

相対論的量子力学（川村）

使用しているのは第1版第1刷.

第 1 章

Dirac 方程式の導出

小さい一様磁場 \boldsymbol{B} に対し $(\boldsymbol{\nabla} + ie/2\hbar(\boldsymbol{B} \times \boldsymbol{x}))^2 = \boldsymbol{\nabla}^2 + ie/\hbar(\boldsymbol{B} \times \boldsymbol{x}) \cdot \boldsymbol{\nabla}$. ($\boldsymbol{\nabla}(\boldsymbol{B} \times \boldsymbol{x}) = 0$, \boldsymbol{B}^2 は無視) なので, 結局 $\boldsymbol{\nabla}^2 + ie/\hbar \boldsymbol{B} \cdot (\boldsymbol{x} \times \boldsymbol{\nabla})$ で角運動量が出てくる.

第 2 章

Dirac 方程式の Lorentz 共変性

$S^{-1}(\Lambda)\sigma^{\mu\nu}S(\Lambda) = \Lambda^\alpha{}_\mu\Lambda^\beta{}_\nu\sigma^{\mu\nu}$ の証明 : (2.16) と (2.23) から従う.

第 3 章

γ 行列に関する基本定理, カイラル表示

■ Γ_i の性質 (1)~(5) 電子のスピンに関する Pauli 行列は次の式で与えられる：

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

さらに, γ 行列

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

と $\tilde{\Gamma}$ 行列

$$\begin{aligned} \tilde{\Gamma}_1 &= I, & \tilde{\Gamma}_2 &= \gamma^0, & \tilde{\Gamma}_5 &= i\gamma^3, \\ \tilde{\Gamma}_3 &= i\gamma^1, & \tilde{\Gamma}_4 &= i\gamma^2, & \tilde{\Gamma}_8 &= -\gamma^0\gamma^3, \\ \tilde{\Gamma}_6 &= -\gamma^0\gamma^1, & \tilde{\Gamma}_7 &= -\gamma^0\gamma^2, & \tilde{\Gamma}_{11} &= i\gamma^3\gamma^1, \\ \tilde{\Gamma}_9 &= i\gamma^1\gamma^2, & \tilde{\Gamma}_{10} &= i\gamma^2\gamma^3, & \tilde{\Gamma}_{12} &= i\gamma^0\gamma^1\gamma^2\gamma^3, \\ \tilde{\Gamma}_{12} &= i\gamma^0\gamma^1\gamma^2\gamma^3, & \tilde{\Gamma}_{13} &= \gamma^1\gamma^2\gamma^3, & \tilde{\Gamma}_{14} &= -i\gamma^0\gamma^2\gamma^3, \\ \tilde{\Gamma}_{14} &= -i\gamma^0\gamma^2\gamma^3, & \tilde{\Gamma}_{15} &= -i\gamma^0\gamma^1\gamma^3, & \tilde{\Gamma}_{16} &= -i\gamma^0\gamma^1\gamma^2, \end{aligned}$$

を考える. $\tilde{\Gamma}$ 行列について次のような性質が成り立つ：

1. 全ての n に対し $(\tilde{\Gamma}_n)^2 = I$.
2. 全ての n, m に対し $\tilde{\Gamma}_n \tilde{\Gamma}_m = \xi_{nm} \tilde{\Gamma}_l$ となる $l = L_{nm}$ が存在し, $\xi_{nm} \in \{\pm 1, \pm i\}$. $n \neq m$ ならば $L_{nm} \neq 1$. さらに, L_{nm} の各行には $1, \dots, 16$ が 1 回ずつ出現する.
3. $\tilde{\Gamma}_n \tilde{\Gamma}_m = \pm \tilde{\Gamma}_m \tilde{\Gamma}_n$.
4. $n \neq 1$ に対して $\tilde{\Gamma}_n \tilde{\Gamma}_m = -\tilde{\Gamma}_m \tilde{\Gamma}_n$ となる m が存在する.
5. $n \neq 1$ に対して $\text{Tr}(\tilde{\Gamma}_n) = 0$.

python でコード^{*1}を書いて確かめる．まず， ξ は

$$\xi = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -i & -i & -i & +i & +i & +i & -1 & -1 & 1 & +i & -i & -1 & 1 & -1 \\ 1 & +i & 1 & +i & -i & -i & -1 & -1 & -i & -1 & +i & -i & -1 & +i & -1 & -1 \\ 1 & +i & -i & 1 & +i & 1 & -i & -1 & +i & -i & -1 & +i & -1 & -1 & -i & 1 \\ 1 & +i & +i & -i & 1 & 1 & 1 & -i & -1 & +i & -i & -i & -1 & 1 & 1 & +i \\ 1 & -i & +i & 1 & 1 & 1 & +i & -i & -i & -1 & +i & -1 & +i & -i & 1 & 1 \\ 1 & -i & -1 & +i & 1 & -i & 1 & +i & +i & -i & -1 & -1 & -i & 1 & +i & -1 \\ 1 & -i & -1 & -1 & +i & +i & -i & 1 & -1 & +i & -i & -1 & +i & -1 & -1 & -i \\ 1 & -1 & +i & -i & -1 & +i & -i & -1 & 1 & +i & -i & -1 & -1 & -1 & -i & +i \\ 1 & -1 & -1 & +i & -i & -1 & +i & -i & -i & 1 & +i & -i & -1 & -1 & -1 & -i \\ 1 & 1 & -i & -1 & +i & -i & -1 & +i & +i & -i & 1 & -1 & -1 & -1 & -i & 1 \\ 1 & -i & +i & -i & +i & -1 & -1 & -1 & -1 & -1 & 1 & +i & -i & -i & +i & -i \\ 1 & +i & -1 & -1 & -1 & -i & +i & -i & -1 & -1 & -1 & -i & 1 & +i & -i & +i \\ 1 & -1 & -i & -1 & 1 & +i & 1 & -1 & +i & -1 & +i & +i & -i & 1 & -i & -i \\ 1 & 1 & -1 & +i & 1 & 1 & -i & -1 & -i & +i & 1 & -i & +i & +i & 1 & -i \\ 1 & -1 & -1 & 1 & -i & 1 & -1 & +i & -1 & -i & -i & +i & -i & +i & +i & 1 \end{pmatrix}$$

となる． L は

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 2 & 1 & 6 & 7 & 8 & 3 & 4 & 5 & 16 & 14 & 15 & 13 & 12 & 10 & 11 & 9 \\ 3 & 6 & 1 & 9 & 11 & 2 & 16 & 15 & 4 & 13 & 5 & 14 & 10 & 12 & 8 & 7 \\ 4 & 7 & 9 & 1 & 10 & 16 & 2 & 14 & 3 & 5 & 13 & 15 & 11 & 8 & 12 & 6 \\ 5 & 8 & 11 & 10 & 1 & 15 & 14 & 2 & 13 & 4 & 3 & 16 & 9 & 7 & 6 & 12 \\ 6 & 3 & 2 & 16 & 15 & 1 & 9 & 11 & 7 & 12 & 8 & 10 & 14 & 13 & 5 & 4 \\ 7 & 4 & 16 & 2 & 14 & 9 & 1 & 10 & 6 & 8 & 12 & 11 & 15 & 5 & 13 & 3 \\ 8 & 5 & 15 & 14 & 2 & 11 & 10 & 1 & 12 & 7 & 6 & 9 & 16 & 4 & 3 & 13 \\ 9 & 16 & 4 & 3 & 13 & 7 & 6 & 12 & 1 & 11 & 10 & 8 & 5 & 15 & 14 & 2 \\ 10 & 14 & 13 & 5 & 4 & 12 & 8 & 7 & 11 & 1 & 9 & 6 & 3 & 2 & 16 & 15 \\ 11 & 15 & 5 & 13 & 3 & 8 & 12 & 6 & 10 & 9 & 1 & 7 & 4 & 16 & 2 & 14 \\ 12 & 13 & 14 & 15 & 16 & 10 & 11 & 9 & 8 & 6 & 7 & 1 & 2 & 3 & 4 & 5 \\ 13 & 12 & 10 & 11 & 9 & 14 & 15 & 16 & 5 & 3 & 4 & 2 & 1 & 6 & 7 & 8 \\ 14 & 10 & 12 & 8 & 7 & 13 & 5 & 4 & 15 & 2 & 16 & 3 & 6 & 1 & 9 & 11 \\ 15 & 11 & 8 & 12 & 6 & 5 & 13 & 3 & 14 & 16 & 2 & 4 & 7 & 9 & 1 & 10 \\ 16 & 9 & 7 & 6 & 12 & 4 & 3 & 13 & 2 & 15 & 14 & 5 & 8 & 11 & 10 & 1 \end{pmatrix}$$

^{*1} https://github.com/Hogemura/python_calc/blob/master/gamma.py

となる。可換・反可換性については

$$\begin{pmatrix} + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\ + & + & - & - & - & - & - & - & + & + & + & + & - & - & + & + & + \\ + & - & + & - & - & + & - & + & - & + & - & - & + & - & + & - & + \\ + & - & - & - & + & + & + & - & + & - & - & - & + & + & + & - & + \\ + & - & - & + & + & + & - & - & - & + & - & + & - & - & + & - & + \\ + & - & + & - & + & - & + & - & - & - & + & + & - & + & - & + & - \\ + & - & + & + & - & - & - & + & + & - & - & + & - & + & + & - & + \\ + & + & - & - & + & - & - & + & + & - & - & + & + & - & - & - & + \\ + & + & + & - & - & + & - & - & - & + & - & + & + & + & - & - & - \\ + & + & - & + & - & - & + & - & - & - & + & + & + & - & + & - & - \\ + & - & - & - & - & + & + & + & + & + & + & + & - & - & - & - & - \\ + & - & + & + & + & - & - & - & + & + & + & - & + & - & - & - & - \\ + & + & - & + & + & - & + & + & - & + & - & - & - & - & + & - & - \\ + & + & + & - & + & + & - & + & - & - & + & - & - & - & - & + & - \\ + & + & + & + & - & + & + & - & + & - & - & - & - & - & - & - & + \end{pmatrix}$$

となる。

第 4 章

Dirac 方程式の解

■(4.17) 慣性系 I での 4 元運動量は $\varepsilon_s p_\mu$ になってる, として計算すればいい (エネルギーも運動量も負).
 $\gamma^0 S(\Lambda)^\dagger \gamma^0 = S^{-1}(\Lambda)$ (2.45).

■(4.70) (4.6)(4.11) (u と $w(\mathbf{p})$ の対応は (4.28)) から分かる.

■(4.73) (4.70) と同様. $\psi = S(\Lambda)w^s(0)\exp(-i\varepsilon_s p_\mu x^\mu/\hbar)$. 入射粒子の運動量は $\mathbf{p} = (0, 0, \hbar k)$ なので (4.11) から定まる $S(\Lambda)$ の (1, 2) 成分は $c\hbar k\sigma^3/(E + mc^2)$. $w^s(0)$ については今回は正エネルギーの粒子を考えているので (4.73) では $w^1(0)$. (4.74) では反射なので $\mathbf{p} = (0, 0, -\hbar k)$. 第 1 項はスピン正なので $w^1(0)$, 第 2 項はスピン負なので $w^2(0)$.

■(4.81) 確率流れは p.29 の最後らへんから $j^\mu = c\bar{\psi}\gamma^\mu\psi = c\psi^\dagger\gamma^0\gamma^\mu\psi = c\psi^\dagger\gamma^0\gamma^0\alpha^\mu\psi = c\psi^\dagger\alpha^\mu\psi$.
 $b_r = b_t = 0$ なんて結局作用するのは α^3 だけ.

第 5 章

Dirac 方程式の非相対論的極限

■(5.10) (1.10) 使えば

$$\begin{pmatrix} 0 & \sigma^i p_i \\ -\sigma^i p_i & 0 \end{pmatrix}^2 = -|\mathbf{p}|^2 I$$

なので

$$\begin{aligned} U &= \sum_{n=0}^{\infty} \frac{1}{n!} (\beta \boldsymbol{\alpha} \cdot \mathbf{p} \theta)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & \sigma^i p_i \\ -\sigma^i p_i & 0 \end{pmatrix}^n \theta^n \\ &= I \left[1 - \frac{(|\mathbf{p}|\theta)^2}{2!} + \frac{(|\mathbf{p}|\theta)^4}{4!} - \dots \right] + \begin{pmatrix} 0 & \frac{\sigma^i p_i}{|\mathbf{p}|} \\ -\frac{\sigma^i p_i}{|\mathbf{p}|} & 0 \end{pmatrix} \left[\frac{(|\mathbf{p}|\theta)}{1!} - \frac{(|\mathbf{p}|\theta)^3}{3!} + \frac{(|\mathbf{p}|\theta)^5}{5!} - \dots \right] \\ &= \cos(|\mathbf{p}|\theta) + \frac{\beta \boldsymbol{\alpha} \cdot \mathbf{p}}{|\mathbf{p}|} \sin(|\mathbf{p}|\theta). \end{aligned}$$

■(5.11) \cos の方は

$$\cos(|\mathbf{p}|\theta)(c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2) = (c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2) \cos(|\mathbf{p}|\theta).$$

\sin は $\beta(\boldsymbol{\alpha} \cdot \mathbf{p}) = -(\boldsymbol{\alpha} \cdot \mathbf{p})\beta$ 使って

$$\frac{\beta \boldsymbol{\alpha} \cdot \mathbf{p}}{|\mathbf{p}|} \sin(|\mathbf{p}|\theta)(c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2) = -(c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta mc^2) \frac{\beta \boldsymbol{\alpha} \cdot \mathbf{p}}{|\mathbf{p}|} \sin(|\mathbf{p}|\theta)$$

なので (5.11) で $e^{iS}H = -He^{iS}$ になる.

第 6 章

水素原子

■(6.11) 要はスピン角運動量と起動角運動量の合成. 正確に書くと $C \times (1, 0) \otimes Y_{l,m-1/2} + D \times (0, 1) \otimes Y_{l,m+1/2}$. C, D は Clebsch - Gordon の表で求まる. JJSakurai(3.8.64) とか.

■(6.32) $-1/2$ 乗を 3 次まで展開.

■(6.42) 合流型超幾何関数が 1 なので積分とっても楽.

■(6.46) $\partial_1 \partial_2 r^{-1}$ みたいな項が無くなるのは, 1s を考えてるから. つまり, 波動関数が完全球対称なので, xy/r^5 をかけて (まず x で) 積分したら 0 になる.

6.1 の計算詳細

原子番号 Z の原子内に存在する電子の Dirac 方程式は

$$\left(-i\hbar c \boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m_e c^2 - k_0 \frac{Ze^2}{r} \right) \psi = E\psi \quad [6.1]$$

で与えられる. 全角運動量の 2 乗 \mathbf{J}^2 , J_z , 軌道角運動量の 2 乗 \mathbf{L}^2 の固有関数は, $j = l \pm 1/2$ に対応して

$$\varphi_{jm}^{(+)} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}} Y_{j-\frac{1}{2}, m-\frac{1}{2}} \\ \sqrt{\frac{j-m}{2j}} Y_{j-\frac{1}{2}, m-\frac{1}{2}} \end{pmatrix}, \quad \varphi_{jm}^{(-)} = \begin{pmatrix} \sqrt{\frac{j-m+1}{2j+2}} Y_{j+\frac{1}{2}, m-\frac{1}{2}} \\ -\sqrt{\frac{j+m+1}{2j+2}} Y_{j+\frac{1}{2}, m+\frac{1}{2}} \end{pmatrix} \quad [6.2]$$

で与えられ,

$$\varphi_{jm}^{(+)} = \boldsymbol{\sigma} \cdot \hat{\mathbf{r}} \varphi_{jm}^{(-)}, \quad \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$$

が成立する. 球面調和関数 Y_{lm} のパリティは $(-1)^l$ なので, これらの関数のパリティはそれぞれ $(-1)^{j-1/2}$ と $(-1)^{j+1/2}$ で与えられる. 1, 2 成分と 3, 4 成分が異なるパリティを持つスピノルは次の様に構成できる:

$$\psi_{jm}^{j-\frac{1}{2}} = \begin{pmatrix} \frac{iG_{j-1/2,j}(r)}{r} \varphi_{jm}^{(+)} \\ \frac{F_{j-1/2,j}(r)}{r} \varphi_{jm}^{(-)} \end{pmatrix}, \quad \psi_{jm}^{j+\frac{1}{2}} = \begin{pmatrix} \frac{iG_{j+1/2,j}(r)}{r} \varphi_{jm}^{(-)} \\ \frac{F_{j+1/2,j}(r)}{r} \varphi_{jm}^{(+)} \end{pmatrix}. \quad [6.3]$$

$l = j \mp 1/2$ に対し, $\varphi_{jm}^{(l)}$ は $j = l + 1/2$ である $\varphi_{jm}^{(+)}$, $j = l - 1/2$ である $\varphi_{jm}^{(-)}$ を表す様に約束すれば

$$\psi_{jm}^{(l)} = \begin{pmatrix} \frac{iG_{lj}(r)}{r} \varphi_{jm}^{(l)} \\ \frac{F_{lj}(r)}{r} (\boldsymbol{\sigma} \cdot \hat{\mathbf{r}}) \varphi_{jm}^{(l)} \end{pmatrix}. \quad [6.4]$$

$(\boldsymbol{\sigma} \cdot \boldsymbol{a})(\boldsymbol{\sigma} \cdot \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{b})I + i\boldsymbol{\sigma} \cdot (\boldsymbol{a} \times \boldsymbol{b})$ を使えば,

$$\begin{aligned}
(\boldsymbol{\sigma} \cdot \boldsymbol{p})f(r)\varphi_{jm}^{(l)} &= (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}})(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}})(\boldsymbol{\sigma} \cdot \boldsymbol{p})f(r)\varphi_{jm}^{(l)} \\
&= \frac{\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}}{r}(\boldsymbol{r} \cdot \boldsymbol{p} + i\boldsymbol{\sigma} \cdot \boldsymbol{L})f(r)\varphi_{jm}^{(l)} \\
&= -i\hbar \frac{\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}}{r} \left[r \frac{df(r)}{dr} + \left\{ 1 \mp \left(j + \frac{1}{2} \right) \right\} f(r) \right] \varphi_{jm}^{(l)} \\
(\boldsymbol{\sigma} \cdot \boldsymbol{p})(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}})f(r)\varphi_{jm}^{(l)} &= \frac{1}{r}(\boldsymbol{r} \cdot \boldsymbol{p} - i\boldsymbol{\sigma} \cdot \boldsymbol{L})f(r)\varphi_{jm}^{(l)} \\
&= i\hbar \frac{1}{r} \left[r \frac{df(r)}{dr} + \left\{ 1 \pm \left(j + \frac{1}{2} \right) \right\} f(r) \right] \varphi_{jm}^{(l)}
\end{aligned}$$

となるので, [6.4] を [6.1] に代入して

$$\left(\frac{E}{\hbar c} - \frac{m_e c}{\hbar} + \frac{Z\alpha}{r} \right) G_{lj}(r) = -\frac{dF_{lj}(r)}{dr} \mp \left(j + \frac{1}{2} \right) \frac{F_{lj}(r)}{r} \quad [6.5]$$

$$\left(\frac{E}{\hbar c} + \frac{m_e c}{\hbar} + \frac{Z\alpha}{r} \right) F_{lj}(r) = \frac{dG_{lj}(r)}{dr} \mp \left(j + \frac{1}{2} \right) \frac{G_{lj}(r)}{r}. \quad [6.6]$$

ここで,

$$\tilde{\lambda} = \sqrt{\left(\frac{m_e c}{\hbar} \right)^2 - \left(\frac{E}{\hbar c} \right)^2}, \quad \rho = 2\tilde{\lambda}r \quad [6.7]$$

$$G(r) = \sqrt{1 + \frac{E}{m_e c^2}} e^{-\rho/2} (F_1(\rho) + F_2(\rho)) \quad [6.8]$$

$$F(r) = \sqrt{1 - \frac{E}{m_e c^2}} e^{-\rho/2} (F_1(\rho) - F_2(\rho)) \quad [6.9]$$

によって $F_1(\rho)$ と $F_2(\rho)$ を定義する. 添字 l, j は省略する. [6.5] に [6.8][6.9] を代入して

$$\begin{aligned}
&\left(\frac{E}{\hbar c} - \frac{m_e c}{\hbar} + \frac{2\tilde{\lambda}Z\alpha}{\rho} \right) \sqrt{\frac{m_e c^2 + E}{m_e c^2 - E}} e^{-\rho/2} (F_1 + F_2) \\
&= -2\tilde{\lambda} \frac{d}{d\rho} \left[e^{-\rho/2} (F_1 - F_2) \right] \mp \left(j + \frac{1}{2} \right) \frac{2\tilde{\lambda}}{\rho} e^{-\rho/2} (F_1 - F_2) \\
&= -2\tilde{\lambda} \left[-\frac{1}{2} e^{-\rho/2} (F_1 - F_2) + e^{-\rho/2} \left(\frac{dF_1}{d\rho} - \frac{dF_2}{d\rho} \right) \right] \mp \left(j + \frac{1}{2} \right) \frac{2\tilde{\lambda}}{\rho} e^{-\rho/2} (F_1 - F_2)
\end{aligned}$$

となるので,

$$\left(\frac{E}{\hbar c} - \frac{m_e c}{\hbar} + \frac{2\tilde{\lambda}Z\alpha}{\rho} \right) \sqrt{\frac{m_e c^2 + E}{m_e c^2 - E}} (F_1 + F_2) = \tilde{\lambda} (F_1 - F_2) - 2\tilde{\lambda} \left(\frac{dF_1}{d\rho} - \frac{dF_2}{d\rho} \right) \mp \left(j + \frac{1}{2} \right) \frac{2\tilde{\lambda}}{\rho} (F_1 - F_2).$$

これに

$$\sqrt{\frac{m_e c^2 + E}{m_e c^2 - E}} = \frac{\tilde{\lambda}}{m_e c/\hbar - E/\hbar c}$$

を代入して

$$-\rho F_1 + \frac{Z\alpha\tilde{\lambda}}{m_e c/\hbar - E/\hbar c} (F_1 + F_2) = -\rho \left(\frac{dF_1}{d\rho} - \frac{dF_2}{d\rho} \right) \mp \left(j + \frac{1}{2} \right) (F_1 - F_2). \quad [6.10]$$

同様に, [6.6] に [6.8][6.9] を代入して,

$$\left(\frac{E}{\hbar c} + \frac{m_e c}{\hbar} + \frac{2\tilde{\lambda} Z \alpha}{\rho} \right) \sqrt{\frac{m_e c^2 - E}{m_e c^2 + E}} e^{-\rho/2} (F_1 - F_2) = 2\tilde{\lambda} \frac{d}{d\rho} \left[e^{-\rho/2} (F_1 + F_2) \right] \mp \left(j + \frac{1}{2} \right) \frac{2\tilde{\lambda}}{\rho} e^{-\rho/2} (F_1 + F_2)$$

となるので,

$$\tilde{\lambda} (F_1 - F_2) + \frac{2\tilde{\lambda} Z \alpha}{\rho} \frac{\tilde{\lambda}}{m_e c / \hbar - E / \hbar c} (F_1 - F_2) = -\tilde{\lambda} (F_1 + F_2) + 2\tilde{\lambda} + 2\tilde{\lambda} \left(\frac{dF_1}{d\rho} + \frac{dF_2}{d\rho} \right) \mp \left(j + \frac{1}{2} \right) (F_1 + F_2)$$

となり,

$$\rho F_1 + \frac{Z \alpha \tilde{\lambda}}{m_e c / \hbar + E / \hbar c} (F_1 - F_2) = \rho \left(\frac{dF_1}{d\rho} + \frac{dF_2}{d\rho} \right) \mp \left(j + \frac{1}{2} \right) (F_1 + F_2). \quad [6.11]$$

[6.10][6.11] を両辺足し引きして

$$\rho \frac{dF_1}{d\rho} = \left(\rho - \frac{E}{\hbar c} \frac{Z \alpha}{\tilde{\lambda}} \right) F_1 + \left[-\frac{m_e c}{\hbar} \frac{Z \alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] F_2 \quad [6.12]$$

$$\rho \frac{dF_2}{d\rho} = \left[\frac{m_e c}{\hbar} \frac{Z \alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] F_1 + \frac{E}{\hbar c} \frac{Z \alpha}{\tilde{\lambda}} F_2 \quad [6.13]$$

[6.12] を両辺微分して

$$\frac{dF_1}{d\rho} + \rho \frac{d^2 F_1}{d\rho^2} = F_1 + \left(\rho - \frac{E}{\hbar c} \frac{Z \alpha}{\tilde{\lambda}} \right) \frac{dF_1}{d\rho} + \left[-\frac{m_e c}{\hbar} \frac{Z \alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] \frac{dF_2}{d\rho}$$

[6.13] を代入して

$$\begin{aligned} &= F_1 + \left(\rho - \frac{E}{\hbar c} \frac{Z \alpha}{\tilde{\lambda}} \right) \frac{dF_1}{d\rho} + \left[-\frac{m_e c}{\hbar} \frac{Z \alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] \left[\frac{m_e c}{\hbar} \frac{Z \alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] \frac{F_1}{\rho} \\ &\quad + \left[-\frac{m_e c}{\hbar} \frac{Z \alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] \frac{E}{\hbar c} \frac{Z \alpha}{\rho \tilde{\lambda}} F_2 \end{aligned}$$

となり, [6.12] を代入すれば

$$\begin{aligned} &= F_1 + \left(\rho - \frac{E}{\hbar c} \frac{Z \alpha}{\tilde{\lambda}} \right) \frac{dF_1}{d\rho} + \left[\left(j + \frac{1}{2} \right)^2 - \left(\frac{m_e c}{\hbar} \frac{Z \alpha}{\tilde{\lambda}} \right)^2 \right] \frac{F_1}{\rho} + \frac{E}{\hbar c} \frac{Z \alpha}{\rho \tilde{\lambda}} \left[\rho \frac{dF_1}{d\rho} - \left(\rho - \frac{E}{\hbar c} \frac{Z \alpha}{\tilde{\lambda}} \right) F_1 \right] \\ &= F_1 - \frac{E}{\hbar c} \frac{Z \alpha}{\tilde{\lambda}} F_1 + \rho \frac{dF_1}{d\rho} + \left[\left(j + \frac{1}{2} \right)^2 - (Z \alpha)^2 \right] \frac{F_1}{\rho} \end{aligned}$$

となる.

$$\gamma = \sqrt{\left(j + \frac{1}{2} \right)^2 - (Z \alpha)^2} \quad [6.14]$$

を導入すれば

$$\rho \frac{d^2 F_1}{d\rho^2} + (1 - \rho) \frac{dF_1}{d\rho} + \left(\frac{Z \alpha}{c \hbar \tilde{\lambda}} - 1 - \frac{\gamma^2}{\rho} \right) F_1 = 0. \quad [6.15]$$

[6.13] を微分して

$$\frac{dF_2}{d\rho} + \rho \frac{d^2 F_2}{d\rho^2} = \left[\frac{m_e c}{\hbar} \frac{Z \alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] \frac{dF_1}{d\rho} + \frac{E}{\hbar c} \frac{Z \alpha}{\tilde{\lambda}} \frac{dF_2}{d\rho}$$

[6.12] を代入して

$$= \left[\frac{m_e c}{\hbar} \frac{Z\alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] \left(1 - \frac{E}{\hbar c} \frac{Z\alpha}{\rho \tilde{\lambda}} \right) F_1 + \left[\frac{m_e c}{\hbar} \frac{Z\alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] \left[-\frac{m_e c}{\hbar} \frac{Z\alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] \frac{F_2}{\rho} + \frac{E}{\hbar c} \frac{Z\alpha}{\tilde{\lambda}} \frac{dF_2}{d\rho}$$

[6.13] を代入すれば,

$$= \left(1 - \frac{E}{\hbar c} \frac{Z\alpha}{\rho \tilde{\lambda}} \right) \left[\rho \frac{dF_2}{d\rho} - \frac{E}{\hbar c} \frac{Z\alpha}{\tilde{\lambda}} F_2 \right] + \left[\left(j + \frac{1}{2} \right)^2 - \left(\frac{m_e c}{\hbar} \frac{Z\alpha}{\tilde{\lambda}} \right)^2 \right] \frac{F_2}{\rho} + \frac{E}{\hbar c} \frac{Z\alpha}{\tilde{\lambda}} \frac{dF_2}{d\rho} \\ = \rho \frac{dF_2}{d\rho} - \frac{E}{\hbar c} \frac{Z\alpha}{\tilde{\lambda}} F_2 + \left[\left(j + \frac{1}{2} \right)^2 - (Z\alpha)^2 \right] \frac{F_2}{\rho}$$

となるので,

$$\rho \frac{d^2 F_2}{d\rho^2} + (1 - \rho) \frac{dF_2}{d\rho} + \left(\frac{Z\alpha E}{c\hbar \tilde{\lambda}} - \frac{\gamma^2}{\rho} \right) F_2 = 0. \quad [6.16]$$

$F_1 = \rho^\gamma \tilde{F}_1$ において,

$$\frac{d\tilde{F}_1}{d\rho} = \gamma \rho^{\gamma-1} \tilde{F}_1 + \rho^\gamma \frac{d\tilde{F}_1}{d\rho}, \quad \frac{d^2 \tilde{F}_1}{d\rho^2} = \gamma(\gamma-1) \rho^{\gamma-2} \tilde{F}_1 + 2\gamma \rho^{\gamma-1} \frac{d\tilde{F}_1}{d\rho} + \rho^\gamma \frac{d^2 \tilde{F}_1}{d\rho^2}$$

を [6.15] に代入すれば

$$\gamma(\gamma-1) \rho^{\gamma-1} \tilde{F}_1 + 2\gamma \rho^\gamma \frac{d\tilde{F}_1}{d\rho} + \rho^{\gamma+1} \frac{d^2 \tilde{F}_1}{d\rho^2} + (1-\rho) \left(\gamma \rho^{\gamma-1} \tilde{F}_1 + \rho^\gamma \frac{d\tilde{F}_1}{d\rho} \right) + \left(\frac{Z\alpha E}{c\hbar \tilde{\lambda}} - 1 - \frac{\gamma^2}{\rho} \right) \rho^{\gamma+1} \tilde{F}_1$$

となり

$$\rho \frac{d^2 \tilde{F}_1}{d\rho^2} + \{(1+2\gamma) - \rho\} \frac{d\tilde{F}_1}{d\rho} - \left(\gamma + 1 - \frac{Z\alpha E}{c\hbar \tilde{\lambda}} \right) \tilde{F}_1.$$

この方程式の解は合流型超幾何関数で

$$F_1 = A \rho^\gamma F \left(\gamma + 1 - \frac{Z\alpha E}{c\hbar \tilde{\lambda}}, 2\gamma + 1; \rho \right) \quad [6.17]$$

となる. $F_2 = \rho^\gamma \tilde{F}_2$ に [6.16] を代入すれば

$$\rho \frac{d^2 \tilde{F}_2}{d\rho^2} + \{(1+2\gamma) - \rho\} \frac{d\tilde{F}_2}{d\rho} - \left(\gamma - \frac{Z\alpha E}{c\hbar \tilde{\lambda}} \right) \tilde{F}_2.$$

となり

$$F_2 = B \rho^\gamma F \left(\gamma - \frac{Z\alpha E}{c\hbar \tilde{\lambda}}, 2\gamma + 1; \rho \right). \quad [6.18]$$

合流型超幾何関数は

$$F(a, c; z) = 1 + \frac{a}{c} z + \dots$$

となるので, [6.17][6.18] から

$$F_1 = A \rho^\gamma \left[1 + \left(\gamma + 1 - \frac{Z\alpha E}{c\hbar \tilde{\lambda}} \right) \frac{\rho}{2\gamma + 1} + \dots \right], \quad F_2 = B \rho^\gamma \left[1 + \left(\gamma - \frac{Z\alpha E}{c\hbar \tilde{\lambda}} \right) \frac{\rho}{2\gamma + 1} + \dots \right]$$

となる。これを [6.13] に代入して ρ^γ の係数を比較すれば

$$B\gamma = \left[\frac{m_e c}{\hbar} \frac{Z\alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right) \right] A + \frac{E}{\hbar c} \frac{Z\alpha}{\tilde{\lambda}} B \quad \therefore \frac{A}{B} = \frac{\gamma - \frac{E}{\hbar c} \frac{Z\alpha}{\tilde{\lambda}}}{\frac{m_e c}{\hbar} \frac{Z\alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right)}.$$

よって,

$$F_1 = \frac{\gamma - \frac{E}{\hbar c} \frac{Z\alpha}{\tilde{\lambda}}}{\frac{m_e c}{\hbar} \frac{Z\alpha}{\tilde{\lambda}} \pm \left(j + \frac{1}{2} \right)} \rho^\gamma F \left(\gamma + 1 - \frac{Z\alpha E}{c\hbar\tilde{\lambda}}, 2\gamma + 1; \rho \right) \quad [6.19]$$

$$F_2 = \rho^\gamma F \left(\gamma - \frac{Z\alpha E}{c\hbar\tilde{\lambda}}, 2\gamma + 1; \rho \right). \quad [6.20]$$

となる ($B = 1$ として, 後から規格化する). また, 以下で

$$n = \frac{Z\alpha E}{c\hbar\tilde{\lambda}} - \gamma + \left(j + \frac{1}{2} \right) \quad [6.21]$$

を定義する. 無限遠点で波動関数が 0 になる必要がある (p.84 に書いてある) ので

$$E = m_e c^2 \left[1 + \frac{Z^2 \alpha^2}{\left\{ n - j - \frac{1}{2} + \sqrt{(j + \frac{1}{2})^2 - Z^2 \alpha^2} \right\}^2} \right]^{-1/2}. \quad [6.22]$$

$n = 1, j = 1/2, m = 1/2$ の時. [6.14] から $\gamma = \sqrt{1 - Z^2 \alpha^2}$, [6.7] から $\tilde{\lambda} = m_e c Z \alpha / \hbar$, [6.22] から $E = \gamma m_e c^2$. [6.19][6.20] から $F_1 = 0$, $F_2 = \rho^\gamma F(0, 2\gamma + 1; \rho) = \rho^\gamma$. [6.8][6.9] から

$$G_{l,1/2} = \sqrt{1 + \gamma} e^{-\rho/2} \rho^\gamma, \quad F_{l,1/2} = -\sqrt{1 - \gamma} e^{-\rho/2} \rho^\gamma. \quad [6.23]$$

[6.2][6.4] に代入すると 1, 2 成分に関しては $j = l + 1/2$ から $l = 0, 3, 4$ 成分に関しては $j = l - 1/2$ から $l = 1$ となることに注意して,

$$\psi_{j=1/2, m=1/2} = N \begin{pmatrix} \frac{iG_{0,1/2}}{r} Y_{0,0} \\ 0 \\ \frac{F_{1,1/2}}{r} \sqrt{\frac{1}{3}} Y_{1,0} \\ -\frac{F_{1,1/2}}{r} \sqrt{\frac{2}{3}} Y_{1,1} \end{pmatrix}. \quad [6.24]$$

規格化条件は

$$\int r^2 \sin \theta |\psi|^2 dr d\theta d\phi = N^2 \int G^2 + F^2 dr = 1$$

で, [6.23] を代入すれば,

$$\frac{N^2(1 + \gamma)}{2\tilde{\gamma}} \int_0^\infty e^{-\rho} \rho^{2\gamma} d\rho + \frac{N^2(1 - \gamma)}{2\tilde{\gamma}} \int_0^\infty e^{-\rho} \rho^{2\gamma} d\rho = 1$$

となり, ガンマ関数の定義

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

から

$$N = \left(\frac{\tilde{\lambda}}{\Gamma(2\gamma + 1)} \right)^{1/2} = \left(\frac{1}{\Gamma(2\gamma + 1)} \right)^{1/2} \left(\frac{m_e c Z \alpha}{\hbar} \right)^{1/2}.$$

[6.24] に代入して $-i$ をかければ

$$\begin{aligned}\psi_{j=1/2, m=1/2} &= -i \left(\frac{1}{\Gamma(2\gamma+1)} \right)^{1/2} \left(\frac{m_e c Z \alpha}{\hbar} \right)^{1/2} \begin{pmatrix} \frac{i G_{0,1/2}}{r} Y_{0,0} \\ 0 \\ \frac{F_{1,1/2}}{r} \sqrt{\frac{1}{3}} Y_{1,0} \\ -\frac{F_{1,1/2}}{r} \sqrt{\frac{2}{3}} Y_{1,1} \end{pmatrix} \\ &= \frac{1}{\sqrt{4\pi}} \left(\frac{2m_e c Z \alpha}{\hbar} \right)^{3/2} \left(\frac{\gamma+1}{2\Gamma(2\gamma+1)} \right)^{1/2} \left(\frac{2m_e c Z \alpha}{\hbar} r \right)^{\gamma-1} \exp \left(-\frac{m_e c Z \alpha}{\hbar} r \right) \begin{pmatrix} 1 \\ 0 \\ i \frac{1-\gamma}{Z\alpha} \cos \theta \\ i \frac{1-\gamma}{Z\alpha} \sin \theta e^{i\phi} \end{pmatrix}.\end{aligned}$$

$n=1, j=1/2, m=-1/2$ のとき. N は共通で 3, 4 成分の符号が逆なので

$$\begin{aligned}\psi_{j=1/2, m=-1/2} &= \frac{1}{\sqrt{4\pi}} \left(\frac{2m_e c Z \alpha}{\hbar} \right)^{3/2} \left(\frac{\gamma+1}{2\Gamma(2\gamma+1)} \right)^{1/2} \left(\frac{2m_e c Z \alpha}{\hbar} r \right)^{\gamma-1} \exp \left(-\frac{m_e c Z \alpha}{\hbar} r \right) \begin{pmatrix} 0 \\ 1 \\ i \frac{1-\gamma}{Z\alpha} \sin \theta e^{-i\phi} \\ -i \frac{1-\gamma}{Z\alpha} \cos \theta \end{pmatrix}.\end{aligned}$$

第 7 章

空孔理論

■(7.14) $\gamma^0 \not{p}^* = \gamma^0 \gamma^\mu p_\mu^* = \gamma^0 \gamma^0 p_0^* + \sum_{i=1}^3 \gamma^0 \gamma^{i*} p_i^* = p_0^* - \sum_{i=1}^3 \gamma^{i*} p_i^* \gamma^0$. γ^{i*} が γ^i と符号逆なのは $i = 2$ だけ. $\not{p}^T \gamma^0 = \gamma^0 p_0 \gamma^0 + \sum_{i=1}^3 p_i \gamma^{iT} \gamma^0$. γ^{iT} が γ^i と符号逆になるのは $i = 1, 3$. 以上から $\gamma^0 \not{p}^* = \not{p}^T \gamma^0$. 上と同様にして $\gamma^0 \not{s}^* = \not{s}^T \gamma^0$ なので $\gamma^0 (\gamma_5 \not{s})^* = \gamma^0 \gamma_5 \not{s}^* = -\gamma_5 \gamma^0 \not{s}^* = -\gamma_5 \not{s}^T \gamma^0$. γ^μ のうち転置で符号変化するのは $1, 3$ で, C との交換で符号変化するのが $0, 2$ なので $C \not{p}^T = -\not{p} C$

■(7.28)

$$T \not{p}^* = i \gamma^1 \gamma^3 (\gamma^0 p_0 - \gamma_1 p_1 + \gamma^2 p_2 - \gamma^3 p_3) = \gamma_0 p_0 i \gamma^1 \gamma^3 + \gamma^1 p_1 i \gamma^1 \gamma^3 + \gamma^2 p_2 i \gamma^1 \gamma^3 + \gamma^3 p_3 i \gamma^1 \gamma^3 = \not{p}' T.$$

第 9 章

伝播理論 — 相対論的電子 —

■(9.16) 射影演算子を $\sum_{r=1,2} w^r \bar{w}^r$ とするやつ：p.50 の w の完全性と直交性から， $w\bar{w}$ を w で展開した波動関数に作用させれば分かる

第 11 章

Coulomb 散乱

11.2.2 のはじめ 「 $|\mathbf{J}_i| = |\bar{\psi}_i c \gamma \psi_i| \dots$ 」: (4.56) と同じ話, 確率流れは 2.2.3 参照.

■(11.20) の証明 まず,

$$\mathbf{p} \cdot \boldsymbol{\sigma} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}.$$

$r = 1$ なら

$$S(\Lambda)w^1(0) = \sqrt{\frac{E + m_e c^2}{2m_e c^2}} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E + m_e c^2} \\ \frac{c(p_x + ip_y)}{E + m_e c^2} \end{pmatrix}$$

及び

$$\bar{w}^1(0)S^{-1}(\Lambda) = \sqrt{\frac{E + m_e c^2}{2m_e c^2}} \begin{pmatrix} 1, 0, -\frac{cp_z}{E + m_e c^2}, -\frac{c(p_x - ip_y)}{E + m_e c^2} \end{pmatrix}.$$

$r = 2$ なら

$$S(\Lambda)w^2(0) = \sqrt{\frac{E + m_e c^2}{2m_e c^2}} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x - ip_y)}{E + m_e c^2} - \frac{cp_z}{E + m_e c^2} \end{pmatrix}$$

及び

$$\bar{w}^2(0)S^{-1}(\Lambda) = \sqrt{\frac{E + m_e c^2}{2m_e c^2}} \begin{pmatrix} 0, 1, -\frac{c(p_x + ip_y)}{E + m_e c^2}, \frac{cp_z}{E + m_e c^2} \end{pmatrix}.$$

$\sum_{r=1,2} (S(\Lambda)w^r(0))_\beta (\bar{w}^r(0)S^{-1}(\Lambda))_\gamma$ を (β, γ) 成分とする行列は

$$\begin{pmatrix} \frac{E + m_e c^2}{2m_e c^2} & 0 & -\frac{p_z}{2m_e c} & -\frac{p_x - ip_y}{2m_e c} \\ 0 & \frac{E + m_e c^2}{2m_e c^2} & -\frac{p_x + ip_y}{2m_e c} & -\frac{p_z}{2m_e c} \\ \frac{p_z}{2m_e c} & \frac{p_x - ip_y}{2m_e c} & -\frac{p^2}{2m} \frac{1}{E + m_e c^2} & 0 \\ \frac{p_x + ip_y}{2m_e c} & \frac{p_z}{2m_e c} & 0 & -\frac{p^2}{2m} \frac{1}{E + m_e c^2} \end{pmatrix}$$

となる.

$$\not{p} = \gamma^\mu p_\mu = \frac{E}{c} \begin{pmatrix} I & \\ & I \end{pmatrix} - \begin{pmatrix} \mathbf{p} \cdot \boldsymbol{\sigma} & \\ & \mathbf{p} \cdot \boldsymbol{\sigma} \end{pmatrix}$$

なので,

$$\frac{\not{p} + m_e c}{2m_e c} = \begin{pmatrix} \frac{m_e c^2 + E}{2m_e c^2} I & -\frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{2m_e c} \\ \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{2m_e c} & \frac{m_e c^2 - E}{2m_e c^2} I \end{pmatrix}.$$

2つの行列は実際に等しくなる.

■(11.32) (7.6) と (7.10).

■(11.36) Griffiths の p.250 参照. より一般には, $[\bar{u}(a)\Gamma_1 u(b)][\bar{u}(d)\Gamma_2 u(b)]^* = [\bar{u}(a)\Gamma_1 u(b)][\bar{u}(d)\bar{\Gamma}_2 u(c)]$ となる.

第 12 章

Compton 散乱

■(12.6)

$$\begin{aligned}
 E_\mu &= -\frac{\partial A_\mu}{\partial t} \\
 &= \frac{\hbar \varepsilon^\mu}{\sqrt{2EV\varepsilon_0}} \left(-\frac{i}{\hbar} E e^{-iqx/\hbar} + \frac{i}{\hbar} E e^{iqx/\hbar} \right) \\
 &= \sqrt{\frac{2E}{V\varepsilon_0}} \varepsilon^\mu \sin(qx/\hbar)
 \end{aligned}$$

なので,

$$\frac{\varepsilon_0}{2} \mathbf{E}^2 = \frac{E}{V} \varepsilon^2 \sin^2(qx/\hbar) = \frac{E}{V} \sin^2(qx/\hbar).$$

$\mathbf{B} = \partial_i A_j \mathbf{e}_k \varepsilon_{ijk}$ なので,

$$\begin{aligned}
 \frac{B^2}{2\mu_0} &= \frac{1}{2\mu_0} \left(\sum_{ijk} \varepsilon_{ijk} \partial_i A_j \mathbf{e}_k \right) \cdot \left(\sum_{lmn} \varepsilon_{lmn} \partial_l A_m \mathbf{e}_n \right) \\
 &= \frac{1}{2\mu_0} \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} \partial_i A_j \partial_l A_m \delta_{kn} \\
 &= \frac{1}{2\mu_0} \sum_{ijklm} \varepsilon_{ijk} \varepsilon_{lmn} \partial_i A_j \partial_l A_m \\
 &= \frac{1}{2\mu_0} \sum_{ijlm} (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_i A_j \partial_l A_m \\
 &= \frac{1}{2\mu_0} \sum_{ij} \partial_i A_j (\partial_i A_j - \partial_j A_i) \\
 &= \frac{2}{\mu_0} \sum_{ij} \frac{\hbar \varepsilon^j}{\sqrt{2EV\varepsilon_0}} \frac{q_i}{\hbar} \sin(qx/\hbar) \left[\frac{\hbar \varepsilon^j}{\sqrt{2EV\varepsilon_0}} \frac{q_i}{\hbar} \sin(qx/\hbar) - \frac{\hbar \varepsilon^i}{\sqrt{2EV\varepsilon_0}} \frac{q_j}{\hbar} \sin(qx/\hbar) \right] \\
 &= \frac{1}{EV\varepsilon_0\mu_0} \sin^2(qx/\hbar) \sum_{ij} (\varepsilon^j \varepsilon^j q_i q_i - \varepsilon^i \varepsilon^j q_i q_j) \\
 &= \frac{1}{EV\varepsilon_0\mu_0} \sin^2(qx/\hbar) \sum_{ij} (\varepsilon^j \varepsilon^j q_i q_i - \varepsilon^i \varepsilon^j q_i q_j) \\
 &= \frac{c^2}{EV} \sin^2(qx/\hbar) \sum_{ij} (\varepsilon^j \varepsilon^j q_i q_i - \varepsilon^i \varepsilon^j q_i q_j).
 \end{aligned}$$

ここで, $\sum_j \varepsilon^j \varepsilon^j = 1$, $q_\mu \varepsilon^\mu = q_1 \varepsilon^1 + \dots = 0$, $0 = q^\mu q_\mu = q_0 q_0 - q_1 q_2 - \dots$ なので,

$$\begin{aligned} &= \frac{c^2}{EV} \sin^2(qx/\hbar) \sum_{i=0}^3 q_i^2 = \frac{c^2}{EV} \sin^2(qx/\hbar) q_0^2 \\ &= \frac{c^2}{EV} \sin^2(qx/\hbar) \left(\frac{E}{c}\right)^2 = \frac{E}{V} \sin^2(qx/\hbar). \end{aligned}$$

■(12.13)

$$\begin{aligned} \int_0^\infty dp_{f0} \delta(p_f^2 - m_e^2 c^2) &= \int_0^\infty dp_{f0} \delta(p_{f0}^2 - |\mathbf{p}_f|^2 - m_e^2 c^2) \\ &= \int_0^\infty dp_{f0} \delta\left(p_{f0}^2 - \frac{E_f^2}{c^2}\right) = \frac{c}{2E_f}. \end{aligned}$$

■(12.21) (11.19) の計算と同様に (11.20)(11.21)(11.36) を使う.

■(12.30) まずは 3 行目第 1 項の計算から. $q\varepsilon = 0$ なので,

$$\text{Tr}[(\not{p}_i + m_e c) \not{\varepsilon}' \not{q} (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{q}] = -\text{Tr}[(\not{p}_i + m_e c) \not{\varepsilon}' \not{q} \not{\varepsilon}' (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{q}].$$

(H.29) を使って,

$$\begin{aligned} &= -2(\varepsilon' q) \text{Tr}[(\not{p}_i + m_e c) \not{\varepsilon}' (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{q}] + \text{Tr}[(\not{p}_i + m_e c) \not{q} \not{\varepsilon}' \not{q}' (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{q}] \\ &= -2(\varepsilon' q) \text{Tr}[(\not{p}_i + m_e c)(-\not{p}_i + m_e c) \not{\varepsilon}' \not{q}' \not{\varepsilon}' \not{q}] + \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{\varepsilon}' \not{q}' \not{\varepsilon}' \not{q}]. \end{aligned}$$

$(\not{p}_i + m_e c)(-\not{p}_i + m_e c) = -(p_i)^2 + (m_e c)^2 = 0$ なので,

$$\begin{aligned} &= 2(\varepsilon q') \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{\varepsilon}' \not{q}' \not{\varepsilon}] - \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{\varepsilon}' \not{q}' \not{\varepsilon}' \not{q}] \\ &= -2(\varepsilon q') \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{\varepsilon}] \\ &\quad - 2(\varepsilon' q') \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{\varepsilon}' \not{q}] + \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{\varepsilon}' \not{q}]. \end{aligned}$$

$\varepsilon' q' = 0$ なので,

$$\begin{aligned} &= -2(\varepsilon q') \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{\varepsilon}] + \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{\varepsilon}' \not{q}] \\ &= -2(\varepsilon q') \text{Tr}[(\not{p}_i + m_e c) \not{q} \not{\varepsilon} (\not{p}_i - m_e c)] + \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{\varepsilon}' \not{q}]. \end{aligned}$$

p.160 定理 1 を使えば,

$$\begin{aligned} &= -2(\varepsilon q') \text{Tr}[\not{p}_i \not{q} \not{\varepsilon} \not{p}_i] + 2(\varepsilon q')(m_e c)^2 \text{Tr}[\not{q} \not{\varepsilon}] + \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{\varepsilon}' \not{q}] \\ &= -2(\varepsilon q') \text{Tr}[\not{q} \not{\varepsilon} \not{p}_i \not{p}_i] + 2(\varepsilon q')(m_e c)^2 \text{Tr}[\not{q} \not{\varepsilon}] + \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{\varepsilon}' \not{q}] \\ &= -2(\varepsilon q')(m_e c)^2 \text{Tr}[\not{q} \not{\varepsilon}] + 2(\varepsilon q')(m_e c)^2 \text{Tr}[\not{q} \not{\varepsilon}] + \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{\varepsilon}' \not{q}] \\ &= \text{Tr}[(\not{p}_i + m_e c) \not{q} (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{\varepsilon}' \not{q}]. \end{aligned}$$

次に 3 行目第 2 項の計算.

$$\begin{aligned} &\text{Tr}[\not{q} \not{\varepsilon}' \not{q} \not{q}' (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{q}] \\ &= 2(\varepsilon' q) \text{Tr}[\not{\varepsilon} \not{q} (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{q}] - \text{Tr}[\not{\varepsilon}' \not{q} \not{q} \not{q}' (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{q}] \\ &= 2(\varepsilon' q) \text{Tr}[\not{q} (\not{p}_i + m_e c) \not{q}' \not{\varepsilon}' \not{q}] \end{aligned}$$

$$\begin{aligned}
& -2(\varepsilon\varepsilon') \text{Tr}[\not{\varepsilon}'\not{q}\not{q}(\not{p}_i + m_e c)\not{q}'] \\
& + \text{Tr}[\not{\varepsilon}'\not{q}\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] \\
& = -2(\varepsilon'q) \text{Tr}[\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] \\
& - 4(\varepsilon\varepsilon')(\varepsilon q) \text{Tr}[\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{q}'] + 2(\varepsilon\varepsilon') \text{Tr}[\not{\varepsilon}'\not{q}\not{q}(\not{p}_i + m_e c)\not{q}'] \\
& + \text{Tr}[\not{q}\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}'].
\end{aligned}$$

$\not{q}\not{q} = 0$, $\varepsilon q = 0$ なので,

$$\begin{aligned}
& = -2(\varepsilon'q) \text{Tr}[\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] - \text{Tr}[\not{q}\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] \\
& = -2(\varepsilon'q) \text{Tr}[\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] \\
& \quad - 2(\varepsilon q) \text{Tr}[\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] + \text{Tr}[\not{q}\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] \\
& = -2(\varepsilon'q) \text{Tr}[\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] \\
& = -2(\varepsilon'q) \text{Tr}[\not{q}\not{p}_i\not{q}'\not{\varepsilon}'] - 2m_e c(\varepsilon'q) \text{Tr}[\not{q}\not{q}'\not{\varepsilon}'] \\
& = -2(\varepsilon'q) \text{Tr}[\not{q}\not{p}_i\not{q}'\not{\varepsilon}'].
\end{aligned}$$

第3項についても同様に,

$$\begin{aligned}
& \text{Tr}[\not{q}'\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] \\
& = \text{Tr}[\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{q}'] \\
& = 2(\varepsilon'q') \text{Tr}[\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{q}'\not{q}'] - \text{Tr}[\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{\varepsilon}'\not{q}'\not{q}'] \\
& = -2(\varepsilon q') \text{Tr}[\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{\varepsilon}'\not{q}'] + \text{Tr}[\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{\varepsilon}'\not{q}'\not{q}'] \\
& = -4(\varepsilon q')(\varepsilon'q') \text{Tr}[\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)] + 2(\varepsilon q') \text{Tr}[\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] \\
& = 2(\varepsilon q') \text{Tr}[\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] \\
& = -2(\varepsilon q') \text{Tr}[\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{q}'] \\
& = -2(\varepsilon q') \text{Tr}[\not{\varepsilon}'\not{q}\not{p}_i\not{q}'].
\end{aligned}$$

以上から,

$$\begin{aligned}
& \text{Tr}[(\not{p}_f + m_e c)\not{\varepsilon}'\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] \\
& = \text{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}'] - 2(\varepsilon'q) \text{Tr}[\not{q}\not{p}_i\not{q}'\not{\varepsilon}'] + 2(\varepsilon q') \text{Tr}[\not{\varepsilon}'\not{q}\not{p}_i\not{q}'] \\
& = 2(\varepsilon\varepsilon') \text{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] - \text{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}'] \\
& \quad - 2(\varepsilon'q) [4(qp_i)(q'\varepsilon') + 4(q\varepsilon')(p_i q') - 4(qq')(p_i \varepsilon')] \\
& \quad + 2(\varepsilon q') [4(\varepsilon q)(p_i q') + 4(\varepsilon q')(qp_i) - 4(\varepsilon p_i)(qq')] \\
& = 2(\varepsilon\varepsilon') \text{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'] - \text{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'] \\
& \quad - 8(\varepsilon'q)^2(p_i q') + 8(\varepsilon q')^2(p_i q) \\
& = 2(\varepsilon\varepsilon') \text{Tr}[\not{p}_i\not{q}\not{p}_i\not{q}'\not{\varepsilon}'] + 2(\varepsilon\varepsilon')(m_e c)^2 \text{Tr}[\not{q}\not{q}'\not{\varepsilon}'] \\
& \quad - \text{Tr}[\not{p}_i\not{q}\not{p}_i\not{q}'] - (m_e c)^2 \text{Tr}[\not{q}\not{q}'] \\
& \quad - 8(\varepsilon'q)^2(p_i q') + 8(\varepsilon q')^2(p_i q) \\
& = 2(\varepsilon\varepsilon') \text{Tr}[\not{p}_i\not{q}\not{p}_i\not{q}'\not{\varepsilon}'] \\
& \quad + 2(\varepsilon\varepsilon')(m_e c)^2 [4(qq')(\varepsilon\varepsilon') + 4(q\varepsilon)(q'\varepsilon') - 4(q\varepsilon')(q'\varepsilon)] \\
& \quad - 4[2(p_i q)(p_i q') - (p_i)^2(qq')] - (m_e c)^2 4(qq') \\
& \quad - 8(\varepsilon'q)^2(p_i q') + 8(\varepsilon q')^2(p_i q)
\end{aligned}$$

$$\begin{aligned}
&= 2(\varepsilon\varepsilon') \text{Tr}[\not{p}_i \not{q} \not{p}_i \not{q}' \not{\varepsilon}' \not{\varepsilon}] \\
&\quad + 8(m_e c)^2 (\varepsilon\varepsilon')^2 (qq') - 8(m_e c)^2 (\varepsilon\varepsilon') (q\varepsilon') (q'\varepsilon) \\
&\quad - 8(p_i q) (p_i q') - 8(\varepsilon' q)^2 (p_i q') + 8(\varepsilon q')^2 (p_i q) \\
&= 2(\varepsilon\varepsilon') 4[(\varepsilon\varepsilon') (p_i q) (p_i q') - (m_e c)^2 (qq') (\varepsilon\varepsilon') + (m_e c)^2 (q\varepsilon') (q'\varepsilon) + (\varepsilon\varepsilon') (p_i q') (p_i q)] \\
&\quad + 8(m_e c)^2 (\varepsilon\varepsilon')^2 (qq') - 8(m_e c)^2 (\varepsilon\varepsilon') (q\varepsilon') (q'\varepsilon) \\
&\quad - 8(p_i q) (p_i q') - 8(\varepsilon' q)^2 (p_i q') + 8(\varepsilon q')^2 (p_i q) \\
&= 8(p_i q) (p_i q') [2(\varepsilon\varepsilon')^2 - 1] - 8(\varepsilon' q)^2 (p_i q') + 8(\varepsilon q')^2 (p_i q).
\end{aligned}$$

電子・陽電子の対消滅の詳しい計算

対消滅過程の S 行列要素は (12.1) と同様に

$$S_{fi} = -i \left(\frac{e}{\hbar} \right)^2 \int d^4 x \int d^4 y \bar{\psi}_f(y) [A(y; q_2) S_F(y; x) A(x; q_1) + A(y; q_1) S_F(y; x) A(x; q_2)] \psi_i(x). \quad [12.1]$$

始状態は運動量 p_- の電子：

$$\psi_i(x) = \sqrt{\frac{m_e c^2}{E_- V}} u(p_-, s_-) e^{-\frac{i}{\hbar} p_- x}. \quad [12.2]$$

終状態は運動量 p_+ の陽電子：

$$\bar{\psi}_f(y) = \sqrt{\frac{m_e c^2}{E_+ V}} \bar{v}(p_+, s_+) e^{-\frac{i}{\hbar} p_+ y}. \quad [12.3]$$

光子の平面波は

$$\begin{aligned}
A^\mu(x; q_1) &= \frac{\hbar \varepsilon_1^\mu}{\sqrt{2E_1 V \varepsilon_0}} (e^{-\frac{i}{\hbar} q_1 x} + e^{\frac{i}{\hbar} q_1 x}), \quad E_1 = q_1^0 c, \\
A^\mu(y; q_2) &= \frac{\hbar \varepsilon_2^\mu}{\sqrt{2E_2 V \varepsilon_0}} (e^{-\frac{i}{\hbar} q_2 y} + e^{\frac{i}{\hbar} q_2 y}), \quad E_2 = q_2^0 c.
\end{aligned} \quad [12.4]$$

Lorenz ゲージを採用する：

$$(q_1)^2 = (q_2)^2 = 0, \quad q_1 \varepsilon_1 = q_2 \varepsilon_2 = 0. \quad [12.5]$$

Feynman 伝播関数は

$$S_F(x'; x) = \int \frac{d^4 p}{(2\pi \hbar)^4} e^{-\frac{i}{\hbar} p(x' - x)} \frac{\hbar}{\not{p} - m_e c} \quad [12.6]$$

で与えられる。これらを [12.1] に代入して,

$$\begin{aligned}
S_{fi} = & -i \left(\frac{e}{\hbar} \right)^2 \frac{\hbar^2}{c\varepsilon_0} \int d^4x \int d^4y \frac{m_e c^2}{V^2} \frac{1}{\sqrt{E_- E_+}} \frac{1}{\sqrt{2q_1^0 q_2^0}} \\
& \times \bar{v}(p_+, s_+) \not{\epsilon}_2 \int \frac{d^4p}{(2\pi\hbar)^4} \frac{\hbar}{\not{p} - m_e c} e^{-\frac{i}{\hbar} p(y-x)} \not{\epsilon}_1 u(p_-, s_-) \\
& \times e^{-\frac{i}{\hbar} p_+ y} (e^{-\frac{i}{\hbar} q_2 y} + e^{\frac{i}{\hbar} q_2 y}) (e^{-\frac{i}{\hbar} q_1 x} + e^{\frac{i}{\hbar} q_1 x}) e^{-\frac{i}{\hbar} p_- x} \\
& - i \left(\frac{e}{\hbar} \right)^2 \frac{\hbar^2}{c\varepsilon_0} \int d^4x \int d^4y \frac{m_e c^2}{V^2} \frac{1}{\sqrt{E_- E_+}} \frac{1}{\sqrt{2q_1^0 q_2^0}} \\
& \times \bar{v}(p_+, s_+) \not{\epsilon}_1 \int \frac{d^4p}{(2\pi\hbar)^4} \frac{\hbar}{\not{p} - m_e c} e^{-\frac{i}{\hbar} p(y-x)} \not{\epsilon}_2 u(p_-, s_-) \\
& \times e^{-\frac{i}{\hbar} p_+ y} (e^{-\frac{i}{\hbar} q_1 y} + e^{\frac{i}{\hbar} q_1 y}) (e^{-\frac{i}{\hbar} q_2 x} + e^{\frac{i}{\hbar} q_2 x}) e^{-\frac{i}{\hbar} p_- x}.
\end{aligned} \tag{12.7}$$

運動量保存 $p_- + p_+ = q_1 + q_2$ を考慮して, (11.7) を使えば

$$S_{fi} = \left(\frac{e}{\hbar} \right)^2 \frac{\hbar^2}{c\varepsilon_0} \frac{m_e c^2}{V^2} \frac{1}{\sqrt{E_+ E_-}} \frac{1}{\sqrt{2q_1^0 q_2^0}} (2\pi\hbar)^4 \delta^4(-p_+ + q_2 - p_- + q_1) M_{fi} \tag{12.8}$$

$$M_{fi} = \bar{v}(p_+, s_+) \left[(-i\not{\epsilon}_2) \frac{i\hbar}{\not{p}_- - \not{q}_1 - m_e c} (-i\not{\epsilon}_1) + (-i\not{\epsilon}_1) \frac{i\hbar}{\not{p}_- - \not{q}_2 - m_e c} (-i\not{\epsilon}_2) \right] u(p_-, s_-) \tag{12.9}$$

となる。散乱断面積は [12.8] から

$$\begin{aligned}
d\sigma &= \frac{|S_{fi}|^2}{v_+/V} \frac{V d^3 q_1}{(2\pi\hbar)^3} \frac{V d^3 q_2}{(2\pi\hbar)^3} \frac{1}{T} \\
&= \frac{e^4 m_e^2 c^5}{(c\varepsilon_0)^2 (2\pi\hbar)^2 E_+ E_- v_+} \delta^4(-p_+ + q_2 - p_- + q_1) |M_{fi}|^2 \frac{d^3 q_1}{2q_1^0} \frac{d^3 q_2}{2q_2^0}
\end{aligned} \tag{12.10}$$

となる ($v_+ E_+ = p_+ c^2$).

はじめに電子が静止している系とする:

$$p_- = (m_e c, 0) \tag{12.11}$$

とする。さらに,

$$\varepsilon_1 p_- = \varepsilon_2 p_- = 0, \quad \varepsilon_1 = (0, \boldsymbol{\varepsilon}_1), \quad \varepsilon_2 = (0, \boldsymbol{\varepsilon}_2), \quad \boldsymbol{\varepsilon}_1 \cdot \mathbf{q}_1 = \boldsymbol{\varepsilon}_2 \cdot \mathbf{q}_2 = 0 \tag{12.12}$$

とする。Dirac 方程式から

$$(\not{p}_- + m_e c) \not{\epsilon}_1 u(p_-, s_-) = \not{\epsilon}_1 (\not{p}_- - m_e c) u(p_-, s_-) = 0 \tag{12.13}$$

$$(\not{p}_- + m_e c) \not{\epsilon}_2 u(p_-, s_-) = \not{\epsilon}_2 (\not{p}_- - m_e c) u(p_-, s_-) = 0 \tag{12.14}$$

なので, 不変振幅 [12.9] は

$$M_{fi} = -i\hbar \bar{v}(p_+, s_+) \left[\frac{\not{\epsilon}_2 \not{q}_1 \not{\epsilon}_1}{2q_1 p_-} + \frac{\not{\epsilon}_1 \not{q}_2 \not{\epsilon}_2}{2q_2 p_-} \right] u(p_-, s_-) \tag{12.15}$$

[12.15] を [12.10] に代入して,

$$\begin{aligned}
d\bar{\sigma} &= \frac{1}{4} \sum_{\pm s_-, \pm s_+} d\sigma \\
&= \frac{e^4}{(2\pi\hbar)^2} \frac{\hbar^2}{(c\varepsilon_0)^2} \frac{m_e^2 c^5}{E_- E_+ v_+} \frac{1}{4} \delta^4(-p_+ + q_2 - p_- + q_1) \\
&\quad \times \sum_{\pm s_-, \pm s_+} \left| \bar{v}(p_+, s_+) \left[\frac{\not{\epsilon}_2 \not{q}_1 \not{\epsilon}_1}{2q_1 p_-} + \frac{\not{\epsilon}_1 \not{q}_2 \not{\epsilon}_2}{2q_2 p_-} \right] u(p_-, s_-) \right|^2 \frac{d^3 q_1}{2|\mathbf{q}_1|} \frac{d^3 q_2}{2|\mathbf{q}_2|}.
\end{aligned} \tag{12.16}$$

和の部分を実算する. Griffiths(7.99) から (11.20)(11.21) と同等の式

$$\sum_{\pm s_-} u_\beta(p_-, s_-) \bar{u}_\gamma(p_-, s_-) = \left(\frac{\not{p}_- + m_e c}{2m_e c} \right)_{\beta\gamma} \tag{12.17}$$

$$\sum_{\pm s_+} v_\delta(p_+, s_+) \bar{v}_\alpha(p_+, s_+) = \left(\frac{\not{p}_+ - m_e c}{2m_e c} \right)_{\delta\alpha} \tag{12.18}$$

と (11.36) を使って, (11.19) と同様に計算すれば

$$\begin{aligned}
&\sum_{\alpha\beta\gamma\delta} \sum_{\pm s_-, \pm s_+} \bar{v}_\alpha(p_+, s_+) \Gamma_{\alpha\beta} u_\beta(p_-, s_-) \bar{u}_\gamma(p_-, s_-) \bar{\Gamma}_{\gamma\delta} v_\delta(p_+, s_+) \\
&= \sum_{\alpha\beta\gamma\delta} \left(\frac{\not{p}_+ - m_e c}{2m_e c} \right)_{\delta\alpha} \Gamma_{\alpha\beta} \left(\frac{\not{p}_- + m_e c}{2m_e c} \right)_{\beta\gamma} \bar{\Gamma}_{\gamma\delta} \\
&= \text{Tr} \left(\frac{\not{p}_+ - m_e c}{2m_e c} \Gamma \frac{\not{p}_- + m_e c}{2m_e c} \bar{\Gamma} \right).
\end{aligned} \tag{12.19}$$

(11.37) から

$$\bar{\Gamma} = \frac{\not{\epsilon}_1 \not{q}_1 \not{\epsilon}_2}{2q_1 p_-} + \frac{\not{\epsilon}_2 \not{q}_2 \not{\epsilon}_1}{2q_2 p_-} \tag{12.20}$$

なので, [12.16] は

$$\begin{aligned}
d\bar{\sigma} &= \frac{e^4}{(2\pi\hbar)^2} \frac{\hbar^2}{(c\varepsilon_0)^2} \frac{m_e^2 c^5}{E_- E_+ v_+} \frac{1}{4} \text{Tr} \left[\frac{\not{p}_+ - m_e c}{2m_e c} \left(\frac{\not{\epsilon}_2 \not{q}_1 \not{\epsilon}_1}{2q_1 p_-} + \frac{\not{\epsilon}_1 \not{q}_2 \not{\epsilon}_2}{2q_2 p_-} \right) \frac{\not{p}_- + m_e c}{2m_e c} \left(\frac{\not{\epsilon}_1 \not{q}_1 \not{\epsilon}_2}{2q_1 p_-} + \frac{\not{\epsilon}_2 \not{q}_2 \not{\epsilon}_1}{2q_2 p_-} \right) \right] \\
&\quad \times \delta^4(-p_+ + q_2 - p_- + q_1) \frac{d^3 q_1}{2|\mathbf{q}_1|} \frac{d^3 q_2}{2|\mathbf{q}_2|} \\
&= \frac{-e^4}{(2\pi\hbar)^2} \frac{\hbar^2}{(c\varepsilon_0)^2} \frac{m_e^2 c^5}{E_- E_+ v_+} \frac{1}{4} \text{Tr} \left[\frac{-\not{p}_+ + m_e c}{2m_e c} \left(\frac{\not{\epsilon}_2 \not{\epsilon}_1 \not{q}_1}{2q_1 p_-} + \frac{\not{\epsilon}_1 \not{\epsilon}_2 \not{q}_2}{2q_2 p_-} \right) \frac{\not{p}_- + m_e c}{2m_e c} \left(\frac{\not{q}_1 \not{\epsilon}_1 \not{\epsilon}_2}{2q_1 p_-} + \frac{\not{q}_2 \not{\epsilon}_2 \not{\epsilon}_1}{2q_2 p_-} \right) \right] \\
&\quad \times \delta^4(-p_+ + q_2 - p_- + q_1) \frac{d^3 q_1}{2|\mathbf{q}_1|} \frac{d^3 q_2}{2|\mathbf{q}_2|}.
\end{aligned} \tag{12.21}$$

これで (12.43) が導出できた.

[12.21] のトレースを実算する. (12.21) と見比べれば,

$$p_i \leftrightarrow p_- \quad p_f \leftrightarrow -p_+ \quad \varepsilon \leftrightarrow \varepsilon_1 \quad \varepsilon' \leftrightarrow \varepsilon_2 \quad q \leftrightarrow q_1 \quad q' \leftrightarrow q_2 \tag{12.22}$$

の対応があることが分かる. まず, (12.28) と同等の式を実算する:

$$\begin{aligned}
\text{Tr}[(-\not{p}_+ + m_e c) \not{\epsilon}_2 \not{\epsilon}_1 \not{q}_1 (\not{p}_- + m_e c) \not{q}_1 \not{\epsilon}_1 \not{\epsilon}_2] &= -8(q_1 p_-)[(q_1 p_+) + 2(q_1 \varepsilon_2)(p_+ \varepsilon_2)] \\
&= -8(q_1 p_-)[(q_2 p_-) + 2(q_1 \varepsilon_2)^2].
\end{aligned} \tag{12.23}$$

次に (12.29) :

$$\begin{aligned}\text{Tr}[(-\not{p}_+ + m_e c)\not{\epsilon}_1\not{\epsilon}_2\not{q}_2(\not{p}_- + m_e c)\not{q}_2\not{\epsilon}_2\not{\epsilon}_1] &= -8(q_2 p_-)[(q_2 p_+) + 2(q_2 \varepsilon_1)(p_+ \varepsilon_1)] \\ &= -8(q_2 p_-)[(q_1 p_-) + 2(q_2 \varepsilon_1)^2].\end{aligned}\quad [12.24]$$

次に (12.30) :

$$\begin{aligned}\text{Tr}[(-\not{p}_+ + m_e c)\not{\epsilon}_2\not{\epsilon}_1\not{q}_1(\not{p}_- + m_e c)\not{q}_2\not{\epsilon}_2\not{\epsilon}_1] \\ &= \text{Tr}[(\not{p}_- - \not{q}_1 - \not{q}_2 + m_e c)\not{\epsilon}_2\not{\epsilon}_1\not{q}_1(\not{p}_- + m_e c)\not{q}_2\not{\epsilon}_2\not{\epsilon}_1] \\ &= \text{Tr}[(\not{p}_- + m_e c)\not{\epsilon}_2\not{\epsilon}_1\not{q}_1(\not{p}_- + m_e c)\not{q}_2\not{\epsilon}_2\not{\epsilon}_1] - \text{Tr}[(\not{q}_1 + \not{q}_2)\not{\epsilon}_2\not{\epsilon}_1\not{q}_1(\not{p}_- + m_e c)\not{q}_2\not{\epsilon}_2\not{\epsilon}_1] \\ &= 8(p_- q_1)(p_- q_2)[2(\varepsilon_1 \varepsilon_2)^2 - 1] + 8(\varepsilon_2 q_1)^2(p_- q_2) + 8(\varepsilon_1 q_2)^2(p_- q_1).\end{aligned}\quad [12.25]$$

同様に, T_4 に対応する部分はこれと等しい. よって, トレースは

$$\begin{aligned}\text{Tr}\left[\frac{-\not{p}_+ + m_e c}{2m_e c}\left(\frac{\not{\epsilon}_2\not{\epsilon}_1\not{q}_1}{2q_1 p_-} + \frac{\not{\epsilon}_1\not{\epsilon}_2\not{q}_2}{2q_2 p_-}\right)\frac{\not{p}_- + m_e c}{2m_e c}\left(\frac{\not{q}_1\not{\epsilon}_1\not{\epsilon}_2}{2q_1 p_-} + \frac{\not{q}_2\not{\epsilon}_2\not{\epsilon}_1}{2q_2 p_-}\right)\right] \\ &= \frac{1}{4m_e^2 c^2}\left[\frac{-8(q_1 p_-)}{4(q_1 p_-)^2}\{(q_2 p_-) + 2(q_1 \varepsilon_2)^2\} + \frac{-8(q_2 p_-)}{4(q_2 p_-)^2}\{(q_1 p_-) + 2(q_2 \varepsilon_1)^2\}\right] \\ &\quad + \frac{1}{4m_e^2 c^2}\frac{2}{4(q_1 p_-)(q_2 p_-)}[8(p_- q_1)(p_- q_2)\{2(\varepsilon_1 \varepsilon_2)^2 - 1\} + 8(\varepsilon_2 q_1)^2(p_- q_2) + 8(\varepsilon_1 q_2)^2(p_- q_1)] \\ &= \frac{1}{4m_e^2 c^2}\left[-2\frac{(q_2 p_-)}{(q_1 p_-)} - 4\frac{(q_1 \varepsilon_2)^2}{(q_1 p_-)} - 2\frac{(q_1 p_-)}{(q_2 p_-)} - 4\frac{(q_2 \varepsilon_1)^2}{(q_2 p_-)} + 4\{2(\varepsilon_1 \varepsilon_2)^2 - 1\} + 4\frac{(q_1 \varepsilon_2)^2}{(q_1 p_-)} + 4\frac{(q_2 \varepsilon_1)^2}{(q_2 p_-)}\right] \\ &= -\frac{1}{2m_e^2 c^2}\left[\frac{|\mathbf{q}_2|}{|\mathbf{q}_1|} + \frac{|\mathbf{q}_1|}{|\mathbf{q}_2|} - 4(\varepsilon_1 \varepsilon_2)^2 + 2\right].\end{aligned}\quad [12.26]$$

これを [12.21] に代入して,

$$\begin{aligned}d\bar{\sigma} &= \frac{e^4}{(2\pi\hbar)^2} \frac{\hbar^2}{(c\varepsilon_0)^2} \frac{m_e^2 c^5}{E_- E_+ v_+} \frac{1}{4} \frac{1}{2m_e^2 c^2} \left[\frac{|\mathbf{q}_2|}{|\mathbf{q}_1|} + \frac{|\mathbf{q}_1|}{|\mathbf{q}_2|} - 4(\varepsilon_1 \varepsilon_2)^2 + 2\right] \delta^4(-p_+ + q_2 - p_- + q_1) \frac{d^3 q_1}{2|\mathbf{q}_1|} \frac{d^3 q_2}{2|\mathbf{q}_2|} \\ &= \frac{\alpha^2 \hbar^2}{2m_e c |\mathbf{p}_+|} \left[\frac{|\mathbf{q}_2|}{|\mathbf{q}_1|} + \frac{|\mathbf{q}_1|}{|\mathbf{q}_2|} - 4(\varepsilon_1 \varepsilon_2)^2 + 2\right] \delta^4(-p_+ + q_2 - p_- + q_1) \frac{d^3 q_1}{2|\mathbf{q}_1|} \frac{d^3 q_2}{2|\mathbf{q}_2|}.\end{aligned}\quad [12.27]$$

[12.27] の積分を計算する.

$$\begin{aligned}&\int \delta^4(-p_+ + q_2 - p_- + q_1) \frac{d^3 q_1}{2|\mathbf{q}_1|} \frac{d^3 q_2}{2|\mathbf{q}_2|} \\ &= \int \delta^3(-\mathbf{p}_+ + \mathbf{q}_2 + \mathbf{q}_1) \delta\left(-\frac{E_+}{c} + |\mathbf{q}_2| - m_e c + |\mathbf{q}_1|\right) \frac{d^3 q_1}{2|\mathbf{q}_1|} \frac{d^3 q_2}{2|\mathbf{q}_2|} \\ &= \int \delta\left(-\frac{E_+}{c} + |\mathbf{p}_+ - \mathbf{q}_1| - m_e c + |\mathbf{q}_1|\right) \frac{d^3 q_1}{4|\mathbf{q}_1||\mathbf{p}_+ - \mathbf{q}_1|} \\ &= \int \delta\left(-\frac{E_+}{c} + \sqrt{|\mathbf{p}_+|^2 + |\mathbf{q}_1|^2 - 2|\mathbf{p}_+||\mathbf{q}_1|\cos\theta} - m_e c + |\mathbf{q}_1|\right) \frac{d^3 q_1}{4|\mathbf{q}_1||\mathbf{p}_+ - \mathbf{q}_1|} \\ &= \int \delta\left(-\frac{E_+}{c} + \sqrt{|\mathbf{p}_+|^2 + |\mathbf{q}_1|^2 - 2|\mathbf{p}_+||\mathbf{q}_1|\cos\theta} - m_e c + |\mathbf{q}_1|\right) \frac{|\mathbf{q}_1|}{4|\mathbf{p}_+ - \mathbf{q}_1|} d\Omega_{q_1}\end{aligned}\quad [12.28]$$

なので, $\mathbf{q}_2 = \mathbf{p}_+ - \mathbf{q}_1$. デルタ関数の引数を $f(|\mathbf{q}_1|)$ とする. $E_+^2 = c^2|\mathbf{p}_+|^2 + m_e^2 c^2$ に注意して, $f = 0$ となるのは,

$$\begin{aligned} |\mathbf{q}_1| &= \frac{(E_+/c + m_e c)^2 - |\mathbf{p}_+|^2}{2(E_+/c + m_e c - |\mathbf{p}_+| \cos \theta)} \\ &= \frac{m_e c(m_e c^2 + E_+)}{(E_+ + m_e c^2 - c|\mathbf{p}_+| \cos \theta)}. \end{aligned} \quad [12.29]$$

このとき,

$$\begin{aligned} |\mathbf{q}_2| &= |\mathbf{p}_+ - \mathbf{q}_1| \\ &= \frac{E_+}{c} + m_e c - |\mathbf{q}_1| \\ &= \frac{E_+ - c|\mathbf{p}_+| \cos \theta}{m_e c^2} |\mathbf{q}_1| \\ &= \frac{E_+ - c|\mathbf{p}_+| \cos \theta}{m_e c^2} \frac{m_e c(m_e c^2 + E_+)}{(E_+ + m_e c^2 - c|\mathbf{p}_+| \cos \theta)}. \end{aligned} \quad [12.30]$$

次に, $\delta(f(|\mathbf{q}_1|))$ を計算する.

$$f'(|\mathbf{q}_1|) = \frac{|\mathbf{q}_1| - |\mathbf{p}_+| \cos \theta + |\mathbf{p}_+ - \mathbf{q}_1|}{|\mathbf{p}_+ - \mathbf{q}_1|}$$

なので,

$$f' \left(\frac{m_e c(m_e c^2 + E_+)}{(E_+ + m_e c^2 - c|\mathbf{p}_+| \cos \theta)} \right) = \frac{E_+ + m_e c^2 - c|\mathbf{p}_+| \cos \theta}{c|\mathbf{q}_2|}.$$

従って,

$$\delta(f(|\mathbf{q}_1|)) = \frac{c|\mathbf{q}_2|}{E_+ + m_e c^2 - c|\mathbf{p}_+| \cos \theta} \delta \left(|\mathbf{q}_1| - \frac{m_e c(m_e c^2 + E_+)}{(E_+ + m_e c^2 - c|\mathbf{p}_+| \cos \theta)} \right) \quad [12.31]$$

これらを [12.28] に代入して,

$$\int \delta^4(-p_+ + q_2 - p_- + q_1) \frac{d^3 q_1}{2|\mathbf{q}_1|} \frac{d^3 q_2}{2|\mathbf{q}_2|} = \frac{1}{4} \frac{m_e c^2(m_e c^2 + E_+)}{(m_e c^2 + E_+ - c|\mathbf{p}_+| \cos \theta)^2} d\Omega_{q_1}. \quad [12.32]$$

[12.27] に代入して,

$$\begin{aligned} \frac{d\bar{\sigma}}{d\Omega_{q_1}} &= \frac{\alpha^2 \hbar^2}{2m_e c|\mathbf{p}_+|} \left[\frac{|\mathbf{q}_2|}{|\mathbf{q}_1|} + \frac{|\mathbf{q}_1|}{|\mathbf{q}_2|} - 4(\varepsilon_1 \varepsilon_2)^2 + 2 \right] \frac{1}{4} \frac{m_e c^2(m_e c^2 + E_+)}{(m_e c^2 + E_+ - c|\mathbf{p}_+| \cos \theta)^2} \\ &= \frac{\hbar^2 \alpha^2 c(m_e c^2 + E_+)}{8|\mathbf{p}_+|(m_e c^2 + E_+ - c|\mathbf{p}_+| \cos \theta)^2} \left[\frac{|\mathbf{q}_2|}{|\mathbf{q}_1|} + \frac{|\mathbf{q}_1|}{|\mathbf{q}_2|} - 4(\varepsilon_1 \varepsilon_2)^2 + 2 \right] \\ &= \frac{\hbar^2 \alpha^2 c(m_e c^2 + E_+)}{8|\mathbf{p}_+|(m_e c^2 + E_+ - c|\mathbf{p}_+| \cos \theta)^2} \left[\frac{E_+ - c|\mathbf{p}_+| \cos \theta}{m_e c^2} + \frac{m_e c^2}{E_+ - c|\mathbf{p}_+| \cos \theta} - 4(\varepsilon_1 \varepsilon_2)^2 + 2 \right]. \end{aligned} \quad [12.33]$$

$|\mathbf{q}_1|$, $|\mathbf{q}_2|$ は [12.29][12.30] で与えられる.

最後に, 全断面積を求める. (12.34) 以降の手続きと同様にすれば良いが, いくつか注意点:

- 2つの光子の偏極について和を取る (Compton 散乱の場合は始状態の光子は平均を取って, 終状態は和を取った), つまり (12.34), (12.35) は2倍になる.

- ε_1 と ε_2 のなす角 ϕ は散乱角 θ と異なる.
- 出てくる光子は区別できないので, 計算結果を $1/2$ する必要がある.

特に, 2 点目については,

$$\begin{aligned}\sin^2 \phi &= \left(\frac{|\mathbf{p}_+|}{|\mathbf{q}_2|} \right) \sin^2 \theta \\ &= \left(\frac{m_e c^2 + E_+ - c|\mathbf{p}_+| \cos \theta}{E_+ - c|\mathbf{p}_+| \cos \theta} \right)^2 \left(\frac{c|\mathbf{p}_+|}{E_+ + m_e c^2} \right)^2 \sin^2 \theta\end{aligned}\quad [12.34]$$

となる. まず, 光子の偏極に関して平均を取れば,

$$\begin{aligned}\frac{d\bar{\sigma}}{d\Omega_{q_1}} &= \frac{\hbar^2 \alpha^2 c (m_e c^2 + E_+)}{2|\mathbf{p}_+| (m_e c^2 + E_+ - c|\mathbf{p}_+| \cos \theta)^2} \left[\frac{E_+ - c|\mathbf{p}_+| \cos \theta}{m_e c^2} + \frac{m_e c^2}{E_+ - c|\mathbf{p}_+| \cos \theta} - 4 \frac{1 + \cos^2 \phi}{4} + 2 \right] \\ &= \frac{\hbar^2 \alpha^2 c (m_e c^2 + E_+)}{2|\mathbf{p}_+| (m_e c^2 + E_+ - c|\mathbf{p}_+| \cos \theta)^2} \left[\frac{E_+ - c|\mathbf{p}_+| \cos \theta}{m_e c^2} + \frac{m_e c^2}{E_+ - c|\mathbf{p}_+| \cos \theta} + \sin^2 \phi \right].\end{aligned}\quad [12.35]$$

従って, 全断面積は,

$$\bar{\sigma} = \frac{1}{2} \int \frac{d\bar{\sigma}}{d\Omega_{q_1}} d\Omega_{q_1} \quad [12.36]$$

$$= \pi \int_0^\pi \frac{d\bar{\sigma}}{d\Omega_{q_1}} \sin \theta d\theta \quad [12.37]$$

$$= \pi \int_{-1}^1 \frac{d\bar{\sigma}}{d\Omega_{q_1}} dz \quad [12.38]$$

$$= \frac{\pi \hbar^2 \alpha^2 c (m_e c^2 + E_+)}{2|\mathbf{p}_+|} (R_1 + R_2 + R_3 + R_4) \quad [12.39]$$

となる. ただし,

$$\begin{aligned}R_1 &= \int_{-1}^1 \frac{1}{(m_e c^2 + E_+ - c|\mathbf{p}_+|z)^2} \frac{E_+ - c|\mathbf{p}_+|z}{m_e c^2} dz \\ &= \frac{1}{m_e c^2} \int_{-1}^1 \frac{dz}{m_e c^2 + E_+ - c|\mathbf{p}_+|z} - \int_{-1}^1 \frac{1}{(m_e c^2 + E_+ - c|\mathbf{p}_+|z)^2} dz, \\ R_2 &= \int_{-1}^1 \frac{1}{(m_e c^2 + E_+ - c|\mathbf{p}_+|z)^2} \frac{m_e c^2}{E_+ - c|\mathbf{p}_+|z} dz \\ &= -\frac{1}{m_e c^2} \int_{-1}^1 \frac{dz}{m_e c^2 + E_+ - c|\mathbf{p}_+|z} - \int_{-1}^1 \frac{1}{(m_e c^2 + E_+ - c|\mathbf{p}_+|z)^2} dz + \frac{1}{m_e c^2} \int_{-1}^1 \frac{dz}{E_+ - c|\mathbf{p}_+|z}, \\ R_3 &= \left(\frac{c|\mathbf{p}_+|}{E_+ + m_e c^2} \right)^2 \int_{-1}^1 \frac{dz}{(E_+ - c|\mathbf{p}_+|z)^2} \\ R_4 &= \left(\frac{c|\mathbf{p}_+|}{E_+ + m_e c^2} \right)^2 \int_{-1}^1 \frac{-z^2 dz}{(E_+ - c|\mathbf{p}_+|z)^2} \\ &= \left(\frac{c|\mathbf{p}_+|}{E_+ + m_e c^2} \right)^2 \left[\int_{-1}^1 -\frac{dz}{c^2 |\mathbf{p}_+|^2} + \frac{2E_+}{c^2 |\mathbf{p}_+|^2} \int_{-1}^1 \frac{dz}{E_+ - c|\mathbf{p}_+|z} - \frac{E_+^2}{c^2 |\mathbf{p}_+|^2} \int_{-1}^1 \frac{dz}{(E_+ - c|\mathbf{p}_+|z)^2} \right] \\ &= -\frac{2}{(E_+ + m_e c^2)^2} + \frac{2E_+}{(E_+ + m_e c^2)^2} \int_{-1}^1 \frac{dz}{E_+ - c|\mathbf{p}_+|z} - \left(\frac{E_+}{E_+ + m_e c^2} \right)^2 \int_{-1}^1 \frac{dz}{(E_+ - c|\mathbf{p}_+|z)^2}\end{aligned}\quad [12.40]$$

と分けて計算する。以上から、

$$\begin{aligned}
R_1 + R_2 + R_3 + R_4 = & -\frac{2}{(E + m_e c^2)^2} \\
& + \left[\frac{2E_+}{(E_+ + m_e c^2)^2} + \frac{1}{m_e c^2} \right] \int_{-1}^1 \frac{dz}{E_+ - c|\mathbf{p}_+|z} \\
& + \frac{c^2|\mathbf{p}_+|^2 - E_+^2}{(E_+ + m_e c^2)^2} \int_{-1}^1 \frac{dz}{(E_+ - c|\mathbf{p}_+|z)^2} \\
& - 2 \int_{-1}^1 \frac{dz}{(m_e c^2 + E_+ - c|\mathbf{p}_+|z)^2}.
\end{aligned} \tag{12.41}$$

第2項は

$$\begin{aligned}
\left[\frac{2E_+}{(E_+ + m_e c^2)^2} + \frac{1}{m_e c^2} \right] \frac{1}{c|\mathbf{p}_+|} \log \left| \frac{E_+ + c|\mathbf{p}_+|}{E_+ - c|\mathbf{p}_+|} \right| &= \left[\frac{2E_+}{(E_+ + m_e c^2)^2} + \frac{1}{m_e c^2} \right] \frac{1}{c|\mathbf{p}_+|} \log \frac{(E_+ + c|\mathbf{p}_+|)^2}{E_+^2 - c^2|\mathbf{p}_+|^2} \\
&= \left[\frac{2E_+}{(E_+ + m_e c^2)^2} + \frac{1}{m_e c^2} \right] \frac{2}{c|\mathbf{p}_+|} \log \frac{E_+ + c|\mathbf{p}_+|}{m_e c^2} \\
&= \frac{E_+^2 + 4E_+ m_e c^2 + m_e^2 c^4}{(E_+ + m_e c^2)^2 m_e c^2} \frac{2}{c|\mathbf{p}_+|} \log \frac{E_+ + c|\mathbf{p}_+|}{m_e c^2}.
\end{aligned} \tag{12.42}$$

第3項は

$$\begin{aligned}
\frac{c^2|\mathbf{p}_+|^2 - E_+^2}{(E_+ + m_e c^2)^2} \int_{-1}^1 \frac{dz}{(E_+ - c|\mathbf{p}_+|z)^2} &= \frac{-m_e^2 c^4}{(E_+ + m_e c^2)^2} \frac{2}{m_e^2 c^4} \\
&= -\frac{2}{(E_+ + m_e c^2)^2}.
\end{aligned} \tag{12.43}$$

第4項は

$$\begin{aligned}
-2 \int_{-1}^1 \frac{1}{(m_e c^2 + E_+ - c|\mathbf{p}_+|z)^2} dz &= -\frac{4}{(E_+ + m_e c^2)^2 - c^2|\mathbf{p}_+|^2} \\
&= -\frac{2}{m_e c^2 (E_+ + m_e c^2)}.
\end{aligned} \tag{12.44}$$

[12.42][12.43][12.44] を [12.41] に代入して、

$$\begin{aligned}
R_1 + R_2 + R_3 + R_4 &= -\frac{2}{(E_+ + m_e c^2)^2} + \frac{E_+^2 + 4E_+ m_e c^2 + m_e^2 c^4}{(E_+ + m_e c^2)^2 m_e c^2} \frac{2}{c|\mathbf{p}_+|} \log \frac{E_+ + c|\mathbf{p}_+|}{m_e c^2} \\
&\quad - \frac{2}{(E_+ + m_e c^2)^2} - \frac{2}{m_e c^2 (E_+ + m_e c^2)} \\
&= \frac{2}{(E_+ + m_e c^2)^2 m_e c^2 c|\mathbf{p}_+|} \left[(E_+^2 + 4m_e c^2 E_+ + m_e^2 c^4) \log \frac{E_+ + c|\mathbf{p}_+|}{m_e c^2} - (E_+ + 3m_e c^2) c|\mathbf{p}_+| \right].
\end{aligned} \tag{12.45}$$

[12.39] に代入して,

$$\begin{aligned}
\bar{\sigma} &= \frac{\pi \hbar^2 \alpha^2 c (m_e c^2 + E_+)}{2 |\mathbf{p}_+|} \frac{2}{(E_+ + m_e c^2)^2 m_e c^2 c |\mathbf{p}_+|} \\
&\quad \times \left[(E_+^2 + 4m_e c^2 E_+ + m_e^2 c^4) \log \frac{E_+ + c |\mathbf{p}_+|}{m_e c^2} - (E_+ + 3m_e c^2) c |\mathbf{p}_+| \right] \\
&= \frac{\pi \hbar^2 \alpha^2}{m_e c^2 |\mathbf{p}_+| (E_+ + m_e c^2)} \left[(E_+^2 + 4m_e c^2 E_+ + m_e^2 c^4) \log \frac{E_+ + c |\mathbf{p}_+|}{m_e c^2} - (E_+ + 3m_e c^2) c |\mathbf{p}_+| \right].
\end{aligned}
\tag{12.46}$$

第 13 章

電子・電子散乱と電子・陽電子散乱

■(13.21) 1 つめの項は

$$\begin{aligned}
 & \sum_{\pm s_1, \pm s'_1} \sum_{\pm s_2, \pm s'_2} \frac{1}{(p_1 - p'_1)^4} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1) \bar{u}(p'_2, s'_2) \gamma^\mu u(p_2, s_2)] \\
 & \quad \times [\bar{u}(p'_1, s'_1) \gamma_\nu u(p_1, s_1) \bar{u}(p'_2, s'_2) \gamma^\nu u(p_2, s_2)]^* \\
 &= \sum_{\pm s_1, \pm s'_1} \sum_{\pm s_2, \pm s'_2} \frac{1}{(p_1 - p'_1)^4} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1)] [\bar{u}(p'_1, s'_1) \gamma_\nu u(p_1, s_1)]^* \\
 & \quad \times [\bar{u}(p'_2, s'_2) \gamma^\mu u(p_2, s_2)] [\bar{u}(p'_2, s'_2) \gamma^\nu u(p_2, s_2)]^* \\
 &= \frac{1}{(p_1 - p'_1)^4} \sum_{\pm s_1, \pm s'_1} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1)] [\bar{u}(p_1, s_1) \bar{\gamma}_\nu u(p'_1, s'_1)] \\
 & \quad \times \sum_{\pm s_2, \pm s'_2} [\bar{u}(p'_2, s'_2) \gamma^\mu u(p_2, s_2)] [\bar{u}(p_2, s_2) \bar{\gamma}^\nu u(p'_2, s'_2)]
 \end{aligned}$$

となり, (11.19) と同様に計算すれば,

$$= \frac{1}{(p_1 - p'_1)^4} \text{Tr} \left(\frac{\not{p}'_1 + m_e c}{2m_e c} \gamma_\mu \frac{\not{p}_1 + m_e c}{2m_e c} \gamma_\nu \right) \text{Tr} \left(\frac{\not{p}'_2 + m_e c}{2m_e c} \gamma^\mu \frac{\not{p}_2 + m_e c}{2m_e c} \gamma^\nu \right).$$

2 つめの項は

$$\begin{aligned}
 & \sum_{\pm s_1, \pm s'_1} \sum_{\pm s_2, \pm s'_2} \frac{1}{(p_1 - p'_1)^2 (p_1 - p'_2)^2} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1) \bar{u}(p'_2, s'_2) \gamma^\mu u(p_2, s_2)] \\
 & \quad \times [\bar{u}(p'_2, s'_2) \gamma_\nu u(p_1, s_1) \bar{u}(p'_1, s'_1) \gamma^\nu u(p_2, s_2)]^* \\
 &= \sum_{\pm s_1, \pm s'_1} \sum_{\pm s_2, \pm s'_2} \frac{1}{(p_1 - p'_1)^2 (p_1 - p'_2)^2} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1)] [\bar{u}(p'_2, s'_2) \gamma^\mu u(p_2, s_2)] \\
 & \quad \times [\bar{u}(p'_2, s'_2) \gamma_\nu u(p_1, s_1)]^* [\bar{u}(p'_1, s'_1) \gamma^\nu u(p_2, s_2)]^* \\
 &= \sum_{\pm s_1, \pm s'_1} \sum_{\pm s_2, \pm s'_2} \frac{1}{(p_1 - p'_1)^2 (p_1 - p'_2)^2} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1)] [\bar{u}(p'_2, s'_2) \gamma^\mu u(p_2, s_2)] \\
 & \quad \times [\bar{u}(p_1, s_1) \bar{\gamma}_\nu u(p'_2, s'_2)] [\bar{u}(p_2, s_2) \bar{\gamma}^\nu u(p'_1, s'_1)] \\
 &= \sum_{\pm s_1, \pm s'_1} \sum_{\pm s_2, \pm s'_2} \frac{1}{(p_1 - p'_1)^2 (p_1 - p'_2)^2} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1)] [\bar{u}(p_1, s_1) \bar{\gamma}_\nu u(p'_2, s'_2)] \\
 & \quad \times [\bar{u}(p'_2, s'_2) \gamma^\mu u(p_2, s_2)] [\bar{u}(p_2, s_2) \bar{\gamma}^\nu u(p'_1, s'_1)]
 \end{aligned}$$

となり, (11.19) と同様に計算すれば,

$$= \frac{1}{(p_1 - p'_1)^2 (p_1 - p'_2)^2} \text{Tr} \left(\frac{\not{p}'_1 + m_e c}{2m_e c} \gamma_\mu \frac{\not{p}_1 + m_e c}{2m_e c} \gamma_\nu \frac{\not{p}'_2 + m_e c}{2m_e c} \gamma^\mu \frac{\not{p}_2 + m_e c}{2m_e c} \gamma^\nu \right).$$

■(13.31) $\delta_{(s)}^\mu$ を $\mu = s$ に対しては 1, $\mu \neq s$ に対しては 0 となる 4 元ベクトルとすれば, $\not{\delta}_{(\nu)} = \gamma_\nu$ となる. これに注意して (H.34), (H.30) を使えば,

$$\begin{aligned} \text{Tr}(\not{p}'_1 \gamma_\mu \not{p}_1 \gamma_\nu \not{p}'_2 \gamma^\mu \not{p}_2 \gamma^\nu) &= -2 \text{Tr}(\not{p}'_1 \not{p}'_2 \gamma_\nu \not{p}_1 \not{p}_2 \gamma^\nu) \\ &= -8(p_1 p_2) \text{Tr}(\not{p}'_1 \not{p}'_2) \\ &= -8(p_1 p_2)(p'_1 p'_2). \end{aligned}$$

■(13.37), (13.38) (13.9) を使う. 図 13.2(a) のパターンだと

$$\begin{aligned} \psi_i(x) &= \sqrt{\frac{m_e c^2}{E_1 V}} u(p_1, s_1) e^{-\frac{i}{\hbar} p_1 x}, \\ \bar{\psi}_f(x) &= \sqrt{\frac{m_e c^2}{E'_1 V}} \bar{u}(p'_1, s'_1) e^{\frac{i}{\hbar} p'_1 x}, \\ \psi_i^{(2)}(y) &= \sqrt{\frac{m_e c^2}{\tilde{E}'_2 V}} v(\bar{p}'_2, \bar{s}'_2) e^{\frac{i}{\hbar} \bar{p}'_2 y}, \\ \bar{\psi}_f^{(2)}(y) &= \sqrt{\frac{m_e c^2}{\tilde{E}_2 V}} \bar{v}(\bar{p}_2, \bar{s}_2) e^{-\frac{i}{\hbar} \bar{p}_2 y}. \end{aligned}$$

図 13.2(b) のパターンだと

$$\begin{aligned} \psi_i(x) &= \sqrt{\frac{m_e c^2}{E_1 V}} u(p_1, s_1) e^{-\frac{i}{\hbar} p_1 x}, \\ \bar{\psi}_f(x) &= \sqrt{\frac{m_e c^2}{\tilde{E}_2 V}} \bar{v}(\bar{p}_2, \bar{s}_2) e^{-\frac{i}{\hbar} \bar{p}_2 y}, \\ \psi_i^{(2)}(y) &= \sqrt{\frac{m_e c^2}{\tilde{E}'_2 V}} v(\bar{p}'_2, \bar{s}'_2) e^{\frac{i}{\hbar} \bar{p}'_2 y}, \\ \bar{\psi}_f^{(2)}(y) &= \sqrt{\frac{m_e c^2}{E'_1 V}} \bar{u}(p'_1, s'_1) e^{\frac{i}{\hbar} p'_1 x}. \end{aligned}$$

[Griffiths](7.99) から (11.20)(11.21) と同等の式

$$\begin{aligned} \sum_{\pm s_-} u_\beta(p_-, s_-) \bar{u}_\gamma(p_-, s_-) &= \left(\frac{\not{p}_- + m_e c}{2m_e c} \right)_{\beta\gamma} \\ \sum_{\pm s_+} v_\delta(p_+, s_+) \bar{v}_\alpha(p_+, s_+) &= \left(\frac{\not{p}_+ - m_e c}{2m_e c} \right)_{\delta\alpha} \end{aligned}$$

■(13.39) (13.20) に対応する式は

$$d\sigma = \frac{16\hbar^2 c \alpha^2}{|\mathbf{v}_1 - \mathbf{v}_2|} \frac{m_e^4 c^8}{E_1 \bar{E}'_2} \delta^4(p'_1 + \bar{p}'_2 - p_1 - \bar{p}_2) |M_{fi}|^2 \frac{d^3 p'_1}{2E'_1} \frac{d^3 \bar{p}_2}{2\bar{E}_2}.$$

高エネルギー極限で重心系から見た場合は (13.22) と同様に,

$$p_1 = (E/c, \mathbf{p}), \quad \bar{p}_2 = (E/c, -\mathbf{p}), \quad p'_1 = (E'/c, \mathbf{p}'), \quad \bar{p}'_2 = (E'/c, -\mathbf{p}'), \quad |\mathbf{v}_1 - \mathbf{v}_2| = \frac{2|\mathbf{p}|c^2}{E}.$$

(13.23) に対応する式は

$$d\bar{\sigma} = \frac{d^3 p'_1}{2E'} \frac{d^3 \bar{p}_2}{2E} \delta^4(p'_1 + \bar{p}_2 - p_1 - \bar{p}_2) F(p'_1, \bar{p}_2), \quad F(p'_1, \bar{p}_2) = \frac{8\hbar^2 \alpha^2 m_e^4 c^7}{|\mathbf{p}|E} |M_{fi}|^2.$$

(13.26) に対応する式は

$$d\bar{\sigma} = d\Omega_1 \frac{\hbar^2 \alpha^2 m_e^4 c^6}{E^2} |M_{fi}|^2.$$

(13.27) 対応する式は

$$p_1 \bar{p}_2 \approx \frac{2E^2}{c^2}, \quad p'_1 \bar{p}'_2 \approx \frac{2E'^2}{c^2}, \quad p_1 \bar{p}'_2 = p'_1 \bar{p}_2 \approx \frac{2EE'}{c^2} \cos^2 \frac{\theta}{2}, \quad p_1 p'_1 = \bar{p}_2 \bar{p}'_2 \approx \frac{2EE'}{c^2} \sin^2 \frac{\theta}{2}.$$

$|M_{fi}|^2$ の第 1 項に対応する部分 (前半は (13.21) の計算; 最後の計算は (13.29) と同じ) は,

$$\begin{aligned} & \frac{1}{4} \sum_{\pm s_1, \pm s'_1} \sum_{\pm \bar{s}_2, \pm \bar{s}'_2} \frac{1}{(p_1 - p'_1)^4} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1) \bar{v}(\bar{p}_2, \bar{s}_2) \gamma^\mu v(\bar{p}'_2, \bar{s}'_2)] \\ & \quad \times [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1) \bar{v}(\bar{p}_2, \bar{s}_2) \gamma^\mu v(\bar{p}'_2, \bar{s}'_2)]^* \\ & = \frac{1}{4} \frac{1}{(p_1 - p'_1)^4} \sum_{\pm s_1, \pm s'_1} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1)] [\bar{u}(p_1, s_1) \bar{\gamma}_\nu u(p'_1, s'_1)] \\ & \quad \times \sum_{\pm \bar{s}_2, \pm \bar{s}'_2} [\bar{v}(\bar{p}_2, \bar{s}_2) \gamma^\mu v(\bar{p}'_2, \bar{s}'_2)] [\bar{v}(\bar{p}'_2, \bar{s}'_2) \bar{\gamma}_\nu v(\bar{p}_2, \bar{s}_2)] \\ & = \frac{1}{4} \frac{1}{(p_1 - p'_1)^4} \text{Tr} \left(\frac{\not{p}'_1 + m_e c}{2m_e c} \gamma_\mu \frac{\not{p}_1 + m_e c}{2m_e c} \gamma_\nu \right) \text{Tr} \left(\frac{\not{\bar{p}}_2 - m_e c}{2m_e c} \gamma^\mu \frac{\not{\bar{p}}'_2 - m_e c}{2m_e c} \gamma^\nu \right) \\ & \approx \frac{1}{8m_e^4 c^4} \frac{1 + \cos^4 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}}. \end{aligned}$$

第 2・3 項は (トレースの展開は (13.31) と同様)

$$\begin{aligned} & -\frac{1}{4} \sum_{\pm s_1, \pm s'_1} \sum_{\pm \bar{s}_2, \pm \bar{s}'_2} \frac{1}{(p_1 - p'_1)^2 (p_1 + \bar{p}_2)^2} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1) \bar{v}(\bar{p}_2, \bar{s}_2) \gamma^\mu v(\bar{p}'_2, \bar{s}'_2)] \\ & \quad \times [\bar{u}(p'_1, s'_1) \gamma_\nu v(\bar{p}'_2, \bar{s}'_2) \bar{v}(\bar{p}_2, \bar{s}_2) \gamma^\nu u(p_1, s_1)]^* \\ & = -\frac{1}{4} \frac{1}{(p_1 - p'_1)^2 (p_1 + \bar{p}_2)^2} \sum_{\pm s_1, \pm s'_1} \sum_{\pm \bar{s}_2, \pm \bar{s}'_2} [\bar{u}(p'_1, s'_1) \gamma_\mu u(p_1, s_1)] [\bar{u}(p_1, s_1) \bar{\gamma}_\nu v(\bar{p}_2, \bar{s}_2)] \\ & \quad \times [\bar{v}(\bar{p}_2, \bar{s}_2) \gamma^\mu v(\bar{p}'_2, \bar{s}'_2)] [\bar{v}(\bar{p}'_2, \bar{s}'_2) \bar{\gamma}_\nu u(p'_1, s'_1)] \\ & = -\frac{1}{4} \frac{1}{(p_1 - p'_1)^2 (p_1 + \bar{p}_2)^2} \text{Tr} \left(\frac{\not{p}'_1 + m_e c}{2m_e c} \gamma_\mu \frac{\not{p}_1 + m_e c}{2m_e c} \gamma_\nu \frac{\not{\bar{p}}_2 - m_e c}{2m_e c} \gamma^\mu \frac{\not{\bar{p}}'_2 - m_e c}{2m_e c} \gamma^\nu \right) \\ & \approx -\frac{1}{4} \frac{1}{(p_1 - p'_1)^2 (p_1 + \bar{p}_2)^2} \frac{1}{16m_e^4 c^4} \text{Tr}(\not{p}'_1 \gamma_\mu \not{p}_1 \gamma^\nu \not{\bar{p}}_2 \gamma^\mu \not{\bar{p}}'_2 \gamma_\nu) \\ & \approx -\frac{1}{4} \left(-4 \frac{EE'}{c^2} \sin^2 \frac{\theta}{2} \right)^{-1} \left(\frac{4E^2}{c^2} \right)^{-1} \frac{-32}{16m_e^4 c^4} (p_1 \bar{p}'_2) (p'_1 \bar{p}_2) \\ & = -\frac{1}{8m_e^4 c^4} \frac{\cos^4 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}}. \end{aligned}$$

第4項は

$$\begin{aligned}
& \frac{1}{4} \sum_{\pm s_1, \pm s'_1} \sum_{\pm \bar{s}_2, \pm \bar{s}'_2} \frac{1}{(p_1 + \bar{p}_2)^4} [\bar{u}(p'_1, s'_1) \gamma_\mu v(\bar{p}'_2, \bar{s}'_2) \bar{v}(\bar{p}_2, \bar{s}_2) \gamma^\mu u(p_1, s_1)] \\
& \quad \times [\bar{u}(p'_1, s'_1) \gamma_\nu v(\bar{p}'_2, \bar{s}'_2) \bar{v}(\bar{p}_2, \bar{s}_2) \gamma^\nu u(p_1, s_1)]^* \\
& = \frac{1}{4} \frac{1}{(p_1 + \bar{p}_2)^4} \sum_{\pm s_1, \pm s'_1} \sum_{\pm \bar{s}_2, \pm \bar{s}'_2} [\bar{u}(p'_1, s'_1) \gamma_\mu v(\bar{p}'_2, \bar{s}'_2)] [\bar{v}(\bar{p}'_2, \bar{s}'_2) \gamma_\nu u(p'_1, s'_1)] \\
& \quad \times [\bar{v}(\bar{p}_2, \bar{s}_2) \gamma^\mu u(p_1, s_1)] [\bar{u}(p_1, s_1) \gamma^\nu v(\bar{p}_2, \bar{s}_2)] \\
& = \frac{1}{4} \frac{1}{(p_1 + \bar{p}_2)^4} \text{Tr} \left(\frac{\not{p}'_1 + m_e c}{2m_e c} \gamma_\mu \frac{\not{\bar{p}}'_2 - m_e c}{2m_e c} \gamma_\nu \right) \text{Tr} \left(\frac{\not{\bar{p}}_2 - m_e c}{2m_e c} \gamma^\mu \frac{\not{p}_1 + m_e c}{2m_e c} \gamma^\nu \right) \\
& \approx \frac{1}{4} \frac{1}{(p_1 + \bar{p}_2)^4} \frac{1}{16m_e^4 c^4} \text{Tr} \left(\not{p}'_1 \gamma_\mu \not{\bar{p}}'_2 \gamma_\nu \right) \text{Tr} \left(\not{\bar{p}}_2 \gamma^\mu \not{p}_1 \gamma^\nu \right) \\
& \approx \frac{1}{4} \frac{1}{(p_1 + \bar{p}_2)^4} \frac{1}{m_e^4 c^4} [p'_{1\mu} \bar{p}'_{2\nu} + p'_{1\nu} \bar{p}'_{2\mu} - (p'_1 \bar{p}'_2) \eta_{\mu\nu}] [\bar{p}_2^\mu p_1^\nu + \bar{p}_2^\nu p_1^\mu - (\bar{p}_2 p_1) \eta^{\mu\nu}] \\
& = \frac{1}{2} \frac{1}{(p_1 + \bar{p}_2)^4} \frac{1}{m_e^4 c^4} [(p'_1 \bar{p}_2)(\bar{p}'_2 p_1) + (p'_1 p_1)(\bar{p}'_2 \bar{p}_2)] \\
& = \frac{1}{8} \frac{1}{m_e^4 c^4} \left(\cos^4 \frac{\theta}{2} + \sin^4 \frac{\theta}{2} \right).
\end{aligned}$$

従って,

$$\begin{aligned}
\frac{d\bar{\sigma}}{d\Omega'_1} &= \frac{c^2 \hbar^2 \alpha^2}{8E^2} \left(\frac{1 + \cos^4 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}} - 2 \times \frac{\cos^4 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} + \cos^4 \frac{\theta}{2} + \sin^4 \frac{\theta}{2} \right) \\
&= \frac{c^2 \hbar^2 \alpha^2}{8E^2} \left[\frac{1 + \cos^4 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}} - 2 \times \frac{\cos^4 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} + \left(\frac{1 + \cos \theta}{2} \right)^2 \left(\frac{1 - \cos \theta}{2} \right)^2 \right] \\
&= \frac{c^2 \hbar^2 \alpha^2}{8E^2} \left(\frac{1 + \cos^4 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}} - 2 \frac{\cos^4 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}} + \frac{1 + \cos^2 \theta}{2} \right).
\end{aligned}$$

第 15 章

高次補正 —その 1—

■(15.3) (11.7)(11.8) を使う．左図の S 行列は

$$\begin{aligned}
 & \bar{u}(p_f, s_f) \int \frac{d^4 p}{(2\pi\hbar)^4} i \not{\epsilon} \frac{i\hbar}{\not{p} - m_e c + i\varepsilon} i\gamma^0 \int d^4 x e^{\frac{i}{\hbar}(p_f \mp k - p)x} \int d^4 y e^{\frac{i}{\hbar}(p - p_i)y} \frac{1}{|y|} u(p_i, s_i) \\
 &= -\bar{u}(p_f, s_f) \int \frac{d^4 p}{(2\pi\hbar)^4} i \not{\epsilon} \frac{i\hbar}{\not{p} - m_e c + i\varepsilon} \gamma^0 (2\pi\hbar)^4 \delta^4(p_f \mp k - p) (2\pi\hbar) \delta(E - E_i) \frac{4\pi\hbar}{|\mathbf{p} - \mathbf{p}_i|^2} u(p_i, s_i) \\
 &= -\bar{u}(p_f, s_f) i \not{\epsilon} \frac{i\hbar}{\not{p}_f + \not{k} - m_e c + i\varepsilon} \gamma^0 2\pi\hbar \delta(E_f + |\mathbf{k}|c - E_i) \frac{4\pi\hbar}{|\mathbf{q}|^2} u(p_i, s_i) \\
 &= -i\bar{u}(p_f, s_f) i \not{\epsilon} \frac{\not{p}_f + \not{k} + m_e c}{(p_f + k)^2 - m_e^2 c^2} \gamma^0 2\pi\hbar \delta(E_f + |\mathbf{k}|c - E_i) \frac{4\pi\hbar^2}{|\mathbf{q}|^2} u(p_i, s_i).
 \end{aligned}$$

右図は

$$\begin{aligned}
 & \bar{u}(p_f, s_f) \int \frac{d^4 p}{(2\pi\hbar)^4} \int d^4 x e^{\frac{i}{\hbar}(p_f - p)x} \frac{1}{|x|} i\gamma^0 \frac{i\hbar}{\not{p} - m_e c + i\varepsilon} i \not{\epsilon} \int d^4 y e^{\frac{i}{\hbar}(p \mp k - p_i)y} u(p_i, s_i) \\
 &= -\bar{u}(p_f, s_f) \int \frac{d^4 p}{(2\pi\hbar)^4} 2\pi\hbar \delta(E_f - E) \frac{4\pi\hbar}{|\mathbf{p}_f - \mathbf{p}|^2} \gamma^0 \frac{i\hbar}{\not{p} - m_e c + i\varepsilon} i \not{\epsilon} (2\pi\hbar)^4 \delta(p \mp k - p_i) u(p_i, s_i) \\
 &= -\bar{u}(p_f, s_f) \gamma^0 \frac{i\hbar}{\not{p}_i - \not{k} - m_e c + i\varepsilon} i \not{\epsilon} 2\pi\hbar \delta(E_f + |\mathbf{k}|c - E_i) \frac{4\pi\hbar}{|\mathbf{q}|^2} u(p_i, s_i) \\
 &= -i\bar{u}(p_f, s_f) \gamma^0 \frac{\not{p}_i - \not{k} + m_e c}{(p_i - k)^2 - m_e^2 c^2 + i\varepsilon} i \not{\epsilon} 2\pi\hbar \delta(E_f + |\mathbf{k}|c - E_i) \frac{4\pi\hbar^2}{|\mathbf{q}|^2} u(p_i, s_i).
 \end{aligned}$$

■(15.10) $|1 + \tilde{F}_1(q^2)|^2$ は通常散乱，真空偏極，頂点補正の断面積． $\int \cdots dk$ は制動放射．

■(15.25)

$$\begin{aligned}
 \int \bar{\psi} \gamma_0 \gamma \cdot \nabla A^0 \psi d^3 x &= \int (\varphi^\dagger \quad \chi^\dagger) \begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \nabla \\ -\boldsymbol{\sigma} \cdot \nabla & 0 \end{pmatrix} A^0 \begin{pmatrix} \varphi \\ \chi \end{pmatrix} d^3 x \\
 &= \int d^3 x [\varphi^\dagger (\boldsymbol{\sigma} \cdot \nabla A^0) \chi - \chi^\dagger (\boldsymbol{\sigma} \cdot \nabla A^0) \varphi] \\
 &= -\frac{i\hbar}{2m_e c} \int d^3 x [\varphi^\dagger (\boldsymbol{\sigma} \cdot \nabla A^0) (\boldsymbol{\sigma} \cdot \nabla \varphi) + (\nabla \varphi^\dagger \cdot \boldsymbol{\sigma}) (\boldsymbol{\sigma} \cdot \nabla A^0) \varphi] \\
 &= -\frac{i\hbar}{2m_e c} \int d^3 x [\varphi^\dagger (\boldsymbol{\sigma} \cdot \nabla A^0) (\boldsymbol{\sigma} \cdot \nabla \varphi) - \varphi^\dagger \boldsymbol{\sigma} \cdot \nabla \{(\boldsymbol{\sigma} \cdot \nabla A^0) \varphi\}].
 \end{aligned}$$

[...] の中身は

$$\begin{aligned}
& \varphi^\dagger (\boldsymbol{\sigma} \cdot \nabla A^0) (\boldsymbol{\sigma} \cdot \nabla \varphi) - \varphi^\dagger \boldsymbol{\sigma} \cdot \nabla \{ (\boldsymbol{\sigma} \cdot \nabla A^0) \varphi \} \\
&= \varphi^\dagger [(\nabla A^0) \cdot (\nabla \varphi) + i \boldsymbol{\sigma} \cdot (\nabla A^0) \times (\nabla \varphi)] \\
&\quad - \varphi^\dagger [\{(\nabla^2 + i \boldsymbol{\sigma} \cdot \nabla \times \nabla) A^0\} \varphi + \{(\nabla \varphi) \cdot (\nabla A^0) + i \boldsymbol{\sigma} \cdot (\nabla \varphi) \times (\nabla A^0)\}] \\
&= -\varphi^\dagger (\nabla^2 A^0) \varphi + 2i \varphi^\dagger \boldsymbol{\sigma} \cdot (\nabla A^0) \times (\nabla \varphi) \\
&= -\varphi^\dagger (\nabla^2 A^0) \varphi + 2i \varphi^\dagger \boldsymbol{\sigma} \cdot \left(\frac{dA^0}{dr} \frac{\mathbf{r}}{r} \right) \times (\nabla \varphi) \\
&= -\varphi^\dagger (\nabla^2 A^0) \varphi - \frac{2}{\hbar} \varphi^\dagger \frac{1}{r} \frac{dA^0}{dr} \boldsymbol{\sigma} \cdot \mathbf{L} \varphi.
\end{aligned}$$

第 16 章

Appendix C

Runge-Lenz-Pauli ベクトル：

$$\mathbf{R} = \frac{1}{2m}(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}) + k_0 Z e^2 \frac{\mathbf{r}}{r} \quad [16.1]$$

の性質を調べる^{*1}。まずは、エルミート性について、

$$\begin{aligned} (\mathbf{L} \times \mathbf{p})^\dagger &= \sum_{ijk} (L_i p_j \mathbf{e}_k \varepsilon_{ijk})^\dagger \\ &= \sum_{ijk} (p_j L_i \mathbf{e}_k \varepsilon_{ijk}) \\ &= \sum_{ijk} (p_i L_j \mathbf{e}_k \varepsilon_{jik}) \\ &= - \sum_{ijk} (p_i L_j \mathbf{e}_k \varepsilon_{ijk}) \\ &= -\mathbf{p} \times \mathbf{L} \end{aligned}$$

から、

$$\mathbf{R}^\dagger = \mathbf{R}. \quad [16.2]$$

次に、ハミルトニアン H と可換であることを示す。

$$\begin{aligned} \mathbf{L} \times \mathbf{p} &= \sum_{ijk} L_i p_j \mathbf{e}_k \varepsilon_{ijk} \\ &= \sum_{ijk} \sum_{mn} r_m p_n \varepsilon_{mni} p_j \mathbf{e}_k \varepsilon_{ijk} \\ &= \sum_{ijk} \sum_{mn} r_m p_n p_j \mathbf{e}_k \varepsilon_{imn} \varepsilon_{ijk} \\ &= \sum_{ijk} \sum_{mn} r_m p_n p_j \mathbf{e}_k (\varepsilon_{ijk})^2 (\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}) \\ &= \sum_{ijk} (r_j p_j p_k - r_k p_j^2) \mathbf{e}_k (\varepsilon_{ijk})^2, \end{aligned}$$

^{*1} <https://adlara.hatenadiary.jp/entry/2016/04/14/141203> を大変参考にした

$$\begin{aligned}
\mathbf{p} \times \mathbf{L} &= \sum_{ijk} p_i L_j \mathbf{e}_k \varepsilon_{ijk} \\
&= \sum_{ijk} \sum_{mn} p_i r_m p_n \varepsilon_{mnj} \mathbf{e}_k \varepsilon_{ijk} \\
&= \sum_{ijk} \sum_{mn} p_i r_m p_n \mathbf{e}_k \varepsilon_{njm} \varepsilon_{ijk} \\
&= \sum_{ijk} \sum_{mn} p_i r_m p_n \mathbf{e}_k (\varepsilon_{ijk})^2 (\delta_{in} \delta_{km} - \delta_{im} \delta_{kn}) \\
&= \sum_{ijk} p_i (r_k p_i - r_i p_k) \mathbf{e}_k (\varepsilon_{ijk})^2
\end{aligned}$$

から,

$$\begin{aligned}
\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L} &= \sum_{ijk} (r_j p_j p_k - r_k p_j^2 - r_k p_i^2 + p_i r_i p_k) \mathbf{e}_k (\varepsilon_{ijk})^2 \\
&= \sum_{ijk} (p_i r_i p_k + r_j p_j p_k + r_k p_k p_k - r_k p_i^2 - r_k p_j^2 - r_k p_k^2) \mathbf{e}_k (\varepsilon_{ijk})^2 \\
&= \sum_{ijk} (r_i p_i p_k + r_j p_j p_k + r_k p_k p_k - i\hbar p_k - r_k \mathbf{p}^2) \mathbf{e}_k (\varepsilon_{ijk})^2 \\
&= \sum_{ijk} [(\mathbf{r} \cdot \mathbf{p}) p_k - i\hbar p_k - r_k \mathbf{p}^2] \mathbf{e}_k (\varepsilon_{ijk})^2. \tag{16.3}
\end{aligned}$$

よって,

$$\begin{aligned}
[\mathbf{R}, H] &= \left[\frac{1}{2m} (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}) + k_0 Z e^2 \frac{\mathbf{r}}{r}, \frac{\mathbf{p}^2}{2m} - k_0 Z e^2 \frac{1}{r} \right] \\
&= \frac{1}{(2m)^2} [\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}, \mathbf{p}^2] + \frac{k_0 Z e^2}{2m} \left[\frac{\mathbf{r}}{r}, \mathbf{p}^2 \right] - \frac{k_0 Z e^2}{2m} \left[\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}, \frac{1}{r} \right] \tag{16.4}
\end{aligned}$$

[16.4] の第1項は, [16.3] から,

$$[\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}, \mathbf{p}^2] = \sum_{ijk} [(\mathbf{r} \cdot \mathbf{p}) p_k, \mathbf{p}^2] - [r_k \mathbf{p}^2, \mathbf{p}^2] \mathbf{e}_k \tag{16.5}$$

となるが, 例えば,

$$[r_1, p_1^2] = r_1 p_1 p_1 - p_1 p_1 r_1 = 2i\hbar p_1,$$

$$[p_i, p_j] = 0$$

などから,

$$\begin{aligned}
&[(\mathbf{r} \cdot \mathbf{p}) p_k, \mathbf{p}^2] \\
&= (r_1 p_1 p_k + r_2 p_3 p_k + r_3 p_3 p_k)(p_1^2 + p_2^2 + p_3^2) - (p_1^2 + p_2^2 + p_3^2)(r_1 p_1 p_k + r_2 p_3 p_k + r_3 p_3 p_k) \\
&= [r_1, p_1^2] p_1 p_k + [r_2, p_2^2] p_2 p_k + [r_3, p_3^2] p_3 p_k \\
&= 2i\hbar \mathbf{p}^2 p_k
\end{aligned}$$

となる. 一方,

$$\begin{aligned}
[r_k \mathbf{p}^2, \mathbf{p}^2] &= r_k \mathbf{p}^4 - \mathbf{p}^2 r_k \mathbf{p}^2 \\
&= r_k \mathbf{p}^4 - (r_k \mathbf{p}^2 - 2i\hbar p_k) \mathbf{p}^2
\end{aligned}$$

$$= 2i\hbar \mathbf{p}^2 p_k$$

なので, [16.5] に代入して,

$$[\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}, \mathbf{p}^2] = 0.$$

次に, [16.4] 第2項は

$$\begin{aligned} \left[\frac{\mathbf{r}}{r}, \mathbf{p}^2 \right] &= (-\hbar^2) \sum_i \left(\frac{\mathbf{r}}{r} \partial_i^2 - \partial_i^2 \frac{\mathbf{r}}{r} \right) \\ &= (-\hbar^2) \sum_i \left[\frac{\mathbf{r}}{r} \partial_i^2 - \partial_i \left(\frac{\mathbf{e}_i}{r} - \frac{r_i}{r^3} \mathbf{r} + \frac{\mathbf{r}}{r} \partial_i \right) \right] \\ &= (-\hbar^2) \sum_i \left[-2 \left(\frac{\mathbf{e}_i}{r} - \frac{r_i}{r^3} \mathbf{r} \right) \partial_i - \left\{ \partial_i \left(\frac{\mathbf{e}_i}{r} - \frac{r_i}{r^3} \mathbf{r} \right) \right\} \right] \\ &= (-\hbar^2) \sum_i \left[-2 \frac{\mathbf{e}_i}{r} \partial_i + 2 \frac{\mathbf{r}}{r^3} r_i \partial_i - \left(-2 \frac{r_i}{r^3} \mathbf{e}_i - \frac{\mathbf{r}}{r^3} + 3 \frac{r_i^2}{r^5} \mathbf{r} \right) \right] \\ &= (-\hbar^2) \left[-\frac{2}{r} \boldsymbol{\partial} + \frac{2\mathbf{r}}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}) - \left(-2 \frac{\mathbf{r}}{r^3} - \frac{3\mathbf{r}}{r^3} + 3 \frac{\mathbf{r}}{r^3} \right) \right] \\ &= 2\hbar^2 \left[\frac{\boldsymbol{\partial}}{r} - \frac{\mathbf{r}}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}) - \frac{\mathbf{r}}{r^3} \right]. \end{aligned}$$

[16.4] 第3項は,

$$\begin{aligned} \left[\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}, \frac{1}{r} \right] &= \sum_{ijk} (\varepsilon_{ijk})^2 \left[\left[(\mathbf{r} \cdot \mathbf{p}) p_k, \frac{1}{r} \right] - i\hbar \left[p_k, \frac{1}{r} \right] - \left[r_k \mathbf{p}^2, \frac{1}{r} \right] \right] \mathbf{e}_k \\ &= 2 \sum_k \left[\left[(\mathbf{r} \cdot \mathbf{p}) p_k, \frac{1}{r} \right] - i\hbar \left[p_k, \frac{1}{r} \right] - \left[r_k \mathbf{p}^2, \frac{1}{r} \right] \right] \mathbf{e}_k \end{aligned} \quad [16.6]$$

となる. [16.6] の \mathbf{e}_k の係数の第1項は,

$$\begin{aligned} \left[(\mathbf{r} \cdot \mathbf{p}) p_k, \frac{1}{r} \right] &= (\mathbf{r} \cdot \mathbf{p}) p_k \frac{1}{r} - \frac{1}{r} (\mathbf{r} \cdot \mathbf{p}) p_k \\ &= -\hbar^2 (\mathbf{r} \cdot \boldsymbol{\partial}) \partial_k \frac{1}{r} + \hbar^2 \frac{1}{r} (\mathbf{r} \cdot \boldsymbol{\partial}) \partial_k \\ &= -\hbar^2 (\mathbf{r} \cdot \boldsymbol{\partial}) \frac{1}{r} \partial_k + \hbar^2 (\mathbf{r} \cdot \boldsymbol{\partial}) \frac{r_k}{r^3} + \frac{\hbar^2}{r} (\mathbf{r} \cdot \boldsymbol{\partial}) \partial_k \\ &= -\hbar^2 \sum_i r_i \partial_i \frac{1}{r} \partial_k + \hbar^2 \sum_i r_i \partial_i \frac{r_k}{r^3} + \frac{\hbar^2}{r} (\mathbf{r} \cdot \boldsymbol{\partial}) \partial_k \\ &= -\hbar^2 \sum_i r_i \left(\frac{1}{r} \partial_i - \frac{r_i}{r^3} \right) \partial_k + \hbar^2 \sum_i r_i \left(\frac{\delta_{ki}}{r^3} - r_k \frac{3r_i}{r^5} \right) + \hbar^2 \frac{r_k}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}) + \frac{\hbar^2}{r} (\mathbf{r} \cdot \boldsymbol{\partial}) \partial_k \\ &= -\hbar^2 \frac{1}{r} \sum_i r_i \partial_i \partial_k + \frac{\hbar^2}{r^3} \sum_i r_i^2 \partial_k + \frac{\hbar^2}{r^3} \sum_i r_i \delta_{ki} \\ &\quad - \hbar^2 r_k \sum_i \frac{3r_i^2}{r^5} + \hbar^2 \frac{r_k}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}) + \frac{\hbar^2}{r} (\mathbf{r} \cdot \boldsymbol{\partial}) \partial_k \\ &= -\frac{\hbar^2}{r} (\mathbf{r} \cdot \boldsymbol{\partial}) \partial_k + \frac{\hbar^2}{r} \partial_k + \hbar^2 \frac{r_k}{r^3} - \hbar^2 \frac{3r_k}{r^3} + \hbar^2 \frac{r_k}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}) + \frac{\hbar^2}{r} (\mathbf{r} \cdot \boldsymbol{\partial}) \partial_k \\ &= -2\hbar^2 \frac{r_k}{r^3} + \frac{\hbar^2}{r} \partial_k + \hbar^2 \frac{r_k}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}), \end{aligned}$$

[16.6] の \mathbf{e}_k の係数の第2項は,

$$\begin{aligned}
\left[p_k, \frac{1}{r} \right] &= p_k \frac{1}{r} - \frac{1}{r} p_k \\
&= -i\hbar \left(\partial_k \frac{1}{r} - \frac{1}{r} \partial_k \right) \\
&= -i\hbar \left(\frac{1}{r} \partial_k - \frac{r_k}{r^3} - \frac{1}{r} \partial_k \right) \\
&= i\hbar \frac{r_k}{r^3},
\end{aligned}$$

[16.6] の \mathbf{e}_k の係数の第3項は,

$$\begin{aligned}
\left[r_k \mathbf{p}^2, \frac{1}{r} \right] &= r_k \mathbf{p}^2 \frac{1}{r} - \frac{1}{r} r_k \mathbf{p}^2 \\
&= -\hbar^2 r_k \sum_i \partial_i^2 \frac{1}{r} + \hbar^2 \frac{r_k}{r} \sum_i \partial_i^2 \\
&= -\hbar^2 r_k \sum_i \partial_i \left(\frac{1}{r} \partial_i - \frac{r_i}{r^3} \right) + \hbar^2 \frac{r_k}{r} \sum_i \partial_i^2 \\
&= -\hbar^2 r_k \sum_i \left(-2 \frac{r_i}{r^3} \partial_i - \frac{1}{r^3} + 3 \frac{r_i^2}{r^5} \right) \\
&= -\hbar^2 r_k \left(-\frac{2}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}) - \frac{3}{r^3} + 3 \frac{1}{r^3} \right) \\
&= \hbar^2 \frac{2r_k}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial})
\end{aligned}$$

となる。よって, [16.6] は,

$$\begin{aligned}
\left[\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}, \frac{1}{r} \right] &= 2 \sum_k \left[-2\hbar^2 \frac{r_k}{r^3} + \frac{\hbar^2}{r} \partial_k + \hbar^2 \frac{r_k}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}) + \hbar^2 \frac{r_k}{r^3} - \hbar^2 \frac{2r_k}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}) \right] \mathbf{e}_k \\
&= 2\hbar^2 \sum_k \left[\frac{\partial_k}{r} - \frac{r_k}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}) - \frac{r_k}{r^3} \right] \mathbf{e}_k \\
&= 2\hbar^2 \left[\frac{\boldsymbol{\partial}}{r} - \frac{\mathbf{r}}{r^3} (\mathbf{r} \cdot \boldsymbol{\partial}) - \frac{\mathbf{r}}{r^3} \right]
\end{aligned}$$

となり, [16.4] 第2項と第3項の和は0. 以上から,

$$[\mathbf{R}, H] = 0. \quad [16.7]$$

次に, 角運動量 \mathbf{L} と Runge-Lenz-Pauli ベクトル \mathbf{R} の交換関係, $[L_i, R_j]$ を調べよう. その準備として, まずは角運動量の交換関係を求める. $L_i = \sum_{kl} \varepsilon_{ikl} r_k p_l$ から,

$$\begin{aligned}
[L_i, L_j] &= \sum_{klmn} \varepsilon_{ikl} \varepsilon_{jmn} (r_k p_l r_m p_n - r_m p_n r_k p_l) \\
&= \sum_{klmn} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) r_k p_l r_m p_n \\
&= \sum_{klmn} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) r_k (r_m p_l - i\hbar \delta_{lm}) p_n \\
&= -i\hbar \sum_{klmn} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) \delta_{lm} r_k p_n + \sum_{klmn} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) r_k r_m p_l p_n
\end{aligned}$$

$$\begin{aligned}
&= -i\hbar \sum_{kmn} (\varepsilon_{ikm} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkm}) r_k p_n + \sum_{klmn} (\varepsilon_{ikl} \varepsilon_{jmn} r_k r_m p_l p_n - \varepsilon_{ikl} \varepsilon_{jmn} r_m r_k p_n p_l) \\
&= -i\hbar \sum_m (\varepsilon_{ijm} \varepsilon_{jmi} r_j p_i - \varepsilon_{imj} \varepsilon_{jim} r_i p_j) \\
&= i\hbar \sum_m (\varepsilon_{ijm})^2 (r_i p_j - r_j p_i) \\
&= i\hbar \sum_k \varepsilon_{ijk} L_k.
\end{aligned} \tag{16.8}$$

さらに、角運動量と運動量については、

$$\begin{aligned}
[L_i, p_j] &= L_i p_j - p_j L_i \\
&= \sum_{kl} \varepsilon_{ikl} (r_k p_l p_j - p_j r_k p_l) \\
&= \sum_l \varepsilon_{ijl} (r_j p_j - p_j r_j) p_l \\
&= i\hbar \sum_k \varepsilon_{ijk} p_k.
\end{aligned} \tag{16.9}$$

また、次の交換関係も示しておく：

$$\begin{aligned}
\left[L_i, \frac{r_j}{r} \right] &= \sum_{kl} \varepsilon_{ikl} \left[r_k p_l, \frac{r_j}{r} \right] \\
&= -i\hbar \sum_{kl} \varepsilon_{ikl} \left[r_k \partial_l, \frac{r_j}{r} \right] \\
&= -i\hbar \sum_{kl} \varepsilon_{ikl} \left(r_k \partial_l \frac{r_j}{r} - \frac{r_j}{r} r_k \partial_l \right) \\
&= -i\hbar \sum_{kl} \varepsilon_{ikl} \left[r_k \left(\frac{\delta_{lj}}{r} - \frac{r_j r_l}{r^3} \right) + r_k \frac{r_j}{r} \partial_l - \frac{r_j}{r} r_k \partial_l \right] \\
&= -i\hbar \sum_{kl} \varepsilon_{ikl} \delta_{lj} \frac{r_k}{r} + i\hbar \sum_{kl} \varepsilon_{ikl} \frac{r_j r_k r_l}{r^3} \\
&= -i\hbar \sum_k \varepsilon_{ikj} \frac{r_k}{r} + i\hbar \varepsilon_{i+1+2} \frac{r_j r_{i+1} r_{i+2}}{r^3} + i\hbar \varepsilon_{i+2+i+1} \frac{r_j r_{i+2} r_{i+1}}{r^3} \\
&= i\hbar \sum_k \varepsilon_{ijk} \frac{r_k}{r}.
\end{aligned} \tag{16.10}$$

以上の式を使えば、

$$\begin{aligned}
[L_i, (\mathbf{L} \times \mathbf{p})_j] &= \sum_{mn} \varepsilon_{jmn} [L_i, L_m p_n] \\
&= \sum_{mn} \varepsilon_{jmn} (L_i L_m p_n - L_m p_n L_i) \\
&= \sum_{kmn} \varepsilon_{jmn} [(L_m L_i + i\hbar \varepsilon_{imk} L_k) p_n - L_m (L_i p_n - i\hbar \varepsilon_{ink} p_k)] \\
&= i\hbar \sum_{kmn} (\varepsilon_{jmn} \varepsilon_{imk} L_k p_n + \varepsilon_{jmn} \varepsilon_{ink} L_m p_k) \\
&= i\hbar \sum_{kmn} (\varepsilon_{jmn} \varepsilon_{imk} L_k p_n + \varepsilon_{jnm} \varepsilon_{imk} L_n p_k) \\
&= i\hbar \sum_{kmn} \varepsilon_{jmn} \varepsilon_{imk} (L_k p_n - L_n p_k)
\end{aligned}$$

$$\begin{aligned}
&= i\hbar \sum_m \varepsilon_{jmi} \varepsilon_{imj} (L_j p_i - L_i p_j) \\
&= -i\hbar \sum_m (\varepsilon_{mji})^2 (L_j p_i - L_i p_j) \\
&= i\hbar \sum_k \varepsilon_{ijk} (\mathbf{L} \times \mathbf{p})_k.
\end{aligned} \tag{16.11}$$

同様に,

$$\begin{aligned}
[L_i, (\mathbf{p} \times \mathbf{L})_j] &= \sum_{mn} \varepsilon_{jmn} [L_i, p_m L_n] \\
&= \sum_{mn} \varepsilon_{jmn} (L_i p_m L_n - p_m L_n L_i) \\
&= \sum_{mnk} \varepsilon_{jmn} [(p_m L_i + i\hbar \varepsilon_{iml} p_k) L_n - p_m (L_i L_n + i\hbar \varepsilon_{nik} L_k)] \\
&= i\hbar \sum_{mnk} (\varepsilon_{jmn} \varepsilon_{imk} p_k L_n - \varepsilon_{jmn} \varepsilon_{nik} p_m L_k) \\
&= i\hbar \sum_{mnk} (\varepsilon_{jmn} \varepsilon_{imk} p_k L_n - \varepsilon_{jnm} \varepsilon_{mik} p_n L_k) \\
&= i\hbar \sum_{mnk} \varepsilon_{jmn} \varepsilon_{imk} (p_k L_n - p_n L_k) \\
&= i\hbar \sum_m \varepsilon_{jmi} \varepsilon_{imj} (p_j L_i - p_i L_j) \\
&= -i\hbar \sum_m (\varepsilon_{mji})^2 (p_j L_i - p_i L_j) \\
&= -i\hbar \sum_k \varepsilon_{ijk} (\mathbf{p} \times \mathbf{L})_k.
\end{aligned} \tag{16.12}$$

よって, [16.10][16.11][16.12] から,

$$\begin{aligned}
[L_i, R_j] &= \left[L_i, \frac{1}{2m} (\mathbf{L} \times \mathbf{p})_j - \frac{1}{2m} (\mathbf{p} \times \mathbf{L})_j + k_0 Z e^2 \frac{r_j}{r} \right] \\
&= \frac{1}{2m} [L_i, (\mathbf{L} \times \mathbf{p})_j] - \frac{1}{2m} [L_i, (\mathbf{p} \times \mathbf{L})_j] + k_0 Z e^2 \left[L_i, \frac{r_j}{r} \right] \\
&= i\hbar \sum_k \varepsilon_{ijk} \left[\frac{1}{2m} (\mathbf{L} \times \mathbf{p})_k - \frac{1}{2m} (\mathbf{p} \times \mathbf{L})_k + k_0 Z e^2 \frac{r_k}{r} \right] \\
&= i\hbar \sum_k \varepsilon_{ijk} R_k.
\end{aligned} \tag{16.13}$$

\mathbf{R} と \mathbf{R} の交換関係 $[R_i, R_j]$ を調べる.

$$\begin{aligned}
&[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j] \\
&= \left[\sum_{kl} \varepsilon_{ikl} (L_k p_l - p_k L_l), \sum_{mn} \varepsilon_{jmn} (L_m p_n - p_m L_n) \right] \\
&= \sum_{klmn} [\varepsilon_{ikl} \varepsilon_{jmn} (L_k p_l - p_k L_l) (L_m p_n - p_m L_n) - \varepsilon_{ikl} \varepsilon_{jmn} (L_m p_n - p_m L_n) (L_k p_l - p_k L_l)] \\
&= \sum_{klmn} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) (L_k p_l - p_k L_l) (L_m p_n - p_m L_n) \\
&= \sum_{klmn} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) (L_k p_l L_m p_n - L_k p_l p_m L_n - p_k L_l L_m p_n + p_k L_l p_m L_n)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \left(p_l L_k + i\hbar \sum_s \varepsilon_{kls} p_s \right) \left(p_n L_m + i\hbar \sum_t \varepsilon_{mnt} p_t \right) \quad (\because [16.9]) \\
&\quad - \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \left(p_l L_k + i\hbar \sum_s \varepsilon_{kls} p_s \right) p_m L_n \\
&\quad - \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l \left(p_n L_m + i\hbar \sum_t \varepsilon_{mnt} p_t \right) \\
&\quad + \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_m L_n \\
&= \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_l L_k p_n L_m + i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} p_s p_n L_m \\
&\quad + i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{mns} p_l L_k p_s - \hbar^2 \sum_{klmnst} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} \varepsilon_{mnt} p_s p_t \\
&\quad - \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_l L_k p_m L_n - i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} p_s p_m L_n \\
&\quad - \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_n L_m - i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{mns} p_k L_l p_s \\
&\quad + \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_m L_n
\end{aligned}$$

となり、最後に得られた式の1項目で l と k 及び m と n を入れ替え ($\varepsilon_{ikl}\varepsilon_{jmn} = \varepsilon_{ilk}\varepsilon_{jnm}$ などに注意), 2項目で l と k を入れ替え, 3項目で m と n を入れ替えると,

$$\begin{aligned}
[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j] &= 4 \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_m L_n \\
&\quad + i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} p_s (p_n L_m - p_m L_n) \\
&\quad + i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{mns} (p_l L_k - p_k L_l) p_s \\
&\quad - \hbar^2 \sum_{klmnst} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} \varepsilon_{mnt} p_s p_t
\end{aligned}$$

が得られる. 右辺第2項で $(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_m L_n$ の添字 m と n を入れ替えれば, $-(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_n L_m$ となるので,

$$\begin{aligned}
[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j] &= 4 \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_m L_n \\
&\quad + 2i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} p_s p_n L_m \\
&\quad + 2i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{mns} p_l L_k p_s \\
&\quad - \hbar^2 \sum_{klmnst} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} \varepsilon_{mnt} p_s p_t \quad [16.14] \\
&\hspace{15em} [16.15]
\end{aligned}$$

[16.14] の第2項について,

$$2i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} p_s p_n L_m = 2i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} \varepsilon_{kli} p_i p_n L_m - \varepsilon_{imn}\varepsilon_{jkl} \varepsilon_{klj} p_j p_n L_m)$$

$$\begin{aligned}
&= 4i\hbar \sum_{mn} (\varepsilon_{jmn} p_i p_n L_m - \varepsilon_{imn} p_j p_n L_m) \\
&= 4i\hbar [p_j (\mathbf{p} \times \mathbf{L})_i - p_i (\mathbf{p} \times \mathbf{L})_j].
\end{aligned}$$

さらに、第2項について同様に、

$$\begin{aligned}
2i\hbar \sum_{klmns} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) \varepsilon_{mns} p_l L_k p_s &= 2i\hbar \sum_{klmn} (\varepsilon_{ikl} \varepsilon_{jmn} \varepsilon_{mnj} p_l L_k p_j - \varepsilon_{imn} \varepsilon_{jkl} \varepsilon_{mni} p_l L_k p_i) \\
&= 4i\hbar \sum_{kl} (\varepsilon_{ikl} p_l L_k p_j - \varepsilon_{jkl} p_l L_k p_i) \\
&= 4i\hbar [(\mathbf{p} \times \mathbf{L})_j p_i - (\mathbf{p} \times \mathbf{L})_i p_j].
\end{aligned}$$

ここで、次の関係式に注目する：

$$\begin{aligned}
[p_i, (\mathbf{p} \times \mathbf{L})_j] &= \sum_{mn} \varepsilon_{jmn} [p_i, p_m L_n] \\
&= \sum_{mn} \varepsilon_{jmn} (p_i p_m L_n - p_m L_n p_i) \\
&= \sum_{mn} \varepsilon_{jmn} \left[p_m p_i L_n - p_m \left(p_i L_n + i\hbar \sum_k \varepsilon_{nik} p_k \right) \right] \\
&= -i\hbar \sum_{mnk} \varepsilon_{jmn} \varepsilon_{nik} p_m p_k \\
&= -i\hbar \sum_n \varepsilon_{jin} \varepsilon_{nij} p_i p_j \\
&= i\hbar \sum_n (\varepsilon_{jin})^2 p_i p_j.
\end{aligned}$$

よって、[16.14] の第2項と第3項の和は、

$$\begin{aligned}
&2i\hbar \sum_{klmns} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) \varepsilon_{kls} p_s p_n L_m + 2i\hbar \sum_{klmns} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) \varepsilon_{mns} p_l L_k p_s \\
&= 4i\hbar [p_j (\mathbf{p} \times \mathbf{L})_i - p_i (\mathbf{p} \times \mathbf{L})_j] + 4i\hbar [(\mathbf{p} \times \mathbf{L})_j p_i - (\mathbf{p} \times \mathbf{L})_i p_j] \\
&= 4i\hbar [p_j, (\mathbf{p} \times \mathbf{L})_i] - 4i\hbar [p_i, (\mathbf{p} \times \mathbf{L})_j] \\
&= 0.
\end{aligned}$$

さらに、[16.14] の第4項について、

$$\begin{aligned}
\sum_{klmnst} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) \varepsilon_{kls} \varepsilon_{mnt} p_s p_t &= \sum_{klmnst} (\varepsilon_{ikl} \varepsilon_{jmn} \varepsilon_{kls} \varepsilon_{mnt} p_s p_t - \varepsilon_{imn} \varepsilon_{jkl} \varepsilon_{kls} \varepsilon_{mnt} p_s p_t) \\
&= \sum_{klmnst} (\varepsilon_{ikl} \varepsilon_{jmn} \varepsilon_{kls} \varepsilon_{mnt} p_s p_t - \varepsilon_{ikl} \varepsilon_{jmn} \varepsilon_{mnt} \varepsilon_{kls} p_t p_s) \\
&= 0.
\end{aligned}$$

以上から、[16.14] の3つの項は0となり、

$$[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j] = 4 \sum_{klmn} (\varepsilon_{ikl} \varepsilon_{jmn} - \varepsilon_{imn} \varepsilon_{jkl}) p_k L_l p_m L_n.$$

ここで、 $\varepsilon_{ikl} = \varepsilon_{ijl} \delta_{jk} + \varepsilon_{ikj} \delta_{jl}$ と変換する（他の部分についても同様、 i が添字に含まれていれば、それ以外の部分を j に変えて、もともとあった文字と δ を作る）。この変換が、表式を変えないことを確認しておこう。

- $l = k = j$ の時: $\varepsilon_{ikl} = \varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl} = 0$
- $l = k \neq j$ の時: $\varepsilon_{ikl} = \varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl} = 0$
- $l = j, k \neq j$ の時: $\varepsilon_{ikl} = \varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl} = \varepsilon_{ikj}$
- $l \neq j, k = j$ の時: $\varepsilon_{ikl} = \varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl} = \varepsilon_{ijl}$
- $l \neq k \neq j$ の時: $\varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl} = 0$. $i = j$ で, $(k, l) = (i+1, i+2), (i+2, i+1)$ の時は, ε_{ikl} が非零となるが, $\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl} = 0$ なので, 変換しても結局得られる結果は同じ 0 なので問題ない.

以上のことを踏まえて, 先ほど得られた式を変換すると,

$$[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j] \quad [16.16]$$

$$= 4 \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_m L_n$$

$$= 4 \sum_{klmn} [(\varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl})(\varepsilon_{jin}\delta_{im} + \varepsilon_{jmi}\delta_{in}) - (\varepsilon_{ijn}\delta_{jm} + \varepsilon_{imj}\delta_{jn})(\varepsilon_{jil}\delta_{ik} + \varepsilon_{jki}\delta_{il})] p_k \quad [16.17]$$

$$\times \left(p_m L_l + i\hbar \sum_s \varepsilon_{lms} p_s \right) L_n$$

$$= 4 \sum_{klmn} (\varepsilon_{ijl}\varepsilon_{jin}\delta_{jk}\delta_{im} + \varepsilon_{ijl}\varepsilon_{jmi}\delta_{jk}\delta_{in} + \varepsilon_{ikj}\varepsilon_{jin}\delta_{jl}\delta_{im} + \varepsilon_{ikj}\varepsilon_{jmi}\delta_{jl}\delta_{in}$$

$$- \varepsilon_{ijn}\varepsilon_{jil}\delta_{jm}\delta_{ik} - \varepsilon_{ijn}\varepsilon_{jki}\delta_{jm}\delta_{il} - \varepsilon_{imj}\varepsilon_{jil}\delta_{jn}\delta_{ik} - \varepsilon_{imj}\varepsilon_{jki}\delta_{jn}\delta_{il})$$

$$\times \left(p_k p_m L_l L_n + i\hbar \sum_s \varepsilon_{lms} p_k p_s L_n \right) \quad [16.18]$$

[16.18] の第 1 項は,

$$4 \sum_{klmn} (\varepsilon_{ijl}\varepsilon_{jin}\delta_{jk}\delta_{im} + \varepsilon_{ijl}\varepsilon_{jmi}\delta_{jk}\delta_{in} + \varepsilon_{ikj}\varepsilon_{jin}\delta_{jl}\delta_{im} + \varepsilon_{ikj}\varepsilon_{jmi}\delta_{jl}\delta_{in}$$

$$- \varepsilon_{ijn}\varepsilon_{jil}\delta_{jm}\delta_{ik} - \varepsilon_{ijn}\varepsilon_{jki}\delta_{jm}\delta_{il} - \varepsilon_{imj}\varepsilon_{jil}\delta_{jn}\delta_{ik} - \varepsilon_{imj}\varepsilon_{jki}\delta_{jn}\delta_{il}) p_k p_m L_l L_n$$

$$= 4 \sum_m (\varepsilon_{ijm})^2 (-p_j p_i L_m^2 + p_m p_j L_m L_i + p_m p_i L_j L_m - p_m^2 L_j L_i$$

$$+ p_i p_j L_m^2 - p_m p_j L_i L_m - p_i p_m L_m L_j + p_m^2 L_i L_j)$$

$$= 4 \sum_m (\varepsilon_{ijm})^2 p_m (p_i [L_j, L_m] + p_j [L_m, L_i] + p_m [L_i, L_j])$$

$$= 4i\hbar \sum_m (\varepsilon_{ijm})^2 p_m \left(p_i \sum_s \varepsilon_{jms} L_s + p_j \sum_s \varepsilon_{mis} L_s + p_m \sum_s \varepsilon_{ijs} L_s \right) \quad (\because [16.8])$$

$$= 4i\hbar \sum_m \varepsilon_{ijm} p_m (p_i L_i + p_j L_j + p_m L_m). \quad [16.19]$$

[16.18] の第 2 項は,

$$4 \sum_{klmn} (\varepsilon_{ijl}\varepsilon_{jin}\delta_{jk}\delta_{im} + \varepsilon_{ijl}\varepsilon_{jmi}\delta_{jk}\delta_{in} + \varepsilon_{ikj}\varepsilon_{jin}\delta_{jl}\delta_{im} + \varepsilon_{ikj}\varepsilon_{jmi}\delta_{jl}\delta_{in}$$

$$- \varepsilon_{ijn}\varepsilon_{jil}\delta_{jm}\delta_{ik} - \varepsilon_{ijn}\varepsilon_{jki}\delta_{jm}\delta_{il} - \varepsilon_{imj}\varepsilon_{jil}\delta_{jn}\delta_{ik} - \varepsilon_{imj}\varepsilon_{jki}\delta_{jn}\delta_{il}) i\hbar \sum_s \varepsilon_{lms} p_k p_s L_n$$

$$= 4i\hbar \sum_{kls} \varepsilon_{ijl}\varepsilon_{jin}\varepsilon_{lis} p_j p_s L_n + 4i\hbar \sum_{lms} \varepsilon_{ijl}\varepsilon_{jmi}\varepsilon_{lms} p_j p_s L_i$$

$$+ 4i\hbar \sum_{kns} \varepsilon_{ikj}\varepsilon_{jin}\varepsilon_{jis} p_k p_s L_n + 4i\hbar \sum_{kms} \varepsilon_{ikj}\varepsilon_{jmi}\varepsilon_{jms} p_k p_s L_i$$

$$[16.20]$$

$$\begin{aligned}
& -4i\hbar \sum_{lns} \varepsilon_{ijn} \varepsilon_{jil} \varepsilon_{ljs} p_i p_s L_n - 4i\hbar \sum_{kns} \varepsilon_{ijn} \varepsilon_{jki} \varepsilon_{ijs} p_k p_s L_n \\
& -4i\hbar \sum_{lms} \varepsilon_{imj} \varepsilon_{jil} \varepsilon_{lms} p_i p_n L_j - 4i\hbar \sum_{kms} \varepsilon_{imj} \varepsilon_{jki} \varepsilon_{ims} p_k p_s L_j \\
& = 4i\hbar \sum_m (-\varepsilon_{ijm} p_j^2 L_m + 0 - \varepsilon_{ijm} p_m^2 L_m - \varepsilon_{ijm} p_m p_i L_i - \varepsilon_{ijm} p_i^2 L_m - \varepsilon_{ijm} p_m^2 L_m - 0 - \varepsilon_{ijm} p_m p_j L_j) \\
& = -4i\hbar \sum_m \varepsilon_{ijm} [p_m (p_i L_i + p_j L_j) + (p_i^2 + p_j^2 + 2p_m^2) L_m].
\end{aligned} \tag{16.22}$$

よって, [16.18][16.19][16.22] から,

$$[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j] = -4i\hbar \mathbf{p}^2 \sum_m \varepsilon_{ijm} L_m. \tag{16.23}$$

次に,

$$\begin{aligned}
\left[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, \frac{r_j}{r}\right] &= \sum_{kl} \varepsilon_{ikl} \left[(L_k p_l - p_k L_l), \frac{r_j}{r}\right] \\
&= \sum_{kl} \varepsilon_{ikl} \left[(L_k p_l + p_l L_k), \frac{r_j}{r}\right] \\
&= 2 \sum_{kl} \varepsilon_{ikl} \left[p_l L_k, \frac{r_j}{r}\right] + i\hbar \sum_{kls} \varepsilon_{ikl} \varepsilon_{kls} \left[p_s, \frac{r_j}{r}\right]
\end{aligned} \tag{16.24}$$

[16.24] 第1項は,

$$\begin{aligned}
& \sum_{kl} \varepsilon_{ikl} \left[p_l L_k, \frac{r_j}{r}\right] \\
&= \sum_{klmn} \varepsilon_{ikl} \varepsilon_{kmn} \left[p_l r_m p_n, \frac{r_j}{r}\right] \\
&= \sum_{klmn} \varepsilon_{ikl} \varepsilon_{kmn} \left(p_l r_m p_n \frac{r_j}{r} - \frac{r_j}{r} p_l r_m p_n\right) \\
&= -\hbar^2 \sum_{klmn} \varepsilon_{ikl} \varepsilon_{kmn} \left(\partial_l r_m \partial_n \frac{r_j}{r} - \frac{r_j}{r} \partial_l r_m \partial_n\right) \\
&= -\hbar^2 \sum_{klmn} \varepsilon_{ikl} \varepsilon_{kmn} \left[\partial_l r_m \left(\frac{r_j}{r} \partial_n + \frac{\delta_{nj}}{r} - \frac{r_j r_n}{r^3}\right) - \frac{r_j}{r} \partial_l r_m \partial_n\right] \\
&= -\hbar^2 \sum_{klmn} \varepsilon_{ikl} \varepsilon_{kmn} \left[\frac{r_j}{r} \partial_l r_m \partial_n + \left(\frac{\delta_{lj}}{r} - \frac{r_l r_j}{r^3}\right) r_m \partial_n + \partial_l r_m \left(\frac{\delta_{nj}}{r} - \frac{r_j r_n}{r^3}\right) - \frac{r_j}{r} \partial_l r_m \partial_n\right] \\
&= -\hbar^2 \sum_{klmn} \varepsilon_{ikl} \varepsilon_{kmn} \left[\left(\frac{\delta_{lj}}{r} - \frac{r_l r_j}{r^3}\right) r_m \partial_n + \partial_l r_m \left(\frac{\delta_{nj}}{r} - \frac{r_j r_n}{r^3}\right)\right] \\
&= -i\hbar \sum_{kl} \varepsilon_{ikl} \left(\frac{\delta_{lj}}{r} - \frac{r_l r_j}{r^3}\right) L_k - i\hbar \sum_{klmn} \varepsilon_{ikl} \varepsilon_{kmn} p_l \frac{r_m}{r} \delta_{nj} + i\hbar \sum_{klmn} \varepsilon_{ikl} \varepsilon_{kmn} \partial_l \frac{r_m r_j r_n}{r^3} \\
&= -i\hbar \sum_k \varepsilon_{ikj} \frac{L_k}{r} + i\hbar \frac{r_j}{r^3} (\mathbf{L} \times \mathbf{r})_i - i\hbar \sum_{kl} \varepsilon_{ikl} \sum_m \varepsilon_{kmj} p_l \frac{r_m}{r}.
\end{aligned} \tag{16.26}$$

最後の式変形で, $\sum_{mn} \varepsilon_{kmn} r_m r_n = (\mathbf{r} \times \mathbf{r})_k = 0$ を使った. [16.24] 第2項は,

$$i\hbar \sum_{kls} \varepsilon_{ikl} \varepsilon_{kls} \left[p_s, \frac{r_j}{r}\right] = -\hbar^2 \sum_{kls} \varepsilon_{ikl} \varepsilon_{kls} \left(\partial_s \frac{r_j}{r} - \frac{r_j}{r} \partial_s\right)$$

$$\begin{aligned}
&= -\hbar^2 \sum_{kls} \varepsilon_{ikl} \varepsilon_{kls} \left(\frac{\delta_{sj}}{r} - \frac{r_j r_s}{r^3} \right) \\
&= -\hbar^2 \sum_{kl} (\varepsilon_{ikl})^2 \left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3} \right) \\
&= -2\hbar^2 \left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3} \right)
\end{aligned} \tag{16.27}$$

よって, [16.24][16.26][16.27] から,

$$\begin{aligned}
&\left[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, \frac{r_j}{r} \right] \\
&= -2i\hbar \sum_k \varepsilon_{ikj} \frac{L_k}{r} + 2i\hbar \frac{r_j}{r^3} (\mathbf{L} \times \mathbf{r})_i - 2i\hbar \sum_{kl} \varepsilon_{ikl} \sum_m \varepsilon_{kmj} p_l \frac{r_m}{r} + 2\hbar^2 \left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3} \right).
\end{aligned}$$

さらに,

$$\begin{aligned}
&\left[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, \frac{r_j}{r} \right] + \left[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j, \frac{r_i}{r} \right] \\
&= -2i\hbar \sum_k \varepsilon_{ikj} \frac{L_k}{r} + 2i\hbar \sum_k \varepsilon_{jki} \frac{L_k}{r} + 2i\hbar \frac{r_j}{r^3} (\mathbf{L} \times \mathbf{r})_i - 2i\hbar \frac{r_i}{r^3} (\mathbf{L} \times \mathbf{r})_j \\
&\quad - 2i\hbar \sum_{kl} \varepsilon_{ikl} \sum_m \varepsilon_{kmj} p_l \frac{r_m}{r} + 2i\hbar \sum_{km} \varepsilon_{ikm} \sum_l \varepsilon_{klj} p_m \frac{r_l}{r} \\
&= 4i\hbar \sum_k \varepsilon_{ijk} \frac{L_k}{r} - 2i\hbar \frac{1}{r^3} [r_i (\mathbf{L} \times \mathbf{r})_j - r_j (\mathbf{L} \times \mathbf{r})_i] - 2i\hbar \sum_{klm} \varepsilon_{ikl} \varepsilon_{kmj} (p_l r_m - p_m r_l) \frac{1}{r}.
\end{aligned} \tag{16.28}$$

[16.28] の第2項は,

$$\begin{aligned}
\frac{1}{r^3} [r_i (\mathbf{L} \times \mathbf{r})_j - r_j (\mathbf{L} \times \mathbf{r})_i] &= \frac{1}{r^3} \sum_l (\varepsilon_{ijl})^2 [r_i (\mathbf{L} \times \mathbf{r})_j - r_j (\mathbf{L} \times \mathbf{r})_i] \\
&= \frac{1}{r^3} \sum_l \varepsilon_{ijl} [\mathbf{r} \times (\mathbf{L} \times \mathbf{r})]_l \\
&= \frac{1}{r^3} \sum_l \varepsilon_{ijl} [\mathbf{r} \times (\mathbf{r}^2 \mathbf{p} - (\mathbf{r} \cdot \mathbf{p}) \mathbf{r})]_l \\
&= \frac{1}{r} \sum_l \varepsilon_{ijl} L_l,
\end{aligned}$$

[16.28] の第3項は,

$$\begin{aligned}
\sum_{klm} \varepsilon_{ikl} \varepsilon_{kmj} (p_l r_m - p_m r_l) \frac{1}{r} &= - \sum_k (\varepsilon_{ijk})^2 (p_j r_i - p_i r_j) \frac{1}{r} \\
&= - \sum_k (\varepsilon_{ijk})^2 (r_i p_j - r_j p_i) \frac{1}{r} \\
&= - \sum_k \varepsilon_{ijk} L_k \frac{1}{r} \\
&= - \frac{1}{r} \sum_k \varepsilon_{ijk} L_k
\end{aligned}$$

となるので, [16.28] は,

$$\left[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, \frac{r_j}{r} \right] + \left[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j, \frac{r_i}{r} \right] = 4i\hbar \sum_k \varepsilon_{ijk} \frac{L_k}{r}. \tag{16.29}$$

以上から, [16.23] と [16.29] を使えば,

$$\begin{aligned}
[R_i, R_j] &= \left[\frac{1}{2m} (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i + k_0 Z e^2 \frac{r_i}{r}, \frac{1}{2m} (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j + k_0 Z e^2 \frac{r_j}{r} \right] \\
&= \frac{1}{4m^2} [(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j] \\
&\quad + \frac{k_0 Z e^2}{2m} \left[\left[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_i, \frac{r_j}{r} \right] + \left[(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})_j, \frac{r_i}{r} \right] \right] \\
&= -i\hbar \frac{\mathbf{p}^2}{m^2} \sum_k \varepsilon_{ijk} L_k + 2i\hbar \frac{k_0 Z e^2}{m} \sum_k \varepsilon_{ijk} \frac{L_k}{r} \\
&= -\frac{2i\hbar}{m} \sum_k \varepsilon_{ijk} \left(\frac{\mathbf{p}^2}{2m} - k_0 Z e^2 \frac{1}{r} \right) L_k \\
&= -\frac{2i\hbar}{m} \sum_k \varepsilon_k H L_k
\end{aligned} \tag{16.30}$$

が得られる.

角運動量 \mathbf{L} と Runge-Lenz-Pauli ベクトル \mathbf{R} の内積を調べる. [16.3] から

$$\begin{aligned}
&\mathbf{L} \cdot (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}) \\
&= \left[\sum_{lmn} \varepsilon_{lmn} r_l p_m \mathbf{e}_n \right] \cdot \left[\sum_{ijk} [(\mathbf{r} \cdot \mathbf{p}) p_k - i\hbar p_k - r_k \mathbf{p}^2] \mathbf{e}_k (\varepsilon_{ijk})^2 \right] \\
&= \sum_{ijklmn} \delta_{kn} \varepsilon_{lmn} (\varepsilon_{ijk})^2 r_l p_m [(\mathbf{r} \cdot \mathbf{p}) p_k - i\hbar p_k - r_k \mathbf{p}^2] \\
&= \sum_{ijklm} \varepsilon_{lmk} (\varepsilon_{ijk})^2 r_l p_m [(\mathbf{r} \cdot \mathbf{p}) p_k - i\hbar p_k - r_k \mathbf{p}^2] \\
&= \sum_{ijk} (\varepsilon_{ijk})^2 [r_i p_j \varepsilon_{ijk} + r_j p_i \varepsilon_{jik}] [(\mathbf{r} \cdot \mathbf{p}) p_k - i\hbar p_k - r_k \mathbf{p}^2] \\
&= \sum_{ijk} (\varepsilon_{ijk})^3 (r_i p_j - r_j p_i) [(\mathbf{r} \cdot \mathbf{p}) p_k - i\hbar p_k - r_k \mathbf{p}^2] \\
&= 2 \sum_{ijk} \varepsilon_{ijk} r_i p_j [(\mathbf{r} \cdot \mathbf{p}) p_k - i\hbar p_k - r_k \mathbf{p}^2] \\
&= 2 \sum_i r_i [p_{i+1} (\mathbf{r} \cdot \mathbf{p}) p_{i-1} - p_{i-1} (\mathbf{r} \cdot \mathbf{p}) p_{i+1}] - 2i\hbar \sum_i r_i (p_{i+1} p_{i-1} - p_{i-1} p_{i+1}) \\
&\quad - 2i\hbar \sum_i r_i (p_{i+1} r_{i-1} - p_{i-1} r_{i+1}) \mathbf{p}^2 \\
&= 2 \sum_i r_i [p_{i+1} (\mathbf{r} \cdot \mathbf{p}) p_{i-1} - p_{i-1} (\mathbf{r} \cdot \mathbf{p}) p_{i+1}] - 2i\hbar \sum_i (r_i r_{i-1} p_{i+1} \mathbf{p}^2 - r_i r_{i+1} p_{i-1} \mathbf{p}^2) \\
&= 2 \sum_{is} [r_i p_{i+1} r_s p_s p_{i-1} - r_i p_{i-1} r_s p_s p_{i+1}] - 2i\hbar \sum_i (r_i r_{i-1} p_{i+1} \mathbf{p}^2 - r_{i-1} r_i p_{i+1} \mathbf{p}^2) \\
&= 2 \sum_{is} [r_i (r_s p_{i+1} - i\hbar \delta_{s,i+1}) p_s p_{i-1} - r_i (r_s p_{i-1} - i\hbar \delta_{s,i-1}) p_s p_{i+1}] \\
&= -2i\hbar \sum_{is} [r_i \delta_{s,i+1} p_s p_{i-1} - r_i \delta_{s,i-1} p_s p_{i+1}] \\
&= -2i\hbar \sum_i [r_i p_{i+1} p_{i-1} - r_i p_{i-1} p_{i+1}] \\
&= 0.
\end{aligned} \tag{16.31}$$

[16.11][16.12] から

$$[L_i, (\mathbf{L} \times \mathbf{p})_i] = [L_i, (\mathbf{p} \times \mathbf{L})_i] = 0$$

なので, [16.33] と合わせて

$$(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L}) \cdot \mathbf{L} = 0. \quad [16.34]$$

\mathbf{R} の第2項に関しては

$$\begin{aligned} \mathbf{L} \cdot \frac{\mathbf{r}}{r} &= \left(\sum_{ijk} r_j p_k \mathbf{e}_i \right) \cdot \left(\sum_l \mathbf{e}_l \frac{r_l}{r} \right) \\ &= \sum_{ijkl} \varepsilon_{ijk} \delta_{il} r_j p_k \frac{r_l}{r} \\ &= \sum_{ijk} \varepsilon_{ijk} r_i r_j p_k \frac{1}{r} \\ &= \frac{i\hbar}{r^2} \sum_{ijk} \varepsilon_{ijk} r_i r_j r_k \\ &= \frac{i\hbar}{r^2} \sum_i r_i (r_{i+1} r_{i-1} - r_{i-1} r_{i+1}) \\ &= 0. \end{aligned} \quad [16.35]$$

さらに,

$$\begin{aligned} \frac{\mathbf{r}}{r} \cdot \mathbf{L} &= \left(\sum_l \mathbf{e}_l \frac{r_l}{r} \right) \cdot \left(\sum_{ijk} r_j p_k \mathbf{e}_i \right) \\ &= \sum_{ijkl} \varepsilon_{ijk} \delta_{li} \frac{r_l}{r} r_j p_k \\ &= \sum_{ijk} \varepsilon_{ijk} \frac{r_i r_j}{r} p_k \\ &= \sum_k \frac{r_{k+1} r_{k-1} - r_{k-1} r_{k+1}}{r} p_k \\ &= 0. \end{aligned} \quad [16.36]$$

[16.33][16.35] から

$$\mathbf{L} \cdot \mathbf{R} = 0.$$

[16.34][16.36] から

$$\mathbf{R} \cdot \mathbf{L} = 0.$$

最後に, \mathbf{R}^2 を計算する. まずは三重積について:

$$\begin{aligned} \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= \sum_{ijk} A_i (\varepsilon_{ijk} B_j C_k) \\ &= \sum_{ijk} (\varepsilon_{kij} A_i B_j) C_k \\ &= (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}. \end{aligned} \quad [16.37]$$

[16.9] から

$$\begin{aligned}
(\mathbf{p} \times \mathbf{L})_i &= \sum_{jk} \varepsilon_{ijk} p_j L_k \\
&= \sum_{jk} \varepsilon_{ijk} \left[L_k p_j - i\hbar \sum_l \varepsilon_{kjl} p_l \right] \\
&= \sum_{jk} \varepsilon_{ijk} L_k p_j - i\hbar \sum_{jkl} \varepsilon_{ijk} \varepsilon_{kjl} p_l \\
&= - \sum_{jk} \varepsilon_{ikj} L_k p_j + i\hbar \sum_{jkl} \varepsilon_{ijk} \varepsilon_{ljk} p_l \\
&= -(\mathbf{L} \times \mathbf{p})_i + i\hbar \sum_{jk} (\varepsilon_{ijk})^2 p_i \\
&= -(\mathbf{L} \times \mathbf{p})_i + 2i\hbar p_i
\end{aligned}$$

となるので,

$$\mathbf{p} \times \mathbf{L} = -\mathbf{L} \times \mathbf{p} + 2i\hbar \mathbf{p}. \quad [16.38]$$

[16.37][16.38] から,

$$\begin{aligned}
\mathbf{p} \cdot (\mathbf{p} \times \mathbf{L}) &= (\mathbf{p} \times \mathbf{p}) \cdot \mathbf{L} \\
&= 0.
\end{aligned} \quad [16.39]$$

$$\begin{aligned}
\mathbf{p} \cdot (\mathbf{L} \times \mathbf{p}) &= \mathbf{p} \cdot (-\mathbf{p} \times \mathbf{L} + 2i\hbar \mathbf{p}) \\
&= 2i\hbar \mathbf{p}^2.
\end{aligned} \quad [16.40]$$

[16.9][16.37] から

$$\begin{aligned}
(\mathbf{p} \times \mathbf{L}) \cdot (\mathbf{p} \times \mathbf{L}) &= \left(\sum_{ijk} \varepsilon_{ijk} p_i L_j \mathbf{e}_k \right) \left(\sum_{lmn} \varepsilon_{lmn} p_l L_m \mathbf{e}_n \right) \\
&= \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} \delta_{kn} p_i L_j p_l L_m \\
&= \sum_{ijklm} \varepsilon_{ijk} \varepsilon_{lmk} p_i L_j p_l L_m \\
&= \sum_{ijlm} (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) p_i L_j p_l L_m \\
&= \sum_{ij} (p_i L_j p_i L_j - p_i L_j p_j L_i) \\
&= \sum_{ij} p_i \left(p_i L_j + i\hbar \sum_k \varepsilon_{jik} p_k \right) L_j - \sum_i p_i (\mathbf{L} \cdot \mathbf{p}) L_i \\
&= \sum_{ij} p_i \left(p_i L_j - i\hbar \sum_k \varepsilon_{ikj} p_k \right) L_j - \sum_i p_i [(\mathbf{r} \times \mathbf{p}) \cdot \mathbf{p}] \\
&= \mathbf{p}^2 \mathbf{L}^2 + i\hbar \sum_{ijk} \varepsilon_{ikj} p_i p_k L_j - \sum_i p_i [\mathbf{r} \cdot (\mathbf{p} \times \mathbf{p})] \\
&= \mathbf{p}^2 \mathbf{L}^2 + i\hbar \sum_{ijk} \varepsilon_{ikj} p_i p_k L_j
\end{aligned}$$

$$\begin{aligned}
&= \mathbf{p}^2 \mathbf{L}^2 + i\hbar(\mathbf{p} \times \mathbf{p}) \cdot \mathbf{L} \\
&= \mathbf{p}^2 \mathbf{L}^2.
\end{aligned} \tag{16.41}$$

[16.38][16.39][16.40][16.41] から

$$\begin{aligned}
(\mathbf{p} \times \mathbf{L}) \cdot (\mathbf{L} \times \mathbf{p}) &= (\mathbf{p} \times \mathbf{L}) \cdot (-\mathbf{p} \times \mathbf{L} + 2i\hbar\mathbf{p}) \\
&= -\mathbf{p}^2 \mathbf{L}^2 + 2i\hbar(\mathbf{p} \times \mathbf{L}) \cdot \mathbf{p} \\
&= -\mathbf{p}^2 \mathbf{L}^2 + 2i\hbar\mathbf{p} \cdot (\mathbf{L} \times \mathbf{p}) \\
&= -\mathbf{p}^2 \mathbf{L}^2 - 4\hbar^2 \mathbf{p}^2,
\end{aligned} \tag{16.42}$$

$$\begin{aligned}
(\mathbf{L} \times \mathbf{p}) \cdot (\mathbf{p} \times \mathbf{L}) &= (-\mathbf{p} \times \mathbf{L} + 2i\hbar\mathbf{p}) \cdot (\mathbf{p} \times \mathbf{L}) \\
&= -\mathbf{p}^2 \mathbf{L}^2 + 2i\hbar\mathbf{p} \cdot (\mathbf{p} \times \mathbf{L}) \\
&= -\mathbf{p}^2 \mathbf{L}^2,
\end{aligned} \tag{16.43}$$

$$\begin{aligned}
(\mathbf{L} \times \mathbf{p}) \cdot (\mathbf{L} \times \mathbf{p}) &= (-\mathbf{p} \times \mathbf{L} + 2i\hbar\mathbf{p}) \cdot (-\mathbf{p} \times \mathbf{L} + 2i\hbar\mathbf{p}) \\
&= \mathbf{p}^2 \mathbf{L}^2 - 2i\hbar(\mathbf{p} \times \mathbf{L}) \cdot \mathbf{p} - 2i\hbar\mathbf{p} \cdot (\mathbf{p} \times \mathbf{L}) - 4\hbar^2 \mathbf{p}^2 \\
&= \mathbf{p}^2 \mathbf{L}^2 - 2i\hbar\mathbf{p} \cdot (\mathbf{L} \times \mathbf{p}) - 4\hbar^2 \mathbf{p}^2 \\
&= \mathbf{p}^2 \mathbf{L}^2.
\end{aligned} \tag{16.44}$$

また,

$$\begin{aligned}
\mathbf{L} \frac{1}{r} &= \sum_{ijk} r_i p_j \mathbf{e}_k \varepsilon_{ijk} \frac{1}{r} + \frac{1}{r} \mathbf{L} \\
&= -i\hbar \sum_{ijk} \varepsilon_{ijk} r_i \mathbf{e}_k \partial_j \frac{1}{r} + \frac{1}{r} \mathbf{L} \\
&= \frac{i\hbar}{r^3} \sum_{ijk} \varepsilon_{ijk} r_i r_j \mathbf{e}_k + \frac{1}{r} \mathbf{L} \\
&= \frac{i\hbar}{r^2} \sum_k (r_{k+1} r_{k-1} - r_{k-1} r_{k+1}) \mathbf{e}_k + \frac{1}{r} \mathbf{L} \\
&= \frac{1}{r} \mathbf{L}
\end{aligned}$$

となるので,

$$\begin{aligned}
\mathbf{L}^2 \frac{1}{r} &= \mathbf{L} \left(\mathbf{L} \frac{1}{r} \right) \\
&= \mathbf{L} \left(\frac{1}{r} \mathbf{L} \right) \\
&= \mathbf{L} \frac{1}{r} + \frac{1}{r} \mathbf{L}^2 \\
&= \frac{1}{r} \mathbf{L}^2.
\end{aligned} \tag{16.45}$$

[16.37][16.38][16.45] から

$$\begin{aligned}
(\mathbf{L} \times \mathbf{p}) \cdot \frac{\mathbf{r}}{r} &= \mathbf{L} \cdot (\mathbf{p} \times \mathbf{r}) \frac{1}{r} \\
&= -\mathbf{L}^2 \frac{1}{r}
\end{aligned} \tag{16.46}$$

$$= -\frac{1}{r}\mathbf{L}^2, \quad [16.47]$$

$$\begin{aligned} (\mathbf{p} \times \mathbf{L}) \cdot \frac{\mathbf{r}}{r} &= (-\mathbf{L} \times \mathbf{p} + 2i\hbar\mathbf{p}) \cdot \frac{\mathbf{r}}{r} \\ &= \mathbf{L}^2 \frac{1}{r} + 2i\hbar\mathbf{p} \cdot \mathbf{r} \frac{1}{r} \\ &= \frac{1}{r}\mathbf{L}^2 + 2i\hbar(\mathbf{r} \cdot \mathbf{p} - 3i\hbar) \frac{1}{r} \\ &= \frac{1}{r}\mathbf{L}^2 + \frac{6\hbar^2}{r} + 2\hbar^2\mathbf{r} \cdot \nabla \frac{1}{r} \\ &= \frac{1}{r}\mathbf{L}^2 + \frac{6\hbar^2}{r} + 2\hbar^2 \frac{\mathbf{r}}{r} \cdot \nabla - \frac{2\hbar^2}{r} \\ &= \frac{1}{r}\mathbf{L}^2 + \frac{4\hbar^2}{r} + \frac{2i\hbar}{r}\mathbf{r} \cdot \mathbf{p}, \end{aligned} \quad [16.48]$$

$$\begin{aligned} \frac{\mathbf{r}}{r} \cdot (\mathbf{p} \times \mathbf{L}) &= \frac{1}{r}(\mathbf{r} \times \mathbf{p}) \cdot \mathbf{L} \\ &= \frac{1}{r}\mathbf{L}^2, \end{aligned} \quad [16.49]$$

$$\begin{aligned} \frac{\mathbf{r}}{r} \cdot (\mathbf{L} \times \mathbf{p}) &= \frac{\mathbf{r}}{r} \cdot (-\mathbf{p} \times \mathbf{L} + 2i\hbar\mathbf{p}) \\ &= -\frac{1}{r}\mathbf{r} \cdot (\mathbf{p} \times \mathbf{L}) + \frac{2i\hbar}{r}\mathbf{r} \cdot \mathbf{p} \\ &= -\frac{1}{r}(\mathbf{r} \times \mathbf{p}) \cdot \mathbf{L} + \frac{2i\hbar}{r}\mathbf{r} \cdot \mathbf{p} \\ &= -\frac{1}{r}\mathbf{L}^2 + \frac{2i\hbar}{r}\mathbf{r} \cdot \mathbf{p}. \end{aligned} \quad [16.50]$$

[16.42][16.41][16.43][16.44] から

$$\begin{aligned} (\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})^2 &= (\mathbf{L} \times \mathbf{p}) \cdot (\mathbf{L} \times \mathbf{p}) - (\mathbf{L} \times \mathbf{p}) \cdot (\mathbf{p} \times \mathbf{L}) - (\mathbf{p} \times \mathbf{L}) \cdot (\mathbf{L} \times \mathbf{p}) - (\mathbf{p} \times \mathbf{L}) \cdot (\mathbf{p} \times \mathbf{L}) \\ &= 4\mathbf{p}^2\mathbf{L}^2 + 4\hbar^2\mathbf{p}^2. \end{aligned} \quad [16.51]$$

[16.47][16.48][16.49][16.50] から

$$(\mathbf{L} \times \mathbf{p}) \cdot \frac{\mathbf{r}}{r} - (\mathbf{p} \times \mathbf{L}) \cdot \frac{\mathbf{r}}{r} + \frac{\mathbf{r}}{r} \cdot (\mathbf{L} \times \mathbf{p}) - \frac{\mathbf{r}}{r} \cdot (\mathbf{p} \times \mathbf{L}) = -\frac{4}{r}\mathbf{L}^2 - \frac{4\hbar^2}{r}. \quad [16.52]$$

[16.51][16.52] から

$$\begin{aligned} \mathbf{R}^2 & \quad [16.53] \\ &= \frac{1}{4m}(\mathbf{L} \times \mathbf{p} - \mathbf{p} \times \mathbf{L})^2 + \frac{k_0Ze^2}{2m} \left[(\mathbf{L} \times \mathbf{p}) \cdot \frac{\mathbf{r}}{r} - (\mathbf{p} \times \mathbf{L}) \cdot \frac{\mathbf{r}}{r} + \frac{\mathbf{r}}{r} \cdot (\mathbf{L} \times \mathbf{p}) - \frac{\mathbf{r}}{r} \cdot (\mathbf{p} \times \mathbf{L}) \right] + (k_0Ze^2)^2 \\ &= \frac{2}{m} \left(\frac{\mathbf{p}^2}{2m} - \frac{k_0Ze^2}{r} \right) (\mathbf{L}^2 + \hbar^2) + (k_0Ze^2)^2 \\ &= \frac{2}{m}H(\mathbf{L}^2 + \hbar^2) + (k_0Ze^2)^2 \end{aligned} \quad [16.54]$$

が得られる。