u_L(k_1)u_R(p_2)\gamma_\nu u_R(k_2)}{4\sqrt{(p_1 \cdot k_1)(p_2 \cdot k_2)}} \bar{u}_R(p_2) \left[\gamma^\mu \frac{\not{p}_1 - \not{k}_2}{u} \gamma^\nu \right] u_R(p_1) \\
 &= -\frac{2}{ut} \bar{u}_R(p_2) [u_L(k_1)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(k_1)] \\
 &\quad \times [u_L(p_1)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(p_1)] \\
 &\quad \times [u_L(p_2)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(p_2)] u_R(p_1) \\
 &+ \frac{2}{ut} \bar{u}_R(p_2) [u_L(k_1)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(k_1)] \\
 &\quad \times [u_L(k_2)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(k_2)] \\
 &\quad \times [u_L(p_2)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(p_2)] u_R(p_1) \\
 &= -\frac{2}{ut} \bar{u}_R(p_2) u_L(k_1) \bar{u}_L(p_1) u_R(p_1) \bar{u}_R(p_1) u_L(p_2) \bar{u}_L(k_2) u_R(p_1) \\
 &\quad + \frac{2}{ut} \bar{u}_R(p_2) u_L(k_1) \bar{u}_L(p_1) u_R(k_2) \bar{u}_R(k_2) u_L(p_2) \bar{u}_L(k_2) u_R(p_1) \\
 &= -\frac{2}{ut} s(p_2, k_1) t(p_1, p_1) s(p_1, p_2) t(k_2, p_1) + \frac{2}{ut} s(p_2, k_1) t(p_1, k_2) s(k_2, p_2) t(k_2, p_1) \\
 &= \frac{2}{ut} s(p_2, k_1) t(p_1, k_2) s(k_2, p_2) t(k_2, p_1).
 \end{aligned} \tag{5.5.37}$$

[5.5.35][5.5.36][5.5.37] から

$$i\mathcal{M}(e_R^- e_L^+ \rightarrow \gamma_- \gamma_+) = -ie^2 \frac{2}{ut} s(p_2, k_1) t(p_1, k_2) s(k_2, p_2) t(k_2, p_1). \tag{5.5.38}$$

$e_R^- e_L^+ \rightarrow \gamma_+ \gamma_-$ の過程を考える.



(a) で定義した偏極ベクトルを使い、不変振幅は

$$\begin{aligned}
 i\mathcal{M} &= -ie^2 \epsilon_{+\mu}^*(k_1) \epsilon_{-\nu}^*(k_2) \bar{u}_R(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{(p_1 - k_1)^2} \gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2}{(p_1 - k_2)^2} \gamma^\nu \right] u_R(p_1) \\
 &= -ie^2 \frac{\bar{u}_R(p_2) \gamma_\mu u_R(k_1) u_L(p_1) \gamma_\nu u_L(k_2)}{4\sqrt{(p_2 \cdot k_1)(p_1 \cdot k_2)}} \bar{u}_R(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{t} \gamma^\mu + \gamma^\mu \frac{\not{p}_1 - \not{k}_2}{u} \gamma^\nu \right] u_R(p_1)
 \end{aligned} \tag{5.5.39}$$

となる*7.

[5.5.39] の第1項は,

$$\begin{aligned}
 & \frac{\bar{u}_R(p_2) \gamma_\mu u_R(k_1) u_L(p_1) \gamma_\nu u_L(k_2)}{4\sqrt{(p_2 \cdot k_1)(p_1 \cdot k_2)}} \bar{u}_L(p_2) \left[\gamma^\nu \frac{\not{p}_1 - \not{k}_1}{t} \gamma^\mu \right] u_L(p_1) \\
 &= -\frac{2}{ut} \bar{u}_R(p_2) [u_L(k_2) \bar{u}_L(p_1) + u_R(p_1) \bar{u}_R(k_2)] \\
 & \quad \times [u_L(p_1) \bar{u}_L(p_1) + u_R(p_1) \bar{u}_R(p_1)] \\
 & \quad \times [u_L(p_2) \bar{u}_L(k_1) + u_R(k_1) \bar{u}_R(p_2)] u_R(p_1) \\
 &+ \frac{2}{ut} \bar{u}_R(p_2) [u_L(k_2) \bar{u}_L(p_1) + u_R(p_1) \bar{u}_R(k_2)] \\
 & \quad \times [u_L(k_1) \bar{u}_L(k_1) + u_R(k_1) \bar{u}_R(k_1)] \\
 & \quad \times [u_L(p_2) \bar{u}_L(k_1) + u_R(k_1) \bar{u}_R(p_2)] u_R(p_1) \\
 &= -\frac{2}{ut} \bar{u}_R(p_2) u_L(k_2) \bar{u}_L(p_1) u_R(p_1) \bar{u}_R(p_1) u_L(p_2) \bar{u}_L(k_1) u_R(p_1) \\
 & \quad + \frac{2}{ut} \bar{u}_R(p_2) u_L(k_2) \bar{u}_L(p_1) u_R(k_1) \bar{u}_R(k_1) u_L(p_2) \bar{u}_L(k_1) u_R(p_1) \\
 &= -\frac{2}{ut} s(p_2, k_2) t(p_1, p_1) s(p_1, p_2) t(k_1, p_1) + \frac{2}{ut} s(p_2, k_2) t(p_1, k_1) s(k_1, p_2) t(k_1, p_1) \\
 &= \frac{2}{ut} s(p_2, k_2) t(p_1, k_1) s(k_1, p_2) t(k_1, p_1).
 \end{aligned} \tag{5.5.40}$$

*7 $\epsilon(k_1)$ の定義に p_2 , $\epsilon(k_2)$ の定義に p_1 を使った

同様に, [5.5.39] の第 2 項は,

$$\begin{aligned}
 & \frac{\bar{u}_R(p_2)\gamma_\mu u_R(k_1)u_L(p_1)\gamma_\nu u_L(k_2)\bar{u}_R(p_2)}{4\sqrt{(p_2 \cdot k_1)(p_1 \cdot k_2)}} \left[\gamma^\mu \frac{\not{p}_1 - \not{k}_2}{u} \gamma^\nu \right] u_R(p_1) \\
 &= -\frac{2}{u^2} \bar{u}_R(p_2) [u_L(p_2)\bar{u}_L(k_1) + u_R(k_1)\bar{u}_R(p_2)] \\
 &\quad \times [u_L(p_1)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(p_1)] \\
 &\quad \times [u_L(k_2)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(k_2)] u_R(p_1) \\
 &+ \frac{2}{u^2} \bar{u}_R(p_2) [u_L(p_2)\bar{u}_L(k_1) + u_R(k_1)\bar{u}_R(p_2)] \\
 &\quad \times [u_L(k_2)\bar{u}_L(k_2) + u_R(k_2)\bar{u}_R(k_2)] \\
 &\quad \times [u_L(k_2)\bar{u}_L(p_1) + u_R(p_1)\bar{u}_R(k_2)] u_R(p_1) \\
 &= -\frac{2}{u^2} \bar{u}_R(p_2) u_L(p_2) \bar{u}_L(k_1) u_R(p_1) \bar{u}_R(p_1) u_L(k_2) \bar{u}_L(p_1) u_R(p_1) \\
 &\quad + \frac{2}{u^2} \bar{u}_R(p_2) u_L(p_2) \bar{u}_L(k_1) u_R(k_2) \bar{u}_R(k_2) u_L(k_2) \bar{u}_L(p_1) u_R(p_1) \\
 &= -\frac{2}{u^2} s(p_2, p_2) t(k_1, p_1) s(p_1, k_2) t(p_1, p_1) + \frac{2}{u^2} s(p_2, p_2) t(k_1, k_2) s(k_2, k_2) t(p_1, p_1) \\
 &= 0.
 \end{aligned} \tag{5.5.41}$$

[5.5.39][5.5.40][5.5.41] から

$$i\mathcal{M}(e_R^- e_L^+ \rightarrow \gamma_+ \gamma_-) = -ie^2 \frac{2}{ut} s(p_2, k_2) t(p_1, k_1) s(k_1, p_2) t(k_1, p_1). \tag{5.5.42}$$

[5.5.30][5.5.34][5.5.38][5.5.42] から,

$$\begin{aligned}
 |\mathcal{M}(e_L^- e_R^+ \rightarrow \gamma_- \gamma_+)|^2 &= |\mathcal{M}(e_R^- e_L^+ \rightarrow \gamma_+ \gamma_-)|^2 = e^4 \frac{64}{u^2 t^2} (p_2 \cdot k_2)(p_1 \cdot k_1)(k_1 \cdot p_2)(k_1 \cdot p_1) \\
 &= 4e^4 \frac{t}{u}, \\
 |\mathcal{M}(e_L^- e_R^+ \rightarrow \gamma_+ \gamma_-)|^2 &= |\mathcal{M}(e_R^- e_L^+ \rightarrow \gamma_- \gamma_+)|^2 = e^4 \frac{64}{u^2 t^2} (p_2 \cdot k_1)(p_1 \cdot k_2)(k_2 \cdot p_2)(k_2 \cdot p_1) \\
 &= 4e^4 \frac{u}{t}.
 \end{aligned}$$

従って,

$$\frac{1}{4} \sum_{\text{spin}} \sum_{\text{polarization}} |\mathcal{M}|^2 = 2e^4 \left(\frac{u}{t} + \frac{t}{u} \right).$$

慣性質量から観測すれば, (4.85) から

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\pi^2 E_{\text{cm}}^2} = \frac{\alpha^2}{2s} \left(\frac{u}{t} + \frac{t}{u} \right) = \frac{\alpha^2}{s} \frac{1 + \cos^2 \theta}{\sin^2 \theta}.$$

従って,

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s} \frac{1 + \cos^2 \theta}{\sin^2 \theta}.$$

Chapter 6

Radiative Corrections: Introduction

6.3 The Electron Vertex Function: Evaluation

(6.44)

電子の運動量 p , $p' = p + q$ について,

$$p'^2 = (p + q)^2 = m^2$$

が成立する. 直ちに

$$2p \cdot q + q^2 = 0$$

が分かる. さらに,

$$x + y + z = 1.$$

よって,

$$\begin{aligned} \ell^2 - \Delta &= (k + yq - zp)^2 + xyq^2 - (1 - z)^2 m^2 \\ &= k^2 + y^2 q^2 + z^2 p^2 + 2yk \cdot q - 2yzp \cdot q - 2zk \cdot p + xyq^2 - (1 - z)^2 m^2 \\ &= k^2 + 2k \cdot (yq - zp) + y^2 q^2 + z^2 p^2 - 2yzp \cdot q + xyq^2 - (1 - z)^2 m^2 \\ &= k^2 + 2k \cdot (yq - zp) + y(x + y)q^2 - 2yzp \cdot q + \{z^2 - (1 - z)^2\} m^2 \\ &= k^2 + 2k \cdot (yq - zp) + y(x + y)q^2 + yzq^2 + (2z - 1)m^2 \\ &= k^2 + 2k \cdot (yq - zp) + y(x + y + z)q^2 + (z - x - y)m^2 \\ &= k^2 + 2k \cdot (yq - zp) + yq^2 + zm^2 - (x + y)m^2 \\ &= k^2 + 2k \cdot (yq - zp) + yq^2 + zp^2 - (x + y)m^2 \\ &= D - i\epsilon. \end{aligned}$$

p.191 中盤の式

(6.38) の積分変数 k を $l = k + yq - zp$ に変更する. (6.38) は

$$2ie^2 \int_0^1 dx dy dz \delta(x + y + z - 1) \int \frac{d^4 \ell}{(2\pi)^4} \frac{\bar{u}(p') \left[\not{k} \gamma^\mu \not{k}' + m^2 \gamma^\mu - 2m(k + k')^\mu \right] u(p)}{D^3} \quad [6.3.1]$$

となる。(積分値が変わらない範囲で; \rightarrow と表記する) 分子を式変形する.

ここで

$$k = \ell - yq + zp =: \ell + a, \quad k' = k + q =: \ell + b$$

とおく.

$$\not{k}\gamma^\mu\not{k}' = (\ell + a)\gamma^\mu(\ell + b) = \not{\ell}\gamma^\mu\not{\ell} + \not{\ell}\gamma^\mu\not{b} + \not{a}\gamma^\mu\not{\ell} + \not{a}\gamma^\mu\not{b}$$

となるが, $\not{\ell}\gamma^\mu\not{b} = \ell_\nu\gamma^\nu\gamma^\mu\not{b}$ なので, (6.45) から上式の第 2, 3 項の積分は 0 である. 従って,

$$\begin{aligned} &\rightarrow \not{\ell}\gamma^\mu\not{\ell} + \not{a}\gamma^\mu\not{b} \\ &= \ell^\nu\gamma^\nu\gamma^\mu\not{\ell} + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\ &= \ell^\nu(2g^{\mu\nu} - \gamma^\mu\gamma^\nu)\not{\ell} + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\ &= 2\ell^\mu\not{\ell} - \gamma^\mu\not{\ell}\not{\ell} + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\ &= 2\ell^\mu\ell^\nu\gamma_\nu - \gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}). \end{aligned}$$

(6.46) から

$$\begin{aligned} &\rightarrow 2\frac{1}{4}g^{\mu\nu}\ell^2\gamma_\nu - \gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\ &= \frac{1}{2}\gamma^\mu\ell^2 - \gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) \\ &= -\frac{1}{2}\gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}). \end{aligned}$$

同様に,

$$-2m(k + k')^\mu = -2m(2\ell^\mu + a^\mu + b^\mu) \rightarrow -2m(a^\mu + b^\mu) = -2m((1-2y)q^\mu + 2zp^\mu).$$

p.192 序盤の式

Dirac 方程式

$$\not{p}u(p) = mu(p), \quad \bar{u}(p')\not{p}' = \bar{u}(p')m, \quad \bar{u}(p')\not{q}u(p) = 0 \quad [6.3.2]$$

に注意する.

$$\bar{u}(p') \left[-\frac{\gamma^\mu}{2}\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu) \right] u(p)$$

を計算する (これ以降は両端のスピンルを省略する). 2 項目は

$$\begin{aligned} &(-y\not{q} + z\not{p})\gamma^\mu[(1-y)\not{q} + z\not{p}] \\ &= [-(y+z)\not{q} + z\not{p}']\gamma^\mu[(1-y)\not{q} + z\not{p}] \\ &= [-(y+z)\not{q} + zm]\gamma^\mu[(1-y)\not{q} + zm] \\ &= -(y+z)(1-y)\not{q}\gamma^\mu\not{q} - z(y+z)m\not{q}\gamma^\mu + z(1-y)m\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\ &= -(y+z)(1-y)(2q^\mu - \gamma^\mu\not{q})\not{q} \\ &\quad - z(y+z)m(2q^\mu - \gamma^\mu\not{q}) + z(1-y)m\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\ &= -2(y+z)(1-y)q^\mu\not{q} + (y+z)(1-y)\gamma^\mu\not{q}\not{q} \end{aligned}$$

$$\begin{aligned}
& -2z(y+z)mq^\mu + z(y+z)m\gamma^\mu \not{q} + z(1-y)m\gamma^\mu \not{q} + z^2m^2\gamma^\mu \\
& = -2(y+z)(1-y)q^\mu \not{q} + (y+z)(1-y)\gamma^\mu q^2 \\
& \quad -2z(y+z)mq^\mu + z(1+z)m\gamma^\mu \not{q} + z^2m^2\gamma^\mu.
\end{aligned}$$

$\bar{u}(p')\not{q}u(p) = 0$ なので,

$$= (y+z)(1-y)\gamma^\mu q^2 - 2z(y+z)mq^\mu + z(1+z)m\gamma^\mu \not{q} + z^2m^2\gamma^\mu.$$

$\gamma^\mu \not{q} = \gamma^\mu (\not{p}' - \not{p}) = \gamma^\mu (\not{p}' - m) = (\gamma^\mu \not{p}' - m\gamma^\mu) = (2p'^\mu - \not{p}'\gamma^\mu - m\gamma^\mu) = 2p'^\mu - 2m\gamma^\mu$ なので,

$$\begin{aligned}
& = (y+z)(1-y)\gamma^\mu q^2 - 2z(y+z)mq^\mu + z(1+z)m(2p'^\mu - 2m\gamma^\mu) + z^2m^2\gamma^\mu \\
& = (y+z)(1-y)\gamma^\mu q^2 - 2z(y+z)mq^\mu + 2z(1+z)mp'^\mu - 2z(1+z)m^2\gamma^\mu + z^2m^2\gamma^\mu \\
& = (1-x)(1-y)\gamma^\mu q^2 - 2z(y+z)mq^\mu + 2z(1+z)mp'^\mu + (-2z-z^2)m^2\gamma^\mu.
\end{aligned} \tag{6.3.3}$$

他の項も併せれば

$$\begin{aligned}
& -\frac{\gamma^\mu}{2}\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu) \\
& = -\frac{\gamma^\mu}{2}\ell^2 \\
& \quad + (1-x)(1-y)\gamma^\mu q^2 - 2z(y+z)mq^\mu + 2z(1+z)mp'^\mu + (-2z-z^2)m^2\gamma^\mu \\
& \quad + m^2\gamma^\mu - 2m((1-2y)q^\mu + 2zp^\mu) \\
& = -\frac{\gamma^\mu}{2}\ell^2 + (1-x)(1-y)\gamma^\mu q^2 + (1-2z-z^2)m^2\gamma^\mu \\
& \quad - 2z(y+z)mq^\mu + 2z(1+z)mp'^\mu - 2(1-2y)mq^\mu - 4zmp^\mu.
\end{aligned} \tag{6.3.4}$$

2行目の部分を計算する：

$$\begin{aligned}
& -2z(y+z)mq^\mu + 2z(1+z)mp'^\mu - 2(1-2y)mq^\mu - 4zmp^\mu \\
& = -4zmp^\mu + 2z(1+z)mp'^\mu - 2z(y+z)mq^\mu - 2(1-2y)mq^\mu \\
& = -4zmp^\mu + 2z(1+z)m(p^\mu + q^\mu) - 2z(y+z)mq^\mu - 2(1-2y)mq^\mu \\
& = 2z(z-1)mp^\mu + 2z(1+z)mq^\mu - 2z(y+z)mq^\mu - 2(1-2y)mq^\mu \\
& = 2z(z-1)mp^\mu + 2z(1-y)mq^\mu - 2(1-2y)mq^\mu \\
& = 2z(z-1)mp^\mu + z(z-1)mq^\mu - z(z-1)mq^\mu + 2z(1-y)mq^\mu - 2(1-2y)mq^\mu \\
& = z(z-1)m(2p^\mu + q^\mu) - z(z-1)mq^\mu + 2z(1-y)mq^\mu - 2(1-2y)mq^\mu \\
& = (p'^\mu + p^\mu) \cdot mz(z-1) + q^\mu \cdot m(-z^2 + z + 2z - 2yz - 2 + 4y) \\
& = (p'^\mu + p^\mu) \cdot mz(z-1) + q^\mu \cdot m(-z^2 + 3z - 2yz - 2 + 4y).
\end{aligned} \tag{6.3.5}$$

最後の部分を計算する：

$$\begin{aligned}
& -z^2 + 3z - 2yz - 2 + 4y = -z(1-x-y) + 3z - 2yz - 2 + 4y \\
& = xz - yz + 4y + 2z - 2 \\
& = xz - yz + 4y + 2(1-x-y) - 2 \\
& = xz - yz - 2x + 2y \\
& = (z-2)(x-y).
\end{aligned} \tag{6.3.6}$$

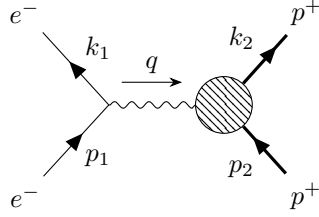
[6.3.4][6.3.5][6.3.6] から

$$-\frac{\gamma^\mu}{2}\ell^2 + (1-x)(1-y)\gamma^\mu q^2 + (1-2z-z^2)m^2\gamma^\mu + (p'^\mu + p^\mu) \cdot mz(z-1) + q^\mu \cdot m(z-2)(x-y).$$

Problems

Problem 6.1: Rosenbluth formula

電子と陽子の散乱を考える。



電子の質量は 0, 陽子の質量は M とする. スピンの平均を取った不変振幅は

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4q^4} \text{Tr}[\gamma_\mu \not{p}_1 \gamma_\nu \not{k}_1] \\ \times \text{Tr} \left[\left(\gamma^\mu (F_1 + F_2) - \frac{(p_2 + k_2)^\mu}{2M} F_2 \right) (\not{p}_2 + M) \left(\gamma^\nu (F_1 + F_2) - \frac{(p_2 + k_2)^\nu}{2M} F_2 \right) (\not{k}_2 + M) \right]$$

である. 1 つめのトレースは次のように計算できる:

$$\text{Tr}[\gamma_\mu \not{p}_1 \gamma_\nu \not{k}_1] = 4[p_{1\mu} k_{1\nu} + k_{1\mu} p_{1\nu} - g_{\mu\nu} (p_1 \cdot k_1)].$$

2 つめのトレースで非零なのはガンマ行列が偶数個含まれる項:

$$\begin{aligned} \text{Tr}[\dots] &= (F_1 + F_2)^2 \text{Tr}(\gamma^\mu \not{p}_2 \gamma^\nu \not{k}_2) + F_2^2 (p_2 + k_2)^\mu (p_2 + k_2)^\nu \\ &\quad + M^2 (F_1 + F_2)^2 \text{Tr}(\gamma^\mu \gamma^\nu) + \frac{F_2^2}{4M^2} (p_2 + k_2)^\mu (p_2 + k_2)^\nu \text{Tr}(\not{p}_2 \not{k}_2) \\ &\quad - \frac{1}{2} F_2 (F_1 + F_2) (p_2 + k_2)^\nu \text{Tr}(\gamma^\mu \not{p}_2) - \frac{1}{2} F_2 (F_1 + F_2) (p_2 + k_2)^\nu \text{Tr}(\gamma^\mu \not{k}_2) \\ &\quad - \frac{1}{2} F_2 (F_1 + F_2) (p_2 + k_2)^\mu \text{Tr}(\gamma^\nu \not{p}_2) - \frac{1}{2} F_2 (F_1 + F_2) (p_2 + k_2)^\mu \text{Tr}(\gamma^\nu \not{k}_2) \\ &= 4(F_1 + F_2)^2 [p_2^\mu k_2^\nu + k_2^\mu p_2^\nu - g^{\mu\nu} (p_2 \cdot k_2)] + F_2^2 (p_2 + k_2)^\mu (p_2 + k_2)^\nu \\ &\quad + 4M^2 (F_1 + F_2)^2 g^{\mu\nu} + \frac{F_2^2}{M^2} (p_2 \cdot k_2) (p_2 + k_2)^\mu (p_2 + k_2)^\nu \\ &\quad - 4F_2 (F_1 + F_2) (p_2 + k_2)^\mu (p_2 + k_2)^\nu \\ &= 4(F_1 + F_2)^2 [p_2^\mu k_2^\nu + k_2^\mu p_2^\nu - g^{\mu\nu} (p_2 \cdot k_2)] \\ &\quad + 4M^2 (F_1 + F_2)^2 g^{\mu\nu} \\ &\quad + \left[F_2^2 + \frac{F_2^2}{M^2} (p_2 \cdot k_2) - 4F_2 (F_1 + F_2) \right] (p_2 + k_2)^\mu (p_2 + k_2)^\nu. \end{aligned}$$

これらの積を求める. 1 項目は

$$\begin{aligned} &[p_{1\mu} k_{1\nu} + k_{1\mu} p_{1\nu} - g_{\mu\nu} (p_1 \cdot k_1)] [p_2^\mu k_2^\nu + k_2^\mu p_2^\nu - g^{\mu\nu} (p_2 \cdot k_2)] \\ &= [p_{1\mu} k_{1\nu} + k_{1\mu} p_{1\nu}] [p_2^\mu k_2^\nu + k_2^\mu p_2^\nu] \\ &= 2(p_1 \cdot p_2) (k_1 \cdot k_2) + 2(p_1 \cdot k_2) (k_1 \cdot p_2). \end{aligned}$$

2 項目は

$$[p_{1\mu}k_{1\nu} + k_{1\mu}p_{1\nu} - g_{\mu\nu}(p_1 \cdot k_1)]g^{\mu\nu} = -2(p_1 \cdot k_1).$$

3 項目は

$$\begin{aligned} & [p_{1\mu}k_{1\nu} + k_{1\mu}p_{1\nu} - g_{\mu\nu}(p_1 \cdot k_1)](p_2 + k_2)^\mu(p_2 + k_2)^\nu \\ &= 2p_1 \cdot (p_2 + k_2)k_1 \cdot (p_2 + k_2) - (p_1 \cdot k_1)(p_2 + k_2)^2 \\ &= 2p_1 \cdot (p_2 + k_2)k_1 \cdot (p_2 + k_2) - 2M^2(p_1 \cdot k_1) - 2(p_1 \cdot k_1)(p_2 \cdot k_2). \end{aligned}$$

以上から,

$$\begin{aligned} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{8e^4}{q^4} (F_1 + F_2)^2 [(p_1 \cdot p_2)(k_1 \cdot k_2) + (p_1 \cdot k_2)(k_1 \cdot p_2) - M^2(p_1 \cdot k_1)] \\ &\quad + \frac{2e^4}{q^4} \left[F_2^2 + \frac{F_2^2}{M^2} (p_2 \cdot k_2) - 4F_2(F_1 + F_2) \right] \\ &\quad \times [p_1 \cdot (p_2 + k_2)k_1 \cdot (p_2 + k_2) - M^2(p_1 \cdot k_1) - (p_1 \cdot k_1)(p_2 \cdot k_2)]. \end{aligned} \quad [6.3.7]$$

ここで, 初めに陽子が静止している実験系から見る:

$$p_1^\mu = (E, 0, 0, E), \quad k_1^\mu = (E', E' \sin \theta, 0, E' \cos \theta), \quad p_2^\mu = (M, 0, 0, 0).$$

$p_1 + p_2 = k_1 + k_2$ から

$$k_2^\mu = (M + E - E', -E' \sin \theta, 0, E - E' \cos \theta), \quad E' = \frac{ME}{M + 2E \sin^2 \theta/2}.$$

$q = p_1 - k_1$ なので,

$$q^2 = p_1^2 + k_1^2 - 2p_1 \cdot k_1 = -2p_1 \cdot k_1.$$

従って,

$$\begin{aligned} (p_1 \cdot p_2)(k_1 \cdot k_2) &= ME(ME' + EE' - EE' \cos \theta) \\ &= MEE' (M + 2E \sin^2 \theta/2) \\ &= M^2 E^2, \\ (p_1 \cdot k_2)(k_1 \cdot p_2) &= (ME - EE' + EE' \cos \theta)ME' \\ &= ME' ME' \\ &= M^2 E'^2, \\ p_1 \cdot k_1 &= -\frac{1}{2} q^2. \end{aligned}$$

以上から, [6.3.7] の第 1 項は

$$\begin{aligned} & \frac{8e^4}{q^4} (F_1 + F_2)^2 [(p_1 \cdot p_2)(k_1 \cdot k_2) + (p_1 \cdot k_2)(k_1 \cdot p_2) - M^2(p_1 \cdot k_1)] \\ &= \frac{4e^4 M^2}{q^4} (F_1 + F_2)^2 [2E^2 + 2E'^2 + q^2]. \end{aligned}$$

$q = k_2 - p_2$ なので,

$$k_2 \cdot p_2 = M^2 - \frac{q^2}{2}.$$

従って,

$$\begin{aligned}
 & F_2^2 + \frac{F_2^2}{M^2}(p_2 \cdot k_2) - 4F_2(F_1 + F_2) \\
 &= F_2^2 + \frac{F_2^2}{M^2} \left(M^2 - \frac{q^2}{2} \right) - 4F_2(F_1 + F_2) \\
 &= -2 \left(2F_1F_2 + F_2^2 + \frac{F_2^2 q^2}{4M^2} \right)
 \end{aligned}$$

及び

$$\begin{aligned}
 & p_1 \cdot (p_2 + k_2)k_1 \cdot (p_2 + k_2) - M^2(p_1 \cdot k_1) - (p_1 \cdot k_1)(p_2 \cdot k_2) \\
 &= M(E + E')M(E + E') + M^2 \frac{q^2}{2} + \left(M^2 - \frac{q^2}{2} \right) \frac{q^2}{2} \\
 &= M^2 \left[(E + E')^2 + q^2 - \frac{q^4}{4M^2} \right]
 \end{aligned}$$

を得る. これらから, [6.3.7] の第 2 項は

$$\begin{aligned}
 & \frac{2e^4}{q^4} \left[F_2^2 + \frac{F_2^2}{M^2}(p_2 \cdot k_2) - 4F_2(F_1 + F_2) \right] \\
 & \quad \times [p_1 \cdot (p_2 + k_2)k_1 \cdot (p_2 + k_2) - M^2(p_1 \cdot k_1) - (p_1 \cdot k_1)(p_2 \cdot k_2)] \\
 &= -\frac{4e^4 M^2}{q^4} \left[(E + E')^2 + q^2 - \frac{q^4}{4M^2} \right] \left(2F_1F_2 + F_2^2 + \frac{F_2^2 q^2}{4M^2} \right) \\
 &= -\frac{4e^4 M^2}{q^4} \left[(E + E')^2 + q^2 - \frac{q^4}{4M^2} \right] \left((F_1 + F_2)^2 - F_1^2 + \frac{F_2^2 q^2}{4M^2} \right).
 \end{aligned}$$

以上から,

$$\begin{aligned}
 \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{4e^4 M^2}{q^4} (F_1 + F_2)^2 [2E^2 + 2E'^2 + q^2] \\
 & \quad - \frac{4e^4 M^2}{q^4} \left[(E + E')^2 + q^2 - \frac{q^4}{4M^2} \right] \left((F_1 + F_2)^2 - F_1^2 + \frac{F_2^2 q^2}{4M^2} \right) \\
 &= \frac{4e^4 M^2}{q^4} (F_1 + F_2)^2 \left[2E^2 + 2E'^2 - (E + E')^2 + \frac{q^4}{4M^2} \right] \\
 & \quad - \frac{4e^4 M^2}{q^4} \left[(E + E')^2 + q^2 - \frac{q^4}{4M^2} \right] \left(-F_1^2 + \frac{F_2^2 q^2}{4M^2} \right) \\
 &= \frac{4e^4 M^2}{q^4} (F_1 + F_2)^2 \left[(E - E')^2 + \frac{q^4}{4M^2} \right] \\
 & \quad + \frac{4e^4 M^2}{q^4} \left[(E + E')^2 + q^2 - \frac{q^4}{4M^2} \right] \left(F_1^2 - \frac{F_2^2 q^2}{4M^2} \right).
 \end{aligned}$$

ここで, $p_2 \cdot k_2 = M^2 + M(E - E') = M^2 - q^2/2$ なのので,

$$E - E' = -\frac{q^2}{2M}.$$

$p_1 \cdot k_1 = -q^2/2 = EE'(1 - \cos \theta) = 2EE' \sin^2 \theta/2$ なのので,

$$q^2 = -4EE' \sin^2 \frac{\theta}{2}.$$

これらを使えば,

$$\begin{aligned}
 \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{4e^4 M^2}{q^4} (F_1 + F_2)^2 \frac{q^4}{2M^2} \\
 &\quad + \frac{4e^4 M^2}{q^4} \left[E^2 + E'^2 + 2EE' - 4EE' \sin^2 \frac{\theta}{2} - \frac{q^4}{4M^2} \right] \left(F_1^2 - \frac{F_2^2 q^2}{4M^2} \right) \\
 &= \frac{4e^4 M^2}{q^4} (F_1 + F_2)^2 \frac{q^4}{2M^2} \\
 &\quad + \frac{4e^4 M^2}{q^4} \left[(E - E')^2 + 4EE' \cos^2 \frac{\theta}{2} - \frac{q^4}{4M^2} \right] \left(F_1^2 - \frac{F_2^2 q^2}{4M^2} \right) \\
 &= \frac{4e^4 M^2}{q^4} (F_1 + F_2)^2 \frac{q^4}{2M^2} + \frac{4e^4 M^2}{q^4} \left(F_1^2 - \frac{F_2^2 q^2}{4M^2} \right) 4EE' \cos^2 \frac{\theta}{2} \\
 &= \frac{4e^4 M^2}{q^4} (F_1 + F_2)^2 \frac{q^2}{2M^2} \left(-4EE' \sin^2 \frac{\theta}{2} \right) + \frac{4e^4 M^2}{q^4} \left(F_1^2 - \frac{F_2^2 q^2}{4M^2} \right) 4EE' \cos^2 \frac{\theta}{2} \\
 &= \frac{16e^4 M^2}{q^4} EE' \left[\left(F_1^2 - \frac{q^2}{4M^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right] \\
 &= \frac{16e^4 M^3 E^2}{q^4 (M + 2E \sin^2 \theta/2)} \left[\left(F_1^2 - \frac{q^2}{4M^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right] \\
 &= \frac{e^4 M (M + 2E \sin^2 \theta/2)}{E^2 \sin^4 \theta/2} \left[\left(F_1^2 - \frac{q^2}{4M^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right] \\
 &= \frac{16\pi^2 \alpha^2 M (M + 2E \sin^2 \theta/2)}{E^2 \sin^4 \theta/2} \left[\left(F_1^2 - \frac{q^2}{4M^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right].
 \end{aligned}$$

(A.56) から

$$\begin{aligned}
 d\sigma &= \frac{1}{4EM} \frac{d^3 k_1 d^3 k_2}{(2\pi)^6 4E_1 E_2} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \\
 &= \frac{1}{4EM} \frac{d^3 k_1}{(2\pi)^2 4E' E_2} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \delta(E + M - E' - E_2) \\
 &= \frac{1}{4EM} \frac{E' d \cos \theta dE'}{(2\pi) 4E_2} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \delta(E + M - E' - E_2).
 \end{aligned}$$

$\mathbf{k}_2 = \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{k}_1$ なので,

$$|\mathbf{k}_2|^2 = E^2 + E'^2 - 2EE' \cos \theta.$$

さらに,

$$E_2^2 = |\mathbf{k}_2|^2 + M^2 = M^2 + E^2 + E'^2 - 2EE' \cos \theta.$$

よって,

$$\begin{aligned}
 \frac{d\sigma}{d \cos \theta} &= \frac{1}{4EM} \int \frac{E' dE'}{(2\pi) 4E_2} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \delta(E + M - E' - E_2) \\
 &= \frac{1}{4EM} \frac{E'}{(2\pi) 4E_2} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \frac{E_2}{E_2 + E' - E \cos \theta} \\
 &= \frac{1}{32\pi EM} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \frac{E'}{E_2 + E' - E \cos \theta}
 \end{aligned}$$

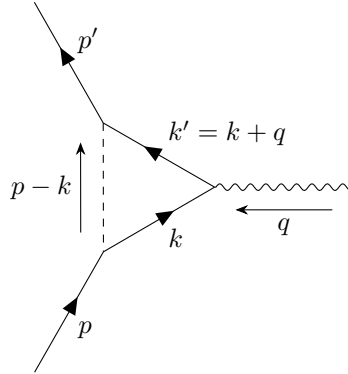
$$\begin{aligned}
 &= \frac{1}{32\pi EM} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \frac{E'}{E + M - E \cos \theta} \\
 &= \frac{1}{32\pi EM} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \frac{E'}{M + 2E \sin^2 \theta/2} \\
 &= \frac{1}{32\pi} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 \frac{1}{(M + 2E \sin^2 \theta/2)^2} \\
 &= \frac{1}{32\pi} \frac{16\pi^2 \alpha^2 M}{E^2 \sin^4 \theta/2 (M + 2E \sin^2 \theta/2)} \left[\left(F_1^2 - \frac{q^2}{4M^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right] \\
 &= \frac{\pi \alpha^2 M}{2E^2 \sin^4 \theta/2 (M + 2E \sin^2 \theta/2)} \left[\left(F_1^2 - \frac{q^2}{4M^2} F_2^2 \right) \cos^2 \frac{\theta}{2} - \frac{q^2}{2M^2} (F_1 + F_2)^2 \sin^2 \frac{\theta}{2} \right].
 \end{aligned}$$

 Problem 6.3: Exotic contribuutons to $g - 2$

(a)

相互作用のハミルトニアンは

$$\mathcal{H}_{\text{int}} = \frac{\lambda}{\sqrt{2}} h \bar{\psi} \psi$$

 なので, これは結合定数 $g = \lambda/\sqrt{2}$ の Yukawa 理論 (p.118).


p.189 以降と同様に頂点補正を求める. 上図に対応する振幅は

$$\bar{u}(p') \delta \Gamma^\mu(p, p') u(p) = \frac{i\lambda^2}{2} \int \frac{d^4 k}{(2\pi)^4} \frac{\bar{u}(p') \left[\not{k}' \gamma^\mu \not{k} + m^2 \gamma^\mu + m(\not{k}' \gamma^\mu + \gamma^\mu \not{k}) \right] u(p)}{[(k-p)^2 - m_h^2](k'^2 - m^2)(k^2 - m^2)}$$

 で与えられる (以降は両端のスピンル $\bar{u}(p') \cdots u(p)$ は省略する).

まず, 分母を求める.

$$\ell = k + yq - zp, \quad D = \ell^2 - \Delta + i\epsilon, \quad \Delta = -xyq^2 + (1-z)^2 m^2 + zm_h^2$$

とおけば, 分母は

$$\int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{D^3} = \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{(\ell^2 - \Delta + i\epsilon)^3}.$$

次に分子を求める.

$$\begin{aligned}
 \not{k}'\gamma^\mu + \gamma^\mu\not{k} &= \not{q}\gamma^\mu + (\not{k}\gamma^\mu + \gamma^\mu\not{k}) = 2k^\mu + \not{q}\gamma^\mu = 2k^\mu + (\not{p}' - \not{p})\gamma^\mu = 2k^\mu + (m - \not{p})\gamma^\mu \\
 &= 2k^\mu + \gamma^\mu(m + \not{p}) - 2p^\mu \\
 &= 2k^\mu + 2m\gamma^\mu - 2p^\mu
 \end{aligned} \tag{6.3.8}$$

となるので, [6.3.3] と同様の計算を行えば

$$\begin{aligned}
 &\not{k}'\gamma^\mu\not{k} + m^2\gamma^\mu + m(\not{k}'\gamma^\mu + \gamma^\mu\not{k}) \\
 &= -\frac{1}{2}\gamma^\mu\ell^2 + [(1-y)\not{q} + z\not{p}]\gamma^\mu(-y\not{q} + z\not{p}) + m^2\gamma^\mu + 2mk^\mu + 2m^2\gamma^\mu - 2mp^\mu
 \end{aligned}$$

となる. 第2項は

$$\begin{aligned}
 &[(1-y)\not{q} + z\not{p}]\gamma^\mu(-y\not{q} + z\not{p}) \\
 &= [(1-y-z)\not{q} + z\not{p}']\gamma^\mu(-y\not{q} + z\not{p}) \\
 &= [(1-y-z)\not{q} + zm]\gamma^\mu(-y\not{q} + zm) \\
 &= -y(1-y-z)\not{q}\gamma^\mu\not{q} + z(1-y-z)m\not{q}\gamma^\mu - yzm\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\
 &= -y(1-y-z)(2q^\mu - \gamma^\mu\not{q})\not{q} + z(1-y-z)m(2q^\mu - \gamma^\mu\not{q}) - yzm\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\
 &= -2y(1-y-z)q^\mu\not{q} + y(1-y-z)\gamma^\mu\not{q}\not{q} + 2z(1-y-z)mq^\mu - z(1-y-z)m\gamma^\mu\not{q} \\
 &\quad - yzm\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\
 &= y(1-y-z)\gamma^\mu q^2 + 2z(1-y-z)mq^\mu - z(1-z)m\gamma^\mu\not{q} + z^2m^2\gamma^\mu \\
 &= y(1-y-z)\gamma^\mu q^2 + 2z(1-y-z)mq^\mu - z(1-z)m(2p'^\mu - 2m\gamma^\mu) + z^2m^2\gamma^\mu \\
 &= y(1-y-z)\gamma^\mu q^2 + 2z(1-y-z)mq^\mu - 2z(1-z)mp'^\mu + z(2-z)m^2\gamma^\mu \\
 &= yx\gamma^\mu q^2 + 2zxm q^\mu - 2z(1-z)mp'^\mu + z(2-z)m^2\gamma^\mu.
 \end{aligned}$$

他の項と併せて, 分子は

$$\begin{aligned}
 &= -\frac{1}{2}\gamma^\mu\ell^2 + yx\gamma^\mu q^2 + 2zxm q^\mu - 2z(1-z)mp'^\mu + z(2-z)m^2\gamma^\mu \\
 &\quad + m^2\gamma^\mu + 2mk^\mu + 2m^2\gamma^\mu - 2mp^\mu \\
 &= \left[-\frac{\ell^2}{2} + (3+2z-z^2)m^2 + xyq^2\right]\gamma^\mu + 2zxm q^\mu - 2z(1-z)m(p^\mu + q^\mu) \\
 &\quad + 2m(\ell^\mu - yq^\mu + zp^\mu) - 2mp^\mu \\
 &= \left[-\frac{\ell^2}{2} + (3+2z-z^2)m^2 + xyq^2\right]\gamma^\mu + 2(z^2-1)mp^\mu + 2(zx-z+z^2-y)m q^\mu \\
 &= \left[-\frac{\ell^2}{2} + (3+2z-z^2)m^2 + xyq^2\right]\gamma^\mu + (z^2-1)m(p^\mu + p'^\mu) + (x-y)(1+z)m q^\mu \\
 &= \left[-\frac{\ell^2}{2} + (3+2z-z^2)m^2 + xyq^2\right]\gamma^\mu + (z^2-1)m(p^\mu + p'^\mu).
 \end{aligned}$$

Gordon 恒等式を使って変形すれば,

$$\begin{aligned}
 &= \left[-\frac{\ell^2}{2} + (3+2z-z^2)m^2 + xyq^2\right]\gamma^\mu + (z^2-1)m(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu) \\
 &= \left[-\frac{\ell^2}{2} + (1+z)^2m^2 + xyq^2\right]\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m}2m^2(1-z^2).
 \end{aligned}$$

(6.33)(6.49) から

$$\begin{aligned} F_2(q^2) &= \frac{i\lambda^2}{2} \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{2}{(\ell^2 - \Delta)^3} \bar{u}(p') 2m^2(1-z^2)u(p) \\ &= \frac{\lambda^2}{2} \frac{2m^2}{(4\pi)^2} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{1-z^2}{-xyq^2 + (1-z)^2m^2 + zm_h^2}. \end{aligned}$$

(6.37) から

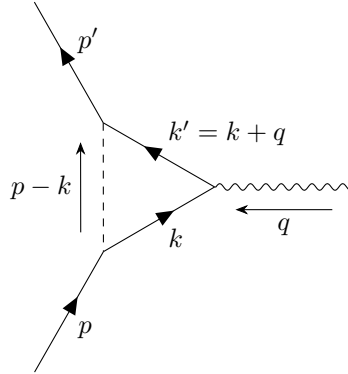
$$\begin{aligned} \frac{g-2}{2} &= F_2(q^2=0) \\ &= \frac{\lambda^2}{2} \frac{2m^2}{(4\pi)^2} \int_0^1 dx dy dz \delta(x+y+z-1) \frac{1-z^2}{(1-z)^2m^2 + zm_h^2} \\ &= \frac{\lambda^2}{(4\pi)^2} \int_0^1 dz \frac{(1-z^2)(1-z)}{(1-z)^2 + z(m_h/m)^2}. \end{aligned}$$

(c)

相互作用のハミルトニアンは

$$\mathcal{H}_{\text{int}} = \frac{i\lambda}{\sqrt{2}} a \bar{\psi} \gamma^5 \psi$$

なので, これは vertex factor が $\lambda\gamma^5/\sqrt{2}$ の QED (p.123).



p.189 以降と同様に頂点補正を求める. 上図に対応する振幅は

$$\begin{aligned} \bar{u}(p') \delta\Gamma^\mu(p, p') u(p) &= -\frac{i\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p') \gamma^5 \left[\not{k}' \gamma^\mu \not{k} + m^2 \gamma^\mu + m(\not{k}' \gamma^\mu + \gamma^\mu \not{k}) \right] \gamma^5 u(p)}{[(k-p)^2 - m_a^2](k'^2 - m^2)(k^2 - m^2)} \\ &= \frac{i\lambda^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p') \left[\not{k}' \gamma^\mu \not{k} + m^2 \gamma^\mu - m(\not{k}' \gamma^\mu + \gamma^\mu \not{k}) \right] u(p)}{[(k-p)^2 - m_a^2](k'^2 - m^2)(k^2 - m^2)} \end{aligned}$$

で与えられる (以降は両端のスピンル $\bar{u}(p') \cdots u(p)$ は省略する).

まず, 分母を求める.

$$\ell = k + yq - zp, \quad D = \ell^2 - \Delta + i\epsilon, \quad \Delta = -xyq^2 + (1-z)^2m^2 + zm_a^2$$

とおけば、分母は

$$\int_0^1 dx dy dz \delta(x + y + z - 1) \frac{2}{D^3} = \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{2}{(\ell^2 - \Delta + i\epsilon)^3}.$$

分子は

$$\begin{aligned} &= -\frac{1}{2}\gamma^\mu \ell^2 + yx\gamma^\mu q^2 + 2zxmq^\mu - 2z(1-z)mp'^\mu + z(2-z)m^2\gamma^\mu \\ &\quad + m^2\gamma^\mu - 2mk^\mu - 2m^2\gamma^\mu + 2mp^\mu \\ &= \left[-\frac{\ell^2}{2} - (z-1)^2m^2 + xyq^2\right]\gamma^\mu + 2zxmq^\mu - 2z(1-z)m(p^\mu + q^\mu) \\ &\quad - 2m(\ell^\mu - yq^\mu + zp^\mu) + 2mp^\mu \\ &= \left[-\frac{\ell^2}{2} - (z-1)^2m^2 + xyq^2\right]\gamma^\mu + 2(z-1)^2mp^\mu + 2(zx - z + z^2 + y)mq^\mu \\ &= \left[-\frac{\ell^2}{2} - (z-1)^2m^2 + xyq^2\right]\gamma^\mu + (z-1)^2m(p^\mu + p'^\mu) + (x-y)(z-1)mq^\mu \\ &= \left[-\frac{\ell^2}{2} - (z-1)^2m^2 + xyq^2\right]\gamma^\mu + (z-1)^2m(p^\mu + p'^\mu) \\ &= \left[-\frac{\ell^2}{2} - (z-1)^2m^2 + xyq^2\right]\gamma^\mu + (z-1)^2m(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu) \\ &= \left[-\frac{\ell^2}{2} + (z-1)^2m^2 + xyq^2\right]\gamma^\mu + \frac{i\sigma^{\mu\nu}q_\nu}{2m}(-2m^2)(z-1)^2. \end{aligned}$$

(6.33)(6.49) から

$$\begin{aligned} F_2(q^2) &= \frac{i\lambda^2}{2} \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{2}{(\ell^2 - \Delta)^3} \bar{u}(p')(-2m^2)(z-1)^2 u(p) \\ &= -\frac{\lambda^2}{2} \frac{2m^2}{(4\pi)^2} \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{(z-1)^2}{-xyq^2 + (1-z)^2m^2 + zm_a^2}. \end{aligned}$$

(6.37) から

$$\begin{aligned} \frac{g-2}{2} &= F_2(q^2=0) \\ &= -\frac{\lambda^2}{2} \frac{2m^2}{(4\pi)^2} \int_0^1 dx dy dz \delta(x + y + z - 1) \frac{(z-1)^2}{(1-z)^2m^2 + zm_a^2} \\ &= -\frac{\lambda^2}{(4\pi)^2} \int_0^1 dz \frac{(1-z)^3}{(1-z)^2 + z(m_a/m)^2}. \end{aligned}$$

Chapter 7

Radiative Corrections: Some Formal Developments

7.2 The LSZ Reduction formula

(7.45)

直前にある 4 点相関関数について証明する. LSZ 簡約公式 (7.42) から

$$\begin{aligned}
 & \left(\prod_1^2 \frac{\sqrt{Z}i}{p_i^2 - m^2} \right) \left(\prod_1^2 \frac{\sqrt{Z}i}{k_i^2 - m^2} \right) \langle p_1 p_2 | S | k_1 k_2 \rangle \\
 &= \left(\prod_1^2 \int d^4 x_i e^{ip_i \cdot x_i} \right) \left(\prod_1^2 \int d^4 y_i e^{ik_i \cdot y_i} \right) \langle \Omega | T \{ \phi(x_1) \phi(x_2) \phi(y_1) \phi(y_2) \} | \Omega \rangle \\
 &= \text{Diagram 1} \\
 &= \text{Diagram 2}
 \end{aligned}$$

Diagram 1: A central shaded circle with four external lines. Top-left line labeled p_1 (incoming), top-right line labeled p_2 (incoming), bottom-left line labeled k_1 (incoming), bottom-right line labeled k_2 (incoming).

Diagram 2: A central circle labeled "Amp" with four external lines. Top-left line labeled p_1 (incoming), top-right line labeled p_2 (incoming), bottom-left line labeled k_1 (incoming), bottom-right line labeled k_2 (incoming). Each of the four external lines is connected to a shaded circle.

(7.44) とその後の式から

$$= \text{Diagram 3} \times \frac{iZ}{p_1^2 - m^2} \frac{iZ}{p_2^2 - m^2} \frac{iZ}{k_1^2 - m^2} \frac{iZ}{k_2^2 - m^2}.$$

Diagram 3: A central circle labeled "Amp" with four external lines. Top-left line labeled p_1 (incoming), top-right line labeled p_2 (incoming), bottom-left line labeled k_1 (incoming), bottom-right line labeled k_2 (incoming).

以上から,

$$\langle \mathbf{p}_1 \mathbf{p}_2 | S | \mathbf{k}_1 \mathbf{k}_2 \rangle = (\sqrt{Z})^4 \text{Amp} .$$

7.3 The Optical Theorem

(7.53)

(7.52) の積分

$$\begin{aligned} i\delta\mathcal{M} &= \frac{\lambda^2}{2} \int \frac{d^3q}{(2\pi)^4} \int dq^0 \frac{1}{(q - k/2)^2 - m^2 + i\epsilon} \frac{1}{(q + k/2)^2 - m^2 + i\epsilon} \\ &= \frac{\lambda^2}{2} \int \frac{d^3q}{(2\pi)^4} \int dq^0 \frac{1}{(q^0 - k^0/2)^2 - E_{\mathbf{q}}^2 + i\epsilon} \frac{1}{(q^0 + k^0/2)^2 - E_{\mathbf{q}}^2 + i\epsilon} \end{aligned}$$

を下半分の半円上で q^0 について実行する. この際に, $q^0 = -k^0/2 + E_{\mathbf{q}} - i\epsilon$ の極の留数のみ取れば,

$$\begin{aligned} i\delta\mathcal{M} &= \frac{\lambda^2}{2} \int \frac{d^3q}{(2\pi)^4} \frac{1}{(-k^0 + E_{\mathbf{q}})^2 - E_{\mathbf{q}}^2} \frac{-2\pi i}{2E_{\mathbf{q}}} \\ &= \frac{\lambda^2}{2} \int \frac{d^3q}{(2\pi)^4} \int dq^0 \frac{1}{(q^0 - k^0/2)^2 - E_{\mathbf{q}}^2} (-2\pi i) \delta((k/2 + q)^2 - m^2) \end{aligned}$$

となる. 従って, 極を限定する操作は

$$\frac{1}{(q + k/2)^2 - m^2 + i\epsilon} \rightarrow -2\pi i \delta((k/2 + q)^2 - m^2)$$

とみなすことに等しい.

(7.55)

まず,

$$\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} = \frac{-2i\epsilon}{x^2 + \epsilon^2} = -2\pi i \delta(x)$$

である. 実際, $a < 0 < b$ で積分すれば,

$$\int_a^b \frac{1}{x^2 + \epsilon^2} dx = \frac{1}{\epsilon} \int_{a/\epsilon}^{b/\epsilon} \frac{1}{x^2 + 1} dx \approx \frac{1}{\epsilon} \int_{-\infty}^{+\infty} \frac{1}{x^2 + 1} dx = \frac{\pi}{\epsilon}$$

となる. $[a, b]$ が 0 を含まなければ積分は 0 となる.

(7.51) のあとで議論したように, $s \pm i\epsilon$ での \mathcal{M} の不連続性を計算する. ϵ は微小なので, $k^0 \pm i\epsilon$ で計算してもよい. 先ほど得られた式

$$i\delta\mathcal{M}(k^0) = \frac{\lambda^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(k^0/2 - q^0)^2 - E_{\mathbf{q}}^2} (-2\pi i) \delta((k/2 + q)^2 - m^2)$$

を使えば,

$$i \text{Disc } \mathcal{M} = i\delta\mathcal{M}(k^0 + i\epsilon) - i\delta\mathcal{M}(k^0 - i\epsilon)$$

$$\begin{aligned}
&= \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} \left[\frac{1}{(k^0/2 - q^0 + i\epsilon/2)^2 - E_{\mathbf{q}}^2} - \frac{1}{(k^0/2 - q^0 - i\epsilon/2)^2 - E_{\mathbf{q}}^2} \right] \\
&\quad \times (-2\pi i) \delta((k/2 + q)^2 - m^2) \\
&= \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} \left[\frac{1}{(k^0/2 - q^0 + i\epsilon/2)^2 - E_{\mathbf{q}}^2} - \frac{1}{(k^0/2 - q^0 - i\epsilon/2)^2 - E_{\mathbf{q}}^2} \right] \\
&\quad \times (-2\pi i) \delta((k/2 + q)^2 - m^2) \\
&= \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} \left[\frac{1}{(k^0/2 - q^0)^2 - E_{\mathbf{q}}^2 + i\epsilon} - \frac{1}{(k^0/2 - q^0)^2 - E_{\mathbf{q}}^2 - i\epsilon} \right] \\
&\quad \times (-2\pi i) \delta((k/2 + q)^2 - m^2) \\
&= \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} (-2\pi i) \delta((k^0/2 - q^0)^2 - E_{\mathbf{q}}^2) (-2\pi i) \delta((k/2 + q)^2 - m^2) \\
&= \frac{\lambda^2}{2} \int \frac{d^4 q}{(2\pi)^4} (-2\pi i) \delta((k/2 - q)^2 - m^2) (-2\pi i) \delta((k/2 + q)^2 - m^2)
\end{aligned}$$

となる。従って、不連続値を求める操作は

$$\frac{1}{(q - k/2)^2 - m^2 + i\epsilon} \rightarrow -2\pi i \delta((k/2 - q)^2 - m^2)$$

とみなすことに等しい。

(7.58)~(7.59)

m の定義は

$$m^2 - m_0^2 - \text{Re } M^2(m^2) = 0.$$

(7.44) と同様に、修正した伝播関数は

$$\frac{i}{p^2 - m_0^2 - \text{Re } M^2(p^2)} \sim \frac{iZ}{p^2 - m^2}$$

で与えられる。従って、

$$\begin{aligned}
\frac{i}{p^2 - m_0^2 - M^2(p^2)} &= \frac{i}{p^2 - m_0^2 - \text{Re } M^2(p^2) - i \text{Im } M^2(p^2)} \\
&= \frac{iZ}{p^2 - m^2 - iZ \text{Im } M^2(p^2)}.
\end{aligned}$$

Problems

Problem 6.3(a) の Yukawa vertex factor を求める際に dimensional regularization をする。これにより、 $\not{\ell} \gamma^\mu \not{\ell}$ の計算結果が変化する。(6.46) から

$$\not{\ell} \gamma^\mu \not{\ell} = \ell_a \ell_b \gamma^a \gamma^\mu \gamma^b \rightarrow \frac{\ell^2}{d} g_{ab} \gamma^a \gamma^\mu \gamma^b = \frac{\ell^2}{d} \gamma^a \gamma^\mu \gamma_a = \frac{2-d}{d} \ell^2 \gamma^\mu.$$

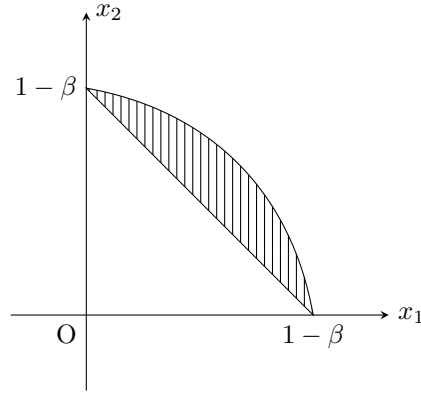
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$\mu^2/s = \beta$ とする. 被積分函数は

$$\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} + \left[\frac{2(x_1 + x_2)}{(1-x_1)(1-x_2)} - \frac{1}{(1-x_1)^2} - \frac{1}{(1-x_2)^2} \right] \beta + \frac{2}{(1-x_1)(1-x_2)} \beta^2.$$

積分範囲は (b) で求めた次の通り.



β の 1 乗以上の項は無視する. 被積分函数のうち β を含まない項は

$$\begin{aligned} I_0 &= \int_0^{1-\beta} dx_1 \int_{1-x_1-\beta}^{1-\beta/(1-x_1)} dx_2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \\ &= \int \frac{dx_1}{1-x_1} \int \frac{x_2^2 + x_1^2}{1-x_2} dx_2 \\ &= \int \frac{dx_1}{1-x_1} \int \left(-x_2 - 1 - \frac{x_1^2 + 1}{x_2 - 1} \right) dx_2 \\ &= \int \frac{dx_1}{1-x_1} \frac{1}{2} \left[-2(x_1^2 + 1) \log(1-x_2) - x_2(x_2 + 2) \right]_{1-x_1-\beta}^{1-\beta/(1-x_1)} \\ &= \int \frac{dx_1}{1-x_1} \frac{1}{2} \left[-2(x_1^2 + 1) \log \frac{\beta}{1-x_1} + 2(x_1^2 + 1) \log(x_1 + \beta) \right] \\ &\quad + \int \frac{dx_1}{1-x_1} \frac{1}{2} \left[- \left(1 - \frac{\beta}{1-x_1} \right) \left(3 - \frac{\beta}{1-x_1} \right) + (1-x_1-\beta)(3-x_1-\beta) \right] \\ &\approx \int \frac{dx_1}{1-x_1} \left[(x_1^2 + 1) \log(1-x_1) - (x_1^2 + 1) \log \beta + (x_1^2 + 1) \log(x_1 + \beta) \right] \\ &\quad + \int \frac{dx_1}{1-x_1} \frac{x_1^2 - 4x_1}{2} + \int \frac{dx_1}{1-x_1} \left[\frac{2\beta}{1-x_1} - \frac{1}{2} \frac{\beta^2}{(1-x_1)^2} \right]. \end{aligned}$$

ここで、 $(4 - 2x_1)\beta + \beta^2$ は積分すれば β^1 以上の次数となるため、無視した。計算を継続する：

$$\begin{aligned}
&= \int_0^{1-\beta} \frac{x_1^2 + 1}{1 - x_1} \log(1 - x_1) dx_1 - \log \beta \int_0^{1-\beta} \frac{x_1^2 + 1}{1 - x_1} dx_1 \\
&\quad + \int_0^{1-\beta} \frac{x_1^2 + 1}{1 - x_1} \log(x_1 + \beta) dx_1 + \frac{1}{2} \int_0^{1-\beta} \frac{x_1^2 - 4x_1}{1 - x_1} dx_1 \\
&\quad + \int_0^{1-\beta} \left[\frac{2\beta}{(1 - x_1)^2} - \frac{1}{2} \frac{\beta^2}{(1 - x_1)^3} \right] dx_1.
\end{aligned}$$

第1項：

$$\begin{aligned}
\int_0^{1-\beta} \frac{x_1^2 + 1}{1 - x_1} \log(1 - x_1) dx_1 &= \int_\beta^1 \frac{x^2 - 2x + 2}{x} \log x dx \\
&= \left[2x - \frac{x^2}{4} + \frac{x(x - 4)}{2} \log x + \log^2 x \right]_\beta^1 \\
&\approx \frac{7}{4} - \log^2 \beta.
\end{aligned}$$

第2項：

$$\begin{aligned}
\log \beta \int_0^{1-\beta} \frac{x_1^2 + 1}{1 - x_1} dx_1 &= \log \beta \int_\beta^1 \frac{x^2 - 2x + 2}{x} dx \\
&= \log \beta \left[\frac{x^2}{2} - 2x + 2 \log x \right]_\beta^1 \\
&\approx -\frac{3}{2} \log \beta - 2 \log^2 \beta.
\end{aligned}$$

第3項：

$$\begin{aligned}
&\int_0^{1-\beta} \frac{x_1^2 + 1}{1 - x_1} \log(x_1 + \beta) dx_1 \\
&= \int_\beta^1 \frac{x^2 - 2x + 2}{x} \log(1 + \beta - x) dx \\
&= \left[\frac{(1 + \beta - x)^2}{2} \log(1 + \beta - x) \right]_\beta^1 + (1 + \beta) \int_\beta^1 \log(1 + \beta - x) dx + \frac{1}{2} \int_\beta^1 (1 + \beta - x) dx \\
&\quad - 2 \int_\beta^1 \log(1 + \beta - x) dx \\
&\quad + 2 \int_\beta^1 \frac{\log(1 + \beta - x)}{x} dx \\
&\approx - \int_\beta^1 \log(1 + \beta - x) dx + \frac{1}{2} \int_\beta^1 (1 + \beta - x) dx + 2 \int_\beta^1 \frac{\log(1 + \beta - x)}{x} dx \\
&= \left[(1 + \beta - x) \log(1 + \beta - x) + x - \frac{1}{4} (1 + \beta - x)^2 \right]_\beta^1 + 2 \int_\beta^1 \frac{\log(1 + \beta - x)}{x} dx \\
&\approx \frac{5}{4} + 2 \int_\beta^1 \frac{\log(1 + \beta - x)}{x} dx \\
&\approx \frac{5}{4} + 2 \int_{\beta/(1+\beta)}^{1/(1+\beta)} \frac{\log(1 + \beta) + \log(1 - x)}{x} dx
\end{aligned}$$

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$$\begin{aligned}
&\approx \frac{5}{4} + 2 \int_{\beta/(1+\beta)}^{1/(1+\beta)} \frac{\log(1-x)}{x} dx \\
&\approx \frac{5}{4} + 2 \int_0^1 \frac{\log(1-x)}{x} dx \\
&= \frac{5}{4} - \frac{\pi^2}{3}.
\end{aligned}$$

第4項：

$$\begin{aligned}
\frac{1}{2} \int_0^{1-\beta} \frac{x_1^2 - 4x_1}{1-x_1} dx_1 &= \frac{1}{2} \int_0^{1-\beta} \left(3 - x - \frac{3}{1-x} \right) dx \\
&= \frac{1}{2} \left[3x - \frac{x^2}{2} + 3 \log(1-x) \right]_0^{1-\beta} \\
&= \frac{5}{4} + \frac{3}{2} \log \beta.
\end{aligned}$$

第5項：

$$\begin{aligned}
\int_0^{1-\beta} \left[\frac{2\beta}{(1-x_1)^2} - \frac{1}{2} \frac{\beta^2}{(1-x_1)^3} \right] dx_1 &= \left[\frac{2\beta}{1-x_1} - \frac{1}{4} \frac{\beta^2}{(1-x_1)^2} \right]_0^{1-\beta} \\
&\approx \frac{7}{4}.
\end{aligned}$$

以上から、

$$\begin{aligned}
I_0 &= \frac{7}{4} - \log^2 \beta - \left(-\frac{3}{2} \log \beta - 2 \log^2 \beta \right) + \frac{5}{4} - \frac{\pi^2}{3} + \frac{5}{4} + \frac{3}{2} \log \beta + \frac{7}{4} \\
&= 6 + 3 \log \beta + \log^2 \beta - \frac{\pi^2}{3}.
\end{aligned} \tag{I.1}$$

被積分函数のうち β^1 を含む項は

$$\begin{aligned}
I_1 &= \beta \int_0^{1-\beta} dx_1 \int_{1-x_1-\beta}^{1-\beta/(1-x_1)} dx_2 \left[\frac{2(x_1+x_2)}{(1-x_1)(1-x_2)} - \frac{1}{(1-x_1)^2} - \frac{1}{(1-x_2)^2} \right] \\
&= \beta \int dx_1 \int dx_2 \left[\frac{2(x_1+x_2)}{(1-x_1)(1-x_2)} - \frac{1}{(1-x_1)^2} - \frac{1}{(1-x_2)^2} \right] \\
&= \beta \int dx_1 \int dx_2 \left[\frac{4-2(1-x_1)-2(1-x_2)}{(1-x_1)(1-x_2)} - \frac{1}{(1-x_1)^2} - \frac{1}{(1-x_2)^2} \right] \\
&= \beta \int dx_1 \int dx_2 \left[\frac{4}{(1-x_1)(1-x_2)} - \frac{2}{1-x_1} - \frac{2}{1-x_2} - \frac{1}{(1-x_1)^2} - \frac{1}{(1-x_2)^2} \right] \\
&= \beta \int dx_1 \int dx_2 \left[\frac{4}{(1-x_1)(1-x_2)} - \frac{4}{1-x_1} - \frac{2}{(1-x_1)^2} \right],
\end{aligned}$$

計算の最後で、積分が x_1, x_2 に関して対称であることを用いた、第1項：

$$\begin{aligned}
&4\beta \int_0^{1-\beta} \frac{dx_1}{1-x_1} \int_{1-x_1-\beta}^{1-\beta/(1-x_1)} \frac{dx_2}{1-x_2} \\
&= 4\beta \int_0^{1-\beta} [-\log(1-x_2)]_{1-x_1-\beta}^{1-\beta/(1-x_1)} \\
&= 4\beta \int_0^{1-\beta} \left[\log(x_1+\beta) - \log \frac{\beta}{1-x_1} \right]
\end{aligned}$$

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$$\begin{aligned}
&= 4\beta \int_0^{1-\beta} \frac{\log(x_1 + \beta)}{1-x_1} dx_1 - 4\beta \log \beta \int_0^{1-\beta} \frac{dx_1}{1-x_1} + 4\beta \int_0^{1-\beta} \frac{\log(1-x_1)}{1-x_1} dx_1 \\
&= 4\beta \int_\beta^1 \frac{\log(1+\beta-x)}{x} dx + 4\beta \log \beta [\log(1-x_1)]_0^{1-\beta} + 2\beta [\log^2(1-x_1)]_0^{1-\beta} \\
&= 4\beta \int_\beta^1 \frac{\log(1+\beta-x)}{x} dx + 4\beta \log^2 \beta + 2\beta \log^2 \beta
\end{aligned}$$

初めの積分については、 I_0 の第 3 項と同様に計算すればよいので、

$$\begin{aligned}
&\approx 4\beta \left(-\frac{\pi^2}{6} \right) + 4\beta \log^2 \beta + 2\beta \log^2 \beta \\
&\approx 0.
\end{aligned}$$

第 2 項：

$$\begin{aligned}
4\beta \int_0^{1-\beta} \frac{dx_1}{1-x_1} \int_{1-x_1-\beta}^{1-\beta/(1-x_1)} dx_2 &= 4\beta \int_0^{1-\beta} \frac{dx_1}{1-x_1} \left(x_1 + \beta - \frac{\beta}{1-x_1} \right) \\
&= 4\beta \int_0^{1-\beta} dx_1 \left[-1 + \frac{\beta+1}{1-x_1} - \frac{\beta}{(1-x_1)^2} \right] \\
&\approx 0.
\end{aligned}$$

第 3 項：

$$\begin{aligned}
2\beta \int_0^{1-\beta} \frac{dx_1}{(1-x_1)^2} \int_{1-x_1-\beta}^{1-\beta/(1-x_1)} dx_2 &= 2\beta \int_0^{1-\beta} \frac{dx_1}{(1-x_1)^2} \left(x_1 + \beta - \frac{\beta}{1-x_1} \right) \\
&= 2\beta \int_0^{1-\beta} dx_1 \left[\frac{1}{x_1-1} + \frac{\beta+1}{(1-x_1)^2} - \frac{\beta}{(1-x_1)^3} \right] \\
&= 2\beta \left[\log(1-x_1) + \frac{\beta+1}{1-x_1} - \frac{\beta}{2(1-x_1)^2} \right]_0^{1-\beta} \\
&= 2\beta \left[\log \beta + \frac{1}{\beta} - \beta - \frac{1}{2\beta} + \frac{\beta}{2} \right] \\
&\approx 1.
\end{aligned}$$

以上から、

$$I_1 \approx 0 - 0 - 1 = -1. \quad [I.2]$$

I_1 の第 1 項の計算結果から、積分のうち β^2 を含む項は 0 である。[I.1][I.2] から、積分は

$$I_0 + I_1 = 5 + 3 \log \beta + \log^2 \beta - \frac{\pi^2}{3}.$$

(e)

(a) で得られた Feynman パラメータの積分

$$\int_0^1 dx \int_0^{1-x} dz \left[\log \frac{z\beta}{z\beta - x(1-x-z)} + \frac{(1-x)(x+z)}{z\beta - x(1-x-z)} \right]$$

を計算する。

発散しない項：

$$\begin{aligned}
 \int_0^1 dx \int_0^{1-x} dz \log(z\beta) &= \int_0^1 dx \int_0^{1-x} dz (\log z + \log \beta) \\
 &= \int_0^1 dx [z \log z - z + z \log \beta]_0^{1-x} \\
 &= \int_0^1 dx [(1-x) \log(1-x) - (1-x) + (1-x) \log \beta] \\
 &= \int_0^1 dx [x \log x - x + x \log \beta] \\
 &= \left[\frac{x^2}{2} \log x - \frac{x^2}{4} - \frac{x^2}{2} + \frac{x^2}{2} \log \beta \right]_0^1 \\
 &= \frac{1}{2} \log \beta - \frac{3}{4}.
 \end{aligned} \tag{I.3}$$

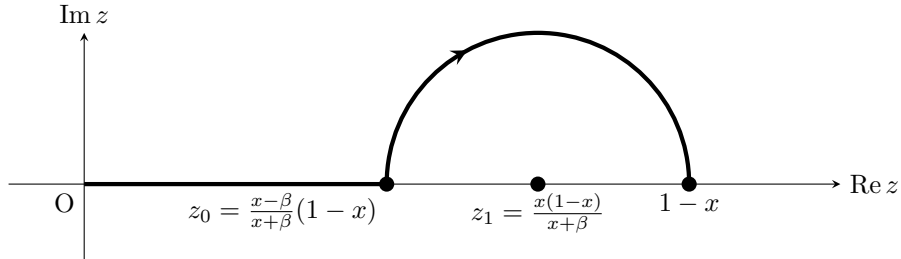
簡単のため,

$$F(z) = z\beta - x(1-x-z) = (x+\beta)z - x(1-x)$$

とおく. $\log F(z)$ には特異点

$$z_1 = \frac{x-\beta}{x+\beta}(1-x)$$

が存在するので, それを回避するように積分する.



線分での積分は*1

$$\begin{aligned}
 &\int_{F=-x(1-x)}^{F=-\beta(1-x)} dz \log F(z) \\
 &= \int_{-x(1-x)}^{-\beta(1-x)} dF \frac{dz}{dF} \log F \\
 &= \frac{1}{x+\beta} \int_{\beta(1-x)}^{x(1-x)} dr (\log r + i\pi) \\
 &= i\pi \frac{x-\beta}{x+\beta}(1-x) + \frac{1}{x+\beta} \int_{\beta(1-x)}^{x(1-x)} \log r \, dr \\
 &= i\pi \frac{x-\beta}{x+\beta}(1-x) + \frac{1}{x+\beta} [r \log r - r]_{\beta(1-x)}^{x(1-x)} \\
 &= i\pi \frac{x-\beta}{x+\beta}(1-x) + \frac{x(1-x)}{x+\beta} [\log x(1-x) - 1] - \beta \frac{1-x}{x+\beta} [\log \beta(1-x) - 1]
 \end{aligned}$$

*1 積分結果のうち実部しか使わないので, 分岐はどう選んでも結構

$$\begin{aligned}
&= i\pi \frac{x-\beta}{x+\beta} (1-x) + \frac{x(1-x)}{x+\beta} [\log x + \log(1-x) - 1] - \beta \frac{1-x}{x+\beta} [\log \beta + \log(1-x) - 1] \\
&= (i\pi - 1) \frac{x-\beta}{x+\beta} (1-x) + \frac{x-\beta}{x+\beta} (1-x) \log(1-x) + \frac{x(1-x)}{x+\beta} \log x - \beta \log \beta \frac{1-x}{x+\beta}.
\end{aligned}$$

半円

$$z = z_1 + \frac{1-x}{x+\beta} \beta e^{i\theta} \quad (\theta: \pi \rightarrow 0), \quad F = (x+\beta)(z-z_1) = \beta(1-x)e^{i\theta}$$

での積分は

$$\begin{aligned}
\int_{\theta=\pi}^{\theta=0} dz \log F &= \int_{\pi}^0 d\theta \frac{dz}{d\theta} \log \beta(1-x)e^{i\theta} \\
&= i\beta \frac{1-x}{x+\beta} \int_{\pi}^0 d\theta e^{i\theta} [i\theta + \log \beta(1-x)] \\
&= \beta \frac{1-x}{x+\beta} \int_0^{\pi} d\theta \theta e^{i\theta} + i\beta \frac{1-x}{x+\beta} \log \beta(1-x) \int_{\pi}^0 d\theta e^{i\theta} \\
&= \beta \frac{1-x}{x+\beta} [(1-i\theta)e^{i\theta}]_0^{\pi} + i\beta \frac{1-x}{x+\beta} \log \beta(1-x) [-ie^{i\theta}]_{\pi}^0 \\
&= \beta \frac{1-x}{x+\beta} [(1-i\theta)e^{i\theta}]_0^{\pi} + i\beta \frac{1-x}{x+\beta} \log \beta(1-x) [-ie^{i\theta}]_{\pi}^0 \\
&= \beta \frac{1-x}{x+\beta} [i\pi - 2] + i\beta \frac{1-x}{x+\beta} \log \beta(1-x) [-2i] \\
&= (i\pi - 2 + 2\log \beta) \beta \frac{1-x}{x+\beta} + 2\beta \frac{1-x}{x+\beta} \log(1-x)
\end{aligned}$$

従って、積分は

$$\begin{aligned}
\int_0^{1-x} dz \log \frac{1}{F} &= -(i\pi - 1) \frac{x-\beta}{x+\beta} (1-x) - \frac{x-\beta}{x+\beta} (1-x) \log(1-x) - \frac{x(1-x)}{x+\beta} \log x + \beta \log \beta \frac{1-x}{x+\beta} \\
&\quad - (i\pi - 2 + 2\log \beta) \beta \frac{1-x}{x+\beta} - 2\beta \frac{1-x}{x+\beta} \log(1-x) \\
&= -(i\pi - 2 + \log \beta) \beta \frac{1-x}{x+\beta} - (i\pi - 1) \frac{x-\beta}{x+\beta} (1-x) \\
&\quad - (1-x) \log(1-x) - \frac{x(1-x)}{x+\beta} \log x.
\end{aligned}$$

これを $x: 0 \rightarrow 1$ で積分する。第1項：

$$\begin{aligned}
-(i\pi - 2 + \log \beta) \beta \int_0^1 dx \frac{1-x}{x+\beta} &= -(i\pi - 2 + \log \beta) \beta \int_0^1 dx \left(-1 + \frac{1+\beta}{x+\beta} \right) \\
&= -(i\pi - 2 + \log \beta) \beta [-x + (1+\beta) \log(x+\beta)]_0^1 \\
&= -(i\pi - 2 + \log \beta) \beta \left[-1 + (1+\beta) \log \frac{1+\beta}{\beta} \right] \\
&\approx 0.
\end{aligned}$$

第2項：

$$\begin{aligned}
-(i\pi - 1) \int_0^1 dx \frac{x-\beta}{x+\beta} (1-x) &= -(i\pi - 1) \int_0^1 dx \frac{-(x+\beta)(x-1-2\beta) - 2\beta(1+\beta)}{x+\beta} \\
&= -(i\pi - 1) \int_0^1 dx \left[-x + 1 + 2\beta - 2\beta(1+\beta) \frac{1}{x+\beta} \right]
\end{aligned}$$

$$\begin{aligned}
&= -(i\pi - 1) \left[-\frac{x^2}{2} + (1 + 2\beta)x - 2\beta(1 + \beta) \log(x + \beta) \right]_0^1 \\
&= -(i\pi - 1) \left[-\frac{1}{2} + (1 + 2\beta) - 2\beta(1 + \beta) \log \frac{1 + \beta}{\beta} \right] \\
&= -(i\pi - 1) \left[-\frac{1}{2} + (1 + 2\beta) - 2\beta(1 + \beta) \log \frac{1 + \beta}{\beta} \right] \\
&\approx \frac{1}{2}(1 - i\pi).
\end{aligned}$$

第3項：

$$\begin{aligned}
-\int_0^1 dx (1-x) \log(1-x) &= -\int_0^1 dx x \log x \\
&= -\left[\frac{x^2}{2} \log x - \frac{x^2}{4} \right]_0^1 \\
&= \frac{1}{4}.
\end{aligned}$$

第4項：

$$\begin{aligned}
-\int_0^1 dx \frac{x(1-x)}{x+\beta} \log x &= \int_0^1 dx \left[x - \beta - 1 + \frac{\beta(\beta+1)}{x+\beta} \right] \log x \\
&= \int_0^1 x \log x dx - (\beta+1) \int_0^1 \log x dx + \beta(\beta+1) \int_0^1 \frac{\log x}{x+\beta} dx \\
&= -\frac{1}{4} + (\beta+1) + \beta(\beta+1) \int_0^1 \frac{\log x}{x+\beta} dx \\
&\approx \frac{3}{4}.
\end{aligned}$$

以上から,

$$\int_0^1 dx \int_0^{1-x} dz \log \frac{1}{F} = 0 + \frac{1}{2}(1 - i\pi) + \frac{1}{4} + \frac{3}{4} = \frac{3 - i\pi}{2}. \quad [\text{I.4}]$$

有理多項式の項も極 z_1 が存在するので, 先程と同じ経路で積分する. 線分での積分は

$$\begin{aligned}
\int_0^{z_0} dz \frac{(1-x)(x+z)}{z\beta - x(1-x-z)} &= \frac{1-x}{x+\beta} \int_0^{z_0} \frac{z+x}{z-z_1} dz \\
&= \frac{1-x}{x+\beta} \int_0^{z_0} \left(1 + \frac{x+z_1}{z-z_1} \right) dz \\
&= \frac{1-x}{x+\beta} \left[z_0 + (x+z_1) \log \left| \frac{z_0 - z_1}{z_1} \right| \right] \\
&= \frac{1-x}{x+\beta} \left[\frac{x-\beta}{x+\beta} (1-x) + (1+\beta) \frac{x}{x+\beta} \log \frac{\beta}{x} \right] \\
&= \left(\frac{1-x}{x+\beta} \right)^2 (x-\beta) + (1+\beta) \log \beta \frac{x(1-x)}{(x+\beta)^2} - (1+\beta) \frac{x(1-x)}{(x+\beta)^2} \log x.
\end{aligned}$$

半円

$$z = z_1 + \frac{1-x}{x+\beta} \beta e^{i\theta} \quad (\theta: \pi \rightarrow 0)$$

での積分は

$$\begin{aligned}
\int dz \frac{(1-x)(x+z)}{z\beta - x(1-x-z)} &= \frac{1-x}{x+\beta} \int_{\theta=\pi}^{\theta=0} \frac{z+x}{z-z_1} \frac{dz}{d\theta} d\theta \\
&= \frac{1-x}{x+\beta} \int_{\pi}^0 i(z+x) d\theta \\
&= i \frac{1-x}{x+\beta} \int_{\pi}^0 \left[(1+\beta) \frac{x}{x+\beta} + \beta \frac{1-x}{x+\beta} e^{i\theta} \right] d\theta \\
&= i \frac{1-x}{x+\beta} \left[-\pi(1+\beta) \frac{x}{x+\beta} - 2i\beta \frac{1-x}{x+\beta} \right] \\
&= -i\pi(1+\beta) \frac{x(1-x)}{(x+\beta)^2} + 2\beta \left(\frac{1-x}{x+\beta} \right)^2.
\end{aligned}$$

以上から,

$$\begin{aligned}
&\int_0^{1-x} dz \frac{(1-x)(x+z)}{z\beta - x(1-x-z)} \\
&= \left(\frac{1-x}{x+\beta} \right)^2 (x-\beta) + (1+\beta) \log \beta \frac{x(1-x)}{(x+\beta)^2} - (1+\beta) \frac{x(1-x)}{(x+\beta)^2} \log x \\
&\quad - i\pi(1+\beta) \frac{x(1-x)}{(x+\beta)^2} + 2\beta \left(\frac{1-x}{x+\beta} \right)^2 \\
&= \frac{(1-x)^2}{x+\beta} + (1+\beta)(\log \beta - i\pi) \frac{x(1-x)}{(x+\beta)^2} - (1+\beta) \frac{x(1-x)}{(x+\beta)^2} \log x.
\end{aligned}$$

これを $x: 0 \rightarrow 1$ で積分する. 第1項:

$$\begin{aligned}
\int_0^1 dx \frac{(1-x)^2}{x+\beta} &= \int_{\beta}^{1+\beta} \frac{(1+\beta-x)^2}{x} \\
&= \int_{\beta}^{1+\beta} \frac{x^2 - 2(1+\beta)x + (1+\beta)^2}{x} \\
&= \left[\frac{x^2}{2} - 2(1+\beta)x + (1+\beta)^2 \log x \right]_{\beta}^{1+\beta} \\
&= \frac{1+2\beta}{2} - 2(1+\beta) + (1+\beta)^2 \log \frac{1+\beta}{\beta} \\
&\approx -\frac{3}{2} - \log \beta.
\end{aligned}$$

第2項:

$$\begin{aligned}
(1+\beta)(\log \beta - i\pi) \int_0^1 dx \frac{x(1-x)}{(x+\beta)^2} &= (1+\beta)(\log \beta - i\pi) \int_{\beta}^{1+\beta} \frac{(x-\beta)(1+\beta-x)}{x^2} \\
&= (1+\beta)(\log \beta - i\pi) \int_{\beta}^{1+\beta} \frac{-x^2 + (1+2\beta)x - \beta(1+\beta)}{x^2} \\
&= (1+\beta)(\log \beta - i\pi) \left[-x + (1+2\beta) \log x + \beta(1+\beta) \frac{1}{x} \right]_{\beta}^{1+\beta} \\
&= (1+\beta)(\log \beta - i\pi) \left[-1 + (1+2\beta) \log \frac{1+\beta}{\beta} - 1 \right] \\
&\approx (\log \beta - i\pi) [-2 - \log \beta]
\end{aligned}$$

$$= -2 \log \beta - \log^2 \beta + i\pi(2 + \log \beta)$$

第 3 項：

$$\begin{aligned} & - \int_0^1 dx \frac{x(1-x)}{(x+\beta)^2} \log x \\ &= - \int_\beta^{1+\beta} dx \frac{(x-\beta)(1+\beta-x)}{x^2} \log(x-\beta) \\ &= - \int_\beta^{1+\beta} dx \frac{-x^2 + (1+2\beta)x - \beta(1+\beta)}{x^2} \log(x-\beta) \\ &= \int_\beta^{1+\beta} dx \log(x-\beta) - (1+2\beta) \int_\beta^{1+\beta} dx \frac{\log(x-\beta)}{x} + \beta(1+\beta) \int_\beta^{1+\beta} dx \frac{\log(x-\beta)}{x^2}. \end{aligned}$$

第 3 項の第 1 項：

$$\int_\beta^{1+\beta} dx \log(x-\beta) = \int_0^1 dx \log x = -1.$$

第 3 項の第 2 項：

$$\begin{aligned} \int_\beta^{1+\beta} dx \frac{\log(x-\beta)}{x} &= \int_1^{1+1/\beta} dx \frac{\log \beta(x-1)}{x} \\ &= \int_1^{1+1/\beta} dx \frac{\log \beta + \log(x-1)}{x} \\ &= \log \beta \int_1^{1+1/\beta} \frac{dx}{x} + \int_1^{1+1/\beta} dx \frac{\log(x-1)}{x} \\ &= \log \beta \log \frac{1+\beta}{\beta} + \int_1^{\beta/(1+\beta)} \left(-\frac{dx}{x^2} \right) x \log \left(\frac{1}{x} - 1 \right) \\ &\approx -\log^2 \beta + \int_{\beta/(1+\beta)}^1 dx \frac{\log(1-x) - \log x}{x} \\ &= -\log^2 \beta + \int_{\beta/(1+\beta)}^1 dx \frac{\log(1-x)}{x} - \int_{\beta/(1+\beta)}^1 dx \frac{\log x}{x} \\ &\approx -\log^2 \beta - \frac{\pi^2}{6} - \left[\frac{1}{2} \log^2 x \right]_{\beta/(1+\beta)}^1 \\ &\approx -\log^2 \beta - \frac{\pi^2}{6} + \frac{1}{2} \log^2 \frac{\beta}{1+\beta} \\ &\approx -\frac{1}{2} \log^2 \beta - \frac{\pi^2}{6}. \end{aligned}$$

第 3 項の第 3 項：

$$\begin{aligned} \beta(1+\beta) \int_\beta^{1+\beta} dx \frac{\log(x-\beta)}{x^2} &= (1+\beta) \int_1^{1+1/\beta} dx \frac{\log \beta(x-1)}{x^2} \\ &= (1+\beta) \int_1^{1+1/\beta} dx \frac{\log \beta + \log(x-1)}{x^2} \\ &= (1+\beta) \log \beta \int_1^{1+1/\beta} \frac{dx}{x^2} + (1+\beta) \int_1^{1+1/\beta} dx \frac{\log(x-1)}{x^2} \end{aligned}$$

$$\begin{aligned}
&= \log \beta + (1 + \beta) \int_1^{\beta/(1+\beta)} \left(-\frac{dx}{x^2} \right) x^2 \log \left(\frac{1}{x} - 1 \right) \\
&= \log \beta + (1 + \beta) \int_{\beta/(1+\beta)}^1 dx \log \frac{1-x}{x} \\
&= \log \beta + (1 + \beta) \int_{\beta/(1+\beta)}^1 dx \log(1-x) - (1 + \beta) \int_{\beta/(1+\beta)}^1 dx \log x \\
&= \log \beta + (1 + \beta) \int_0^{1/(1+\beta)} dx \log x - (1 + \beta) \int_{\beta/(1+\beta)}^1 dx \log x \\
&= \log \beta + (1 + \beta) [x \log x - x]_0^{1/(1+\beta)} - (1 + \beta) [x \log x - x]_{\beta/(1+\beta)}^1 \\
&\approx \log \beta.
\end{aligned}$$

従って、第 3 項は

$$- \int_0^1 dx \frac{x(1-x)}{(x+\beta)^2} \log x = -1 + \frac{1}{2} \log^2 \beta + \frac{\pi^2}{6} + \log \beta.$$

以上から、 x での積分結果は

$$\begin{aligned}
&\int_0^{1-x} dz \frac{(1-x)(x+z)}{z\beta - x(1-x-z)} \\
&= -\frac{3}{2} - \log \beta - 2 \log \beta - \log^2 \beta + i\pi(2 + \log \beta) - 1 + \frac{1}{2} \log^2 \beta + \frac{\pi^2}{6} + \log \beta \quad [\text{I.5}] \\
&= -\frac{5}{2} - \frac{1}{2} \log^2 \beta - 2 \log \beta + \frac{\pi^2}{6} + i\pi(2 + \log \beta).
\end{aligned}$$

[I.3][I.4][I.5] から、積分は

$$\begin{aligned}
&\frac{1}{2} \log \beta - \frac{3}{4} + \frac{3-i\pi}{2} - \frac{5}{2} - \frac{1}{2} \log^2 \beta - 2 \log \beta + \frac{\pi^2}{6} + i\pi(2 + \log \beta) \\
&= -\frac{7}{4} - \frac{1}{2} \log^2 \beta - \frac{3}{2} \log \beta + \frac{\pi^2}{6} + \frac{i\pi}{2} (3 + 2 \log \beta).
\end{aligned}$$

Part II

Renormalization

Chapter 9

Functional Methods

9.5 Functional Quantization of Spinor Fields

Functional determinant

$i\not{D} - m$ の固有値と固有ベクトルを

$$(i\not{D} - m)\psi_i = b_i\psi_i$$

とする. (9.76) を Wick 回転して

$$\begin{aligned} \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[- \int d^4x \bar{\psi} (i\not{D} - m) \psi \right] &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp \left[- \sum_i \int d^4x \bar{\psi}_i b_i \psi_i \right] \\ &= \prod_i \int \mathcal{D}\bar{\psi}_i \mathcal{D}\psi_i \exp \left[- \int d^4x \bar{\psi}_i b_i \psi_i \right] \\ &= \prod_i \prod_x \int d\bar{\psi}_i(x) d\psi_i(x) \exp \left[- \bar{\psi}_i(x) b_i(x) \psi_i(x) \right] \\ &= \prod_i \prod_x b_i(x). \end{aligned}$$

最後に (9.67) を使った. これは $i\not{D} - m$ の固有値の積なので, $\det(i\not{D} - m)$ と表せる.

Problems

Problem 9.1: Scalar QED

伝播関数を計算しておく. スカラー粒子は

$$\begin{aligned} (-\partial^2 - m^2 + i\epsilon)D_F(x-y) &= (-\partial^2 - m^2 + i\epsilon) \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \\ &= i \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \\ &= i \delta^{(4)}(x-y). \end{aligned}$$

光子は (9.58) から

$$(\partial^2 g_{\mu\nu} - i\epsilon g_{\mu\nu})_x D_F^{\nu\rho}(x-y) = (\partial^2 g_{\mu\nu} - i\epsilon g_{\mu\nu})_x \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2 + i\epsilon} g^{\nu\rho} e^{-ik(x-y)}$$

$$= i\delta_\mu{}^\rho \delta^{(4)}(x-y).$$

相互作用を含むラグランジアンは

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - m^2\phi^*\phi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\phi^* - ieA_\mu\phi^*)(\partial_\mu\phi + ieA_\mu\phi) - m^2\phi^*\phi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu\phi^*)(\partial_\mu\phi) + ieA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) - m^2\phi^*\phi + e^2A^2|\phi|^2 \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^*(\partial^2 + m^2)\phi + ieA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + e^2A^2|\phi|^2\end{aligned}$$

(最後は部分積分を行い、表面項を無視した)。このうち、自由場は

$$\mathcal{L}_0 = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \phi^*(\partial^2 + m^2)\phi.$$

自由場の生成関数を次のように定義する：

$$\begin{aligned}Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*] &= \int \mathcal{D}A \int \mathcal{D}\phi \int \mathcal{D}\phi^* \exp \left[i \int d^4x (\mathcal{L}_0 + A_\mu J_{\text{em}}^\mu - J_s^* \phi + \phi^* J_s) \right] \\ &= \int \mathcal{D}\phi \int \mathcal{D}\phi^* \exp \left[i \int d^4x (\phi^*(-\partial^2 - m^2 + i\epsilon)\phi - J_s^* \phi + \phi^* J_s) \right] \\ &\quad \times \int \mathcal{D}A \exp \left[i \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu J_{\text{em}}^\mu \right) \right].\end{aligned}$$

第1項を計算する (x についての積分は省略)。 (9.36) 以降の計算と同様に

$$\phi(x) = \phi'(x) + i \int d^4y D_F(x-y) J_s(y)$$

とおけば,

$$\begin{aligned}&\phi^*(x)(-\partial^2 - m^2 + i\epsilon)\phi(x) - J_s x(x)^* \phi(x) + \phi^*(x) J_s(x) \\ &= \left[\phi'^*(x) - i \int d^4y D_F(x-y) J_s^*(y) \right] (-\partial^2 - m^2 + i\epsilon)_x \left[\phi'(x) + i \int d^4y D_F(x-y) J_s(y) \right] \\ &\quad - J_s^*(x) \left[\phi'(x) + i \int d^4y D_F(x-y) J_s(y) \right] + J_s(x) \left[\phi'^*(x) - i \int d^4y D_F(x-y) J_s^*(y) \right] \\ &= \phi'^*(x)(-\partial^2 - m^2 + i\epsilon)\phi'(x) - i \int d^4y D_F(x-y) J_s^*(y)(-\partial^2 - m^2 + i\epsilon)\phi'(x) \\ &\quad - \left[\phi'^*(x) - i \int d^4y D_F(x-y) J_s^*(y) \right] \int d^4y \delta^{(4)}(x-y) J_s(y) \\ &\quad - J_s^*(x)\phi'(x) + J_s(x)\phi'^*(x) - 2i \int d^4y J_s^*(x) D_F(x-y) J_s(y) \\ &= \phi'^*(x)(-\partial^2 - m^2 + i\epsilon)\phi'(x) - i \int d^4y [(-\partial^2 - m^2 + i\epsilon)_x D_F(x-y)] J_s^*(y)\phi'(x) \\ &\quad - \left[\phi'^*(x) - i \int d^4y D_F(x-y) J_s^*(y) \right] J_s(x) \\ &\quad - J_s^*(x)\phi'(x) + J_s(x)\phi'^*(x) - 2i \int d^4y J_s^*(x) D_F(x-y) J_s(y) \\ &= \phi'^*(x)(-\partial^2 - m^2 + i\epsilon)\phi'(x) + \int d^4y \delta^{(4)}(x-y) J_s^*(y)\phi'(x)\end{aligned}$$

$$\begin{aligned}
 & - \left[\phi'^*(x) - i \int d^4 y D_F(x-y) J_s^*(y) \right] J_s(x) \\
 & - J_s^*(x) \phi'(x) + J_s(x) \phi'^*(x) - 2i \int d^4 y J_s^*(x) D_F(x-y) J_s(y) \\
 & = \phi'^*(x) (-\partial^2 - m^2 + i\epsilon) \phi'(x) - i \int d^4 y J_s^*(x) D_F(x-y) J_s(y).
 \end{aligned}$$

第2項を計算する (x についての積分は省略).

$$A_\mu(x) = A'_\mu(x) + i \int d^4 y D_{\mu\nu}^F(x-y) J_{\text{em}}^\nu(y)$$

とおけば,

$$\begin{aligned}
 & -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + A_\mu J_{\text{em}}^\mu \\
 & = \frac{1}{2} A_\mu(x) (\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu}) A_\nu(x) + A_\mu(x) J_{\text{em}}^\mu(x) \\
 & = \frac{1}{2} \left[A'_\mu(x) + i \int d^4 y D_{\mu\sigma}^F(x-y) J_{\text{em}}^\sigma(y) \right] (\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})_x \left[A'_\nu(x) + i \int d^4 y D_{\nu\rho}^F(x-y) J_{\text{em}}^\rho(y) \right] \\
 & \quad + \left[A'_\mu(x) + i \int d^4 y D_{\mu\nu}^F(x-y) J_{\text{em}}^\nu(y) \right] J_{\text{em}}^\mu(x) \\
 & = \frac{1}{2} A'_\mu(x) (\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu}) A'_\nu(x) + \frac{i}{2} \int d^4 y D_{\mu\sigma}^F(x-y) J_{\text{em}}^\sigma(y) (\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu}) A'_\nu(x) \\
 & \quad - \frac{1}{2} \left[A'_\mu(x) + i \int d^4 y D_{\mu\sigma}^F(x-y) J_{\text{em}}^\sigma(y) \right] J_{\text{em}}^\mu(x) \\
 & \quad + \left[A'_\mu(x) + i \int d^4 y D_{\mu\nu}^F(x-y) J_{\text{em}}^\nu(y) \right] J_{\text{em}}^\mu(x) \\
 & = \frac{1}{2} A'_\mu(x) (\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu}) A'_\nu(x) + \frac{i}{2} \int d^4 y (\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu})_x D_{\mu\sigma}^F(x-y) J_{\text{em}}^\sigma(y) A'_\nu(x) \\
 & \quad - \frac{1}{2} \left[A'_\mu(x) + i \int d^4 y D_{\mu\sigma}^F(x-y) J_{\text{em}}^\sigma(y) \right] J_{\text{em}}^\mu(x) \\
 & \quad + \left[A'_\mu(x) + i \int d^4 y D_{\mu\nu}^F(x-y) J_{\text{em}}^\nu(y) \right] J_{\text{em}}^\mu(x) \\
 & = \frac{1}{2} A'_\mu(x) (\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu}) A'_\nu(x) - \frac{1}{2} \int d^4 y \delta^\nu_\sigma \delta^{(4)}(x-y) J_{\text{em}}^\sigma(y) A'_\nu(x) \\
 & \quad - \frac{1}{2} \left[A'_\mu(x) + i \int d^4 y D_{\mu\sigma}^F(x-y) J_{\text{em}}^\sigma(y) \right] J_{\text{em}}^\mu(x) \\
 & \quad + \left[A'_\mu(x) + i \int d^4 y D_{\mu\nu}^F(x-y) J_{\text{em}}^\nu(y) \right] J_{\text{em}}^\mu(x) \\
 & = \frac{1}{2} A'_\mu(x) (\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu}) A'_\nu(x) - \frac{1}{2} J_{\text{em}}^\nu(x) A'_\nu(x) \\
 & \quad - \frac{1}{2} \left[A'_\mu(x) + i \int d^4 y D_{\mu\sigma}^F(x-y) J_{\text{em}}^\sigma(y) \right] J_{\text{em}}^\mu(x) \\
 & \quad + \left[A'_\mu(x) + i \int d^4 y D_{\mu\nu}^F(x-y) J_{\text{em}}^\nu(y) \right] J_{\text{em}}^\mu(x) \\
 & = \frac{1}{2} A'_\mu(x) (\partial^2 g^{\mu\nu} - i\epsilon g^{\mu\nu}) A'_\nu(x) + \frac{i}{2} \int d^4 y J_{\text{em}}^\mu(x) D_{\mu\nu}^F(x-y) J_{\text{em}}^\nu(y).
 \end{aligned}$$

以上から,

$$Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*]$$

$$= Z_{\text{free}}[0, 0, 0] \exp \left[\int d^4x \int d^4y \left(J_s^*(x) D_F(x-y) J_s(y) - \frac{1}{2} J_{\text{em}}^\mu(x) D_{\mu\nu}^F(x-y) J_{\text{em}}^\nu(y) \right) \right]$$

を得る.

相互作用項

$$\mathcal{L}_{\text{int}} = ieA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + e^2A^2|\phi|^2$$

を含めた生成関数は

$$\begin{aligned} Z[J_{\text{em}}, J_s, J_s^*] &= Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*] \int \mathcal{D}A \int \mathcal{D}\phi \int \mathcal{D}\phi^* \exp \left[i \int d^4x \mathcal{L}_{\text{int}} \right] \\ &= Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*] \int \mathcal{D}A \int \mathcal{D}\phi \int \mathcal{D}\phi^* \exp \left[i \int d^4x e^2 A^2 |\phi|^2 \right] \\ &\quad \times \exp \left[i \int d^4x ieA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) \right] \\ &\approx Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*] \int \mathcal{D}A \int \mathcal{D}\phi \int \mathcal{D}\phi^* \left[1 + ie^2 \int d^4x A(x)^2 |\phi(x)|^2 \right] \\ &\quad \times \left[1 - e \int d^4x A^\mu(x) \{ \phi(x) \partial_\mu \phi^*(x) - \phi^*(x) \partial_\mu \phi(x) \} \right] \end{aligned}$$

となる.

(9.35) と同様に汎関数微分をすれば伝播関数が求まる. 例えば,

$$\langle 0 | T \phi(x_1) \phi^*(x_2) | 0 \rangle = \frac{1}{Z_{\text{free}}[0, 0, 0]} \left(+i \frac{\delta}{\delta J_s^*(x_2)} \right) \left(-i \frac{\delta}{\delta J_s(x_1)} \right) Z_{\text{free}}[J_{\text{em}}, J_s, J_s^*] \Big|_{J=0}.$$

分母を計算する. (A.55) を使えば

$$\begin{aligned}
& \gamma^\nu (\not{k}' + m) \gamma^\mu (\not{k} + m) \gamma_\nu \\
&= \gamma^\nu \not{k}' \gamma^\mu \not{k} \gamma_\nu + m \gamma^\nu \not{k}' \gamma^\mu \gamma_\nu + m \gamma^\nu \gamma^\mu \not{k} \gamma_\nu + m^2 \gamma^\nu \gamma^\mu \gamma_\nu \\
&= -2 \not{k} \gamma^\mu \not{k}' + (4-d) \not{k}' \gamma^\mu \not{k} - (d-2) m^2 \gamma^\mu + m[4(k+k')^\mu - (4-d)(\not{k}' \gamma^\mu + \gamma^\mu \not{k})].
\end{aligned} \tag{10.3.1}$$

簡単のため,

$$k = \ell + a, \quad k' = \ell + b, \quad a = -yq + zp, \quad b = (1-y)q + zp$$

とおく. ℓ^1 の項は積分すれば 0 なので無視し, [6.3.2] を使えば,

$$\not{q} \gamma^\mu \not{q} = \not{q} (\gamma^\mu \gamma^\nu) q_\nu = \not{q} (2g^{\mu\nu} - \gamma^\nu \gamma^\mu) q_\nu = 2q^\mu \not{q} - \not{q} \not{q} \gamma^\mu = -q^2 \gamma^\mu.$$

これらを使えば,

$$\begin{aligned}
\not{k} \gamma^\mu \not{k}' &= (\ell + \phi) \gamma^\mu (\ell + \not{b}) \\
&= \ell \gamma^\mu \ell + \phi \gamma^\mu \not{b} \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + (-y \not{q} + z \not{p}) \gamma^\mu [(1-y) \not{q} + z \not{p}] \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + [-y \not{q} + z(\not{p}' - \not{q})] \gamma^\mu [(1-y) \not{q} + z \not{m}] \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + [-(y+z) \not{q} + z \not{p}'] \gamma^\mu [(1-y) \not{q} + z \not{m}] \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + [-(1-x) \not{q} + z \not{m}] \gamma^\mu [(1-y) \not{q} + z \not{m}] \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu - (1-x)(1-y) \not{q} \gamma^\mu \not{q} - (1-x) z m \not{q} \gamma^\mu + (1-y) z m \gamma^\mu \not{q} + z^2 m^2 \gamma^\mu \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + (1-x)(1-y) q^2 \gamma^\mu - (1-x) z m \not{q} \gamma^\mu + (1-y) z m \gamma^\mu \not{q} + z^2 m^2 \gamma^\mu \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + (1-x)(1-y) q^2 \gamma^\mu - (1-x) z m (2q^\mu - \gamma^\mu \not{q}) + (1-y) z m \gamma^\mu \not{q} + z^2 m^2 \gamma^\mu \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + (1-x)(1-y) q^2 \gamma^\mu - 2(1-x) z m q^\mu + (2-x-y) z m \gamma^\mu \not{q} + z^2 m^2 \gamma^\mu \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + (1-x)(1-y) q^2 \gamma^\mu - 2(1-x) z m q^\mu + (1+z) z m \gamma^\mu \not{q} + z^2 m^2 \gamma^\mu
\end{aligned}$$

および

$$\begin{aligned}
\not{k}' \gamma^\mu \not{k} &= (\ell + \not{b}) \gamma^\mu (\ell + \phi) \\
&= \ell \gamma^\mu \ell + \not{b} \gamma^\mu \phi \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + [(1-y) \not{q} + z \not{p}] \gamma^\mu (-y \not{q} + z \not{p}) \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + [(1-y) \not{q} + z(\not{p}' - \not{q})] \gamma^\mu (-y \not{q} + z \not{m}) \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + [(1-y-z) \not{q} + z \not{p}'] \gamma^\mu (-y \not{q} + z \not{m}) \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + [x \not{q} + z \not{m}] \gamma^\mu (-y \not{q} + z \not{m}) \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu - xy \not{q} \gamma^\mu \not{q} + xzm \not{q} \gamma^\mu - yzm \gamma^\mu \not{q} + z^2 m^2 \gamma^\mu \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + xy q^2 \gamma^\mu + xzm \not{q} \gamma^\mu - yzm \gamma^\mu \not{q} + z^2 m^2 \gamma^\mu
\end{aligned}$$

$$\begin{aligned}
&= \frac{2-d}{d} \ell^2 \gamma^\mu + xyq^2 \gamma^\mu + xzm(2q^\mu - \gamma^\mu \not{q}) - yzm\gamma^\mu \not{q} + z^2 m^2 \gamma^\mu \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + xyq^2 \gamma^\mu + 2xzmq^\mu - (x+y)zm\gamma^\mu \not{q} + z^2 m^2 \gamma^\mu \\
&= \frac{2-d}{d} \ell^2 \gamma^\mu + xyq^2 \gamma^\mu + 2xzmq^\mu - (1-z)zm\gamma^\mu \not{q} + z^2 m^2 \gamma^\mu
\end{aligned}$$

が得られる。したがって、[10.3.1] の第 1, 2 項は

$$\begin{aligned}
&-2\cancel{k}\gamma^\mu \cancel{k}' + (4-d)\cancel{k}'\gamma^\mu \cancel{k} \\
&= -2 \left[\frac{2-d}{d} \ell^2 \gamma^\mu + (1-x)(1-y)q^2 \gamma^\mu - 2(1-x)zmq^\mu + (1+z)zm\gamma^\mu \not{q} + z^2 m^2 \gamma^\mu \right] \\
&\quad + (4-d) \left[\frac{2-d}{d} \ell^2 \gamma^\mu + xyq^2 \gamma^\mu + 2xzmq^\mu - (1-z)zm\gamma^\mu \not{q} + z^2 m^2 \gamma^\mu \right] \\
&= \frac{(2-d)^2}{d} \ell^2 \gamma^\mu - 2(1-x)(1-y)q^2 \gamma^\mu + (4-d)xyq^2 \gamma^\mu \\
&\quad - [2(1+z) + (4-d)(1-z)]zm\gamma^\mu \not{q} + [4(1-x) + 2(4-d)x]zmq^\mu + (2-d)z^2 m^2 \gamma^\mu \\
&= \frac{(2-d)^2}{d} \ell^2 \gamma^\mu - 2(1-x)(1-y)q^2 \gamma^\mu + (4-d)xyq^2 \gamma^\mu \\
&\quad - [2(1+z) + (4-d)(1-z)]zm(2p'^\mu - 2m\gamma^\mu) + [4(1-x) + 2(4-d)x]zm(p'^\mu - p^\mu) \\
&\quad + (2-d)z^2 m^2 \gamma^\mu.
\end{aligned} \tag{10.3.2}$$

[10.3.1] の第 4 項は

$$\begin{aligned}
4m(k^\mu + k'^\mu) &= 4m[(1-2y)q^\mu + 2zp^\mu] \\
&= 4m[(1-2y)(p'^\mu - p^\mu) + 2zp^\mu] \\
&= 4m[(1-2y)p'^\mu + (-1+2y+2z)p^\mu] \\
&= 4m[(1-2y)p'^\mu + (1-2x)p^\mu].
\end{aligned} \tag{10.3.3}$$

[10.3.1] の第 5 項は、[6.3.8] から

$$\begin{aligned}
-(4-d)m(\cancel{k}'\gamma^\mu + \gamma^\mu \cancel{k}) &= -(4-d)m(2k^\mu + 2m\gamma^\mu - 2p^\mu) \\
&= -(4-d)m(-2yq^\mu + 2zp^\mu + 2m\gamma^\mu - 2p^\mu) \\
&= -(4-d)m[-2y(p'^\mu - p^\mu) + 2zp^\mu + 2m\gamma^\mu - 2p^\mu] \\
&= -(4-d)m[-2yp'^\mu + 2(y+z-1)p^\mu + 2m\gamma^\mu] \\
&= -(4-d)m[-2yp'^\mu - 2xp^\mu + 2m\gamma^\mu].
\end{aligned} \tag{10.3.4}$$

[10.3.2][10.3.3][10.3.4] から [10.3.1] は

$$\begin{aligned}
&-2\cancel{k}\gamma^\mu \cancel{k}' + (4-d)\cancel{k}'\gamma^\mu \cancel{k} - (d-2)m^2 \gamma^\mu + m[4(k+k')^\mu - (4-d)(\cancel{k}'\gamma^\mu + \gamma^\mu \cancel{k})] \\
&= \frac{(2-d)^2}{d} \ell^2 \gamma^\mu - 2(1-x)(1-y)q^2 \gamma^\mu + (4-d)xyq^2 \gamma^\mu \\
&\quad - [2(1+z) + (4-d)(1-z)]zm(2p'^\mu - 2m\gamma^\mu) + [4(1-x) + 2(4-d)x]zm(p'^\mu - p^\mu) \\
&\quad + (2-d)z^2 m^2 \gamma^\mu - (d-2)m^2 \gamma^\mu \\
&\quad + 4m[(1-2y)p'^\mu + (1-2x)p^\mu] - (4-d)m[-2yp'^\mu - 2xp^\mu + 2m\gamma^\mu]
\end{aligned}$$

となり, p, p' を含まない項と含む項に分ければ

$$\begin{aligned}
&= \frac{(2-d)^2}{d} \ell^2 \gamma^\mu - 2(1-x)(1-y)q^2 \gamma^\mu + (4-d)xyq^2 \gamma^\mu \\
&\quad + 2[2(1+z) + (4-d)(1-z)]zm^2 \gamma^\mu + (2-d)z^2m^2 \gamma^\mu - (d-2)m^2 \gamma^\mu - 2(4-d)m^2 \gamma^\mu \\
&\quad - 2[2(1+z) + (4-d)(1-z)]zmp'^\mu + [4(1-x) + 2(4-d)x]zm(p'^\mu - p^\mu) \\
&\quad + 4m[(1-2y)p'^\mu + (1-2x)p^\mu] + 2(4-d)m[yp'^\mu + xp^\mu].
\end{aligned} \tag{10.3.5}$$

p, p' を含まない項のうち, m を含む項は $m^2 \gamma^\mu \times$

$$\begin{aligned}
&2[2(1+z) + (4-d)(1-z)]z + (2-d)z^2 - (d-2) - 2(4-d) \\
&= 4(1+z)z + 2(4-d)(1-z)z + (4-d)z^2 - 2z^2 + (4-d) - 2 - 2(4-d) \\
&= 2(z^2 + 2z - 1) - (4-d)(z-1)^2.
\end{aligned}$$

p を含む項は $mp^\mu \times$

$$\begin{aligned}
&-4(1-x)z - 2(4-d)xz + 4(1-2x) + 2(4-d)x = 4[1-2x-z+xz] + 2(4-d)x(1-z) \\
&= 4[1-2y-z+yz] + 2(4-d)y(1-z).
\end{aligned}$$

p' を含む項は $mp'^\mu \times$

$$\begin{aligned}
&-4(1+z)z - 2(4-d)(1-z)z + 4(1-x)z + 2(4-d)xz + 4(1-2y) + 2(4-d)y \\
&= 4[1-2y-xz-z^2] + 2(4-d)[xz+y-z+z^2] \\
&= 4[1-2y-xz-z(1-x-y)] + 2(4-d)[xz+y-z+z(1-x-y)] \\
&= 4[1-2y-z+yz] + 2(4-d)y(1-z).
\end{aligned}$$

さらに,

$$\begin{aligned}
\int_0^1 dz \int_0^{1-z} dy (1-2y-z+yz) &= \int_0^1 dz \left[(1-z) - (1-z)^2 - (1-z)z + \frac{1}{2}(1-z)^2 z \right] \\
&= \int_0^1 dz \int_0^{1-z} dy \frac{1}{2} z(1-z)
\end{aligned}$$

及び

$$\int_0^1 dz \int_0^{1-z} dy y(1-z) = \int_0^1 dz \frac{1}{2}(1-z)^3 = \int_0^1 dz \int_0^{1-z} dy \frac{1}{2}(1-z)^2$$

なので, p, p' を含む項は

$$[4(1-2y-z+yz) + 2(4-d)y(1-z)]m(p^\mu + p'^\mu) = [2z(1-z) + (4-d)(1-z)^2]m(p^\mu + p'^\mu).$$

以上から,

$$\begin{aligned}
[10.3.5] &= \frac{(2-d)^2}{d} \ell^2 \gamma^\mu - 2(1-x)(1-y)q^2 \gamma^\mu + (4-d)xyq^2 \gamma^\mu \\
&\quad + [2(z^2 + 2z - 1) - (4-d)(z-1)^2]m^2 \gamma^\mu \\
&\quad + [2z(1-z) + (4-d)(1-z)^2]m(p^\mu + p'^\mu) \\
&= \frac{(2-d)^2}{d} \ell^2 \gamma^\mu - 2(1-x)(1-y)q^2 \gamma^\mu + (4-d)xyq^2 \gamma^\mu \\
&\quad + [2(z^2 + 2z - 1) - (4-d)(z-1)^2]m^2 \gamma^\mu
\end{aligned}$$

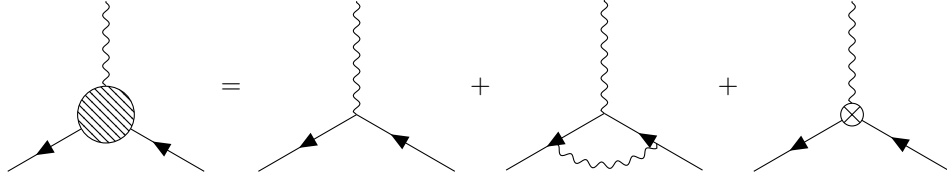
$$\begin{aligned}
& + [2z(1-z) + (4-d)(1-z)^2] m(2m\gamma^\mu - i\sigma^{\mu\nu}q_\nu) \\
& = \frac{(2-d)^2}{d} \ell^2 \gamma^\mu - 2(1-x)(1-y)q^2 \gamma^\mu + (4-d)xyq^2 \gamma^\mu \\
& \quad + [2(-z^2 + 4z - 1) + (4-d)(z-1)^2] m^2 \gamma^\mu \\
& \quad - [2z(1-z) + (4-d)(1-z)^2] im\sigma^{\mu\nu}q_\nu \\
& = \frac{(2-d)^2}{d} \ell^2 \gamma^\mu \\
& \quad - (q^2[2(1-x)(1-y) - \epsilon xy] + m^2[2(1-4z+z^2) - \epsilon(1-z)^2]) \gamma^\mu \\
& \quad - [2z(1-z) + (4-d)(1-z)^2] im\sigma^{\mu\nu}q_\nu.
\end{aligned}$$

(6.33) より, 初め 2 項が F_1 への寄与. したがって,

$$\begin{aligned}
\delta F_1(q^2) & = -2ie^2 \int_0^1 dx dy dz \delta(x+y+z-1) \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^3} \frac{(2-d)^2}{d} \ell^2 \\
& \quad + 2ie^2 \int_0^1 dx dy dz \delta(x+y+z-1) \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^3} (q^2[\dots] + m^2[\dots]) \\
& = \frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx dy dz \delta(x+y+z-1) \left[\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-d/2}} \frac{(2-\epsilon)^2}{2} \right. \\
& \quad \left. + \frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-d/2}} (q^2[2(1-x)(1-y) - \epsilon xy] + m^2[2(1-4z+z^2) - \epsilon(1-z)^2]) \right].
\end{aligned}$$

(10.46)

(10.39) で定義したように,



$$-ie\Gamma^\mu(p', p) = -ie\gamma^\mu - ie \left[\gamma^\mu \delta F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2) \right] - ie\gamma^\mu \delta_1.$$

Problems

Problem 10.1: One-loop structure of QED

ガンマ行列 8 個の積を計算する.

$$\begin{aligned}
\text{Tr}[\gamma^\mu \not{k} \gamma^\nu \not{k} \gamma^\rho \not{k} \gamma^\sigma \not{k}] & = \text{Tr}[\gamma^\mu \not{k} (2k^\nu - \not{k} \gamma^\nu) \gamma^\rho \not{k} (2k^\sigma - \not{k} \gamma^\sigma)] \\
& = 4k^\nu k^\sigma \text{Tr}[\gamma^\mu \not{k} \gamma^\rho \not{k}] - 2k^\nu \text{Tr}[\gamma^\mu \not{k} \gamma^\rho \not{k} \not{k} \gamma^\sigma] \\
& \quad - 2k^\sigma \text{Tr}[\gamma^\mu \not{k} \not{k} \gamma^\nu \gamma^\rho \not{k}] + \text{Tr}[\gamma^\mu \not{k} \not{k} \gamma^\nu \gamma^\rho \not{k} \not{k} \gamma^\sigma] \\
& = 4k^\nu k^\sigma \text{Tr}[\gamma^\mu \not{k} \gamma^\rho \not{k}] - 2k^2 k^\nu \text{Tr}[\gamma^\mu \not{k} \gamma^\rho \gamma^\sigma] \\
& \quad - 2k^2 k^\sigma \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \not{k}] + k^4 \text{Tr}[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma]
\end{aligned}$$

$$\begin{aligned}
 &= 16k^\nu k^\sigma (2k^\mu k^\rho - k^2 g^{\mu\rho}) - 8k^2 k^\nu (k^\mu g^{\rho\sigma} - g^{\mu\rho} k^\sigma + g^{\mu\sigma} k^\rho) \\
 &\quad - 8k^2 k^\sigma (g^{\mu\nu} k^\rho - g^{\mu\rho} k^\nu + k^\mu g^{\nu\rho}) + 4k^4 (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\
 &= 32k^\mu k^\nu k^\rho k^\sigma - 8k^2 (k^\mu k^\nu g^{\rho\sigma} + k^\rho k^\sigma g^{\mu\nu} + k^\mu k^\sigma g^{\nu\rho} + k^\nu k^\rho g^{\mu\sigma}) \\
 &\quad + 4k^4 (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}).
 \end{aligned}$$

(A.41)(A.42) から

$$\begin{aligned}
 &\rightarrow 3k^4 (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) - 2k^4 (g^{\mu\nu} g^{\rho\sigma} + g^{\rho\sigma} g^{\mu\nu} + g^{\mu\sigma} g^{\nu\rho} + g^{\nu\rho} g^{\mu\sigma}) \\
 &\quad + 4k^4 (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\
 &= 3k^4 (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) - 4k^4 (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\
 &\quad + 4k^4 (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\
 &= 3k^4 (g^{\mu\nu} g^{\rho\sigma} - 2g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}).
 \end{aligned}$$

Chapter 11

Renormalization and Symmetry

11.4 Computation of Effective Action

(11.67)

汎関数微分について、Euler-Lagrange 方程式を導く際の変分と同様にして

$$\begin{aligned}\frac{\delta \mathcal{L}[\phi, \dots]}{\delta \phi} &= \frac{\mathcal{L}[\phi + \delta \phi, \dots] - \mathcal{L}[\phi, \dots]}{\delta \phi} \\ &= \frac{1}{\delta \phi} \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) + \dots \right] \delta \phi \\ &= \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right).\end{aligned}$$

$(\partial_\mu \phi_{\text{cl}}^i)^2/2$ を 2 階変分すれば、 $-\partial^2 \delta^{ij}$ が得られる。

Chapter 12

The Renormalization Group

12.1 Wilson's Approach to Renormalization Theory

(12.8)

汎関数積分のために (12.7) を和の形に書いておく：

$$\int \mathcal{L}_0 = \frac{1}{2} \sum_{|k|=b\Lambda}^{\Lambda} \frac{k^2}{(2\pi)^d} |\hat{\phi}(k)|^2.$$

(9.23) と同様にして

$$\int \mathcal{D}\hat{\phi} \exp\left(-\int \mathcal{L}_0\right) = \prod_{b\Lambda \leq |k| < \Lambda} \sqrt{(2\pi)^d \frac{\pi}{k^2}}.$$

$b\Lambda \leq |k| < \Lambda$ でない場合は $\hat{\phi}(k) = 0$. 以下では $b\Lambda \leq |k| < \Lambda$ の場合を考える. (9.26) と同様にして

$$\begin{aligned} & \int \mathcal{D}\hat{\phi} \exp\left(-\int \mathcal{L}_0\right) \hat{\phi}(k) \hat{\phi}(p) \\ &= \left(\prod_{b\Lambda \leq |l| < \Lambda} \int d\text{Re } \hat{\phi}(l) d\text{Im } \hat{\phi}(l) \right) \left(\text{Re } \hat{\phi}(k) + i \text{Im } \hat{\phi}(k) \right) \left(\text{Re } \hat{\phi}(p) + i \text{Im } \hat{\phi}(p) \right) \\ & \quad \times \exp\left[-\frac{1}{2} \sum_{|l|=b\Lambda}^{\Lambda} \frac{l^2}{(2\pi)^d} (\text{Re } \hat{\phi}(l))^2\right] \exp\left[-\frac{1}{2} \sum_{|l|=b\Lambda}^{\Lambda} \frac{l^2}{(2\pi)^d} (\text{Im } \hat{\phi}(l))^2\right]. \end{aligned}$$

(9.26) の後で説明されているように, 積分が非零となるのは $k + p = 0$ の場合のみ.

$$\begin{aligned} & \int \mathcal{D}\hat{\phi} \exp\left(-\int \mathcal{L}_0\right) \hat{\phi}(k) \hat{\phi}(-k) \\ &= \left(\prod_{b\Lambda \leq |l| < \Lambda} \int d\text{Re } \hat{\phi}(l) d\text{Im } \hat{\phi}(l) \right) \left[(\text{Re } \hat{\phi}(k))^2 + (\text{Im } \hat{\phi}(k))^2 \right] \\ & \quad \times \exp\left[-\frac{1}{2} \sum_{|l|=b\Lambda}^{\Lambda} \frac{l^2}{(2\pi)^d} (\text{Re } \hat{\phi}(l))^2\right] \exp\left[-\frac{1}{2} \sum_{|l|=b\Lambda}^{\Lambda} \frac{l^2}{(2\pi)^d} (\text{Im } \hat{\phi}(l))^2\right] \\ &= 2 \int d\text{Re } \hat{\phi}(k) (\text{Re } \hat{\phi}(k))^2 \exp\left[-\frac{1}{2} \frac{k^2}{(2\pi)^d} (\text{Re } \hat{\phi}(k))^2\right] \int d\text{Im } \hat{\phi}(k) \exp\left[-\frac{1}{2} \frac{k^2}{(2\pi)^d} (\text{Im } \hat{\phi}(k))^2\right] \end{aligned}$$

$$\begin{aligned}
& \times \prod_{\substack{b\Lambda \leq |l| < \Lambda \\ l \neq k}} \int d\operatorname{Re} \hat{\phi}(l) \exp \left[-\frac{1}{2} \frac{l^2}{(2\pi)^d} (\operatorname{Re} \hat{\phi}(l))^2 \right] \int d\operatorname{Im} \hat{\phi}(l) \exp \left[-\frac{1}{2} \frac{l^2}{(2\pi)^d} (\operatorname{Im} \hat{\phi}(l))^2 \right] \\
& = \frac{(2\pi)^d}{k^2} \prod_{b\Lambda \leq |k| < \Lambda} \sqrt{(2\pi)^d \frac{\pi}{k^2}}.
\end{aligned}$$

以上から

$$\overline{\hat{\phi}(k)} \hat{\phi}(p) = \frac{(2\pi)^d}{k^2} \delta^{(d)}(k+p) \Theta(k).$$

(12.10)

(12.8) から

$$\begin{aligned}
\overline{\hat{\phi}(x)} \hat{\phi}(x) &= \int \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{-i(k+p) \cdot x} \overline{\hat{\phi}(k)} \hat{\phi}(p) \\
&= \int \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{-i(k+p) \cdot x} \frac{(2\pi)^d}{k^2} \delta^{(d)}(k+p) \Theta(k) \\
&= \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2}.
\end{aligned}$$

よって,

$$\begin{aligned}
-\int d^d x \frac{\lambda}{4} \phi(x) \phi(x) \overline{\hat{\phi}(x)} \hat{\phi}(x) &= -\int d^d x \frac{\lambda}{4} \phi(x) \phi(x) \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \\
&= -\frac{\mu}{2} \int d^d x \phi(x) \phi(x) \\
&= -\frac{\mu}{2} \int d^d x \int \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{-i(k+p) \cdot x} \phi(k) \phi(p) \\
&= -\frac{\mu}{2} \int \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \delta^{(d)}(k+p) \phi(k) \phi(p) \\
&= -\frac{\mu}{2} \int \frac{d^d k}{(2\pi)^d} \phi(k) \phi(-k).
\end{aligned}$$

(12.14)

(12.8) から

$$\begin{aligned}
\overline{\hat{\phi}(x)} \hat{\phi}(y) &= \int \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{-i(k \cdot x + p \cdot y)} \overline{\hat{\phi}(k)} \hat{\phi}(p) \\
&= \int \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{-i(k \cdot x + p \cdot y)} \frac{(2\pi)^d}{k^2} \delta^{(d)}(k+p) \Theta(k) \\
&= \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} e^{-ik \cdot (x-y)} \frac{1}{k^2}.
\end{aligned}$$

$\exp(-\lambda \phi^2 \hat{\phi}^2/4)$ の 2 次の展開

$$\frac{1}{2} \int d^d x \frac{\lambda}{4} \phi(x) \phi(x) \hat{\phi}(x) \hat{\phi}(x) \int d^d y \frac{\lambda}{4} \phi(y) \phi(y) \hat{\phi}(y) \hat{\phi}(y)$$

を考える． $\hat{\phi}$ の縮約には

$$\overbrace{\hat{\phi}(x)\hat{\phi}(x)}\overbrace{\hat{\phi}(y)\hat{\phi}(y)} = \left(\text{diagram of a loop with two external lines} \right)^2$$

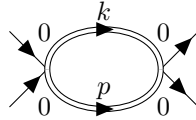
および

$$\overbrace{\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y)} = \text{diagram of a bubble with four external lines}$$

の 2 通りがある． 2 つ目の縮約は 2 通りあるので，

$$\begin{aligned} & \frac{\lambda^2}{16} \int d^d x d^d y \phi(x) \phi(x) \phi(y) \phi(y) \overbrace{\hat{\phi}(x)\hat{\phi}(x)\hat{\phi}(y)\hat{\phi}(y)} \\ &= \frac{\lambda^2}{16} \int d^d x d^d y \phi^2(x) \phi^2(y) \int_{\substack{b\Lambda \leq |k| < \Lambda \\ b\Lambda \leq |p| < \Lambda}} \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} e^{-i(k+p) \cdot (x-y)} \frac{1}{k^2} \frac{1}{p^2} \\ &= \frac{\lambda^2}{16} \int_{\substack{b\Lambda \leq |k| < \Lambda \\ b\Lambda \leq |p| < \Lambda}} \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \frac{1}{k^2} \frac{1}{p^2} \int d^d x \phi^2(x) e^{-i(k+p) \cdot x} \int d^d y \phi^2(y) e^{i(k+p) \cdot y} \\ &= \frac{\lambda^2}{16} \int_{\substack{b\Lambda \leq |k| < \Lambda \\ b\Lambda \leq |p| < \Lambda}} \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \frac{1}{k^2} \frac{1}{p^2} \mathcal{F}[\phi^2](-k-p) \mathcal{F}[\phi^2](k+p) \\ &= \frac{\lambda^2}{16} \int_{\substack{b\Lambda \leq |k| < \Lambda \\ b\Lambda \leq |p| < \Lambda}} \frac{d^d k}{(2\pi)^d} \frac{d^d p}{(2\pi)^d} \frac{1}{k^2} \frac{1}{p^2} |\mathcal{F}[\phi^2](k+p)|^2. \end{aligned}$$

ϕ の運動量に関する条件から， $\mathcal{F}[\phi^2](k+p) \approx \mathcal{F}[\phi^2](0) \delta^{(d)}(k+p)$.



よって，

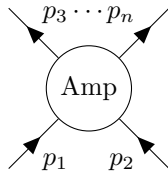
$$\begin{aligned} & \approx \frac{1}{(2\pi)^d} \frac{\lambda^2}{16} \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 |\mathcal{F}[\phi^2](0)|^2 \\ & \approx \frac{\lambda^2}{16} \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \int_{-\infty}^{\infty} \frac{d^d p}{(2\pi)^d} |\mathcal{F}[\phi^2](p)|^2 \\ & = \frac{\lambda^2}{16} \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \|\mathcal{F}[\phi^2]\|^2 \\ & = \frac{\lambda^2}{16} \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \|\phi^2\|^2 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\lambda^2}{16} \int_{b\Lambda \leq |k| < \Lambda} \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \int d^d x \phi^4(x) \\
 &= -\frac{1}{4!} \int d^d x \zeta \phi^4.
 \end{aligned}$$

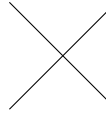
12.2 The Callan-Symanzik Equation

(12.52)

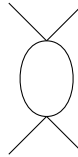
n 頂点の Green 関数を考える. (7.42)(12.35) から

$$\begin{aligned}
 \langle \Omega | T \{ \phi(p_1) \phi(p_2) \cdots \phi(p_n) \} | \Omega \rangle &= Z^{-n/2} \langle \Omega | T \{ \phi_0(p_1) \phi_0(p_2) \cdots \phi_0(p_n) \} | \Omega \rangle \\
 &= Z^{-n/2} \prod_{i=1}^n \frac{\sqrt{Z} i}{p_i^2} \langle \mathbf{p} | S | \mathbf{p} \rangle \\
 &= \left(\prod_{i=1}^n \frac{i}{p_i^2} \right) \text{Amp}
 \end{aligned}$$


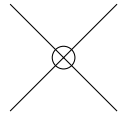
となる^{*1}. tree-level は,


 $= -ig.$

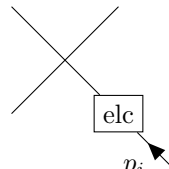
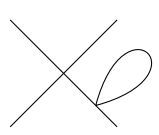
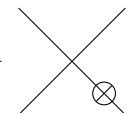
1PI-loop は, (10.20) などで計算したように, $\log(-p^2)$ の発散を持つ:


 $+ \cdots = -iB \log \frac{\Lambda^2}{-p^2}.$

vertex counterterm は,


 $= -i\delta_g.$

運動量 p_i の外線に対する external leg correction は


 $=$

 $+$


$$= (-ig) A_i \log \frac{\Lambda^2}{-p^2} + (-ig) \times i p_i^2 \delta_{Z_i} \frac{i}{p_i^2}$$

^{*1} くりこみした量でダイアグラムを計算するので, (7.45) の右辺の \sqrt{Z} は不要

$$= (-ig) \left(A_i \log \frac{\Lambda^2}{-p^2} - \delta_{Z_i} \right)$$

となる (g^1 の項のみ考えるので, i について和を取れば良い).

(12.57)

$G^{(2,0)}(p)$ を求める. 電子の自己エネルギー (7.15) を使えば (12.49) と同様に

$$\begin{aligned} G^{(2,0)}(p) &= \text{---} + \text{---} \text{---} \text{---} + \text{---} \text{---} \text{---} \otimes \text{---} \\ &= \frac{i}{\not{p}} + \frac{i}{\not{p}} [-i\Sigma_2] \frac{i}{\not{p}} + \frac{i}{\not{p}} i(\not{p}\delta_2) \frac{i}{\not{p}} \\ &= \frac{i}{\not{p}} + \frac{i}{\not{p}} [-i\Sigma_2] \frac{i}{\not{p}} - \frac{i}{\not{p}} \delta_2. \end{aligned}$$

(12.50) と同様に, Callan-Syazmzik 方程式から

$$-\frac{i}{\not{p}} M \frac{\partial}{\partial M} \delta_2 + 2\gamma_2 \frac{i}{\not{p}} = 0$$

なので

$$\gamma_2 = \frac{1}{2} M \frac{\partial}{\partial M} \delta_2.$$

$G^{(0,2)}(q)$ を求める. 真空偏極 (7.71)(7.73) を使えば (12.49) と同様に

$$\begin{aligned} G^{(0,2)}(q) &= \mu \text{---} \nu + \mu \text{---} \text{---} \text{---} \text{---} \nu + \mu \text{---} \text{---} \text{---} \otimes \text{---} \nu \\ &= \frac{-i}{q^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \\ &\quad + \frac{-i}{q^2} \left(\delta^\mu{}_\rho - \frac{q^\mu q_\rho}{q^2} \right) i(g^{\rho\sigma} q^2 - q^\rho q^\sigma) \Pi_2 \frac{-i}{q^2} \left(\delta^\nu{}_\sigma - \frac{q^\nu q_\sigma}{q^2} \right) \\ &\quad + \frac{-i}{q^2} \left(\delta^\mu{}_\rho - \frac{q^\mu q_\rho}{q^2} \right) (-i)(g^{\rho\sigma} q^2 - q^\rho q^\sigma) \delta_3 \frac{-i}{q^2} \left(\delta^\nu{}_\sigma - \frac{q^\nu q_\sigma}{q^2} \right) \\ &= \frac{-i}{q^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \\ &\quad + \frac{-i}{q^2} (g^{\mu\nu} q^2 - q^\mu q^\nu) i \Pi_2 \frac{-i}{q^2} + \frac{-i}{q^2} (g^{\mu\nu} q^2 - q^\mu q^\nu) (-i) \delta_3 \frac{-i}{q^2} \\ &= \frac{-i}{q^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) [1 + \Pi_2 - \delta_3]. \end{aligned}$$

(12.50) と同様に, Callan-Syazmzik 方程式から

$$-M \frac{\partial}{\partial M} \delta_3 + 2\gamma_3 = 0$$

なので

$$\gamma_3 = \frac{1}{2} M \frac{\partial}{\partial M} \delta_3.$$

(12.58)

$G^{(2,1)}(p_1, p_2, q)$ を求める．頂点補正 (6.38) を使えば (12.52) と同様に

$$\begin{aligned}
 G^{(2,1)}(p_1, p_2, q) = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} \\
 & + \text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} \\
 = & \frac{i}{\not{p}_1} (-ie\gamma^\mu) \frac{i}{\not{p}_2} \frac{-i}{q^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) + \frac{i}{\not{p}_1} (-ie\delta\Gamma^\mu) \frac{i}{\not{p}_2} \frac{-i}{q^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \\
 & + \frac{i}{\not{p}_1} (-ie\gamma^\mu \delta_1) \frac{i}{\not{p}_2} \frac{-i}{q^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \\
 & + \frac{i}{\not{p}_1} (-ie\gamma^\mu) (\Pi_2 - \delta_3) \frac{i}{\not{p}_2} \frac{-i}{q^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \\
 & + \frac{i}{\not{p}_1} \left(1 + \Sigma_2 \frac{1}{\not{p}_1} - \delta_2 \right) (-ie\gamma^\mu) \frac{i}{\not{p}_2} \frac{-i}{q^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right) \\
 & + \frac{i}{\not{p}_1} (-ie\gamma^\mu) \left(1 + \Sigma_2 \frac{1}{\not{p}_2} - \delta_2 \right) \frac{i}{\not{p}_2} \frac{-i}{q^2} \left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} \right).
 \end{aligned}$$

(12.53) と同様に, Callan-Syazmzik 方程式から

$$M \frac{\partial}{\partial M} (\delta_1 - 2\delta_2 - \delta_3) + \frac{\beta}{e} + 2\gamma_2 + \gamma_3 = 0.$$

(12.57) を代入して

$$\beta = eM \frac{\partial}{\partial M} \left(-\delta_1 + \delta_2 + \frac{\delta_3}{2} \right).$$

12.4 Renormalization of Local Operators

(12.112)

相互作用項は

$$\delta\mathcal{L} = \frac{g}{\sqrt{2}} W_\mu \bar{\psi} \gamma^\mu (1 - \gamma^5) \psi$$

なので

$$\begin{aligned}
 \text{Feynman diagram 1} &= \frac{ig}{\sqrt{2}}\gamma^\mu(1-\gamma^5), \\
 \text{Feynman diagram 2} &= \frac{-ig^{\mu\nu}}{q^2 - m_W^2 + i\epsilon}, \\
 \text{Feynman diagram 3} &= i\not{p}\delta_Z.
 \end{aligned}$$

(12.57) で γ_2 を導出したのと同様に

$$\gamma = \frac{1}{2}M \frac{\partial}{\partial M} \delta_Z.$$

$G^{(m,1)}$ は

$$\begin{aligned}
 &(\text{tree level}) + (\text{1PI loop}) + (\text{external leg corrections}) + (\text{operator counterterm}) \\
 &= (\text{TL}) + (\text{TL}) \times B + (\text{TL}) \times m(A - \delta_Z) + (\text{TL}) \times \delta_{\mathcal{O}}
 \end{aligned}$$

と表せる (A はフェルミオンの自己エネルギー, $-\delta_Z$ はフェルミオンの相殺項 $i\not{p}\delta_Z$ と伝播関数 i/\not{p} の積) ので, Callan-Symanzik 方程式から

$$M \frac{\partial}{\partial M} (\delta_{\mathcal{O}} - m\delta_Z) + m\gamma + \gamma_{\mathcal{O}} = 0.$$

よって,

$$\gamma_{\mathcal{O}} = M \frac{\partial}{\partial M} \left(-\delta_{\mathcal{O}} + \frac{m}{2}\delta_Z \right).$$

12.5 Evolution of Mass Parameters

(12.123)

$C \rightarrow \rho M^{4-d}$ と変換する. 変換前の Callan-Symanzik 方程式は

$$\left[M \frac{\partial}{\partial M} + \gamma C \frac{\partial}{\partial C} + \cdots \right] G(M, C) = 0.$$

変換後は

$$\begin{aligned}
 M \frac{\partial}{\partial M} G(M, \rho M^{4-d}) &= M \frac{\partial}{\partial M} G(M, C) + M \frac{\partial \rho M^{4-d}}{\partial M} \frac{\partial}{\partial \rho M^{4-d}} G(M, \rho M^{4-d}) \\
 &= M \frac{\partial}{\partial M} G(M, C) + (4-d) \frac{\partial}{\partial \rho} G(M, \rho M^{4-d}).
 \end{aligned}$$

及び

$$\gamma C \frac{\partial}{\partial C} G(M, C) = \gamma \rho \frac{\partial}{\partial \rho} G(M, \rho M^{4-d})$$

となるので, Callan-Symanzik 方程式は

$$\left[M \frac{\partial}{\partial M} + (\gamma + d - 4) \rho \frac{\partial}{\partial \rho} + \cdots \right] G(M, \rho M^{4-d}) = 0$$

と修正される.

(12.131)

$d = 4$ の Callan-Symanzik 方程式は既知なので、 d 次元を考える際は、これに帰着させれば良い。 d 次元の場合は \mathcal{L} を (12.129) の様に変形すれば、形式的に $d = 4$ となる。すなわち、 $d = 4$ のラグランジアン $\mathcal{L}^{(4)}$ の各項に適当な M の冪乗をかけることによって、 d 次元の場合を表すことができる。

質量項の補正

d 次元のラグランジアンの質量項は

$$\frac{1}{2}\rho_m M^2 \phi_M^2$$

である ($d = 4$ の場合と同じ)。 $d = 4$ の Callan-Symanzik 方程式 (12.119)

$$\left[M \frac{\partial}{\partial M} + \gamma_{\phi^2}^{(4)} m^2 \frac{\partial}{\partial m^2} + \cdots \right] G(M, m^2) = 0$$

で $m^2 \rightarrow \rho_m M^2$ とすれば d 次元の方程式が得られる。(12.123) と同様に

$$M \frac{\partial}{\partial M} G(M, \rho_m M^2) = M \frac{\partial}{\partial M} G(M, m^2) + 2\rho_m \frac{\partial}{\partial \rho_m} G(M, \rho_m M^2)$$

及び

$$\gamma_{\phi^2}^{(4)} m^2 \frac{\partial}{\partial m^2} G(M, m^2) = \gamma_{\phi^2}^{(4)} \rho_m \frac{\partial}{\partial \rho_m} G(M, \rho_m M^2)$$

なので、Callan-Symanzik 方程式は

$$\left[M \frac{\partial}{\partial M} + (\gamma_{\phi^2}^{(4)} - 2)\rho_m \frac{\partial}{\partial \rho_m} + \cdots \right] G(M, \rho_m M^2) = 0$$

と修正される。すなわち質量のベータ関数は

$$\beta_m = (\gamma_{\phi^2}^{(4)} - 2)\rho_m.$$

相互作用項の補正

$d = 4$ での相互作用項を

$$\frac{\lambda}{4} \phi_M^4 \rightarrow \frac{\lambda}{4} M^{4-d} \phi_M^4$$

とすれば d 次元の相互作用項を表すことができる。 $d = 4$ の Callan-Symanzik 方程式 (12.119)

$$\left[M \frac{\partial}{\partial M} + \beta^{(4)} \frac{\partial}{\partial \lambda} + \cdots \right] G(M, \lambda) = 0$$

で $\lambda \rightarrow M^{4-d}\lambda$ とすれば d 次元の方程式が得られる。(12.123) と同様に

$$M \frac{\partial}{\partial M} G(M, M^{4-d}\lambda) = M \frac{\partial}{\partial M} G(M, \lambda) + (4-d)\lambda \frac{\partial}{\partial \lambda} G(M, M^{4-d}\lambda)$$

及び

$$\beta^{(4)} \frac{\partial}{\partial \lambda} G(M, \lambda) = \beta^{(4)} \frac{\partial}{\partial \lambda} G(M, M^{4-d}\lambda)$$

なので, Callan-Symanzik 方程式は

$$\left[M \frac{\partial}{\partial M} + [(d-4)\lambda + \beta^{(4)}] \frac{\partial}{\partial \lambda} + \dots \right] G(M, M^{4-d}\lambda) = 0$$

と修正される. すなわち相互作用のベータ関数は

$$\beta = (d-4)\lambda + \beta^{(4)}.$$

一般作用素の補正

$d = 4$ での一般作用素を

$$\rho_i M^{4-d_i} \mathcal{O}_M^i(x) \rightarrow \rho_i M^{d-d_i} \mathcal{O}_M^i(x)$$

とすれば d 次元の相互作用項を表すことができる. $d = 4$ の Callan-Symanzik 方程式 (12.123)

$$\left[M \frac{\partial}{\partial M} + (d_i - 4 + \gamma_i^{(4)}) \rho_i \frac{\partial}{\partial \rho_i} + \dots \right] G(M, \rho_i M^{4-d}) = 0$$

で $M^{4-d_i} \rightarrow M^{d-d_i}$ とすれば d 次元の方程式が得られる. (12.123) と同様に

$$M \frac{\partial}{\partial M} G(M, \rho_i M^{d-d_i}) = M \frac{\partial}{\partial M} G(M, \rho_i M^{4-d_i}) + (d-4) \rho_i \frac{\partial}{\partial \rho_i} G(M, \rho_i M^{d-d_i})$$

及び

$$(d_i - 4 + \gamma_i^{(4)}) \rho_i \frac{\partial}{\partial \rho_i} G(M, \rho_i M^{4-d}) = (d_i - 4 + \gamma_i^{(4)}) \rho_i \frac{\partial}{\partial \rho_i} G(M, \rho_i M^{d-d_i})$$

なので, Callan-Symanzik 方程式は

$$\left[M \frac{\partial}{\partial M} + (d_i - d + \gamma_i^{(4)}) \rho_i \frac{\partial}{\partial \rho_i} + \dots \right] G(M, M^{d-d_i} \rho_i) = 0$$

と修正される. すなわち一般作用素のベータ関数は

$$\beta_i = (d_i - d + \gamma_i^{(4)}) \rho_i.$$

Problems

Problem 12.2: Beta function of the Gross-Neveu model

Gross-Neveu 模型は

$$\mathcal{L} = \sum_{i=1}^N \bar{\psi}_i (i \not{\partial}) \psi_i + \frac{g^2}{2} \left(\sum_{i=1}^N \bar{\psi}_i \psi_i \right)^2$$

で与えられる. $d = 2$ の Dirac 行列は

$$(\gamma^0)_{\alpha\beta} = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad (\gamma^1)_{\alpha\beta} = \begin{pmatrix} & i \\ i & \end{pmatrix}$$

である (スピノルの添字を $\alpha = 0, 1$ などのギリシャ文字で表す).

伝播函数は

$$i\alpha \xrightarrow{p} j\beta = \left(\frac{i}{\not{p}} \right)_{\beta\alpha} \delta_{ij}$$

で与えられる。ガンマ行列は対角成分を持たないので、 $\alpha = \beta$ なら伝播函数は 0 である。

4 点相関函数

$$\begin{array}{c}
 k\gamma \quad l\delta \\
 \swarrow \quad \searrow \\
 \text{---} \bigcirc \text{---} \\
 \nwarrow \quad \nearrow \\
 i\alpha \quad j\beta
 \end{array} = \langle \Omega | T \{ \psi_{i\alpha}(x_1) \psi_{j\beta}(x_2) \bar{\psi}_{k\gamma}(x_3) \bar{\psi}_{l\delta}(x_4) \} | \Omega \rangle$$

を求める。(4.31) から、1 次の展開は

$$\langle 0 | T \{ \psi_{i\alpha}(x_1) \psi_{j\beta}(x_2) \bar{\psi}_{k\gamma}(x_3) \bar{\psi}_{l\delta}(x_4) \left(i \frac{g^2}{2} \right) \int d^4x \sum_{mn} \sum_{\rho\sigma} \bar{\psi}_{m\rho}(x) \psi_{m\rho}(x) \bar{\psi}_{n\sigma}(x) \psi_{n\sigma}(x) \} | 0 \rangle$$

となる。縮約の方法には 4 通りある：

$$1. \overbrace{\psi_{i\alpha} \bar{\psi}_{k\gamma} \bar{\psi}_{m\rho} \psi_{m\rho}} \text{ なら}$$

$$m = i, \quad m = k, \quad \rho \neq \alpha, \quad \rho \neq \gamma; \quad n = j, \quad n = l, \quad \sigma \neq \beta, \quad \sigma \neq \delta \quad \therefore \quad \delta_{ik} \delta_{jl} \delta_{\alpha\gamma} \delta_{\beta\delta}.$$

$$2. \overbrace{\psi_{j\beta} \bar{\psi}_{l\delta} \bar{\psi}_{m\rho} \psi_{m\rho}} \text{ なら}$$

$$m = j, \quad m = l, \quad \rho \neq \beta, \quad \rho \neq \delta; \quad n = i, \quad n = k, \quad \sigma \neq \alpha, \quad \sigma \neq \gamma \quad \therefore \quad \delta_{ik} \delta_{jl} \delta_{\alpha\gamma} \delta_{\beta\delta}.$$

$$3. \overbrace{\psi_{i\alpha} \bar{\psi}_{l\delta} \bar{\psi}_{m\rho} \psi_{m\rho}} \text{ なら}$$

$$m = i, \quad m = l, \quad \rho \neq \alpha, \quad \rho \neq \delta; \quad n = j, \quad n = k, \quad \sigma \neq \beta, \quad \sigma \neq \gamma \quad \therefore \quad \delta_{il} \delta_{jk} \delta_{\alpha\delta} \delta_{\beta\gamma}.$$

$$4. \overbrace{\psi_{j\beta} \bar{\psi}_{k\gamma} \bar{\psi}_{m\rho} \psi_{m\rho}} \text{ なら}$$

$$m = j, \quad m = k, \quad \rho \neq \beta, \quad \rho \neq \gamma; \quad n = i, \quad n = l, \quad \sigma \neq \alpha, \quad \sigma \neq \delta \quad \therefore \quad \delta_{il} \delta_{jk} \delta_{\alpha\delta} \delta_{\beta\gamma}.$$

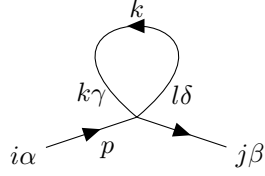
以上から、

$$\begin{array}{c}
 k\gamma \quad l\delta \\
 \swarrow \quad \searrow \\
 \text{---} \bullet \text{---} \\
 \nwarrow \quad \nearrow \\
 i\alpha \quad j\beta
 \end{array} = ig^2 (\delta_{ik} \delta_{jl} \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{il} \delta_{jk} \delta_{\alpha\delta} \delta_{\beta\gamma}).$$

相殺項は

$$\begin{array}{c}
 k\gamma \quad l\delta \\
 \swarrow \quad \searrow \\
 \text{---} \bigotimes \text{---} \\
 \nwarrow \quad \nearrow \\
 i\alpha \quad j\beta
 \end{array} = 2ig (\delta_{ik} \delta_{jl} \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{il} \delta_{jk} \delta_{\alpha\delta} \delta_{\beta\gamma}) \delta_g.$$

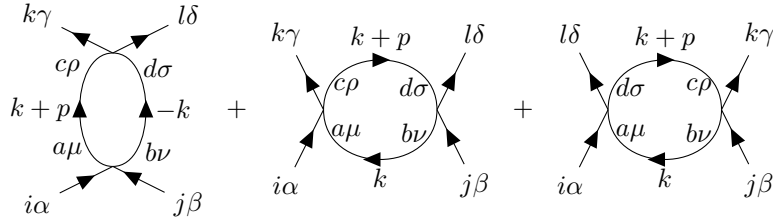
δ_f を求める.



$$= -ig^2 \sum_{kl} \sum_{\gamma\delta} \int \frac{d^d k}{(2\pi)^d} (\dots) \left(\frac{i}{\not{k}} \right)_{\gamma\delta} \delta_{kl} = 0$$

なので, $\delta_f = 0$.

δ_g を求める. 頂点の 1 ループは



から成る. $\log(-p^2)$ の発散項にのみ興味があるので, それ以外の項は無視する.

1 つ目は

$$V_s = -\frac{(ig^2)^2}{2} \sum_{abcd} \sum_{\mu\nu\rho\sigma} \int \frac{d^d k}{(2\pi)^d} (\delta_{kc} \delta_{ld} \delta_{\gamma\rho} \delta_{\delta\sigma} + \delta_{kd} \delta_{lc} \delta_{\gamma\sigma} \delta_{\delta\rho})$$

$$\times \left(\frac{i}{\not{k} + \not{p}} \right)_{\rho\mu} \delta_{ac} \left(\frac{i}{-\not{k}} \right)_{\sigma\nu} \delta_{bd} (\delta_{ia} \delta_{jb} \delta_{\alpha\mu} \delta_{\beta\nu} + \delta_{ib} \delta_{ja} \delta_{\alpha\nu} \delta_{\beta\mu}).$$

和を計算すれば

$$\begin{aligned} & \sum_{abcd} \sum_{\mu\nu\rho\sigma} (\delta_{kc} \delta_{ld} \delta_{\gamma\rho} \delta_{\delta\sigma} + \delta_{kd} \delta_{lc} \delta_{\gamma\sigma} \delta_{\delta\rho}) \left(\frac{i}{\not{k} + \not{p}} \right)_{\rho\mu} \delta_{ac} \left(\frac{i}{-\not{k}} \right)_{\sigma\nu} \delta_{bd} \\ & \times (\delta_{ia} \delta_{jb} \delta_{\alpha\mu} \delta_{\beta\nu} + \delta_{ib} \delta_{ja} \delta_{\alpha\nu} \delta_{\beta\mu}) \\ &= \sum_{ab} \sum_{\mu\nu\rho\sigma} (\delta_{ka} \delta_{lb} \delta_{\gamma\rho} \delta_{\delta\sigma} + \delta_{kb} \delta_{la} \delta_{\gamma\sigma} \delta_{\delta\rho}) \left(\frac{i}{\not{k} + \not{p}} \right)_{\rho\mu} \left(\frac{i}{-\not{k}} \right)_{\sigma\nu} \\ & \times (\delta_{ia} \delta_{jb} \delta_{\alpha\mu} \delta_{\beta\nu} + \delta_{ib} \delta_{ja} \delta_{\alpha\nu} \delta_{\beta\mu}) \\ &= \sum_{ab} \sum_{\mu\nu\rho\sigma} (\delta_{ka} \delta_{lb} \delta_{\gamma\rho} \delta_{\delta\sigma}) \left(\frac{i}{\not{k} + \not{p}} \right)_{\rho\mu} \left(\frac{i}{-\not{k}} \right)_{\sigma\nu} (\delta_{ia} \delta_{jb} \delta_{\alpha\mu} \delta_{\beta\nu}) \\ & + \sum_{ab} \sum_{\mu\nu\rho\sigma} (\delta_{ka} \delta_{lb} \delta_{\gamma\rho} \delta_{\delta\sigma}) \left(\frac{i}{\not{k} + \not{p}} \right)_{\rho\mu} \left(\frac{i}{-\not{k}} \right)_{\sigma\nu} (\delta_{ib} \delta_{ja} \delta_{\alpha\nu} \delta_{\beta\mu}) \\ & + \sum_{ab} \sum_{\mu\nu\rho\sigma} (\delta_{kb} \delta_{la} \delta_{\gamma\sigma} \delta_{\delta\rho}) \left(\frac{i}{\not{k} + \not{p}} \right)_{\rho\mu} \left(\frac{i}{-\not{k}} \right)_{\sigma\nu} (\delta_{ia} \delta_{jb} \delta_{\alpha\mu} \delta_{\beta\nu}) \\ & + \sum_{ab} \sum_{\mu\nu\rho\sigma} (\delta_{kb} \delta_{la} \delta_{\gamma\sigma} \delta_{\delta\rho}) \left(\frac{i}{\not{k} + \not{p}} \right)_{\rho\mu} \left(\frac{i}{-\not{k}} \right)_{\sigma\nu} (\delta_{ib} \delta_{ja} \delta_{\alpha\nu} \delta_{\beta\mu}) \\ &= (\delta_{ik} \delta_{jl}) \left(\frac{1}{\not{k} + \not{p}} \right)_{\gamma\alpha} \left(\frac{1}{\not{k}} \right)_{\delta\beta} + (\delta_{il} \delta_{jk}) \left(\frac{1}{\not{k} + \not{p}} \right)_{\gamma\beta} \left(\frac{1}{\not{k}} \right)_{\delta\alpha} \\ & + (\delta_{il} \delta_{jk}) \left(\frac{1}{\not{k} + \not{p}} \right)_{\delta\alpha} \left(\frac{1}{\not{k}} \right)_{\gamma\beta} + (\delta_{ik} \delta_{jl}) \left(\frac{1}{\not{k} + \not{p}} \right)_{\delta\beta} \left(\frac{1}{\not{k}} \right)_{\gamma\alpha} \end{aligned}$$

なので,

$$\begin{aligned}
 V_s &= g^4 (\delta_{ik} \delta_{jl}) \int \frac{d^d k}{(2\pi)^d} \frac{(\not{k} + \not{p})_{\gamma\alpha} (\not{k})_{\delta\beta} + (\not{k})_{\gamma\alpha} (\not{k} + \not{p})_{\delta\beta}}{2k^2(k+p)^2} \\
 &\quad + g^4 (\delta_{il} \delta_{jk}) \int \frac{d^d k}{(2\pi)^d} \frac{(\not{k} + \not{p})_{\gamma\beta} (\not{k})_{\delta\alpha} + (\not{k})_{\gamma\beta} (\not{k} + \not{p})_{\delta\alpha}}{k^2(k+p)^2} \\
 &\sim g^4 \delta_{ik} \delta_{jl} (\gamma^\mu)_{\gamma\alpha} (\gamma^\nu)_{\delta\beta} \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{k^2(k+p)^2} + g^4 \delta_{il} \delta_{jk} (\gamma^\mu)_{\gamma\beta} (\gamma^\nu)_{\delta\alpha} \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{k^2(k+p)^2} \\
 &= g^4 [\delta_{ik} \delta_{jl} (\gamma^\mu)_{\gamma\alpha} (\gamma^\nu)_{\delta\beta} + \delta_{il} \delta_{jk} (\gamma^\mu)_{\gamma\beta} (\gamma^\nu)_{\delta\alpha}] \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2(k+p)^2} \frac{k^2 g_{\mu\nu}}{d} \\
 &= g^4 [\delta_{ik} \delta_{jl} (\gamma^\mu)_{\gamma\alpha} (\gamma_\mu)_{\delta\beta} + \delta_{il} \delta_{jk} (\gamma^\mu)_{\gamma\beta} (\gamma_\mu)_{\delta\alpha}] \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k+p)^2} \\
 &= g^4 [\delta_{ik} \delta_{jl} (\gamma^\mu)_{\gamma\alpha} (\gamma_\mu)_{\delta\beta} + \delta_{il} \delta_{jk} (\gamma^\mu)_{\gamma\beta} (\gamma_\mu)_{\delta\alpha}] \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2},
 \end{aligned}$$

$k_\mu k_\nu$ の変換に (A.41) を使った. 2 つ目は

$$\begin{aligned}
 V_t &= (ig^2)^2 \sum_{abcd} \sum_{\mu\nu\rho\sigma} \int \frac{d^d k}{(2\pi)^d} (\delta_{ik} \delta_{ac} \delta_{\alpha\gamma} \delta_{\mu\rho} + \delta_{ic} \delta_{ak} \delta_{\alpha\rho} \delta_{\mu\gamma}) \\
 &\quad \times \left(\frac{1}{\not{k} + \not{p}} \right)_{\sigma\rho} \delta_{cd} \left(\frac{1}{\not{k}} \right)_{\mu\nu} \delta_{ab} (\delta_{jl} \delta_{db} \delta_{\beta\delta} \delta_{\sigma\nu} + \delta_{jb} \delta_{dl} \delta_{\beta\nu} \delta_{\sigma\delta}).
 \end{aligned}$$

和を計算すれば

$$\begin{aligned}
 &\sum_{abcd} \sum_{\mu\nu\rho\sigma} (\delta_{ik} \delta_{ac} \delta_{\alpha\gamma} \delta_{\mu\rho} + \delta_{ic} \delta_{ak} \delta_{\alpha\rho} \delta_{\mu\gamma}) \left(\frac{1}{\not{k} + \not{p}} \right)_{\sigma\rho} \delta_{cd} \left(\frac{1}{\not{k}} \right)_{\mu\nu} \delta_{ab} \\
 &\quad \times (\delta_{jl} \delta_{db} \delta_{\beta\delta} \delta_{\sigma\nu} + \delta_{jb} \delta_{dl} \delta_{\beta\nu} \delta_{\sigma\delta}) \\
 &= \sum_{ac} \sum_{\mu\nu\rho\sigma} (\delta_{ik} \delta_{ac} \delta_{\alpha\gamma} \delta_{\mu\rho} + \delta_{ic} \delta_{ak} \delta_{\alpha\rho} \delta_{\mu\gamma}) \left(\frac{1}{\not{k} + \not{p}} \right)_{\sigma\rho} \left(\frac{1}{\not{k}} \right)_{\mu\nu} \\
 &\quad \times (\delta_{jl} \delta_{ca} \delta_{\beta\delta} \delta_{\sigma\nu} + \delta_{ja} \delta_{cl} \delta_{\beta\nu} \delta_{\sigma\delta}) \\
 &= \sum_{ac} \sum_{\mu\nu\rho\sigma} (\delta_{ik} \delta_{ac} \delta_{\alpha\gamma} \delta_{\mu\rho}) \left(\frac{1}{\not{k} + \not{p}} \right)_{\sigma\rho} \left(\frac{1}{\not{k}} \right)_{\mu\nu} (\delta_{jl} \delta_{ca} \delta_{\beta\delta} \delta_{\sigma\nu}) \\
 &\quad + \sum_{ac} \sum_{\mu\nu\rho\sigma} (\delta_{ik} \delta_{ac} \delta_{\alpha\gamma} \delta_{\mu\rho}) \left(\frac{1}{\not{k} + \not{p}} \right)_{\sigma\rho} \left(\frac{1}{\not{k}} \right)_{\mu\nu} (\delta_{ja} \delta_{cl} \delta_{\beta\nu} \delta_{\sigma\delta}) \\
 &\quad + \sum_{ac} \sum_{\mu\nu\rho\sigma} (\delta_{ic} \delta_{ak} \delta_{\alpha\rho} \delta_{\mu\gamma}) \left(\frac{1}{\not{k} + \not{p}} \right)_{\sigma\rho} \left(\frac{1}{\not{k}} \right)_{\mu\nu} (\delta_{jl} \delta_{ca} \delta_{\beta\delta} \delta_{\sigma\nu}) \\
 &\quad + \sum_{ac} \sum_{\mu\nu\rho\sigma} (\delta_{ic} \delta_{ak} \delta_{\alpha\rho} \delta_{\mu\gamma}) \left(\frac{1}{\not{k} + \not{p}} \right)_{\sigma\rho} \left(\frac{1}{\not{k}} \right)_{\mu\nu} (\delta_{ja} \delta_{cl} \delta_{\beta\nu} \delta_{\sigma\delta}) \\
 &= N (\delta_{ik} \delta_{jl} \delta_{\alpha\gamma} \delta_{\beta\delta}) \text{Tr} \left(\frac{1}{\not{k} + \not{p}} \frac{1}{\not{k}} \right) + (\delta_{ik} \delta_{jl} \delta_{\alpha\gamma}) \left(\frac{1}{\not{k} + \not{p}} \frac{1}{\not{k}} \right)_{\delta\beta} \\
 &\quad + (\delta_{ik} \delta_{jl} \delta_{\beta\delta}) \left(\frac{1}{\not{k} + \not{p}} \frac{1}{\not{k}} \right)_{\gamma\alpha} + (\delta_{il} \delta_{jk}) \left(\frac{1}{\not{k}} \right)_{\delta\alpha} \left(\frac{1}{\not{k} + \not{p}} \right)_{\gamma\beta} \\
 &\sim (2N + 2) (\delta_{ik} \delta_{jl} \delta_{\alpha\gamma} \delta_{\beta\delta}) \frac{k^2 + k \cdot p}{k^2(k+p)^2} + (\delta_{il} \delta_{jk}) \left(\frac{1}{\not{k}} \right)_{\delta\alpha} \left(\frac{1}{\not{k} + \not{p}} \right)_{\gamma\beta}
 \end{aligned}$$

なので

$$\begin{aligned} V_t &= -g^4(2N+2)\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta} \int \frac{d^d k}{(2\pi)^d} \frac{k^2 + k \cdot p}{k^2(k+p)^2} - g^4\delta_{il}\delta_{jk}(\gamma^\mu)_{\gamma\beta}(\gamma^\nu)_{\delta\alpha} \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu k_\nu}{k^4} \\ &= -g^4(2N+2)\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta} \int \frac{d^d k}{(2\pi)^d} \frac{k^2 + k \cdot p}{k^2(k+p)^2} - g^4\delta_{il}\delta_{jk}(\gamma^\mu)_{\gamma\beta}(\gamma_\mu)_{\delta\alpha} \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2}. \end{aligned}$$

3 つ目は V_t で $k\gamma \leftrightarrow l\delta$ を入れ替えたもの :

$$V_u = -g^4(2N+2)\delta_{ik}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma} \int \frac{d^d k}{(2\pi)^d} \frac{k^2 + k \cdot p}{k^2(k+p)^2} - g^4\delta_{ik}\delta_{jl}(\gamma^\mu)_{\delta\beta}(\gamma_\mu)_{\gamma\alpha} \frac{1}{d} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2}.$$

以上から

$$\begin{aligned} V_s + V_t + V_u &= -g^4(2N+2) (\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}) \int \frac{d^d k}{(2\pi)^d} \frac{k^2 + p \cdot k}{k^2(k+p)^2} \\ &= -g^4(2N+2) (\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}) \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2 + \Delta}{(\ell^2 - \Delta)^2} \\ &\approx ig^4(2N+2) (\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}) \int_0^1 dx \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \Gamma(1-d/2) \left(\frac{1}{\Delta}\right)^{1-d/2}. \end{aligned}$$

$\epsilon = 2 - d$ とすれば

$$\begin{aligned} \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \Gamma(1-d/2) \left(\frac{1}{\Delta}\right)^{1-d/2} &= \frac{1}{4\pi} (1 - \epsilon/2) \Gamma(\epsilon/2) \left(\frac{4\pi}{\Delta}\right)^{\epsilon/2} \\ &\approx \frac{1}{4\pi} \left(1 - \frac{\epsilon}{2}\right) \left(\frac{2}{\epsilon} - \gamma\right) \left(1 - \frac{\epsilon}{2} \log \frac{\Delta}{4\pi}\right) \\ &\approx \frac{1}{4\pi} \left(\frac{2}{\epsilon} - \gamma - \log \frac{\Delta}{4\pi}\right) \\ &\approx -\frac{1}{4\pi} \log(-p^2) \end{aligned}$$

なので,

$$V_s + V_t + V_u = -\frac{ig^4}{4\pi} (2N+2) (\delta_{ik}\delta_{jl}\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{il}\delta_{jk}\delta_{\alpha\delta}\delta_{\beta\gamma}) \log(-p^2).$$

$p^2 = -M^2$ で, これと相殺項の和が 0 なので,

$$\delta_g = \frac{g^3}{4\pi} (N+1) \log M^2 = \frac{g^3}{2\pi} (N+1) \log M.$$

Chapter 13

Critical Exponents and Scalar Field Theory

13.3 The Nonlinear Sigma Model

Figure 13.1

非線形シグマ模型 (13.73):

$$\mathcal{L} = \frac{1}{2g^2} |\partial_\mu \vec{\pi}|^2 + \frac{1}{2g^2} (\vec{\pi} \cdot \partial_\mu \vec{\pi})^2$$

の 4 点相関関数を求める．伝播関数は

$$\begin{aligned} \overline{\pi^i(x)} \pi^j(y) &= \int \frac{d^2 k}{(2\pi)^2} \frac{ig^2}{p^2} \delta^{ij} e^{-ik \cdot (x-y)}, \\ \overline{\pi^i(p)} \pi^j(q) &= (2\pi)^2 \delta^{(2)}(p+q) \frac{ig^2}{p^2} \delta^{ij}. \end{aligned}$$

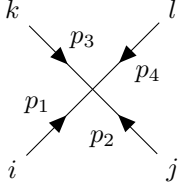
(4.31) から

$$\begin{aligned} &\langle \Omega | T \{ \pi^i(x_1) \pi^j(x_2) \pi^k(x_3) \pi^l(x_4) \} | \Omega \rangle \\ &\approx \frac{i}{2g^2} \sum_{rs} \int d^2 x \langle 0 | T \{ \pi^i(x_1) \pi^j(x_2) \pi^k(x_3) \pi^l(x_4) \pi^r(x) \partial_\mu \pi^r(x) \pi^s(x) \partial^\mu \pi^s(x) \} | 0 \rangle \\ &= \frac{i}{2g^2} \left(\prod_{i=1}^4 \frac{d^2 p_i}{(2\pi)^2} e^{-ip_i \cdot x_i} \right) \left(\prod_{i=1}^4 \frac{d^2 k_i}{(2\pi)^2} \right) \sum_{rs} \int d^2 x e^{-i(k_1+k_2+k_3+k_4) \cdot x} \\ &\quad \times \langle 0 | T \{ \pi^i(p_1) \pi^j(p_2) \pi^k(p_3) \pi^l(p_4) \pi^r(k_1) (-ik_{2\mu}) \pi^r(k_2) \pi^s(k_3) (-ik_4^\mu) \pi^s(k_4) \} | 0 \rangle \\ &= -\frac{i}{2g^2} \left(\prod_{i=1}^4 \frac{d^2 p_i}{(2\pi)^2} e^{-ip_i \cdot x_i} \right) \left(\prod_{i=1}^4 \frac{d^2 k_i}{(2\pi)^2} \right) (k_2 \cdot k_4) \sum_{rs} (2\pi)^2 \delta^{(2)}(k_1+k_2+k_3+k_4) \\ &\quad \times \langle 0 | T \{ \pi^i(p_1) \pi^j(p_2) \pi^k(p_3) \pi^l(p_4) \pi^r(k_1) \pi^r(k_2) \pi^s(k_3) \pi^s(k_4) \} | 0 \rangle. \end{aligned}$$

縮約は全部で 24 通りある．例えば， $\delta^{ij} \delta^{kl}$ を与える項は (p_1, p_2) と (k_1, k_2) を縮約する（4 通り）か， (p_1, p_2) と (k_3, k_4) を縮約する（4 通り），合計 8 通り．伝播関数を代入すれば

$$-\frac{i}{g^2} \left(\prod_{i=1}^4 \frac{d^2 p_i}{(2\pi)^2} e^{-ip_i \cdot x_i} \frac{ig^2}{p_i^2} \right) (2\pi)^2 \delta^{(2)}(p_1+p_2+p_3+p_4) \delta^{ij} \delta^{kl} (p_1+p_2)(p_3+p_4)$$

を得る. $\delta^{ik}\delta^{jl}$, $\delta^{il}\delta^{jk}$ についても同様に計算すれば



$$= -\frac{i}{g^2} \left[(p_1 + p_2)(p_3 + p_4) \delta^{ij} \delta^{kl} + (p_1 + p_3)(p_2 + p_4) \delta^{ik} \delta^{jl} + (p_1 + p_4)(p_2 + p_3) \delta^{il} \delta^{jk} \right].$$

(13.96)

(7.81) を使えば

$$\begin{aligned} \langle \phi_a(0) \phi_b(0) \rangle &= \int_{\substack{b\Lambda \leq |p| < \Lambda \\ b\Lambda \leq |q| < \Lambda}} \frac{d^2 p}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} \langle \phi_a(p) \phi_b(q) \rangle = \int_{\substack{b\Lambda \leq |p| < \Lambda \\ b\Lambda \leq |q| < \Lambda}} \frac{d^2 p}{(2\pi)^2} \frac{d^2 q}{(2\pi)^2} (2\pi)^2 \delta^{(2)}(p+q) \frac{g^2}{p^2} \delta_{ab} \\ &= \int_{b\Lambda \leq |p| < \Lambda} \frac{d^2 p}{(2\pi)^2} \frac{g^2}{p^2} \delta_{ab} = \int_{b\Lambda \leq |p| < \Lambda} \frac{d|p|}{2\pi} \frac{g^2}{|p|} \delta_{ab} = \delta_{ab} \frac{g^2}{2\pi} \log \frac{1}{b}. \end{aligned}$$

(13.109)

$\beta(T)$ を $T \sim T_*$ で展開して

$$\beta(T) \approx \left[\frac{d\beta}{dT} \Big|_{T=T_*} \right] (T - T_*).$$

(12.73) から

$$\frac{\partial \bar{T}}{\partial \log p/M} = \beta(\bar{T}) \approx \left[\frac{d\beta}{dT} \Big|_{T=T_*} \right] (\bar{T} - T_*) = \left[\frac{d\beta}{dT} \Big|_{T=T_*} \right] \bar{\rho}_T.$$

(13.114)

(13.113) で部分積分を実行して

$$\begin{aligned} &\exp \left[-\frac{1}{2g_0^2} \int d^d x \{ (\partial_\mu n)^2 + i\alpha(n^2 - 1) \} \right] \\ &= \exp \left[-\frac{1}{2g_0^2} \int d^d x \{ -\vec{n} \cdot (\partial^2 \vec{n}) + i\alpha(n^2 - 1) \} \right] \\ &= \exp \left[-\frac{1}{2g_0^2} \int d^d x \{ \vec{n} \cdot (-\partial^2 + i\alpha) \vec{n} - i\alpha \} \right] \\ &= \exp \left[-\frac{1}{2g_0^2} \int d^d x \vec{n} \cdot (-\partial^2 + i\alpha) \vec{n} \right] \exp \left[\frac{i}{2g_0^2} \int d^d x \alpha \right]. \end{aligned}$$

ここで, \vec{n} を適当に変換して

$$Z = \int \mathcal{D}\alpha \mathcal{D}\vec{n} \exp \left[-\int d^d x \vec{n} \cdot (-\partial^2 + i\alpha) \vec{n} \right] \exp \left[\frac{i}{2g_0^2} \int d^d x \alpha \right]$$

としてよい. (9.24) から

$$\begin{aligned} & \int \mathcal{D}\vec{n} \exp \left[- \int d^d x \vec{n} \cdot (-\partial^2 + i\alpha) \vec{n} \right] \\ &= \int \mathcal{D}n^1 \cdots \mathcal{D}n^N \exp \left[- \int d^d x n^1 (-\partial^2 + i\alpha) n^1 \right] \times \cdots \times \exp \left[- \int d^d x n^N (-\partial^2 + i\alpha) n^N \right] \\ &= [\det(-\partial^2 + i\alpha)]^{-N/2} \end{aligned}$$

なので,

$$Z = \int \mathcal{D}\alpha [\det(-\partial^2 + i\alpha)]^{-N/2} \exp \left[\frac{i}{2g_0^2} \int d^d x \alpha \right].$$

2 つ目の表式は (9.77) を使えば得られる.

(13.115)

$(-\partial^2 + i\alpha)$ の固有値は固有ベクトル $|k\rangle$ に対し $k^2 + i\alpha$ で与えられるので,

$$\mathrm{Tr}\{\log(-\partial^2 + i\alpha)\} = \frac{1}{V} \sum_k \log(k^2 + i\alpha).$$

ここで, (9.22) と同様に離散 Fourier 変換を行った. (13.114) の指数関数の引数が $\alpha(x)$ に関し極小なので,

$$\begin{aligned} 0 &= \frac{\delta}{\delta\alpha(x)} \left[-\frac{N}{2} \mathrm{Tr}\{\log(-\partial^2 + i\alpha)\} + \frac{i}{2g_0^2} \int d^d y \alpha(y) \right] \\ &= \frac{\delta}{\delta\alpha(x)} \left[-\frac{N}{2} \frac{1}{V} \sum_k \log(k^2 + i\alpha) \right] + \frac{i}{2g_0^2} \int d^d y \delta(x-y) \\ &= -\frac{N}{2V} \sum_k \frac{1}{k^2 + i\alpha} \frac{\delta}{\delta\alpha(x)} (k^2 + i\alpha) + \frac{i}{2g_0^2} \\ &= -\frac{iN}{2V} \sum_k \frac{1}{k^2 + i\alpha} + \frac{i}{2g_0^2} \\ &\rightarrow -\frac{iN}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + i\alpha} + \frac{i}{2g_0^2}. \end{aligned}$$

Problems

Problem 13.3: The $\mathbb{C}P^N$ model

(a)

$\mathbb{C}P^1$ モデルのスカラーを

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

とする. $\vec{n} = z^\dagger \vec{\sigma} z$ を計算すれば

$$\begin{aligned} n_1 &= (z_1^*, z_2^*) \sigma^1 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1^*, z_2^*) \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 z_2^* + z_1^* z_2, \\ n_2 &= (z_1^*, z_2^*) \sigma^2 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1^*, z_2^*) \begin{pmatrix} & -i \\ i & \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = i(z_1 z_2^* - z_1^* z_2), \end{aligned}$$

$$n_3 = (z_1^*, z_2^*) \sigma^3 \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = (z_1^*, z_2^*) \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = |z_1|^2 - |z_2|^2.$$

この結果を使えば

$$n_1^2 + n_2^2 + n_3^2 = (|z_1|^2 + |z_2|^2)^2$$

なので, 規格化条件 $n_1^2 + n_2^2 + n_3^2 = 1$ は $|z_1|^2 + |z_2|^2 = 1$ と等しい. 非線形シグマ模型のラグランジアン (13.67) に代入して

$$\begin{aligned} & (\partial_\mu n^1)(\partial^\mu n^1) + (\partial_\mu n^2)(\partial^\mu n^2) + (\partial_\mu n^3)(\partial^\mu n^3) \\ &= [\partial_\mu(z_1 z_2^*) + \partial_\mu(z_1^* z_2)][\partial^\mu(z_1 z_2^*) + \partial^\mu(z_1^* z_2)] \\ &\quad - [\partial_\mu(z_1 z_2^*) - \partial_\mu(z_1^* z_2)][\partial^\mu(z_1 z_2^*) - \partial^\mu(z_1^* z_2)] \\ &\quad + [\partial_\mu(z_1 z_1^*) - \partial_\mu(z_2 z_2^*)][\partial^\mu(z_1 z_1^*) - \partial^\mu(z_2 z_2^*)] \\ &= 4\partial_\mu(z_1 z_2^*)\partial^\mu(z_1^* z_2) + \partial_\mu(z_1 z_1^*)\partial^\mu(z_1 z_1^*) + \partial_\mu(z_2 z_2^*)\partial^\mu(z_2 z_2^*) - 2\partial_\mu(z_1 z_1^*)\partial^\mu(z_2 z_2^*) \\ &= 4|z_1|^2(\partial_\mu z_2)(\partial^\mu z_2^*) + 4|z_2|^2(\partial_\mu z_1)(\partial^\mu z_1^*) + 4z_1 z_2(\partial_\mu z_1^*)(\partial^\mu z_2^*) + 4z_1^* z_2^*(\partial_\mu z_1)(\partial^\mu z_2) \\ &\quad + 2|z_1|^2(\partial_\mu z_1)(\partial^\mu z_1^*) + z_1^2(\partial_\mu z_1^*)(\partial^\mu z_1^*) + z_1^{*2}(\partial_\mu z_1)(\partial^\mu z_1) \\ &\quad + 2|z_2|^2(\partial_\mu z_2)(\partial^\mu z_2^*) + z_2^2(\partial_\mu z_2^*)(\partial^\mu z_2^*) + z_2^{*2}(\partial_\mu z_2)(\partial^\mu z_2) \\ &\quad - 2z_1 z_2(\partial_\mu z_1^*)(\partial^\mu z_2^*) - 2z_1^* z_2^*(\partial_\mu z_1)(\partial^\mu z_2^*) - 2z_1 z_2^*(\partial_\mu z_1^*)(\partial^\mu z_2) - 2z_1^* z_2^*(\partial_\mu z_1)(\partial^\mu z_2) \\ &= 4|z_1|^2(\partial_\mu z_2)(\partial^\mu z_2^*) + 4|z_2|^2(\partial_\mu z_1)(\partial^\mu z_1^*) + 2|z_1|^2(\partial_\mu z_1)(\partial^\mu z_1^*) + 2|z_2|^2(\partial_\mu z_2)(\partial^\mu z_2^*) \\ &\quad + z_1^2(\partial_\mu z_1^*)(\partial^\mu z_1^*) + z_1^{*2}(\partial_\mu z_1)(\partial^\mu z_1) + z_2^2(\partial_\mu z_2^*)(\partial^\mu z_2^*) + z_2^{*2}(\partial_\mu z_2)(\partial^\mu z_2) \\ &\quad + 2z_1 z_2(\partial_\mu z_1^*)(\partial^\mu z_2^*) + 2z_1^* z_2^*(\partial_\mu z_1)(\partial^\mu z_2) - 2z_1^* z_2(\partial_\mu z_1)(\partial^\mu z_2^*) - 2z_1 z_2^*(\partial_\mu z_1^*)(\partial^\mu z_2) \\ &= 4(\partial_\mu z_1)(\partial^\mu z_1^*) + 4(\partial_\mu z_2)(\partial^\mu z_2^*) - 2|z_1|^2(\partial_\mu z_1)(\partial^\mu z_1^*) - 2|z_2|^2(\partial_\mu z_2)(\partial^\mu z_2^*) \\ &\quad - 2z_1^* z_2(\partial_\mu z_1)(\partial^\mu z_2^*) - 2z_1 z_2^*(\partial_\mu z_1^*)(\partial^\mu z_2) \\ &\quad + (z_1^* \partial_\mu z_1 + z_2^* \partial_\mu z_2)(z_1^* \partial^\mu z_1 + z_2^* \partial^\mu z_2) + (z_1 \partial_\mu z_1^* + z_2 \partial_\mu z_2^*)(z_1 \partial^\mu z_1^* + z_2 \partial^\mu z_2^*) \\ &= 4(\partial_\mu z_1)(\partial^\mu z_1^*) + 4(\partial_\mu z_2)(\partial^\mu z_2^*) - 2(z_1^* \partial_\mu z_1 + z_2^* \partial_\mu z_2)(z_1 \partial^\mu z_1^* + z_2 \partial^\mu z_2^*) \\ &\quad + (z_1^* \partial_\mu z_1 + z_2^* \partial_\mu z_2)(z_1^* \partial^\mu z_1 + z_2^* \partial^\mu z_2) + (z_1 \partial_\mu z_1^* + z_2 \partial_\mu z_2^*)(z_1 \partial^\mu z_1^* + z_2 \partial^\mu z_2^*). \end{aligned}$$

$|z_1|^2 + |z_2|^2 = 1$ を微分して

$$z_1 \partial_\mu z_1^* + z_1^* \partial_\mu z_1 + z_2 \partial_\mu z_2^* + z_2^* \partial_\mu z_2 = 0$$

なので,

$$\begin{aligned} &= 4(\partial_\mu z_1)(\partial^\mu z_1^*) + 4(\partial_\mu z_2)(\partial^\mu z_2^*) - 2(z_1^* \partial_\mu z_1 + z_2^* \partial_\mu z_2)(z_1 \partial^\mu z_1^* + z_2 \partial^\mu z_2^*) \\ &\quad + 2(z_1^* \partial_\mu z_1 + z_2^* \partial_\mu z_2)(z_1^* \partial^\mu z_1 + z_2^* \partial^\mu z_2) \\ &= 4(\partial_\mu z_1)(\partial^\mu z_1^*) + 4(\partial_\mu z_2)(\partial^\mu z_2^*) \\ &\quad - 2(z_1^* \partial_\mu z_1 + z_2^* \partial_\mu z_2)(z_1 \partial^\mu z_1^* + z_2 \partial^\mu z_2^* - z_1^* \partial^\mu z_1 - z_2^* \partial^\mu z_2) \\ &= 4(\partial_\mu z_1)(\partial^\mu z_1^*) + 4(\partial_\mu z_2)(\partial^\mu z_2^*) - 4(z_1^* \partial_\mu z_1 + z_2^* \partial_\mu z_2)(z_1 \partial^\mu z_1^* + z_2 \partial^\mu z_2^*). \end{aligned}$$

以上から

$$|\partial_\mu \vec{n}|^2 = 4[(\partial_\mu \mathbf{z})(\partial^\mu \mathbf{z}^*) + (\mathbf{z}^* \cdot \partial_\mu \mathbf{z})(\mathbf{z} \cdot \partial^\mu \mathbf{z}^*)].$$

Final Project II: The Coleman-Weinberg Potential

(a)

Coleman-Weinberg 模型：

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^\dagger(D_\mu\phi) + \mu^2\phi^\dagger\phi - \frac{\lambda}{6}(\phi^\dagger\phi)^2.$$

ポテンシャル項は

$$\phi_0 = \mu\sqrt{\frac{3}{\lambda}}$$

で最小値を取る.

$$\phi(x) = \phi_0 + \frac{1}{\sqrt{2}}[\sigma(x) + i\pi(x)]$$

とする.

$$\begin{aligned} D_\mu\phi(x) &= (\partial_\mu + ieA_\mu) \left[\phi_0 + \frac{1}{\sqrt{2}}[\sigma(x) + i\pi(x)] \right] \\ &= \frac{1}{\sqrt{2}}\partial_\mu\sigma + \frac{i}{\sqrt{2}}\partial_\mu\pi + ieA_\mu\phi_0 + \frac{i}{\sqrt{2}}eA_\mu\sigma(x) - \frac{1}{\sqrt{2}}eA_\mu\pi(x) \\ &= \left[\frac{1}{\sqrt{2}}\partial_\mu\sigma - \frac{1}{\sqrt{2}}eA_\mu\pi(x) \right] + \frac{i}{\sqrt{2}}\partial_\mu\pi + ieA_\mu\phi_0 + \frac{i}{\sqrt{2}}eA_\mu\sigma(x) \end{aligned}$$

なので,

$$\begin{aligned} (D_\mu\phi)^\dagger(D_\mu\phi) &= \left[\frac{1}{\sqrt{2}}\partial_\mu\sigma - \frac{1}{\sqrt{2}}eA_\mu\pi(x) \right]^2 + \left[\frac{1}{\sqrt{2}}\partial_\mu\pi + eA_\mu\phi_0 + \frac{1}{\sqrt{2}}eA_\mu\sigma(x) \right]^2 \\ &= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}e^2A_\mu A^\mu\pi^2 - e\pi A^\mu\partial_\mu\sigma \\ &\quad + \frac{1}{2}(\partial_\mu\pi)^2 + \phi_0^2e^2A_\mu A^\mu + \frac{1}{2}e^2A_\mu A^\mu\sigma^2 \\ &\quad + \sqrt{2}\phi_0eA^\mu\partial_\mu\pi + \sqrt{2}\phi_0e^2A^\mu A_\mu\sigma + e\sigma A^\mu\partial_\mu\pi \\ &= \frac{1}{2}(\sqrt{2}\phi_0)^2e^2A_\mu A^\mu + \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 + \frac{1}{2}e^2A_\mu A^\mu\pi^2 + \frac{1}{2}e^2A_\mu A^\mu\sigma^2 \\ &\quad + e\sigma A^\mu\partial_\mu\pi - e\pi A^\mu\partial_\mu\sigma + \sqrt{2}\phi_0eA^\mu\partial_\mu\pi + \sqrt{2}\phi_0e^2A^\mu A_\mu\sigma. \end{aligned}$$

さらに

$$\phi^\dagger\phi = \left(\phi_0 + \frac{\sigma}{\sqrt{2}} \right)^2 + \left(\frac{\pi}{\sqrt{2}} \right)^2 = \phi_0^2 + \sqrt{2}\phi_0\sigma + \frac{\sigma^2}{2} + \frac{\pi^2}{2}$$

および

$$\begin{aligned}
(\phi^\dagger \phi)^2 &= \left[\left(\phi_0 + \frac{\sigma}{\sqrt{2}} \right)^2 + \left(\frac{\pi}{\sqrt{2}} \right)^2 \right]^2 \\
&= \left(\phi_0 + \frac{\sigma}{\sqrt{2}} \right)^4 + \pi^2 \left(\phi_0 + \frac{\sigma}{\sqrt{2}} \right)^2 + \frac{\pi^4}{4} \\
&= \phi_0^4 + (\sqrt{2}\phi_0)^3 \sigma + 3\phi_0^2 \sigma^2 + \sqrt{2}\phi_0 \sigma^3 + \frac{\sigma^4}{4} + \pi^2 \left(\phi_0^2 + \sqrt{2}\phi_0 \sigma + \frac{\sigma^2}{2} \right) + \frac{\pi^4}{4}.
\end{aligned}$$

以上から

$$\begin{aligned}
\mathcal{L} &\rightarrow -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2}(\sqrt{2}\phi_0)^2 e^2 A_\mu A^\mu \\
&\quad + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{1}{2}(\partial_\mu \pi)^2 + \frac{1}{2}e^2 A_\mu A^\mu \pi^2 + \frac{1}{2}e^2 A_\mu A^\mu \sigma^2 \\
&\quad + e\sigma A^\mu \partial_\mu \pi - e\pi A^\mu \partial_\mu \sigma + \sqrt{2}\phi_0 e A^\mu \partial_\mu \pi + \sqrt{2}\phi_0 e^2 A^\mu A_\mu \sigma \\
&\quad + \mu^2 \left(\sqrt{2}\phi_0 \sigma + \frac{\sigma^2}{2} + \frac{\pi^2}{2} \right) \\
&\quad - \frac{\lambda}{6} \left[(\sqrt{2}\phi_0)^3 \sigma + 3\phi_0^2 \sigma^2 + \sqrt{2}\phi_0 \sigma^3 + \frac{\sigma^4}{4} + \pi^2 \left(\phi_0^2 + \sqrt{2}\phi_0 \sigma + \frac{\sigma^2}{2} \right) + \frac{\pi^4}{4} \right] \\
&= -\frac{1}{4}(F_{\mu\nu})^2 + \frac{1}{2} \left(\frac{6\mu^2 e^2}{\lambda} \right) A_\mu A^\mu + \frac{1}{2}(\partial_\mu \sigma)^2 - \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{2}(\partial_\mu \pi)^2 \\
&\quad - \frac{\lambda}{6} \left[\frac{\sigma^4}{4} + \mu\sqrt{\frac{6}{\lambda}}\sigma^3 + \frac{\pi^2 \sigma^2}{2} + \mu\sqrt{\frac{6}{\lambda}}\pi^2 \sigma + \frac{\pi^4}{4} \right] \\
&\quad + \frac{1}{2}e^2 A_\mu A^\mu \pi^2 + \frac{1}{2}e^2 A_\mu A^\mu \sigma^2 \\
&\quad + e\sigma A^\mu \partial_\mu \pi - e\pi A^\mu \partial_\mu \sigma + \mu\sqrt{\frac{6}{\lambda}}e A^\mu \partial_\mu \pi + \mu\sqrt{\frac{6}{\lambda}}e^2 A^\mu A_\mu \sigma.
\end{aligned}$$

Part III

Non-Abelian Gauge Theories

Chapter 15

Non-Abelian Gauge Invariance

15.3 The Gauge-Invariant Wilson Loop

(15.62)

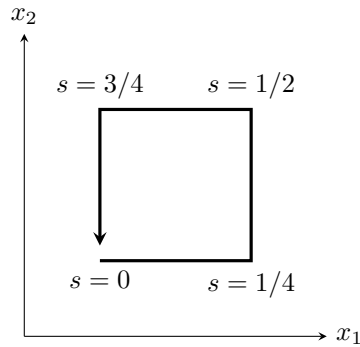
(15.56) を ϵ の 1 次まで展開すれば,

$$U_P(x, x) \approx 1 + ig \oint ds \frac{dx^\mu}{ds} A_\mu^a(x(s)) t^a \approx 1 + ig \left[\frac{\partial A_2^a}{\partial x^1} \epsilon^2 - \frac{\partial A_1^a}{\partial x^2} \epsilon^2 \right] t^a.$$

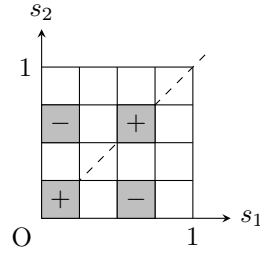
2 次の展開は

$$\begin{aligned} & -\frac{g^2}{2} \int_0^1 ds_1 \int_0^1 ds_2 \frac{dx^\mu}{ds_1} \frac{dx^\nu}{ds_2} A_\mu^a(x(s_1)) A_\nu^b(x(s_2)) P\{t^a t^b\} \\ & \approx -\frac{g^2}{2} A_\mu^a(x) A_\nu^b(x) \int_0^1 ds_1 \int_0^1 ds_2 \frac{dx^\mu}{ds_1} \frac{dx^\nu}{ds_2} P\{t^a t^b\} \end{aligned}$$

で与えられる. $s_1 > s_2$ なら $P\{t^a t^b\} = t^a t^b$ である.



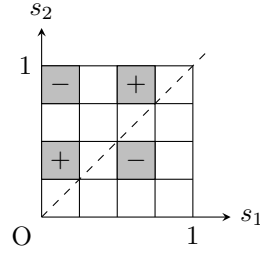
$\mu = \nu = 1$ の場合, $dx^1/ds \neq 0$ となるのは $0 < s < 1/4$ と $1/2 < s < 3/4$ のみ.



$s_1 > s_2$ の領域では積分が 0 になり, $s_1 < s_2$ の領域でも同様に積分は 0 となる.

$\mu = \nu = 2$ の場合, $dx^2/ds \neq 0$ となるのは $1/4 < s < 1/2$ と $3/4 < s < 1$ のみ. $s_1 > s_2$ の領域では積分が 0 になり, $s_1 < s_2$ の領域でも同様に積分は 0 となる.

$\mu = 1, \nu = 2$ の場合,



$s_1 > s_2$ の領域では積分は

$$-16\epsilon^2 \frac{1}{16} t^a t^b = -\epsilon^2 t^a t^b.$$

$s_1 < s_2$ の領域では積分は

$$+16\epsilon^2 \frac{1}{16} t^b t^a = \epsilon^2 t^b t^a.$$

(15.44) から

$$\epsilon^2 \frac{g^2}{2} A_1^a A_2^b [t^a, t^b] = \frac{i}{2} g^2 \epsilon^2 f^{abc} A_1^a A_2^b t^c.$$

$\mu = 2, \nu = 1$ の場合も同様なので,

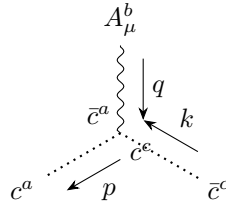
$$\begin{aligned} U_P(x, x) &\approx 1 + ig \left[\frac{\partial A_2^a}{\partial x^1} \epsilon^2 - \frac{\partial A_1^a}{\partial x^2} \epsilon^2 \right] t^a + ig \epsilon^2 f^{abc} A_1^a A_2^b t^c \\ &= 1 + ig \epsilon^2 \left[\partial_1 A_2^c - \partial_2 A_1^c + g f^{abc} A_1^a A_2^b \right] t^c. \end{aligned}$$

Chapter 16

Quantization of Non-Abelian Gauge Theories

16.3 Ghosts and Unitarity

Figure 16.5



(16.32) から

$$\begin{aligned}
 \mathcal{L}_{\text{ghost}} &= \bar{c}^a (-g \partial^\mu f^{abc} A_\mu^b) c^c + \dots \\
 &= -g f^{abc} \bar{c}^a (\partial^\mu A_\mu^b) c^c - g f^{abc} \bar{c}^a A_\mu^b (\partial^\mu c^c) + \dots \\
 &= -g f^{abc} \bar{c}^a (-i q^\mu A_\mu^b) c^c - g f^{abc} \bar{c}^a A_\mu^b (-i k^\mu c^c) + \dots \\
 &= -g f^{abc} \bar{c}^a (-i p^\mu A_\mu^b) c^c + \dots
 \end{aligned}$$

となるので、頂点は

$$i\mathcal{L} \rightarrow -g f^{abc} p_\mu.$$

16.5 One-Loop Divergences of Non-Abelian Gauge Theory

p.522 最後の式

フェルミオンは $d(r)$ 個の Dirac スピノル ψ_i からなる： $\psi = (\psi_1, \dots, \psi_{d(r)})$. Lie 代数の生成子はサイズが $d(r) \times d(r)$ の行列 $d(G)$ 個の集合： $\{t^a \mid 1 \leq a \leq d(G)\}$.

(16.34) のうち、ボソンとフェルミオンの頂点は

$$\mathcal{L}_{\bar{\psi}\psi A} = g \bar{\psi}^i \gamma^\mu \psi^j A_\mu^b (t^b)_{ij}$$

$$\begin{aligned}
&= N^{\mu\nu} \\
&\quad - \frac{1-\xi}{p^2} [p^\mu(q-p)^\sigma + p^\sigma(2p+q)^\mu + g^{\sigma\mu}p \cdot (-p-2q)] \\
&\quad \times [p^\nu(p-q)_\sigma + p_\sigma(-2p-q)^\nu + \delta^\nu_\sigma p \cdot (p+2q)] \\
&\quad - \frac{1-\xi}{(p+q)^2} [g^{\mu\rho}(p+q) \cdot (q-p) + (p+q)^\rho(2p+q)^\mu + (p+q)^\mu(-p-2q)^\rho] \\
&\quad \times [\delta^\nu_\rho(p+q) \cdot (p-q) + (p+q)_\rho(-2p-q)^\nu + (p+q)^\nu(p+2q)_\rho] \\
&\quad + \frac{(1-\xi)^2}{p^2(p+q)^2} [p^\mu(p+q) \cdot (q-p) + p \cdot (p+q)(2p+q)^\mu + (p+q)^\mu p \cdot (-p-2q)] \\
&\quad \times [p^\nu(p+q) \cdot (p-q) + p \cdot (p+q)(-2p-q)^\nu + (p+q)^\nu p \cdot (p+2q)] \\
&= N^{\mu\nu} \\
&\quad - \frac{1-\xi}{p^2} [p^\mu(q-p)^\sigma + p^\sigma(2p+q)^\mu + g^{\sigma\mu}p \cdot (-p-2q)] \\
&\quad \times [p^\nu(p-q)_\sigma + p_\sigma(-2p-q)^\nu + \delta^\nu_\sigma p \cdot (p+2q)] \\
&\quad - \frac{1-\xi}{(p+q)^2} [(p+q)^\mu(-p-2q)^\sigma + (p+q)^\sigma(2p+q)^\mu + g^{\sigma\mu}(p+q) \cdot (q-p)] \\
&\quad \times [(p+q)^\nu(p+2q)_\sigma + (p+q)_\sigma(-2p-q)^\nu + \delta^\nu_\sigma(p+q) \cdot (p-q)] \\
&\quad + \frac{(1-\xi)^2}{p^2(p+q)^2} [p^\mu(p+q) \cdot (q-p) + p \cdot (p+q)(2p+q)^\mu + (p+q)^\mu p \cdot (-p-2q)] \\
&\quad \times [p^\nu(p+q) \cdot (p-q) + p \cdot (p+q)(-2p-q)^\nu + (p+q)^\nu p \cdot (p+2q)].
\end{aligned}$$

ここで、第2項の $[\dots][\dots]$ を $p \mapsto p+q$ および $q \mapsto -q$ とすれば第3項の $[\dots][\dots]$ となる。さらに、

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^4(p+q)^2} \mapsto \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p+q)^4 p^2}$$

となるので、第2項と第3項の積分は等しい。

第4項を計算する。(6.42)より分母は

$$\begin{aligned}
\int \frac{d^d p}{(2\pi)^d} \frac{1}{p^4(p+q)^4} &= 6 \int_0^1 dx dy \delta(x+y-1) \int \frac{d^d p}{(2\pi)^d} \frac{xy}{[yp^2 + x(p+q)^2]^4} \\
&= 6 \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \frac{x(1-x)}{[(1-x)p^2 + x(p+q)^2]^4} \\
&= 6 \int_0^1 dx \int \frac{d^d P}{(2\pi)^d} \frac{x(1-x)}{[P^2 - \Delta]^4} \\
&\quad (P = p+xq, \quad \Delta = -x(1-x)q^2).
\end{aligned}$$

分子は

$$\begin{aligned}
&p^\mu(p+q) \cdot (q-p) + p \cdot (p+q)(2p+q)^\mu + (p+q)^\mu p \cdot (-p-2q) \\
&= p^\mu(q^2 - p^2) + (p^2 + p \cdot q)(2p+q)^\mu + (p+q)^\mu(-p^2 - 2p \cdot q) \\
&= q^2 p^\mu - (p \cdot q)q^\mu
\end{aligned}$$

及び

$$\begin{aligned}
&p^\nu(p+q) \cdot (p-q) + p \cdot (p+q)(-2p-q)^\nu + (p+q)^\nu p \cdot (p+2q) \\
&= p^\nu(p^2 - q^2) + (p^2 + p \cdot q)(-2p-q)^\nu + (p+q)^\nu(p^2 + 2p \cdot q)
\end{aligned}$$

$$= -q^2 p^\nu + (p \cdot q) q^\nu$$

をかけて

$$\begin{aligned}
& [q^2 p^\mu - (p \cdot q) q^\mu] [-q^2 p^\nu + (p \cdot q) q^\nu] \\
&= -q^4 p^\mu p^\nu - (p \cdot q)^2 q^\mu q^\nu + q^2 (p \cdot q) (p^\mu q^\nu + q^\mu p^\nu) \\
&\rightarrow -q^4 [P^\mu P^\nu + x^2 q^\mu q^\nu] - [P \cdot q - x q^2]^2 q^\mu q^\nu \\
&\quad + q^2 [P \cdot q - x q^2] [P^\mu q^\nu + q^\mu P^\nu - 2x q^\mu q^\nu] \\
&\rightarrow -q^4 [P^\mu P^\nu + x^2 q^\mu q^\nu] - [(P \cdot q)^2 + x^2 q^4] q^\mu q^\nu \\
&\quad + q^2 [P^\rho q_\rho (P^\mu q^\nu + q^\mu P^\nu) + 2x^2 q^2 q^\mu q^\nu] \\
&= -q^4 P^\mu P^\nu - x^2 q^4 q^\mu q^\nu - [(P \cdot q)^2 + x^2 q^4] q^\mu q^\nu \\
&\quad + q^2 q^\nu q_\rho P^\mu P^\rho + q^2 q^\mu q_\rho P^\nu P^\rho + 2x^2 q^4 q^\mu q^\nu \\
&= -q^4 P^\mu P^\nu - q_\rho q_\sigma q^\mu q^\nu P^\rho P^\sigma + q^2 q^\nu q_\rho P^\mu P^\rho + q^2 q^\mu q_\rho P^\nu P^\rho \\
&\rightarrow -q^4 \frac{g^{\mu\nu}}{d} P^2 - q_\rho q_\sigma q^\mu q^\nu \frac{g^{\rho\sigma}}{d} P^2 + q^2 q^\nu q_\rho \frac{g^{\mu\rho}}{d} P^2 + q^2 q^\mu q_\rho \frac{g^{\nu\rho}}{d} P^2 \\
&= -q^4 \frac{g^{\mu\nu}}{d} P^2 - q^2 q^\mu q^\nu \frac{1}{d} P^2 + q^2 q^\nu q^\mu \frac{1}{d} P^2 + q^2 q^\mu q^\nu \frac{1}{d} P^2 \\
&= -(q^2 g^{\mu\nu} - q^\mu q^\nu) \frac{q^2}{d} P^2.
\end{aligned}$$

よって、第4項は

$$-3g^2(1-\xi)^2 C_2(G) \delta^{ab} (q^2 g^{\mu\nu} - q^\mu q^\nu) \frac{q^2}{d} \int_0^1 dx x(1-x) \int \frac{d^d P}{(2\pi)^d} \frac{P^2}{[P^2 - \Delta]^4} \quad [16.5.1]$$

となる (有限値).

第2項 (と第3項は等しい) を計算する. (6.40) から分母は

$$\begin{aligned}
& \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^4 (p+q)^2} \\
&= \int_0^1 dx dy \delta(x+y-1) \int \frac{d^d p}{(2\pi)^d} \frac{2y}{[(1-x)p^2 + x(p+q)^2]^3} \\
&= 2 \int_0^1 dx (1-x) \int \frac{d^d P}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3}, \\
& (P = p + xq, \quad \Delta = -x(1-x)q^2).
\end{aligned}$$

分子は

$$\begin{aligned}
& [p^\mu (q-p)^\sigma + p^\sigma (2p+q)^\mu + g^{\sigma\mu} p \cdot (-p-2q)] \\
& \times [p^\nu (p-q)_\sigma + p_\sigma (-2p-q)^\nu + \delta^\nu_\sigma p \cdot (p+2q)] \\
&= -p^\mu p^\nu (q-p)^2 - g^{\mu\nu} (p^2 + 2p \cdot q)^2 - p^2 (q+2p)^\mu (q+2p)^\nu \\
&\quad + p \cdot (p-q) (2p+q)^\mu p^\nu + p \cdot (p+2q) (q-p)^\mu p^\nu + p \cdot (p+2q) p^\mu (2p+q)^\nu \\
&\quad + ((\mu \leftrightarrow \nu)) \\
&= -(P^\mu - xq^\mu) (P^\nu - xq^\nu) [P - (x+1)q]^2 \\
&\quad - g^{\mu\nu} [(P-xq)^2 + 2(P-xq) \cdot q]^2 \\
&\quad - (P-xq)^2 (2P + (1-2x)q)^\mu (2P + (1-2x)q)^\nu
\end{aligned}$$

$$\begin{aligned}
& + (P - xq) \cdot (P - (x + 1)q)(2P + (1 - 2x)q)^\mu (P - xq)^\nu \\
& + (P - xq) \cdot (P + (2 - x)q)(-P + (x + 1)q)^\mu (P - xq)^\nu \\
& + (P - xq) \cdot (P + (2 - x)q)(P - xq)^\mu (2P + (1 - 2x)q)^\nu \\
& + ((\mu \leftrightarrow \nu)) \\
\rightarrow & -(P^\mu P^\nu - xP^\mu q^\nu - xq^\mu P^\nu + x^2 q^\mu q^\nu) [P^2 - 2(x + 1)P \cdot q + (x + 1)^2 q^2] \\
& - g^{\mu\nu} [P^2 - 2(x - 1)P \cdot q + x(x - 2)q^2]^2 \\
& - (P^2 - 2xP \cdot q + x^2 q^2)[4P^\mu P^\nu + 2(1 - 2x)(P^\mu q^\nu + q^\mu P^\nu) + (1 - 2x)^2 q^\mu q^\nu] \\
& + (P^2 - (2x + 1)P \cdot q + x(x + 1)q^2)(2P^\mu P^\nu - 2xP^\mu q^\nu + (1 - 2x)q^\mu P^\nu - x(1 - 2x)q^\mu q^\nu) \\
& + (P^2 + 2(1 - x)P \cdot q - x(2 - x)q^2)(-P^\mu P^\nu + xP^\mu q^\nu + (x + 1)q^\mu P^\nu - x(x + 1)q^\mu q^\nu) \\
& + (P^2 + 2(1 - x)P \cdot q - x(2 - x)q^2)(2P^\mu P^\nu + (1 - 2x)P^\mu q^\nu - 2xq^\mu P^\nu - x(1 - 2x)q^\mu q^\nu) \\
& + ((\mu \leftrightarrow \nu)) \\
\rightarrow & -P^2 P^\mu P^\nu - (x + 1)^2 q^2 P^\mu P^\nu - x^2 q^\mu q^\nu P^2 \\
& - x^2(x + 1)^2 q^2 q^\mu q^\nu - 2x(x + 1)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& - g^{\mu\nu} [(P^2)^2 + 4(x - 1)^2 q_\rho q_\sigma P^\rho P^\sigma + 2x(x - 2)q^2 P^2 + x^2(x - 2)^2 q^4] \\
& - 4P^2 P^\mu P^\nu - 4x^2 q^2 P^\mu P^\nu - (1 - 2x)^2 q^\mu q^\nu P^2 \\
& - x^2(1 - 2x)^2 q^2 q^\mu q^\nu + 4x(1 - 2x)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& + (P^2 - (2x + 1)P \cdot q + x(x + 1)q^2)[4P^\mu P^\nu + (1 - 4x)(P^\mu q^\nu + q^\mu P^\nu) - 2x(1 - 2x)q^\mu q^\nu] \\
& + (P^2 + 2(1 - x)P \cdot q - x(2 - x)q^2)[-2P^\mu P^\nu + (2x + 1)(P^\mu q^\nu + q^\mu P^\nu) - 2x(x + 1)q^\mu q^\nu] \\
& + (P^2 + 2(1 - x)P \cdot q - x(2 - x)q^2)[4P^\mu P^\nu + (1 - 4x)(P^\mu q^\nu + q^\mu P^\nu) - 2x(1 - 2x)q^\mu q^\nu] \\
= & -P^2 P^\mu P^\nu - (x + 1)^2 q^2 P^\mu P^\nu - x^2 q^\mu q^\nu P^2 \\
& - x^2(x + 1)^2 q^2 q^\mu q^\nu - 2x(x + 1)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& - g^{\mu\nu} [(P^2)^2 + 4(x - 1)^2 q_\rho q_\sigma P^\rho P^\sigma + 2x(x - 2)q^2 P^2 + x^2(x - 2)^2 q^4] \\
& - 4P^2 P^\mu P^\nu - 4x^2 q^2 P^\mu P^\nu - (1 - 2x)^2 q^\mu q^\nu P^2 \\
& - x^2(1 - 2x)^2 q^2 q^\mu q^\nu + 4x(1 - 2x)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& + (2P^2 + (1 - 4x)P \cdot q + x(2x - 1)q^2)[4P^\mu P^\nu + (1 - 4x)(P^\mu q^\nu + q^\mu P^\nu) - 2x(1 - 2x)q^\mu q^\nu] \\
& + (P^2 + 2(1 - x)P \cdot q - x(2 - x)q^2)[-2P^\mu P^\nu + (2x + 1)(P^\mu q^\nu + q^\mu P^\nu) - 2x(x + 1)q^\mu q^\nu] \\
\rightarrow & -P^2 P^\mu P^\nu - (x + 1)^2 q^2 P^\mu P^\nu - x^2 q^\mu q^\nu P^2 \\
& - x^2(x + 1)^2 q^2 q^\mu q^\nu - 2x(x + 1)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& - g^{\mu\nu} [(P^2)^2 + 4(x - 1)^2 q_\rho q_\sigma P^\rho P^\sigma + 2x(x - 2)q^2 P^2 + x^2(x - 2)^2 q^4] \\
& - 4P^2 P^\mu P^\nu - 4x^2 q^2 P^\mu P^\nu - (1 - 2x)^2 q^\mu q^\nu P^2 \\
& - x^2(1 - 2x)^2 q^2 q^\mu q^\nu + 4x(1 - 2x)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& + 8P^2 P^\mu P^\nu + 4x(2x - 1)q^2 P^\mu P^\nu - 4x(1 - 2x)q^\mu q^\nu P^2 \\
& + 2x^2(1 - 2x)^2 q^2 q^\mu q^\nu + (1 - 4x)^2 q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& - 2P^2 P^\mu P^\nu + 2x(2 - x)q^2 P^\mu P^\nu - 2x(x + 1)q^\mu q^\nu P^2 \\
& + 2x^2(2 - x)(x + 1)q^2 q^\mu q^\nu + 2(1 - x)(2x + 1)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
= & -g^{\mu\nu} [(P^2)^2 + 4(x - 1)^2 q_\rho q_\sigma P^\rho P^\sigma + 2x(x - 2)q^2 P^2 + x^2(x - 2)^2 q^4] \\
& - P^2 P^\mu P^\nu - (x + 1)^2 q^2 P^\mu P^\nu - x^2 q^\mu q^\nu P^2 \\
& - x^2(x + 1)^2 q^2 q^\mu q^\nu - 2x(x + 1)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu)
\end{aligned}$$

$$\begin{aligned}
& -4P^2 P^\mu P^\nu - 4x^2 q^2 P^\mu P^\nu - (1-2x)^2 q^\mu q^\nu P^2 \\
& -x^2(1-2x)^2 q^2 q^\mu q^\nu + 4x(1-2x)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& + 8P^2 P^\mu P^\nu + 4x(2x-1)q^2 P^\mu P^\nu - 4x(1-2x)q^\mu q^\nu P^2 \\
& + 2x^2(1-2x)^2 q^2 q^\mu q^\nu + (1-4x)^2 q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& - 2P^2 P^\mu P^\nu + 2x(2-x)q^2 P^\mu P^\nu - 2x(x+1)q^\mu q^\nu P^2 \\
& + 2x^2(2-x)(x+1)q^2 q^\mu q^\nu + 2(1-x)(2x+1)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& = -g^{\mu\nu} \left[(P^2)^2 + 4(x-1)^2 q_\rho q_\sigma P^\rho P^\sigma + 2x(x-2)q^2 P^2 + x^2(x-2)^2 q^4 \right] \\
& + P^2 P^\mu P^\nu + (x^2 - 2x - 1)q^2 P^\mu P^\nu + (x^2 - 2x - 1)q^\mu q^\nu P^2 \\
& + x^2(x-2)^2 q^2 q^\mu q^\nu + (2x^2 - 4x + 3)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu) \\
& \rightarrow -g^{\mu\nu} \left[(P^2)^2 + 4(x-1)^2 q_\rho q_\sigma \frac{g^{\rho\sigma}}{d} P^2 + 2x(x-2)q^2 P^2 + x^2(x-2)^2 q^4 \right] \\
& + P^2 P^\mu P^\nu + (x^2 - 2x - 1)q^2 \frac{g^{\mu\nu}}{d} P^2 + (x^2 - 2x - 1)q^\mu q^\nu P^2 \\
& + x^2(x-2)^2 q^2 q^\mu q^\nu + (2x^2 - 4x + 3)q_\rho \left(q^\nu \frac{g^{\mu\rho}}{d} P^2 + q^\mu \frac{g^{\nu\rho}}{d} P^2 \right) \\
& = -g^{\mu\nu} \left[(P^2)^2 + \left\{ \frac{4}{d}(x-1)^2 + 2x(x-2) \right\} q^2 P^2 + x^2(x-2)^2 q^4 \right] \\
& + P^2 P^\mu P^\nu + (x^2 - 2x - 1)q^2 \frac{g^{\mu\nu}}{d} P^2 + (x^2 - 2x - 1)q^\mu q^\nu P^2 \\
& + x^2(x-2)^2 q^2 q^\mu q^\nu + \frac{2}{d}(2x^2 - 4x + 3)q^\mu q^\nu P^2 \\
& = P^2 P^\mu P^\nu - g^{\mu\nu}(P^2)^2 - g^{\mu\nu} \left\{ \frac{1}{d}(3x^2 - 6x + 5) + 2x(x-2) \right\} q^2 P^2 \\
& + \left\{ \frac{2}{d}(2x^2 - 4x + 3) + (x^2 - 2x - 1) \right\} q^\mu q^\nu P^2 - x^2(x-2)^2 q^2 (g^{\mu\nu} q^2 - q^\mu q^\nu).
\end{aligned}$$

最後の項は積分すれば有限値

$$2g^2(1-\xi)C_2(G)\delta^{ab}(g^{\mu\nu}q^2 - q^\mu q^\nu) \int_0^1 dx (1-x) \int \frac{d^d P}{(2\pi)^d} \frac{-x^2(x-2)^2 q^2}{[P^2 - \Delta]^3} \quad [16.5.2]$$

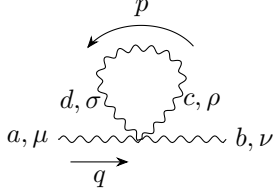
になるので、これ以降は考えない。

以上から (16.62) に加わる発散項は

$$\begin{aligned}
& 2g^2(1-\xi)C_2(G)\delta^{ab} \int_0^1 dx (1-x) \int \frac{d^d P}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\
& \times \left[P^2 P^\mu P^\nu - g^{\mu\nu}(P^2)^2 - g^{\mu\nu} \left\{ \frac{1}{d}(3x^2 - 6x + 5) + 2x(x-2) \right\} q^2 P^2 \right. \\
& \left. + \left\{ \frac{2}{d}(2x^2 - 4x + 3) + (x^2 - 2x - 1) \right\} q^\mu q^\nu P^2 \right] \quad [16.5.3]
\end{aligned}$$

(16.65) の修正

(16.63) は



$$\begin{aligned}
&= \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{-i}{p^2} \left[g_{\rho\sigma} - (1-\xi) \frac{p_\rho p_\sigma}{p^2} \right] \delta^{cd} (-ig^2) \\
&\quad \times [f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})] \\
&= (16.65) + \frac{g^2}{2} (1-\xi) \int \frac{d^d p}{(2\pi)^d} \frac{p_\rho p_\sigma}{p^4} \delta^{cd} \\
&\quad \times [f^{abe} f^{cde} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ace} f^{bde} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho}) + f^{ade} f^{bce} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma})] \\
&= (16.65) + \frac{g^2}{2} (1-\xi) \int \frac{d^d p}{(2\pi)^d} \frac{p_\rho p_\sigma}{p^4} f^{ace} f^{bce} [g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\nu\rho} + g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma}] \\
&= (16.65) + g^2 (1-\xi) C_2(G) \delta^{ab} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^4} [g^{\mu\nu} p^2 - p^\mu p^\nu] \\
&= (16.65) + g^2 (1-\xi) C_2(G) \delta^{ab} \int \frac{d^d p}{(2\pi)^d} \frac{g^{\mu\nu} p^2}{p^4} - g^2 (1-\xi) C_2(G) \delta^{ab} \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{p^4}.
\end{aligned}$$

第2項の積分は

$$\begin{aligned}
&\int \frac{d^d p}{(2\pi)^d} \frac{p^2}{p^4} \\
&= \int \frac{d^d p}{(2\pi)^d} \frac{p^2}{p^4} \frac{(p+q)^2}{(p+q)^2} \\
&= 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{p^2 (p+q)^2}{[P^2 - \Delta]^3} \\
&= 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{(P-xq)^2 (P+(1-x)q)^2}{[P^2 - \Delta]^3} \\
&= 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\
&\quad \times [P^2 - 2xP \cdot q + x^2 q^2] [P^2 + 2(1-x)P \cdot q + (1-x)^2 q^2] \\
&\rightarrow 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\
&\quad \times [(P^2)^2 + (2x^2 - 2x + 1)q^2 P^2 - 4x(1-x)q_\rho q_\sigma P^\rho P^\sigma + x^2(1-x)^2 q^4] \\
&\rightarrow 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\
&\quad \times \left[(P^2)^2 + (2x^2 - 2x + 1)q^2 P^2 - 4x(1-x)q_\rho q_\sigma \frac{g^{\rho\sigma}}{d} P^2 + x^2(1-x)^2 q^4 \right] \\
&\rightarrow 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3}
\end{aligned}$$

$$\times \left[(P^2)^2 + \left\{ (2x^2 - 2x + 1) - \frac{4}{d}x(1-x) \right\} q^2 P^2 + x^2(1-x)^2 q^4 \right].$$

第 3 項の積分は

$$\begin{aligned} & \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{p^4} \\ &= \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{p^4} \frac{(p+q)^2}{(p+q)^2} \\ &= 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu (p+q)^2}{[P^2 - \Delta]^3} \\ &= 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{(P-xq)^\mu (P-xq)^\nu (P+(1-x)q)^2}{[P^2 - \Delta]^3} \\ &= 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ &\quad \times [P^\mu P^\nu - x(P^\mu q^\nu + q^\mu P^\nu) + x^2 q^\mu q^\nu] [P^2 + 2(1-x)q \cdot P + (1-x)^2 q^2] \\ &\rightarrow 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ &\quad \times [P^2 P^\mu P^\nu + (1-x)^2 q^2 P^\mu P^\nu + x^2 q^\mu q^\nu P^2 \\ &\quad + x^2(1-x)^2 q^\mu q^\nu q^2 - 2x(1-x)q_\rho P^\rho (P^\mu q^\nu + q^\mu P^\nu)] \\ &\rightarrow 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ &\quad \times \left[P^2 P^\mu P^\nu + (1-x)^2 q^2 \frac{g^{\mu\nu}}{d} P^2 + x^2 q^\mu q^\nu P^2 \right. \\ &\quad \left. + x^2(1-x)^2 q^\mu q^\nu q^2 - 2x(1-x)q_\rho \left(q^\nu \frac{g^{\mu\rho}}{d} P^2 + q^\mu \frac{g^{\nu\rho}}{d} P^2 \right) \right] \\ &= 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ &\quad \times \left[P^2 P^\mu P^\nu + (1-x)^2 q^2 \frac{g^{\mu\nu}}{d} P^2 + x^2 q^\mu q^\nu P^2 + x^2(1-x)^2 q^\mu q^\nu q^2 - \frac{4}{d}x(1-x)q^\mu q^\nu P^2 \right] \\ &\rightarrow 2 \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ &\quad \times \left[P^2 P^\mu P^\nu + g^{\mu\nu} \frac{1}{d} (1-x)^2 q^2 P^2 + \left\{ x^2 - \frac{4}{d}x(1-x) \right\} q^\mu q^\nu P^2 + x^2(1-x)^2 q^\mu q^\nu q^2 \right]. \end{aligned}$$

以上から (16.65) に加わる項は

$$\begin{aligned} & 2g^2(1-\xi)C_2(G)\delta^{ab} \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ &\quad \times \left[g^{\mu\nu} (P^2)^2 + g^{\mu\nu} \left\{ (2x^2 - 2x + 1) - \frac{4}{d}x(1-x) \right\} q^2 P^2 + g^{\mu\nu} x^2(1-x)^2 q^4 \right] \\ &- 2g^2(1-\xi)C_2(G)\delta^{ab} \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ &\quad \times \left[P^2 P^\mu P^\nu + g^{\mu\nu} \frac{1}{d} (1-x)^2 q^2 P^2 + \left\{ x^2 - \frac{4}{d}x(1-x) \right\} q^\mu q^\nu P^2 + x^2(1-x)^2 q^\mu q^\nu q^2 \right] \\ &= 2g^2(1-\xi)C_2(G)\delta^{ab} \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \end{aligned}$$

$$\times \left[-P^2 P^\mu P^\nu + g^{\mu\nu} (P^2)^2 + g^{\mu\nu} \left\{ (2x^2 - 2x + 1) + \frac{1}{d}(3x^2 - 2x - 1) \right\} q^2 P^2 \right. \\ \left. - \left\{ x^2 - \frac{4}{d}x(1-x) \right\} q^\mu q^\nu P^2 + x^2(1-x)^2 q^2 (q^2 g^{\mu\nu} - q^\mu q^\nu) \right].$$

最後の項

$$2g^2(1-\xi)C_2(G)\delta^{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu) \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{x^2(1-x)^2 q^2}{[P^2 - \Delta]^3} \quad [16.5.4]$$

は有限なので、これ以降考えない。以上から、(16.65)に加わる発散項は

$$2g^2(1-\xi)C_2(G)\delta^{ab} \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ \times \left[-P^2 P^\mu P^\nu + g^{\mu\nu} (P^2)^2 + g^{\mu\nu} \left\{ (2x^2 - 2x + 1) + \frac{1}{d}(3x^2 - 2x - 1) \right\} q^2 P^2 \right. \\ \left. - \left\{ x^2 - \frac{4}{d}x(1-x) \right\} q^\mu q^\nu P^2 \right]. \quad [16.5.5]$$

[16.5.3] と [16.5.5] を足して、ゲージによる修正 (のうち発散する部分) は

$$2g^2(1-\xi)C_2(G)\delta^{ab} \int_0^1 dx (1-x) \int \frac{d^d P}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ \times \left[P^2 P^\mu P^\nu - g^{\mu\nu} (P^2)^2 - g^{\mu\nu} \left\{ \frac{1}{d}(3x^2 - 6x + 5) + 2x(x-2) \right\} q^2 P^2 \right. \\ \left. + \left\{ \frac{2}{d}(2x^2 - 4x + 3) + (x^2 - 2x - 1) \right\} q^\mu q^\nu P^2 \right] \\ + 2g^2(1-\xi)C_2(G)\delta^{ab} \int_0^1 dx (1-x) \int \frac{d^d p}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ \times \left[-P^2 P^\mu P^\nu + g^{\mu\nu} (P^2)^2 + g^{\mu\nu} \left\{ (2x^2 - 2x + 1) + \frac{1}{d}(3x^2 - 2x - 1) \right\} q^2 P^2 \right. \\ \left. - \left\{ x^2 - \frac{4}{d}x(1-x) \right\} q^\mu q^\nu P^2 \right] \\ = 2g^2(1-\xi)C_2(G)\delta^{ab} \int_0^1 dx (1-x) \int \frac{d^d P}{(2\pi)^d} \frac{1}{[P^2 - \Delta]^3} \\ \times \left[g^{\mu\nu} \left\{ (2x+1) + \frac{2}{d}(2x-3) \right\} q^2 P^2 - \left\{ (2x+1) + \frac{2}{d}(2x-3) \right\} q^\mu q^\nu P^2 \right] \\ = 2g^2(1-\xi)C_2(G)\delta^{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu) \int_0^1 dx (1-x) \left\{ (2x+1) + \frac{2}{d}(2x-3) \right\} \\ \times \int \frac{d^d P}{(2\pi)^d} \frac{P^2}{[P^2 - \Delta]^3} \\ = 2g^2(1-\xi)C_2(G)\delta^{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu) \int_0^1 dx (1-x) \left\{ (2x+1) + \frac{2}{d}(2x-3) \right\} \\ \times \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(2-d/2)}{\Gamma(3)} \left(\frac{1}{\Delta} \right)^{2-d/2} \\ = \frac{2ig^2}{(4\pi)^{d/2}} (1-\xi)C_2(G)\delta^{ab}(q^2 g^{\mu\nu} - q^\mu q^\nu) \\ \times \int_0^1 dx (1-x) \left\{ (2x+1) + \frac{2}{d}(2x-3) \right\} \frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}$$

$$\approx i(q^2 g^{\mu\nu} - q^\mu q^\nu) \delta^{ab} \left[\frac{-g^2}{(4\pi)^2} \frac{-1+\xi}{2} C_2(G) \Gamma(2-d/2) \right].$$

(16.71) と比べて

$$-\frac{5}{3} \mapsto -\frac{5}{3} + \frac{-1+\xi}{2} = -\frac{13}{6} + \frac{\xi}{2}$$

とすれば良いことが分かる.

(16.88)

(10.36) の直前で定義したように,

$$\psi_0 = \sqrt{Z_2} \psi, \quad A_0^\mu = \sqrt{Z_3} A^\mu.$$

(16.86) の第 2 項は

$$\bar{\psi}_0(i\partial - m_0)\psi_0 = Z_2 \bar{\psi}(i\partial - m_0)\psi = \bar{\psi}(i\partial - m)\psi + \bar{\psi}(i\delta_2\partial - \delta_m)\psi.$$

右辺の第 1 項は (16.34), 第 2 項は (16.87) に含まれる. 他も同様.

16.6 Asymptotic Freedom: The Background Field Method

(16.99)

ラグランジアン (16.98) が不変なことを確かめる. 随伴表現 $(t^b)_{ac} = if^{abc}$ を使う.

$$\tilde{A}_\mu^a = A_\mu^a + \mathcal{A}_\mu^a, \quad \tilde{D}_\mu^{ac} = D_\mu^{ac} + f^{abc} \mathcal{A}_\mu^b = \partial_\mu \delta^{ac} + f^{abc} \tilde{A}_\mu^b$$

とする. \tilde{A}_μ^a の変換は

$$\begin{aligned} \tilde{A}_\mu^a &\rightarrow A_\mu^a + \mathcal{A}_\mu^a + (D_\mu \beta)^a - f^{abc} \beta^b \mathcal{A}_\mu^c \\ &= A_\mu^a + \mathcal{A}_\mu^a + D_\mu^{ac} \beta^c - f^{acb} \beta^c \mathcal{A}_\mu^b \\ &= A_\mu^a + \mathcal{A}_\mu^a + (\partial_\mu \delta^{ac} + f^{abc} A_\mu^b) \beta^c + f^{abc} \beta^c \mathcal{A}_\mu^b \\ &= A_\mu^a + \mathcal{A}_\mu^a + (\partial_\mu \delta^{ac} + f^{abc} A_\mu^b + f^{abc} \mathcal{A}_\mu^b) \beta^c \\ &= \tilde{A}_\mu^a + (\partial_\mu \delta^{ac} + f^{abc} \tilde{A}_\mu^b) \beta^c \\ &= \tilde{A}_\mu^a + \tilde{D}_\mu^{ac} \beta^c \\ &= \tilde{A}_\mu^a + (\tilde{D}_\mu \beta)^a \\ &= \tilde{A}_\mu^a + \partial_\mu \beta^a + f^{abc} \tilde{A}_\mu^b \beta^c. \end{aligned}$$

fermion

フェルミオンの変換は

$$\psi^a \rightarrow \psi^a + i\beta^b (t^b \psi)_a = \psi^a + i\beta^b (t^b)_{ac} \psi^c = \psi^a - f^{abc} \beta^b \psi^c.$$

(15.30) と同様にフェルミオンの共変微分はフェルミオンと同じ変換になる. 実際,

$$(\tilde{D}_\mu \psi)^a = \tilde{D}_\mu^{ac} \psi^c$$

$$\begin{aligned}
&= (\partial_\mu \delta^{ac} + f^{abc} \tilde{A}_\mu^b) \psi^c \\
&= \partial_\mu \psi^a + f^{abc} \tilde{A}_\mu^b \psi^c \\
&\rightarrow \partial_\mu (\psi^a - f^{abc} \beta^b \psi^c) + f^{abc} (\tilde{A}_\mu^b + \partial_\mu \beta^b + f^{bde} \tilde{A}_\mu^d \beta^e) (\psi^c - f^{cfg} \beta^f \psi^g) \\
&\approx \partial_\mu \psi^a - f^{abc} (\partial_\mu \beta^b) \psi^c - f^{abc} \beta^b \partial_\mu \psi^c \\
&\quad + f^{abc} \left[\tilde{A}_\mu^b \psi^c + (\partial_\mu \beta^b) \psi^c + f^{bde} \tilde{A}_\mu^d \beta^e \psi^c - \tilde{A}_\mu^b f^{cfg} \beta^f \psi^g \right] \\
&= \partial_\mu \psi^a - f^{abc} \beta^b \partial_\mu \psi^c + f^{abc} \left[\tilde{A}_\mu^b \psi^c + f^{bde} \tilde{A}_\mu^d \beta^e \psi^c - \tilde{A}_\mu^b f^{cfg} \beta^f \psi^g \right] \\
&= \partial_\mu \psi^a + f^{abc} \tilde{A}_\mu^b \psi^c - f^{abc} \beta^b \partial_\mu \psi^c + f^{abc} f^{bde} \tilde{A}_\mu^d \beta^e \psi^c - f^{abc} f^{cfg} \tilde{A}_\mu^b \beta^f \psi^g \\
&= (\tilde{D}_\mu \psi)^a - f^{abc} \beta^b \partial_\mu \psi^c + f^{abc} f^{bde} \tilde{A}_\mu^d \beta^e \psi^c - f^{adb} f^{bec} \tilde{A}_\mu^d \beta^e \psi^c \\
&= (\tilde{D}_\mu \psi)^a - f^{abc} \beta^b \partial_\mu \psi^c + (f^{abc} f^{bde} + f^{ebc} f^{adb}) \tilde{A}_\mu^d \beta^e \psi^c.
\end{aligned}$$

Jacobi 恒等式 (15.70) から

$$\begin{aligned}
&= (\tilde{D}_\mu \psi)^a - f^{abc} \beta^b \partial_\mu \psi^c - f^{dbc} f^{eab} \tilde{A}_\mu^d \beta^e \psi^c \\
&= (\tilde{D}_\mu \psi)^a - f^{abc} \beta^b \partial_\mu \psi^c - f^{cde} f^{abc} \tilde{A}_\mu^d \beta^b \psi^e \\
&= (\tilde{D}_\mu \psi)^a - f^{abc} \beta^b (\partial_\mu \psi^c + f^{cde} \tilde{A}_\mu^d \psi^e) \\
&= (\tilde{D}_\mu \psi)^a - f^{abc} \beta^b (\tilde{D}_\mu \psi)^c \\
&= (\delta^{ac} - f^{abc} \beta^b) (\tilde{D}_\mu \psi)^c.
\end{aligned}$$

(13.98) のフェルミオンの項は

$$\begin{aligned}
\bar{\psi}^a (i \not{D}^{ac} + \mathcal{A}_\mu^b \gamma^\mu (t^b)_{ac}) \psi^c &= \bar{\psi}^a (i \not{D}^{ac} + i f^{abc} \mathcal{A}_\mu^b \gamma^\mu) \psi^c \\
&= \bar{\psi}^a (i D_\mu^{ac} \gamma^\mu + i f^{abc} \mathcal{A}_\mu^b \gamma^\mu) \psi^c \\
&= \bar{\psi}^a i (D_\mu^{ac} + f^{abc} \mathcal{A}_\mu^b) \gamma^\mu \psi^c \\
&= \bar{\psi}^a i \gamma^\mu \tilde{D}_\mu^{ac} \psi^c \\
&= \bar{\psi}^a i \gamma^\mu (\tilde{D}_\mu \psi)^a \\
&\rightarrow \bar{\psi}^d (\delta^{ad} - f^{abd} \beta^b) (\delta^{ac} - f^{abc} \beta^b) i \gamma^\mu (\tilde{D}_\mu \psi)_c \\
&\approx \bar{\psi}^d (\delta^{cd} - f^{cbd} \beta^b - f^{dbc} \beta^b) i \gamma^\mu (\tilde{D}_\mu \psi)_c \\
&= \bar{\psi}^d \delta^{cd} i \gamma^\mu (\tilde{D}_\mu \psi)_c \\
&= \bar{\psi}^c i \gamma^\mu (\tilde{D}_\mu \psi)_c
\end{aligned}$$

なので不変.

ghost

ゴーストの変換はフェルミオンと同じ :

$$c^a \rightarrow c^a - f^{abc} \beta^b c^c, \quad (\tilde{D}_\mu c)^a \rightarrow (\tilde{D}_\mu c)^a - f^{abc} \beta^b (\tilde{D}_\mu c)^c$$

なので

$$\begin{aligned}
&(D^2)^{ac} c^c + (D^\mu)^{ab} f^{bdc} \mathcal{A}_\mu^d c^c \\
&= (D^\mu)^{ab} D_\mu^{bc} c^c + (D^\mu)^{ab} f^{bdc} \mathcal{A}_\mu^d c^c \\
&= (D^\mu)^{ab} (D_\mu^{bc} + f^{bdc} \mathcal{A}_\mu^d) c^c
\end{aligned}$$

$$\begin{aligned}
&= (D^\mu)^{ab} \tilde{D}_\mu^{bc} c^c \\
&= (\partial_\mu \delta^{ab} + f^{adb} A_\mu^d) (\tilde{D}^\mu c)^b \\
&\rightarrow [\partial_\mu \delta^{ab} + f^{adb} (A_\mu^d + \partial_\mu \beta^d + f^{def} A_\mu^e \beta^f)] (\delta^{bg} - f^{bhg} \beta^h) (\tilde{D}^\mu c)^g \\
&= [\partial_\mu \delta^{ab} + f^{adb} A_\mu^d + f^{adb} (\partial_\mu \beta^d + f^{def} A_\mu^e \beta^f)] (\delta^{bg} - f^{bhg} \beta^h) (\tilde{D}^\mu c)^g \\
&= [D_\mu^{ab} + f^{adb} (\partial_\mu \beta^d) + f^{adb} f^{def} A_\mu^e \beta^f] (\delta^{bg} - f^{bhg} \beta^h) (\tilde{D}^\mu c)^g \\
&\approx [D_\mu^{ag} + f^{adg} (\partial_\mu \beta^d) + f^{adg} f^{def} A_\mu^e \beta^f - f^{bhg} (D_\mu^{ab}) \beta^h] (\tilde{D}^\mu c)^g \\
&= [D_\mu^{ag} + f^{adg} (\partial_\mu \beta^d) + f^{adg} f^{def} A_\mu^e \beta^f - f^{bhg} (\partial_\mu \delta^{ab} + f^{acb} A_\mu^c) \beta^h] (\tilde{D}^\mu c)^g \\
&= [D_\mu^{ag} + f^{adg} (\partial_\mu \beta^d) + f^{adg} f^{def} A_\mu^e \beta^f - f^{ahg} (\partial_\mu) \beta^h - f^{bhg} f^{acb} A_\mu^c \beta^h] (\tilde{D}^\mu c)^g \\
&= [D_\mu^{ag} + f^{adg} (\partial_\mu \beta^d) + f^{adg} f^{def} A_\mu^e \beta^f - f^{ahg} (\partial_\mu \beta^h) - f^{ahg} \beta^h \partial_\mu - f^{bhg} f^{acb} A_\mu^c \beta^h] (\tilde{D}^\mu c)^g \\
&= [D_\mu^{ag} + f^{abg} f^{bef} A_\mu^e \beta^f - f^{ahg} \beta^h \partial_\mu - f^{bfh} f^{aeb} A_\mu^e \beta^f] (\tilde{D}^\mu c)^g \\
&= [D_\mu^{ag} + (f^{abg} f^{bef} + f^{fbg} f^{aeb}) A_\mu^e \beta^f - f^{ahg} \beta^h \partial_\mu] (\tilde{D}^\mu c)^g \\
&= [D_\mu^{ag} - f^{ebg} f^{fab} A_\mu^e \beta^f - f^{ahg} \beta^h \partial_\mu] (\tilde{D}^\mu c)^g.
\end{aligned}$$

(16.98) のゴーストの項は

$$\begin{aligned}
&\bar{c}^a [(D^2)^{ac} c^c + (D^\mu)^{ab} f^{bcd} \mathcal{A}_\mu^d c^c] \\
&\rightarrow \bar{c}^c (\delta^{ac} - f^{adc} \beta^d) [D_\mu^{ag} - f^{ebg} f^{fab} A_\mu^e \beta^f - f^{ahg} \beta^h \partial_\mu] (\tilde{D}^\mu c)^g \\
&\approx \bar{c}^c (D_\mu^{cg} - f^{ebg} f^{fcb} A_\mu^e \beta^f - f^{chg} \beta^h \partial_\mu - f^{adc} \beta^d D_\mu^{ag}) (\tilde{D}^\mu c)^g \\
&= \bar{c}^c [D_\mu^{cg} - f^{ebg} f^{fcb} A_\mu^e \beta^f - f^{cdg} \beta^d \partial_\mu - f^{adc} \beta^d (\partial_\mu \delta^{ag} + f^{abg} A_\mu^b)] (\tilde{D}^\mu c)^g \\
&= \bar{c}^c [D_\mu^{cg} - f^{ebg} f^{fcb} A_\mu^e \beta^f - f^{adc} f^{abg} A_\mu^b \beta^d] (\tilde{D}^\mu c)^g \\
&= \bar{c}^c [D_\mu^{cg} - f^{ebg} f^{fcb} A_\mu^e \beta^f - f^{bfc} f^{beg} A_\mu^e \beta^f] (\tilde{D}^\mu c)^g \\
&= \bar{c}^c D_\mu^{cg} (\tilde{D}^\mu c)^g
\end{aligned}$$

なので不変.

boson

ボソンの変換はゴーストと同じ

$$\mathcal{A}_\mu^a \rightarrow \mathcal{A}_\mu^a - f^{abc} \beta^b \mathcal{A}_\mu^c, \quad (D^\mu \mathcal{A}_\mu)^a \rightarrow (D^\mu \mathcal{A}_\mu)^a - f^{abc} \beta^b (D^\mu \mathcal{A}_\mu)^c$$

なので, ゲージ依存の項 $(D^\mu \mathcal{A}_\mu)^a)^2$ も不変.

$$\tilde{F}_{\mu\nu}^a = \partial_\mu \tilde{A}_\nu^a - \partial_\nu \tilde{A}_\mu^a + f^{abc} \tilde{A}_\mu^b \tilde{A}_\nu^c = F_{\mu\nu}^a + (D_\mu \mathcal{A}_\nu)^a - (D_\nu \mathcal{A}_\mu)^a + f^{abc} \mathcal{A}_\mu^b \mathcal{A}_\nu^c$$

の変換は

$$\begin{aligned}
\tilde{F}_{\mu\nu}^a &= \partial_\mu \tilde{A}_\nu^a - \partial_\nu \tilde{A}_\mu^a + f^{abc} \tilde{A}_\mu^b \tilde{A}_\nu^c \\
&\rightarrow \partial_\mu (\tilde{A}_\nu^a + \partial_\nu \beta^a + f^{abc} \tilde{A}_\nu^b \beta^c) - \partial_\nu (\tilde{A}_\mu^a + \partial_\mu \beta^a + f^{abc} \tilde{A}_\mu^b \beta^c) \\
&\quad + f^{abc} (\tilde{A}_\mu^b + \partial_\mu \beta^b + f^{bde} \tilde{A}_\mu^d \beta^e) (\tilde{A}_\nu^c + \partial_\nu \beta^c + f^{cfg} \tilde{A}_\nu^f \beta^g) \\
&\approx \partial_\mu \tilde{A}_\nu^a - \partial_\nu \tilde{A}_\mu^a + f^{abc} \beta^c (\partial_\mu \tilde{A}_\nu^b - \partial_\nu \tilde{A}_\mu^b) + f^{abc} (\tilde{A}_\nu^b \partial_\mu - \tilde{A}_\mu^b \partial_\nu) \beta^c \\
&\quad + f^{abc} \tilde{A}_\mu^b \tilde{A}_\nu^c + f^{abc} \tilde{A}_\mu^b (\partial_\nu \beta^c + f^{cfg} \tilde{A}_\nu^f \beta^g) + f^{abc} \tilde{A}_\nu^c (\partial_\mu \beta^b + f^{bde} \tilde{A}_\mu^d \beta^e) \\
&= \partial_\mu \tilde{A}_\nu^a - \partial_\nu \tilde{A}_\mu^a + f^{abc} \tilde{A}_\mu^b \tilde{A}_\nu^c
\end{aligned}$$

$$\begin{aligned}
& + f^{abc} \beta^c (\partial_\mu \tilde{A}_\nu^b - \partial_\nu \tilde{A}_\mu^b) + f^{abc} f^{c f g} \tilde{A}_\mu^b \tilde{A}_\nu^f \beta^g + f^{abc} f^{b d e} \tilde{A}_\mu^d \tilde{A}_\nu^e \beta^c \\
& = \tilde{F}_{\mu\nu}^a + f^{abc} \beta^c (\partial_\mu \tilde{A}_\nu^b - \partial_\nu \tilde{A}_\mu^b) + f^{abc} f^{c f g} \tilde{A}_\mu^b \tilde{A}_\nu^f \beta^g + f^{a c f} f^{c b g} \tilde{A}_\mu^b \tilde{A}_\nu^f \beta^g \\
& = \tilde{F}_{\mu\nu}^a + f^{abc} \beta^c (\partial_\mu \tilde{A}_\nu^b - \partial_\nu \tilde{A}_\mu^b) + (f^{abc} f^{c f g} + f^{g b c} f^{a f c}) \tilde{A}_\mu^b \tilde{A}_\nu^f \beta^g \\
& = \tilde{F}_{\mu\nu}^a - f^{abc} \beta^b (\partial_\mu \tilde{A}_\nu^c - \partial_\nu \tilde{A}_\mu^c) - f^{f b c} f^{g a c} \tilde{A}_\mu^b \tilde{A}_\nu^f \beta^g \\
& = \tilde{F}_{\mu\nu}^a - f^{abc} \beta^b (\partial_\mu \tilde{A}_\nu^c - \partial_\nu \tilde{A}_\mu^c + f^{c d e} \tilde{A}_\mu^d \tilde{A}_\nu^e) \\
& = \tilde{F}_{\mu\nu}^a - f^{abc} \beta^b \tilde{F}_{\mu\nu}^c.
\end{aligned}$$

(13.98) のボソンの項は

$$\begin{aligned}
(\tilde{F}_{\mu\nu}^a)^2 & \rightarrow (\tilde{F}_{\mu\nu}^a - f^{abc} \beta^b \tilde{F}_{\mu\nu}^c)(\tilde{F}_a^{\mu\nu} - f^{ade} \beta^d \tilde{F}_e^{\mu\nu}) \\
& \approx (\tilde{F}_{\mu\nu}^a)^2 - f^{abc} \beta^b \tilde{F}_{\mu\nu}^c \tilde{F}_a^{\mu\nu} - f^{ade} \beta^d \tilde{F}_{\mu\nu}^a \tilde{F}_e^{\mu\nu} \\
& = (\tilde{F}_{\mu\nu}^a)^2 - f^{abc} \beta^b \tilde{F}_{\mu\nu}^c \tilde{F}_a^{\mu\nu} - f^{eba} \beta^b \tilde{F}_{\mu\nu}^e \tilde{F}_a^{\mu\nu} \\
& = (\tilde{F}_{\mu\nu}^a)^2
\end{aligned}$$

なので不変.

(16.122)

ボソン \mathcal{A}_μ^a の場合を考える. (16.109) で定義したように

$$(\Delta_{G,1})_{ac}^{\mu\nu} = -(D^2)_{ac} g^{\mu\nu} + 2F_{\rho\sigma}^b (\mathcal{J}^{\rho\sigma})^{\mu\nu} (t^b)_{ac}.$$

これを $(a, \mu; c, \nu)$ を添字とする $4d(r) \times 4d(r)$ 行列と考える.

$(\Delta_{G,1})_{ac}^{\mu\nu}$ の固有値と固有函数 (p でラベルする) を

$$(\Delta_{G,1})_{ac}^{\mu\nu}(x) \phi_\nu^{c(p)}(x) = V_{(p)}(x) \phi_\mu^{a(p)}(x)$$

とおけば,

$$\det \Delta_{G,1} = \prod_x \det \Delta_{G,1}(x) = \prod_x \prod_p V_{(p)}(x).$$

よって

$$\begin{aligned}
\log \det \Delta_{G,1} & = \sum_x \sum_p \log V_{(p)}(x) = \sum_x \text{Tr} \log \Delta_{G,1}(x) = \int d^d x \text{Tr} \log \Delta_{G,1}(x) \\
& = \int d^d x (\log \Delta_{G,1}(x))_{aa}^{\mu\mu} \\
& \sim \int d^d x \left[\log \left\{ 1 + (-\partial^2)^{-1} (\Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})}) \right\} \right]_{aa}^{\mu\mu} \\
& \approx \int d^d x \left[(-\partial^2)^{-1} (\Delta^{(1)} + \Delta^{(2)} + \Delta^{(\mathcal{J})}) + \dots \right]_{aa}^{\mu\mu}.
\end{aligned}$$

このうち $\Delta^{(1)}$ の 2 乗を含む項は

$$\begin{aligned}
& - \frac{1}{2} \int d^d x \langle x | \left[(-\partial^2)^{-1} \Delta^{(1)} (-\partial^2)^{-1} \Delta^{(1)} \right]_{cc}^{\mu\mu} | x \rangle \\
& = - \frac{g^{\mu\nu} g_{\nu\mu}}{2} \int d^d x \langle x | (-\partial^2)^{-1} \Delta_{cd}^{(1)} (-\partial^2)^{-1} \Delta_{dc}^{(1)} | x \rangle
\end{aligned}$$

$$\begin{aligned}
&= -2 \int d^d x \int \frac{d^d p}{(2\pi)^d} \langle x | (-\partial^2)^{-1} \Delta_{cd}^{(1)} (-\partial^2)^{-1} \Delta_{dc}^{(1)} e^{-ip \cdot x} | p \rangle \\
&= -2 \int d^d x \int \frac{d^d p}{(2\pi)^d} \langle x | (-\partial^2)^{-1} \Delta_{cd}^{(1)} (-\partial^2)^{-1} [2A_\nu^b i\partial^\nu + (i\partial^\nu A_\nu^b)] (t^b)_{dc} e^{-ip \cdot x} | p \rangle \\
&= -2 \int d^d x \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \langle x | (-\partial^2)^{-1} \Delta_{cd}^{(1)} (-\partial^2)^{-1} e^{-ik \cdot x} (2p+k)^\nu A_\nu^b(k) e^{-ip \cdot x} | p \rangle (t^b)_{dc} \\
&= -2 \int d^d x \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \langle x | (-\partial^2)^{-1} \Delta_{cd}^{(1)} (-\partial^2)^{-1} e^{-i(p+k) \cdot x} | p \rangle (2p+k)^\nu A_\nu^b(k) (t^b)_{dc} \\
&= -2 \int d^d x \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \langle x | (-\partial^2)^{-1} \Delta_{cd}^{(1)} \frac{1}{(p+k)^2} e^{-i(p+k) \cdot x} | p \rangle (2p+k)^\nu A_\nu^b(k) (t^b)_{dc} \\
&= -2 \int d^d x \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \langle x | (-\partial^2)^{-1} [2A_\mu^a i\partial^\mu + (i\partial^\mu A_\mu^a)] (t^a)_{cd} e^{-i(p+k) \cdot x} | p \rangle \\
&\quad \times \frac{(2p+k)^\nu}{(p+k)^2} A_\nu^b(k) (t^b)_{dc} \\
&= -2 \int d^d x \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \langle x | (-\partial^2)^{-1} e^{-iq \cdot x} A_\mu^a(q) e^{-i(p+k) \cdot x} | p \rangle \\
&\quad \times (2p+2k+q)^\mu \frac{(2p+k)^\nu}{(p+k)^2} A_\nu^b(k) (t^a)_{cd} (t^b)_{dc} \\
&= -2 \int d^d x \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \langle x | e^{-i(p+k+q) \cdot x} | p \rangle \frac{(2p+2k+q)^\mu}{(p+k+q)^2} \\
&\quad \times \frac{(2p+k)^\nu}{(p+k)^2} A_\mu^a(q) A_\nu^b(k) (t^a)_{cd} (t^b)_{dc} \\
&= -2 \int d^d x \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} \langle x | p \rangle e^{-i(p+k+q) \cdot x} \frac{(2p+2k+q)^\mu}{(p+k+q)^2} \\
&\quad \times \frac{(2p+k)^\nu}{(p+k)^2} A_\mu^a(q) A_\nu^b(k) (t^a)_{cd} (t^b)_{dc} \\
&= -2 \int d^d x \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} e^{-i(k+q) \cdot x} \frac{(2p+2k+q)^\mu}{(p+k+q)^2} \\
&\quad \times \frac{(2p+k)^\nu}{(p+k)^2} A_\mu^a(q) A_\nu^b(k) (t^a)_{cd} (t^b)_{dc} \\
&= -2 \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \int \frac{d^d q}{(2\pi)^d} (2\pi)^d \delta^{(d)}(k+q) \frac{(2p+2k+q)^\mu}{(p+k+q)^2} \\
&\quad \times \frac{(2p+k)^\nu}{(p+k)^2} A_\mu^a(q) A_\nu^b(k) (t^a)_{cd} (t^b)_{dc} \\
&= -2 \int \frac{d^d p}{(2\pi)^d} \int \frac{d^d k}{(2\pi)^d} \frac{(2p+k)^\mu}{p^2} \frac{(2p+k)^\nu}{(p+k)^2} A_\mu^a(-k) A_\nu^b(k) \text{Tr}(t^a t^b).
\end{aligned}$$

$\Delta^{(\mathcal{J})}$ の 2 乗を含む項で, Lorentz 添字についてトレースを取ると $(\mathcal{J}^{\rho\sigma})^{\lambda\kappa} (\mathcal{J}^{\alpha\beta})_{\kappa\lambda} = \text{Tr}[\mathcal{J}^{\rho\sigma} \mathcal{J}^{\alpha\beta}]$ を得る.

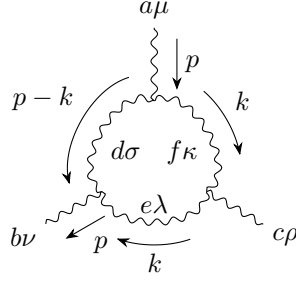
フェルミオンの場合は, Lorentz 添字 $\mu = 0, \dots, 3$ の代わりに Dirac スピノルの添字 $\alpha = 1, \dots, 4$ を考える.

Problems

Problem 16.3: Counterterm relations

(b)

3 ボソン頂点 3 つのダイアグラムを考える.



分母は $k^4(k-p)^2$ なので, $k = \ell + xp$ で, ℓ^2 の項のみが発散する. 分子は^{*1}

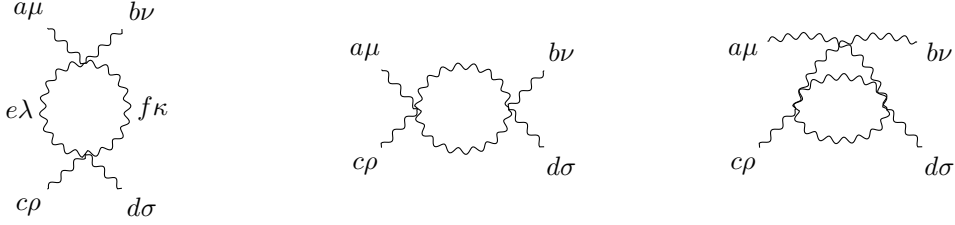
$$\begin{aligned}
 (\text{Num}) &= 2k^2 g^{\mu\nu} k^\rho + 4k^2 g^{\mu\nu} p^\rho + 2k^2 g^{\mu\rho} k^\nu - 3k^2 g^{\mu\rho} p^\nu \\
 &\quad + 2k^2 g^{\nu\rho} k^\mu - 3k^2 g^{\nu\rho} p^\mu - 8(k \cdot p) g^{\mu\nu} p^\rho - 2(k \cdot p) g^{\mu\rho} k^\nu \\
 &\quad + 3(k \cdot p) g^{\mu\rho} p^\nu - 2(k \cdot p) g^{\nu\rho} k^\mu + 3(k \cdot p) g^{\nu\rho} p^\mu + 10p^2 g^{\mu\nu} k^\rho \\
 &\quad - p^2 g^{\mu\rho} k^\nu - p^2 g^{\nu\rho} k^\mu + 18k^\mu k^\nu k^\rho - 9k^\mu k^\rho p^\nu \\
 &\quad + 3k^\mu p^\nu p^\rho - 9k^\nu k^\rho p^\mu + 3k^\nu p^\mu p^\rho - 6k^\rho p^\mu p^\nu \\
 &\sim 2k^2 g^{\mu\nu} k^\rho + 4k^2 g^{\mu\nu} p^\rho + 2k^2 g^{\mu\rho} k^\nu - 3k^2 g^{\mu\rho} p^\nu \\
 &\quad + 2k^2 g^{\nu\rho} k^\mu - 3k^2 g^{\nu\rho} p^\mu - 2(k \cdot p) g^{\mu\nu} k^\nu \\
 &\quad - 2(k \cdot p) g^{\nu\rho} k^\mu \\
 &\quad + 18k^\mu k^\nu k^\rho - 9k^\mu k^\rho p^\nu \\
 &\quad - 9k^\nu k^\rho p^\mu \\
 &\sim 2g^{\mu\nu}(\ell + xp)^2(\ell + xp)^\rho + 4g^{\mu\nu}(\ell + xp)^2 p^\rho + 2g^{\mu\rho}(\ell + xp)^2(\ell + xp)^\nu - 3g^{\mu\rho}(\ell + xp)^2 p^\nu \\
 &\quad + 2g^{\nu\rho}(\ell + xp)^2(\ell + xp)^\mu - 3g^{\nu\rho}(\ell + xp)^2 p^\mu - 2g^{\mu\rho} p \cdot (\ell + xp)(\ell + xp)^\nu \\
 &\quad - 2g^{\nu\rho} p \cdot (\ell + xp)(\ell + xp)^\mu \\
 &\quad + 18(\ell + xp)^\mu(\ell + xp)^\nu(\ell + xp)^\rho - 9(\ell + xp)^\mu(\ell + xp)^\rho p^\nu \\
 &\quad - 9(\ell + xp)^\nu(\ell + xp)^\rho p^\mu \\
 &\sim 2g^{\mu\nu}[xp^\rho \ell^2 + 2x(p \cdot \ell)\ell^\rho] + 4g^{\mu\nu} p^\rho \ell^2 + 2g^{\mu\rho}[xp^\nu \ell^2 + 2x(p \cdot \ell)\ell^\nu] - 3g^{\mu\rho} p^\nu \ell^2 \\
 &\quad + 2g^{\nu\rho}[xp^\mu \ell^2 + 2x(p \cdot \ell)\ell^\mu] - 3g^{\nu\rho} p^\mu \ell^2 - 2g^{\mu\rho}(p \cdot \ell)\ell^\nu - 2g^{\nu\rho}(p \cdot \ell)\ell^\mu \\
 &\quad + 18x(p^\mu \ell^\nu \ell^\rho + p^\nu \ell^\rho \ell^\mu + p^\rho \ell^\mu \ell^\nu) - 9p^\nu \ell^\mu \ell^\rho - 9p^\mu \ell^\nu \ell^\rho \\
 &\sim 3xg^{\mu\nu} p^\rho \ell^2 + 4g^{\mu\nu} p^\rho \ell^2 + 3xg^{\mu\rho} p^\nu \ell^2 - 3g^{\mu\rho} p^\nu \ell^2 \\
 &\quad + 3xg^{\nu\rho} p^\mu \ell^2 - 3g^{\nu\rho} p^\mu \ell^2 - \frac{1}{2}g^{\mu\rho} p^\nu \ell^2 - \frac{1}{2}g^{\nu\rho} p^\mu \ell^2 \\
 &\quad + \frac{18x-9}{4}g^{\nu\rho} p^\mu \ell^2 + \frac{18x-9}{4}g^{\mu\rho} p^\nu \ell^2 + \frac{9}{2}xg^{\mu\nu} p^\rho \ell^2
 \end{aligned}$$

^{*1} ./src/py/NonAbelian_1loop.ipynb

$$= \frac{15x+8}{2} g^{\mu\nu} p^\rho \ell^2 + \frac{30x-23}{4} (g^{\mu\rho} p^\nu + g^{\nu\rho} p^\mu) \ell^2.$$

(c)

4 ボソン頂点 2 つのダイアグラムを考える.



1 つ目のダイアグラムは

$$\begin{aligned} & [f_{abg} f_{efg} (-g_{\mu\kappa} g_{\nu\lambda} + g_{\mu\lambda} g_{\nu\kappa}) + f_{aeg} f_{bfg} (g_{\kappa\lambda} g_{\mu\nu} - g_{\mu\kappa} g_{\nu\lambda}) + f_{afg} f_{beg} (g_{\kappa\lambda} g_{\mu\nu} - g_{\mu\lambda} g_{\nu\kappa})] \\ & \times [f_{cdh} f_{efh} (-g_{\rho\kappa} g_{\sigma\lambda} + g_{\rho\lambda} g_{\sigma\kappa}) + f_{ech} f_{fdh} (g_{\kappa\lambda} g_{\rho\sigma} - g_{\rho\kappa} g_{\sigma\lambda}) + f_{edh} f_{fch} (g_{\kappa\lambda} g_{\rho\sigma} - g_{\rho\lambda} g_{\sigma\kappa})] \\ & = 2f_{abg} f_{cdh} f_{efg} f_{efh} g^{\mu\rho} g^{\nu\sigma} - 2f_{abg} f_{cdh} f_{efg} f_{efh} g^{\mu\sigma} g^{\nu\rho} \\ & + f_{abg} f_{ech} f_{efg} f_{fdh} g^{\mu\rho} g^{\nu\sigma} - f_{abg} f_{ech} f_{efg} f_{fdh} g^{\mu\sigma} g^{\nu\rho} \\ & - f_{abg} f_{edh} f_{efg} f_{fch} g^{\mu\rho} g^{\nu\sigma} + f_{abg} f_{edh} f_{efg} f_{fch} g^{\mu\sigma} g^{\nu\rho} \\ & + f_{aeg} f_{bfg} f_{cdh} f_{efh} g^{\mu\rho} g^{\nu\sigma} - f_{aeg} f_{bfg} f_{cdh} f_{efh} g^{\mu\sigma} g^{\nu\rho} \\ & + 2f_{aeg} f_{bfg} f_{ech} f_{fdh} g^{\mu\nu} g^{\rho\sigma} + f_{aeg} f_{bfg} f_{ech} f_{fdh} g^{\mu\rho} g^{\nu\sigma} \\ & + 2f_{aeg} f_{bfg} f_{edh} f_{fch} g^{\mu\nu} g^{\rho\sigma} + f_{aeg} f_{bfg} f_{edh} f_{fch} g^{\mu\sigma} g^{\nu\rho} \\ & - f_{afg} f_{beg} f_{cdh} f_{efh} g^{\mu\rho} g^{\nu\sigma} + f_{afg} f_{beg} f_{cdh} f_{efh} g^{\mu\sigma} g^{\nu\rho} \\ & + 2f_{afg} f_{beg} f_{ech} f_{fdh} g^{\mu\nu} g^{\rho\sigma} + f_{afg} f_{beg} f_{ech} f_{fdh} g^{\mu\sigma} g^{\nu\rho} \\ & + 2f_{afg} f_{beg} f_{edh} f_{fch} g^{\mu\nu} g^{\rho\sigma} + f_{afg} f_{beg} f_{edh} f_{fch} g^{\mu\rho} g^{\nu\sigma} \\ & = 2f_{abg} f_{cdh} f_{efg} f_{efh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + (f_{abg} f_{ech} f_{efg} f_{fdh} - f_{abg} f_{edh} f_{efg} f_{fch}) (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + (f_{aeg} f_{bfg} f_{cdh} f_{efh} - f_{afg} f_{beg} f_{cdh} f_{efh}) (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + (f_{aeg} f_{bfg} f_{ech} f_{fdh} + f_{afg} f_{beg} f_{edh} f_{fch}) (2g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma}) \\ & + (f_{aeg} f_{bfg} f_{edh} f_{fch} + f_{afg} f_{beg} f_{ech} f_{fdh}) (2g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\ & = 2f_{abg} f_{cdh} f_{efg} f_{efh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + 2f_{abg} f_{ech} f_{efg} f_{fdh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + 2f_{aeg} f_{bfg} f_{cdh} f_{efh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\ & + 2f_{aeg} f_{bfg} f_{ech} f_{fdh} (2g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma}) \\ & + 2f_{aeg} f_{bfg} f_{edh} f_{fch} (2g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho}) \end{aligned}$$

1 行目は

$$2f_{abg} f_{cdh} f_{efg} f_{efh} = 2f_{abg} f_{cdh} C_2(G) \delta^{gh} = 2f_{abg} f_{cdg} C_2(G).$$

(16.79) から

$$i \operatorname{Tr}(t_G^a t_G^b t_G^c) = f^{ade} f^{ebf} f^{fdc} = -\frac{1}{2} f^{abc} C_2(G) \quad [16.6.1]$$

なので、2行目は

$$2f^{abg}f^{ech}f^{efg}f^{fdh} = -2f^{abg}f^{ceh}f^{hdf}f^{feg} = f^{abg}f^{cdg}C_2(G).$$

同様に3行目は

$$2f^{aeg}f^{bfg}f^{cdh}f^{efh} = -2f^{cdh}f^{aeg}f^{gbf}f^{feh} = f^{cdh}f^{abh}C_2(G).$$

以上から、

$$\begin{aligned} &= 4C_2(G)f^{abg}f^{cdg}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + 2f^{aeg}f^{bfg}f^{ech}f^{fdh}(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) \\ &\quad + 2f^{aeg}f^{bfg}f^{edh}f^{fch}(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\ &= 4C_2(G)f^{abg}f^{cdg}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + 2\text{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) \\ &\quad + 2\text{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}). \end{aligned}$$

随伴表現は

$$(t_G^\dagger)_{ac} = (t_G^b)_{ca}^* = (if_{cba})^* = -if_{cba} = if_{abc} = (t_G^b)_{ac}$$

なので t_G^a は Hermite 行列. さらに $t_G^a t_G^b t_G^c t_G^d$ は実数なので

$$\begin{aligned} \text{Tr}(t_G^a t_G^b t_G^c t_G^d) &= (t_G^a)_{ij}(t_G^b)_{jk}(t_G^c)_{kl}(t_G^d)_{li} = (t_G^a)_{ij}^*(t_G^b)_{jk}^*(t_G^c)_{kl}^*(t_G^d)_{li}^* \\ &= (t_G^a)_{ji}(t_G^b)_{kj}(t_G^c)_{lk}(t_G^d)_{il} \\ &= \text{Tr}(t_G^a t_G^d t_G^c t_G^b). \end{aligned} \tag{16.6.2}$$

[16.6.1] と併せて

$$\begin{aligned} C_2(G)f^{abg}f^{cdg} &= -2i\text{Tr}(t_G^a t_G^b t_G^c t_G^d)f^{cdg} = -2\text{Tr}(t_G^a t_G^b [t_G^c, t_G^d]) \\ &= -2\text{Tr}(t_G^a t_G^b t_G^c t_G^d) + 2\text{Tr}(t_G^a t_G^b t_G^d t_G^c). \end{aligned} \tag{16.6.3}$$

よって、

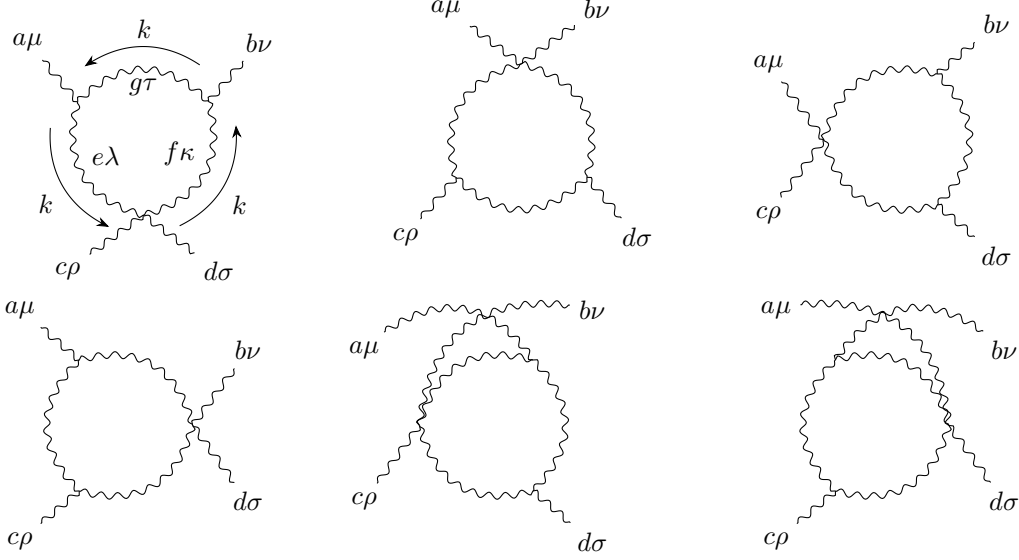
$$\begin{aligned} &= 4[-2\text{Tr}(t_G^a t_G^b t_G^c t_G^d) + 2\text{Tr}(t_G^a t_G^b t_G^d t_G^c)](g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + 2\text{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma}) \\ &\quad + 2\text{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho}) \\ &= 2\text{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 5g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + 2\text{Tr}(t_G^a t_G^b t_G^d t_G^c)(2g^{\mu\nu}g^{\rho\sigma} + 5g^{\mu\rho}g^{\nu\sigma} - 4g^{\mu\sigma}g^{\nu\rho}). \end{aligned}$$

残りのダイアグラムは $((b\nu \leftrightarrow c\rho)), ((b\nu \leftrightarrow d\sigma))$ によって得られる. 合計は

$$\begin{aligned} &= 2\text{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 5g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + 2\text{Tr}(t_G^a t_G^c t_G^b t_G^d)(2g^{\mu\rho}g^{\nu\sigma} - 4g^{\mu\nu}g^{\rho\sigma} + 5g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + 2\text{Tr}(t_G^a t_G^b t_G^c t_G^d)(2g^{\mu\sigma}g^{\nu\rho} - 4g^{\mu\rho}g^{\nu\sigma} + 5g^{\mu\nu}g^{\rho\sigma}) \\ &\quad + 2\text{Tr}(t_G^a t_G^b t_G^d t_G^c)(2g^{\mu\nu}g^{\rho\sigma} + 5g^{\mu\rho}g^{\nu\sigma} - 4g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + 2\text{Tr}(t_G^a t_G^c t_G^b t_G^d)(2g^{\mu\rho}g^{\nu\sigma} + 5g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\sigma}g^{\nu\rho}) \\ &\quad + 2\text{Tr}(t_G^a t_G^c t_G^b t_G^d)(2g^{\mu\sigma}g^{\nu\rho} + 5g^{\mu\rho}g^{\nu\sigma} - 4g^{\mu\nu}g^{\rho\sigma}) \\ &= 2\text{Tr}(t_G^a t_G^b t_G^c t_G^d)[7g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + 7g^{\mu\sigma}g^{\nu\rho}] \end{aligned}$$

$$\begin{aligned}
 &+ 2 \operatorname{Tr}(t_G^a t_G^b t_G^d t_G^c) [7g^{\mu\nu} g^{\rho\sigma} + 7g^{\mu\rho} g^{\nu\sigma} - 8g^{\mu\sigma} g^{\nu\rho}] \\
 &+ 2 \operatorname{Tr}(t_G^a t_G^c t_G^b t_G^d) [-8g^{\mu\nu} g^{\rho\sigma} + 7g^{\mu\rho} g^{\nu\sigma} + 7g^{\mu\sigma} g^{\nu\rho}].
 \end{aligned}$$

4 ボソン頂点 1 つと 3 ボソン頂点 2 つのダイアグラムを考える.



1 つ目のダイアグラムは $f^{aeg} f^{bgf} \times (efcd) \times$

$$\begin{aligned}
 &(-2g_{\lambda\tau} k_\mu + g_{\mu\lambda} k_\tau + g_{\mu\tau} k_\lambda) (-2g_{\kappa\tau} k_\nu + g_{\nu\kappa} k_\tau + g_{\nu\tau} k_\kappa) \\
 &= g_{\mu\lambda} g_{\nu\kappa} k^2 + 4g_{\kappa\lambda} k_\mu k_\nu - 2g_{\mu\kappa} k_\lambda k_\nu - g_{\mu\lambda} k_\kappa k_\nu + g_{\mu\nu} k_\kappa k_\lambda - g_{\nu\kappa} k_\lambda k_\mu - 2g_{\nu\lambda} k_\kappa k_\mu \\
 &\sim g_{\mu\lambda} g_{\nu\kappa} k^2 + g_{\kappa\lambda} g_{\mu\nu} k^2 - \frac{1}{2} g_{\mu\kappa} g_{\nu\lambda} k^2 - \frac{1}{4} g_{\mu\lambda} g_{\nu\kappa} k^2 + \frac{1}{4} g_{\mu\nu} g_{\lambda\kappa} k^2 - \frac{1}{4} g_{\nu\kappa} g_{\mu\lambda} k^2 - \frac{1}{2} g_{\nu\lambda} g_{\mu\kappa} k^2 \\
 &= \frac{5}{4} g_{\mu\nu} g_{\lambda\kappa} k^2 + \frac{1}{2} g_{\mu\lambda} g_{\nu\kappa} k^2 - g_{\mu\kappa} g_{\nu\lambda} k^2.
 \end{aligned}$$

従って, k^2 の係数は $f^{aeg} f^{bgf} / 4 \times$

$$\begin{aligned}
 &[f^{cdh} f^{efh} (-g^{\rho\kappa} g^{\sigma\lambda} + g^{\rho\lambda} g^{\sigma\kappa}) + f^{ech} f^{fdh} (g^{\kappa\lambda} g^{\rho\sigma} - g^{\rho\kappa} g^{\sigma\lambda}) + f^{edh} f^{fch} (g^{\kappa\lambda} g^{\rho\sigma} - g^{\rho\lambda} g^{\sigma\kappa})] \\
 &\quad \times (5g^{\mu\nu} g^{\lambda\kappa} + 2g^{\mu\lambda} g^{\nu\kappa} - 4g^{\mu\kappa} g^{\nu\lambda}) \\
 &= 6f^{cdh} f^{efh} g^{\mu\rho} g^{\nu\sigma} - 6f^{cdh} f^{efh} g^{\mu\sigma} g^{\nu\rho} + 13f^{ech} f^{fdh} g^{\mu\nu} g^{\rho\sigma} + 4f^{ech} f^{fdh} g^{\mu\rho} g^{\nu\sigma} \\
 &\quad - 2f^{ech} f^{fdh} g^{\mu\sigma} g^{\nu\rho} + 13f^{edh} f^{fch} g^{\mu\nu} g^{\rho\sigma} - 2f^{edh} f^{fch} g^{\mu\rho} g^{\nu\sigma} + 4f^{edh} f^{fch} g^{\mu\sigma} g^{\nu\rho} \\
 &= 6f^{cdh} f^{efh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
 &\quad + f^{ech} f^{fdh} (13g^{\mu\nu} g^{\rho\sigma} + 4g^{\mu\rho} g^{\nu\sigma} - 2g^{\mu\sigma} g^{\nu\rho}) \\
 &\quad + f^{edh} f^{fch} (13g^{\mu\nu} g^{\rho\sigma} - 2g^{\mu\rho} g^{\nu\sigma} + 4g^{\mu\sigma} g^{\nu\rho}).
 \end{aligned}$$

[16.6.1][16.6.3] を使えば

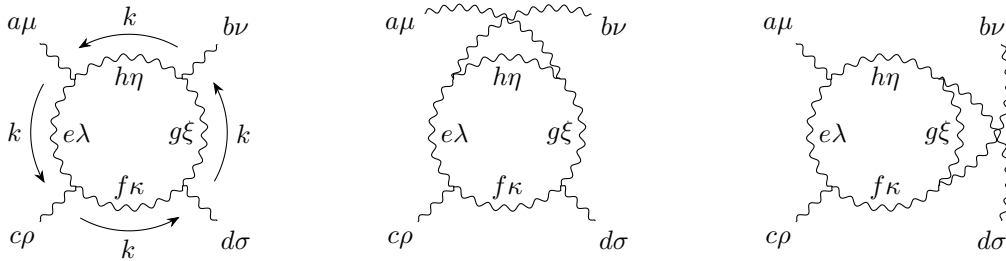
$$\begin{aligned}
 &\frac{3}{2} f^{aeg} f^{bgf} f^{cdh} f^{efh} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) \\
 &\quad + \frac{1}{4} f^{aeg} f^{bgf} f^{ech} f^{fdh} (13g^{\mu\nu} g^{\rho\sigma} + 4g^{\mu\rho} g^{\nu\sigma} - 2g^{\mu\sigma} g^{\nu\rho}) \\
 &\quad + \frac{1}{4} f^{aeg} f^{bgf} f^{edh} f^{fch} (8g^{\mu\nu} g^{\rho\sigma} - 2g^{\mu\rho} g^{\nu\sigma} + 4g^{\mu\sigma} g^{\nu\rho})
 \end{aligned}$$

$$\begin{aligned}
 &= -\frac{3}{4}C_2(G)f^{abh}f^{cdh}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad -\frac{1}{4}\text{Tr}(t_G^at_G^bt_G^ct_G^d)(13g^{\mu\nu}g^{\rho\sigma} + 4g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad -\frac{1}{4}\text{Tr}(t_G^at_G^bt_G^ct_G^d)(13g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}) \\
 &= \frac{3}{2}[\text{Tr}(t_G^at_G^bt_G^ct_G^d) - \text{Tr}(t_G^at_G^bt_G^dt_G^c)](g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad -\frac{1}{4}\text{Tr}(t_G^at_G^bt_G^ct_G^d)(13g^{\mu\nu}g^{\rho\sigma} + 4g^{\mu\rho}g^{\nu\sigma} - 2g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad -\frac{1}{4}\text{Tr}(t_G^at_G^bt_G^ct_G^d)(13g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}) \\
 &= \frac{1}{4}\text{Tr}(t_G^at_G^bt_G^ct_G^d)(-13g^{\mu\nu}g^{\rho\sigma} + 8g^{\mu\rho}g^{\nu\sigma} - 10g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad +\frac{1}{4}\text{Tr}(t_G^at_G^bt_G^dt_G^c)(-13g^{\mu\nu}g^{\rho\sigma} - 10g^{\mu\rho}g^{\nu\sigma} + 8g^{\mu\sigma}g^{\nu\rho}).
 \end{aligned}$$

残りのダイアグラムは $((a\mu, b\nu \leftrightarrow c\rho, d\sigma))$, $((a\mu \leftrightarrow d\sigma))$, $((b\nu \leftrightarrow c\rho))$, $((b\nu \leftrightarrow d\sigma))$, $((a\mu \leftrightarrow c\rho))$ の置換によって得られる。[16.6.2] に注意して全て足せば (1, 2 個目及び 3, 4 個目及び 5, 6 個目はそれぞれ同じ値になる),

$$\begin{aligned}
 &\frac{1}{2}\text{Tr}(t_G^at_G^bt_G^ct_G^d)(-13g^{\mu\nu}g^{\rho\sigma} + 8g^{\mu\rho}g^{\nu\sigma} - 10g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad +\frac{1}{2}\text{Tr}(t_G^at_G^ct_G^bt_G^d)(-13g^{\nu\sigma}g^{\mu\rho} + 8g^{\rho\sigma}g^{\nu\mu} - 10g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad +\frac{1}{2}\text{Tr}(t_G^at_G^bt_G^ct_G^d)(-13g^{\mu\sigma}g^{\nu\rho} + 8g^{\mu\rho}g^{\nu\sigma} - 10g^{\mu\nu}g^{\rho\sigma}) \\
 &\quad +\frac{1}{2}\text{Tr}(t_G^at_G^bt_G^dt_G^c)(-13g^{\mu\nu}g^{\rho\sigma} - 10g^{\mu\rho}g^{\nu\sigma} + 8g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad +\frac{1}{2}\text{Tr}(t_G^at_G^bt_G^dt_G^c)(-13g^{\nu\sigma}g^{\mu\rho} - 10g^{\rho\sigma}g^{\mu\nu} + 8g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad +\frac{1}{2}\text{Tr}(t_G^at_G^ct_G^bt_G^d)(-13g^{\mu\sigma}g^{\nu\rho} - 10g^{\mu\rho}g^{\nu\sigma} + 8g^{\mu\nu}g^{\rho\sigma}) \\
 &= \text{Tr}(t_G^at_G^bt_G^ct_G^d)(-23g^{\mu\nu}g^{\rho\sigma} + 16g^{\mu\rho}g^{\nu\sigma} - 23g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad +\text{Tr}(t_G^at_G^bt_G^dt_G^c)(-23g^{\mu\nu}g^{\rho\sigma} - 23g^{\mu\rho}g^{\nu\sigma} + 16g^{\mu\sigma}g^{\nu\rho}) \\
 &\quad +\text{Tr}(t_G^at_G^ct_G^bt_G^d)(16g^{\mu\nu}g^{\rho\sigma} - 23g^{\mu\rho}g^{\nu\sigma} - 23g^{\mu\sigma}g^{\nu\rho}).
 \end{aligned}$$

3 ボソン頂点 4 つのダイアグラムを考える。



1 つ目のダイアグラムは $f^{aeh}f^{ecf}fgfdfbhg \times$

$$\begin{aligned}
 &(-2g^{\lambda\eta}k^\mu + g^{\mu\eta}k^\lambda + g^{\mu\lambda}k^\eta)(-2g^{\kappa\lambda}k^\rho + g^{\rho\kappa}k^\lambda + g^{\rho\lambda}k^\kappa) \\
 &\quad \times (-2g^{\kappa\xi}k^\sigma + g^{\sigma\kappa}k^\xi + g^{\sigma\xi}k^\kappa)(g^{\nu\eta}k^\xi + g^{\nu\xi}k^\eta - 2g^{\xi\eta}k^\nu)
 \end{aligned}$$

$$= k^4 g^{\mu\nu} g^{\rho\sigma} + k^4 g^{\mu\rho} g^{\nu\sigma} + 3k^2 g^{\mu\nu} k^\rho k^\sigma + 3k^2 g^{\mu\rho} k^\nu k^\sigma + 3k^2 g^{\nu\sigma} k^\mu k^\rho + 3k^2 g^{\rho\sigma} k^\mu k^\nu + 34k^\mu k^\nu k^\rho k^\sigma$$

となる。(A.47)(A.48)を参考にすれば

$$k^\mu k^\nu k^\rho k^\sigma \rightarrow \frac{1}{24}(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$$

として良いことが分かる。よって、 k^4 の係数は $\text{Tr}(t_G^a t_G^b t_G^d t_G^c) \times$

$$\begin{aligned} & g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + \frac{3}{4} g^{\mu\nu} g^{\rho\sigma} + \frac{3}{4} k^2 g^{\mu\rho} g^{\nu\sigma} + \frac{3}{4} g^{\nu\sigma} g^{\mu\rho} + \frac{3}{4} g^{\rho\sigma} g^{\mu\nu} + \frac{17}{12} (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\ &= \frac{47}{12} g^{\mu\nu} g^{\rho\sigma} + \frac{47}{12} g^{\mu\rho} g^{\nu\sigma} + \frac{17}{12} g^{\mu\sigma} g^{\nu\rho}. \end{aligned}$$

残りのダイアグラムは $((a\mu \leftrightarrow b\nu)), ((b\nu \leftrightarrow d\sigma))$ によって得られる。

Chapter 17

Quantum Chromodynamics

17.4 Hard-Scattering Processes in Hadron Collisions

(17.58)

Jacobian を計算するコード.

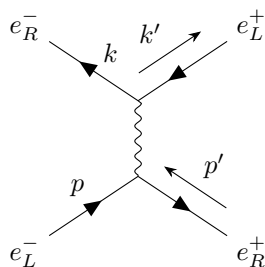
```
import sympy
p = sympy.Symbol('p')
y_3 = sympy.Symbol('y_3')
y_4 = sympy.Symbol('y_4')
y = (y_3 - y_4) / 2
Y = (y_3 + y_4) / 2

x_1 = 2 * p * sympy.cosh(y) * sympy.exp(Y)
x_2 = 2 * p * sympy.cosh(y) * sympy.exp(-Y)
t = -2 * p**2 * sympy.cosh(y) * sympy.exp(-y)

O = sympy.Matrix([x_1, x_2, t])
N = sympy.Matrix([y_3, y_4, p])
J = sympy.det(O.jacobian(N))
print(sympy.simplify(J))
```

(17.67)

$e_L^- e_R^+ \rightarrow e_R^- e_L^+$ の過程を考える.



Section 5.2 と同様に、射影演算子 $(1 \pm \gamma^5)/2$ を使えば

$$\begin{aligned} i\mathcal{M} &= ie^2 \bar{u}(k) \gamma^\mu \frac{1 - \gamma^5}{2} v(k') \frac{1}{(p + p')^2} \bar{v}(p') \gamma_\mu \frac{1 - \gamma^5}{2} u(p) \\ &= \frac{ie^2}{s} \bar{u}(k) \gamma^\mu \frac{1 - \gamma^5}{2} v(k') \bar{v}(p') \gamma_\mu \frac{1 - \gamma^5}{2} u(p) \end{aligned}$$

なので

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{e^4}{s^2} \bar{u}(k) \gamma^\mu \frac{1 - \gamma^5}{2} v(k') \bar{v}(k') \gamma^\nu \frac{1 - \gamma^5}{2} u(k) \\ &\quad \times \bar{v}(p') \gamma_\mu \frac{1 - \gamma^5}{2} u(p) \bar{u}(p) \gamma_\nu \frac{1 - \gamma^5}{2} v(p'). \end{aligned}$$

入射電子・陽電子に関してはスピンを平均，散乱電子・陽電子に関してはスピンを合計して，

$$\begin{aligned} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 &= \frac{e^4}{4s^2} \text{Tr} \left(\not{k} \gamma^\mu \frac{1 - \gamma^5}{2} \not{k}' \gamma^\nu \frac{1 - \gamma^5}{2} \right) \text{Tr} \left(\not{p}' \gamma_\mu \frac{1 - \gamma^5}{2} \not{p} \gamma_\nu \frac{1 - \gamma^5}{2} \right) \\ &= \frac{e^4}{4s^2} \text{Tr} \left(\not{k} \gamma^\mu \not{k}' \gamma^\nu \frac{1 - \gamma^5}{2} \right) \text{Tr} \left(\not{p}' \gamma_\mu \not{p} \gamma_\nu \frac{1 - \gamma^5}{2} \right) \\ &= \frac{e^4}{4s^2} E^2 (1 - \cos \theta)^2 \\ &= \frac{e^4}{4} \left(\frac{t}{s} \right)^2. \end{aligned}$$

トレースの計算は (5.23) の結果を利用した。(4.85) から

$$\frac{d\sigma}{d \cos \theta} = \frac{|\mathcal{M}|^2}{32\pi s} = \frac{e^2}{128\pi s} \left(\frac{t}{s} \right)^2 = \frac{\pi \alpha^2}{8s} \left(\frac{t}{s} \right)^2.$$

さらに

$$t = -E^2(1 - \cos \theta)^2 - E^2 \sin^2 \theta = E^2(2 \cos \theta - 2) = \frac{s}{2}(\cos \theta - 1)$$

なので

$$\frac{d\sigma}{dt} = \frac{2}{s} \frac{d\sigma}{d \cos \theta} = \frac{\pi \alpha^2}{4s^2} \left(\frac{t}{s} \right)^2.$$

Problems

Problem 17.4: The gluon splitting function

$P_{g \leftarrow g}(z,)$ の規格化条件を求める。(17.39) と同様に

$$\int_0^1 dz z \left[\sum_f \{f_f(z, Q) + f_{\bar{f}}(z, Q)\} + f_g(z, Q) \right] = 1.$$

(17.128) を積分して

$$\frac{d}{d \log Q} \int_0^1 dx x f_g(x, Q)$$

$$\begin{aligned}
 &= \frac{\alpha_s}{\pi} \int_0^1 dx x \int_x^1 \frac{dz}{z} \left[P_{gq}(z) \sum_f \{f_f(x/z) + f_{\bar{f}}(x/z)\} + P_{gg}(z) f_g(x/z) \right] \\
 &= \frac{\alpha_s}{\pi} \int_0^1 ds s \int_0^1 dz z \left[P_{gq}(z) \sum_f \{f_f(s) + f_{\bar{f}}(s)\} + P_{gg}(z) f_g(s) \right]
 \end{aligned}$$

となる ($s = x/z$ とした). 他の式も同様にして

$$\begin{aligned}
 \frac{d}{d \log Q} \int_0^1 dx x f_f(x, Q) &= \frac{\alpha_s}{\pi} \int_0^1 ds s \int_0^1 dz z [P_{qg}(z) f_f(s) + P_{qg}(z) f_g(s)], \\
 \frac{d}{d \log Q} \int_0^1 dx x f_{\bar{f}}(x, Q) &= \frac{\alpha_s}{\pi} \int_0^1 ds s \int_0^1 dz z [P_{qg}(z) f_{\bar{f}}(s) + P_{qg}(z) f_g(s)].
 \end{aligned}$$

以上から,

$$\begin{aligned}
 0 &= \frac{d}{d \log Q} \int_0^1 dx x \left[\sum_f \{f_f(x, Q) + f_{\bar{f}}(x, Q)\} + f_g(x, Q) \right] \\
 &= \frac{\alpha_s}{\pi} \int_0^1 ds s \int_0^1 dz z \left[\{P_{gq}(z) + P_{qg}(z)\} \sum_f \{f_f(s) + f_{\bar{f}}(s)\} + \{2n_f P_{qg}(z) + P_{gg}(z)\} f_g(s) \right] \\
 &= \frac{\alpha_s}{\pi} \int_0^1 ds s \{f_f(s) + f_{\bar{f}}(s)\} \int_0^1 dz z \{P_{gq}(z) + P_{qg}(z)\} \\
 &\quad + \frac{\alpha_s}{\pi} \int_0^1 ds s f_g(s) \int_0^1 dz z \{2n_f P_{qg}(z) + P_{gg}(z)\} \\
 &= -\frac{\alpha_s}{\pi} \int_0^1 ds s f_g(s) \int_0^1 dz z \{P_{gq}(z) + P_{qg}(z)\} + \frac{\alpha_s}{\pi} \int_0^1 ds s f_g(s) \int_0^1 dz z \{2n_f P_{qg}(z) + P_{gg}(z)\}
 \end{aligned}$$

なので,

$$\begin{aligned}
 \int_0^1 dz z \{2n_f P_{qg}(z) + P_{gg}(z)\} &= \int_0^1 dz z \{P_{gq}(z) + P_{qg}(z)\} \\
 &= 3 \int_0^1 dz \left[1 + (1-z)^2 + \frac{z+z^3}{(1-z)_+} \right] + 2 \\
 &= 3 \int_0^1 dz \left[z^2 - 2z + 2 + \frac{z^3 + z - 2}{1-z} \right] + 2 \\
 &= 3 \int_0^1 dz [z^2 - 2z + 2 - (z^2 + z + 2)] + 2 \\
 &= -4 \int_0^1 z dz + 2 \\
 &= 0.
 \end{aligned}$$

Chapter 18

Operator Products and Effective Vertices

18.5 Operator Analysis of Deep Inelastic Scattering

(18.136)

$$\bar{u}(P)\gamma^\mu u(P) = \text{Tr}[\bar{u}(P)\gamma^\mu u(P)] = \text{Tr}[\gamma^\mu u(P)\bar{u}(P)].$$

スピンのに関して平均を取って

$$\rightarrow \frac{1}{2} \text{Tr}[\gamma^\mu \not{P}] = 2P^\mu.$$

(18.208)

(18.207) から

$$\frac{d}{d \log Q^2} M_n^+ = \frac{d}{d \log Q^2} \int_0^1 dx x^{n-1} \sum_f (f_f + f_{\bar{f}})(x).$$

(17.128) から

$$= \frac{\alpha_s}{2\pi} \int_0^1 dx x^{n-1} \int_x^1 \frac{dz}{z} \left[P_{q \leftarrow q}(z) \sum_f (f_f + f_{\bar{f}})(x/z) + 2P_{q \leftarrow g}(z) f_g(x/z) \right].$$

(17.17)(18.199) から

$$\begin{aligned} &= \frac{2}{b_0 \log(Q^2/\Lambda^2)} \int_0^1 dz z^{n-1} P_{q \leftarrow q}(z) \int_0^1 dy y^{n-1} \sum_f (f_f + f_{\bar{f}})(y) \\ &\quad + \frac{2}{b_0 \log(Q^2/\Lambda^2)} \int_0^1 dz z^{n-1} 2P_{q \leftarrow g}(z) \int_0^1 dy y^{n-1} f_g(y) \\ &= \frac{2}{b_0 \log(Q^2/\Lambda^2)} \left[M_n^+ \int_0^1 dz z^{n-1} P_{q \leftarrow q}(z) + M_{gn} \int_0^1 dz z^{n-1} 2P_{q \leftarrow g}(z) \right]. \end{aligned}$$

(17.129)(18.181)(18.203) から

$$= \frac{2}{b_0 \log(Q^2/\Lambda^2)} \left[\frac{a_{ff}^n}{4} M_n^+ + \frac{a_{fg}^n}{4} M_{gn} \right].$$

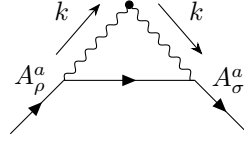
Problems

Problem 18.3: Anomalous dimensions of gluon twist-2 operators (計算合わない)

[5] を参考にして修正予定.

a_{gf}^n の 1 ループ補正

Figure 18.15 (a) のダイアグラムを計算する.



(18.174) から

$$\mathcal{O}_g^{(n)\mu_1 \dots \mu_n} \rightarrow -(\partial^{\mu_1} A^\nu - \partial^\nu A^{\mu_1})(i\partial^{\mu_2}) \dots (i\partial^{\mu_{n-1}})(\partial^{\mu_n} A_\nu - \partial_\nu A^{\mu_n}).$$

$\partial^{\mu_1} A^\nu$ と A_σ および $\partial^{\mu_n} A_\nu$ と A_ρ を縮約した場合は,

$$\mathcal{O} = -(ik^{\mu_1})(k^{\mu_2} \dots k^{\mu_{n-1}})(-ik^{\mu_n})$$

なので,

$$-(ig)^2 \int \frac{d^d k}{(2\pi)^d} t^a \gamma^\nu \frac{i}{\not{p} - \not{k}} t^a \gamma_\nu \left(\frac{-i}{k^2} \right)^2 k^{\mu_1} \dots k^{\mu_n}.$$

(6.42)(A.34)(A.37) から $\ell = k - xp$ として

$$\begin{aligned} &= 3ig^2 \int \frac{d^d \ell}{(2\pi)^d} \gamma^\nu \gamma_\alpha \gamma_\nu \int_0^1 dx 2(1-x) \frac{1}{(\ell^2 - \Delta)^3} [\ell - (1-x)p]^\alpha (\ell + xp)^{\mu_1} \dots (\ell + xp)^{\mu_n} \\ &= \frac{8}{3} (2-d) ig^2 \int_0^1 dx (1-x) \gamma_\alpha \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^3} [\ell - (1-x)p]^\alpha (\ell + xp)^{\mu_1} \dots (\ell + xp)^{\mu_n}. \end{aligned}$$

このうち (18.161) の \mathcal{O}_f の形 $\gamma^{\mu_1} p^{\mu_2} \dots p^{\mu_n}$ を持つ項は

$$\begin{aligned} &\rightarrow \frac{8}{3} (2-d) ig^2 n \int_0^1 dx (1-x) \gamma_\alpha \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^3} \ell^\alpha \ell^{\mu_1} x^{n-1} p^{\mu_2} \dots p^{\mu_n} \\ &= \frac{8}{3} \frac{2-d}{d} ig^2 n \int_0^1 dx (1-x) x^{n-1} \gamma_\alpha \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta)^3} g^{\alpha\mu_1} p^{\mu_2} \dots p^{\mu_n} \\ &= \frac{8}{3} \frac{2-d}{d} ig^2 n \int_0^1 dx (1-x) x^{n-1} \frac{i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(2-d/2)}{\Gamma(3)} \left(\frac{1}{\Delta} \right)^{2-d/2} \gamma^{\mu_1} p^{\mu_2} \dots p^{\mu_n} \\ &= 3 \frac{g^2}{(4\pi)^2} n \int_0^1 dx (1-x) x^{n-1} \gamma^{\mu_1} p^{\mu_2} \dots p^{\mu_n} \frac{2}{\epsilon} \\ &= 3 \frac{g^2}{(4\pi)^2} \frac{1}{n+1} \gamma^{\mu_1} p^{\mu_2} \dots p^{\mu_n} \frac{2}{\epsilon}. \end{aligned}$$

$-\partial^\nu A^{\mu_1}$ と A_σ および $-\partial_\nu A^{\mu_n}$ と A_ρ を縮約した場合は,

$$-(ig)^2 \int \frac{d^d k}{(2\pi)^d} t^a \gamma^{\mu_1} \frac{i}{\not{p} - \not{k}} t^a \gamma^{\mu_n} \left(\frac{-i}{k^2} \right)^2 k^\nu k^{\mu_2} \dots k^{\mu_{n-1}} k_\nu$$

$$\begin{aligned}
 &= 3ig^2 \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} [\ell - (1-x)p]^\alpha (\ell + xp)^{\mu_2} \cdots (\ell + xp)^{\mu_{n-1}} \gamma^{\mu_1} \gamma_\alpha \gamma^{\mu_n} \\
 &\rightarrow 3ig^2 \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} [\ell - (1-x)p]^\alpha (\ell + xp)^{\mu_2} \cdots (\ell + xp)^{\mu_{n-1}} \gamma^{\mu_1} \gamma_\alpha \gamma^{\mu_n} \\
 &\rightarrow 3ig^2 \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} [\ell - (1-x)p]^\alpha (\ell + xp)^{\mu_2} \cdots (\ell + xp)^{\mu_{n-1}} (\delta_\alpha^{\mu_n} \gamma^{\mu_1} + \delta_\alpha^{\mu_1} \gamma^{\mu_n}) \\
 &\rightarrow \frac{8}{3} ig^2 \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} \gamma^{\mu_n} [\ell - (1-x)p]^{\mu_1} (\ell + xp)^{\mu_2} \cdots (\ell + xp)^{\mu_{n-1}} \\
 &= -\frac{8}{3} ig^2 \int_0^1 dx (1-x) x^{n-2} \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} \gamma^{\mu_n} p^{\mu_1} \cdots p^{\mu_{n-1}} \\
 &= \frac{8}{3} \frac{g^2}{(4\pi)^2} \frac{1}{n(n-1)} p^{\mu_1} \cdots p^{\mu_{n-1}} \gamma^{\mu_n} \frac{2}{\epsilon}.
 \end{aligned}$$

$\partial^{\mu_1} A^\nu$ と A_σ および $-\partial_\nu A^{\mu_n}$ と A_ρ を縮約した場合は,

$$\begin{aligned}
 &(ig)^2 \int \frac{d^d k}{(2\pi)^d} t^a \gamma^\nu \frac{i}{\not{p} - \not{k}} t^a \gamma^{\mu_n} \left(\frac{-i}{k^2} \right)^2 k^{\mu_1} \cdots k^{\mu_{n-1}} k_\nu \\
 &= -3ig^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} (k-p)^\alpha k^{\mu_1} \cdots k^{\mu_{n-1}} k_\nu \gamma^\nu \gamma_\alpha \gamma^{\mu_n}.
 \end{aligned}$$

$-\partial^\nu A^{\mu_1}$ と A_σ および $\partial^{\mu_n} A_\nu$ と A_ρ を縮約した場合は,

$$\begin{aligned}
 &(ig)^2 \int \frac{d^d k}{(2\pi)^d} t^a \gamma^{\mu_1} \frac{i}{\not{p} - \not{k}} t^a \gamma_\nu \left(\frac{-i}{k^2} \right)^2 k^{\mu_2} \cdots k^{\mu_n} k^\nu \\
 &\rightarrow -3ig^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} (k-p)^\alpha k^{\mu_1} \cdots k^{\mu_{n-1}} k_\nu \gamma^{\mu_n} \gamma_\alpha \gamma^\nu.
 \end{aligned}$$

以上 2 つを足して,

$$\begin{aligned}
 &-3ig^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} (k-p)^\alpha k^{\mu_1} \cdots k^{\mu_{n-1}} k_\nu (\gamma^{\mu_n} \gamma_\alpha \gamma^\nu + \gamma^\nu \gamma_\alpha \gamma^{\mu_n}) \\
 &= -\frac{8}{3} ig^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} (k-p)^\alpha k^{\mu_1} \cdots k^{\mu_{n-1}} k_\nu (\delta_\alpha^\nu \gamma^{\mu_n} + \delta_\alpha^{\mu_n} \gamma^\nu - g^{\mu_n \nu} \gamma_\alpha) \\
 &\rightarrow -\frac{8}{3} ig^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} k^{\mu_1} \cdots k^{\mu_{n-1}} \gamma^{\mu_n} k \cdot (k-p) \\
 &\quad - \frac{8}{3} ig^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} k^{\mu_1} \cdots k^{\mu_{n-1}} (k^{\mu_n} - p^{\mu_n}) \not{k} \\
 &\quad + \frac{8}{3} ig^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} k^{\mu_1} \cdots k^{\mu_n} (\not{k} - \not{p}) \\
 &= -\frac{8}{3} ig^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} k^{\mu_1} \cdots k^{\mu_{n-1}} \gamma^{\mu_n} k \cdot (k-p) \\
 &\quad + \frac{8}{3} ig^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} k^{\mu_1} \cdots k^{\mu_{n-1}} p^{\mu_n} \not{k} \\
 &= -\frac{8}{3} ig^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2} \frac{1}{(k-p)^2} k^{\mu_1} \cdots k^{\mu_{n-1}} \gamma^{\mu_n} \\
 &= +\frac{8}{3} ig^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} k^{\mu_1} \cdots k^{\mu_{n-1}} \gamma^{\mu_n} k \cdot p
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{8}{3} i g^2 \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{k^2} \right)^2 \frac{1}{(k-p)^2} k^{\mu_1} \dots k^{\mu_{n-1}} p^{\mu_n} \not{k} \\
 & \rightarrow -\frac{8}{3} i g^2 \int dx x^{n-1} \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} p^{\mu_1} \dots p^{\mu_{n-1}} \gamma^{\mu_n} \\
 & + \frac{16}{3} \frac{n-1}{d} i g^2 \int dx (1-x) x^{n-2} \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta)^3} p^{\mu_1} \dots p^{\mu_{n-1}} \gamma^{\mu_n} \\
 & + \frac{16}{3} \frac{n-1}{d} i g^2 \int dx (1-x) x^{n-2} \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta)^3} p^{\mu_1} \dots p^{\mu_{n-1}} \gamma^{\mu_n} \\
 & = \frac{8}{3} \frac{g^2}{(4\pi)^2} \int dx x^{n-1} p^{\mu_1} \dots p^{\mu_{n-1}} \gamma^{\mu_n} \frac{2}{\epsilon} \\
 & - \frac{8}{3} (n-1) \frac{g^2}{(4\pi)^2} \int dx (1-x) x^{n-2} p^{\mu_1} \dots p^{\mu_{n-1}} \gamma^{\mu_n} \frac{2}{\epsilon} \\
 & = 0.
 \end{aligned}$$

縮約を逆にしたものも含めて, 1 ループの補正は

$$\begin{aligned}
 & \frac{8}{3} \frac{g^2}{(4\pi)^2} \frac{1}{n+1} \gamma^{\mu_1} p^{\mu_2} \dots p^{\mu_n} \frac{2}{\epsilon} + \frac{16}{3} \frac{g^2}{(4\pi)^2} \frac{1}{n(n-1)} p^{\mu_1} \dots p^{\mu_{n-1}} \gamma^{\mu_n} \frac{2}{\epsilon} \\
 & = \frac{8}{3} \frac{g^2}{(4\pi)^2} \frac{n^2 + n + 2}{n(n^2 - 1)} \gamma^{\mu_1} p^{\mu_2} \dots p^{\mu_n} \frac{2}{\epsilon} \\
 & \rightarrow \frac{8}{3} \frac{g^2}{(4\pi)^2} \frac{n^2 + n + 2}{n(n^2 - 1)} \mathcal{O}_f^{(n)\mu_1 \dots \mu_n} \left(\frac{2}{\epsilon} - \log M^2 + \dots \right).
 \end{aligned}$$

a_{gg}^n の 1 ループ補正

\mathcal{O}_g のゲージ場への作用

$$\begin{aligned}
 \langle \Omega | A_b^\sigma \mathcal{O}_g^{(n)} A_a^\rho | \Omega \rangle & = \langle 0 | A_b^\sigma \exp \left(i \int d^4 x \mathcal{L} \right) \mathcal{O}_g^{(n)} A_a^\rho | 0 \rangle \\
 & = \langle 0 | A_b^\sigma \mathcal{O}_g^{(n)} A_a^\rho | 0 \rangle + \langle 0 | A_b^\sigma \left(i \int d^4 x \mathcal{L} \right) \mathcal{O}_g^{(n)} A_a^\rho | 0 \rangle + \dots
 \end{aligned}$$

を g^2 のオーダーで考える.

(18.167) の後の議論と同様に, 随伴表現を使えば

$$(iD^{\mu_j})_{cd} = i\partial_{\mu_j} \delta_{cd} - g A_e^{\mu_j} (t_G^e)_{cd} = i\partial_{\mu_j} \delta_{cd} + ig A_e^{\mu_j} f^{cde}$$

である. さらに (16.2) から

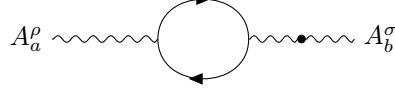
$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc} A_\mu^b A_\nu^c.$$

(18.174) は

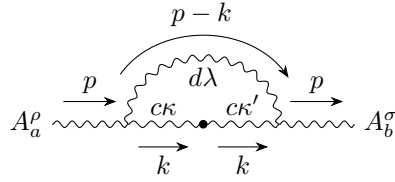
$$\begin{aligned}
 \mathcal{O}_g^{(n)\mu_1 \dots \mu_n} & = -(\partial^{\mu_1} A_c^\nu - \partial^\nu A_c^{\mu_1})(i\partial^{\mu_2}) \dots (i\partial^{\mu_{n-1}})(\partial^{\mu_n} A_\nu^c - \partial_\nu A_c^{\mu_n}) \\
 & - \sum_{j=2}^{n-1} (\partial^{\mu_1} A_c^\nu - \partial^\nu A_c^{\mu_1})(i\partial^{\mu_2}) \dots (ig A_e^{\mu_j} f^{cde}) \dots (i\partial^{\mu_{n-1}})(\partial^{\mu_n} A_\nu^d - \partial_\nu A_d^{\mu_n}) \\
 & - (\partial^{\mu_1} A_c^\nu - \partial^\nu A_c^{\mu_1})(i\partial^{\mu_2}) \dots (i\partial^{\mu_{n-1}}) g f^{cde} A_d^{\mu_n} A_\nu^e \\
 & - \dots
 \end{aligned}$$

となる. 第 1 項は 2 つのゲージ場と縮約でき, 第 2 項は 3 つのゲージ場と縮約できる.

\mathcal{O}_g の第 1 項は g^0 オーダーなので, $e^{i\mathcal{L}}$ の展開のうち g^2 のオーダーのものを考えれば良い. これは fermion 頂点 2 つ, 3-boson 頂点 2 つ, もしくは 4-boson 頂点 1 つである. fermion 頂点 2 つの場合は



となるが, これは external leg correction δ_3 なので, $\delta_{\mathcal{O}}$ には含めない. 4-boson 頂点は \mathcal{O}_g の形を持たないので, 計算には含めない. 結局, $\delta_{\mathcal{O}}$ の計算に含めるのは 3-boson 頂点 2 つからなる Figure 18.15 (b) 1 つ目の diagram である.



頂点のゲージ場 $A_c^\kappa, A_c^{\kappa'}$ と

$$\mathcal{O}_g^{(n)\mu_1 \dots \mu_n} \rightarrow (i\partial^{\mu_1} A_c^\nu - i\partial^\nu A_c^{\mu_1})(i\partial^{\mu_2}) \dots (i\partial^{\mu_{n-1}})(i\partial^{\mu_n} A_\nu^c - i\partial_\nu A_c^{\mu_n})$$

のゲージ場との縮約を計算する.

$\partial^{\mu_1} \overline{A_c^\nu} A_c^{\kappa'}, \partial^{\mu_n} \overline{A_\nu^c} A_c^\kappa$ の項は

$$\begin{aligned} & g^2 f^{adc} f^{bcd} \int \frac{d^d k}{(2\pi)^d} \left(\frac{-i}{k^2} \right)^2 \frac{-i}{(k-p)^2} \\ & \times [2k^2 g^{\rho\sigma} - 2(k \cdot p) g^{\rho\sigma} + 5p^2 g^{\rho\sigma} + 10k^\rho k^\sigma - 5k^\rho p^\sigma - 5k^\sigma p^\rho - 2p^\rho p^\sigma] k^{\mu_1} \dots k^{\mu_n} \\ & \rightarrow -3ig^2 \delta^{ab} \int_0^1 dx 2(1-x) \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta)^3} \left[2x^n - \frac{2n}{d} x^{n-1} + \frac{10}{d} x^n \right] g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n} \\ & = -3ig^2 \delta^{ab} \int_0^1 dx (1-x) (9x^n - nx^{n-1}) \frac{i}{(4\pi)^2} g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n} \frac{2}{\epsilon} \\ & = 3\delta^{ab} \frac{g^2}{(4\pi)^2} \left(-\frac{9}{n+2} + \frac{8}{n+1} \right) g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n} \frac{2}{\epsilon}. \end{aligned}$$

$-\partial^\nu \overline{A_c^{\mu_1}} A_c^{\kappa'}, -\partial_\nu \overline{A_c^{\mu_n}} A_c^\kappa$ の項は

$$\begin{aligned} & g^2 f^{adc} f^{bcd} \int \frac{d^d k}{(2\pi)^d} \left(\frac{-i}{k^2} \right)^2 \frac{-i}{(k-p)^2} N^{\rho\sigma\mu_1\mu_n} k^\nu k^{\mu_2} \dots k^{\mu_{n-1}} k_\nu \\ & = -3ig^2 \delta^{ab} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p)^2} N^{\rho\sigma\mu_1\mu_n} k^{\mu_2} \dots k^{\mu_{n-1}}. \end{aligned}$$

$N^{\rho\sigma\mu_1\mu_n}$ のうち $g^{\rho\sigma} p^{\mu_1} p^{\mu_n}$ の形の項は

$$\begin{aligned} N^{\rho\sigma\mu_1\mu_n} &= k^2 g^{\rho\mu_1} g^{\sigma\mu_n} + 2(k \cdot p) g^{\rho\mu_1} g^{\sigma\mu_n} + p^2 g^{\rho\mu_1} g^{\sigma\mu_n} \\ &+ g^{\rho\sigma} k^{\mu_n} k^{\mu_1} - 2g^{\rho\sigma} k^{\mu_n} p^{\mu_1} - 2g^{\rho\sigma} k^{\mu_1} p^{\mu_n} + 4g^{\rho\sigma} p^{\mu_n} p^{\mu_1} \\ &- 2g^{\rho\mu_n} k^\sigma k^{\mu_1} + 4g^{\rho\mu_n} k^\sigma p^{\mu_1} + g^{\rho\mu_n} k^{\mu_1} p^\sigma - 2g^{\rho\mu_n} p^\sigma p^{\mu_1} \\ &- g^{\rho\mu_1} k^\sigma k^{\mu_n} - 4g^{\rho\mu_1} k^\sigma p^{\mu_n} + 2g^{\rho\mu_1} k^{\mu_n} p^\sigma - g^{\rho\mu_1} p^\sigma p^{\mu_n} \end{aligned}$$

$$\begin{aligned}
 & -g^{\sigma\mu_n} k^\rho k^{\mu_1} - 4g^{\sigma\mu_n} k^\rho p^{\mu_1} + 2g^{\sigma\mu_n} k^{\mu_1} p^\rho - g^{\sigma\mu_n} p^\rho p^{\mu_1} \\
 & - 2g^{\sigma\mu_1} k^\rho k^{\mu_n} + 4g^{\sigma\mu_1} k^\rho p^{\mu_n} + g^{\sigma\mu_1} k^{\mu_n} p^\rho - 2g^{\sigma\mu_1} p^\rho p^{\mu_n} \\
 & + 4g^{\mu_1\mu_n} k^\rho k^\sigma - 2g^{\mu_1\mu_n} k^\rho p^\sigma - 2g^{\mu_1\mu_n} k^\sigma p^\rho + g^{\mu_1\mu_n} p^\rho p^\sigma \\
 & \rightarrow g^{\rho\sigma} k^{\mu_n} k^{\mu_1} - 2g^{\rho\sigma} k^{\mu_n} p^{\mu_1} - 2g^{\rho\sigma} k^{\mu_1} p^{\mu_n} + 4g^{\rho\sigma} p^{\mu_n} p^{\mu_1} \\
 & \rightarrow (x^2 - 4x + 4)g^{\rho\sigma} p^{\mu_1} p^{\mu_n}
 \end{aligned}$$

なので,

$$\begin{aligned}
 & = -3ig^2\delta^{ab} \int_0^1 dx x^{n-2} (x^2 - 4x + 4) \int \frac{d^d\ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n} \\
 & = 3\delta^{ab} \frac{g^2}{(4\pi)^2} \left(\frac{1}{n+1} - \frac{4}{n} + \frac{4}{n-1} \right) g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n} \frac{2}{\epsilon}.
 \end{aligned}$$

$\partial^{\mu_1} \overline{A_c^\nu} A_c^{\kappa'}$, $-\partial_\nu \overline{A_c^{\mu_n}} A_c^\kappa$ の項は

$$\begin{aligned}
 & -g^2 f^{adc} f^{bcd} \int \frac{d^d k}{(2\pi)^d} \left(\frac{-i}{k^2} \right)^2 \frac{-i}{(k-p)^2} N^{\rho\sigma\nu\mu_n} k^{\mu_1} \dots k^{\mu_{n-1}} k_\nu \\
 & = 3ig^2\delta^{ab} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4 (k-p)^2} N^{\rho\sigma\nu\mu_n} k^{\mu_1} \dots k^{\mu_{n-1}} k_\nu.
 \end{aligned}$$

$-\partial^\nu \overline{A_c^{\mu_1}} A_c^{\kappa'}$, $\partial^{\mu_n} \overline{A_\nu^c} A_c^\kappa$ の項は

$$\begin{aligned}
 & -g^2 f^{adc} f^{bcd} \int \frac{d^d k}{(2\pi)^d} \left(\frac{-i}{k^2} \right)^2 \frac{-i}{(k-p)^2} N^{\rho\sigma\mu_1\nu} k^{\mu_2} \dots k^{\mu_n} k^\nu \\
 & \rightarrow 3ig^2\delta^{ab} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^4 (k-p)^2} N^{\rho\sigma\mu_n\nu} k^{\mu_1} \dots k^{\mu_{n-1}} k_\nu.
 \end{aligned}$$

$N^{\rho\sigma(\nu\mu_n)}$ のうち $g^{\rho\sigma} p^{\mu_1} p^{\mu_n}$ の形の項は

$$\begin{aligned}
 & N^{\rho\sigma\nu\mu_n} + N^{\rho\sigma\mu_n\nu} \\
 & = k^2 g^{\rho\mu_n} g^{\sigma\nu} + k^2 g^{\rho\nu} g^{\sigma\mu_n} + 2(k \cdot p) g^{\rho\mu_n} g^{\sigma\nu} + 2(k \cdot p) g^{\rho\nu} g^{\sigma\mu_n} + p^2 g^{\rho\mu_n} g^{\sigma\nu} + p^2 g^{\rho\nu} g^{\sigma\mu_n} \\
 & + 2g^{\rho\sigma} k^{\mu_n} k^\nu - 4g^{\rho\sigma} k^{\mu_n} p^\nu - 4g^{\rho\sigma} k^\nu p^{\mu_n} + 8g^{\rho\sigma} p^{\mu_n} p^\nu \\
 & - 3g^{\rho\mu_n} k^\sigma k^\nu + 3g^{\rho\mu_n} k^\nu p^\sigma - 3g^{\rho\mu_n} p^\sigma p^\nu \\
 & - 3g^{\rho\nu} k^\sigma k^{\mu_n} + 3g^{\rho\nu} k^{\mu_n} p^\sigma - 3g^{\rho\nu} p^\sigma p^{\mu_n} \\
 & - 3g^{\sigma\mu_n} k^\rho k^\nu + 3g^{\sigma\mu_n} k^\nu p^\rho - 3g^{\sigma\mu_n} p^\rho p^\nu \\
 & - 3g^{\sigma\nu} k^\rho k^{\mu_n} + 3g^{\sigma\nu} k^{\mu_n} p^\rho - 3g^{\sigma\nu} p^\rho p^{\mu_n} \\
 & + 8g^{\mu_n\nu} k^\rho k^\sigma - 4g^{\mu_n\nu} k^\rho p^\sigma - 4g^{\mu_n\nu} k^\sigma p^\rho + 2g^{\mu_n\nu} p^\rho p^\sigma \\
 & \rightarrow 2g^{\rho\sigma} k^{\mu_n} k^\nu - 4g^{\rho\sigma} k^{\mu_n} p^\nu - 4g^{\rho\sigma} k^\nu p^{\mu_n} + 8g^{\rho\sigma} p^{\mu_n} p^\nu - 3g^{\rho\nu} k^\sigma k^{\mu_n} - 3g^{\sigma\nu} k^\rho k^{\mu_n} + 8g^{\mu_n\nu} k^\rho k^\sigma.
 \end{aligned}$$

従って,

$$\begin{aligned}
 & (N^{\rho\sigma\nu\mu_n} + N^{\rho\sigma\mu_n\nu}) k^{\mu_1} \dots k^{\mu_{n-1}} k_\nu \\
 & \rightarrow 2k^2 g^{\rho\sigma} k^{\mu_1} \dots k^{\mu_n} - 4(k \cdot p) g^{\rho\sigma} k^{\mu_1} \dots k^{\mu_n} \\
 & - 4k^2 g^{\rho\sigma} k^{\mu_1} \dots k^{\mu_{n-1}} p^{\mu_n} + 8(k \cdot p) g^{\rho\sigma} k^{\mu_1} \dots k^{\mu_{n-1}} p^{\mu_n} \\
 & + 2k^\rho k^\sigma k^{\mu_1} \dots k^{\mu_n} \\
 & = \left[2x^n - \frac{4n}{d} x^{n-1} - 4x^{n-1} + \frac{8(n-1)}{d} x^{n-2} + \frac{2}{d} x^n \right] \ell^2 g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n}
 \end{aligned}$$

$$= \left[\frac{5}{2}x^n - (n+4)x^{n-1} + 2(n-1)x^{n-2} \right] \ell^2 g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n}.$$

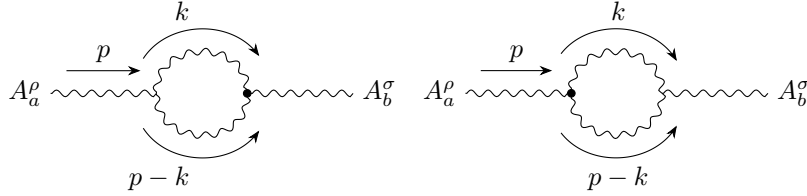
以上から

$$\begin{aligned} &= 3ig^2 \delta^{ab} \int_0^1 dx (1-x) [5x^n - 2(n+4)x^{n-1} + 4(n-1)x^{n-2}] \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^2}{(\ell^2 - \Delta)^3} g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n} \\ &= -3\delta^{ab} \frac{g^2}{(4\pi)^2} \left(-\frac{5}{n+2} + \frac{11}{n+1} - \frac{4}{n} \right) g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n} \frac{2}{\epsilon}. \end{aligned}$$

これらを全て足せば、(縮約を逆にしたものも含めて 2 倍, Taylor 展開の係数で 1/2 倍)

$$3\delta^{ab} \frac{g^2}{(4\pi)^2} \left[-\frac{4}{n+2} - \frac{2}{n+1} + \frac{4}{n-1} \right] g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n} \frac{2}{\epsilon}. \quad [18.5.1]$$

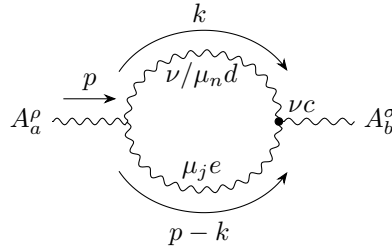
\mathcal{O}_g の第 2 項は g^1 オーダーなので, $e^{i\mathcal{L}}$ の展開のうち g^1 のオーダーのものを考えれば良い. これは 3-boson 頂点 1 つであり, 次の diagram で表現される.



\mathcal{O}_g の形から, 2 つの diagram は同じ振幅を与えるので, 左側のみ考える.

$$\mathcal{O}_g \rightarrow (i\partial^{\mu_1} A_c^\nu - i\partial^\nu A_c^{\mu_1})(i\partial^{\mu_2} \dots (igA_e^{\mu_j} f^{cde}) \dots (i\partial^{\mu_{n-1}})(i\partial^{\mu_n} A_\nu^d - i\partial_\nu A_d^{\mu_n})$$

のゲージ場のうち, A_b^σ と縮約した際に, $g^{\rho\sigma}$ を与えるのは $\partial^{\mu_1} A_c^\nu$ (もしくは $\partial^{\mu_n} A_\nu^d$) である.



$\partial^{\mu_n} A_\nu^d$ を縮約した項は

$$\begin{aligned} &igf^{cde} \delta^{bc} \int \frac{d^d k}{(2\pi)^d} g f^{ade} [\delta_\nu^\rho (p+k)^{\mu_j} + \delta_\nu^{\mu_j} (p-2k)^\rho + g^{\mu_j\rho} (k-2p)_\nu] g^{\nu\sigma} \\ &\times \frac{-i}{k^2} \frac{-i}{(p-k)^2} p^{\mu_1} \dots p^{\mu_{j-1}} k^{\mu_j+1} \dots k^{\mu_n} \\ &\rightarrow -3ig^2 g^{\rho\sigma} \delta^{ab} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k-p)^2} p^{\mu_1} \dots p^{\mu_{j-1}} (k+p)^{\mu_j} k^{\mu_j+1} \dots k^{\mu_n} \\ &= -3ig^2 g^{\rho\sigma} \delta^{ab} \int_0^1 dx (1+x) x^{n-j} \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^2} p^{\mu_1} \dots p^{\mu_n} \\ &= 3\delta^{ab} \frac{g^2}{(4\pi)^2} \left(\frac{1}{n-j+1} + \frac{1}{n-j+2} \right) g^{\rho\sigma} p^{\mu_1} \dots p^{\mu_n} \frac{2}{\epsilon}. \end{aligned}$$

$-\partial_\nu A_d^{\mu n}$ を縮約した項は 0.

$F^{\mu_1\nu}, F^{\mu_n\nu}$ からゲージ場を 2 つ取り出した項はそれぞれ $j = 1, n$ を与える. さらに diagram の対称性は $S = 2$ なので,

$$\begin{aligned} & -3\delta^{ab} \frac{g^2}{(4\pi)^2} 2 \cdot 2 \cdot \frac{1}{2} \sum_{j=1}^n \left(\frac{1}{n-j+1} + \frac{1}{n-j+2} \right) g^{\rho\sigma} p^{\mu_1} \cdots p^{\mu_n} \frac{2}{\epsilon} \\ & = -3\delta^{ab} \frac{g^2}{(4\pi)^2} \left(4 \sum_{j=2}^n \frac{1}{j} + \frac{2}{n+1} + 2 \right) g^{\rho\sigma} p^{\mu_1} \cdots p^{\mu_n} \frac{2}{\epsilon}. \end{aligned} \quad [18.5.2]$$

[18.5.1][18.5.2] を足して,

$$-\delta_{\mathcal{O}} = 3 \frac{g^2}{(4\pi)^2} \left[-2 - 4 \sum_{j=2}^n \frac{1}{j} - \frac{4}{n+2} - \frac{4}{n+1} + \frac{4}{n-1} \right] \left(\frac{2}{\epsilon} - \log M^2 \right).$$

(18.23)(16.74) から

$$\begin{aligned} \gamma_{\mathcal{O}} &= M \frac{\partial}{\partial M} (-\delta_{\mathcal{O}} + \delta_3) \\ &= \frac{6g^2}{(4\pi)^2} \left(\frac{1}{3} + \frac{2}{9} n_f + 4 \sum_{j=2}^n \frac{1}{j} + \frac{4}{n+2} + \frac{4}{n+1} - \frac{4}{n-1} \right). \end{aligned}$$

Chapter 19

Perturbation Theory Anomalies

19.1 The Axial Current in Two Dimensions

(19.15)

ψ は反交換することに注意して, (4.31) から

$$\begin{aligned}
 \langle j^\mu(q) \rangle &= \int d^2x e^{iq \cdot x} \langle j^\mu(x) \rangle \\
 &= \int d^2x e^{iq \cdot x} \langle \Omega | \bar{\psi}(x) \gamma^\mu \psi(x) | \Omega \rangle \\
 &\sim i \int d^2x e^{iq \cdot x} \int d^2y \langle 0 | \bar{\psi}(x) \gamma^\mu \psi(x) \mathcal{L}(y) | 0 \rangle \\
 &= -ie \int d^2x \int d^2y e^{iq \cdot x} \langle 0 | \bar{\psi}(x) \gamma^\mu \psi(x) A_\nu(y) \bar{\psi}(y) \gamma^\nu \psi(y) | 0 \rangle \\
 &= -ie \int d^2x \int d^2y e^{iq \cdot x} \langle 0 | \overbrace{\bar{\psi}_\alpha(x) (\gamma^\mu)_{\alpha\beta} \psi_\beta(x) A_\nu(y) \bar{\psi}_\gamma(y) (\gamma^\nu)_{\gamma\delta} \psi_\delta(y)} | 0 \rangle \\
 &= ie \int d^2x \int d^2y e^{iq \cdot x} \int \frac{d^2k}{(2\pi)^2} \left(\frac{1}{\not{k}} \right)_{\delta\alpha} e^{-ik \cdot (y-x)} \int \frac{d^2k'}{(2\pi)^2} \left(\frac{1}{\not{k}'} \right)_{\beta\gamma} e^{-ik' \cdot (x-y)} \\
 &\quad \times (\gamma^\mu)^{\alpha\beta} (\gamma^\nu)^{\gamma\delta} \int \frac{d^2p}{(2\pi)^2} A_\nu(p) e^{-p \cdot y} \\
 &= ie \int \frac{d^2k}{(2\pi)^2} \text{Tr} \left[\frac{1}{\not{k}} \gamma^\mu \frac{1}{\not{k} + \not{q}} \gamma^\nu \right] A_\nu(q).
 \end{aligned}$$

これを (7.71) と比べれば良い.

19.3 Goldstone Bosons and Chiral Symmetries in QCD

(19.84)

Lagrangian は

$$\mathcal{L} = u_L^\dagger i \bar{\sigma}^\mu D_\mu u_L + d_L^\dagger i \bar{\sigma}^\mu D_\mu d_L + (R).$$

$U(1)$ 微小変換は

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \rightarrow e^{-i\epsilon} \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \begin{pmatrix} e^{-i\epsilon} u_L \\ e^{-i\epsilon} d_L \end{pmatrix} \approx \begin{pmatrix} (1 - i\epsilon) u_L \\ (1 - i\epsilon) d_L \end{pmatrix}$$

で与えられる。この変換で \mathcal{L} は不変なので、付随するカレントは

$$\epsilon j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu u_L} \Delta u_L + \frac{\partial \mathcal{L}}{\partial \partial_\mu d_L} \Delta d_L = u_L^\dagger i \bar{\sigma}^\mu (-i \epsilon u_L) + d_L^\dagger i \bar{\sigma}^\mu (-i \epsilon d_L).$$

一方で

$$Q_L = \begin{pmatrix} \frac{1-\gamma^5}{2} \\ \frac{1+\gamma^5}{2} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} \frac{1-\gamma^5}{2} u \\ \frac{1+\gamma^5}{2} d \end{pmatrix}$$

に対し

$$\begin{aligned} \bar{Q}_L \gamma^\mu Q_L &= \begin{pmatrix} \bar{u} \frac{1+\gamma^5}{2} & \bar{d} \frac{1+\gamma^5}{2} \end{pmatrix} \begin{pmatrix} \gamma^\mu \frac{1-\gamma^5}{2} u \\ \gamma^\mu \frac{1+\gamma^5}{2} d \end{pmatrix} \\ &= \bar{u} \frac{1+\gamma^5}{2} \gamma^\mu \frac{1-\gamma^5}{2} u + (d) \\ &= \begin{pmatrix} u_R^\dagger & u_L^\dagger \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u_L \\ u_R \end{pmatrix} + (d) \\ &= u_L^\dagger \bar{\sigma}^\mu u_L + (d) \end{aligned}$$

なので $j^\mu = \bar{Q}_L \gamma^\mu Q_L$.

$SU(2)$ 微小変換は

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix} \rightarrow e^{-i\epsilon_a \tau^a} \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \begin{pmatrix} e^{-i\epsilon_a \tau^a} u_L \\ e^{-i\epsilon_a \tau^a} d_L \end{pmatrix} \approx \begin{pmatrix} (1 - i\epsilon_a \tau^a) u_L \\ (1 - i\epsilon_a \tau^a) d_L \end{pmatrix}$$

で与えられる。後は同様。

Chapter 20

Gauge Theories with Spontaneous Symmetry Breaking

20.1 The Higgs Mechanism

(20.27)

\mathbb{C}^2 scalar を \mathbb{R}^4 vector として表す. (21.38) から

$$\mathbb{C}^2 \ni \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} \mapsto \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ h \end{pmatrix} \in \mathbb{R}^4 \quad [20.1.1]$$

とする. この対応により T^a は 4×4 行列となる. (20.14) から

$$T^a = -it^a = -\frac{i}{2}\sigma^a.$$

$a = 1$ なら

$$\begin{aligned} T^1 \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ h \end{pmatrix} &= -\frac{i}{2}\sigma^1 \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} & -i \\ -i & \end{pmatrix} \begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} \\ &= \frac{1}{2\sqrt{2}} \begin{pmatrix} \phi^3 - ih \\ -\phi^1 + i\phi^2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} h \\ -\phi^3 \\ \phi^2 \\ -\phi^1 \end{pmatrix} \end{aligned}$$

なので

$$T^1 = \frac{1}{2} \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}.$$

$a = 2$ なら

$$T^2 \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ h \end{pmatrix} = -\frac{i}{2}\sigma^2 \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} -h - i\phi^3 \\ -\phi^2 - i\phi^1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \phi^3 \\ h \\ -\phi^1 \\ -\phi^2 \end{pmatrix}$$

なので

$$T^2 = \frac{1}{2} \begin{pmatrix} & 1 & \\ -1 & & 1 \\ & -1 & \end{pmatrix}.$$

$a = 3$ なら

$$\begin{aligned} T^3 \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ h \end{pmatrix} &= -\frac{i}{2} \sigma^3 \frac{1}{\sqrt{2}} \begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -i & \\ & i \end{pmatrix} \begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} \\ &= \frac{1}{2\sqrt{2}} \begin{pmatrix} -\phi^1 + i\phi^2 \\ -\phi^3 + ih \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\phi^2 \\ \phi^1 \\ h \\ -\phi^3 \end{pmatrix} \end{aligned}$$

なので

$$T^3 = \frac{1}{2} \begin{pmatrix} & -1 & \\ 1 & & \\ & -1 & 1 \end{pmatrix}.$$

以上から

$$T^1 = \frac{1}{2} \begin{pmatrix} & & 1 \\ & -1 & \\ -1 & 1 & \end{pmatrix}, \quad T^2 = \frac{1}{2} \begin{pmatrix} & 1 & \\ -1 & & 1 \\ & -1 & \end{pmatrix}, \quad T^3 = \frac{1}{2} \begin{pmatrix} & -1 & \\ 1 & & \\ & -1 & 1 \end{pmatrix}. \quad [20.1.2]$$

これらを

$$\begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ 0 \end{pmatrix} \in \mathbb{R}^3$$

に制限すれば

$$T^1 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad T^2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \quad T^3 = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}. \quad [20.1.3]$$

特に, $a = 1, 2, 3$ に対して

$$(T^a)_{bc} = \epsilon^{bac}.$$

(20.27) 右辺は $\phi^c \in \mathbb{R}^3$ に変換した vector. $\phi \in \mathbb{C}^2$ は $SU(2)$ の基本表現に属するが, \mathbb{R}^3 vector への変換によって (見かけ上) 随伴表現に属している.

20.2 The Glashow-Weinberg-Salam Theory of Weak Interactions

(20.80)

(20.63)(20.64) から

$$W_\mu^\pm = \frac{1}{\sqrt{2}}(A_\mu^1 \mp iA_\mu^2), \quad Z_\mu^0 = \frac{1}{\sqrt{g^2 + g'^2}}(gA_\mu^3 - g'B_\mu), \quad A_\mu = \frac{1}{\sqrt{g^2 + g'^2}}(g'A_\mu^3 + gB_\mu).$$

(20.68)(20.70) から

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}, \quad \cos \theta_w = \frac{g}{\sqrt{g^2 + g'^2}}, \quad \sin \theta_w = \frac{g'}{\sqrt{g^2 + g'^2}}.$$

さらに

$$A_\mu^3 = A_\mu \sin \theta_w + Z_\mu^0 \cos \theta_w, \quad B_\mu = A_\mu \cos \theta_w - Z_\mu^0 \sin \theta_w,$$

E_L の項は

$$\begin{aligned} & (\bar{\nu}_L \quad \bar{e}_L) i\gamma^\mu \left(-igA_\mu^a \tau^a + i\frac{1}{2}g'B_\mu \right) \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ &= (\bar{\nu}_L \quad \bar{e}_L) i\gamma^\mu \left[-i\frac{g}{\sqrt{2}} \begin{pmatrix} W_\mu^- & W_\mu^+ \end{pmatrix} - i\frac{g}{2}A_\mu^3 \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} + i\frac{g'}{2}B_\mu \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right] \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \\ &= \frac{g}{\sqrt{2}}W_\mu^+ (\bar{\nu}_L \gamma^\mu e_L) + \frac{g}{\sqrt{2}}W_\mu^- (\bar{e}_L \gamma^\mu \nu_L) \\ &\quad + \frac{1}{2}\bar{\nu}_L \gamma^\mu (gA_\mu^3 - g'B_\mu) \nu_L - \frac{1}{2}\bar{e}_L \gamma^\mu (gA_\mu^3 + g'B_\mu) e_L. \end{aligned}$$

第 1, 2 項は $J_W^{\mu+}, J_W^{\mu-}$ の第 1 項を与える. 第 3 項は

$$\frac{1}{2}\bar{\nu}_L \gamma^\mu (gA_\mu^3 - g'B_\mu) \nu_L = \frac{\sqrt{g^2 + g'^2}}{2} Z_\mu^0 \bar{\nu}_L \gamma^\mu \nu_L = \frac{g}{2} \frac{Z_\mu^0}{\cos \theta_w} (\bar{\nu}_L \gamma^\mu \nu_L)$$

なので J_Z^μ の第 1 項を与える. 第 4 項は

$$\begin{aligned} & -\frac{1}{2}\bar{e}_L \gamma^\mu (gA_\mu^3 + g'B_\mu) e_L \\ &= -\frac{\sqrt{g^2 + g'^2}}{2} (A_\mu^3 \cos \theta_w + B_\mu \sin \theta_w) (\bar{e}_L \gamma^\mu e_L) \\ &= -\frac{\sqrt{g^2 + g'^2}}{2} \left[2A_\mu \frac{e}{\sqrt{g^2 + g'^2}} + Z_\mu^0 (1 - 2 \sin^2 \theta_w) \right] (\bar{e}_L \gamma^\mu e_L) \\ &= -eA_\mu (\bar{e}_L \gamma^\mu e_L) + g \frac{Z_\mu^0}{\cos \theta_w} \left(-\frac{1}{2} + \sin^2 \theta \right) (\bar{e}_L \gamma^\mu e_L) \end{aligned}$$

なので J_{EM}^μ の第 1 項の左成分と J_Z^μ の第 2 成分を与える.

e_R の項は

$$\begin{aligned} \bar{e}_R i\gamma^\mu (ig'B_\mu) e_R &= g'(Z_\mu^0 \sin \theta_w - A_\mu \cos \theta_w) (\bar{e}_R \gamma^\mu e_R) \\ &= g \frac{\sin^2 \theta_w}{\cos \theta_w} Z_\mu^0 (\bar{e}_R \gamma^\mu e_R) - eA_\mu (\bar{e}_R \gamma^\mu e_R) \end{aligned}$$

なので J_Z^μ の第 3 項と J_{EM}^μ の第 1 項の右成分を与える. u_R は $Y = 2/3$, d_R は $Y = -1/3$ とすれば良い.
 Q_L の項は

$$\begin{aligned} & (\bar{u}_L \quad \bar{d}_L) i\gamma^\mu \left(-igA_\mu^a \tau^a - i\frac{1}{6}g'B_\mu \right) \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\ &= (\bar{u}_L \quad \bar{d}_L) i\gamma^\mu \left[-i\frac{g}{\sqrt{2}} \begin{pmatrix} W_\mu^- & W_\mu^+ \end{pmatrix} - i\frac{g}{2}A_\mu^3 \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} - i\frac{g'}{6}B_\mu \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right] \begin{pmatrix} u_L \\ d_L \end{pmatrix} \\ &= \frac{g}{\sqrt{2}}W_\mu^+ (\bar{u}_L \gamma^\mu d_L) + \frac{g}{\sqrt{2}}W_\mu^- (\bar{d}_L \gamma^\mu u_L) \\ &\quad + \frac{1}{2}\bar{u}_L \gamma^\mu \left(gA_\mu^3 + \frac{1}{3}g'B_\mu \right) u_L - \frac{1}{2}\bar{d}_L \gamma^\mu \left(gA_\mu^3 - \frac{1}{3}g'B_\mu \right) d_L. \end{aligned}$$

第 1, 2 項は $J_W^{+\mu}$, $J_W^{-\mu}$ の第 2 項を与える. 第 3 項は

$$\begin{aligned} & \frac{1}{2}\bar{u}_L \gamma^\mu \left(gA_\mu^3 + \frac{1}{3}g'B_\mu \right) u_L \\ &= \frac{\sqrt{g^2 + g'^2}}{2} \left(A_\mu^3 \cos \theta_w + \frac{1}{3}B_\mu \sin \theta_w \right) (\bar{u}_L \gamma^\mu u_L) \\ &= \frac{g}{2 \cos \theta_w} \left[\frac{4}{3} \cos \theta_w \sin \theta_w A_\mu + Z_\mu^0 \left(\cos^2 \theta_w - \frac{\sin^2 \theta_w}{3} \right) \right] (\bar{u}_L \gamma^\mu u_L) \\ &= \frac{2}{3}eA_\mu (\bar{u}_L \gamma^\mu u_L) + \frac{g}{2 \cos \theta_w} Z_\mu^0 \left(1 - \frac{4}{3} \sin^2 \theta_w \right) (\bar{u}_L \gamma^\mu u_L) \end{aligned}$$

なので J_{EM}^μ の第 2 項と J_Z^μ の第 4 項を与える. 第 4 項は

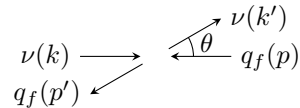
$$\begin{aligned} & -\frac{1}{2}\bar{d}_L \gamma^\mu \left(gA_\mu^3 - \frac{1}{3}g'B_\mu \right) d_L \\ &= \frac{\sqrt{g^2 + g'^2}}{2} \left(-A_\mu^3 \cos \theta_w + \frac{1}{3}B_\mu \sin \theta_w \right) (\bar{d}_L \gamma^\mu d_L) \\ &= \frac{g}{2 \cos \theta_w} \left[-\frac{2}{3} \cos \theta_w \sin \theta_w A_\mu - Z_\mu^0 \left(\cos^2 \theta_w + \frac{\sin^2 \theta_w}{3} \right) \right] (\bar{d}_L \gamma^\mu d_L) \\ &= -\frac{1}{3}eA_\mu (\bar{d}_L \gamma^\mu d_L) + \frac{g}{2 \cos \theta_w} Z_\mu^0 \left(-1 + \frac{2}{3} \sin^2 \theta_w \right) (\bar{d}_L \gamma^\mu d_L) \end{aligned}$$

なので J_{EM}^μ の第 3 項と J_Z^μ の第 5 項を与える.

Problems

Problem 20.4: Neutral-current deep inelastic scattering

parton レベルの散乱は次のようになる.



Mandelstam variable は

$$p \cdot k = p' \cdot k' = \frac{\hat{s}}{2}, \quad p \cdot p' = k \cdot k' = -\frac{\hat{t}}{2}, \quad p \cdot k' = p' \cdot k = -\frac{\hat{u}}{2}.$$

(17.31)(20.80) から effective Lagrangian は

$$\Delta\mathcal{L} = \frac{g^2}{\cos^2\theta_w} \frac{1}{m_Z^2} \left[\bar{\nu}\gamma_\mu \frac{1}{2} \frac{1-\gamma^5}{2} \nu \right] \left[\bar{q}\gamma^\mu (T^3 - \sin^2\theta_w Q) \frac{1\mp\gamma^5}{2} q \right].$$

$\nu + u_L \rightarrow \nu + u_L$

不変振幅は

$$i\mathcal{M} = \frac{g^2}{m_Z^2 \cos^2\theta_w} \frac{\frac{1}{2} - \frac{2}{3}\sin^2\theta_w}{2} \left[\bar{\nu}(k')\gamma_\mu \frac{1-\gamma^5}{2} \nu(k) \right] \left[\bar{u}(p')\gamma^\mu \frac{1-\gamma^5}{2} u(p) \right]$$

なので (A.27) から

$$\begin{aligned} |\mathcal{M}|^2 &= \sum_{\text{spins}} |\mathcal{M}|^2 \\ &= \frac{1}{4} \left(\frac{g^2}{m_Z^2 \cos^2\theta_w} \right)^2 \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w \right)^2 \text{Tr} \left[\not{k}' \gamma_\mu \frac{1-\gamma^5}{2} \not{k} \gamma_\nu \right] \text{Tr} \left[\not{p}' \gamma^\mu \frac{1-\gamma^5}{2} \not{p} \gamma^\nu \right] \\ &= \left(\frac{g^2}{m_Z^2 \cos^2\theta_w} \right)^2 \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w \right)^2 [k^\mu k'^\nu + k'^\mu k^\nu - (k \cdot k')g^{\mu\nu} + ik'_\alpha k_\beta \epsilon^{\alpha\mu\beta\nu}] \\ &\quad \times [p_\mu p'_\nu + p'_\mu p_\nu - (p \cdot p')g_{\mu\nu} + ip'^\gamma p^\delta \epsilon_{\gamma\mu\delta\nu}]. \end{aligned}$$

運動量の積は (A.30) から

$$\begin{aligned} &[k^\mu k'^\nu + k'^\mu k^\nu - (k \cdot k')g^{\mu\nu}][p_\mu p'_\nu + p'_\mu p_\nu - (p \cdot p')g_{\mu\nu}] - k'_\alpha k_\beta p'^\gamma p^\delta \epsilon^{\alpha\mu\beta\nu} \epsilon_{\gamma\mu\delta\nu} \\ &= 2(p \cdot k)(p' \cdot k') + 2(p \cdot k')(p' \cdot k) + 2(p \cdot k)(p' \cdot k') - 2(p \cdot k')(p' \cdot k) \\ &= 4(p \cdot k)(p' \cdot k') \end{aligned}$$

なので,

$$|\mathcal{M}(\nu u_L)|^2 = 4 \left(\frac{g^2}{m_Z^2 \cos^2\theta_w} \right)^2 \left(\frac{1}{2} - \frac{2}{3}\sin^2\theta_w \right)^2 (p \cdot k)(p' \cdot k').$$

$\nu + u_R \rightarrow \nu + u_R$

不変振幅は

$$i\mathcal{M} = \frac{g^2}{m_Z^2 \cos^2\theta_w} \frac{-\frac{2}{3}\sin^2\theta_w}{2} \left[\bar{\nu}(k')\gamma_\mu \frac{1-\gamma^5}{2} \nu(k) \right] \left[\bar{u}(p')\gamma^\mu \frac{1+\gamma^5}{2} u(p) \right]$$

なので (A.27) から

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{1}{4} \left(\frac{g^2}{m_Z^2 \cos^2\theta_w} \right)^2 \left(-\frac{2}{3}\sin^2\theta_w \right)^2 \text{Tr} \left[\not{k}' \gamma_\mu \frac{1-\gamma^5}{2} \not{k} \gamma_\nu \right] \text{Tr} \left[\not{p}' \gamma^\mu \frac{1+\gamma^5}{2} \not{p} \gamma^\nu \right] \\ &= \left(\frac{g^2}{m_Z^2 \cos^2\theta_w} \right)^2 \left(-\frac{2}{3}\sin^2\theta_w \right)^2 [k^\mu k'^\nu + k'^\mu k^\nu - (k \cdot k')g^{\mu\nu} + ik'_\alpha k_\beta \epsilon^{\alpha\mu\beta\nu}] \\ &\quad \times [p_\mu p'_\nu + p'_\mu p_\nu - (p \cdot p')g_{\mu\nu} - ip'^\gamma p^\delta \epsilon_{\gamma\mu\delta\nu}]. \end{aligned}$$

運動量の積は (A.30) から

$$[k^\mu k'^\nu + k'^\mu k^\nu - (k \cdot k')g^{\mu\nu}][p_\mu p'_\nu + p'_\mu p_\nu - (p \cdot p')g_{\mu\nu}] + k'_\alpha k_\beta p'^\gamma p^\delta \epsilon^{\alpha\mu\beta\nu} \epsilon_{\gamma\mu\delta\nu}$$

$$\begin{aligned}
 &= 2(p \cdot k)(p' \cdot k') + 2(p \cdot k')(p' \cdot k) - 2(p \cdot k)(p' \cdot k') + 2(p \cdot k')(p' \cdot k) \\
 &= 4(p \cdot k')(p' \cdot k)
 \end{aligned}$$

なので,

$$|\mathcal{M}(\nu u_R)|^2 = 4 \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left(-\frac{2}{3} \sin^2 \theta_w \right)^2 (p \cdot k')(p' \cdot k).$$

同様にして

$$\begin{aligned}
 |\mathcal{M}(\nu d_L)|^2 &= 4 \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right)^2 (p \cdot k)(p' \cdot k'), \\
 |\mathcal{M}(\nu d_R)|^2 &= 4 \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left(\frac{1}{3} \sin^2 \theta_w \right)^2 (p \cdot k')(p' \cdot k).
 \end{aligned}$$

antiquark

$p \leftrightarrow p'$ とすればよいので,

$$\begin{aligned}
 |\mathcal{M}(\nu \bar{u}_L)|^2 &= 4 \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2 (p' \cdot k)(p \cdot k'), \\
 |\mathcal{M}(\nu \bar{u}_R)|^2 &= 4 \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left(-\frac{2}{3} \sin^2 \theta_w \right)^2 (p' \cdot k')(p \cdot k), \\
 |\mathcal{M}(\nu \bar{d}_L)|^2 &= 4 \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right)^2 (p' \cdot k)(p \cdot k'), \\
 |\mathcal{M}(\nu \bar{d}_R)|^2 &= 4 \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left(\frac{1}{3} \sin^2 \theta_w \right)^2 (p' \cdot k')(p \cdot k).
 \end{aligned}$$

cross section

(17.27) から

$$y = \frac{\hat{s} + \hat{u}}{\hat{s}} = -\frac{\hat{t}}{\hat{s}}.$$

(14.1)(14.2)(14.3) と同様に

$$\frac{d\sigma}{dy} = -\hat{s} \frac{d\sigma}{d\hat{t}} = -2 \frac{d\sigma}{d \cos \theta} = -\frac{1}{16\pi \hat{s}} |\mathcal{M}|^2.$$

up quark の散乱断面積は

$$\begin{aligned}
 \frac{d\sigma(\nu u)}{dy} &= -\frac{1}{16\pi \hat{s}} \frac{|\mathcal{M}(\nu u_L)|^2 + |\mathcal{M}(\nu u_R)|^2}{2} \\
 &= -\frac{1}{8\pi \hat{s}} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2 \frac{\hat{s}^2}{4} + \left(-\frac{2}{3} \sin^2 \theta_w \right)^2 \frac{\hat{u}^2}{4} \right] \\
 &= -\frac{\hat{s}}{32\pi} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2 + \left(-\frac{2}{3} \sin^2 \theta_w \right)^2 (1-y)^2 \right] \\
 &= -\frac{sx}{32\pi} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2 + \left(-\frac{2}{3} \sin^2 \theta_w \right)^2 (1-y)^2 \right].
 \end{aligned}$$

down quark の散乱断面積は

$$\begin{aligned}
 \frac{d\sigma(\nu d)}{dy} &= -\frac{1}{16\pi\hat{s}} \frac{|\mathcal{M}(\nu d_L)|^2 + |\mathcal{M}(\nu d_R)|^2}{2} \\
 &= -\frac{1}{8\pi\hat{s}} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right)^2 \frac{\hat{s}^2}{4} + \left(\frac{1}{3} \sin^2 \theta_w \right)^2 \frac{\hat{u}^2}{4} \right] \\
 &= -\frac{sx}{32\pi} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right)^2 + \left(\frac{1}{3} \sin^2 \theta_w \right)^2 (1-y)^2 \right].
 \end{aligned}$$

up antiquark の散乱断面積は

$$\begin{aligned}
 \frac{d\sigma(\nu \bar{u})}{dy} &= -\frac{1}{16\pi\hat{s}} \frac{|\mathcal{M}(\nu \bar{u}_L)|^2 + |\mathcal{M}(\nu \bar{u}_R)|^2}{2} \\
 &= -\frac{1}{8\pi\hat{s}} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2 \frac{\hat{u}^2}{4} + \left(-\frac{2}{3} \sin^2 \theta_w \right)^2 \frac{\hat{s}^2}{4} \right] \\
 &= -\frac{sx}{32\pi} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2 (1-y)^2 + \left(-\frac{2}{3} \sin^2 \theta_w \right)^2 \right].
 \end{aligned}$$

down antiquark の散乱断面積は

$$\begin{aligned}
 \frac{d\sigma(\nu \bar{d})}{dy} &= -\frac{1}{16\pi\hat{s}} \frac{|\mathcal{M}(\nu \bar{d}_L)|^2 + |\mathcal{M}(\nu \bar{d}_R)|^2}{2} \\
 &= -\frac{1}{8\pi\hat{s}} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right)^2 \frac{\hat{u}^2}{4} + \left(\frac{1}{3} \sin^2 \theta_w \right)^2 \frac{\hat{s}^2}{4} \right] \\
 &= -\frac{sx}{32\pi} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right)^2 (1-y)^2 + \left(\frac{1}{3} \sin^2 \theta_w \right)^2 \right].
 \end{aligned}$$

$\bar{\nu} + q \rightarrow \bar{\nu} + q$ の場合は $k \leftrightarrow k'$ となるので, \hat{s}^2 と \hat{u}^2 を入れ替えればよい. 従って,

$$\begin{aligned}
 \frac{d\sigma(\bar{\nu} u)}{dy} &= -\frac{sx}{32\pi} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2 (1-y)^2 + \left(-\frac{2}{3} \sin^2 \theta_w \right)^2 \right], \\
 \frac{d\sigma(\bar{\nu} d)}{dy} &= -\frac{sx}{32\pi} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right)^2 (1-y)^2 + \left(\frac{1}{3} \sin^2 \theta_w \right)^2 \right], \\
 \frac{d\sigma(\bar{\nu} \bar{u})}{dy} &= -\frac{sx}{32\pi} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2 + \left(-\frac{2}{3} \sin^2 \theta_w \right)^2 (1-y)^2 \right], \\
 \frac{d\sigma(\bar{\nu} \bar{d})}{dy} &= -\frac{sx}{32\pi} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 \left[\left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right)^2 + \left(\frac{1}{3} \sin^2 \theta_w \right)^2 (1-y)^2 \right].
 \end{aligned}$$

なお (20.73)(20.91) から

$$\frac{1}{32\pi} \left(\frac{g^2}{m_Z^2 \cos^2 \theta_w} \right)^2 = \frac{G_F^2}{\pi}.$$

Chapter 21

Quantization of Spontaneously Broken Gauge Theories

21.1 The R_ξ Gauges

(21.20)

(21.18) の相互作用項のうち, φ を含む項は (21.3)(21.5) より

$$\bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^* \psi_L = \bar{\psi} \frac{1 + \gamma^5}{2} \psi \frac{i\varphi}{\sqrt{2}} + \bar{\psi} \frac{1 - \gamma^5}{2} \psi \frac{-i\varphi}{\sqrt{2}} = -\frac{1}{\sqrt{2}} \varphi (\bar{\psi} \gamma^5 \psi).$$

(21.39)

$SU(2) \times U_Y(1)$ 基本表現に属する ϕ の共変微分は (20.60) で与えられる :

$$\begin{aligned} (D_\mu \phi)_i &= \partial_\mu \phi_i - ig A_\mu^a (\tau^a)_{ij} \phi_j - ig' B_\mu \frac{1}{2} \phi_i \\ &= \partial_\mu \phi_i + g A_\mu^a [-i(\tau^a)_{ij}] \phi_j + g' B_\mu \left[-\frac{i}{2} \delta_{ij} \right] \phi_j. \end{aligned}$$

これを (21.33) の形に合わせる.

$$(D_\mu \phi)_i = \partial_\mu \phi_i + g A_\mu^a (T^a)_{ij} \phi_j \quad (a = 1, 2, 3, Y)$$

とすれば $a = 1, 2, 3$ に対しては

$$A_\mu^a = A_\mu^a, \quad T^a = -i\tau^a = T^a.$$

$a = Y$ の場合は*¹

$$A_\mu^Y = B_\mu, \quad T^Y = -\frac{i}{2}.$$

ここで [20.1.2] の導出と同様にすれば T^Y の変換は

$$T^Y \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi^3 \\ h \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -i & & & \\ & -i & & \\ & & h + i\phi^3 & \\ & & & \end{pmatrix} \begin{pmatrix} -\phi^2 - i\phi^1 \\ h + i\phi^3 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} -\phi^1 + i\phi^2 \\ \phi^3 - ih \end{pmatrix} = \begin{pmatrix} -\phi^2 \\ \phi^1 \\ -h \\ \phi^3 \end{pmatrix}$$

*1 p. 742 の $g^2 FF^T$ の計算の後ろに書かれているように $g \rightarrow g'$ と解釈する

なので

$$T^Y = \frac{1}{2} \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1 & -1 \end{pmatrix}. \quad [21.1.1]$$

\mathbb{C}^2 から \mathbb{R}^4 に変換して計算する. [20.1.1] から

$$\phi_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 0 \\ 0 \\ v \end{pmatrix}$$

である. [20.1.2] を使えば

$$T^1 \phi^0 = \frac{1}{2} \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad T^2 \phi^0 = \frac{1}{2} \begin{pmatrix} 0 \\ v \\ 0 \\ 0 \end{pmatrix}, \quad T^3 \phi^0 = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ v \\ 0 \end{pmatrix}, \quad T^Y \phi^0 = \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ -v \\ 0 \end{pmatrix}.$$

第 4 成分 (真空期待値 v と Higgs 場 h) が 0 なので無視できる. (21.36) の定義から

$$F^a_i = T^a_{ij} \phi_{0j} = \frac{v}{2} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 & -1 \end{pmatrix}, \quad (a = 1, 2, 3, Y; i = 1, 2, 3).$$

$a = Y$ の場合は $g \rightarrow g'$ とするので

$$g F^a_i = \frac{v}{2} \begin{pmatrix} g & & \\ & g & \\ & & g & -g' \end{pmatrix}.$$

21.2 The Goldstone Boson Equivalence Theorem

(21.54)

Feynman-'t Hooft gauge ($\xi = 1$) とすれば

$$\overline{A^a_\mu(k) A^b_\nu(q)} = \frac{-ig_{\mu\nu}}{k^2 - m_W^2} \delta^{ab} (2\pi)^4 \delta^{(4)}(k + q).$$

(20.63) の定義から

$$\begin{aligned} \overline{W^+_\mu(k) W^-_\nu(q)} &= \frac{-ig_{\mu\nu}}{k^2 - m_W^2} (2\pi)^4 \delta^{(4)}(k + q), \\ \overline{W^+_\mu(k) W^+_\nu(q)} &= \overline{W^-_\mu(k) W^-_\nu(q)} = 0. \end{aligned}$$

(21.56)

ゲージ群 $SU(2) \times U_Y(1)$ の生成子 T^a ($a = 1, 2, 3, Y$) に対応して ghost は 4 つ存在する. ここで

$$c^+ = \frac{c^1 + ic^2}{\sqrt{2}}, \quad c^- = \frac{c^1 - ic^2}{\sqrt{2}}, \quad c^Z = c^3 \cos \theta_w - c^Y \sin \theta_w, \quad c^A = c^3 \sin \theta_w + c^Y \cos \theta_w,$$

Fig 21.8: Feynman rules of Goldstone bosons

$$\bar{c}^+ = \frac{\bar{c}^1 - i\bar{c}^2}{\sqrt{2}}, \quad \bar{c}^- = \frac{\bar{c}^1 + i\bar{c}^2}{\sqrt{2}}, \quad \bar{c}^Z = \bar{c}^3 \cos \theta_w - \bar{c}^Y \sin \theta_w, \quad \bar{c}^A = \bar{c}^3 \sin \theta_w + \bar{c}^Y \cos \theta_w$$

とする. Lagragian は (21.52) から

$$\begin{aligned} \mathcal{L}_{\text{ghost}} &= \bar{c}^a [-(\partial_\mu D^\mu)^{ab} - g^2 (T^a \phi_0) \cdot (T^b \phi)] c^b \\ &= \bar{c}^a [-\partial^2 \delta^{ac} - g f^{abc} \partial_\mu A_\mu^b - g^2 (T^a \phi_0) \cdot (T^c \phi)] c^c. \end{aligned}$$

[20.1.2][21.1.1] から構造定数を求めれば $f^{12Y} = 1$. これを使って計算すれば*2

$$\begin{aligned} \overline{c^+(k)c^+(q)} &= \overline{c^-(k)c^-(q)} = \frac{i}{k^2 - m_W^2} (2\pi)^4 \delta^{(4)}(k - q), \\ \overline{c^Z(k)c^Z(q)} &= \frac{i}{k^2 - m_Z^2} (2\pi)^4 \delta^{(4)}(k - q), \\ \overline{c^A(k)c^A(q)} &= \frac{i}{k^2} (2\pi)^4 \delta^{(4)}(k - q). \end{aligned} \tag{21.2.1}$$

Fig 21.8: Feynman rules of Goldstone bosons

Goldstone boson の propagator は (21.55) から

$$\overline{\phi_i(k)\phi_j(q)} = \frac{i}{k^2 - m^2} \delta_{ij} (2\pi)^4 \delta^{(4)}(k + q).$$

(21.79) の定義から

$$\begin{aligned} \overline{\phi_+(k)\phi_-(q)} &= \frac{i}{k^2 - m_W^2} (2\pi)^4 \delta^{(4)}(k + q), \\ \overline{\phi_+(k)\phi_+(q)} &= \overline{\phi_-(k)\phi_-(q)} = 0. \end{aligned}$$

(21.38)(21.79) から $SU(2)$ scalar 場は

$$\phi = \begin{pmatrix} -i\phi^- \\ (v + h + i\phi^3)/\sqrt{2} \end{pmatrix}, \quad \phi^\dagger = \begin{pmatrix} i\phi^+ & \frac{v + h - i\phi^3}{\sqrt{2}} \end{pmatrix}$$

となる.

(20.60) に (20.65)(20.66) の結果を適用 ($Y = 1/2$) して

$$D_\mu = \partial_\mu - i \frac{g}{\sqrt{2}} (W_\mu^+ T^+ + W_\mu^- T^-) - i \frac{1}{\sqrt{g^2 + g'^2}} Z_\mu^0 (g^2 T^3 - g'^2/2) - i \frac{gg'}{\sqrt{g^2 + g'^2}} A_\mu (T^3 + 1/2).$$

(20.68) を代入して

$$D_\mu = \partial_\mu - i \frac{g}{\sqrt{2}} (W_\mu^+ T^+ + W_\mu^- T^-) - i \frac{1}{\sqrt{g^2 + g'^2}} Z_\mu^0 (g^2 T^3 - g'^2/2) - ie A_\mu (T^3 + 1/2).$$

ϕ の共変微分は

$$D_\mu \phi = \partial_\mu \phi - i \frac{g}{\sqrt{2}} (W_\mu^+ T^+ + W_\mu^- T^-) \phi - i \frac{1}{\sqrt{g^2 + g'^2}} Z_\mu^0 (g^2 T^3 - g'^2/2) \phi - ie A_\mu (T^3 + 1/2) \phi$$

*2 `./src/py/ghost.ipynb`

Fig 21.8: Feynman rules of Goldstone bosons

$$\begin{aligned}
&= \partial_\mu \phi - i \frac{g}{\sqrt{2}} \begin{pmatrix} W_\mu^- & W_\mu^+ \end{pmatrix} \phi - \frac{i}{2} Z_\mu^0 \begin{pmatrix} \frac{g^2 - g'^2}{\sqrt{g^2 + g'^2}} & -\sqrt{g^2 + g'^2} \end{pmatrix} \phi - ie A_\mu \begin{pmatrix} 1 & 0 \end{pmatrix} \phi \\
&= \begin{pmatrix} -i \partial_\mu \phi^- \\ \partial_\mu (h + i \phi^3) / \sqrt{2} \end{pmatrix} - i \frac{g}{\sqrt{2}} \begin{pmatrix} W_\mu^+ (v + h + i \phi^3) / \sqrt{2} \\ -i W_\mu^- \phi^- \end{pmatrix} \\
&\quad - \frac{i}{2} Z_\mu^0 \begin{pmatrix} -i \frac{g^2 - g'^2}{\sqrt{g^2 + g'^2}} \phi^- \\ -\sqrt{g^2 + g'^2} (v + h + i \phi^3) / \sqrt{2} \end{pmatrix} - ie A_\mu \begin{pmatrix} -i \phi^- \\ 0 \end{pmatrix}.
\end{aligned}$$

(20.70) より

$$\begin{aligned}
(D_\mu \phi)_1 &= -i \partial_\mu \phi^- - i \frac{g}{2} W_\mu^+ (v + h + i \phi^3) - \frac{1}{2} \frac{g^2 - g'^2}{\sqrt{g^2 + g'^2}} Z_\mu^0 \phi^- - e A_\mu \phi^- \\
&= -i \partial_\mu \phi^- - i \frac{g}{2} W_\mu^+ (v + h + i \phi^3) - \frac{g}{\cos \theta_w} \left(\frac{1}{2} - \sin^2 \theta_w \right) Z_\mu^0 \phi^- - e A_\mu \phi^-, \\
(D_\mu \phi)_2 &= \frac{1}{\sqrt{2}} \partial_\mu (h + i \phi^3) - \frac{g}{\sqrt{2}} g W_\mu^- \phi^- + \frac{i}{2\sqrt{2}} \sqrt{g^2 + g'^2} Z_\mu^0 (v + h + i \phi^3) \\
&= \frac{1}{\sqrt{2}} \partial_\mu (h + i \phi^3) - \frac{g}{\sqrt{2}} g W_\mu^- \phi^- + \frac{i}{2\sqrt{2}} \frac{g}{\cos \theta_w} Z_\mu^0 (v + h + i \phi^3).
\end{aligned}$$

(20.111) の Lagrangian は (20.63)(20.112)(20.114) から

$$\begin{aligned}
\mathcal{L}_{\text{Higgs}} &= |D_\mu \phi|^2 + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 \\
&= (D_\mu \phi)_1 (D^\mu \phi)_1^\dagger + (D_\mu \phi)_2 (D^\mu \phi)_2^\dagger + \frac{m_h^2}{2} \phi^\dagger \phi - \frac{g^2}{8} \frac{m_h^2}{m_W^2} (\phi^\dagger \phi)^2
\end{aligned} \tag{21.2.2}$$

で与えられる.

頂点の ϕ^- には運動量 p が入る ($e^{-ip \cdot x}$) ので, $\partial^\mu \phi^- \rightarrow -ip^\mu \phi^-$. よって $\phi^- \phi^+ A$ 頂点は

$$\begin{aligned}
i \mathcal{L}_{\text{Higgs}} &= i(-i \partial^\mu \phi^-)(-e A_\mu \phi^+) + i(i \partial^\mu \phi^+)(-e A_\mu \phi^-) + \dots \\
&\rightarrow i(-p^\mu)(-e) + i(-p'^\mu)(-e) \\
&= ie(p + p')^\mu.
\end{aligned} \tag{21.2.3}$$

頂点の ϕ^- には運動量 p が入る ($e^{-ip \cdot x}$) ので, $\partial^\mu \phi^- \rightarrow -ip^\mu \phi^-$. よって $\phi^- \phi^+ Z^0$ 頂点は

$$\begin{aligned}
i \mathcal{L}_{\text{Higgs}} &= i(-i \partial^\mu \phi^-) \frac{-g}{\cos \theta_w} \left(\frac{1}{2} - \sin^2 \theta_w \right) Z_\mu^0 \phi^+ + i(i \partial^\mu \phi^+) \frac{-g}{\cos \theta_w} \left(\frac{1}{2} - \sin^2 \theta_w \right) Z_\mu^0 \phi^- + \dots \\
&\rightarrow i(-p^\mu) \frac{-g}{\cos \theta_w} \left(\frac{1}{2} - \sin^2 \theta_w \right) + i(-p'^\mu) \frac{-g}{\cos \theta_w} \left(\frac{1}{2} - \sin^2 \theta_w \right) \\
&= \frac{ig(p + p')^\mu}{\cos \theta_w} \left(\frac{1}{2} - \sin^2 \theta_w \right).
\end{aligned}$$

(21.108)

./src/py/problem_21_2_LL.ipynb から

$$\begin{aligned}
i\mathcal{M}(e_L^- e_R^+ \rightarrow W_L^+ W_L^-) &= \frac{ie^2 (-2E^2 - m_W^2) \sqrt{E^2 - m_W^2} \sin \theta}{E (4E^2 \cos^2 \theta_w - m_W^2)} \\
&+ \frac{2iEe^2 \sqrt{E^2 - m_W^2} (2E^2 + m_W^2) \sin \theta}{m_W^2 (2E^2 \cos 2\theta_w + 2E^2 - m_W^2) \tan^2 \theta_w} \\
&+ \frac{iEe^2 (2E^3 \cos \theta - 2E^2 \sqrt{E^2 - m_W^2} - m_W^2 \sqrt{E^2 - m_W^2}) \sin \theta}{m_W^2 (2E^2 - 2E \sqrt{E^2 - m_W^2} \cos \theta - m_W^2) \sin^2 \theta_w} \\
&= \frac{i\beta e^2 (-2E^2 - m_W^2) \sin \theta}{4E^2 \cos^2 \theta_w - m_W^2} + \frac{2i\beta E^2 e^2 (2E^2 + m_W^2) \sin \theta}{m_W^2 (4E^2 \cos^2 \theta_w - m_W^2) \tan^2 \theta_w} \\
&+ \frac{iE^2 e^2 (2E^2 \cos \theta - 2\beta E^2 - m_W^2 \beta) \sin \theta}{m_W^2 (2E^2 - 2\beta E^2 \cos \theta - m_W^2) \sin^2 \theta_w}
\end{aligned}$$

(20.73) から

$$\begin{aligned}
&= -\frac{i\beta e^2 (2E^2 + m_W^2) m_Z^2 \sin \theta}{m_W^2 (4E^2 - m_Z^2)} + \frac{2i\beta E^2 e^2 (2E^2 + m_W^2) \sin \theta}{m_W^2 (4E^2 - m_Z^2) \sin^2 \theta_w} \\
&+ \frac{iE^2 e^2 (2E^2 \cos \theta - 2\beta E^2 - m_W^2 \beta) \sin \theta}{m_W^2 (2E^2 - 2\beta E^2 \cos \theta - m_W^2) \sin^2 \theta_w} \\
&= -i\beta e^2 \sin \theta \frac{m_Z^2}{m_W^2} \frac{2E^2 + m_W^2}{4E^2 - m_Z^2} + i\beta e^2 \sin \theta \frac{1}{2 \sin^2 \theta_w} \frac{4E^2}{4E^2 - m_Z^2} \frac{m_W^2 + 2E^2}{m_W^2} \\
&+ \frac{ie^2 (2E^2 \cos \theta - 2\beta E^2 - m_W^2 \beta) \sin \theta}{m_W^2 (1 + \beta^2 - 2\beta \cos \theta) \sin^2 \theta_w} \\
&= -i\beta e^2 \sin \theta \frac{m_Z^2}{m_W^2} \frac{2E^2 + m_W^2}{4E^2 - m_Z^2} + i\beta e^2 \sin \theta \frac{1}{2 \sin^2 \theta_w} \frac{4E^2}{4E^2 - m_Z^2} \left(1 + \frac{m_Z^2}{2m_W^2} + \frac{4E^2 - m_Z^2}{2m_W^2} \right) \\
&+ \frac{ie^2 (2E^2 \cos \theta - 2\beta E^2 - m_W^2 \beta) \sin \theta}{m_W^2 (1 + \beta^2 - 2\beta \cos \theta) \sin^2 \theta_w} \\
&= -i\beta e^2 \sin \theta \frac{m_Z^2}{m_W^2} \frac{2E^2 + m_W^2}{4E^2 - m_Z^2} + i\beta e^2 \sin \theta \frac{1}{2 \sin^2 \theta_w} \frac{4E^2}{4E^2 - m_Z^2} \left(1 + \frac{m_Z^2}{2m_W^2} \right) \\
&+ i\beta e^2 \sin \theta \frac{1}{\sin^2 \theta_w} \frac{E^2}{m_W^2} + \frac{ie^2 (2E^2 \cos \theta - 2\beta E^2 - m_W^2 \beta) \sin \theta}{m_W^2 (1 + \beta^2 - 2\beta \cos \theta) \sin^2 \theta_w} \\
&= -i\beta e^2 \sin \theta \frac{m_Z^2}{m_W^2} \frac{s/2 + m_W^2}{s - m_Z^2} + i\beta e^2 \sin \theta \frac{1}{2 \sin^2 \theta_w} \frac{s}{s - m_Z^2} \left(1 + \frac{m_Z^2}{2m_W^2} \right) \\
&+ i\beta e^2 \sin \theta \frac{1}{2 \sin^2 \theta_w} 2 \left[\frac{E^2}{m_W^2} + \frac{1}{\beta m_W^2} \frac{2E^2 \cos \theta - 2\beta E^2 - m_W^2 \beta}{1 + \beta^2 - 2\beta \cos \theta} \right].
\end{aligned}$$

第3項は

$$\begin{aligned}
[\cdots] &= \frac{E^2}{m_W^2} + \frac{1}{\beta m_W^2} \frac{2E^2 \cos \theta - 2\beta E^2 - m_W^2 \beta}{1 + \beta^2 - 2\beta \cos \theta} \\
&= \frac{\beta E^2 (1 + \beta^2 - 2\beta \cos \theta) + (2E^2 \cos \theta - 2\beta E^2 - \beta m_W^2)}{\beta m_W^2 (1 + \beta^2 - 2\beta \cos \theta)} \\
&= \frac{\beta E^2 + \beta^3 E^2 - 2\beta^2 E^2 \cos \theta + 2E^2 \cos \theta - 2\beta E^2 - \beta m_W^2}{\beta m_W^2 (1 + \beta^2 - 2\beta \cos \theta)}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{\beta^3 E^2 - \beta E^2 + 2(1 - \beta^2)E^2 \cos \theta - \beta m_W^2}{\beta m_W^2(1 + \beta^2 - 2\beta \cos \theta)} \\
 &= \frac{\beta^3 E^2 - \beta E^2 + 2m_W^2 \cos \theta - \beta m_W^2}{\beta m_W^2(1 + \beta^2 - 2\beta \cos \theta)} \\
 &= \frac{-\beta(1 - \beta^2)E^2 + m_W^2/\beta - (1 + \beta^2 - 2\beta \cos \theta)m_W^2/\beta}{\beta m_W^2(1 + \beta^2 - 2\beta \cos \theta)} \\
 &= \frac{-\beta m_W^2 + m_W^2/\beta}{\beta m_W^2(1 + \beta^2 - 2\beta \cos \theta)} - \frac{1}{\beta^2} \\
 &= \frac{1 - \beta^2}{\beta^2(1 + \beta^2 - 2\beta \cos \theta)} - \frac{1}{\beta^2} \\
 &= \frac{m_W^2}{E^2 \beta^2(1 + \beta^2 - 2\beta \cos \theta)} - \frac{1}{\beta^2} \\
 &= \frac{4m_W^2}{s\beta^2(1 + \beta^2 - 2\beta \cos \theta)} - \frac{1}{\beta^2}.
 \end{aligned}$$

さらに

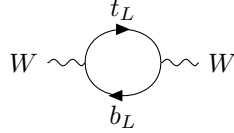
$$ie^2 \bar{v}_L(k_+ - k_-)u_L \frac{1}{s} = -i\beta e^2 \sin \theta$$

を併せれば、右辺を得る.

21.3 One-Loop Corrections to the Weak-Interaction Gauge Theory

(21.153)

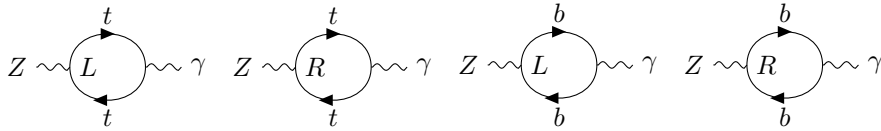
$\Pi_{WW} = g^2 \Pi_{11}$ は



で与えられる. (20.79)(20.80)(21.150) から

$$\Pi_{11}(q_X^2) = -\frac{3}{2} \frac{4}{(4\pi)^2} \left[\left(\frac{q_X^2}{6} - \frac{m_t^2 + m_b^2}{4} \right) E - q_X^2 b_2(btX) + \frac{m_t^2 b_1(btX) + m_b^2 b_1(tbX)}{2} \right].$$

$\Pi_{\gamma Z} = (eg/\cos \theta_w)[\Pi_{3Q} - \sin^2 \theta_w \Pi_{QQ}]$ は



で与えられる. (20.79)(20.80) から (t, t, L) の diagram は

$$\frac{2}{3} \left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right) \times (\Pi_{LL} + \Pi_{LR})$$

となる. (21.149)(21.150)(21.151) から, 全ての diagram の合計は

$$\Pi_{3Q} - \sin^2 \theta_w \Pi_{QQ} = \frac{2}{3} \left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w - \frac{2}{3} \sin^2 \theta_w \right) 3[\Pi_{LL}(ttX) + \Pi_{LR}(ttX)]$$

$$\begin{aligned}
& -\frac{1}{3} \left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w + \frac{1}{3} \sin^2 \theta_w \right) 3[\Pi_{LL}(bbX) + \Pi_{LR}(bbX)] \\
& = -\frac{12}{(4\pi)^2} \left(\frac{1}{3} - \frac{8}{9} \sin^2 \theta_w \right) \left[\frac{q_X^2}{6} E - q_X^2 b_2(ttX) \right] \\
& \quad - \frac{12}{(4\pi)^2} \left(\frac{1}{6} - \frac{2}{9} \sin^2 \theta_w \right) \left[\frac{q_X^2}{6} E - q_X^2 b_2(bbX) \right] \\
& = -\frac{6}{(4\pi)^2} \left[\frac{q_X^2}{6} E - \frac{2}{3} q_X^2 b_2(ttX) - \frac{1}{3} q_X^2 b_2(bbX) \right] - \sin^2 \theta_w \Pi_{QQ}.
\end{aligned}$$

よって

$$\Pi_{3Q}(q_X^2) = -\frac{6}{(4\pi)^2} \left[\frac{q_X^2}{6} E - \frac{2}{3} q_X^2 b_2(ttX) - \frac{1}{3} q_X^2 b_2(bbX) \right].$$

$\Pi_{ZZ} = (g/\cos \theta_w)^2 [\Pi_{33} - 2 \sin^2 \theta_w \Pi_{3Q} + \sin^4 \theta_w \Pi_{QQ}]$ は

$$\begin{aligned}
& \Pi_{33} - 2 \sin^2 \theta_w \Pi_{3Q} + \sin^4 \theta_w \Pi_{QQ} \\
& = \left[\left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right)^2 + \left(-\frac{2}{3} \sin^2 \theta_w \right)^2 \right] 3\Pi_{LL}(ttX) \\
& \quad + 2 \left(\frac{1}{2} - \frac{2}{3} \sin^2 \theta_w \right) \left(-\frac{2}{3} \sin^2 \theta_w \right) 3\Pi_{LR}(ttX) \\
& \quad + \left[\left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right)^2 + \left(\frac{1}{3} \sin^2 \theta_w \right)^2 \right] 3\Pi_{LL}(bbX) \\
& \quad + 2 \left(-\frac{1}{2} + \frac{1}{3} \sin^2 \theta_w \right) \left(\frac{1}{3} \sin^2 \theta_w \right) 3\Pi_{LR}(bbX) \\
& = -\frac{12}{(4\pi)^2} \left(\frac{1}{4} - \frac{2}{3} \sin^2 \theta_w + \frac{8}{9} \sin^4 \theta_w \right) \left[\left(\frac{q_X^2}{6} - \frac{m_t^2}{2} \right) E - q_X^2 b_2(ttX) + m_t^2 b_1(ttX) \right] \\
& \quad - \frac{12}{(4\pi)^2} \left(-\frac{1}{3} \sin^2 \theta_w + \frac{4}{9} \sin^4 \theta_w \right) [m_t^2 E - 2m_t^2 b_1(ttX)] \\
& \quad - \frac{12}{(4\pi)^2} \left(\frac{1}{4} - \frac{1}{3} \sin^2 \theta_w + \frac{2}{9} \sin^4 \theta_w \right) \left[\left(\frac{q_X^2}{6} - \frac{m_b^2}{2} \right) E - q_X^2 b_2(bbX) + m_b^2 b_1(bbX) \right] \\
& \quad - \frac{12}{(4\pi)^2} \left(-\frac{1}{6} \sin^2 \theta_w + \frac{1}{9} \sin^4 \theta_w \right) [m_b^2 E - 2m_b^2 b_1(bbX)] \\
& = -\frac{6}{(4\pi)^2} \left[\left(\frac{q_X^2}{6} - \frac{m_t^2 + m_b^2}{4} \right) E - q_X^2 \frac{b_2(ttX) + b_2(bbX)}{2} + \frac{m_t^2 b_1(ttX) + m_b^2 b_1(bbX)}{2} \right] \\
& \quad - 2 \sin^2 \theta_w \Pi_{3Q} + \sin^4 \theta_w \Pi_{QQ}.
\end{aligned}$$

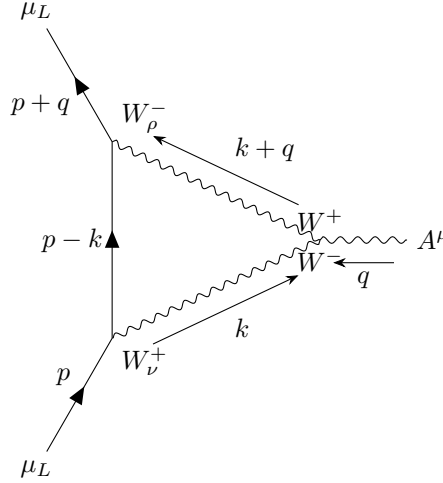
よって

$$\Pi_{33}(q_X^2) = \frac{-6}{(4\pi)^2} \left[\left(\frac{q_X^2}{6} - \frac{m_t^2 + m_b^2}{4} \right) E - q_X^2 \frac{b_2(ttX) + b_2(bbX)}{2} + \frac{m_t^2 b_1(ttX) + m_b^2 b_1(bbX)}{2} \right].$$

Problems

Problem 21.1: Weak interaction contribution to the muon $g - 2$

WW diagram



頂点の 1 loop 補正は

$$\begin{aligned}
 & \bar{u}(p+q)(-ie\delta\Gamma^\mu - ie\delta\Gamma_5^\mu)u(p) \\
 &= \int \frac{d^d k}{(2\pi)^d} \bar{u}(p+q) \frac{ig}{\sqrt{2}} \gamma_\rho \frac{i}{\not{p} - \not{k}} \frac{ig}{\sqrt{2}} \gamma_\nu \frac{1 - \gamma^5}{2} u(p) \frac{-i}{k^2 - m_W^2} \frac{-i}{(k+q)^2 - m_W^2} \\
 & \quad \times ie[g^{\nu\rho}(2k+q)^\mu + g^{\mu\nu}(-k+q)^\rho + g^{\mu\rho}(-k-2q)^\nu] \\
 &= -\frac{eg^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p)^2(k^2 - m_W^2)((k+q)^2 - m_W^2)} \\
 & \quad \times \bar{u}(p+q) [\gamma^\nu(\not{p} - \not{k})\gamma_\nu(2k+q)^\mu + (\not{k} - \not{q})(\not{k} - \not{p})\gamma^\mu + \gamma^\mu(\not{k} - \not{p})(\not{k} + 2\not{q})] \frac{1 - \gamma^5}{2} u(p).
 \end{aligned}$$

γ^5 を含まない項は

$$\begin{aligned}
 \delta\Gamma^\mu &= -i\frac{g^2}{4} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p)^2(k^2 - m_W^2)((k+q)^2 - m_W^2)} \\
 & \quad \times [\gamma^\nu(\not{p} - \not{k})\gamma_\nu(2k+q)^\mu + (\not{k} - \not{q})(\not{k} - \not{p})\gamma^\mu + \gamma^\mu(\not{k} - \not{p})(\not{k} + 2\not{q})].
 \end{aligned}$$

分母は (A.39) から

$$-\frac{ig^2}{2} \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(1-x-y-z) \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta)^3}.$$

ただし,

$$k = \ell + (zp - yq), \quad \Delta = -y(1-y)q^2 - z(1-z)p^2 - 2yzp \cdot q + (1-z)m_W^2.$$

ここで $p^2 = m^2$,

$$2p \cdot q = (p+q)^2 - p^2 - q^2 = m^2 - m^2 - q^2 = -q^2 \quad [21.3.1]$$

なので

$$\begin{aligned}\Delta &= -y(1-y)q^2 - z(1-z)m^2 + yzq^2 + (1-z)m_W^2 \\ &= -y(1-y-z)q^2 - z(1-z)m^2 + (1-z)m_W^2 \\ &= -xyq^2 - z(1-z)m^2 + (1-z)m_W^2.\end{aligned}$$

Dirac 方程式から

$$\not{p}u(p) = mu(p), \quad \bar{u}(p+q)(\not{p} + \not{q}) = \bar{u}(p+q)m.$$

分子は (A.55) から

$$\begin{aligned}A_{WW} &= \gamma^\nu(\not{p} - \not{k})\gamma_\nu(2k+q)^\mu + (\not{k} - \not{q})(\not{k} - \not{p})\gamma^\mu + \gamma^\mu(\not{k} - \not{p})(\not{k} + 2\not{q}) \\ &= (d-2)(\not{k} - \not{p})(2k+q)^\mu + (\not{k} - \not{q})(\not{k} - \not{p})\gamma^\mu + \gamma^\mu(\not{k} - \not{p})(\not{k} + 2\not{q}).\end{aligned}$$

$k = \ell + (zp - yq)$ を代入すると, (A.44)(A.45) より分子の ℓ^2 の項は発散し, ℓ^0 の定数項 A_{WW}^0 が収束する.
第 1 項の定数項は

$$\begin{aligned}&-2[(1-z)\not{p} + y\not{q}][2zp^\mu + (1-2y)q^\mu] \\ &\sim -2(1-z)m[2zp^\mu + (1-2y)q^\mu] \\ &= -2(1-z)m[z(p^\mu + p'^\mu) + (1-2y-z)q^\mu].\end{aligned}$$

第 2 項の定数項は

$$\begin{aligned}&-[z\not{p} - (1+y)\not{q}][(1-z)\not{p} + y\not{q}]\gamma^\mu \\ &\sim -[z(m - \not{q}) - (1+y)\not{q}][(1-z)\not{p}\gamma^\mu + y\not{q}\gamma^\mu] \\ &\sim -[zm - (1+y+z)\not{q}][(1-z)(2p^\mu - \gamma^\mu\not{p}) + y\not{q}\gamma^\mu] \\ &\sim -[zm - (1+y+z)\not{q}][2(1-z)p^\mu - (1-z)m\gamma^\mu + y\not{q}\gamma^\mu] \\ &= -2z(1-z)m p^\mu + z(1-z)m^2\gamma^\mu - yzm\not{q}\gamma^\mu \\ &\quad + 2(1-z)(1+y+z)p^\mu\not{q} - (1-z)(1+y+z)m\not{q}\gamma^\mu + y(1+y+z)q^2\gamma^\mu \\ &\sim [y(1+y+z)q^2 + z(1-z)m^2]\gamma^\mu - 2z(1-z)m p^\mu - [(1-z)(1+y+z) + yz]m\not{q}\gamma^\mu \\ &= [y(1+y+z)q^2 + z(1-z)m^2]\gamma^\mu - 2z(1-z)m p^\mu - (1+y-z^2)m\not{q}\gamma^\mu.\end{aligned}$$

ここで

$$\not{q}\gamma^\mu \sim (m - \not{p})\gamma^\mu = m\gamma^\mu - \not{p}\gamma^\mu = m\gamma^\mu - (2p^\mu - \gamma^\mu\not{p}) \sim m\gamma^\mu - (2p^\mu - \gamma^\mu m) = 2m\gamma^\mu - 2p^\mu$$

なので

$$\begin{aligned}&\sim [y(1+y+z)q^2 + z(1-z)m^2]\gamma^\mu - 2z(1-z)m p^\mu - (1+y-z^2)m(2m\gamma^\mu - 2p^\mu) \\ &= [y(1+y+z)q^2 - (2+2y-z-z^2)m^2]\gamma^\mu + 2(1+y-z)m p^\mu \\ &= [y(1+y+z)q^2 - (2+2y-z-z^2)m^2]\gamma^\mu + (1+y-z)m(p^\mu + p'^\mu) - (1+y-z)m q^\mu.\end{aligned}$$

第 3 項の定数項は

$$\begin{aligned}&-\gamma^\mu[(1-z)\not{p} + y\not{q}][z\not{p} + (2-y)\not{q}] \\ &= -\gamma^\mu[z(1-z)\not{p}^2 + y(2-y)\not{q}^2 + yz\not{q}\not{p} + (1-z)(2-y)\not{p}\not{q}] \\ &\sim -\gamma^\mu[z(1-z)m^2 + y(2-y)q^2 + yz\not{q}\not{p} + (2-y-2z+yz)\not{p}\not{q}]\end{aligned}$$

$$\begin{aligned}
 &= -\gamma^\mu [z(1-z)m^2 + y(2-y)q^2 + yz(\not{q}\not{p} + \not{p}\not{q}) + (2-y-2z)\not{p}\not{q}] \\
 &= -\gamma^\mu [z(1-z)m^2 + y(2-y)q^2 + yz(2p \cdot q) + (2-y-2z)\not{p}\not{q}] \\
 &= [-z(1-z)m^2 - y(2-y)q^2] \gamma^\mu - yz(2p \cdot q)\gamma^\mu - (2-y-2z)\gamma^\mu \not{p}\not{q}.
 \end{aligned}$$

[21.3.1] および

$$\begin{aligned}
 \gamma^\mu \not{p}\not{q} &= (2p^\mu - \not{p}\gamma^\mu)\not{q} \sim -\not{p}\gamma^\mu \not{q} \sim (\not{q} - m)\gamma^\mu \not{q} = (\not{q} - m)(2q^\mu - \not{q}\gamma^\mu) \\
 &= 2q^\mu \not{q} - 2mq^\mu - \not{q}\not{q}\gamma^\mu + m\not{q}\gamma^\mu \sim -2mq^\mu - q^2\gamma^\mu + m(2m\gamma^\mu - 2p^\mu) \\
 &= (2m^2 - q^2)\gamma^\mu - 2mp'^\mu
 \end{aligned}$$

から

$$\begin{aligned}
 &\sim [-z(1-z)m^2 - y(2-y)q^2] \gamma^\mu + yzq^2\gamma^\mu - (2-y-2z)[(2m^2 - q^2)\gamma^\mu - 2mp'^\mu] \\
 &= [(-4 + 2y + 3z + z^2)m^2 + (2 - 3y - 2z + y^2 + yz)q^2] \gamma^\mu + 2(2-y-2z)mp'^\mu \\
 &= [(-4 + 2y + 3z + z^2)m^2 + (2 - 3y - 2z + y^2 + yz)q^2] \gamma^\mu \\
 &\quad + (2-y-2z)m(p^\mu + p'^\mu) + (2-y-2z)mq^\mu.
 \end{aligned}$$

以上を全て足して,

$$\begin{aligned}
 A_{WW}^0 &\sim -2(1-z)m[z(p^\mu + p'^\mu) + (1-2y-z)q^\mu] \\
 &\quad + [y(1+y+z)q^2 - (2+2y-z-z^2)m^2]\gamma^\mu + (1+y-z)m(p^\mu + p'^\mu) - (1+y-z)mq^\mu \\
 &\quad + [(-4 + 2y + 3z + z^2)m^2 + (2 - 3y - 2z + y^2 + yz)q^2] \gamma^\mu \\
 &\quad + (2-y-2z)m(p^\mu + p'^\mu) + (2-y-2z)mq^\mu \\
 &= [-2(3-2z+z^2)m^2 + y(2-2y-2z+2y^2+2yz)q^2] \gamma^\mu \\
 &\quad + (1-z)(3-2z)m(p^\mu + p'^\mu) - (2z-1)(1-2y-z)mq^\mu \\
 &= [-2(3-2z+z^2)m^2 + y(2-2y-2z+2y^2+2yz)q^2] \gamma^\mu \\
 &\quad + (1-z)(3-2z)m(p^\mu + p'^\mu) - (2z-1)(x-y)mq^\mu.
 \end{aligned}$$

被積分関数の分母は $x \leftrightarrow y$ について対称なので,

$$A_{WW}^0 \rightarrow [-2(3-2z+z^2)m^2 + y(2-2y-2z+2y^2+2yz)q^2] \gamma^\mu + (1-z)(3-2z)m(p^\mu + p'^\mu). \quad [21.3.2]$$

Feynman rules of Goldstone bosons

(20.98) から lepton, neutrino と $SU(2)$ scalar の相互作用は

$$\Delta\mathcal{L}_e = -\lambda_e \bar{E}_L \phi e_R + (\text{h.c.})$$

(21.38)(21.79)(20.75) を代入して

$$\begin{aligned}
 -\lambda_e \bar{E}_L \phi e_R &= -\lambda_e (\bar{\nu}_L \quad \bar{e}_L) \begin{pmatrix} -i\phi^- \\ (v+h+i\phi^3)/\sqrt{2} \end{pmatrix} e_R \\
 &= i\lambda_e \bar{\nu}_L e_R \phi^- - \frac{\lambda_e}{\sqrt{2}} \bar{e}_L e_R (v+h+i\phi^3).
 \end{aligned}$$

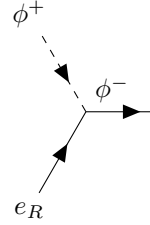
(20.63)(20.100) から

$$\lambda_e = \frac{g}{\sqrt{2}} \frac{m_e}{m_W}.$$

よって

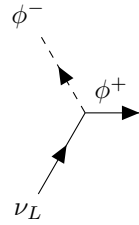
$$\Delta\mathcal{L}_e = i\frac{g}{\sqrt{2}}\frac{m_e}{m_W}\bar{\nu}_L e_R \phi^- - \frac{g}{2}\frac{m_e}{m_W}\bar{e}_L e_R(v+h+i\phi^3) + (\text{h.c.}) \quad [21.3.3]$$

$e\nu\phi^-$ の頂点は



$$= -\frac{g}{\sqrt{2}}\frac{m_e}{m_W}\frac{1+\gamma^5}{2}.$$

$e\nu\phi^+$ の頂点は



$$= \frac{g}{\sqrt{2}}\frac{m_e}{m_W}\frac{1-\gamma^5}{2}.$$

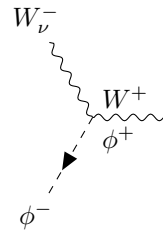
[21.2.2] から

$$\mathcal{L}_{\text{Higgs}} = ie\frac{gv}{2}A^\mu\phi^+W_\mu^+ - ie\frac{gv}{2}A^\mu\phi^-W_\mu^-.$$

(20.63) を代入して

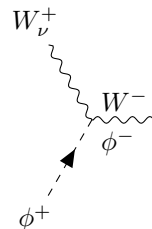
$$i\mathcal{L}_{\text{Higgs}} = -em_W A^\mu\phi^+W_\mu^+ + em_W A^\mu\phi^-W_\mu^-.$$

$A\phi^+W^+$ の頂点は



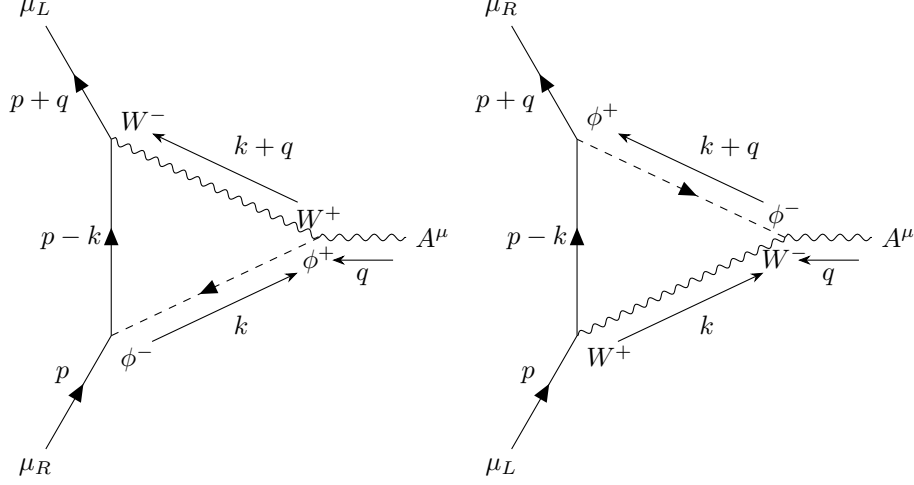
$$A_\mu = -em_W g^{\mu\nu}. \quad [21.3.4]$$

$A\phi^-W^-$ の頂点は



$$A_\mu = em_W g^{\mu\nu}. \quad [21.3.5]$$

$\phi W + W\phi$ diagrams



頂点の 1 loop 補正は

$$\begin{aligned}
 & \bar{u}(p+q)(-ie\delta\Gamma^\mu - ie\delta\Gamma_5^\mu)u(p) \\
 &= \int \frac{d^d k}{(2\pi)^d} \bar{u}(p+q) \frac{ig}{\sqrt{2}} \gamma^\mu \frac{i}{\not{p} - \not{k}} \frac{-g}{\sqrt{2}} \frac{m}{m_W} \frac{1+\gamma^5}{2} u(p) \frac{-i}{(k+q)^2 - m_W^2} \frac{i}{k^2 - m_W^2} (-em_W) \\
 &+ \int \frac{d^d k}{(2\pi)^d} \bar{u}(p+q) \frac{g}{\sqrt{2}} \frac{m}{m_W} \frac{1+\gamma^5}{2} \frac{i}{\not{p} - \not{k}} \frac{ig}{\sqrt{2}} \gamma^\mu u(p) \frac{i}{(k+q)^2 - m_W^2} \frac{-i}{k^2 - m_W^2} em_W \\
 &= -\frac{eg^2 m_\mu}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p)^2 (k^2 - m_W^2) ((k+q)^2 - m_W^2)} \\
 &\quad \times \bar{u}(p+q) \left[\gamma^\mu (\not{k} - \not{p}) \frac{1+\gamma^5}{2} + \frac{1+\gamma^5}{2} (\not{k} - \not{p}) \gamma^\mu \right] u(p).
 \end{aligned}$$

γ^5 を含まない項は

$$\begin{aligned}
 \delta\Gamma^\mu &= -\frac{ig^2}{4} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p)^2 (k^2 - m_W^2) ((k+q)^2 - m_W^2)} m [\gamma^\mu (\not{k} - \not{p}) + (\not{k} - \not{p}) \gamma^\mu] \\
 &= -\frac{ig^2}{4} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p)^2 (k^2 - m_W^2) ((k+q)^2 - m_W^2)} 2m [k^\mu - p^\mu].
 \end{aligned}$$

分子の定数項は

$$\begin{aligned}
 A_{\phi W}^0 &= 2m [-(1-z)p^\mu - yq^\mu] \\
 &= -2(1-z)mp^\mu - 2ymq^\mu \\
 &= -(1-z)m(p^\mu + p'^\mu) + (1-2y-z)mq^\mu \\
 &= -(1-z)m(p^\mu + p'^\mu) + (z-y)q^\mu \\
 &\rightarrow -(1-z)m(p^\mu + p'^\mu).
 \end{aligned}$$

[21.3.2] と併せて

$$\begin{aligned}
 A_{WW}^0 + A_{\phi W}^0 &= [-2(3-2z+z^2)m^2 + y(2-2y-2z+2y^2+2yz)q^2] \gamma^\mu \\
 &\quad + (1-z)(3-2z)m(p^\mu + p'^\mu) - (1-z)m(p^\mu + p'^\mu)
 \end{aligned}$$

$$= [-2(3 - 2z + z^2)m^2 + y(2 - 2y - 2z + 2y^2 + 2yz)q^2] \gamma^\mu + 2(1 - z)(2 - z)m(p^\mu + p'^\mu).$$

Gordon 恒等式 (6.32) を使えば

$$= [\cdots] \gamma^\mu - 4(1 - z)(2 - z)m^2 \frac{i\sigma^{\mu\nu} q_\nu}{2m}.$$

(6.33) から

$$\begin{aligned} F_2(q^2) &= \frac{ig^2}{2} \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(1 - x - y - z) \int \frac{d^4 \ell}{(2\pi)^4} \frac{4(1 - z)(2 - z)m^2}{(\ell^2 - \Delta)^3} \\ &= \frac{ig^2}{2} \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(1 - x - y - z) \frac{-i}{2(4\pi)^2} \frac{4(1 - z)(2 - z)m^2}{\Delta} \\ &= \frac{ig^2}{2} \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(1 - x - y - z) \frac{-i}{2(4\pi)^2} \frac{4(1 - z)(2 - z)m^2}{-xyq^2 - z(1 - z)m^2 + (1 - z)m_W^2}. \end{aligned}$$

よって

$$\begin{aligned} F_2(q^2 = 0) &= \frac{g^2}{(4\pi)^2} \frac{m^2}{m_W^2} \int_0^1 dz \int_0^{1-z} dy \frac{2 - z}{1 - z(m/m_W^2)} \\ &\approx \frac{g^2}{(4\pi)^2} \frac{m^2}{m_W^2} \int_0^1 dz \int_0^{1-z} dy (2 - z) \\ &= \frac{g^2}{(4\pi)^2} \frac{m^2}{m_W^2} \frac{5}{6}. \end{aligned}$$

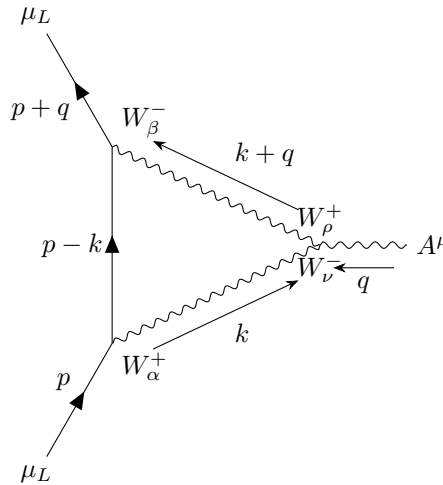
(20.91) から

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}$$

なので, (6.37) から

$$a_\mu(\nu) = F_2(q^2 = 0) = -\frac{G_F m^2}{8\pi^2 \sqrt{2}} \frac{10}{3}.$$

general R_ξ gauge (未計算)



頂点の 1 loop 補正は

$$\begin{aligned}
 & \bar{u}(p+q)(-ie\delta\Gamma^\mu - ie\delta\Gamma_5^\mu)u(p) \\
 &= \int \frac{d^d k}{(2\pi)^d} \bar{u}(p+q) \frac{ig}{\sqrt{2}} \gamma^\beta \frac{i}{\not{p}-\not{k}} \frac{ig}{\sqrt{2}} \gamma^\alpha \frac{1-\gamma^5}{2} u(p) \\
 & \quad \times \frac{-i}{k^2 - m_W^2} \left[g_{\alpha\nu} - \frac{k_\alpha k_\nu}{k^2 - \xi m_W^2} (1-\xi) \right] \frac{-i}{(k+q)^2 - m_W^2} \left[g_{\beta\rho} - \frac{(k+q)_\beta (k+q)_\rho}{(k+q)^2 - \xi m_W^2} (1-\xi) \right] \\
 & \quad \times ie[g^{\nu\rho}(2k+q)^\mu + g^{\mu\nu}(-k+q)^\rho + g^{\mu\rho}(-k-2q)^\nu] \\
 &= -\frac{eg^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k-p)^2(k^2 - m_W^2)((k+q)^2 - m_W^2)} \bar{u}(p+q) A_{WW} \frac{1-\gamma^5}{2} u(p) \\
 & \quad - \frac{eg^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1-\xi}{(k-p)^2(k^2 - m_W^2)((k+q)^2 - m_W^2)((k+q)^2 - \xi m_W^2)} \\
 & \quad \times \bar{u}(p+q) A_{12} \frac{1-\gamma^5}{2} u(p) \\
 & \quad - \frac{eg^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1-\xi}{(k-p)^2(k^2 - m_W^2)(k^2 - \xi m_W^2)((k+q)^2 - m_W^2)} \\
 & \quad \times \bar{u}(p+q) A_{21} \frac{1-\gamma^5}{2} u(p) \\
 & \quad - \frac{eg^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{(1-\xi)^2}{(k-p)^2(k^2 - m_W^2)(k^2 - \xi m_W^2)((k+q)^2 - m_W^2)((k+q)^2 - \xi m_W^2)} \\
 & \quad \times \bar{u}(p+q) A_{22} \frac{1-\gamma^5}{2} u(p).
 \end{aligned}$$

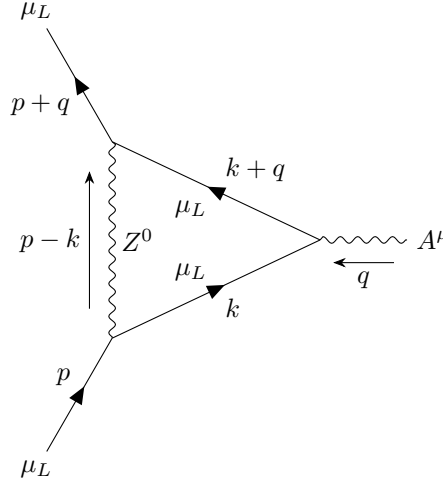
ただし

$$\begin{aligned}
 A_{12} &= -(\not{k} + \not{q})(\not{k} - \not{p}) \left[(\not{k} + \not{q})(-2k - q)^\mu + \gamma^\mu(k + q) \cdot (k - q) + (\not{k} + 2\not{q})(k + q)^\mu \right] \\
 A_{21} &= -\not{k}(\not{k} - \not{p})\not{k}(-2k - q)^\mu + (\not{k} - \not{q})(\not{k} - \not{p})\not{k}k^\mu + \gamma^\mu(\not{k} - \not{p})\not{k}k \cdot (k + 2q) \\
 A_{22} &= -(\not{k} + \not{q})(\not{k} - \not{p})\not{k} [k \cdot (k + q)(-2k - q)^\mu + k^\mu(k + q) \cdot (k - q) + (k + q)^\mu k \cdot (k + 2q)].
 \end{aligned}$$

ZZ diagram

$\mu\mu Z^0$ の頂点は (20.80) から

$$\begin{aligned}
 & \frac{ig}{\cos\theta_w} \gamma^\mu \left[\left(-\frac{1}{2} + \sin^2\theta_w \right) \frac{1-\gamma^5}{2} + \sin^2\theta_w \frac{1+\gamma^5}{2} \right] \\
 &= \frac{ig}{4\cos\theta_w} \gamma^\mu (4\sin^2\theta_w - 1 + \gamma^5)
 \end{aligned}$$



頂点の 1 loop 補正は

$$\begin{aligned}
 & \bar{u}(p+q)(\delta\Gamma^\mu + \delta\Gamma_5^\mu)u(p) \\
 &= \int \frac{d^d k}{(2\pi)^d} \bar{u}(p+q) \frac{ig}{4\cos\theta_w} \gamma^\nu (4\sin^2\theta_w - 1 + \gamma^5) \frac{i}{\not{k} + \not{q} - m} \gamma^\mu \frac{i}{\not{k} - m} \\
 & \quad \times \frac{ig}{4\cos\theta_w} \gamma_\nu (4\sin^2\theta_w - 1 + \gamma^5) u(p) \frac{-i}{(p-k)^2 - m_Z^2} \\
 &= -\frac{i}{16} \left(\frac{g}{\cos\theta_w} \right)^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)((k+q)^2 - m^2)((p-k)^2 - m_Z^2)} \\
 & \quad \times \bar{u}(p+q) \gamma^\nu (4\sin^2\theta_w - 1 + \gamma^5) (\not{k} + \not{q} + m) \gamma^\mu (\not{k} + m) \gamma_\nu (4\sin^2\theta_w - 1 + \gamma^5) u(p).
 \end{aligned}$$

γ^5 を含まない項は

$$\begin{aligned}
 \delta\Gamma^\mu &= -\frac{i}{16} \left(\frac{g}{\cos\theta_w} \right)^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - m^2)((k+q)^2 - m^2)((p-k)^2 - m_Z^2)} \\
 & \quad \times [(4\sin^2\theta_w - 1)^2 \gamma^\nu (\not{k} + \not{q} + m) \gamma^\mu (\not{k} + m) \gamma_\nu + \gamma^\nu (\not{k} + \not{q} - m) \gamma^\mu (\not{k} - m) \gamma_\nu].
 \end{aligned}$$

分母は

$$-\frac{i}{8} \left(\frac{g}{\cos\theta_w} \right)^2 \int_0^1 dx \int_0^1 dy \int_0^1 dz \delta(1-x-y-z) \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^3}.$$

ただし

$$\begin{aligned}
 k &= \ell + (zp - yq), \\
 \Delta &= -z(1-z)p^2 - y(1-y)q^2 + (x+y)m^2 + zm_Z^2 - 2yzp \cdot q \\
 &= -z(1-z)m^2 - y(1-y)q^2 + (1-z)m^2 + zm_Z^2 + yzq^2 \\
 &= -xyq^2 + (1-z)^2 m^2 + zm_Z^2.
 \end{aligned}$$

分子第 1 項の定数項は

$$\begin{aligned}
 &= \gamma^\nu [z\not{p} + (1-y)\not{q} + m] \gamma^\mu [z\not{p} - y\not{q} + m] \gamma_\nu \\
 &= \gamma^\nu [z\not{p} + (1-y)\not{q}] \gamma^\mu [z\not{p} - y\not{q}] \gamma_\nu \\
 & \quad + m\gamma^\nu [z\not{p} + (1-y)\not{q}] \gamma^\mu \gamma_\nu + m\gamma^\nu \gamma^\mu [z\not{p} - y\not{q}] \gamma_\nu + m^2 \gamma^\nu \gamma^\mu \gamma_\nu
 \end{aligned}$$

$$\begin{aligned}
 &= -2[z\not{p} - y\not{q}]\gamma^\mu[z\not{p} + (1-y)\not{q}] \\
 &\quad + 4m[zp + (1-y)q]^\mu + 4m[zp - yq]^\mu - 2m^2\gamma^\mu \\
 &\sim -2[(y+z)\not{p} - ym]\gamma^\mu[zm + (1-y)\not{q}] + 4m[2zp^\mu + (1-2y)q^\mu] - 2m^2\gamma^\mu \\
 &= -2(y+z)zm\not{p}\gamma^\mu - 2(1-y)(y+z)\not{p}\gamma^\mu\not{q} + 2yzm^2\gamma^\mu + 2y(1-y)m\gamma^\mu\not{q} \\
 &\quad + 8zmp^\mu + 4(1-2y)mq^\mu - 2m^2\gamma^\mu \\
 &= 2(yz-1)m^2\gamma^\mu + 8zmp^\mu + 4(1-2y)mq^\mu \\
 &\quad - 2(y+z)zm\not{p}\gamma^\mu - 2(1-y)(y+z)\not{p}\gamma^\mu\not{q} + 2y(1-y)m\gamma^\mu\not{q}.
 \end{aligned}$$

ここで

$$\begin{aligned}
 \not{p}\gamma^\mu &= 2p^\mu - \gamma^\mu\not{p} \sim 2p^\mu - m\gamma^\mu, \\
 \not{p}\gamma^\mu\not{q} &\sim (m - \not{q})\gamma^\mu\not{q} = (m - \not{q})(2q^\mu - \not{q}\gamma^\mu) = -2q^\mu\not{q} + 2mq^\mu + \not{q}\not{q}\gamma^\mu - m\not{q}\gamma^\mu \\
 &\sim 2mq^\mu + q^2\gamma^\mu - m(2m\gamma^\mu - 2p^\mu) \\
 &= (q^2 - 2m^2)\gamma^\mu + 2mp'^\mu, \\
 \gamma^\mu\not{q} &= 2q^\mu - \not{q}\gamma^\mu \sim 2q^\mu - (m - \not{p})\gamma^\mu \\
 &= 2q^\mu - m\gamma^\mu + \not{p}\gamma^\mu = 2q^\mu - m\gamma^\mu + 2p^\mu - \gamma^\mu\not{p} = 2q^\mu - m\gamma^\mu + 2p^\mu - m\gamma^\mu \\
 &= 2p'^\mu - 2m\gamma^\mu
 \end{aligned}$$

を代入して

$$\begin{aligned}
 &\sim 2(yz-1)m^2\gamma^\mu + 8zmp^\mu + 4(1-2y)mq^\mu \\
 &\quad - 2(y+z)zm(2p^\mu - m\gamma^\mu) - 2(1-y)(y+z)[(q^2 - 2m^2)\gamma^\mu + 2mp'^\mu] + 2y(1-y)m(2p'^\mu - 2m\gamma^\mu) \\
 &= [\cdots]\gamma^\mu + 4(2-y-z)zmp^\mu - 4(1-y)zmp'^\mu + 4(1-2y)mq^\mu \\
 &= [\cdots]\gamma^\mu + 2(1-z)zm(p^\mu + p'^\mu) + 2(2-4y-3z+2yz+z^2)mq^\mu \\
 &= [\cdots]\gamma^\mu + 2(1-z)zm(p^\mu + p'^\mu) + 2(x-y)(x+y+1)mq^\mu \\
 &\rightarrow [\cdots]\gamma^\mu + 2(1-z)zm(p^\mu + p'^\mu).
 \end{aligned}$$

分子第 2 項の定数項は

$$\begin{aligned}
 &= \gamma^\nu[z\not{p} + (1-y)\not{q} - m]\gamma^\mu[z\not{p} - y\not{q} - m]\gamma_\nu \\
 &= \gamma^\nu[z\not{p} + (1-y)\not{q}]\gamma^\mu[z\not{p} - y\not{q}]\gamma_\nu \\
 &\quad - m\gamma^\nu[z\not{p} + (1-y)\not{q}]\gamma^\mu\gamma_\nu - m\gamma^\nu\gamma^\mu[z\not{p} - y\not{q}]\gamma_\nu + m^2\gamma^\nu\gamma^\mu\gamma_\nu \\
 &= -2[z\not{p} - y\not{q}]\gamma^\mu[z\not{p} + (1-y)\not{q}] \\
 &\quad - 4m[zp + (1-y)q]^\mu - 4m[zp - yq]^\mu - 2m^2\gamma^\mu \\
 &\sim -2[(y+z)\not{p} - ym]\gamma^\mu[zm + (1-y)\not{q}] - 4m[2zp^\mu + (1-2y)q^\mu] - 2m^2\gamma^\mu \\
 &= -2(y+z)zm\not{p}\gamma^\mu - 2(1-y)(y+z)\not{p}\gamma^\mu\not{q} + 2yzm^2\gamma^\mu + 2y(1-y)m\gamma^\mu\not{q} \\
 &\quad - 8zmp^\mu - 4(1-2y)mq^\mu - 2m^2\gamma^\mu \\
 &= 2(yz-1)m^2\gamma^\mu - 8zmp^\mu - 4(1-2y)mq^\mu \\
 &\quad - 2(y+z)zm\not{p}\gamma^\mu - 2(1-y)(y+z)\not{p}\gamma^\mu\not{q} + 2y(1-y)m\gamma^\mu\not{q} \\
 &\sim 2(yz-1)m^2\gamma^\mu - 8zmp^\mu - 4(1-2y)mq^\mu \\
 &\quad - 2(y+z)zm(2p^\mu - m\gamma^\mu) - 2(1-y)(y+z)[(q^2 - 2m^2)\gamma^\mu + 2mp'^\mu] + 2y(1-y)m(2p'^\mu - 2m\gamma^\mu) \\
 &= [\cdots]\gamma^\mu - 4(2+y+z)zmp^\mu - 4(1-y)zmp'^\mu - 4(1-2y)mq^\mu
 \end{aligned}$$

$$\begin{aligned}
 &= [\cdots] \gamma^\mu - 4(3+z)zm p^\mu - 4(1-2y+z-yz)mq^\mu \\
 &= [\cdots] \gamma^\mu - 2(3+z)zm(p^\mu + p'^\mu) - 2(2-4y-z-2yz-z^2)mq^\mu \\
 &= [\cdots] \gamma^\mu - 2(3+z)zm(p^\mu + p'^\mu) + 2(x-y)(x+y-3)mq^\mu \\
 &\rightarrow [\cdots] \gamma^\mu - 2(3+z)zm(p^\mu + p'^\mu).
 \end{aligned}$$

以上から分子の定数項は

$$\begin{aligned}
 A_{ZZ}^0 &= [\cdots] \gamma^\mu + (4\sin^2 \theta_w - 1)^2 2(1-z)zm(p^\mu + p'^\mu) - 2(3+z)zm(p^\mu + p'^\mu) \\
 &= [\cdots] \gamma^\mu + 2[(4\sin^2 \theta_w - 1)^2(1-z)z - (3+z)z] m(p^\mu + p'^\mu) \\
 &= [\cdots] \gamma^\mu - 4[(4\sin^2 \theta_w - 1)^2(1-z)z - (3+z)z] m^2 \frac{i\sigma^{\mu\nu} q_\nu}{2m}.
 \end{aligned}$$

よって

$$\begin{aligned}
 F_2(q^2) &= \frac{i}{16} \left(\frac{g}{\cos \theta_w} \right)^2 \int_0^1 dz \int_0^{1-z} dy \int \frac{d^d \ell}{(2\pi)^d} \frac{4[(4\sin^2 \theta_w - 1)^2(1-z)z - (3+z)z] m^2}{(\ell^2 - \Delta)^3} \\
 &= \frac{i}{8} \left(\frac{g}{\cos \theta_w} \right)^2 \int_0^1 dz \int_0^{1-z} dy \frac{-i}{2(4\pi)^2} \frac{4[(4\sin^2 \theta_w - 1)^2(1-z)z - (3+z)z] m^2}{\Delta} \\
 &= \frac{i}{8} \left(\frac{g}{\cos \theta_w} \right)^2 \int_0^1 dz \int_0^{1-z} dy \frac{-i}{2(4\pi)^2} \frac{4[(4\sin^2 \theta_w - 1)^2(1-z)z - (3+z)z] m^2}{-xyq^2 + (1-z)^2 m^2 + zm_Z^2}.
 \end{aligned}$$

$m_Z \gg m$ なので

$$\begin{aligned}
 a_\mu(Z) &= F_2(0) \\
 &\approx \frac{i}{8} \left(\frac{g}{\cos \theta_w} \right)^2 \int_0^1 dz \int_0^{1-z} dy \frac{-i}{2(4\pi)^2} \frac{4[(4\sin^2 \theta_w - 1)^2(1-z)z - (3+z)z] m^2}{zm_Z^2} \\
 &\approx \frac{1}{4(4\pi)^2} \frac{m^2}{m_Z^2} \left(\frac{g}{\cos \theta_w} \right)^2 \int_0^1 dz \int_0^{1-z} dy [(4\sin^2 \theta_w - 1)^2(1-z) - (3+z)] \\
 &= \frac{1}{4(4\pi)^2} \frac{m^2}{m_Z^2} \left(\frac{g}{\cos \theta_w} \right)^2 \left[\frac{1}{3}(4\sin^2 \theta_w - 1)^2 - \frac{5}{3} \right].
 \end{aligned}$$

(20.73)(20.91) から

$$= \frac{G_F m^2}{8\pi^2 \sqrt{2}} \left[\frac{16}{3} \sin^4 \theta_w - \frac{8}{3} \sin^2 \theta_w - \frac{4}{3} \right].$$

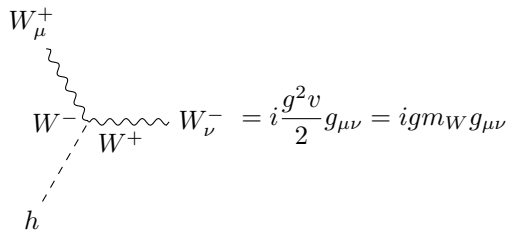
Problem 21.4: Dependence of radiative corrections on the Higgs boson mass

Feynman rules of Higgs bosons

[21.2.2] から

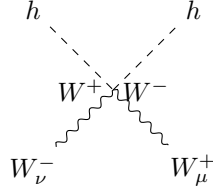
$$i\mathcal{L}_{\text{Higgs}} = i\frac{g^2}{4} W_\mu^+ W^- \nu(v+h+i\phi^3)(v+h-i\phi^3)g^{\mu\nu} + \cdots.$$

W^+W^-h の頂点は



$$W^- \begin{array}{c} W^+ \\ \text{---} \end{array} W_\nu^- = i\frac{g^2 v}{2} g_{\mu\nu} = igm_W g_{\mu\nu}. \quad [21.3.6]$$

W^+W^-hh の頂点は h の縮約が 2 通りあることに注意して,

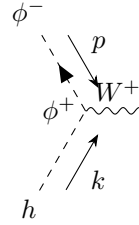


$$= 2 \cdot i \frac{g^2}{4} g_{\mu\nu} = i \frac{g^2}{2} g_{\mu\nu}.$$

[21.2.2] から

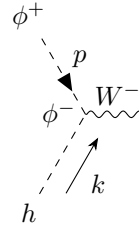
$$i\mathcal{L}_{\text{Higgs}} = i \frac{g}{2} [(\partial_\mu \phi^+) W_\nu^+ h - (\partial_\mu h) W_\nu^+ \phi^+ + (\partial_\mu \phi^-) W_\nu^- h - (\partial_\mu h) W_\nu^- \phi^-] g^{\mu\nu} + \dots$$

$W^-\phi^-h$ の頂点は



$$= i \frac{g}{2} (-ip_\mu + ik_\mu) = \frac{g}{2} (p - k)_\mu. \quad [21.3.7]$$

$W^+\phi^+h$ の頂点は

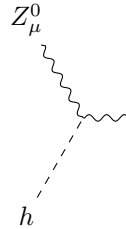


$$= i \frac{g}{2} (-ip_\mu + ik_\mu) = \frac{g}{2} (p - k)_\mu. \quad [21.3.8]$$

[21.2.2] から

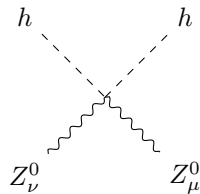
$$i\mathcal{L}_{\text{Higgs}} = i \frac{1}{8 \cos^2 \theta_w} g^2 Z_\mu^0 Z_\nu^0 (v + h + i\phi^3)(v + h - i\phi^3) g^{\mu\nu} + \dots$$

$Z^0 Z^0 h$ の頂点は Z^0 の縮約が 2 通りあることに注意して,



$$= i \frac{g^2 v}{2 \cos^2 \theta_w} g_{\mu\nu} = i \frac{gm_Z}{\cos \theta_w} g_{\mu\nu}.$$

$Z^0 Z^0 hh$ の頂点は Z^0 の縮約が 2 通り, h の縮約が 2 通りあることに注意して,

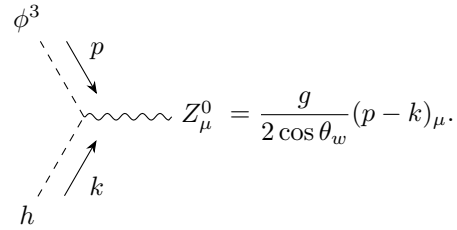


$$= i \frac{g^2}{2 \cos^2 \theta_w} g_{\mu\nu}.$$

[21.2.2] から

$$\begin{aligned}
 i\mathcal{L}_{\text{Higgs}} &= \frac{g}{4\cos\theta_w} [\partial_\mu(h + i\phi^3)(v + h - i\phi^3) - \partial_\mu(h - i\phi^3)(v + h + i\phi^3)] Z_\nu^0 g^{\mu\nu} + \dots \\
 &= \frac{ig}{2\cos\theta_w} [\partial_\mu\phi^3 h - \phi^3 \partial_\mu h] Z_\nu^0 g^{\mu\nu} + \dots
 \end{aligned}$$

$Z^0\phi^3h$ の頂点は



$$Z_\mu^0 = \frac{g}{2\cos\theta_w} (p - k)_\mu.$$

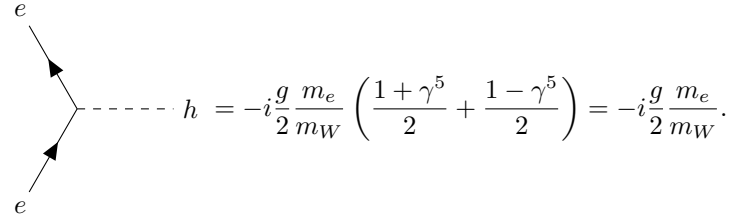
Final Project III: Dicays of the Higgs Boson

(a)

[21.3.3] から

$$\Delta\mathcal{L}_e = i\frac{g}{\sqrt{2}}\frac{m_e}{m_W}\bar{\nu}_L e_R \phi^- - \frac{g}{2}\frac{m_e}{m_W}\bar{e}_L e_R(v+h+i\phi^3) + (\text{h.c.})$$

$h\bar{e}e$ の頂点は



$$h = -i\frac{g}{2}\frac{m_e}{m_W}\left(\frac{1+\gamma^5}{2} + \frac{1-\gamma^5}{2}\right) = -i\frac{g}{2}\frac{m_e}{m_W}.$$

(20.101) から quark と $SU(2)$ scalar の相互作用は

$$\Delta\mathcal{L}_q = -\lambda_d\bar{Q}_L\phi d_R - \lambda_u\epsilon^{ab}\bar{Q}_{La}\phi_b^\dagger u_R + (\text{h.c.})$$

$\Delta\mathcal{L}_q$ の第 1 項は (21.38)(21.79)(20.75) を代入して

$$\begin{aligned} -\lambda_d\bar{Q}_L\phi d_R &= -\lambda_d\begin{pmatrix}\bar{u}_L & \bar{d}_L\end{pmatrix}\begin{pmatrix}-i\phi^- \\ (v+h+i\phi^3)/\sqrt{2}\end{pmatrix}d_R \\ &= i\lambda_d\bar{u}_L d_R \phi^- - \frac{\lambda_d}{\sqrt{2}}\bar{d}_L d_R(v+h+i\phi^3). \end{aligned}$$

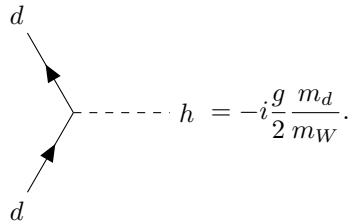
(20.63)(20.103) から

$$\lambda_d = \frac{g}{\sqrt{2}}\frac{m_d}{m_W}.$$

よって

$$-\lambda_d\bar{Q}_L\phi d_R = i\frac{g}{\sqrt{2}}\frac{m_d}{m_W}\bar{u}_L d_R \phi^- - \frac{g}{2}\frac{m_d}{m_W}\bar{d}_L d_R(v+h+i\phi^3) + (\text{h.c.})$$

$h\bar{d}d$ の頂点は

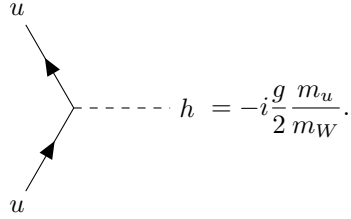


$$h = -i\frac{g}{2}\frac{m_d}{m_W}.$$

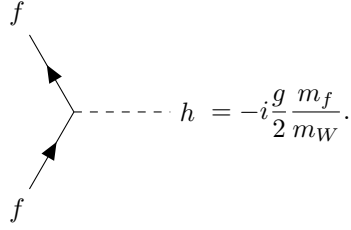
(b)

 $\Delta\mathcal{L}_q$ の第 2 項は

$$\begin{aligned}
-\lambda_u \epsilon^{ab} \bar{Q}_{La} \phi_b^\dagger u_R &= -\lambda_u (\bar{u}_L \quad \bar{d}_L) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} i\phi^+ \\ (v+h-i\phi^3)/\sqrt{2} \end{pmatrix} u_R \\
&= i\lambda_u \bar{d}_L u_R \phi^+ - \frac{\lambda_u}{\sqrt{2}} \bar{u}_L u_R (v+h-i\phi^3) \\
&= i \frac{g}{\sqrt{2}} \frac{m_u}{m_W} \bar{d}_L u_R \phi^+ - \frac{g}{2} \frac{m_u}{m_W} \frac{\lambda_u}{\sqrt{2}} \bar{u}_L u_R (v+h-i\phi^3)
\end{aligned}$$

 $h\bar{u}u$ の頂点は

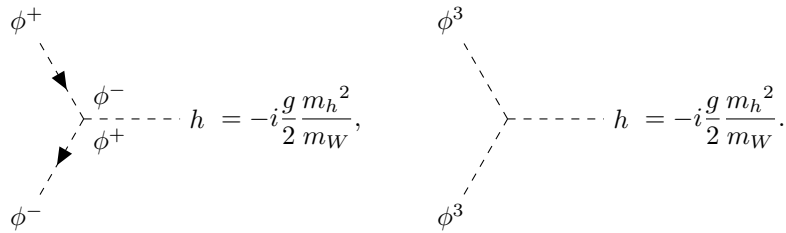
以上から, lepton, quark いずれの場合も



(b)

[21.2.2] と (20.63) から

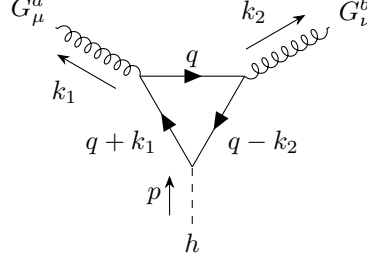
$$\begin{aligned}
\mathcal{L}_{\text{Higgs}} &= -\frac{g^2}{8} \frac{m_h^2}{m_W^2} \left[\phi^+ \phi^- + \frac{(v+h)^2 + (\phi^3)^2}{2} \right]^2 + \dots \\
&= -\frac{g^2}{8} \frac{m_h^2}{m_W^2} [2v\phi^+ \phi^- h + 2v(\phi^3)^2 h] + \dots \\
&= -\frac{g^2}{2} \frac{m_h^2}{m_W^2} \phi^+ \phi^- h - \frac{g^2}{2} \frac{m_h^2}{m_W^2} (\phi^3)^2 h + \dots
\end{aligned}$$

 $h\phi\phi$ の頂点は

[III.9]

(c)

(c)



運動量について

$$p^\mu = (m_h, 0, 0, 0), \quad k_1^\mu = \frac{m_h}{2}(1, \sin \theta, 0, \cos \theta), \quad k_2^\mu = \frac{m_h}{2}(1, -\sin \theta, 0, -\cos \theta),$$

$$k_1^2 = k_2^2 = 0, \quad k_1 \cdot k_2 = \frac{m_h^2}{2}, \quad \epsilon_1 \cdot k_1 = \epsilon_1 \cdot k_2 = 0.$$

不変振幅は

$$i\mathcal{M}_1 = -i \frac{g}{2} \frac{m_q}{m_W} \int \frac{d^d q}{(2\pi)^d} (-1) \text{Tr} \left[i g_s \not{\epsilon}_1^* t^a \frac{i}{\not{q} - m_q} i g_s \not{\epsilon}_2^* t^b \frac{i}{\not{q} - \not{k}_2 - m_q} \frac{i}{\not{q} + \not{k}_1 - m_q} \right]$$

$$= -\frac{g g_s^2}{2} \frac{m_q}{m_W} \text{Tr}(t^a t^b) \int \frac{d^d q}{(2\pi)^d} \text{Tr} \left[\not{\epsilon}_1^* \frac{1}{\not{q} - m_q} \not{\epsilon}_2^* \frac{1}{\not{q} - \not{k}_2 - m_q} \frac{1}{\not{q} + \not{k}_1 - m_q} \right].$$

分母は (A.39) から

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 - m_q^2} \frac{1}{(q + k_1)^2 - m_q^2} \frac{1}{(q - k_2)^2 - m_q^2}$$

$$= \int_0^1 dx \int_0^{1-x} dy \frac{2}{(\ell^2 - \Delta)^3}.$$

ただし

$$q = \ell - (x k_1 - y k_2),$$

$$\Delta = -x(1-x)k_1^2 - y(1-y)k_2^2 - 2xyk_1 \cdot k_2 + m_q^2 = m_q^2 - xym_h^2.$$

分子

$$N = \text{Tr} [\not{\epsilon}_1^* (\not{q} + m_q) \not{\epsilon}_2^* (\not{q} - \not{k}_2 + m_q) (\not{q} + \not{k}_1 + m_q)]$$

のうち ℓ^2 となる項は (A.27)(A.41) から

$$N_2 = m_q \text{Tr} [\not{\epsilon}_1^* \not{\epsilon}_2^* \not{\ell} \not{\ell}] + m_q \text{Tr} [\not{\epsilon}_1^* \not{\ell} \not{\epsilon}_2^* \not{\ell}] + m_q \text{Tr} [\not{\epsilon}_1^* \not{\ell} \not{\epsilon}_2^* \not{\ell}]$$

$$= m_q \ell^2 \text{Tr} [\not{\epsilon}_1^* \not{\epsilon}_2^*] + 2m_q 4\epsilon_{1\mu}^* \epsilon_{2\nu}^* [\ell^\mu \ell^\nu - g^{\mu\nu} \ell^2 + \ell^\mu \ell^\nu]$$

$$= 4m_q \epsilon_{1\mu}^* \epsilon_{2\nu}^* \ell^2 g^{\mu\nu} + 8m_q \epsilon_{1\mu}^* \epsilon_{2\nu}^* [2\ell^\mu \ell^\nu - g^{\mu\nu} \ell^2]$$

$$= 4m_q \epsilon_{1\mu}^* \epsilon_{2\nu}^* [4\ell^\mu \ell^\nu - \ell^2 g^{\mu\nu}]$$

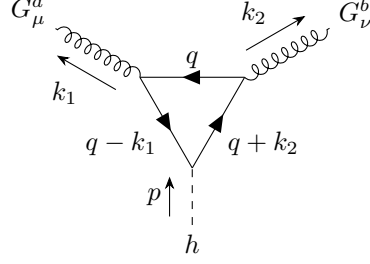
$$\rightarrow 4m_q \epsilon_{1\mu}^* \epsilon_{2\nu}^* \left[\frac{4}{d} - 1 \right] \ell^2 g^{\mu\nu}$$

$$= 4(\epsilon_1^* \cdot \epsilon_2^*) m_q \left[\frac{4}{d} - 1 \right] \ell^2.$$

(f)

定数項は*3

$$N_0 = 2(\epsilon_1^* \cdot \epsilon_2^*) m_q (2xy m_h^2 - m_h^2 + 2m_q^2).$$



不変振幅は

$$\begin{aligned} i\mathcal{M}_2 &= -i \frac{g}{2} \frac{m_q}{m_W} \int \frac{d^d q}{(2\pi)^d} (-1) \text{Tr} \left[i g_s \not{\epsilon}_1^* t^a \frac{i}{\not{q} - m_q} i g_s \not{\epsilon}_2^* t^b \frac{i}{\not{q} + \not{k}_2 - m_q} \frac{i}{\not{q} - \not{k}_1 - m_q} \right] \\ &= -\frac{g g_s^2}{2} \frac{m_q}{m_W} \text{Tr}(t^a t^b) \int \frac{d^d q}{(2\pi)^d} \text{Tr} \left[\not{\epsilon}_1^* \frac{1}{\not{q} - m_q} \not{\epsilon}_2^* \frac{1}{\not{q} + \not{k}_2 - m_q} \frac{1}{\not{q} - \not{k}_1 - m_q} \right]. \end{aligned}$$

分母は (A.39) から

$$\begin{aligned} &\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 - m_q^2} \frac{1}{(q - k_1)^2 - m_q^2} \frac{1}{(q + k_2)^2 - m_q^2} \\ &= \int_0^1 dx \int_0^{1-x} dy \frac{2}{(\ell^2 - \Delta)^3}. \end{aligned}$$

ただし

$$\begin{aligned} q &= \ell + (xk_1 - yk_2), \\ \Delta &= -x(1-x)k_1^2 - y(1-y)k_2^2 - 2xyk_1 \cdot k_2 + m_q^2 = m_q^2 - xym_h^2. \end{aligned}$$

分子

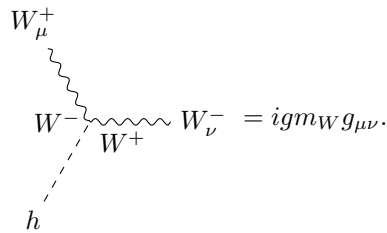
$$N = \text{Tr} [\not{\epsilon}_1^* (\not{q} + m_q) \not{\epsilon}_2^* (\not{q} + \not{k}_2 + m_q) (\not{q} - \not{k}_1 + m_q)]$$

のうち ℓ^2 となる項は先程と全く同じ。定数項も計算すれば一致する。

(f)

Feynman rules of Higgs boson and photon

[21.3.6] から



*3 ./src/py/fp3_c.ipynb

[21.3.7][21.3.8] から

$$W_\mu^\pm = \frac{g}{2}(p-k)_\mu.$$

[III.9] から

$$h = -i\frac{g m_h^2}{2 m_W}.$$

[21.2.3] から

$$= ie(p+p')^\mu.$$

[21.3.4][21.3.5] から

$$A_\mu = \pm e m_W g^{\mu\nu}.$$

[21.2.2] から

$$\mathcal{L}_{\text{Higgs}} = i\frac{ge}{2}W_\mu^+ A^\mu h\phi^+ - i\frac{ge}{2}W_\mu^- A^\mu h\phi^- + e^2 A_\mu A^\mu \phi^+ \phi^- + \dots$$

なので

$$= \mp \frac{ge}{2} g_{\mu\nu}, \quad = 2ie^2 g_{\mu\nu}.$$

$SU(2)$ の構造定数は ϵ^{abc} なので (16.6) から

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}g^2(\epsilon^{ab}A_\kappa^a A_\lambda^b)(\epsilon^{cd}A^{\kappa c}A^{\lambda d}) + \dots$$

$(abcd) = (1133)$ となるのは

- $a = 1, b = 3, c = 1, d = 3$

- $a = 3, b = 1, c = 3, d = 1$
- $a = 1, b = 3, c = 3, d = 1$
- $a = 3, b = 1, c = 1, d = 3$

の場合, $(abcd) = (2233)$ も同様に計算すれば

$$-\frac{g^2}{2} (A_\kappa^1 A^{\kappa 1} A_\lambda^3 A^{\lambda 3} + A_\kappa^2 A^{\kappa 2} A_\lambda^3 A^{\lambda 3} - A_\kappa^1 A^{\kappa 3} A_\lambda^1 A^{\lambda 3} - A_\kappa^2 A^{\kappa 3} A_\lambda^2 A^{\lambda 3}).$$

(20.63)(20.64)(20.72) から

$$A_\mu^1 = \frac{W_\mu^+ + W_\mu^-}{\sqrt{2}}, \quad A_\mu^2 = i \frac{W_\mu^+ - W_\mu^-}{\sqrt{2}}, \quad A_\mu^3 = \frac{1}{\sin \theta_w} A_\mu$$

なので,

$$= g^2 (A^\kappa W_\kappa^+ A^\lambda W_\lambda^- - W_\kappa^+ W^{-\kappa} A_\lambda A^\lambda).$$

$A^\mu A^\nu W_\rho^+ W_\sigma^-$ との縮約を考える. 第 1 項は

- $\kappa = \mu, \lambda = \nu, \lambda = \rho, \kappa = \sigma$
- $\kappa = \nu, \lambda = \mu, \lambda = \rho, \kappa = \sigma$

の縮約が可能で, それぞれ $\delta_\sigma^\mu \delta_\rho^\nu$ および $\delta_\rho^\mu \delta_\sigma^\nu$ となる. 第 2 項は A^λ の縮約が 2 通りあるので $2g^{\mu\nu} g_{\rho\sigma}$.

以上から

$$\begin{array}{c} A_\mu \\ \text{---} \\ W_\rho^+ \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ W^+ \\ \text{---} \end{array} \begin{array}{c} A^\nu \\ \text{---} \\ W^- \\ \text{---} \\ W_\sigma^- \end{array} = ie^2 (\delta_\rho^\mu \delta_\sigma^\nu + \delta_\sigma^\mu \delta_\rho^\nu - 2g^{\mu\nu} g_{\rho\sigma}).$$

Feynman rules of ghosts

[21.2.1] の計算^{*4}を続行する.

$$\begin{aligned} \mathcal{L}_{\text{ghost}} &= -ie\bar{c}^+ \partial^\mu A_\mu c^+ + ie\bar{c}^- \partial^\mu A_\mu c^- + \cdots \\ &\rightarrow ie(\partial^\mu \bar{c}^+) A_\mu c^+ - ie(\partial^\mu \bar{c}^-) A_\mu c^- + \cdots \end{aligned}$$

から

$$\begin{array}{c} A_\mu \\ \text{---} \\ \bar{c}^\pm c^\pm \\ \text{---} \\ c^\pm \end{array} \begin{array}{c} \text{---} \\ p \\ \text{---} \\ \bar{c}^\pm \end{array} = \mp i e p_\mu.$$

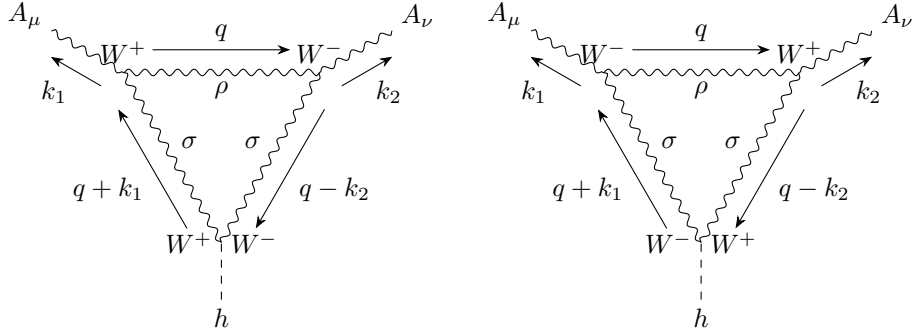
$$\mathcal{L}_{\text{ghost}} = -\frac{g}{2} m_W \bar{c}^+ c^+ h - \frac{g}{2} m_W \bar{c}^- c^- h + \cdots$$

^{*4} ./src/py.ghost.ipynb

から

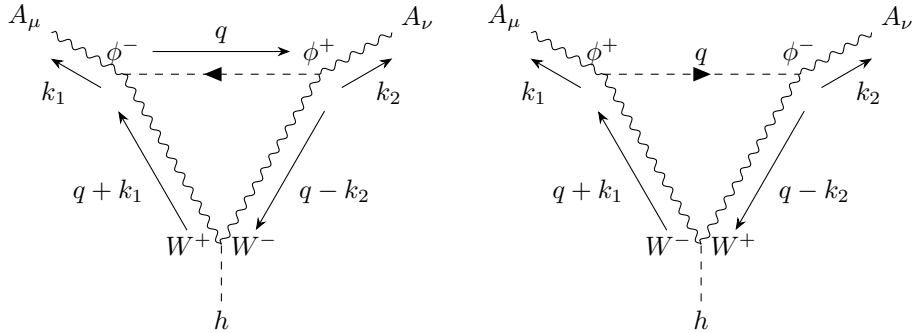
$$\begin{array}{c}
 h \\
 \vdots \\
 \bar{c}^\pm \quad c^\pm \\
 \vdots \\
 c^\pm \quad \bar{c}^\pm
 \end{array}
 = -i \frac{g}{2} m_W.$$

3-boson vertex diagrams



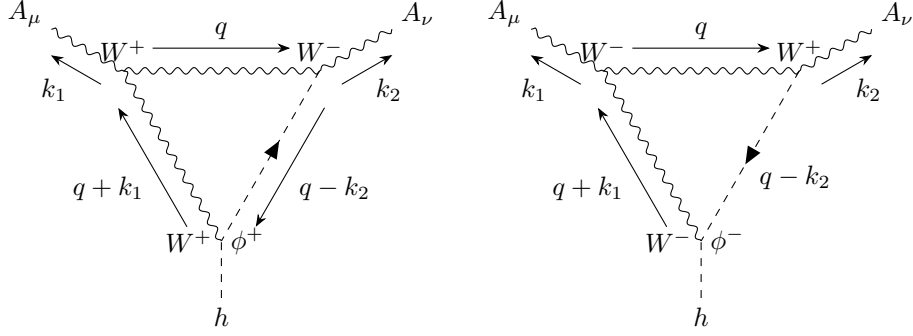
不変振幅は*⁵(A.41)(A.44)(A.45) から

$$\begin{aligned}
 & 2igm_W (ie)^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) (-i)^3 2 \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{(\text{Num})}{(\ell^2 - \Delta)^3} \\
 &= 4gm_W e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{(-4xy + x + y - 5) k_1 \cdot k_2 g^{\mu\nu} + (4d - 6) \ell^\mu \ell^\nu + 2\ell^2 g^{\mu\nu}}{(\ell^2 - \Delta)^3} \\
 &= gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} 4 \left[(-4xy + x + y - 5) \frac{m_h^2}{2} + \left(6 - \frac{6}{d} \right) \ell^2 \right] \frac{1}{(\ell^2 - \Delta)^3} \\
 &= \frac{i}{(4\pi)^2} gm_W e^2 \epsilon_1^* \cdot \epsilon_2^* \int_0^1 dx dy \\
 &\quad \times \left[(4xy - x - y + 5) \frac{m_h^2}{\Delta} + 4 \left(6 - \frac{6}{d} \right) \frac{d}{4} \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta} \right)^{2-d/2} \right].
 \end{aligned}$$

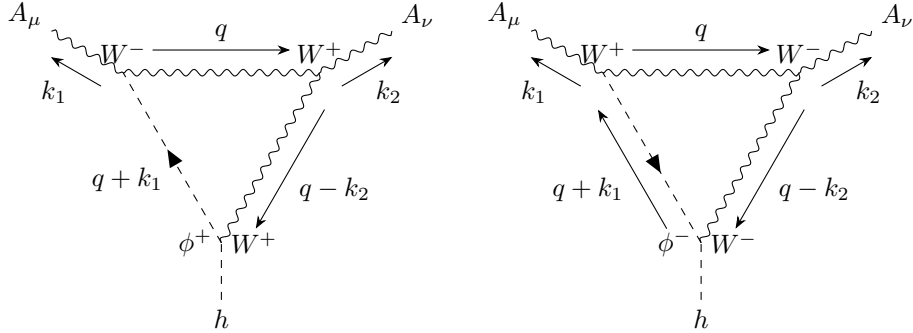


*⁵ ./src/py/NonAbelian_1loop.ipynb

$$\begin{aligned}
& 2igm_W em_W (-em_W) g^{\mu\nu} \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) (-i)^2 i 2 \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^3} \\
& = -4gm_W^3 e^2 \epsilon_1^* \cdot \epsilon_2^* \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^3} \\
& = \frac{i}{(4\pi)^2} gm_W e^2 \epsilon_1^* \cdot \epsilon_2^* \int_0^1 dx dy \frac{2m_W^2}{\Delta}.
\end{aligned}$$

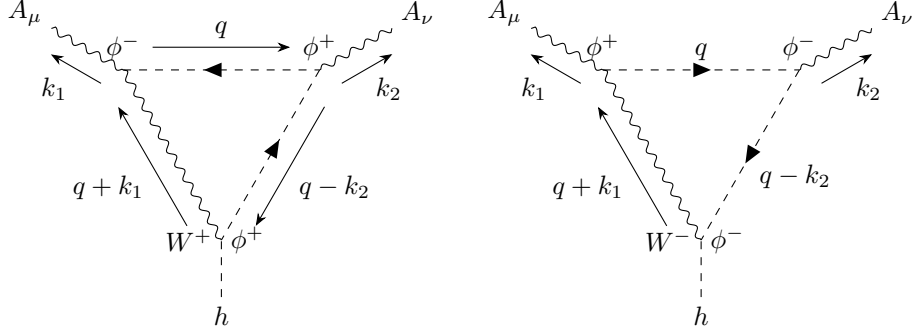


$$\begin{aligned}
& 2ie \frac{g}{2} em_W \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) (-i)^2 i 2 \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{(\text{Num})}{(\ell^2 - \Delta)^3} \\
& = 2gm_W e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{(2xy - 2x + 2y - 2)k_1 \cdot k_2 g^{\mu\nu} + \ell^\mu \ell^\nu - \ell^2 g^{\mu\nu}}{(\ell^2 - \Delta)^3} \\
& = gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} 2 \left[(2xy - 2x + 2y - 2) \frac{m_h^2}{2} + \left(\frac{1}{d} - 1 \right) \ell^2 \right] \frac{1}{(\ell^2 - \Delta)^3} \\
& = \frac{i}{(4\pi)^2} gm_W e^2 \epsilon_1^* \cdot \epsilon_2^* \int_0^1 dx dy \\
& \quad \times \left[-(xy - x + y - 1) \frac{m_h^2}{\Delta} + 2 \left(\frac{1}{d} - 1 \right) \frac{d}{4} \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta} \right)^{2-d/2} \right].
\end{aligned}$$

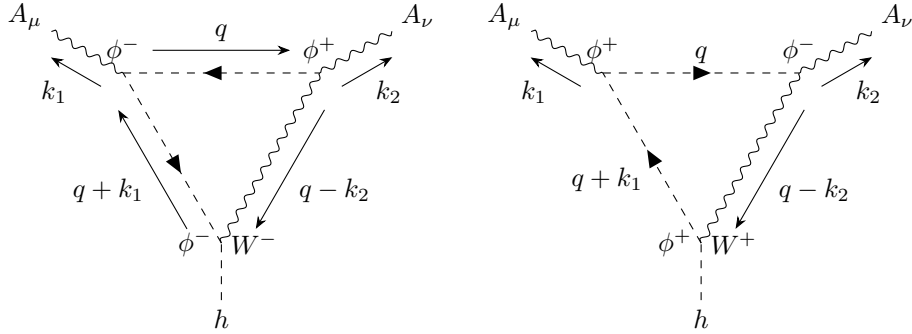


$$\begin{aligned}
& 2ie \frac{g}{2} em_W \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) (-i)^2 i 2 \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{(\text{Num})}{(\ell^2 - \Delta)^3} \\
& = 2gm_W e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{(2xy + 2x - 2y - 2)k_1 \cdot k_2 g^{\mu\nu} + \ell^\mu \ell^\nu - \ell^2 g^{\mu\nu}}{(\ell^2 - \Delta)^3}
\end{aligned}$$

$$\begin{aligned}
&= gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} 2 \left[(2xy + 2x - 2y - 2) \frac{m_h^2}{2} + \left(\frac{1}{d} - 1 \right) \ell^2 \right] \frac{1}{(\ell^2 - \Delta)^3} \\
&= \frac{i}{(4\pi)^2} gm_W e^2 \epsilon_1^* \cdot \epsilon_2^* \int_0^1 dx dy \\
&\quad \times \left[-(xy + x - y - 1) \frac{m_h^2}{\Delta} + 2 \left(\frac{1}{d} - 1 \right) \frac{d}{4} \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta} \right)^{2-d/2} \right].
\end{aligned}$$

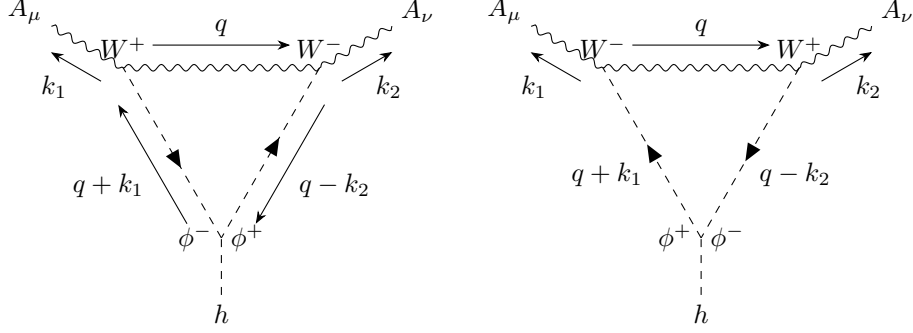


$$\begin{aligned}
&2ie \frac{g}{2} em_W \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) i^2 (-i) 2 \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{(q - k_1 - 2k_2)^\mu (-2q - k_2)^\nu}{(\ell^2 - \Delta)^3} \\
&= 4gm_W e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{(\ell^2 - \Delta)^3} \\
&= gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{4}{d} \frac{\ell^2}{(\ell^2 - \Delta)^3} \\
&= \frac{i}{(4\pi)^2} gm_W e^2 \epsilon_1^* \cdot \epsilon_2^* \int_0^1 dx dy \frac{4}{d} \frac{d}{4} \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta} \right)^{2-d/2}.
\end{aligned}$$

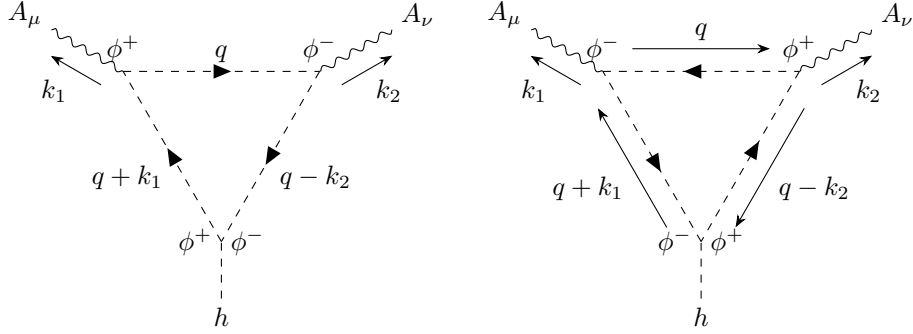


$$\begin{aligned}
&2ie \frac{g}{2} em_W \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) i^2 (-i) 2 \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{(-q - 2k_1 - k_2)^\mu (2q + k_1)^\nu}{(\ell^2 - \Delta)^3} \\
&= 4gm_W e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{(\ell^2 - \Delta)^3} \\
&= gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{4}{d} \frac{\ell^2}{(\ell^2 - \Delta)^3}
\end{aligned}$$

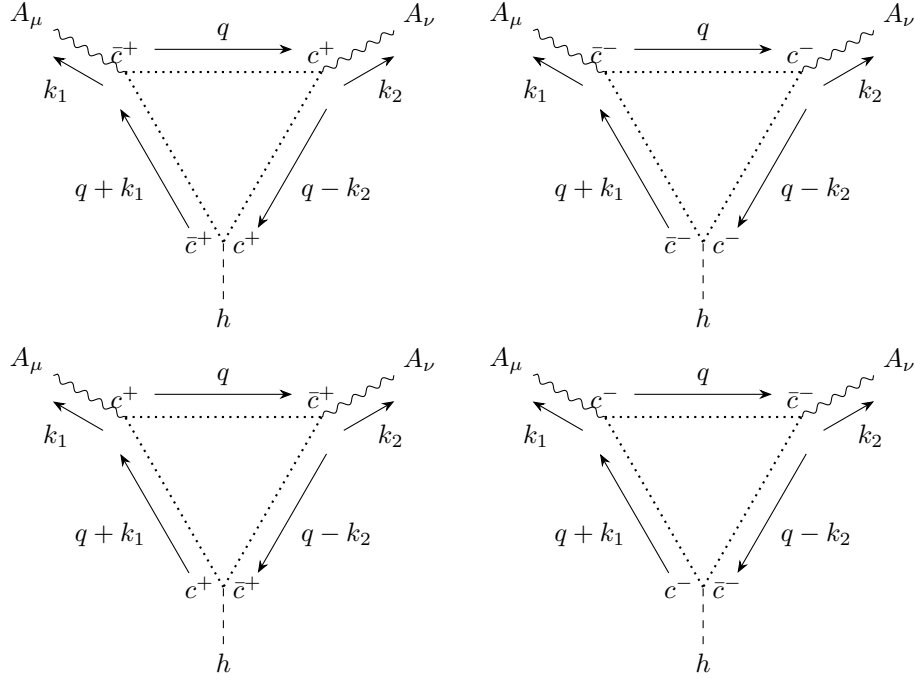
$$= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon_1^* \cdot \epsilon_2^* \int_0^1 dx dy \frac{4}{d} \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{4\pi}{\Delta}\right)^{2-d/2}.$$



$$\begin{aligned} & 2 \left(-i \frac{g m_h^2}{2 m_W} \right) e m_W (-e m_W) \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) g^{\mu\nu} i^2 (-i) 2 \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^3} \\ &= -2 g m_W m_h^2 e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta)^3} \\ &= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon_1^* \cdot \epsilon_2^* \int_0^1 dx dy \frac{m_h^2}{\Delta} \end{aligned}$$



$$\begin{aligned} & 2 \left(-i \frac{g m_h^2}{2 m_W} \right) (ie)^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) i^3 2 \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{(2q + k_1)^\mu (2q - k_2)^\nu}{(\ell^2 - \Delta)^3} \\ &= 8 g \frac{m_h^2}{m_W} e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{(\ell^2 - \Delta)^3} \\ &= g m_W e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{m_h^2}{m_W^2} \frac{8}{d} \frac{\ell^2}{(\ell^2 - \Delta)^3} \\ &= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx dy \frac{m_h^2}{m_W^2} \frac{8}{d} \frac{4}{4} \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{4\pi}{\Delta}\right)^{2-d/2}. \end{aligned}$$

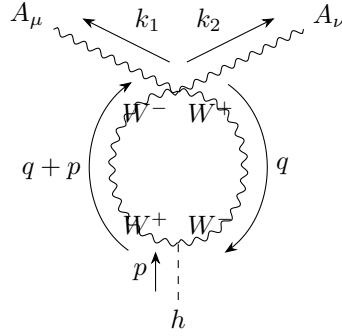


$$\begin{aligned}
& 4 \left(-i \frac{g}{2} m_W \right) (ie)^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) i^3 (-2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{q^\mu (q - k_2)^\nu}{(\ell^2 - \Delta)^3} \\
&= -4 g m_W e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{\ell^\mu \ell^\nu}{(\ell^2 - \Delta)^3} \\
&= -g m_W e^2 \epsilon_\mu^*(k_1) \epsilon_\nu^*(k_2) \int_0^1 dx dy \int \frac{d^d \ell}{(2\pi)^d} \frac{4}{d} \frac{\ell^2}{(\ell^2 - \Delta)^3} \\
&= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx dy \frac{4}{d} \frac{d}{4} \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta} \right)^{2-d/2}.
\end{aligned}$$

以上を全て足して $\Delta = m_W^2 - xym_h^2$ を代入すれば

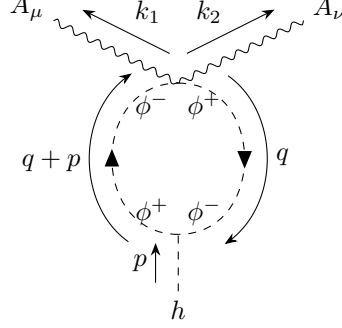
$$\begin{aligned}
i\mathcal{M}_{333} &= \frac{i}{(4\pi)^2} gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \\
&\quad \times \left[\frac{2m_W^2 + (2xy - x - y + 8)m_h^2}{\Delta} + \left(5d - 4 + \frac{2m_h^2}{m_W^2} \right) \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta} \right)^{2-d/2} \right] \\
&= \frac{i}{(4\pi)^2} gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \\
&\quad \times \left[\frac{2m_W^2 + (2xy - x - y + 8)m_h^2}{m_W^2 - xym_h^2} - 10 \left(2 - \frac{d}{2} \right) \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta} \right)^{2-d/2} \right. \\
&\quad \left. + \left(16 + \frac{2m_h^2}{m_W^2} \right) \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta} \right)^{2-d/2} \right] \\
&= \frac{i}{(4\pi)^2} gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \\
&\quad \times \left[\frac{2m_W^2 + (2xy - x - y + 8)m_h^2}{m_W^2 - xym_h^2} - 10 \right. \\
&\quad \left. + \left(16 + \frac{2m_h^2}{m_W^2} \right) \left(\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log(m_W^2 - xym_h^2) \right) \right] \\
&= \frac{i}{(4\pi)^2} gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \\
&\quad \times \left[\frac{(4xy - x - y + 8)m_h^2}{m_W^2 - xym_h^2} - 8 + \left(16 + \frac{2m_h^2}{m_W^2} \right) \left(\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log(m_W^2 - xym_h^2) \right) \right].
\end{aligned}
\tag{III.10}$$

3 + 4-boson vertex diagrams



$$\begin{aligned}
&- (2d - 2) gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta')^2} \\
&= - \frac{i}{(4\pi)^2} gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx (2d - 2) \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta'} \right)^{2-d/2} \\
&= - \frac{i}{(4\pi)^2} gm_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \\
&\quad \times \left[-4 \left(2 - \frac{d}{2} \right) \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta'} \right)^{2-d/2} + 6 \Gamma \left(2 - \frac{d}{2} \right) \left(\frac{4\pi}{\Delta'} \right)^{2-d/2} \right]
\end{aligned}$$

$$= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \left[-4 + 6\Gamma\left(2 - \frac{d}{2}\right) \left(\frac{4\pi}{\Delta'}\right)^{2-d/2} \right].$$

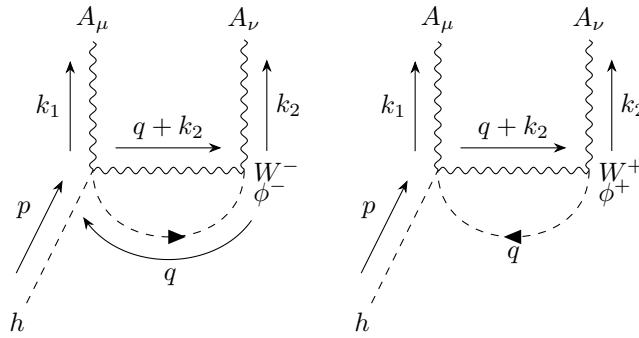


$$\begin{aligned} & -g \frac{m_h^2}{m_W} e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta')^2} \\ & = -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \frac{m_h^2}{m_W^2} \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{4\pi}{\Delta'}\right)^{2-d/2}. \end{aligned}$$

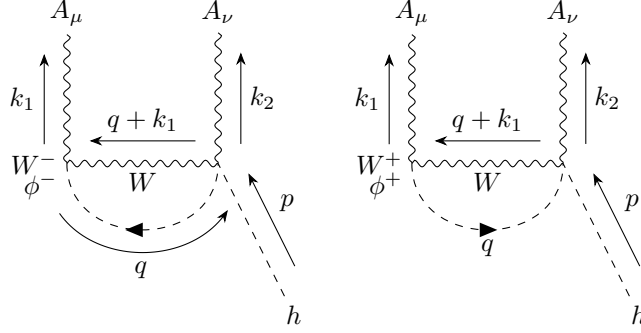
以上から $\Delta' = m_W^2 - x(1-x)m_h^2$ を代入すれば

$$\begin{aligned} i\mathcal{M}_{34} &= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \left[-4 + \left(6 + \frac{m_h^2}{m_W^2}\right) \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{4\pi}{\Delta'}\right)^{2-d/2} \right] \\ &= -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \\ &\quad \times \left[-4 + \left(6 + \frac{m_h^2}{m_W^2}\right) \left[\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log(m_W^2 - x(1-x)m_h^2) \right] \right]. \end{aligned} \quad [\text{III.11}]$$

4 + 3-boson vertex diagrams



$$\begin{aligned} & -2\frac{g}{2} m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta'')^2} \\ & = -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{4\pi}{\Delta''}\right)^{2-d/2} \\ & = -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \left[\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log(m_W^2) \right] \end{aligned}$$



$$\begin{aligned}
& -2\frac{g}{2}m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int \frac{d^d \ell}{(2\pi)^d} \frac{1}{(\ell^2 - \Delta'')^2} \\
& = -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{4\pi}{\Delta''}\right)^{2-d/2} \\
& = -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \left[\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log(m_W^2) \right]
\end{aligned}$$

以上から

$$i\mathcal{M}_{43} = -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) 2 \left[\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log(m_W^2) \right]. \quad [\text{III.12}]$$

[III.10][III.11][III.12] から

$$\begin{aligned}
i\mathcal{M} &= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \frac{(4xy - x - y + 8)m_h^2}{m_W^2 - xym_h^2} - 8 \\
&+ \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \left(16 + \frac{2m_h^2}{m_W^2} \right) \left(\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log m_W^2 \right) \\
&+ \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \left(16 + \frac{2m_h^2}{m_W^2} \right) (-\log(1 - xym_h^2/m_W^2)) \\
&- \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx (-4) \\
&- \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \left(6 + \frac{m_h^2}{m_W^2} \right) \left(\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log m_W^2 \right) \\
&- \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \left(6 + \frac{m_h^2}{m_W^2} \right) [-\log(1 - x(1-x)m_h^2/m_W^2)] \\
&- \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) 2 \left(\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log m_W^2 \right) \\
&= \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \frac{(4xy - x - y + 8)m_h^2}{m_W^2 - xym_h^2} \\
&+ \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \left(8 + \frac{m_h^2}{m_W^2} \right) \left(\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log m_W^2 \right) \\
&+ \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \left(16 + \frac{2m_h^2}{m_W^2} \right) (-\log(1 - xym_h^2/m_W^2)) \\
&- \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \left(6 + \frac{m_h^2}{m_W^2} \right) \left(\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log m_W^2 \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \left(6 + \frac{m_h^2}{m_W^2} \right) [-\log(1 - x(1-x)m_h^2/m_W^2)] \\
& -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) 2 \left(\frac{2}{\varepsilon} - \gamma + \log 4\pi - \log m_W^2 \right) \\
& = \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \frac{(4xy - x - y + 8)m_h^2}{m_W^2 - xy m_h^2} \\
& -\frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \int_0^{1-x} dy \left(16 + \frac{2m_h^2}{m_W^2} \right) \log \left(1 - xy \frac{m_h^2}{m_W^2} \right) \\
& + \frac{i}{(4\pi)^2} g m_W e^2 \epsilon^*(k_1) \cdot \epsilon^*(k_2) \int_0^1 dx \left(6 + \frac{m_h^2}{m_W^2} \right) \log \left(1 - x(1-x) \frac{m_h^2}{m_W^2} \right).
\end{aligned}$$

$m_h \ll m_W$ なら

$$\begin{aligned}
& \int_0^1 dx \int_0^{1-x} dy \frac{(4xy - x - y + 8)m_h^2}{m_W^2 - xy m_h^2} \\
& - \left(16 + \frac{2m_h^2}{m_W^2} \right) \int_0^1 dx \int_0^{1-x} dy \log \left(1 - xy \frac{m_h^2}{m_W^2} \right) \\
& + \left(6 + \frac{m_h^2}{m_W^2} \right) \int_0^1 dx \log \left(1 - x(1-x) \frac{m_h^2}{m_W^2} \right) \\
& \approx \int_0^1 dx \int_0^{1-x} dy (4xy - x - y + 8) \frac{m_h^2}{m_W^2} \\
& + 16 \int_0^1 dx \int_0^{1-x} dy xy \frac{m_h^2}{m_W^2} - 6 \int_0^1 dx x(1-x) \frac{m_h^2}{m_W^2} \\
& = \frac{23}{6} \frac{m_h^2}{m_W^2} + \frac{2}{3} \frac{m_h^2}{m_W^2} - \frac{m_h^2}{m_W^2} \\
& = \frac{7}{2} \frac{m_h^2}{m_W^2}.
\end{aligned}$$