相対論的量子力学（川村）

使用しているのは第 1 版第 1 刷.

第 1 章

Dirac 方程式の導出

小さい一様磁場 $\boldsymbol{B}$ に対し $(\boldsymbol{\nabla}+ie/2\hbar(\boldsymbol{B}\times\boldsymbol{x}))^2=\boldsymbol{\nabla}^2+ie/\hbar(\boldsymbol{B}\times\boldsymbol{x})\cdot\boldsymbol{\nabla}$．($\boldsymbol{\nabla}(\boldsymbol{B}\times\boldsymbol{x})=0$，$\boldsymbol{B}^2$ は無視) なので，結局 $\boldsymbol{\nabla}^2+ie/\hbar\boldsymbol{B}\cdot(\boldsymbol{x}\times\boldsymbol{\nabla})$ で角運動量が出てくる．

第 2 章

Dirac 方程式の Lorentz 共変性

$S^{-1}(\Lambda)\sigma^{\mu\nu}S(\Lambda) = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\sigma^{\mu\nu}$ の証明：(2.16) と (2.23) から従う.

第 3 章

γ 行列に関する基本定理，カイラル表示

■Γ_i の性質 (1)〜(5)　電子のスピンに関する Pauli 行列は次の式で与えられる：

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

さらに，γ 行列

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

と $\tilde{\Gamma}$ 行列

$$\begin{aligned}
&\tilde{\Gamma}_1 = I, && \tilde{\Gamma}_2 = \gamma^0, && \\
&\tilde{\Gamma}_3 = i\gamma^1, && \tilde{\Gamma}_4 = i\gamma^2, && \tilde{\Gamma}_5 = i\gamma^3, \\
&\tilde{\Gamma}_6 = -\gamma^0\gamma^1, && \tilde{\Gamma}_7 = -\gamma^0\gamma^2, && \tilde{\Gamma}_8 = -\gamma^0\gamma^3, \\
&\tilde{\Gamma}_9 = i\gamma^1\gamma^2, && \tilde{\Gamma}_{10} = i\gamma^2\gamma^3, && \tilde{\Gamma}_{11} = i\gamma^3\gamma^1, \\
&\tilde{\Gamma}_{12} = i\gamma^0\gamma^1\gamma^2\gamma^3, && \tilde{\Gamma}_{13} = \gamma^1\gamma^2\gamma^3, && \\
&\tilde{\Gamma}_{14} = -i\gamma^0\gamma^2\gamma^3, && \tilde{\Gamma}_{15} = -i\gamma^0\gamma^1\gamma^3, && \tilde{\Gamma}_{16} = -i\gamma^0\gamma^1\gamma^2,
\end{aligned}$$

を考える．$\tilde{\Gamma}$ 行列について次のような性質が成り立つ：

1. 全ての n に対し $(\tilde{\Gamma}_n)^2 = I$.
2. 全ての n, m に対し $\tilde{\Gamma}_n\tilde{\Gamma}_m = \xi_{nm}\tilde{\Gamma}_l$ となる $l = L_{nm}$ が存在し，$\xi_{nm} \in \{\pm 1, \pm i\}$．$n \neq m$ ならば $L_{nm} \neq 1$．さらに，L_{nm} の各行には $1, \ldots 16$ が 1 回ずつ出現する．
3. $\tilde{\Gamma}_n\tilde{\Gamma}_m = \pm\tilde{\Gamma}_m\tilde{\Gamma}_n$.
4. $n \neq 1$ に対して $\tilde{\Gamma}_n\tilde{\Gamma}_m = -\tilde{\Gamma}_m\tilde{\Gamma}_n$ となる m が存在する．
5. $n \neq 1$ に対して $\mathrm{Tr}(\tilde{\Gamma}_n) = 0$.

python でコード[*1]を書いて確かめる．まず，ξ は

$$\xi = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & -i & -i & -i & +i & +i & +i & -1 & -1 & 1 & +i & -i & -1 & 1 & -1 \\
1 & +i & 1 & +i & -i & -i & -1 & -1 & -i & -1 & +i & -i & -1 & +i & -1 & -1 \\
1 & +i & -i & 1 & +i & 1 & -i & -1 & +i & -i & -1 & +i & -1 & -1 & -i & 1 \\
1 & +i & +i & -i & 1 & 1 & 1 & -i & -1 & +i & -i & -i & -1 & 1 & 1 & +i \\
1 & -i & +i & 1 & 1 & 1 & +i & -i & -i & -1 & +i & -1 & +i & -i & 1 & 1 \\
1 & -i & -1 & +i & 1 & -i & 1 & +i & +i & -i & -1 & -1 & -i & 1 & +i & -1 \\
1 & -i & -1 & -1 & +i & +i & -i & 1 & -1 & +i & -i & -1 & +i & -1 & -1 & -i \\
1 & -1 & +i & -i & -1 & +i & -i & -1 & 1 & +i & -i & -1 & -1 & -i & +i & -1 \\
1 & -1 & -1 & +i & -i & -1 & +i & -i & -i & 1 & +i & -1 & -1 & -1 & -i & +i \\
1 & 1 & -i & -1 & +i & -i & -1 & +i & +i & -i & 1 & -1 & -1 & -i & 1 & +i \\
1 & -i & +i & -i & +i & -1 & -1 & -1 & -1 & -1 & -1 & 1 & +i & -i & +i & -i \\
1 & +i & -1 & -1 & -1 & -i & +i & -i & -1 & -1 & -1 & -i & 1 & +i & -i & +i \\
1 & -1 & -i & -1 & 1 & +i & 1 & -1 & +i & -1 & +i & +i & -i & 1 & -i & -i \\
1 & 1 & -1 & +i & 1 & 1 & -i & -1 & -i & +i & 1 & -i & +i & +i & 1 & -i \\
1 & -1 & -1 & 1 & -i & 1 & -1 & +i & -1 & -i & -i & +i & -i & +i & +i & 1
\end{pmatrix}$$

となる．L は

$$L = \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\
2 & 1 & 6 & 7 & 8 & 3 & 4 & 5 & 16 & 14 & 15 & 13 & 12 & 10 & 11 & 9 \\
3 & 6 & 1 & 9 & 11 & 2 & 16 & 15 & 4 & 13 & 5 & 14 & 10 & 12 & 8 & 7 \\
4 & 7 & 9 & 1 & 10 & 16 & 2 & 14 & 3 & 5 & 13 & 15 & 11 & 8 & 12 & 6 \\
5 & 8 & 11 & 10 & 1 & 15 & 14 & 2 & 13 & 4 & 3 & 16 & 9 & 7 & 6 & 12 \\
6 & 3 & 2 & 16 & 15 & 1 & 9 & 11 & 7 & 12 & 8 & 10 & 14 & 13 & 5 & 4 \\
7 & 4 & 16 & 2 & 14 & 9 & 1 & 10 & 6 & 8 & 12 & 11 & 15 & 5 & 13 & 3 \\
8 & 5 & 15 & 14 & 2 & 11 & 10 & 1 & 12 & 7 & 6 & 9 & 16 & 4 & 3 & 13 \\
9 & 16 & 4 & 3 & 13 & 7 & 6 & 12 & 1 & 11 & 10 & 8 & 5 & 15 & 14 & 2 \\
10 & 14 & 13 & 5 & 4 & 12 & 8 & 7 & 11 & 1 & 9 & 6 & 3 & 2 & 16 & 15 \\
11 & 15 & 5 & 13 & 3 & 8 & 12 & 6 & 10 & 9 & 1 & 7 & 4 & 16 & 2 & 14 \\
12 & 13 & 14 & 15 & 16 & 10 & 11 & 9 & 8 & 6 & 7 & 1 & 2 & 3 & 4 & 5 \\
13 & 12 & 10 & 11 & 9 & 14 & 15 & 16 & 5 & 3 & 4 & 2 & 1 & 6 & 7 & 8 \\
14 & 10 & 12 & 8 & 7 & 13 & 5 & 4 & 15 & 2 & 16 & 3 & 6 & 1 & 9 & 11 \\
15 & 11 & 8 & 12 & 6 & 5 & 13 & 3 & 14 & 16 & 2 & 4 & 7 & 9 & 1 & 10 \\
16 & 9 & 7 & 6 & 12 & 4 & 3 & 13 & 2 & 15 & 14 & 5 & 8 & 11 & 10 & 1
\end{pmatrix}$$

[*1] `./src/py/p_34.py`

となる．可換・反可換性については

$$
\begin{pmatrix}
+ & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
+ & + & - & - & - & - & - & - & + & + & + & - & - & + & + & + \\
+ & - & + & - & - & - & + & + & - & + & - & - & + & - & + & + \\
+ & - & - & + & - & + & - & + & - & - & + & - & + & + & - & + \\
+ & - & - & - & + & + & + & - & + & - & - & - & + & + & + & - \\
+ & - & - & + & + & + & - & - & - & + & - & + & - & - & + & + \\
+ & - & + & - & + & - & + & - & - & - & + & + & - & + & - & + \\
+ & - & + & + & - & - & - & + & + & - & - & + & - & + & + & - \\
+ & + & - & - & + & - & - & + & + & - & - & + & + & - & - & + \\
+ & + & + & - & - & + & - & - & - & + & - & + & + & + & - & - \\
+ & + & - & + & - & - & + & - & - & - & + & + & + & - & + & - \\
+ & - & - & - & - & + & + & + & + & + & + & + & - & - & - & - \\
+ & - & + & + & + & - & - & - & + & + & + & - & + & - & - & - \\
+ & + & - & + & + & - & + & + & - & + & - & - & - & + & - & - \\
+ & + & + & - & + & + & - & + & - & - & + & - & - & - & + & - \\
+ & + & + & + & - & + & + & - & + & - & - & - & - & - & - & +
\end{pmatrix}
$$

となる．

第 4 章

Dirac 方程式の解

■(4.17)　慣性系 I での 4 元運動量は $\varepsilon_s p_\mu$ になってる，として計算すればいい（エネルギーも運動量も負）．$\gamma^0 S(\Lambda)^\dagger \gamma^0 = S^{-1}(\Lambda)$ (2.45)．

■(4.70)　(4.6)(4.11)（u と $w(\boldsymbol{p})$ の対応は (4.28)）から分かる．

■(4.73)　(4.70) と同様．$\psi = S(\Lambda) w^s(0) \exp(-i\varepsilon_s p_\mu x^\mu/\hbar)$．入射粒子の運動量は $\boldsymbol{p} = (0, 0, \hbar k)$ なので (4.11) から定まる $S(\Lambda)$ の $(1, 2)$ 成分は $c\hbar k\sigma^3/(E + mc^2)$．$w^s(0)$ については今回は正エネルギーの粒子を考えているので (4.73) では $w^1(0)$．(4.74) では反射なので $\boldsymbol{p} = (0, 0, -\hbar k)$．第 1 項はスピン正なので $w^1(0)$，第 2 項はスピン負なので $w^2(0)$．

■(4.81)　確率流れは p.29 の最後らへんから $j^\mu = c\bar{\psi}\gamma^\mu\psi = c\psi^\dagger\gamma^0\gamma^\mu\psi = c\psi^\dagger\gamma^0\gamma^0\alpha^\mu\psi = c\psi^\dagger\alpha^\mu\psi$．$b_r = b_t = 0$ なんで結局作用するのは α^3 だけ．

第 5 章

Dirac 方程式の非相対論的極限

■(5.10)　(1.10) 使えば

$$\begin{pmatrix} 0 & \sigma^i p_i \\ -\sigma^i p_i & 0 \end{pmatrix}^2 = -|\boldsymbol{p}|^2 I$$

なので

$$\begin{aligned} U &= \sum_{n=0}^{\infty} \frac{1}{n!} (\beta \boldsymbol{\alpha} \cdot \boldsymbol{p} \theta)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & \sigma^i p_i \\ -\sigma^i p_i & 0 \end{pmatrix}^n \theta^n \\ &= I \left[1 - \frac{(|\boldsymbol{p}|\theta)^2}{2!} + \frac{(|\boldsymbol{p}|\theta)^4}{4!} - \cdots \right] + \begin{pmatrix} 0 & \frac{\sigma^i p_i}{|\boldsymbol{p}|} \\ -\frac{\sigma^i p_i}{|\boldsymbol{p}|} & 0 \end{pmatrix} \left[\frac{(|\boldsymbol{p}|\theta)}{1!} - \frac{(|\boldsymbol{p}|\theta)^3}{3!} + \frac{(|\boldsymbol{p}|\theta)^5}{5!} - \cdots \right] \\ &= \cos(|\boldsymbol{p}|\theta) + \frac{\beta \boldsymbol{\alpha} \cdot \boldsymbol{p}}{|\boldsymbol{p}|} \sin(|\boldsymbol{p}|\theta). \end{aligned}$$

■(5.11)　cos の方は

$$\cos(|\boldsymbol{p}|\theta)(c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m c^2) = (c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m c^2) \cos(|\boldsymbol{p}|\theta).$$

sin は $\beta(\boldsymbol{\alpha} \cdot \boldsymbol{p}) = -(\boldsymbol{\alpha} \cdot \boldsymbol{p})\beta$ 使って

$$\frac{\beta \boldsymbol{\alpha} \cdot \boldsymbol{p}}{|\boldsymbol{p}|} \sin(|\boldsymbol{p}|\theta)(c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m c^2) = -(c\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m c^2) \frac{\beta \boldsymbol{\alpha} \cdot \boldsymbol{p}}{|\boldsymbol{p}|} \sin(|\boldsymbol{p}|\theta)$$

なので (5.11) で $e^{iS} H = -H e^{iS}$ になる.

第 6 章

水素原子

■(6.11)　要はスピン角運動量と軌道角運動量の合成．正確に書くと $C \times (1,0) \otimes Y_{l,m-1/2} + D \times (0,1) \otimes Y_{l,m+1/2}$．$C, D$ は Clebsh-Gordon の表で求まる．JJSakurai(3.8.64) とか.

■(6.32)　$-1/2$ 乗を 3 次まで展開.

■(6.42)　合流型超幾何函数が 1 なので積分とっても楽.

■(6.46)　$\partial_1 \partial_2 r^{-1}$ みたいな項が無くなるのは，1s を考えてるから．つまり，波動関数が完全球対称なので，xy/r^5 をかけて（まず x で）積分したら 0 になる.

6.1 の計算詳細

原子番号 Z の原子内に存在する電子の Dirac 方程式は

$$\left(-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m_e c^2 - k_0\frac{Ze^2}{r}\right)\psi = E\psi \tag{6.0.1}$$

で与えられる．全角運動量の 2 乗 $\boldsymbol{J}^2$，J_z，軌道角運動量の 2 乗 $\boldsymbol{L}^2$ の固有関数は，$j = l \pm 1/2$ に対応して

$$\varphi_{jm}^{(+)} = \begin{pmatrix} \sqrt{\frac{j+m}{2j}}Y_{j-\frac{1}{2},m-\frac{1}{2}} \\ \sqrt{\frac{j-m}{2j}}Y_{j-\frac{1}{2},m-\frac{1}{2}} \end{pmatrix}, \quad \varphi_{jm}^{(-)} = \begin{pmatrix} \sqrt{\frac{j-m+1}{2j+2}}Y_{j+\frac{1}{2},m-\frac{1}{2}} \\ -\sqrt{\frac{j+m+1}{2j+2}}Y_{j+\frac{1}{2},m+\frac{1}{2}} \end{pmatrix} \tag{6.0.2}$$

で与えられ，

$$\varphi_{jm}^{(+)} = \boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}\varphi_{jm}^{(-)}, \quad \hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$$

が成立する．球面調和関数 Y_{lm} のパリティは $(-1)^l$ なので，これらの関数のパリティはそれぞれ $(-1)^{j-1/2}$ と $(-1)^{j+1/2}$ で与えられる．1, 2 成分と 3, 4 成分が異なるパリティを持つスピノルは次の様に構成できる：

$$\psi_{jm}^{j-\frac{1}{2}} = \begin{pmatrix} \dfrac{iG_{j-1/2,j}(r)}{r}\varphi_{jm}^{(+)} \\ \dfrac{F_{j-1/2,j}(r)}{r}\varphi_{jm}^{(-)} \end{pmatrix}, \quad \psi_{jm}^{j+\frac{1}{2}} = \begin{pmatrix} \dfrac{iG_{j+1/2,j}(r)}{r}\varphi_{jm}^{(-)} \\ \dfrac{F_{j+1/2,j}(r)}{r}\varphi_{jm}^{(+)} \end{pmatrix}. \tag{6.0.3}$$

$l = j \mp 1/2$ に対し，$\varphi_{jm}^{(l)}$ は $j = l+1/2$ である $\varphi_{jm}^{(+)}$，$j = l-1/2$ である $\varphi_{jm}^{(-)}$ を表す様に約束すれば

$$\psi_{jm}^{(l)} = \begin{pmatrix} \dfrac{iG_{lj}(r)}{r}\varphi_{jm}^{(l)} \\ \dfrac{F_{lj}(r)}{r}(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\varphi_{jm}^{(l)} \end{pmatrix}. \tag{6.0.4}$$

$(\boldsymbol{\sigma}\cdot\boldsymbol{a})(\boldsymbol{\sigma}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\boldsymbol{b})I + i\boldsymbol{\sigma}\cdot(\boldsymbol{a}\times\boldsymbol{b})$ を使えば，

$$\begin{aligned}
(\boldsymbol{\sigma}\cdot\boldsymbol{p})f(r)\varphi^{(l)}_{jm} &= (\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\sigma}\cdot\boldsymbol{p})f(r)\varphi^{(l)}_{jm} \\
&= \frac{\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}}{r}(\boldsymbol{r}\cdot\boldsymbol{p} + i\boldsymbol{\sigma}\cdot\boldsymbol{L})f(r)\varphi^{(l)}_{jm} \\
&= -i\hbar\frac{\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}}}{r}\left[r\frac{df(r)}{dr} + \left\{1 \mp \left(j+\frac{1}{2}\right)\right\}f(r)\right]\varphi^{(l)}_{jm} \\
(\boldsymbol{\sigma}\cdot\boldsymbol{p})(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})f(r)\varphi^{(l)}_{jm} &= \frac{1}{r}(\boldsymbol{r}\cdot\boldsymbol{p} - i\boldsymbol{\sigma}\cdot\boldsymbol{L})f(r)\varphi^{(l)}_{jm} \\
&= i\hbar\frac{1}{r}\left[r\frac{df(r)}{dr} + \left\{1 \pm \left(j+\frac{1}{2}\right)\right\}f(r)\right]\varphi^{(l)}_{jm}
\end{aligned}$$

となるので，[6.0.4] を [6.0.1] に代入して

$$\left(\frac{E}{\hbar c} - \frac{m_e c}{\hbar} + \frac{Z\alpha}{r}\right)G_{lj}(r) = -\frac{dF_{lj}(r)}{dr} \mp \left(j+\frac{1}{2}\right)\frac{F_{lj}(r)}{r} \qquad [6.0.5]$$

$$\left(\frac{E}{\hbar c} + \frac{m_e c}{\hbar} + \frac{Z\alpha}{r}\right)F_{lj}(r) = \frac{dG_{lj}(r)}{dr} \mp \left(j+\frac{1}{2}\right)\frac{G_{lj}(r)}{r}. \qquad [6.0.6]$$

ここで，

$$\tilde{\lambda} = \sqrt{\left(\frac{m_e c}{\hbar}\right)^2 - \left(\frac{E}{c\hbar}\right)^2}, \quad \rho = 2\tilde{\lambda}r \qquad [6.0.7]$$

$$G(r) = \sqrt{1+\frac{E}{m_e c^2}}e^{-\rho/2}(F_1(\rho) + F_2(\rho)) \qquad [6.0.8]$$

$$F(r) = \sqrt{1-\frac{E}{m_e c^2}}e^{-\rho/2}(F_1(\rho) - F_2(\rho)) \qquad [6.0.9]$$

によって $F_1(\rho)$ と $F_2(\rho)$ を定義する．添字 l, j は省略する．[6.0.5] に [6.0.8][6.0.9] を代入して

$$\begin{aligned}
&\left(\frac{E}{\hbar c} - \frac{m_e c}{\hbar} + \frac{2\tilde{\lambda}Z\alpha}{\rho}\right)\sqrt{\frac{m_e c^2 + E}{m_e c^2 - E}}e^{-\rho/2}(F_1 + F_2) \\
&= -2\tilde{\lambda}\frac{d}{d\rho}\left[e^{-\rho/2}(F_1 - F_2)\right] \mp \left(j+\frac{1}{2}\right)\frac{2\tilde{\lambda}}{\rho}e^{-\rho/2}(F_1 - F_2) \\
&= -2\tilde{\lambda}\left[-\frac{1}{2}e^{-\rho/2}(F_1 - F_2) + e^{-\rho/2}\left(\frac{dF_1}{d\rho} - \frac{dF_2}{d\rho}\right)\right] \mp \left(j+\frac{1}{2}\right)\frac{2\tilde{\lambda}}{\rho}e^{-\rho/2}(F_1 - F_2)
\end{aligned}$$

となるので，

$$\left(\frac{E}{\hbar c} - \frac{m_e c}{\hbar} + \frac{2\tilde{\lambda}Z\alpha}{\rho}\right)\sqrt{\frac{m_e c^2 + E}{m_e c^2 - E}}(F_1 + F_2) = \tilde{\lambda}(F_1 - F_2) - 2\tilde{\lambda}\left(\frac{dF_1}{d\rho} - \frac{dF_2}{d\rho}\right) \mp \left(j+\frac{1}{2}\right)\frac{2\tilde{\lambda}}{\rho}(F_1 - F_2).$$

これに

$$\sqrt{\frac{m_e c^2 + E}{m_e c^2 - E}} = \frac{\tilde{\lambda}}{m_e c/\hbar - E/\hbar c}$$

を代入して

$$-\rho F_1 + \frac{Z\alpha\tilde{\lambda}}{m_e c/\hbar - E/\hbar c}(F_1 + F_2) = -\rho\left(\frac{dF_1}{d\rho} - \frac{dF_2}{d\rho}\right) \mp \left(j+\frac{1}{2}\right)(F_1 - F_2). \qquad [6.0.10]$$

同様に，[6.0.6] に [6.0.8][6.0.9] を代入して，

$$\left(\frac{E}{\hbar c}+\frac{m_e c}{\hbar}+\frac{2\tilde{\lambda}Z\alpha}{\rho}\right)\sqrt{\frac{m_e c^2-E}{m_e c^2+E}}e^{-\rho/2}(F_1-F_2)=2\tilde{\lambda}\frac{d}{d\rho}\left[e^{-\rho/2}(F_1+F_2)\right]\mp\left(j+\frac{1}{2}\right)\frac{2\tilde{\lambda}}{\rho}e^{-\rho/2}(F_1+F_2)$$

となるので，

$$\tilde{\lambda}(F_1-F_2)+\frac{2\tilde{\lambda}Z\alpha}{\rho}\frac{\tilde{\lambda}}{m_e c/\hbar-E/\hbar c}(F_1-F_2)=-\tilde{\lambda}(F_1+F_2)+2\tilde{\lambda}+2\tilde{\lambda}\left(\frac{dF_1}{d\rho}+\frac{dF_2}{d\rho}\right)\mp\left(j+\frac{1}{2}\right)(F_1+F_2)$$

となり，

$$\rho F_1+\frac{Z\alpha\tilde{\lambda}}{m_e c/\hbar+E/\hbar c}(F_1-F_2)=\rho\left(\frac{dF_1}{d\rho}+\frac{dF_2}{d\rho}\right)\mp\left(j+\frac{1}{2}\right)(F_1+F_2). \tag{6.0.11}$$

[6.0.10][6.0.11] を両辺足し引きして

$$\rho\frac{dF_1}{d\rho}=\left(\rho-\frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}\right)F_1+\left[-\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}}\pm\left(j+\frac{1}{2}\right)\right]F_2 \tag{6.0.12}$$

$$\rho\frac{dF_2}{d\rho}=\left[\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}}\pm\left(j+\frac{1}{2}\right)\right]F_1+\frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}F_2 \tag{6.0.13}$$

[6.0.12] を両辺微分して

$$\frac{dF_1}{d\rho}+\rho\frac{d^2F_1}{d\rho^2}=F_1+\left(\rho-\frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}\right)\frac{dF_1}{d\rho}+\left[-\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}}\pm\left(j+\frac{1}{2}\right)\right]\frac{dF_2}{d\rho}$$

[6.0.13] を代入して

$$\begin{aligned}&=F_1+\left(\rho-\frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}\right)\frac{dF_1}{d\rho}+\left[-\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}}\pm\left(j+\frac{1}{2}\right)\right]\left[\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}}\pm\left(j+\frac{1}{2}\right)\right]\frac{F_1}{\rho}\\&\quad+\left[-\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}}\pm\left(j+\frac{1}{2}\right)\right]\frac{E}{\hbar c}\frac{Z\alpha}{\rho\tilde{\lambda}}F_2\end{aligned}$$

となり，[6.0.12] を代入すれば

$$\begin{aligned}&=F_1+\left(\rho-\frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}\right)\frac{dF_1}{d\rho}+\left[\left(j+\frac{1}{2}\right)^2-\left(\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}}\right)^2\right]\frac{F_1}{\rho}+\frac{E}{\hbar c}\frac{Z\alpha}{\rho\tilde{\lambda}}\left[\rho\frac{dF_1}{d\rho}-\left(\rho-\frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}\right)F_1\right]\\&=F_1-\frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}F_1+\rho\frac{dF_1}{d\rho}+\left[\left(j+\frac{1}{2}\right)^2-(Z\alpha)^2\right]\frac{F_1}{\rho}\end{aligned}$$

となる.

$$\gamma=\sqrt{\left(j+\frac{1}{2}\right)^2-(Z\alpha)^2} \tag{6.0.14}$$

を導入すれば

$$\rho\frac{d^2F_1}{d\rho^2}+(1-\rho)\frac{dF_1}{d\rho}+\left(\frac{Z\alpha}{c\hbar\tilde{\lambda}}-1-\frac{\gamma^2}{\rho}\right)F_1=0. \tag{6.0.15}$$

[6.0.13] を微分して

$$\frac{dF_2}{d\rho}+\rho\frac{d^2F_2}{d\rho^2}=\left[\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}}\pm\left(j+\frac{1}{2}\right)\right]\frac{dF_1}{d\rho}+\frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}\frac{dF_2}{d\rho}$$

[6.0.12] を代入して

$$
\begin{aligned}
&= \left[\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}} \pm \left(j+\frac{1}{2}\right)\right]\left(1-\frac{E}{\hbar c}\frac{Z\alpha}{\rho\tilde{\lambda}}\right)F_1 + \left[\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}} \pm \left(j+\frac{1}{2}\right)\right]\left[-\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}} \pm \left(j+\frac{1}{2}\right)\right]\frac{F_2}{\rho} \\
&\quad + \frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}\frac{dF_2}{d\rho}
\end{aligned}
$$

[6.0.13] を代入すれば,

$$
\begin{aligned}
&= \left(1-\frac{E}{\hbar c}\frac{Z\alpha}{\rho\tilde{\lambda}}\right)\left[\rho\frac{dF_2}{d\rho} - \frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}F_2\right] + \left[\left(j+\frac{1}{2}\right)^2 - \left(\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}}\right)^2\right]\frac{F_2}{\rho} + \frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}\frac{dF_2}{d\rho} \\
&= \rho\frac{dF_2}{d\rho} - \frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}F_2 + \left[\left(j+\frac{1}{2}\right)^2 - (Z\alpha)^2\right]\frac{F_2}{\rho}
\end{aligned}
$$

となるので,

$$
\rho\frac{d^2F_2}{d\rho^2} + (1-\rho)\frac{dF_2}{d\rho} + \left(\frac{Z\alpha E}{c\hbar\tilde{\lambda}} - \frac{\gamma^2}{\rho}\right)F_2 = 0. \qquad [6.0.16]
$$

$F_1 = \rho^\gamma \tilde{F}_1$ とおいて,

$$
\frac{d\tilde{F}_1}{d\rho} = \gamma\rho^{\gamma-1}\tilde{F}_1 + \rho^\gamma\frac{d\tilde{F}_1}{d\rho}, \quad \frac{d^2\tilde{F}_2}{d\rho^2} = \gamma(\gamma-1)\rho^{\gamma-2}\tilde{F}_1 + 2\gamma\rho^{\gamma-1}\frac{d\tilde{F}_1}{d\rho} + \rho^\gamma\frac{d^2\tilde{F}_1}{d\rho^2}
$$

を [6.0.15] に代入すれば

$$
\gamma(\gamma-1)\rho^{\gamma-1}\tilde{F}_1 + 2\gamma\rho^\gamma\frac{d\tilde{F}_1}{d\rho} + \rho^{\gamma+1}\frac{d^2\tilde{F}_1}{d\rho^2} + (1-\rho)\left(\gamma\rho^{\gamma-1}\tilde{F}_1 + \rho^\gamma\frac{d\tilde{F}_1}{d\rho}\right) + \left(\frac{Z\alpha E}{c\hbar\tilde{\lambda}} - 1 - \frac{\gamma^2}{\rho}\right)\rho^{\gamma+1}\tilde{F}_1
$$

となり

$$
\rho\frac{d^2\tilde{F}_1}{d\rho^2} + \{(1+2\gamma)-\rho\}\frac{d\tilde{F}_1}{d\rho} - \left(\gamma+1-\frac{Z\alpha E}{c\hbar\tilde{\lambda}}\right)\tilde{F}_1.
$$

この方程式の解は合流型超幾何函数で

$$
F_1 = A\rho^\gamma F\left(\gamma+1-\frac{Z\alpha E}{c\hbar\tilde{\lambda}}, 2\gamma+1; \rho\right) \qquad [6.0.17]
$$

となる．$F_2 = \rho^\gamma\tilde{F}_2$ に [6.0.16] を代入すれば

$$
\rho\frac{d^2\tilde{F}_2}{d\rho^2} + \{(1+2\gamma)-\rho\}\frac{d\tilde{F}_2}{d\rho} - \left(\gamma-\frac{Z\alpha E}{c\hbar\tilde{\lambda}}\right)\tilde{F}_2.
$$

となり

$$
F_2 = B\rho^\gamma F\left(\gamma-\frac{Z\alpha E}{c\hbar\tilde{\lambda}}, 2\gamma+1; \rho\right). \qquad [6.0.18]
$$

合流型超幾何函数は

$$
F(a,c;z) = 1 + \frac{a}{c}z + \cdots
$$

となるので，[6.0.17][6.0.18] から

$$
F_1 = A\rho^\gamma\left[1+\left(\gamma+1-\frac{Z\alpha E}{c\hbar\tilde{\lambda}}\right)\frac{\rho}{2\gamma+1}+\cdots\right], \quad F_2 = B\rho^\gamma\left[1+\left(\gamma-\frac{Z\alpha E}{c\hbar\tilde{\lambda}}\right)\frac{\rho}{2\gamma+1}+\cdots\right]
$$

となる．これを [6.0.13] に代入して ρ^γ の係数を比較すれば

$$B\gamma = \left[\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}} \pm \left(j+\frac{1}{2}\right)\right] A + \frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}} B \quad \therefore \frac{A}{B} = \frac{\gamma - \frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}}{\frac{m_e c}{\hbar}\frac{Z\alpha}{\tilde{\lambda}} \pm \left(j+\frac{1}{2}\right)}.$$

よって，

$$F_1 = \frac{\gamma - \frac{E}{\hbar c}\frac{Z\alpha}{\tilde{\lambda}}}{\frac{m_e c}{\hbar}\frac{Z\alpha E}{\tilde{\lambda}} \pm \left(j+\frac{1}{2}\right)} \rho^\gamma F\left(\gamma + 1 - \frac{Z\alpha E}{c\hbar\tilde{\lambda}}, 2\gamma+1; \rho\right) \qquad [6.0.19]$$

$$F_2 = \rho^\gamma F\left(\gamma - \frac{Z\alpha E}{c\hbar\tilde{\lambda}}, 2\gamma+1; \rho\right). \qquad [6.0.20]$$

となる（$B=1$ として，後から規格化する）．また，以下で

$$n = \frac{Z\alpha E}{c\hbar\tilde{\lambda}} - \gamma + \left(j + \frac{1}{2}\right) \qquad [6.0.21]$$

を定義する．無限遠点で波動関数が 0 になる必要がある（p.84 に書いてある）ので

$$E = m_e c^2 \left[1 + \frac{Z^2\alpha^2}{\left\{n - j - \frac{1}{2} + \sqrt{(j+\frac{1}{2})^2 - Z^2\alpha^2}\right\}^2}\right]^{-1/2}. \qquad [6.0.22]$$

$n=1, j=1/2, m=1/2$ の時．[6.0.14] から $\gamma = \sqrt{1-Z^2\alpha^2}$，[6.0.7] から $\tilde{\lambda} = m_e c Z\alpha/\hbar$，[6.0.22] から $E = \gamma m_e c^2$．[6.0.19][6.0.20] から $F_1 = 0, \quad F_2 = \rho^\gamma F(0, 2\gamma+1; \rho) = \rho^\gamma$．[6.0.8][6.0.9] から

$$G_{l,1/2} = \sqrt{1+\gamma} e^{-\rho/2}\rho^\gamma, \quad F_{l,1/2} = -\sqrt{1-\gamma} e^{-\rho/2}\rho^\gamma. \qquad [6.0.23]$$

[6.0.2][6.0.4] に代入すると 1, 2 成分に関しては $j = l + 1/2$ から $l = 0$，3, 4 成分に関しては $j = l - 1/2$ から $l = 1$ となることに注意して，

$$\psi_{j=1/2,m=1/2} = N \begin{pmatrix} \frac{iG_{0,1/2}}{r} Y_{0,0} \\ 0 \\ \frac{F_{1,1/2}}{r}\sqrt{\frac{1}{3}} Y_{1,0} \\ -\frac{F_{1,1/2}}{r}\sqrt{\frac{2}{3}} Y_{1,1} \end{pmatrix}. \qquad [6.0.24]$$

規格化条件は

$$\int r^2 \sin\theta |\psi|^2 \, dr d\theta d\phi = N^2 \int G^2 + F^2 \, dr = 1$$

で，[6.0.23] を代入すれば，

$$\frac{N^2(1+\gamma)}{2\tilde{\gamma}} \int_0^\infty e^{-\rho}\rho^{2\gamma} \, d\rho + \frac{N^2(1-\gamma)}{2\tilde{\gamma}} \int_0^\infty e^{-\rho}\rho^{2\gamma} \, d\rho = 1$$

となり，ガンマ函数の定義

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

から

$$N = \left(\frac{\tilde{\lambda}}{\Gamma(2\gamma+1)}\right)^{1/2} = \left(\frac{1}{\Gamma(2\gamma+1)}\right)^{1/2} \left(\frac{m_e c Z\alpha}{\hbar}\right)^{1/2}.$$

[6.0.24] に代入して $-i$ をかければ

$$\psi_{j=1/2,m=1/2} = -i\left(\frac{1}{\Gamma(2\gamma+1)}\right)^{1/2}\left(\frac{m_e cZ\alpha}{\hbar}\right)^{1/2}\begin{pmatrix} \frac{iG_{0,1/2}}{r}Y_{0,0} \\ 0 \\ \frac{F_{1,1/2}}{r}\sqrt{\frac{1}{3}}Y_{1,0} \\ -\frac{F_{1,1/2}}{r}\sqrt{\frac{2}{3}}Y_{1,1} \end{pmatrix}$$

$$= \frac{1}{\sqrt{4\pi}}\left(\frac{2m_e cZ\alpha}{\hbar}\right)^{3/2}\left(\frac{\gamma+1}{2\Gamma(2\gamma+1)}\right)^{1/2}\left(\frac{2m_e cZ\alpha}{\hbar}r\right)^{\gamma-1}\exp\left(-\frac{m_e cZ\alpha}{\hbar}r\right)\begin{pmatrix} 1 \\ 0 \\ i\frac{1-\gamma}{Z\alpha}\cos\theta \\ i\frac{1-\gamma}{Z\alpha}\sin\theta e^{i\phi} \end{pmatrix}.$$

$n=1, j=1/2, m=-1/2$ のとき．N は共通で 3, 4 成分の符号が逆なので

$$\psi_{j=1/2,m=-1/2}$$

$$= \frac{1}{\sqrt{4\pi}}\left(\frac{2m_e cZ\alpha}{\hbar}\right)^{3/2}\left(\frac{\gamma+1}{2\Gamma(2\gamma+1)}\right)^{1/2}\left(\frac{2m_e cZ\alpha}{\hbar}r\right)^{\gamma-1}\exp\left(-\frac{m_e cZ\alpha}{\hbar}r\right)\begin{pmatrix} 0 \\ 1 \\ i\frac{1-\gamma}{Z\alpha}\sin\theta e^{-i\phi} \\ -i\frac{1-\gamma}{Z\alpha}\cos\theta \end{pmatrix}.$$

第 7 章

空孔理論

■(7.14)　$\gamma^0 \not{p}^* = \gamma^0 \gamma^\mu p_\mu^* = \gamma^0 \gamma^0 p_0^* + \sum_{i=1}^3 \gamma^0 \gamma^{i*} p_i^* = p_0^* - \sum_{i=1}^3 \gamma^{i*} p_i^* \gamma^0$．$\gamma^{i*}$ が γ^i と符号逆なのは $i=2$ だけ．$\not{p}^T \gamma^0 = \gamma^0 p_0 \gamma^0 + \sum_{i=1}^3 p_i \gamma^{iT} \gamma^0$．$\gamma^{iT}$ が γ^i と符号逆になるのは $i=1,3$．以上から $\gamma^0 \not{p}^* = \not{p}^T \gamma^0$．上と同様にして $\gamma^0 \not{s}^* = \not{s}^T \gamma^0$ なんで $\gamma^0 (\gamma_5 \not{s})^* = \gamma^0 \gamma_5 \not{s}^* = -\gamma_5 \gamma^0 \not{s}^* = -\gamma^5 \not{s}^T \gamma^0$．$\gamma^\mu$ のうち転置で符号変化するのは $1,3$ で，C との交換で符号変化するのが $0,2$ なので $C \not{p}^T = -\not{p} C$

■(7.28)

$$T \not{p}^* = i\gamma^1\gamma^3(\gamma^0 p_0 - \gamma_1 p_1 + \gamma^2 p_2 - \gamma^3 p_3) = \gamma_0 p_0 i\gamma^1\gamma^3 + \gamma^1 p_1 i\gamma^1\gamma^3 + \gamma^2 p_2 i\gamma^1\gamma^3 + \gamma^3 p_3 i\gamma^1\gamma^3 = \not{p}' T.$$

第 9 章

伝播理論　—相対論的電子 —

■(9.16)　射影演算子を $\sum_{r=1,2} w^r \overline{w}^r$ とするやつ：p.50 の w の完全性と直交性から，$w\overline{w}$ を w で展開した波動関数に作用させれば分かる

第 11 章

Coulomb 散乱

11.2.2 のはじめ「$|\boldsymbol{J}_i| = |\overline{\psi}_i c\gamma\psi_i|...$」: (4.56) と同じ話. 確率流れは 2.2.3 参照.

■(11.20) の証明　まず,

$$\boldsymbol{p}\cdot\boldsymbol{\sigma} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}.$$

$r = 1$ なら

$$S(\Lambda)w^1(0) = \sqrt{\frac{E + m_e c^2}{2m_e c^2}} \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+m_e c^2} \\ \frac{c(p_x+ip_y)}{E+m_e c^2} \end{pmatrix}$$

及び

$$\overline{w}^1(0)S^{-1}(\Lambda) = \sqrt{\frac{E + m_e c^2}{2m_e c^2}} \left(1, 0, -\frac{cp_z}{E + m_e c^2}, -\frac{c(p_x - ip_y)}{E + m_e c^2}\right).$$

$r = 2$ なら

$$S(\Lambda)w^2(0) = \sqrt{\frac{E + m_e c^2}{2m_e c^2}} \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+m_e c^2} - \frac{cp_z}{E+m_e c^2} \end{pmatrix}$$

及び

$$\overline{w}^2(0)S^{-1}(\Lambda) = \sqrt{\frac{E + m_e c^2}{2m_e c^2}} \left(0, 1, -\frac{c(p_x + ip_y)}{E + m_e c^2}, \frac{cp_z}{E + m_e c^2}\right).$$

$\sum_{r=1,2}(S(\Lambda)w^r(0))_\beta(\overline{w}^r(0)S^{-1}(\Lambda))_\gamma$ を (β, γ) 成分とする行列は

$$\begin{pmatrix} \frac{E+m_e c^2}{2m_e c^2} & 0 & -\frac{p_z}{2m_e c} & -\frac{p_x-ip_y}{2m_e c} \\ 0 & \frac{E+m_e c^2}{2m_e c^2} & -\frac{p_x+ip_y}{2m_e c} & -\frac{p_z}{2m_e c} \\ \frac{p_z}{2m_e c} & \frac{p_x-ip_y}{2m_e c} & -\frac{p^2}{2m}\frac{1}{E+m_e c^2} & 0 \\ \frac{p_x+ip_y}{2m_e c} & \frac{p_z}{2m_e c} & 0 & -\frac{p^2}{2m}\frac{1}{E+m_e c^2} \end{pmatrix}$$

となる.

$$\not{p} = \gamma^\mu p_\mu = \frac{E}{c}\begin{pmatrix} I & \\ & I \end{pmatrix} - \begin{pmatrix} \boldsymbol{p}\cdot\boldsymbol{\sigma} & \\ & \boldsymbol{p}\cdot\boldsymbol{\sigma} \end{pmatrix}$$

なので,

$$\frac{\not{p} + m_e c}{2m_e c} = \begin{pmatrix} \frac{m_e c^2+E}{2m_e c^2}I & -\frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{2m_e c} \\ \frac{\boldsymbol{p}\cdot\boldsymbol{\sigma}}{2m_e c} & \frac{m_e c^2-E}{2m_e c^2}I \end{pmatrix}.$$

2 つの行列は実際に等しくなる．

■(11.32)　(7.6) と (7.10).

■(11.36)　Griffiths の p.250 参照．より一般には，$[\overline{u}(a)\Gamma_1 u(b)][\overline{u}(d)\Gamma_2 u(b)]^* = [\overline{u}(a)\Gamma_1 u(b)][\overline{u}(d)\overline{\Gamma}_2 u(c)]$ となる．

第 12 章

Compton 散乱

■(12.6)

$$
\begin{aligned}
E_\mu &= -\frac{\partial A_\mu}{\partial t} \\
&= \frac{\hbar \varepsilon^\mu}{\sqrt{2EV\varepsilon_0}} \left(-\frac{i}{\hbar} E e^{-iqx/\hbar} + \frac{i}{\hbar} E e^{iqx/\hbar} \right) \\
&= \sqrt{\frac{2E}{V\varepsilon_0}} \varepsilon^\mu \sin(qx/\hbar)
\end{aligned}
$$

なので,

$$
\frac{\varepsilon_0}{2} \boldsymbol{E}^2 = \frac{E}{V} \boldsymbol{\varepsilon}^2 \sin^2(qx/\hbar) = \frac{E}{V} \sin^2(qx/\hbar).
$$

$\boldsymbol{B} = \partial_i A_j \boldsymbol{e}_k \varepsilon_{ijk}$ なので,

$$
\begin{aligned}
\frac{\boldsymbol{B}^2}{2\mu_0} &= \frac{1}{2\mu_0} \left(\sum_{ijk} \varepsilon_{ijk} \partial_i A_j \boldsymbol{e}_k \right) \cdot \left(\sum_{lmn} \varepsilon_{lmn} \partial_l A_m \boldsymbol{e}_n \right) \\
&= \frac{1}{2\mu_0} \sum_{ijklmn} \varepsilon_{ijk} \varepsilon_{lmn} \partial_i A_j \partial_l A_m \delta_{kn} \\
&= \frac{1}{2\mu_0} \sum_{ijklm} \varepsilon_{ijk} \varepsilon_{lmn} \partial_i A_j \partial_l A_m \\
&= \frac{1}{2\mu_0} \sum_{ijlm} (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \partial_i A_j \partial_l A_m \\
&= \frac{1}{2\mu_0} \sum_{ij} \partial_i A_j (\partial_i A_j - \partial_j A_i) \\
&= \frac{2}{\mu_0} \sum_{ij} \frac{\hbar \varepsilon^j}{\sqrt{2EV\varepsilon_0}} \frac{q_i}{\hbar} \sin(qx/\hbar) \left[\frac{\hbar \varepsilon^j}{\sqrt{2EV\varepsilon_0}} \frac{q_i}{\hbar} \sin(qx/\hbar) - \frac{\hbar \varepsilon^i}{\sqrt{2EV\varepsilon_0}} \frac{q_j}{\hbar} \sin(qx/\hbar) \right] \\
&= \frac{1}{EV\varepsilon_0\mu_0} \sin^2(qx/\hbar) \sum_{ij} (\varepsilon^j \varepsilon^j q_i q_i - \varepsilon^i \varepsilon^j q_i q_j) \\
&= \frac{1}{EV\varepsilon_0\mu_0} \sin^2(qx/\hbar) \sum_{ij} (\varepsilon^j \varepsilon^j q_i q_i - \varepsilon^i \varepsilon^j q_i q_j) \\
&= \frac{c^2}{EV} \sin^2(qx/\hbar) \sum_{ij} (\varepsilon^j \varepsilon^j q_i q_i - \varepsilon^i \varepsilon^j q_i q_j).
\end{aligned}
$$

ここで，$\sum_j \varepsilon^j\varepsilon^j = 1$，$q_\mu\varepsilon^\mu = q_1\varepsilon^1 + \cdots = 0$，$0 = q^\mu q_\mu = q_0 q_0 - q_1 q_2 - \cdots$ なので，

$$
\begin{aligned}
&= \frac{c^2}{EV}\sin^2(qx/\hbar)\sum_{i=0}^{3} {q_i}^2 = \frac{c^2}{EV}\sin^2(qx/\hbar){q_0}^2 \\
&= \frac{c^2}{EV}\sin^2(qx/\hbar)(\frac{E}{c})^2 = \frac{E}{V}\sin^2(qx/\hbar).
\end{aligned}
$$

■(12.13)

$$
\begin{aligned}
\int_0^\infty dp_{f0}\,\delta({p_f}^2 - {m_\mathrm{e}}^2c^2) &= \int_0^\infty dp_{f0}\,\delta({p_{f0}}^2 - |\boldsymbol{p}_f|^2 - {m_\mathrm{e}}^2c^2) \\
&= \int_0^\infty dp_{f0}\,\delta\left({p_{f0}}^2 - \frac{{E_f}^2}{c^2}\right) = \frac{c}{2E_f}.
\end{aligned}
$$

■(12.21)　(11.19) の計算と同様に (11.20)(11.21)(11.36) を使う．

■(12.30)　まずは 3 行目第 1 項の計算から．$q\varepsilon = 0$ なので，

$$
\mathrm{Tr}[(\not{p}_i + m_e c)\not{\varepsilon}'\not{\varepsilon}\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}] = -\,\mathrm{Tr}[(\not{p}_i + m_e c)\not{\varepsilon}'\not{q}\not{\varepsilon}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}].
$$

(H.29) を使って，

$$
\begin{aligned}
&= -2(\varepsilon' q)\,\mathrm{Tr}[(\not{p}_i + m_e c)\not{\varepsilon}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}] + \mathrm{Tr}[(\not{p}_i + m_e c)\not{q}\not{\varepsilon}'\not{\varepsilon}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}] \\
&= -2(\varepsilon' q)\,\mathrm{Tr}[(\not{p}_i + m_e c)(-\not{p}_i + m_e c)\not{\varepsilon}\not{q}'\not{\varepsilon}'\not{\varepsilon}] + \mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{\varepsilon}'\not{\varepsilon}\not{q}'\not{\varepsilon}'\not{\varepsilon}].
\end{aligned}
$$

$(\not{p}_i + m_e c)(-\not{p}_i + m_e c) = -(p_i)^2 + (m_e c)^2 = 0$ なので，

$$
\begin{aligned}
&= 2(\varepsilon q')\,\mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{\varepsilon}'\not{\varepsilon}'\not{\varepsilon}] - \mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{\varepsilon}'\not{q}'\not{\varepsilon}\not{\varepsilon}'\not{\varepsilon}] \\
&= -2(\varepsilon q')\,\mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{\varepsilon}] \\
&\quad - 2(\varepsilon' q')\,\mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{\varepsilon}\not{\varepsilon}'\not{\varepsilon}] + \mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}\not{\varepsilon}'\not{\varepsilon}].
\end{aligned}
$$

$\varepsilon' q' = 0$ なので，

$$
\begin{aligned}
&= -2(\varepsilon q')\,\mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{\varepsilon}] + \mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}\not{\varepsilon}'\not{\varepsilon}] \\
&= -2(\varepsilon q')\,\mathrm{Tr}[(\not{p}_i + m_e c)\not{q}\not{\varepsilon}(\not{p}_i - m_e c)] + \mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}\not{\varepsilon}'\not{\varepsilon}].
\end{aligned}
$$

p.160 定理 1 を使えば，

$$
\begin{aligned}
&= -2(\varepsilon q')\,\mathrm{Tr}[\not{p}_i\not{q}\not{\varepsilon}\not{p}_i] + 2(\varepsilon q')(m_e c)^2\,\mathrm{Tr}[\not{q}\not{\varepsilon}] + \mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}\not{\varepsilon}'\not{\varepsilon}] \\
&= -2(\varepsilon q')\,\mathrm{Tr}[\not{q}\not{\varepsilon}\not{p}_i\not{p}_i] + 2(\varepsilon q')(m_e c)^2\,\mathrm{Tr}[\not{q}\not{\varepsilon}] + \mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}\not{\varepsilon}'\not{\varepsilon}] \\
&= -2(\varepsilon q')(m_e c)^2\,\mathrm{Tr}[\not{q}\not{\varepsilon}] + 2(\varepsilon q')(m_e c)^2\,\mathrm{Tr}[\not{q}\not{\varepsilon}] + \mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}\not{\varepsilon}'\not{\varepsilon}] \\
&= \mathrm{Tr}[(\not{p}_i + m_e c)\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}\not{\varepsilon}'\not{\varepsilon}].
\end{aligned}
$$

次に 3 行目第 2 項の計算．

$$
\begin{aligned}
&\mathrm{Tr}[\not{q}\not{\varepsilon}'\not{\varepsilon}\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}] \\
&= 2(\varepsilon' q)\,\mathrm{Tr}[\not{\varepsilon}\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}] - \mathrm{Tr}[\not{\varepsilon}'\not{q}\not{\varepsilon}\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}] \\
&= 2(\varepsilon' q)\,\mathrm{Tr}[\not{q}(\not{p}_i + m_e c)\not{q}'\not{\varepsilon}'\not{\varepsilon}\not{\varepsilon}]
\end{aligned}
$$

$$
\begin{aligned}
&- 2(\varepsilon\varepsilon')\,\mathrm{Tr}[\slashed{\varepsilon}'\slashed{q}\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'] \\
&+ \mathrm{Tr}[\slashed{\varepsilon}'\slashed{q}\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}\slashed{\varepsilon}'] \\
= {}&-2(\varepsilon' q)\,\mathrm{Tr}[\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'] \\
&- 4(\varepsilon\varepsilon')(\varepsilon q)\,\mathrm{Tr}[\slashed{\varepsilon}'\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'] + 2(\varepsilon\varepsilon')\,\mathrm{Tr}[\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'] \\
&+ \mathrm{Tr}[\slashed{q}\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}\slashed{\varepsilon}'\slashed{\varepsilon}'].
\end{aligned}
$$

$\slashed{q}\slashed{q} = 0$, $\varepsilon q = 0$ なので,

$$
\begin{aligned}
&= -2(\varepsilon' q)\,\mathrm{Tr}[\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'] - \mathrm{Tr}[\slashed{q}\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}] \\
&= -2(\varepsilon' q)\,\mathrm{Tr}[\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'] \\
&\quad - 2(\varepsilon q)\,\mathrm{Tr}[\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}] + \mathrm{Tr}[\slashed{q}\slashed{q}\slashed{\varepsilon}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}] \\
&= -2(\varepsilon' q)\,\mathrm{Tr}[\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'] \\
&= -2(\varepsilon' q)\,\mathrm{Tr}[\slashed{q}\slashed{p}_i\slashed{q}'\slashed{\varepsilon}'] - 2m_e c(\varepsilon' q)\,\mathrm{Tr}[\slashed{q}\slashed{q}'\slashed{\varepsilon}'] \\
&= -2(\varepsilon' q)\,\mathrm{Tr}[\slashed{q}\slashed{p}_i\slashed{q}'\slashed{\varepsilon}'].
\end{aligned}
$$

第 3 項についても同様に,

$$
\begin{aligned}
&\mathrm{Tr}[\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}] \\
&= \mathrm{Tr}[\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}'] \\
&= 2(\varepsilon' q')\,\mathrm{Tr}[\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{\varepsilon}\slashed{q}'] - \mathrm{Tr}[\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{\varepsilon}'\slashed{q}'\slashed{\varepsilon}\slashed{q}'] \\
&= -2(\varepsilon q')\,\mathrm{Tr}[\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{\varepsilon}'\slashed{q}'] + \mathrm{Tr}[\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{\varepsilon}'\slashed{q}'\slashed{q}'\slashed{\varepsilon}] \\
&= -4(\varepsilon q')(\varepsilon' q')\,\mathrm{Tr}[\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)] + 2(\varepsilon q')\,\mathrm{Tr}[\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'] \\
&= 2(\varepsilon q')\,\mathrm{Tr}[\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}'] \\
&= -2(\varepsilon q')\,\mathrm{Tr}[\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'] \\
&= -2(\varepsilon q')\,\mathrm{Tr}[\slashed{\varepsilon}\slashed{q}\slashed{p}_i\slashed{q}'].
\end{aligned}
$$

以上から,

$$
\begin{aligned}
&\mathrm{Tr}[(\slashed{p}_f + m_e c)\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}] \\
&= \mathrm{Tr}[(\slashed{p}_i + m_e c)\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{\varepsilon}'\slashed{\varepsilon}] - 2(\varepsilon' q)\,\mathrm{Tr}[\slashed{q}\slashed{p}_i\slashed{q}'\slashed{\varepsilon}'] + 2(\varepsilon q')\,\mathrm{Tr}[\slashed{\varepsilon}\slashed{q}\slashed{p}_i\slashed{q}'] \\
&= 2(\varepsilon\varepsilon')\,\mathrm{Tr}[(\slashed{p}_i + m_e c)\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}] - \mathrm{Tr}[(\slashed{p}_i + m_e c)\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}'\slashed{\varepsilon}\slashed{\varepsilon}] \\
&\quad - 2(\varepsilon' q)\,[4(qp_i)(q'\varepsilon') + 4(q\varepsilon')(p_i q') - 4(qq')(p_i\varepsilon')] \\
&\quad + 2(\varepsilon q')\,[4(\varepsilon q)(p_i q') + 4(\varepsilon q')(qp_i) - 4(\varepsilon p_i)(qq')] \\
&= 2(\varepsilon\varepsilon')\,\mathrm{Tr}[(\slashed{p}_i + m_e c)\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}] - \mathrm{Tr}[(\slashed{p}_i + m_e c)\slashed{q}(\slashed{p}_i + m_e c)\slashed{q}'] \\
&\quad - 8(\varepsilon' q)^2(p_i q') + 8(\varepsilon q')^2(p_i q) \\
&= 2(\varepsilon\varepsilon')\,\mathrm{Tr}[\slashed{p}_i\slashed{q}\slashed{p}_i\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}] + 2(\varepsilon\varepsilon')(m_e c)^2\,\mathrm{Tr}[\slashed{q}\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}] \\
&\quad - \mathrm{Tr}[\slashed{p}_i\slashed{q}\slashed{p}_i\slashed{q}'] - (m_e c)^2\,\mathrm{Tr}[\slashed{q}\slashed{q}'] \\
&\quad - 8(\varepsilon' q)^2(p_i q') + 8(\varepsilon q')^2(p_i q) \\
&= 2(\varepsilon\varepsilon')\,\mathrm{Tr}[\slashed{p}_i\slashed{q}\slashed{p}_i\slashed{q}'\slashed{\varepsilon}'\slashed{\varepsilon}] \\
&\quad + 2(\varepsilon\varepsilon')(m_e c)^2[4(qq')(\varepsilon\varepsilon') + 4(q\varepsilon)(q'\varepsilon') - 4(q\varepsilon')(q'\varepsilon)] \\
&\quad - 4[2(p_i q)(p_i q') - (p_i)^2(qq')] - (m_e c)^2 4(qq') \\
&\quad - 8(\varepsilon' q)^2(p_i q') + 8(\varepsilon q')^2(p_i q)
\end{aligned}
$$

$$
\begin{aligned}
&= 2(\varepsilon\varepsilon')\,\mathrm{Tr}[\not{p}_i\not{q}\not{p}_i\not{q}'\not{\varepsilon}'\not{\varepsilon}] \\
&\quad + 8(m_e c)^2(\varepsilon\varepsilon')^2(qq') - 8(m_e c)^2(\varepsilon\varepsilon')(q\varepsilon')(q'\varepsilon) \\
&\quad - 8(p_i q)(p_i q') - 8(\varepsilon' q)^2(p_i q') + 8(\varepsilon q')^2(p_i q) \\
&= 2(\varepsilon\varepsilon')4[(\varepsilon\varepsilon')(p_i q)(p_i q') - (m_e c)^2(qq')(\varepsilon\varepsilon') + (m_e c)^2(q\varepsilon')(q'\varepsilon) + (\varepsilon\varepsilon')(p_i q')(p_i q)] \\
&\quad + 8(m_e c)^2(\varepsilon\varepsilon')^2(qq') - 8(m_e c)^2(\varepsilon\varepsilon')(q\varepsilon')(q'\varepsilon) \\
&\quad - 8(p_i q)(p_i q') - 8(\varepsilon' q)^2(p_i q') + 8(\varepsilon q')^2(p_i q) \\
&= 8(p_i q)(p_i q')[2(\varepsilon\varepsilon')^2 - 1] - 8(\varepsilon' q)^2(p_i q') + 8(\varepsilon q')^2(p_i q).
\end{aligned}
$$

電子・陽電子の対消滅の詳しい計算

対消滅過程の S 行列要素は (12.1) と同様に

$$
S_{fi} = -i\left(\frac{e}{\hbar}\right)^2 \int d^4x \int d^4y\, \overline{\psi}_f(y)[\not{A}(y;q_2)S_F(y;x)\not{A}(x;q_1) + \not{A}(y;q_1)S_F(y;x)\not{A}(x;q_2)]\psi_i(x). \quad [12.0.1]
$$

始状態は運動量 p_- の電子：

$$
\psi_i(x) = \sqrt{\frac{m_e c^2}{E_- V}}\, u(p_-, s_-) e^{-\frac{i}{\hbar}p_- x}. \quad [12.0.2]
$$

終状態は運動量 p_+ の陽電子：

$$
\overline{\psi}_f(y) = \sqrt{\frac{m_e c^2}{E_+ V}}\, \overline{v}(p_+, s_+) e^{-\frac{i}{\hbar}p_+ x}. \quad [12.0.3]
$$

光子の平面波は

$$
\begin{aligned}
A^\mu(x;q_1) &= \frac{\hbar\varepsilon_1^\mu}{\sqrt{2E_1 V\varepsilon_0}}(e^{-\frac{i}{\hbar}q_1 x} + e^{\frac{i}{\hbar}q_1 x}), \quad E_1 = q_1^0 c, \\
A^\mu(y;q_2) &= \frac{\hbar\varepsilon_2^\mu}{\sqrt{2E_2 V\varepsilon_0}}(e^{-\frac{i}{\hbar}q_2 y} + e^{\frac{i}{\hbar}q_2 y}), \quad E_2 = q_2^0 c.
\end{aligned} \quad [12.0.4]
$$

Lorenz ゲージを採用する：

$$
(q_1)^2 = (q_2)^2 = 0, \quad q_1\varepsilon_1 = q_2\varepsilon_2 = 0. \quad [12.0.5]
$$

Feynman 伝播函数は

$$
S_F(x';x) = \int \frac{d^4p}{(2\pi\hbar)^4} e^{-\frac{i}{\hbar}p(x'-x)} \frac{\hbar}{\not{p} - m_e c} \quad [12.0.6]
$$

で与えられる．これらを [12.0.1] に代入して，

$$
\begin{aligned}
S_{fi} = & -i\left(\frac{e}{\hbar}\right)^2 \frac{\hbar^2}{c\varepsilon_0} \int d^4x \int d^4y \frac{m_e c^2}{V^2} \frac{1}{\sqrt{E_- E_+}} \frac{1}{\sqrt{2q_1^0 q_2^0}} \\
& \times \overline{v}(p_+, s_+) \not{\varepsilon}_2 \int \frac{d^4p}{(2\pi\hbar)^4} \frac{\hbar}{\not{p} - m_e c} e^{-\frac{i}{\hbar}p(y-x)} \not{\varepsilon}_1 u(p_-, s_-) \\
& \times e^{-\frac{i}{\hbar}p_+ y} (e^{-\frac{i}{\hbar}q_2 y} + e^{\frac{i}{\hbar}q_2 y})(e^{-\frac{i}{\hbar}q_1 x} + e^{\frac{i}{\hbar}q_1 x}) e^{-\frac{i}{\hbar}p_- x} \\
& - i\left(\frac{e}{\hbar}\right)^2 \frac{\hbar^2}{c\varepsilon_0} \int d^4x \int d^4y \frac{m_e c^2}{V^2} \frac{1}{\sqrt{E_- E_+}} \frac{1}{\sqrt{2q_1^0 q_2^0}} \\
& \times \overline{v}(p_+, s_+) \not{\varepsilon}_1 \int \frac{d^4p}{(2\pi\hbar)^4} \frac{\hbar}{\not{p} - m_e c} e^{-\frac{i}{\hbar}p(y-x)} \not{\varepsilon}_2 u(p_-, s_-) \\
& \times e^{-\frac{i}{\hbar}p_+ y} (e^{-\frac{i}{\hbar}q_1 y} + e^{\frac{i}{\hbar}q_1 y})(e^{-\frac{i}{\hbar}q_2 x} + e^{\frac{i}{\hbar}q_2 x}) e^{-\frac{i}{\hbar}p_- x}.
\end{aligned}
\tag{12.0.7}
$$

運動量保存 $p_- + p_+ = q_1 + q_2$ を考慮して，(11.7) を使えば

$$
S_{fi} = \left(\frac{e}{\hbar}\right)^2 \frac{\hbar^2}{c\varepsilon_0} \frac{m_e c^2}{V^2} \frac{1}{\sqrt{E_+ E_-}} \frac{1}{\sqrt{2q_1^0 q_2^0}} (2\pi\hbar)^4 \delta^4(-p_+ + q_2 - p_- + q_1) M_{fi} \tag{12.0.8}
$$

$$
M_{fi} = \overline{v}(p_+, s_+) \left[(-i\not{\varepsilon}_2) \frac{i\hbar}{\not{p}_- - \not{q}_1 - m_e c} (-i\not{\varepsilon}_1) + (-i\not{\varepsilon}_1) \frac{i\hbar}{\not{p}_- - \not{q}_2 - m_e c} (-i\not{\varepsilon}_2) \right] u(p_-, s_-) \tag{12.0.9}
$$

となる．散乱断面積は [12.0.8] から

$$
\begin{aligned}
d\sigma &= \frac{|S_{fi}|^2}{v_+/V} \frac{V d^3 q_1}{(2\pi\hbar)^3} \frac{V d^3 q_2}{(2\pi\hbar)^3} \frac{1}{T} \\
&= \frac{e^4 {m_e}^2 c^5}{(c\varepsilon_0)^2 (2\pi\hbar)^2 E_+ E_- v_+} \delta^4(-p_+ + q_2 - p_- + q_1) |M_{fi}|^2 \frac{d^3 q_1}{2q_1^0} \frac{d^3 q_2}{2q_2^0}
\end{aligned}
\tag{12.0.10}
$$

となる（$v_+ E_+ = p_+ c^2$）．

はじめに電子が静止している系とする：

$$
p_- = (m_e c, 0) \tag{12.0.11}
$$

とする．さらに，

$$
\varepsilon_1 p_- = \varepsilon_2 p_- = 0, \quad \varepsilon_1 = (0, \boldsymbol{\varepsilon}_1), \quad \varepsilon_2 = (0, \boldsymbol{\varepsilon}_2), \quad \boldsymbol{\varepsilon_1} \cdot \boldsymbol{q_1} = \boldsymbol{\varepsilon_2} \cdot \boldsymbol{q_2} = 0 \tag{12.0.12}
$$

とする．Dirac 方程式から

$$
(\not{p}_- + m_e c)\not{\varepsilon}_1 u(p_-, s_-) = \not{\varepsilon}_1 (\not{p}_- - m_e c) u(p_-, s_-) = 0 \tag{12.0.13}
$$

$$
(\not{p}_- + m_e c)\not{\varepsilon}_2 u(p_-, s_-) = \not{\varepsilon}_2 (\not{p}_- - m_e c) u(p_-, s_-) = 0 \tag{12.0.14}
$$

なので，不変振幅 [12.0.9] は

$$
M_{fi} = -i\hbar \overline{v}(p_+, s_+) \left[\frac{\not{\varepsilon}_2 \not{q}_1 \not{\varepsilon}_1}{2q_1 p_-} + \frac{\not{\varepsilon}_1 \not{q}_2 \not{\varepsilon}_2}{2q_2 p_-} \right] u(p_-, s_-) \tag{12.0.15}
$$

[12.0.15] を [12.0.10] に代入して，

$$
\begin{aligned}
d\overline{\sigma} &= \frac{1}{4}\sum_{\pm s_-,\pm s_+} d\sigma \\
&= \frac{e^4}{(2\pi\hbar)^2}\frac{\hbar^2}{(c\varepsilon_0)^2}\frac{m_e^2c^5}{E_-E_+v_+}\frac{1}{4}\delta^4(-p_+ + q_2 - p_- + q_1) \\
&\quad\times \sum_{\pm s_-,\pm s_+}\left|\overline{v}(p_+,s_+)\left[\frac{\not{\varepsilon}_2\not{q}_1\not{\varepsilon}_1}{2q_1p_-}+\frac{\not{\varepsilon}_1\not{q}_2\not{\varepsilon}_2}{2q_2p_-}\right]u(p_-,s_-)\right|^2\frac{d^3q_1}{2|\boldsymbol{q}_1|}\frac{d^3q_2}{2|\boldsymbol{q}_2|}.
\end{aligned}
\tag{12.0.16}
$$

和の部分を計算する．Griffiths(7.99) から (11.20)(11.21) と同等の式

$$
\sum_{\pm s_-} u_\beta(p_-,s_-)\overline{u}_\gamma(p_-,s_-) = \left(\frac{\not{p}_- + m_ec}{2m_ec}\right)_{\beta\gamma}
\tag{12.0.17}
$$

$$
\sum_{\pm s_+} v_\delta(p_+,s_+)\overline{v}_\alpha(p_+,s_+) = \left(\frac{\not{p}_+ - m_ec}{2m_ec}\right)_{\delta\alpha}
\tag{12.0.18}
$$

と (11.36) を使って，(11.19) と同様に計算すれば

$$
\begin{aligned}
&\sum_{\alpha\beta\gamma\delta}\sum_{\pm s_-,\pm s_+}\overline{v}_\alpha(p_+,s_+)\Gamma_{\alpha\beta}u_\beta(p_-,s_-)\overline{u}_\gamma(p_-,s_-)\overline{\Gamma}_{\gamma\delta}v_\delta(p_+,s_+) \\
&= \sum_{\alpha\beta\gamma\delta}\left(\frac{\not{p}_+ - m_ec}{2m_ec}\right)_{\delta\alpha}\Gamma_{\alpha\beta}\left(\frac{\not{p}_- + m_ec}{2m_ec}\right)_{\beta\gamma}\overline{\Gamma}_{\gamma\delta} \\
&= \mathrm{Tr}\left(\frac{\not{p}_+ - m_ec}{2m_ec}\Gamma\frac{\not{p}_- + m_ec}{2m_ec}\overline{\Gamma}\right).
\end{aligned}
\tag{12.0.19}
$$

(11.37) から

$$
\overline{\Gamma} = \frac{\not{\varepsilon}_1\not{q}_1\not{\varepsilon}_2}{2q_1p_-}+\frac{\not{\varepsilon}_2\not{q}_2\not{\varepsilon}_1}{2q_2p_-}
\tag{12.0.20}
$$

なので，[12.0.16] は

$$
\begin{aligned}
d\overline{\sigma} &= \frac{e^4}{(2\pi\hbar)^2}\frac{\hbar^2}{(c\varepsilon_0)^2}\frac{m_e^2c^5}{E_-E_+v_+}\frac{1}{4}\mathrm{Tr}\left[\frac{\not{p}_+ - m_ec}{2m_ec}\left(\frac{\not{\varepsilon}_2\not{q}_1\not{\varepsilon}_1}{2q_1p_-}+\frac{\not{\varepsilon}_1\not{q}_2\not{\varepsilon}_2}{2q_2p_-}\right)\frac{\not{p}_- + m_ec}{2m_ec}\left(\frac{\not{\varepsilon}_1\not{q}_1\not{\varepsilon}_2}{2q_1p_-}+\frac{\not{\varepsilon}_2\not{q}_2\not{\varepsilon}_1}{2q_2p_-}\right)\right] \\
&\quad\times\delta^4(-p_+ + q_2 - p_- + q_1)\frac{d^3q_1}{2|\boldsymbol{q}_1|}\frac{d^3q_2}{2|\boldsymbol{q}_2|} \\
&= \frac{-e^4}{(2\pi\hbar)^2}\frac{\hbar^2}{(c\varepsilon_0)^2}\frac{m_e^2c^5}{E_-E_+v_+}\frac{1}{4}\mathrm{Tr}\left[\frac{-\not{p}_+ + m_ec}{2m_ec}\left(\frac{\not{\varepsilon}_2\not{\varepsilon}_1\not{q}_1}{2q_1p_-}+\frac{\not{\varepsilon}_1\not{\varepsilon}_2\not{q}_2}{2q_2p_-}\right)\frac{\not{p}_- + m_ec}{2m_ec}\left(\frac{\not{q}_1\not{\varepsilon}_1\not{\varepsilon}_2}{2q_1p_-}+\frac{\not{q}_2\not{\varepsilon}_2\not{\varepsilon}_1}{2q_2p_-}\right)\right] \\
&\quad\times\delta^4(-p_+ + q_2 - p_- + q_1)\frac{d^3q_1}{2|\boldsymbol{q}_1|}\frac{d^3q_2}{2|\boldsymbol{q}_2|}.
\end{aligned}
\tag{12.0.21}
$$

これで (12.43) が導出できた．

[12.0.21] のトレースを計算する．(12.21) と見比べれば，

$$
p_i \leftrightarrow p_- \quad p_f \leftrightarrow -p_+ \quad \varepsilon \leftrightarrow \varepsilon_1 \quad \varepsilon' \leftrightarrow \varepsilon_2 \quad q \leftrightarrow q_1 \quad q' \leftrightarrow q_2
\tag{12.0.22}
$$

の対応があることが分かる．まず，(12.28) と同等の式を計算する：

$$
\begin{aligned}
\mathrm{Tr}[(-\not{p}_+ + m_ec)\not{\varepsilon}_2\not{\varepsilon}_1\not{q}_1(\not{p}_- + m_ec)\not{q}_1\not{\varepsilon}_1\not{\varepsilon}_2] &= -8(q_1p_-)[(q_1p_+) + 2(q_1\varepsilon_2)(p_+\varepsilon_2)] \\
&= -8(q_1p_-)[(q_2p_-) + 2(q_1\varepsilon_2)^2].
\end{aligned}
\tag{12.0.23}
$$

次に (12.29)：

$$\begin{aligned}\mathrm{Tr}[(-\not{p}_+ + m_e c)\not{\varepsilon}_1\not{\varepsilon}_2\not{q}_2(\not{p}_- + m_e c)\not{q}_2\not{\varepsilon}_2\not{\varepsilon}_1] &= -8(q_2p_-)[(q_2p_+) + 2(q_2\varepsilon_1)(p_+\varepsilon_1)] \\ &= -8(q_2p_-)[(q_1p_-) + 2(q_2\varepsilon_1)^2].\end{aligned} \tag{12.0.24}$$

次に (12.30)：

$$\begin{aligned}&\mathrm{Tr}[(-\not{p}_+ + m_e c)\not{\varepsilon}_2\not{\varepsilon}_1\not{q}_1(\not{p}_- + m_e c)\not{q}_2\not{\varepsilon}_2\not{\varepsilon}_1] \\ &= \mathrm{Tr}[(\not{p}_- - \not{q}_1 - \not{q}_2 + m_e c)\not{\varepsilon}_2\not{\varepsilon}_1\not{q}_1(\not{p}_- + m_e c)\not{q}_2\not{\varepsilon}_2\not{\varepsilon}_1] \\ &= \mathrm{Tr}[(\not{p}_- + m_e c)\not{\varepsilon}_2\not{\varepsilon}_1\not{q}_1(\not{p}_- + m_e c)\not{q}_2\not{\varepsilon}_2\not{\varepsilon}_1] - \mathrm{Tr}[(\not{q}_1 + \not{q}_2)\not{\varepsilon}_2\not{\varepsilon}_1\not{q}_1(\not{p}_- + m_e c)\not{q}_2\not{\varepsilon}_2\not{\varepsilon}_1] \\ &= 8(p_-q_1)(p_-q_2)[2(\varepsilon_1\varepsilon_2)^2 - 1] + 8(\varepsilon_2q_1)^2(p_-q_2) + 8(\varepsilon_1q_2)^2(p_-q_1).\end{aligned} \tag{12.0.25}$$

同様に，T_4 に対応する部分はこれと等しい．よって，トレースは

$$\begin{aligned}&\mathrm{Tr}\left[\frac{-\not{p}_+ + m_e c}{2m_e c}\left(\frac{\not{\varepsilon}_2\not{\varepsilon}_1\not{q}_1}{2q_1p_-} + \frac{\not{\varepsilon}_1\not{\varepsilon}_2\not{q}_2}{2q_2p_-}\right)\frac{\not{p}_- + m_e c}{2m_e c}\left(\frac{\not{q}_1\not{\varepsilon}_1\not{\varepsilon}_2}{2q_1p_-} + \frac{\not{q}_2\not{\varepsilon}_2\not{\varepsilon}_1}{2q_2p_-}\right)\right] \\ &= \frac{1}{4{m_e}^2c^2}\left[\frac{-8(q_1p_-)}{4(q_1p_-)^2}\{(q_2p_-) + 2(q_1\varepsilon_2)^2\} + \frac{-8(q_2p_-)}{4(q_2p_-)^2}\{(q_1p_-) + 2(q_2\varepsilon_1)^2\}\right] \\ &\quad + \frac{1}{4{m_e}^2c^2}\frac{2}{4(q_1p_-)(q_2p_-)}[8(p_-q_1)(p_-q_2)\{2(\varepsilon_1\varepsilon_2)^2 - 1\} + 8(\varepsilon_2q_1)^2(p_-q_2) + 8(\varepsilon_1q_2)^2(p_-q_1)] \\ &= \frac{1}{4{m_e}^2c^2}\left[-2\frac{(q_2p_-)}{(q_1p_-)} - 4\frac{(q_1\varepsilon_2)^2}{(q_1p_-)} - 2\frac{(q_1p_-)}{(q_2p_-)} - 4\frac{(q_2\varepsilon_1)^2}{(q_2p_-)} + 4\{2(\varepsilon_1\varepsilon_2)^2 - 1\} + 4\frac{(q_1\varepsilon_2)^2}{(q_1p_-)} + 4\frac{(q_2\varepsilon_1)^2}{(q_2p_-)}\right] \\ &= -\frac{1}{2{m_e}^2c^2}\left[\frac{|\boldsymbol{q}_2|}{|\boldsymbol{q}_1|} + \frac{|\boldsymbol{q}_1|}{|\boldsymbol{q}_2|} - 4(\varepsilon_1\varepsilon_2)^2 + 2\right].\end{aligned} \tag{12.0.26}$$

これを [12.0.21] に代入して，

$$\begin{aligned}d\overline{\sigma} &= \frac{e^4}{(2\pi\hbar)^2}\frac{\hbar^2}{(c\varepsilon_0)^2}\frac{m_e^2c^5}{E_-E_+v_+}\frac{1}{4}\frac{1}{2{m_e}^2c^2}\left[\frac{|\boldsymbol{q}_2|}{|\boldsymbol{q}_1|} + \frac{|\boldsymbol{q}_1|}{|\boldsymbol{q}_2|} - 4(\varepsilon_1\varepsilon_2)^2 + 2\right]\delta^4(-p_+ + q_2 - p_- + q_1)\frac{d^3q_1}{2|\boldsymbol{q}_1|}\frac{d^3q_2}{2|\boldsymbol{q}_2|} \\ &= \frac{\alpha^2\hbar^2}{2m_ec|\boldsymbol{p}_+|}\left[\frac{|\boldsymbol{q}_2|}{|\boldsymbol{q}_1|} + \frac{|\boldsymbol{q}_1|}{|\boldsymbol{q}_2|} - 4(\varepsilon_1\varepsilon_2)^2 + 2\right]\delta^4(-p_+ + q_2 - p_- + q_1)\frac{d^3q_1}{2|\boldsymbol{q}_1|}\frac{d^3q_2}{2|\boldsymbol{q}_2|}.\end{aligned} \tag{12.0.27}$$

[12.0.27] の積分を計算する．

$$\begin{aligned}&\int \delta^4(-p_+ + q_2 - p_- + q_1)\frac{d^3q_1}{2|\boldsymbol{q}_1|}\frac{d^3q_2}{2|\boldsymbol{q}_2|} \\ &= \int \delta^3(-\boldsymbol{p}_+ + \boldsymbol{q}_2 + \boldsymbol{q}_1)\delta\left(-\frac{E_+}{c} + |\boldsymbol{q}_2| - m_ec + |\boldsymbol{q}_1|\right)\frac{d^3q_1}{2|\boldsymbol{q}_1|}\frac{d^3q_2}{2|\boldsymbol{q}_2|} \\ &= \int \delta\left(-\frac{E_+}{c} + |\boldsymbol{p}_+ - \boldsymbol{q}_1| - m_ec + |\boldsymbol{q}_1|\right)\frac{d^3q_1}{4|\boldsymbol{q}_1||\boldsymbol{p}_+ - \boldsymbol{q}_1|} \\ &= \int \delta\left(-\frac{E_+}{c} + \sqrt{|\boldsymbol{p}_+|^2 + |\boldsymbol{q}_1|^2 - 2|\boldsymbol{p}_+||\boldsymbol{q}_1|\cos\theta} - m_ec + |\boldsymbol{q}_1|\right)\frac{d^3q_1}{4|\boldsymbol{q}_1||\boldsymbol{p}_+ - \boldsymbol{q}_1|} \\ &= \int \delta\left(-\frac{E_+}{c} + \sqrt{|\boldsymbol{p}_+|^2 + |\boldsymbol{q}_1|^2 - 2|\boldsymbol{p}_+||\boldsymbol{q}_1|\cos\theta} - m_ec + |\boldsymbol{q}_1|\right)\frac{|\boldsymbol{q}_1|}{4|\boldsymbol{p}_+ - \boldsymbol{q}_1|}\,d\Omega_{q_1}\end{aligned} \tag{12.0.28}$$

なので，$\boldsymbol{q}_2 = \boldsymbol{p}_+ - \boldsymbol{q}_1$．デルタ函数の引数を $f(|\boldsymbol{q}_1|)$ とする．${E_+}^2 = c^2|\boldsymbol{p}_+|^2 + {m_e}^2c^2$ に注意して，$f = 0$ となるのは，

$$
\begin{aligned}
|\boldsymbol{q}_1| &= \frac{(E_+/c + m_e c)^2 - |\boldsymbol{p}_+|^2}{2(E_+/c + m_e c - |\boldsymbol{p}_+|\cos\theta)} \\
&= \frac{m_e c(m_e c^2 + E_+)}{(E_+ + m_e c^2 - c|\boldsymbol{p}_+|\cos\theta)}.
\end{aligned}
\tag{12.0.29}
$$

このとき，

$$
\begin{aligned}
|\boldsymbol{q}_2| &= |\boldsymbol{p}_+ - \boldsymbol{q}_1| \\
&= \frac{E_+}{c} + m_e c - |\boldsymbol{q}_1| \\
&= \frac{E_+ - c|\boldsymbol{p}_+|\cos\theta}{m_e c^2}|\boldsymbol{q}_1| \\
&= \frac{E_+ - c|\boldsymbol{p}_+|\cos\theta}{m_e c^2}\frac{m_e c(m_e c^2 + E_+)}{(E_+ + m_e c^2 - c|\boldsymbol{p}_+|\cos\theta)}.
\end{aligned}
\tag{12.0.30}
$$

次に，$\delta(f(|\boldsymbol{q}_1|))$ を計算する．

$$
f'(|\boldsymbol{q}_1|) = \frac{|\boldsymbol{q}_1| - |\boldsymbol{p}_+|\cos\theta + |\boldsymbol{p}_+ - \boldsymbol{q}_1|}{|\boldsymbol{p}_+ - \boldsymbol{q}_1|}
$$

なので，

$$
f'\left(\frac{m_e c(m_e c^2 + E_+)}{(E_+ + m_e c^2 - c|\boldsymbol{p}_+|\cos\theta)}\right) = \frac{E_+ + m_e c^2 - c|\boldsymbol{p}_+|\cos\theta}{c|\boldsymbol{q}_2|}.
$$

従って，

$$
\delta(f(|\boldsymbol{q}_1|)) = \frac{c|\boldsymbol{q}_2|}{E_+ + m_e c^2 - c|\boldsymbol{p}_+|\cos\theta}\delta\left(|\boldsymbol{q}_1| - \frac{m_e c(m_e c^2 + E_+)}{(E_+ + m_e c^2 - c|\boldsymbol{p}_+|\cos\theta)}\right)
\tag{12.0.31}
$$

これらを [12.0.28] に代入して，

$$
\int \delta^4(-p_+ + q_2 - p_- + q_1)\frac{d^3q_1}{2|\boldsymbol{q}_1|}\frac{d^3q_2}{2|\boldsymbol{q}_2|} = \frac{1}{4}\frac{m_e c^2(m_e c^2 + E_+)}{(m_e c^2 + E_+ - c|\boldsymbol{p}_+|\cos\theta)^2}\,d\Omega_{q_1}.
\tag{12.0.32}
$$

[12.0.27] に代入して，

$$
\begin{aligned}
\frac{d\overline{\sigma}}{d\Omega_{q_1}} &= \frac{\alpha^2\hbar^2}{2m_e c|\boldsymbol{p}_+|}\left[\frac{|\boldsymbol{q}_2|}{|\boldsymbol{q}_1|} + \frac{|\boldsymbol{q}_1|}{|\boldsymbol{q}_2|} - 4(\varepsilon_1\varepsilon_2)^2 + 2\right]\frac{1}{4}\frac{m_e c^2(m_e c^2 + E_+)}{(m_e c^2 + E_+ - c|\boldsymbol{p}_+|\cos\theta)^2} \\
&= \frac{\hbar^2\alpha^2 c(m_e c^2 + E_+)}{8|\boldsymbol{p}_+|(m_e c^2 + E_+ - c|\boldsymbol{p}_+|\cos\theta)^2}\left[\frac{|\boldsymbol{q}_2|}{|\boldsymbol{q}_1|} + \frac{|\boldsymbol{q}_1|}{|\boldsymbol{q}_2|} - 4(\varepsilon_1\varepsilon_2)^2 + 2\right] \\
&= \frac{\hbar^2\alpha^2 c(m_e c^2 + E_+)}{8|\boldsymbol{p}_+|(m_e c^2 + E_+ - c|\boldsymbol{p}_+|\cos\theta)^2}\left[\frac{E_+ - c|\boldsymbol{p}_+|\cos\theta}{m_e c^2} + \frac{m_e c^2}{E_+ - c|\boldsymbol{p}_+|\cos\theta} - 4(\varepsilon_1\varepsilon_2)^2 + 2\right].
\end{aligned}
\tag{12.0.33}
$$

$|\boldsymbol{q}_1|$，$|\boldsymbol{q}_2|$ は [12.0.29][12.0.30] で与えられる．

最後に，全断面積を求める．(12.34) 以降の手続きと同様にすれば良いが，いくつか注意点：

- 2 つの光子の偏極について和を取る（Compton 散乱の場合は始状態の光子は平均を取って，終状態は和を取った），つまり (12.34), (12.35) は 2 倍になる．

- ε_1 と ε_2 のなす角 ϕ は散乱角 θ と異なる．
- 出てくる光子は区別できないので，計算結果を 1/2 する必要がある．

特に，2 点目については，

$$\begin{aligned}\sin^2\phi &= \left(\frac{|\boldsymbol{p}_+|}{|\boldsymbol{q}_2|}\right)\sin^2\theta \\ &= \left(\frac{m_ec^2+E_+-c|\boldsymbol{p}_+|\cos\theta}{E_+-c|\boldsymbol{p}_+|\cos\theta}\right)^2\left(\frac{c|\boldsymbol{p}_+|}{E_++m_ec^2}\right)^2\sin^2\theta\end{aligned} \tag{12.0.34}$$

となる．まず，光子の偏極に関して平均を取れば，

$$\begin{aligned}\frac{d\overline{\sigma}}{d\Omega_{q_1}} &= \frac{\hbar^2\alpha^2c(m_ec^2+E_+)}{2|\boldsymbol{p}_+|(m_ec^2+E_+-c|\boldsymbol{p}_+|\cos\theta)^2}\left[\frac{E_+-c|\boldsymbol{p}_+|\cos\theta}{m_ec^2}+\frac{m_ec^2}{E_+-c|\boldsymbol{p}_+|\cos\theta}-4\frac{1+\cos^2\phi}{4}+2\right] \\ &= \frac{\hbar^2\alpha^2c(m_ec^2+E_+)}{2|\boldsymbol{p}_+|(m_ec^2+E_+-c|\boldsymbol{p}_+|\cos\theta)^2}\left[\frac{E_+-c|\boldsymbol{p}_+|\cos\theta}{m_ec^2}+\frac{m_ec^2}{E_+-c|\boldsymbol{p}_+|\cos\theta}+\sin^2\phi\right].\end{aligned} \tag{12.0.35}$$

従って，全断面積は，

$$\overline{\sigma} = \frac{1}{2}\int\frac{d\overline{\sigma}}{d\Omega_{q_1}}\,d\Omega_{q_1} \tag{12.0.36}$$

$$= \pi\int_0^\pi\frac{d\overline{\sigma}}{d\Omega_{q_1}}\sin\theta\,d\theta \tag{12.0.37}$$

$$= \pi\int_{-1}^1\frac{d\overline{\sigma}}{d\Omega_{q_1}}\,dz \tag{12.0.38}$$

$$= \frac{\pi\hbar^2\alpha^2c(m_ec^2+E_+)}{2|\boldsymbol{p}_+|}(R_1+R_2+R_3+R_4) \tag{12.0.39}$$

となる．ただし，

$$\begin{aligned}R_1 &= \int_{-1}^1\frac{1}{(m_ec^2+E_+-c|\boldsymbol{p}_+|z)^2}\frac{E_+-c|\boldsymbol{p}_+|z}{m_ec^2}\,dz \\ &= \frac{1}{m_ec^2}\int_{-1}^1\frac{dz}{m_ec^2+E_+-c|\boldsymbol{p}_+|z}-\int_{-1}^1\frac{1}{(m_ec^2+E_+-c|\boldsymbol{p}_+|z)^2}\,dz, \\ R_2 &= \int_{-1}^1\frac{1}{(m_ec^2+E_+-c|\boldsymbol{p}_+|z)^2}\frac{m_ec^2}{E_+-c|\boldsymbol{p}_+|z}\,dz \\ &= -\frac{1}{m_ec^2}\int_{-1}^1\frac{dz}{m_ec^2+E_+-c|\boldsymbol{p}_+|z}-\int_{-1}^1\frac{1}{(m_ec^2+E_+-c|\boldsymbol{p}_+|z)^2}\,dz+\frac{1}{m_ec^2}\int_{-1}^1\frac{dz}{E_+-c|\boldsymbol{p}_+|z}, \\ R_3 &= \left(\frac{c|\boldsymbol{p}_+|}{E_++m_ec^2}\right)^2\int_{-1}^1\frac{dz}{(E_+-c|\boldsymbol{p}_+|z)^2} \\ R_4 &= \left(\frac{c|\boldsymbol{p}_+|}{E_++m_ec^2}\right)^2\int_{-1}^1\frac{-z^2\,dz}{(E_+-c|\boldsymbol{p}_+|z)^2} \\ &= \left(\frac{c|\boldsymbol{p}_+|}{E_++m_ec^2}\right)^2\left[\int_{-1}^1-\frac{dz}{c^2|\boldsymbol{p}_+|^2}+\frac{2E_+}{c^2|\boldsymbol{p}_+|^2}\int_{-1}^1\frac{dz}{E_+-c|\boldsymbol{p}_+|z}-\frac{{E_+}^2}{c^2|\boldsymbol{p}_+|^2}\int_{-1}^1\frac{dz}{(E_+-c|\boldsymbol{p}_+|z)^2}\right] \\ &= -\frac{2}{(E+m_ec^2)^2}+\frac{2E_+}{(E_++m_ec^2)^2}\int_{-1}^1\frac{dz}{E_+-c|\boldsymbol{p}_+|z}-\left(\frac{E_+}{E_++m_ec^2}\right)^2\int_{-1}^1\frac{dz}{(E_+-c|\boldsymbol{p}_+|z)^2}\end{aligned} \tag{12.0.40}$$

と分けて計算する．以上から，

$$
\begin{aligned}
R_1+R_2+R_3+R_4 = &-\frac{2}{(E+m_ec^2)^2} \\
&+\left[\frac{2E_+}{(E_++m_ec^2)^2}+\frac{1}{m_ec^2}\right]\int_{-1}^{1}\frac{dz}{E_+-c|\boldsymbol{p}_+|z} \\
&+\frac{c^2|\boldsymbol{p}_+|^2-{E_+}^2}{(E_++m_ec^2)^2}\int_{-1}^{1}\frac{dz}{(E_+-c|\boldsymbol{p}_+|z)^2} \\
&-2\int_{-1}^{1}\frac{dz}{(m_ec^2+E_+-c|\boldsymbol{p}_+|z)^2}.
\end{aligned}
\qquad [12.0.41]
$$

第 2 項は

$$
\begin{aligned}
\left[\frac{2E_+}{(E_++m_ec^2)^2}+\frac{1}{m_ec^2}\right]\frac{1}{c|\boldsymbol{p}_+|}\log\left|\frac{E_++c|\boldsymbol{p}_+|}{E_+-c|\boldsymbol{p}_+|}\right| &= \left[\frac{2E_+}{(E_++m_ec^2)^2}+\frac{1}{m_ec^2}\right]\frac{1}{c|\boldsymbol{p}_+|}\log\frac{(E_++c|\boldsymbol{p}_+|)^2}{{E_+}^2-c^2|\boldsymbol{p}_+|^2} \\
&= \left[\frac{2E_+}{(E_++m_ec^2)^2}+\frac{1}{m_ec^2}\right]\frac{2}{c|\boldsymbol{p}_+|}\log\frac{E_++c|\boldsymbol{p}_+|}{m_ec^2} \\
&= \frac{{E_+}^2+4E_+m_ec^2+{m_e}^2c^4}{(E_++m_ec^2)^2m_ec^2}\frac{2}{c|\boldsymbol{p}_+|}\log\frac{E_++c|\boldsymbol{p}_+|}{m_ec^2}.
\end{aligned}
\qquad [12.0.42]
$$

第 3 項は

$$
\begin{aligned}
\frac{c^2|\boldsymbol{p}_+|^2-{E_+}^2}{(E_++m_ec^2)^2}\int_{-1}^{1}\frac{dz}{(E_+-c|\boldsymbol{p}_+|z)^2} &= \frac{-{m_e}^2c^4}{(E_++m_ec^2)^2}\frac{2}{{m_e}^2c^4} \\
&= -\frac{2}{(E_++m_ec^2)^2}.
\end{aligned}
\qquad [12.0.43]
$$

第 4 項は

$$
\begin{aligned}
-2\int_{-1}^{1}\frac{1}{(m_ec^2+E_+-c|\boldsymbol{p}_+|z)^2}\,dz &= -\frac{4}{(E_++m_ec^2)^2-c^2|\boldsymbol{p}_+|^2} \\
&= -\frac{2}{m_ec^2(E_++m_ec^2)}.
\end{aligned}
\qquad [12.0.44]
$$

[12.0.42][12.0.43][12.0.44] を [12.0.41] に代入して，

$$
\begin{aligned}
&R_1+R_2+R_3+R_4 \\
&= -\frac{2}{(E_++m_ec^2)^2}+\frac{{E_+}^2+4E_+m_ec^2+{m_e}^2c^4}{(E_++m_ec^2)^2m_ec^2}\frac{2}{c|\boldsymbol{p}_+|}\log\frac{E_++c|\boldsymbol{p}_+|}{m_ec^2} \\
&\quad -\frac{2}{(E_++m_ec^2)^2}-\frac{2}{m_ec^2(E_++m_ec^2)} \\
&= \frac{2}{(E_++m_ec^2)^2m_ec^2c|\boldsymbol{p}_+|}\left[({E_+}^2+4m_ec^2E_++{m_e}^2c^4)\log\frac{E_++c|\boldsymbol{p}_+|}{m_ec^2}-(E_++3m_ec^2)c|\boldsymbol{p}_+|\right].
\end{aligned}
\qquad [12.0.45]
$$

[12.0.39] に代入して，

$$
\begin{aligned}
\overline{\sigma} &= \frac{\pi\hbar^2\alpha^2 c(m_e c^2 + E_+)}{2|\boldsymbol{p}_+|} \frac{2}{(E_+ + m_e c^2)^2 m_e c^2 c|\boldsymbol{p}_+|} \\
&\qquad \times \left[({E_+}^2 + 4m_e c^2 E_+ + {m_e}^2 c^4) \log \frac{E_+ + c|\boldsymbol{p}_+|}{m_e c^2} - (E_+ + 3m_e c^2) c|\boldsymbol{p}_+| \right] \\
&= \frac{\pi\hbar^2\alpha^2}{m_e c^2 |\boldsymbol{p}_+|(E_+ + m_e c^2)} \left[({E_+}^2 + 4m_e c^2 E_+ + {m_e}^2 c^4) \log \frac{E_+ + c|\boldsymbol{p}_+|}{m_e c^2} - (E_+ + 3m_e c^2) c|\boldsymbol{p}_+| \right].
\end{aligned}
\tag{12.0.46}
$$

第 13 章

電子・電子散乱と電子・陽電子散乱

■(13.21)　1 つめの項は

$$\begin{aligned}
&\sum_{\pm s_1,\pm s_1'}\sum_{\pm s_2,\pm s_2'}\frac{1}{(p_1-p_1')^4}\left[\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right]\\
&\qquad\qquad\times\left[\overline{u}(p_1',s_1')\gamma_\nu u(p_1,s_1)\overline{u}(p_2',s_2')\gamma^\nu u(p_2,s_2)\right]^*\\
&=\sum_{\pm s_1,\pm s_1'}\sum_{\pm s_2,\pm s_2'}\frac{1}{(p_1-p_1')^4}\left[\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right]\left[\overline{u}(p_1',s_1')\gamma_\nu u(p_1,s_1)\right]^*\\
&\qquad\qquad\times\left[\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right]\left[\overline{u}(p_2',s_2')\gamma^\nu u(p_2,s_2)\right]^*\\
&=\frac{1}{(p_1-p_1')^4}\sum_{\pm s_1,\pm s_1'}\left[\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right]\left[\overline{u}(p_1,s_1)\overline{\gamma}_\nu u(p_1',s_1')\right]\\
&\qquad\times\sum_{\pm s_2,\pm s_2'}\left[\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right]\left[\overline{u}(p_2,s_2)\overline{\gamma}^\nu u(p_2',s_2')\right]
\end{aligned}$$

となり，(11.19) と同様に計算すれば，

$$=\frac{1}{(p_1-p_1')^4}\,\mathrm{Tr}\left(\frac{\not{p}_1'+m_ec}{2m_ec}\gamma_\mu\frac{\not{p}_1+m_ec}{2m_ec}\gamma_\nu\right)\mathrm{Tr}\left(\frac{\not{p}_2'+m_ec}{2m_ec}\gamma^\mu\frac{\not{p}_2+m_ec}{2m_ec}\gamma^\nu\right).$$

2 つめの項は

$$\begin{aligned}
&\sum_{\pm s_1,\pm s_1'}\sum_{\pm s_2,\pm s_2'}\frac{1}{(p_1-p_1')^2(p_1-p_2')^2}\left[\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right]\\
&\qquad\qquad\times\left[\overline{u}(p_2',s_2')\gamma_\nu u(p_1,s_1)\overline{u}(p_1',s_1')\gamma^\nu u(p_2,s_2)\right]^*\\
&=\sum_{\pm s_1,\pm s_1'}\sum_{\pm s_2,\pm s_2'}\frac{1}{(p_1-p_1')^2(p_1-p_2')^2}\left[\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right]\left[\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right]\\
&\qquad\qquad\times\left[\overline{u}(p_2',s_2')\gamma_\nu u(p_1,s_1)\right]^*\left[\overline{u}(p_1',s_1')\gamma^\nu u(p_2,s_2)\right]^*\\
&=\sum_{\pm s_1,\pm s_1'}\sum_{\pm s_2,\pm s_2'}\frac{1}{(p_1-p_1')^2(p_1-p_2')^2}\left[\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right]\left[\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right]\\
&\qquad\qquad\times\left[\overline{u}(p_1,s_1)\overline{\gamma}_\nu u(p_2',s_2')\right]\left[\overline{u}(p_2,s_2)\overline{\gamma}^\nu u(p_1',s_1')\right]\\
&=\sum_{\pm s_1,\pm s_1'}\sum_{\pm s_2,\pm s_2'}\frac{1}{(p_1-p_1')^2(p_1-p_2')^2}\left[\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right]\left[\overline{u}(p_1,s_1)\overline{\gamma}_\nu u(p_2',s_2')\right]\\
&\qquad\qquad\times\left[\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right]\left[\overline{u}(p_2,s_2)\overline{\gamma}^\nu u(p_1',s_1')\right]
\end{aligned}$$

となり，(11.19) と同様に計算すれば，

$$= \frac{1}{(p_1-p_1')^2(p_1-p_2')^2}\,\mathrm{Tr}\left(\frac{\not{p}_1'+m_ec}{2m_ec}\gamma_\mu\frac{\not{p}_1+m_ec}{2m_ec}\gamma_\nu\frac{\not{p}_2'+m_ec}{2m_ec}\gamma^\mu\frac{\not{p}_2+m_ec}{2m_ec}\gamma^\nu\right).$$

■(13.31)　${\delta_{(s)}}^\mu$ を $\mu=s$ に対しては 1，$\mu\neq s$ に対しては 0 となる 4 元ベクトルとすれば，$\not{\delta}_{(\nu)}=\gamma_\nu$ となる．これに注意して (H.34), (H.30) を使えば，

$$\begin{aligned}
\mathrm{Tr}(\not{p}_1'\gamma_\mu\not{p}_1\gamma_\nu\not{p}_2'\gamma^\mu\not{p}_2\gamma^\nu) &= -2\,\mathrm{Tr}(\not{p}_1'\not{p}_2'\gamma_\nu\not{p}_1\not{p}_2\gamma^\nu)\\
&= -8(p_1p_2)\,\mathrm{Tr}(\not{p}_1'\not{p}_2')\\
&= -8(p_1p_2)(p_1'p_2').
\end{aligned}$$

■(13.37), (13.38)　(13.9) を使う．図 13.2(a) のパターンだと

$$\begin{aligned}
\psi_i(x) &= \sqrt{\frac{m_ec^2}{E_1V}}u(p_1,s_1)e^{-\frac{i}{\hbar}p_1x},\\
\overline{\psi}_f(x) &= \sqrt{\frac{m_ec^2}{E_1'V}}\overline{u}(p_1',s_1')e^{\frac{i}{\hbar}p_1'x},\\
\psi_i^{(2)}(y) &= \sqrt{\frac{m_ec^2}{\tilde{E}_2'V}}v(\overline{p}_2',\overline{s}_2')e^{\frac{i}{\hbar}\overline{p}_2'y},\\
\overline{\psi}_f^{(2)}(y) &= \sqrt{\frac{m_ec^2}{\tilde{E}_2V}}\overline{v}(\overline{p}_2,\overline{s}_2)e^{-\frac{i}{\hbar}\overline{p}_2y}.
\end{aligned}$$

図 13.2(b) のパターンだと

$$\begin{aligned}
\psi_i(x) &= \sqrt{\frac{m_ec^2}{E_1V}}u(p_1,s_1)e^{-\frac{i}{\hbar}p_1x},\\
\overline{\psi}_f(x) &= \sqrt{\frac{m_ec^2}{\tilde{E}_2V}}\overline{v}(\overline{p}_2,\overline{s}_2)e^{-\frac{i}{\hbar}\overline{p}_2y},\\
\psi_i^{(2)}(y) &= \sqrt{\frac{m_ec^2}{\tilde{E}_2'V}}v(\overline{p}_2',\overline{s}_2')e^{\frac{i}{\hbar}\overline{p}_2'y},\\
\overline{\psi}_f^{(2)}(y) &= \sqrt{\frac{m_ec^2}{E_1'V}}\overline{u}(p_1',s_1')e^{\frac{i}{\hbar}p_1'x}.
\end{aligned}$$

[Griffiths](7.99) から (11.20)(11.21) と同等の式

$$\begin{aligned}
\sum_{\pm s_-}u_\beta(p_-,s_-)\overline{u}_\gamma(p_-,s_-) &= \left(\frac{\not{p}_-+m_ec}{2m_ec}\right)_{\beta\gamma}\\
\sum_{\pm s_+}v_\delta(p_+,s_+)\overline{v}_\alpha(p_+,s_+) &= \left(\frac{\not{p}_+-m_ec}{2m_ec}\right)_{\delta\alpha}
\end{aligned}$$

■(13.39)　(13.20) に対応する式は

$$d\overline{\sigma} = \frac{16\hbar^2c\alpha^2}{|\boldsymbol{v}_1-\boldsymbol{v}_2|}\frac{{m_e}^4c^8}{E_1\overline{E}_2'}\,\delta^4(p_1'+\overline{p}_2'-p_1-\overline{p}_2)\overline{|M_{fi}|^2}\frac{d^3p_1'}{2E_1'}\frac{d^3\overline{p}_2}{2\overline{E}_2}.$$

高エネルギー極限で重心系から見た場合は (13.22) と同様に,

$$p_1 = (E/c, \boldsymbol{p}), \quad \overline{p}_2 = (E/c, -\boldsymbol{p}), \quad p_1' = (E'/c, \boldsymbol{p}'), \quad \overline{p}_2' = (E'/c, -\boldsymbol{p}'), \quad |\boldsymbol{v}_1 - \boldsymbol{v}_2| = \frac{2|\boldsymbol{p}|c^2}{E}.$$

(13.23) に対応する式は

$$d\overline{\sigma} = \frac{d^3 p_1'}{2E'} \frac{d^3 \overline{p}_2}{2E} \, \delta^4(p_1' + \overline{p}_2' - p_1 - \overline{p}_2) F(p_1', \overline{p}_2), \quad F(p_1', \overline{p}_2) = \frac{8\hbar^2 \alpha^2 {m_e}^4 c^7}{|\boldsymbol{p}| E} |M_{fi}|^2.$$

(13.26) に対応する式は

$$d\overline{\sigma} = d\Omega_1' \frac{\hbar^2 \alpha^2 {m_e}^4 c^6}{E^2} |M_{fi}|^2.$$

(13.27) 対応する式は

$$p_1 \overline{p}_2 \approx \frac{2E^2}{c^2}, \quad p_1' \overline{p}_2' \approx \frac{2E'^2}{c^2}, \quad p_1 \overline{p}_2' = p_1' \overline{p}_2 \approx \frac{2EE'}{c^2} \cos^2 \frac{\theta}{2}, \quad p_1 p_1' = \overline{p}_2 \overline{p}_2' \approx \frac{2EE'}{c^2} \sin^2 \frac{\theta}{2}.$$

$|M_{fi}|^2$ の第 1 項に対応する部分（前半は (13.21) の計算；最後の計算は (13.29) と同じ）は,

$$\begin{aligned}
&\frac{1}{4} \sum_{\pm s_1, \pm s_1'} \sum_{\pm \overline{s}_2, \pm \overline{s}_2'} \frac{1}{(p_1 - p_1')^4} [\overline{u}(p_1', s_1') \gamma_\mu u(p_1, s_1) \overline{v}(\overline{p}_2, \overline{s}_2) \gamma^\mu v(\overline{p}_2', \overline{s}_2')] \\
&\qquad\qquad \times [\overline{u}(p_1', s_1') \gamma_\mu u(p_1, s_1) \overline{v}(\overline{p}_2, \overline{s}_2) \gamma^\mu v(\overline{p}_2', \overline{s}_2')]^* \\
&= \frac{1}{4} \frac{1}{(p_1 - p_1')^4} \sum_{\pm s_1, \pm s_1'} [\overline{u}(p_1', s_1') \gamma_\mu u(p_1, s_1)][\overline{u}(p_1, s_1) \overline{\gamma}_\nu u(p_1', s_1')] \\
&\qquad\qquad \times \sum_{\pm \overline{s}_2, \pm \overline{s}_2'} [\overline{v}(\overline{p}_2, \overline{s}_2) \gamma^\mu v(\overline{p}_2', \overline{s}_2')][\overline{v}(\overline{p}_2', \overline{s}_2') \overline{\gamma}^\nu v(\overline{p}_2, \overline{s}_2)] \\
&= \frac{1}{4} \frac{1}{(p_1 - p_1')^4} \operatorname{Tr} \left(\frac{\not{p}_1' + m_e c}{2m_e c} \gamma_\mu \frac{\not{p}_1 + m_e c}{2m_e c} \gamma_\nu \right) \operatorname{Tr} \left(\frac{\overline{\not{p}}_2 - m_e c}{2m_e c} \gamma^\mu \frac{\overline{\not{p}}_2' - m_e c}{2m_e c} \gamma^\nu \right) \\
&\approx \frac{1}{8{m_e}^4 c^4} \frac{1 + \cos^4 \frac{\theta}{2}}{\sin^4 \frac{\theta}{2}}.
\end{aligned}$$

第 2・3 項は（トレースの展開は (13.31) と同様）

$$\begin{aligned}
&-\frac{1}{4} \sum_{\pm s_1, \pm s_1'} \sum_{\pm \overline{s}_2, \pm \overline{s}_2'} \frac{1}{(p_1 - p_1')^2 (p_1 + \overline{p}_2)^2} [\overline{u}(p_1', s_1') \gamma_\mu u(p_1, s_1) \overline{v}(\overline{p}_2, \overline{s}_2) \gamma^\mu v(\overline{p}_2', \overline{s}_2')] \\
&\qquad\qquad \times [\overline{u}(p_1', s_1') \gamma_\nu v(\overline{p}_2', \overline{s}_2') \overline{v}(\overline{p}_2, \overline{s}_2) \gamma^\nu u(p_1, s_1)]^* \\
&= -\frac{1}{4} \frac{1}{(p_1 - p_1')^2 (p_1 + \overline{p}_2)^2} \sum_{\pm s_1, \pm s_1'} \sum_{\pm \overline{s}_2, \pm \overline{s}_2'} [\overline{u}(p_1', s_1') \gamma_\mu u(p_1, s_1)][\overline{u}(p_1, s_1) \overline{\gamma}^\nu v(\overline{p}_2, \overline{s}_2)] \\
&\qquad\qquad \times [\overline{v}(\overline{p}_2, \overline{s}_2) \gamma^\mu v(\overline{p}_2', \overline{s}_2')][\overline{v}(\overline{p}_2', \overline{s}_2') \overline{\gamma}_\nu u(p_1', s_1')] \\
&= -\frac{1}{4} \frac{1}{(p_1 - p_1')^2 (p_1 + \overline{p}_2)^2} \operatorname{Tr} \left(\frac{\not{p}_1' + m_e c}{2m_e c} \gamma_\mu \frac{\not{p}_1 + m_e c}{2m_e c} \gamma^\nu \frac{\overline{\not{p}}_2 - m_e c}{2m_e c} \gamma^\mu \frac{\overline{\not{p}}_2' - m_e c}{2m_e c} \gamma_\nu \right) \\
&\approx -\frac{1}{4} \frac{1}{(p_1 - p_1')^2 (p_1 + \overline{p}_2)^2} \frac{1}{16{m_e}^4 c^4} \operatorname{Tr}(\not{p}_1' \gamma_\mu \not{p}_1 \gamma^\nu \overline{\not{p}}_2 \gamma^\mu \overline{\not{p}}_2' \gamma_\nu) \\
&\approx -\frac{1}{4} \left(-4 \frac{EE'}{c^2} \sin^2 \frac{\theta}{2} \right)^{-1} \left(\frac{4E^2}{c^2} \right)^{-1} \frac{-32}{16{m_e}^4 c^4} (p_1 \overline{p}_2')(p_1' \overline{p}_2) \\
&= -\frac{1}{8{m_e}^4 c^4} \frac{\cos^4 \frac{\theta}{2}}{\sin^2 \frac{\theta}{2}}.
\end{aligned}$$

第 4 項は

$$
\begin{aligned}
&\frac{1}{4}\sum_{\pm s_1,\pm s_1'}\sum_{\pm \overline{s}_2,\pm \overline{s}_2'}\frac{1}{(p_1+\overline{p}_2)^4}[\overline{u}(p_1',s_1')\gamma_\mu v(\overline{p}_2',\overline{s}_2')\overline{v}(\overline{p}_2,\overline{s}_2)\gamma^\mu u(p_1,s_1)] \\
&\qquad\qquad\qquad\qquad \times[\overline{u}(p_1',s_1')\gamma_\nu v(\overline{p}_2',\overline{s}_2')\overline{v}(\overline{p}_2,\overline{s}_2)\gamma^\nu u(p_1,s_1)]^* \\
&=\frac{1}{4}\frac{1}{(p_1+\overline{p}_2)^4}\sum_{\pm s_1,\pm s_1'}\sum_{\pm \overline{s}_2,\pm \overline{s}_2'}[\overline{u}(p_1',s_1')\gamma_\mu v(\overline{p}_2',\overline{s}_2')][\overline{v}(\overline{p}_2',\overline{s}_2')\gamma_\nu u(p_1',s_1')] \\
&\qquad\qquad\qquad\qquad \times[\overline{v}(\overline{p}_2,\overline{s}_2)\gamma^\mu u(p_1,s_1)][\overline{u}(p_1,s_1)\gamma^\nu v(\overline{p}_2,\overline{s}_2)] \\
&=\frac{1}{4}\frac{1}{(p_1+\overline{p}_2)^4}\operatorname{Tr}\left(\frac{\not{p}_1'+m_ec}{2m_ec}\gamma_\mu\frac{\overline{\not{p}}_2'-m_ec}{2m_ec}\gamma_\nu\right)\operatorname{Tr}\left(\frac{\overline{\not{p}}_2-m_ec}{2m_ec}\gamma^\mu\frac{\not{p}_1+m_ec}{2m_ec}\gamma^\nu\right) \\
&\approx\frac{1}{4}\frac{1}{(p_1+\overline{p}_2)^4}\frac{1}{16{m_e}^4c^4}\operatorname{Tr}\left(\not{p}_1'\gamma_\mu\overline{\not{p}}_2'\gamma_\nu\right)\operatorname{Tr}\left(\overline{\not{p}}_2\gamma^\mu\not{p}_1\gamma^\nu\right) \\
&\approx\frac{1}{4}\frac{1}{(p_1+\overline{p}_2)^4}\frac{1}{{m_e}^4c^4}[p_{1\mu}'\overline{p}_{2\nu}'+p_{1\nu}'\overline{p}_{2\mu}'-(p_1'\overline{p}_2')\eta_{\mu\nu}][{\overline{p}_2}^\mu {p_1}^\nu+{\overline{p}_2}^\nu {p_1}^\mu-(\overline{p}_2p_1)\eta^{\mu\nu}] \\
&=\frac{1}{2}\frac{1}{(p_1+\overline{p}_2)^4}\frac{1}{{m_e}^4c^4}[(p_1'\overline{p}_2)(\overline{p}_2'p_1)+(p_1'p_1)(\overline{p}_2'\overline{p}_2)] \\
&=\frac{1}{8}\frac{1}{{m_e}^4c^4}\left(\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}\right).
\end{aligned}
$$

従って，

$$
\begin{aligned}
\frac{d\overline{\sigma}}{d\Omega_1'}&=\frac{c^2\hbar^2\alpha^2}{8E^2}\left(\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}-2\times\frac{\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}+\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}\right) \\
&=\frac{c^2\hbar^2\alpha^2}{8E^2}\left[\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}-2\times\frac{\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}+\left(\frac{1+\cos\theta}{2}\right)^2\left(\frac{1-\cos\theta}{2}\right)^2\right] \\
&=\frac{c^2\hbar^2\alpha^2}{8E^2}\left(\frac{1+\cos^4\frac{\theta}{2}}{\sin^4\frac{\theta}{2}}-2\frac{\cos^4\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}+\frac{1+\cos^2\theta}{2}\right).
\end{aligned}
$$

第 15 章

高次補正　—その 1—

■(15.3)　(11.7)(11.8) を使う．左図の S 行列は

$$
\begin{aligned}
&\overline{u}(p_f, s_f) \int \frac{d^4p}{(2\pi\hbar)^4} i\not{\epsilon} \frac{i\hbar}{\not{p} - m_e c + i\varepsilon} i\gamma^0 \int d^4x\, e^{\frac{i}{\hbar}(p_f \mp k - p)x} \int d^4y\, e^{\frac{i}{\hbar}(p - p_i)y} \frac{1}{|\boldsymbol{y}|} u(p_i, s_i) \\
&= -\overline{u}(p_f, s_f) \int \frac{d^4p}{(2\pi\hbar)^4} \not{\epsilon} \frac{i\hbar}{\not{p} - m_e c + i\varepsilon} \gamma^0 (2\pi\hbar)^4\, \delta^4(p_f \mp k - p)(2\pi\hbar)\, \delta(E - E_i) \frac{4\pi\hbar}{|\boldsymbol{p} - \boldsymbol{p}_i|^2} u(p_i, s_i) \\
&= -\overline{u}(p_f, s_f) \not{\epsilon} \frac{i\hbar}{\not{p}_f + \not{k} - m_e c + i\varepsilon} \gamma^0 2\pi\hbar\, \delta(E_f + |\boldsymbol{k}|c - E_i) \frac{4\pi\hbar}{|\boldsymbol{q}|^2} u(p_i, s_i) \\
&= -i\overline{u}(p_f, s_f) \not{\epsilon} \frac{\not{p}_f + \not{k} + m_e c}{(p_f + k)^2 - {m_e}^2 c^2} \gamma^0 2\pi\hbar\, \delta(E_f + |\boldsymbol{k}|c - E_i) \frac{4\pi\hbar^2}{|\boldsymbol{q}|^2} u(p_i, s_i).
\end{aligned}
$$

右図は

$$
\begin{aligned}
&\overline{u}(p_f, s_f) \int \frac{d^4p}{(2\pi\hbar)^4} \int d^4x\, e^{\frac{i}{\hbar}(p_f - p)x} \frac{1}{|\boldsymbol{x}|} i\gamma^0 \frac{i\hbar}{\not{p} - m_e c + i\varepsilon} i\not{\epsilon} \int d^4y\, e^{\frac{i}{\hbar}(p \mp k - p_i)y} u(p_i, s_i) \\
&= -\overline{u}(p_f, s_f) \int \frac{d^4p}{(2\pi\hbar)^4} 2\pi\hbar\, \delta(E_f - E) \frac{4\pi\hbar}{|\boldsymbol{p}_f - \boldsymbol{p}|^2} \gamma^0 \frac{i\hbar}{\not{p} - m_e c + i\varepsilon} \not{\epsilon} (2\pi\hbar)^4\, \delta(p \mp k - p_i) u(p_i, s_i) \\
&= -\overline{u}(p_f, s_f) \gamma^0 \frac{i\hbar}{\not{p}_i - \not{k} - m_e c + i\varepsilon} \not{\epsilon} 2\pi\hbar\, \delta(E_f + |\boldsymbol{k}|c - E_i) \frac{4\pi\hbar}{|\boldsymbol{q}|^2} u(p_i, s_i) \\
&= -i\overline{u}(p_f, s_f) \gamma^0 \frac{\not{p}_i - \not{k} + m_e c + i\varepsilon}{(p_i - k)^2 - {m_e}^2 c^2 + i\varepsilon} \not{\epsilon} 2\pi\hbar\, \delta(E_f + |\boldsymbol{k}|c - E_i) \frac{4\pi\hbar^2}{|\boldsymbol{q}|^2} u(p_i, s_i).
\end{aligned}
$$

■(15.10)　$|1 + \tilde{F}_1(q^2)|^2$ は通常散乱，真空偏極，頂点補正の断面積．$\int \cdots dk$ は制動放射．

■(15.25)

$$
\begin{aligned}
\int \overline{\psi} \gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{\nabla} A^0 \psi\, d^3x &= \int \begin{pmatrix} \varphi^\dagger & \chi^\dagger \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \\ -\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} & 0 \end{pmatrix} A^0 \begin{pmatrix} \varphi \\ \chi \end{pmatrix} d^3x \\
&= \int d^3x\, [\varphi^\dagger (\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} A^0) \chi - \chi^\dagger (\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} A^0) \varphi] \\
&= -\frac{i\hbar}{2m_e c} \int d^3x\, [\varphi^\dagger (\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} A^0)(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \varphi) + (\boldsymbol{\nabla} \varphi^\dagger \cdot \boldsymbol{\sigma})(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} A^0) \varphi] \\
&= -\frac{i\hbar}{2m_e c} \int d^3x\, [\varphi^\dagger (\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} A^0)(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \varphi) - \varphi^\dagger \boldsymbol{\sigma} \cdot \boldsymbol{\nabla} \{(\boldsymbol{\sigma} \cdot \boldsymbol{\nabla} A^0) \varphi\}].
\end{aligned}
$$

$[\cdots]$ の中身は

$$
\begin{aligned}
&\varphi^\dagger(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}A^0)(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\varphi)-\varphi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\{(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}A^0)\varphi\}\\
&=\varphi^\dagger[(\boldsymbol{\nabla}A^0)\cdot(\boldsymbol{\nabla}\varphi)+i\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}A^0)\times(\boldsymbol{\nabla}\varphi)]\\
&\quad-\varphi^\dagger[\{(\boldsymbol{\nabla}^2+i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\times\boldsymbol{\nabla})A^0\}\varphi+\{(\boldsymbol{\nabla}\varphi)\cdot(\boldsymbol{\nabla}A^0)+i\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\varphi)\times(\boldsymbol{\nabla}A^0)\}]\\
&=-\varphi^\dagger(\boldsymbol{\nabla}^2A^0)\varphi+2i\varphi^\dagger\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}A^0)\times(\boldsymbol{\nabla}\varphi)\\
&=-\varphi^\dagger(\boldsymbol{\nabla}^2A^0)\varphi+2i\varphi^\dagger\boldsymbol{\sigma}\cdot\left(\frac{dA^0}{dr}\frac{\boldsymbol{r}}{r}\right)\times(\boldsymbol{\nabla}\varphi)\\
&=-\varphi^\dagger(\boldsymbol{\nabla}^2A^0)\varphi-\frac{2}{\hbar}\varphi^\dagger\frac{1}{r}\frac{dA^0}{dr}\boldsymbol{\sigma}\cdot\boldsymbol{L}\varphi.
\end{aligned}
$$

第 16 章

Appendix C

Runge-Lenz-Pauli ベクトル：

$$\boldsymbol{R} = \frac{1}{2m}(\boldsymbol{L}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{L}) + k_0 Ze^2\frac{\boldsymbol{r}}{r} \tag{16.0.1}$$

の性質を調べる[*1]．まずは，エルミート性について，

$$\begin{aligned}
(\boldsymbol{L}\times\boldsymbol{p})^\dagger &= \sum_{ijk}(L_i p_j \boldsymbol{e}_k \varepsilon_{ijk})^\dagger \\
&= \sum_{ijk}(p_j L_i \boldsymbol{e}_k \varepsilon_{ijk}) \\
&= \sum_{ijk}(p_i L_j \boldsymbol{e}_k \varepsilon_{jik}) \\
&= -\sum_{ijk}(p_i L_j \boldsymbol{e}_k \varepsilon_{ijk}) \\
&= -\boldsymbol{p}\times\boldsymbol{L}
\end{aligned}$$

から，

$$\boldsymbol{R}^\dagger = \boldsymbol{R}. \tag{16.0.2}$$

次に，ハミルトニアン H と可換であることを示す．

$$\begin{aligned}
\boldsymbol{L}\times\boldsymbol{p} &= \sum_{ijk} L_i p_j \boldsymbol{e}_k \varepsilon_{ijk} \\
&= \sum_{ijk}\sum_{mn} r_m p_n \varepsilon_{mni} p_j \boldsymbol{e}_k \varepsilon_{ijk} \\
&= \sum_{ijk}\sum_{mn} r_m p_n p_j \boldsymbol{e}_k \varepsilon_{imn} \varepsilon_{ijk} \\
&= \sum_{ijk}\sum_{mn} r_m p_n p_j \boldsymbol{e}_k (\varepsilon_{ijk})^2 (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) \\
&= \sum_{ijk}(r_j p_j p_k - r_k {p_j}^2)\boldsymbol{e}_k (\varepsilon_{ijk})^2,
\end{aligned}$$

[*1] `https://adhara.hatenadiary.jp/entry/2016/04/14/141203` を大変参考にした

$$
\begin{aligned}
\boldsymbol{p} \times \boldsymbol{L} &= \sum_{ijk} p_i L_j \boldsymbol{e}_k \varepsilon_{ijk} \\
&= \sum_{ijk} \sum_{mn} p_i r_m p_n \varepsilon_{mnj} \boldsymbol{e}_k \varepsilon_{ijk} \\
&= \sum_{ijk} \sum_{mn} p_i r_m p_n \boldsymbol{e}_k \varepsilon_{njm} \varepsilon_{ijk} \\
&= \sum_{ijk} \sum_{mn} p_i r_m p_n \boldsymbol{e}_k (\varepsilon_{ijk})^2 (\delta_{in}\delta_{km} - \delta_{im}\delta_{kn}) \\
&= \sum_{ijk} p_i (r_k p_i - r_i p_k) \boldsymbol{e}_k (\varepsilon_{ijk})^2
\end{aligned}
$$

から，

$$
\begin{aligned}
\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L} &= \sum_{ijk} (r_j p_j p_k - r_k {p_j}^2 - r_k {p_i}^2 + p_i r_i p_k) \boldsymbol{e}_k (\varepsilon_{ijk})^2 \\
&= \sum_{ijk} (p_i r_i p_k + r_j p_j p_k + r_k p_k p_k - r_k {p_i}^2 - r_k {p_j}^2 - r_k {p_k}^2) \boldsymbol{e}_k (\varepsilon_{ijk})^2 \\
&= \sum_{ijk} (r_i p_i p_k + r_j p_j p_k + r_k p_k p_k - i\hbar p_k - r_k \boldsymbol{p}^2) \boldsymbol{e}_k (\varepsilon_{ijk})^2 \\
&= \sum_{ijk} \left[(\boldsymbol{r} \cdot \boldsymbol{p}) p_k - i\hbar p_k - r_k \boldsymbol{p}^2 \right] \boldsymbol{e}_k (\varepsilon_{ijk})^2. \qquad [16.0.3]
\end{aligned}
$$

よって，

$$
\begin{aligned}
[\boldsymbol{R}, H] &= \left[\frac{1}{2m} (\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L}) + k_0 Z e^2 \frac{\boldsymbol{r}}{r}, \frac{\boldsymbol{p}^2}{2m} - k_0 Z e^2 \frac{1}{r} \right] \\
&= \frac{1}{(2m)^2} [\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L}, \boldsymbol{p}^2] + \frac{k_0 Z e^2}{2m} \left[\frac{\boldsymbol{r}}{r}, \boldsymbol{p}^2 \right] - \frac{k_0 Z e^2}{2m} \left[\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L}, \frac{1}{r} \right] \qquad [16.0.4]
\end{aligned}
$$

[16.0.4] の第 1 項は，[16.0.3] から，

$$
[\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L}, \boldsymbol{p}^2] = \sum_{ijk} \left[[(\boldsymbol{r} \cdot \boldsymbol{p}) p_k, \boldsymbol{p}^2] - [r_k \boldsymbol{p}^2, \boldsymbol{p}^2] \right] \boldsymbol{e}_k \qquad [16.0.5]
$$

となるが，例えば，

$$
[r_1, {p_1}^2] = r_1 p_1 p_1 - p_1 p_1 r_1 = 2i\hbar p_1,
$$

$$
[p_i, p_j] = 0
$$

などから，

$$
\begin{aligned}
&[(\boldsymbol{r} \cdot \boldsymbol{p}) p_k, \boldsymbol{p}^2] \\
&= (r_1 p_1 p_k + r_2 p_3 p_k + r_3 p_3 p_k)({p_1}^2 + {p_2}^2 + {p_3}^2) - ({p_1}^2 + {p_2}^2 + {p_3}^2)(r_1 p_1 p_k + r_2 p_3 p_k + r_3 p_3 p_k) \\
&= [r_1, {p_1}^2] p_1 p_k + [r_2, {p_2}^2] p_2 p_k + [r_3, {p_3}^2] p_3 p_k \\
&= 2i\hbar \boldsymbol{p}^2 p_k
\end{aligned}
$$

となる．一方，

$$
\begin{aligned}
[r_k \boldsymbol{p}^2, \boldsymbol{p}^2] &= r_k \boldsymbol{p}^4 - \boldsymbol{p}^2 r_k \boldsymbol{p}^2 \\
&= r_k \boldsymbol{p}^4 - (r_k \boldsymbol{p}^2 - 2i\hbar p_k) \boldsymbol{p}^2
\end{aligned}
$$

$$= 2i\hbar \boldsymbol{p}^2 p_k$$

なので，[16.0.5] に代入して，

$$[\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L},\boldsymbol{p}^2]=0.$$

次に，[16.0.4] 第 2 項は

$$\begin{aligned}
\left[\frac{\boldsymbol{r}}{r},\boldsymbol{p}^2\right] &= (-\hbar^2)\sum_i\left(\frac{\boldsymbol{r}}{r}{\partial_i}^2-{\partial_i}^2\frac{\boldsymbol{r}}{r}\right)\\
&= (-\hbar^2)\sum_i\left[\frac{\boldsymbol{r}}{r}{\partial_i}^2-\partial_i\left(\frac{\boldsymbol{e}_i}{r}-\frac{r_i}{r^3}\boldsymbol{r}+\frac{\boldsymbol{r}}{r}\partial_i\right)\right]\\
&= (-\hbar^2)\sum_i\left[-2\left(\frac{\boldsymbol{e}_i}{r}-\frac{r_i}{r^3}\boldsymbol{r}\right)\partial_i-\left\{\partial_i\left(\frac{\boldsymbol{e}_i}{r}-\frac{r_i}{r^3}\boldsymbol{r}\right)\right\}\right]\\
&= (-\hbar^2)\sum_i\left[-2\frac{\boldsymbol{e}_i}{r}\partial_i+2\frac{\boldsymbol{r}}{r^3}r_i\partial_i-\left(-2\frac{r_i}{r^3}\boldsymbol{e}_i-\frac{\boldsymbol{r}}{r^3}+3\frac{{r_i}^2}{r^5}\boldsymbol{r}\right)\right]\\
&= (-\hbar^2)\left[-\frac{2}{r}\boldsymbol{\partial}+\frac{2\boldsymbol{r}}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial})-\left(-2\frac{\boldsymbol{r}}{r^3}-\frac{3\boldsymbol{r}}{r^3}+3\frac{\boldsymbol{r}}{r^3}\right)\right]\\
&= 2\hbar^2\left[\frac{\boldsymbol{\partial}}{r}-\frac{\boldsymbol{r}}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial})-\frac{\boldsymbol{r}}{r^3}\right].
\end{aligned}$$

[16.0.4] 第 3 項は，

$$\begin{aligned}
\left[\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L},\frac{1}{r}\right] &= \sum_{ijk}(\varepsilon_{ijk})^2\left[\left[(\boldsymbol{r}\cdot\boldsymbol{p})p_k,\frac{1}{r}\right]-i\hbar\left[p_k,\frac{1}{r}\right]-\left[r_k\boldsymbol{p}^2,\frac{1}{r}\right]\right]\boldsymbol{e}_k\\
&= 2\sum_k\left[\left[(\boldsymbol{r}\cdot\boldsymbol{p})p_k,\frac{1}{r}\right]-i\hbar\left[p_k,\frac{1}{r}\right]-\left[r_k\boldsymbol{p}^2,\frac{1}{r}\right]\right]\boldsymbol{e}_k \qquad [16.0.6]
\end{aligned}$$

となる．[16.0.6] の $\boldsymbol{e}_k$ の係数の第 1 項は，

$$\begin{aligned}
\left[(\boldsymbol{r}\cdot\boldsymbol{p})p_k,\frac{1}{r}\right] &= (\boldsymbol{r}\cdot\boldsymbol{p})p_k\frac{1}{r}-\frac{1}{r}(\boldsymbol{r}\cdot\boldsymbol{p})p_k\\
&= -\hbar^2(\boldsymbol{r}\cdot\boldsymbol{\partial})\partial_k\frac{1}{r}+\hbar^2\frac{1}{r}(\boldsymbol{r}\cdot\boldsymbol{\partial})\partial_k\\
&= -\hbar^2(\boldsymbol{r}\cdot\boldsymbol{\partial})\frac{1}{r}\partial_k+\hbar^2(\boldsymbol{r}\cdot\boldsymbol{\partial})\frac{r_k}{r^3}+\frac{\hbar^2}{r}(\boldsymbol{r}\cdot\boldsymbol{\partial})\partial_k\\
&= -\hbar^2\sum_i r_i\partial_i\frac{1}{r}\partial_k+\hbar^2\sum_i r_i\partial_i\frac{r_k}{r^3}+\frac{\hbar^2}{r}(\boldsymbol{r}\cdot\boldsymbol{\partial})\partial_k\\
&= -\hbar^2\sum_i r_i\left(\frac{1}{r}\partial_i-\frac{r_i}{r^3}\right)\partial_k+\hbar^2\sum_i r_i\left(\frac{\delta_{ki}}{r^3}-r_k\frac{3r_i}{r^5}\right)+\hbar^2\frac{r_k}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial})+\frac{\hbar^2}{r}(\boldsymbol{r}\cdot\boldsymbol{\partial})\partial_k\\
&= -\hbar^2\frac{1}{r}\sum_i r_i\partial_i\partial_k+\frac{\hbar^2}{r^3}\sum_i r^{i2}\partial_k+\frac{\hbar^2}{r^3}\sum_i r_i\delta_{ki}\\
&\quad -\hbar^2 r_k\sum_i\frac{3{r_i}^2}{r^5}+\hbar^2\frac{r_k}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial})+\frac{\hbar^2}{r}(\boldsymbol{r}\cdot\boldsymbol{\partial})\partial_k\\
&= -\frac{\hbar^2}{r}(\boldsymbol{r}\cdot\boldsymbol{\partial})\partial_k+\frac{\hbar^2}{r}\partial_k+\hbar^2\frac{r_k}{r^3}-\hbar^2\frac{3r_k}{r^3}+\hbar^2\frac{r_k}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial})+\frac{\hbar^2}{r}(\boldsymbol{r}\cdot\boldsymbol{\partial})\partial_k\\
&= -2\hbar^2\frac{r_k}{r^3}+\frac{\hbar^2}{r}\partial_k+\hbar^2\frac{r_k}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial}),
\end{aligned}$$

[16.0.6] の $\boldsymbol{e}_k$ の係数の第 2 項は，

$$\begin{aligned}\left[p_k, \frac{1}{r}\right] &= p_k\frac{1}{r} - \frac{1}{r}p_k \\ &= -i\hbar\left(\partial_k\frac{1}{r} - \frac{1}{r}\partial_k\right) \\ &= -i\hbar\left(\frac{1}{r}\partial_k - \frac{r_k}{r^3} - \frac{1}{r}\partial_k\right) \\ &= i\hbar\frac{r_k}{r^3},\end{aligned}$$

[16.0.6] の $\boldsymbol{e}_k$ の係数の第 3 項は，

$$\begin{aligned}\left[r_k\boldsymbol{p}^2, \frac{1}{r}\right] &= r_k\boldsymbol{p}^2\frac{1}{r} - \frac{1}{r}r_k\boldsymbol{p}^2 \\ &= -\hbar^2 r_k\sum_i {\partial_i}^2\frac{1}{r} + \hbar^2\frac{r_k}{r}\sum_i {\partial_i}^2 \\ &= -\hbar^2 r_k\sum_i \partial_i\left(\frac{1}{r}\partial_i - \frac{r_i}{r^3}\right) + \hbar^2\frac{r_k}{r}\sum_i {\partial_i}^2 \\ &= -\hbar^2 r_k\sum_i\left(-2\frac{r_i}{r^3}\partial_i - \frac{1}{r^3} + 3\frac{{r_i}^2}{r^5}\right) \\ &= -\hbar^2 r_k\left(-\frac{2}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial}) - \frac{3}{r^3} + 3\frac{1}{r^3}\right) \\ &= \hbar^2\frac{2r_k}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial})\end{aligned}$$

となる．よって，[16.0.6] は，

$$\begin{aligned}\left[\boldsymbol{L}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{L}, \frac{1}{r}\right] &= 2\sum_k\left[-2\hbar^2\frac{r_k}{r^3} + \frac{\hbar^2}{r}\partial_k + \hbar^2\frac{r_k}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial}) + \hbar^2\frac{r_k}{r^3} - \hbar^2\frac{2r_k}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial})\right]\boldsymbol{e}_k \\ &= 2\hbar^2\sum_k\left[\frac{\partial_k}{r} - \frac{r_k}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial}) - \frac{r_k}{r^3}\right]\boldsymbol{e}_k \\ &= 2\hbar^2\left[\frac{\boldsymbol{\partial}}{r} - \frac{\boldsymbol{r}}{r^3}(\boldsymbol{r}\cdot\boldsymbol{\partial}) - \frac{\boldsymbol{r}}{r^3}\right]\end{aligned}$$

となり，[16.0.4] 第 2 項と第 3 項の和は 0．以上から，

$$[\boldsymbol{R}, H] = 0. \qquad [16.0.7]$$

次に，角運動量 $\boldsymbol{L}$ と Runge-Lenz-Pauli ベクトル $\boldsymbol{R}$ の交換関係，$[L_i, R_j]$ を調べよう．その準備として，まずは角運動量の交換関係を求める．$L_i = \sum_{kl}\varepsilon_{ikl}r_kp_l$ から，

$$\begin{aligned}[L_i, L_j] &= \sum_{klmn}\varepsilon_{ikl}\varepsilon_{jmn}(r_kp_lr_mp_n - r_mp_nx_kp_l) \\ &= \sum_{klmn}(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})r_kp_lr_mp_n \\ &= \sum_{klmn}(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})r_k\left(r_mp_l - i\hbar\delta_{lm}\right)p_n \\ &= -i\hbar\sum_{klmn}(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})\delta_{lm}r_kp_n + \sum_{klmn}(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})r_kr_mp_lp_n\end{aligned}$$

$$
\begin{aligned}
&= -i\hbar \sum_{kmn} (\varepsilon_{ikm}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkm}) r_k p_n + \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} r_k r_m p_l p_n - \varepsilon_{ikl}\varepsilon_{jmn} r_m r_k p_n p_l) \\
&= -i\hbar \sum_{m} (\varepsilon_{ijm}\varepsilon_{jmi} r_j p_i - \varepsilon_{imj}\varepsilon_{jim} r_i p_j) \\
&= i\hbar \sum_{m} (\varepsilon_{ijm})^2 (r_i p_j - r_j p_i) \\
&= i\hbar \sum_{k} \varepsilon_{ijk} L_k.
\end{aligned}
\tag{16.0.8}
$$

さらに，角運動量と運動量については，

$$
\begin{aligned}
[L_i, p_j] &= L_i p_j - p_j L_i \\
&= \sum_{kl} \varepsilon_{ikl} (r_k p_l p_j - p_j r_k p_l) \\
&= \sum_{l} \varepsilon_{ijl} (r_j p_j - p_j r_j) p_l \\
&= i\hbar \sum_{k} \varepsilon_{ijk} p_k.
\end{aligned}
\tag{16.0.9}
$$

また，次の交換関係も示しておく：

$$
\begin{aligned}
\left[L_i, \frac{r_j}{r}\right] &= \sum_{kl} \varepsilon_{ikl} \left[r_k p_l, \frac{r_j}{r}\right] \\
&= -i\hbar \sum_{kl} \varepsilon_{ikl} \left[r_k \partial_l, \frac{r_j}{r}\right] \\
&= -i\hbar \sum_{kl} \varepsilon_{ikl} \left(r_k \partial_l \frac{r_j}{r} - \frac{r_j}{r} r_k \partial_l\right) \\
&= -i\hbar \sum_{kl} \varepsilon_{ikl} \left[r_k \left(\frac{\delta_{lj}}{r} - \frac{r_j r_l}{r^3}\right) + r_k \frac{r_j}{r} \partial_l - \frac{r_j}{r} r_k \partial_l\right] \\
&= -i\hbar \sum_{kl} \varepsilon_{ikl} \delta_{lj} \frac{r_k}{r} + i\hbar \sum_{kl} \varepsilon_{ikl} \frac{r_j r_k r_l}{r^3} \\
&= -i\hbar \sum_{k} \varepsilon_{ikj} \frac{r_k}{r} + i\hbar \varepsilon_{i\ i+1\ i+2} \frac{r_j r_{i+1} r_{i+2}}{r^3} + i\hbar \varepsilon_{i\ i+2\ i+1} \frac{r_j r_{i+2} r_{i+1}}{r^3} \\
&= i\hbar \sum_{k} \varepsilon_{ijk} \frac{r_k}{r}.
\end{aligned}
\tag{16.0.10}
$$

以上の式を使えば，

$$
\begin{aligned}
[L_i, (\boldsymbol{L} \times \boldsymbol{p})_j] &= \sum_{mn} \varepsilon_{jmn} [L_i, L_m p_n] \\
&= \sum_{mn} \varepsilon_{jmn} (L_i L_m p_n - L_m p_n L_i) \\
&= \sum_{kmn} \varepsilon_{jmn} [(L_m L_i + i\hbar \varepsilon_{imk} L_k) p_n - L_m (L_i p_n - i\hbar \varepsilon_{ink} p_k)] \\
&= i\hbar \sum_{kmn} (\varepsilon_{jmn}\varepsilon_{imk} L_k p_n + \varepsilon_{jmn}\varepsilon_{ink} L_m p_k) \\
&= i\hbar \sum_{kmn} (\varepsilon_{jmn}\varepsilon_{imk} L_k p_n + \varepsilon_{jnm}\varepsilon_{imk} L_n p_k) \\
&= i\hbar \sum_{kmn} \varepsilon_{jmn}\varepsilon_{imk} (L_k p_n - L_n p_k)
\end{aligned}
$$

$$
\begin{aligned}
&= i\hbar \sum_m \varepsilon_{jmi}\varepsilon_{imj}(L_j p_i - L_i p_j) \\
&= -i\hbar \sum_m (\varepsilon_{mji})^2 (L_j p_i - L_i p_j) \\
&= i\hbar \sum_k \varepsilon_{ijk}(\boldsymbol{L}\times\boldsymbol{p})_k. && [16.0.11]
\end{aligned}
$$

同様に,

$$
\begin{aligned}
[L_i, (\boldsymbol{p}\times\boldsymbol{L})_j] &= \sum_{mn}\varepsilon_{jmn}[L_i, p_m L_n] \\
&= \sum_{mn}\varepsilon_{jmn}(L_i p_m L_n - p_m L_n L_i) \\
&= \sum_{mnk}\varepsilon_{jmn}[(p_m L_i + i\hbar\varepsilon_{iml}p_k)L_n - p_m(L_i L_n + i\hbar\varepsilon_{nik}L_k)] \\
&= i\hbar \sum_{mnk}(\varepsilon_{jmn}\varepsilon_{imk}p_k L_n - \varepsilon_{jmn}\varepsilon_{nik}p_m L_k) \\
&= i\hbar \sum_{mnk}(\varepsilon_{jmn}\varepsilon_{imk}p_k L_n - \varepsilon_{jnm}\varepsilon_{mik}p_n L_k) \\
&= i\hbar \sum_{mnk}\varepsilon_{jmn}\varepsilon_{imk}(p_k L_n - p_n L_k) \\
&= i\hbar \sum_m \varepsilon_{jmi}\varepsilon_{imj}(p_j L_i - p_i L_j) \\
&= -i\hbar \sum_m (\varepsilon_{mji})^2 (p_j L_i - p_i L_j) \\
&= -i\hbar \sum_k \varepsilon_{ijk}(\boldsymbol{p}\times\boldsymbol{L})_k. && [16.0.12]
\end{aligned}
$$

よって，[16.0.10][16.0.11][16.0.12] から,

$$
\begin{aligned}
[L_i, R_j] &= \left[L_i, \frac{1}{2m}(\boldsymbol{L}\times\boldsymbol{p})_j - \frac{1}{2m}(\boldsymbol{p}\times\boldsymbol{L})_j + k_0 Z e^2 \frac{r_j}{r}\right] \\
&= \frac{1}{2m}[L_i, (\boldsymbol{L}\times\boldsymbol{p})_j] - \frac{1}{2m}[L_i, (\boldsymbol{p}\times\boldsymbol{L})_j] + k_0 Z e^2 \left[L_i, \frac{r_j}{r}\right] \\
&= i\hbar \sum_k \varepsilon_{ijk}\left[\frac{1}{2m}(\boldsymbol{L}\times\boldsymbol{p})_k - \frac{1}{2m}(\boldsymbol{p}\times\boldsymbol{L})_k + k_0 Z e^2 \frac{r_k}{r}\right] \\
&= i\hbar \sum_k \varepsilon_{ijk} R_k. && [16.0.13]
\end{aligned}
$$

$\boldsymbol{R}$ と $\boldsymbol{R}$ の交換関係 $[R_i, R_j]$ を調べる.

$$
\begin{aligned}
&[(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_i, (\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_j] \\
&= \left[\sum_{kl}\varepsilon_{ikl}(L_k p_l - p_k L_l), \sum_{mn}\varepsilon_{jmn}(L_m p_n - p_m L_n)\right] \\
&= \sum_{klmn}[\varepsilon_{ikl}\varepsilon_{jmn}(L_k p_l - p_k L_l)(L_m p_n - p_m L_n) - \varepsilon_{ikl}\varepsilon_{jmn}(L_m p_n - p_m L_n)(L_k p_l - p_k L_l)] \\
&= \sum_{klmn}(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})(L_k p_l - p_k L_l)(L_m p_n - p_m L_n) \\
&= \sum_{klmn}(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})(L_k p_l L_m p_n - L_k p_l p_m L_n - p_k L_l L_m p_n + p_k L_l p_m L_n)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \left(p_l L_k + i\hbar \sum_s \varepsilon_{kls} p_s \right) \left(p_n L_m + i\hbar \sum_t \varepsilon_{mnt} p_t \right) \quad (\because [16.0.9]) \\
&\quad - \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \left(p_l L_k + i\hbar \sum_s \varepsilon_{kls} p_s \right) p_m L_n \\
&\quad - \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l \left(p_n L_m + i\hbar \sum_t \varepsilon_{mnt} p_t \right) \\
&\quad + \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_m L_n \\
&= \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_l L_k p_n L_m + i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} p_s p_n L_m \\
&\quad + i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{mns} p_l L_k p_s - \hbar^2 \sum_{klmnst} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls}\varepsilon_{mnt} p_s p_t \\
&\quad - \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_l L_k p_m L_n - i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} p_s p_m L_n \\
&\quad - \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_n L_m - i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{mns} p_k L_l p_s \\
&\quad + \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_m L_n
\end{aligned}
$$

となり，最後に得られた式の 1 項目で l と k 及び m と n を入れ替え（$\varepsilon_{ikl}\varepsilon_{jmn} = \varepsilon_{ilk}\varepsilon_{jnm}$ などに注意），2 項目で l と k を入れ替え，3 項目で m と n を入れ替えると，

$$
\begin{aligned}
[(\boldsymbol{L}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{L})_i, (\boldsymbol{L}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{L})_j] &= 4 \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_m L_n \\
&\quad + i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} p_s (p_n L_m - p_m L_n) \\
&\quad + i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{mns} (p_l L_k - p_k L_l) p_s \\
&\quad - \hbar^2 \sum_{klmnst} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls}\varepsilon_{mnt} p_s p_t
\end{aligned}
$$

が得られる．右辺第 2 項で $(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_m L_n$ の添字 m と n を入れ替えれば，$-(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_n L_m$ となるので，

$$
\begin{aligned}
[(\boldsymbol{L}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{L})_i, (\boldsymbol{L}\times\boldsymbol{p} - \boldsymbol{p}\times\boldsymbol{L})_j] &= 4 \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_m L_n \\
&\quad + 2i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} p_s p_n L_m \\
&\quad + 2i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{mns} p_l L_k p_s \\
&\quad - \hbar^2 \sum_{klmnst} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls}\varepsilon_{mnt} p_s p_t \qquad [16.0.14]
\end{aligned}
$$

$$[16.0.15]$$

[16.0.14] の第 2 項について，

$$
2i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) \varepsilon_{kls} p_s p_n L_m = 2i\hbar \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn}\varepsilon_{kli} p_i p_n L_m - \varepsilon_{imn}\varepsilon_{jkl}\varepsilon_{klj} p_j p_n L_m)
$$

$$
\begin{aligned}
&= 4i\hbar \sum_{mn} (\varepsilon_{jmn} p_i p_n L_m - \varepsilon_{imn} p_j p_n L_m) \\
&= 4i\hbar [p_j (\boldsymbol{p} \times \boldsymbol{L})_i - p_i (\boldsymbol{p} \times \boldsymbol{L})_j].
\end{aligned}
$$

さらに，第 2 項について同様に，

$$
\begin{aligned}
2i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})\varepsilon_{mns} p_l L_k p_s &= 2i\hbar \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn}\varepsilon_{mnj} p_l L_k p_j - \varepsilon_{imn}\varepsilon_{jkl}\varepsilon_{mni} p_l L_k p_i) \\
&= 4i\hbar \sum_{kl} (\varepsilon_{ikl} p_l L_k p_j - \varepsilon_{jkl} p_l L_k p_i) \\
&= 4i\hbar [(\boldsymbol{p} \times \boldsymbol{L})_j p_i - (\boldsymbol{p} \times \boldsymbol{L})_i p_j].
\end{aligned}
$$

ここで，次の関係式に注目する：

$$
\begin{aligned}
[p_i, (\boldsymbol{p} \times \boldsymbol{L})_j] &= \sum_{mn} \varepsilon_{jmn} [p_i, p_m L_n] \\
&= \sum_{mn} \varepsilon_{jmn} (p_i p_m L_n - p_m L_n p_i) \\
&= \sum_{mn} \varepsilon_{jmn} \left[p_m p_i L_n - p_m \left(p_i L_n + i\hbar \sum_k \varepsilon_{nik} p_k \right) \right] \\
&= -i\hbar \sum_{mnk} \varepsilon_{jmn} \varepsilon_{nik} p_m p_k \\
&= -i\hbar \sum_{n} \varepsilon_{jin} \varepsilon_{nij} p_i p_j \\
&= i\hbar \sum_{n} (\varepsilon_{jin})^2 p_i p_j.
\end{aligned}
$$

よって，[16.0.14] の第 2 項と第 3 項の和は，

$$
\begin{aligned}
&2i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})\varepsilon_{kls} p_s p_n L_m + 2i\hbar \sum_{klmns} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})\varepsilon_{mns} p_l L_k p_s \\
&= 4i\hbar [p_j (\boldsymbol{p} \times \boldsymbol{L})_i - p_i (\boldsymbol{p} \times \boldsymbol{L})_j] + 4i\hbar [(\boldsymbol{p} \times \boldsymbol{L})_j p_i - (\boldsymbol{p} \times \boldsymbol{L})_i p_j] \\
&= 4i\hbar [p_j, (\boldsymbol{p} \times \boldsymbol{L})_i] - 4i\hbar [p_i, (\boldsymbol{p} \times \boldsymbol{L})_j] \\
&= 0.
\end{aligned}
$$

さらに，[16.0.14] の第 4 項について，

$$
\begin{aligned}
\sum_{klmnst} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})\varepsilon_{kls}\varepsilon_{mnt} p_s p_t &= \sum_{klmnst} (\varepsilon_{ikl}\varepsilon_{jmn}\varepsilon_{kls}\varepsilon_{mnt} p_s p_t - \varepsilon_{imn}\varepsilon_{jkl}\varepsilon_{kls}\varepsilon_{mnt} p_s p_t) \\
&= \sum_{klmnst} (\varepsilon_{ikl}\varepsilon_{jmn}\varepsilon_{kls}\varepsilon_{mnt} p_s p_t - \varepsilon_{ikl}\varepsilon_{jmn}\varepsilon_{mnt}\varepsilon_{kls} p_t p_s) \\
&= 0.
\end{aligned}
$$

以上から，[16.0.14] の 3 つの項は 0 となり，

$$
[(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_i, (\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_j] = 4 \sum_{klmn} (\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl}) p_k L_l p_m L_n.
$$

ここで，$\varepsilon_{ikl} = \varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl}$ と変換する（他の部分についても同様．i が添字に含まれていれば，それ以外の部分を j に変えて，もともとあった文字と δ を作る）．この変換が，表式を変えないことを確認しておこう．

- $l = k = j$ の時：$\varepsilon_{ikl} = \varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl} = 0$
- $l = k =\neq j$ の時：$\varepsilon_{ikl} = \varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl} = 0$
- $l = j$,$k \neq j$ の時：$\varepsilon_{ikl} = \varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl} = \varepsilon_{ikj}$
- $l \neq j$,$k = j$ の時：$\varepsilon_{ikl} = \varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl} = \varepsilon_{ijl}$
- $l \neq k \neq j$ の時：$\varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl} = 0$．$i = j$ で，$(k, l) = (i+1, i+2), (i+2, 1+1)$ の時は，ε_{ikl} が非零となるが，$\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl} = 0$ なので，変換しても結局得られる結果は同じ 0 なので問題ない．

以上のことを踏まえて，先ほど得られた式を変換すると，

$$
\begin{aligned}
&[(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_i, (\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_j] && [16.0.16]\\
&= 4\sum_{klmn}(\varepsilon_{ikl}\varepsilon_{jmn} - \varepsilon_{imn}\varepsilon_{jkl})p_k L_l p_m L_n \\
&= 4\sum_{klmn}[(\varepsilon_{ijl}\delta_{jk} + \varepsilon_{ikj}\delta_{jl})(\varepsilon_{jin}\delta_{im} + \varepsilon_{jmi}\delta_{in}) - (\varepsilon_{ijn}\delta_{jm} + \varepsilon_{imj}\delta_{jn})(\varepsilon_{jil}\delta_{ik} + \varepsilon_{jki}\delta_{il})]p_k && [16.0.17]\\
&\qquad \times \left(p_m L_l + i\hbar\sum_s \varepsilon_{lms}p_s\right)L_n \\
&= 4\sum_{klmn}(\varepsilon_{ijl}\varepsilon_{jin}\delta_{jk}\delta_{im} + \varepsilon_{ijl}\varepsilon_{jmi}\delta_{jk}\delta_{in} + \varepsilon_{ikj}\varepsilon_{jin}\delta_{jl}\delta_{im} + \varepsilon_{ikj}\varepsilon_{jmi}\delta_{jl}\delta_{in} \\
&\quad - \varepsilon_{ijn}\varepsilon_{jil}\delta_{jm}\delta_{ik} - \varepsilon_{ijn}\varepsilon_{jki}\delta_{jm}\delta_{il} - \varepsilon_{imj}\varepsilon_{jil}\delta_{jn}\delta_{ik} - \varepsilon_{imj}\varepsilon_{jki}\delta_{jn}\delta_{il}) \\
&\quad \times \left(p_k p_m L_l L_n + i\hbar\sum_s \varepsilon_{lms}p_k p_s L_n\right) && [16.0.18]
\end{aligned}
$$

[16.0.18] の第 1 項は，

$$
\begin{aligned}
&4\sum_{klmn}(\varepsilon_{ijl}\varepsilon_{jin}\delta_{jk}\delta_{im} + \varepsilon_{ijl}\varepsilon_{jmi}\delta_{jk}\delta_{in} + \varepsilon_{ikj}\varepsilon_{jin}\delta_{jl}\delta_{im} + \varepsilon_{ikj}\varepsilon_{jmi}\delta_{jl}\delta_{in} \\
&\qquad - \varepsilon_{ijn}\varepsilon_{jil}\delta_{jm}\delta_{ik} - \varepsilon_{ijn}\varepsilon_{jki}\delta_{jm}\delta_{il} - \varepsilon_{imj}\varepsilon_{jil}\delta_{jn}\delta_{ik} - \varepsilon_{imj}\varepsilon_{jki}\delta_{jn}\delta_{il})p_k p_m L_l L_n \\
&= 4\sum_m (\varepsilon_{ijm})^2(-p_j p_i {L_m}^2 + p_m p_j L_m L_i + p_m p_i L_j L_m - {p_m}^2 L_j L_i \\
&\qquad + p_i p_j {L_m}^2 - p_m p_j L_i L_m - p_i p_m L_m L_j + {p_m}^2 L_i L_j) \\
&= 4\sum_m (\varepsilon_{ijm})^2 p_m (p_i[L_j, L_m] + p_j[L_m, L_i] + p_m[L_i, L_j]) \\
&= 4i\hbar\sum_m (\varepsilon_{ijm})^2 p_m \left(p_i\sum_s \varepsilon_{jms}L_s + p_j\sum_s \varepsilon_{mis}L_s + p_m\sum_s \varepsilon_{ijs}L_s\right) \quad (\because [16.0.8]) \\
&= 4i\hbar\sum_m \varepsilon_{ijm}p_m(p_i L_i + p_j L_j + p_m L_m). && [16.0.19]
\end{aligned}
$$

[16.0.18] の第 2 項は，

$$
\begin{aligned}
&4\sum_{klmn}(\varepsilon_{ijl}\varepsilon_{jin}\delta_{jk}\delta_{im} + \varepsilon_{ijl}\varepsilon_{jmi}\delta_{jk}\delta_{in} + \varepsilon_{ikj}\varepsilon_{jin}\delta_{jl}\delta_{im} + \varepsilon_{ikj}\varepsilon_{jmi}\delta_{jl}\delta_{in} \\
&\qquad - \varepsilon_{ijn}\varepsilon_{jil}\delta_{jm}\delta_{ik} - \varepsilon_{ijn}\varepsilon_{jki}\delta_{jm}\delta_{il} - \varepsilon_{imj}\varepsilon_{jil}\delta_{jn}\delta_{ik} - \varepsilon_{imj}\varepsilon_{jki}\delta_{jn}\delta_{il})i\hbar\sum_s \varepsilon_{lms}p_k p_s L_n \\
&= 4i\hbar\sum_{kls}\varepsilon_{ijl}\varepsilon_{jin}\varepsilon_{lis}p_j p_s L_n + 4i\hbar\sum_{lms}\varepsilon_{ijl}\varepsilon_{jmi}\varepsilon_{lms}p_j p_s L_i && [16.0.20]\\
&\quad + 4i\hbar\sum_{kns}\varepsilon_{ikj}\varepsilon_{jin}\varepsilon_{jis}p_k p_s L_n + 4i\hbar\sum_{kms}\varepsilon_{ikj}\varepsilon_{jmi}\varepsilon_{jms}p_k p_s L_i
\end{aligned}
$$

$$
\begin{aligned}
&-4i\hbar\sum_{lns}\varepsilon_{ijn}\varepsilon_{jil}\varepsilon_{ljs}p_i p_s L_n - 4i\hbar\sum_{kns}\varepsilon_{ijn}\varepsilon_{jki}\varepsilon_{ijs}p_k p_s L_n \qquad [16.0.21]\\
&-4i\hbar\sum_{lms}\varepsilon_{imj}\varepsilon_{jil}\varepsilon_{lms}p_i p_n L_j - 4i\hbar\sum_{kms}\varepsilon_{imj}\varepsilon_{jki}\varepsilon_{ims}p_k p_s L_j\\
&= 4i\hbar\sum_m(-\varepsilon_{ijm}{p_j}^2 L_m + 0 - \varepsilon_{ijm}{p_m}^2 L_m - \varepsilon_{ijm}p_m p_i L_i - \varepsilon_{ijm}{p_i}^2 L_m - \varepsilon_{ijm}{p_m}^2 L_m - 0 - \varepsilon_{ijm}p_m p_j L_j)\\
&= -4i\hbar\sum_m\varepsilon_{ijm}[p_m(p_i L_i + p_j L_j) + ({p_i}^2 + {p_j}^2 + 2{p_m}^2)L_m]. \qquad [16.0.22]
\end{aligned}
$$

よって，[16.0.18][16.0.19][16.0.22] から，

$$
[(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_i, (\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_j] = -4i\hbar\boldsymbol{p}^2\sum_m\varepsilon_{ijm}L_m. \qquad [16.0.23]
$$

次に，

$$
\begin{aligned}
\left[(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_i, \frac{r_j}{r}\right] &= \sum_{kl}\varepsilon_{ikl}\left[(L_k p_l - p_k L_l), \frac{r_j}{r}\right]\\
&= \sum_{kl}\varepsilon_{ikl}\left[(L_k p_l + p_l L_k), \frac{r_j}{r}\right]\\
&= 2\sum_{kl}\varepsilon_{ikl}\left[p_l L_k, \frac{r_j}{r}\right] + i\hbar\sum_{kls}\varepsilon_{ikl}\varepsilon_{kls}\left[p_s, \frac{r_j}{r}\right] \qquad [16.0.24]
\end{aligned}
$$

[16.0.24] 第 1 項は，

$$
\begin{aligned}
&\sum_{kl}\varepsilon_{ikl}\left[p_l L_k, \frac{r_j}{r}\right] \qquad [16.0.25]\\
&= \sum_{klmn}\varepsilon_{ikl}\varepsilon_{kmn}\left[p_l r_m p_n, \frac{r_j}{r}\right]\\
&= \sum_{klmn}\varepsilon_{ikl}\varepsilon_{kmn}\left(p_l r_m p_n\frac{r_j}{r} - \frac{r_j}{r}p_l r_m p_n\right)\\
&= -\hbar^2\sum_{klmn}\varepsilon_{ikl}\varepsilon_{kmn}\left(\partial_l r_m\partial_n\frac{r_j}{r} - \frac{r_j}{r}\partial_l r_m\partial_n\right)\\
&= -\hbar^2\sum_{klmn}\varepsilon_{ikl}\varepsilon_{kmn}\left[\partial_l r_m\left(\frac{r_j}{r}\partial_n + \frac{\delta_{nj}}{r} - \frac{r_j r_n}{r^3}\right) - \frac{r_j}{r}\partial_l r_m\partial_n\right]\\
&= -\hbar^2\sum_{klmn}\varepsilon_{ikl}\varepsilon_{kmn}\left[\frac{r_j}{r}\partial_l r_m\partial_n + \left(\frac{\delta_l j}{r} - \frac{r_l r_j}{r^3}\right)r_m\partial_n + \partial_l r_m\left(\frac{\delta_{nj}}{r} - \frac{r_j r_n}{r^3}\right) - \frac{r_j}{r}\partial_l r_m\partial_n\right]\\
&= -\hbar^2\sum_{klmn}\varepsilon_{ikl}\varepsilon_{kmn}\left[\left(\frac{\delta_l j}{r} - \frac{r_l r_j}{r^3}\right)r_m\partial_n + \partial_l r_m\left(\frac{\delta_{nj}}{r} - \frac{r_j r_n}{r^3}\right)\right]\\
&= -i\hbar\sum_{kl}\varepsilon_{ikl}\left(\frac{\delta_{lj}}{r} - \frac{r_l r_j}{r^3}\right)L_k - i\hbar\sum_{klmn}\varepsilon_{ikl}\varepsilon_{kmn}p_l\frac{r_m}{r}\delta_{nj} + i\hbar\sum_{klmn}\varepsilon_{ikl}\varepsilon_{kmn}\partial_l\frac{r_m r_j r_n}{r^3}\\
&= -i\hbar\sum_k\varepsilon_{ikj}\frac{L_k}{r} + i\hbar\frac{r_j}{r^3}(\boldsymbol{L}\times\boldsymbol{r})_i - i\hbar\sum_{kl}\varepsilon_{ikl}\sum_m\varepsilon_{kmj}p_l\frac{r_m}{r}. \qquad [16.0.26]
\end{aligned}
$$

最後の式変形で，$\sum_{mn}\varepsilon_{kmn}r_m r_n = (\boldsymbol{r}\times\boldsymbol{r})_k = 0$ を使った．[16.0.24] 第 2 項は，

$$
i\hbar\sum_{kls}\varepsilon_{ikl}\varepsilon_{kls}\left[p_s, \frac{r_j}{r}\right] = -\hbar^2\sum_{kls}\varepsilon_{ikl}\varepsilon_{kls}\left(\partial_s\frac{r_j}{r} - \frac{r_j}{r}\partial_s\right)
$$

$$
\begin{aligned}
&= -\hbar^2 \sum_{kls} \varepsilon_{ikl}\varepsilon_{kls} \left(\frac{\delta_{sj}}{r} - \frac{r_j r_s}{r^3} \right) \\
&= -\hbar^2 \sum_{kl} (\varepsilon_{ikl})^2 \left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3} \right) \\
&= -2\hbar^2 \left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3} \right) \qquad [16.0.27]
\end{aligned}
$$

よって，[16.0.24][16.0.26][16.0.27] から，

$$
\begin{aligned}
&\left[(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_i, \frac{r_j}{r} \right] \\
&\quad = -2i\hbar \sum_k \varepsilon_{ikj} \frac{L_k}{r} + 2i\hbar \frac{r_j}{r^3} (\boldsymbol{L} \times \boldsymbol{r})_i - 2i\hbar \sum_{kl} \varepsilon_{ikl} \sum_m \varepsilon_{kmj} p_l \frac{r_m}{r} + 2\hbar^2 \left(\frac{\delta_{ij}}{r} - \frac{r_i r_j}{r^3} \right).
\end{aligned}
$$

さらに，

$$
\begin{aligned}
&\left[(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_i, \frac{r_j}{r} \right] + \left[(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_j, \frac{r_i}{r} \right] \\
&= -2i\hbar \sum_k \varepsilon_{ikj} \frac{L_k}{r} + 2i\hbar \sum_k \varepsilon_{jki} \frac{L_k}{r} + 2i\hbar \frac{r_j}{r^3} (\boldsymbol{L} \times \boldsymbol{r})_i - 2i\hbar \frac{r_i}{r^3} (\boldsymbol{L} \times \boldsymbol{r})_j \\
&\quad - 2i\hbar \sum_{kl} \varepsilon_{ikl} \sum_m \varepsilon_{kmj} p_l \frac{r_m}{r} + 2i\hbar \sum_{km} \varepsilon_{ikm} \sum_l \varepsilon_{klj} p_m \frac{r_l}{r} \\
&= 4i\hbar \sum_k \varepsilon_{ijk} \frac{L_k}{r} - 2i\hbar \frac{1}{r^3} [r_i (\boldsymbol{L} \times \boldsymbol{r})_j - r_j (\boldsymbol{L} \times \boldsymbol{r})_i] - 2i\hbar \sum_{klm} \varepsilon_{ikl}\varepsilon_{kmj} (p_l r_m - p_m r_l) \frac{1}{r}. \qquad [16.0.28]
\end{aligned}
$$

[16.0.28] の第 2 項は，

$$
\begin{aligned}
\frac{1}{r^3} [r_i (\boldsymbol{L} \times \boldsymbol{r})_j - r_j (\boldsymbol{L} \times \boldsymbol{r})_i] &= \frac{1}{r^3} \sum_l (\varepsilon_{ijl})^2 [r_i (\boldsymbol{L} \times \boldsymbol{r})_j - r_j (\boldsymbol{L} \times \boldsymbol{r})_i] \\
&= \frac{1}{r^3} \sum_l \varepsilon_{ijl} [\boldsymbol{r} \times (\boldsymbol{L} \times \boldsymbol{r})]_l \\
&= \frac{1}{r^3} \sum_l \varepsilon_{ijl} [\boldsymbol{r} \times (\boldsymbol{r}^2 \boldsymbol{p} - (\boldsymbol{r} \cdot \boldsymbol{p}) \boldsymbol{r})]_l \\
&= \frac{1}{r} \sum_l \varepsilon_{ijl} L_l,
\end{aligned}
$$

[16.0.28] の第 3 項は，

$$
\begin{aligned}
\sum_{klm} \varepsilon_{ikl}\varepsilon_{kmj} (p_l r_m - p_m r_l) \frac{1}{r} &= -\sum_k (\varepsilon_{ijk})^2 (p_j r_i - p_i r_j) \frac{1}{r} \\
&= -\sum_k (\varepsilon_{ijk})^2 (r_i p_j - r_j p_i) \frac{1}{r} \\
&= -\sum_k \varepsilon_{ijk} L_k \frac{1}{r} \\
&= -\frac{1}{r} \sum_k \varepsilon_{ijk} L_k
\end{aligned}
$$

となるので，[16.0.28] は，

$$
\left[(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_i, \frac{r_j}{r} \right] + \left[(\boldsymbol{L} \times \boldsymbol{p} - \boldsymbol{p} \times \boldsymbol{L})_j, \frac{r_i}{r} \right] = 4i\hbar \sum_k \varepsilon_{ijk} \frac{L_k}{r}. \qquad [16.0.29]
$$

以上から，[16.0.23] と [16.0.29] を使えば，

$$
\begin{aligned}
[R_i, R_j] &= \left[\frac{1}{2m}(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_i + k_0 Ze^2\frac{r_i}{r}, \frac{1}{2m}(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_j + k_0 Ze^2\frac{r_j}{r}\right] \\
&= \frac{1}{4m^2}[(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_i, (\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_j] \qquad [16.0.30] \\
&\quad + \frac{k_0 Ze^2}{2m}\left[\left[(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_i, \frac{r_j}{r}\right] + \left[(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})_j, \frac{r_i}{r}\right]\right] \\
&= -i\hbar\frac{\boldsymbol{p}^2}{m^2}\sum_k \varepsilon_{ijk}L_k + 2i\hbar\frac{k_0 Ze^2}{m}\sum_k \varepsilon_{ijk}\frac{L_k}{r} \\
&= -\frac{2i\hbar}{m}\sum_k \varepsilon_{ijk}\left(\frac{\boldsymbol{p}^2}{2m} - k_0 Ze^2\frac{1}{r}\right)L_k \\
&= -\frac{2i\hbar}{m}\sum_k \varepsilon_k H L_k \qquad [16.0.31]
\end{aligned}
$$

が得られる．

角運動量 $\boldsymbol{L}$ と Runge-Lenz-Pauli ベクトル $\boldsymbol{R}$ の内積を調べる．[16.0.3] から

$$
\begin{aligned}
&\boldsymbol{L}\cdot(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L}) \qquad [16.0.32] \\
&= \left[\sum_{lmn}\varepsilon_{lmn}r_l p_m \boldsymbol{e}_n\right]\cdot\left[\sum_{ijk}[(\boldsymbol{r}\cdot\boldsymbol{p})p_k - i\hbar p_k - r_k\boldsymbol{p}^2]\boldsymbol{e}_k(\varepsilon_{ijk})^2\right] \\
&= \sum_{ijklmn}\delta_{kn}\varepsilon_{lmn}(\varepsilon_{ijk})^2 r_l p_m[(\boldsymbol{r}\cdot\boldsymbol{p})p_k - i\hbar p_k - r_k\boldsymbol{p}^2] \\
&= \sum_{ijklm}\varepsilon_{lmk}(\varepsilon_{ijk})^2 r_l p_m[(\boldsymbol{r}\cdot\boldsymbol{p})p_k - i\hbar p_k - r_k\boldsymbol{p}^2] \\
&= \sum_{ijk}(\varepsilon_{ijk})^2[r_i p_j\varepsilon_{ijk} + r_j p_i\varepsilon_{jik}][(\boldsymbol{r}\cdot\boldsymbol{p})p_k - i\hbar p_k - r_k\boldsymbol{p}^2] \\
&= \sum_{ijk}(\varepsilon_{ijk})^3(r_i p_j - r_j p_i)[(\boldsymbol{r}\cdot\boldsymbol{p})p_k - i\hbar p_k - r_k\boldsymbol{p}^2] \\
&= 2\sum_{ijk}\varepsilon_{ijk}r_i p_j[(\boldsymbol{r}\cdot\boldsymbol{p})p_k - i\hbar p_k - r_k\boldsymbol{p}^2] \\
&= 2\sum_i r_i[p_{i+1}(\boldsymbol{r}\cdot\boldsymbol{p})p_{i-1} - p_{i-1}(\boldsymbol{r}\cdot\boldsymbol{p})p_{i+1}] - 2i\hbar\sum_i r_i(p_{i+1}p_{i-1} - p_{i-1}p_{i+1}) \\
&\quad - 2i\hbar\sum_i r_i(p_{i+1}r_{i-1} - p_{i-1}r_{i+1})\boldsymbol{p}^2 \\
&= 2\sum_i r_i[p_{i+1}(\boldsymbol{r}\cdot\boldsymbol{p})p_{i-1} - p_{i-1}(\boldsymbol{r}\cdot\boldsymbol{p})p_{i+1}] - 2i\hbar\sum_i(r_i r_{i-1}p_{i+1}\boldsymbol{p}^2 - r_i r_{i+1}p_{i-1}\boldsymbol{p}^2) \\
&= 2\sum_{is}[r_i p_{i+1}r_s p_s p_{i-1} - r_i p_{i-1}r_s p_s p_{i+1}] - 2i\hbar\sum_i(r_i r_{i-1}p_{i+1}\boldsymbol{p}^2 - r_{i-1}r_i p_{i+1}\boldsymbol{p}^2) \\
&= 2\sum_{is}[r_i(r_s p_{i+1} - i\hbar\delta_{s,i+1})p_s p_{i-1} - r_i(r_s p_{i-1} - i\hbar\delta_{s,i-1})p_s p_{i+1}] \\
&= -2i\hbar\sum_{is}[r_i\delta_{s,i+1}p_s p_{i-1} - r_i\delta_{s,i-1}p_s p_{i+1}] \\
&= -2i\hbar\sum_i[r_i p_{i+1}p_{i-1} - r_i p_{i-1}p_{i+1}] \\
&= 0. \qquad [16.0.33]
\end{aligned}
$$

[16.0.11][16.0.12] から

$$[L_i, (\boldsymbol{L}\times\boldsymbol{p})_i] = [L_i, (\boldsymbol{p}\times\boldsymbol{L})_i] = 0$$

なので，[16.0.33] と合わせて

$$(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})\cdot\boldsymbol{L} = 0. \qquad [16.0.34]$$

$\boldsymbol{R}$ の第 2 項に関しては

$$\begin{aligned}
\boldsymbol{L}\cdot\frac{\boldsymbol{r}}{r} &= \left(\sum_{ijk} r_j p_k \boldsymbol{e}_i\right)\cdot\left(\sum_l \boldsymbol{e}_l\frac{r_l}{r}\right)\\
&= \sum_{ijkl}\varepsilon_{ijk}\delta_{il} r_j p_k \frac{r_l}{r}\\
&= \sum_{ijk}\varepsilon_{ijk} r_i r_j p_k \frac{1}{r}\\
&= \frac{i\hbar}{r^2}\sum_{ijk}\varepsilon_{ijk} r_i r_j r_k\\
&= \frac{i\hbar}{r^2}\sum_i r_i(r_{i+1}r_{i-1} - r_{i-1}r_{i+1})\\
&= 0.
\end{aligned} \qquad [16.0.35]$$

さらに，

$$\begin{aligned}
\frac{\boldsymbol{r}}{r}\cdot\boldsymbol{L} &= \left(\sum_l \boldsymbol{e}_l\frac{r_l}{r}\right)\cdot\left(\sum_{ijk} r_j p_k \boldsymbol{e}_i\right)\\
&= \sum_{ijkl}\varepsilon_{ijk}\delta_{li}\frac{r_l}{r} r_j p_k\\
&= \sum_{ijk}\varepsilon_{ijk}\frac{r_i r_j}{r} p_k\\
&= \sum_k \frac{r_{k+1}r_{k-1} - r_{k-1}r_{k+1}}{r} p_k\\
&= 0.
\end{aligned} \qquad [16.0.36]$$

[16.0.33][16.0.35] から

$$\boldsymbol{L}\cdot\boldsymbol{R} = 0.$$

[16.0.34][16.0.36] から

$$\boldsymbol{R}\cdot\boldsymbol{L} = 0.$$

最後に，$\boldsymbol{R}^2$ を計算する．まずは三重積について：

$$\begin{aligned}
\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) &= \sum_{ijk} A_i(\varepsilon_{ijk}B_j C_k)\\
&= \sum_{ijk}(\varepsilon_{kij}A_i B_j)C_k\\
&= (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C}.
\end{aligned} \qquad [16.0.37]$$

[16.0.9] から

$$
\begin{aligned}
(\boldsymbol{p}\times\boldsymbol{L})_i &= \sum_{jk}\varepsilon_{ijk}p_jL_k \\
&= \sum_{jk}\varepsilon_{ijk}\left[L_kp_j - i\hbar\sum_{l}\varepsilon_{kjl}p_l\right] \\
&= \sum_{jk}\varepsilon_{ijk}L_kp_j - i\hbar\sum_{jkl}\varepsilon_{ijk}\varepsilon_{kjl}p_l \\
&= -\sum_{jk}\varepsilon_{ikj}L_kp_j + i\hbar\sum_{jkl}\varepsilon_{ijk}\varepsilon_{ljk}p_l \\
&= -(\boldsymbol{L}\times\boldsymbol{p})_i + i\hbar\sum_{jk}(\varepsilon_{ijk})^2p_i \\
&= -(\boldsymbol{L}\times\boldsymbol{p})_i + 2i\hbar p_i
\end{aligned}
$$

となるので,

$$\boldsymbol{p}\times\boldsymbol{L} = -\boldsymbol{L}\times\boldsymbol{p} + 2i\hbar\boldsymbol{p}. \tag{16.0.38}$$

[16.0.37][16.0.38] から,

$$
\begin{aligned}
\boldsymbol{p}\cdot(\boldsymbol{p}\times\boldsymbol{L}) &= (\boldsymbol{p}\times\boldsymbol{p})\cdot\boldsymbol{L} \\
&= 0. \qquad [16.0.39] \\
\boldsymbol{p}\cdot(\boldsymbol{L}\times\boldsymbol{p}) &= \boldsymbol{p}\cdot(-\boldsymbol{p}\times\boldsymbol{L} + 2i\hbar\boldsymbol{p}) \\
&= 2i\hbar\boldsymbol{p}^2. \qquad [16.0.40]
\end{aligned}
$$

[16.0.9][16.0.37] から

$$
\begin{aligned}
(\boldsymbol{p}\times\boldsymbol{L})\cdot(\boldsymbol{p}\times\boldsymbol{L}) &= \left(\sum_{ijk}\varepsilon_{ijk}p_iL_j\boldsymbol{e}_k\right)\left(\sum_{lmn}\varepsilon_{lmn}p_lL_m\boldsymbol{e}_n\right) \\
&= \sum_{ijklmn}\varepsilon_{ijk}\varepsilon_{lmn}\delta_{kn}p_iL_jp_lL_m \\
&= \sum_{ijklm}\varepsilon_{ijk}\varepsilon_{lmk}p_iL_jp_lL_m \\
&= \sum_{ijlm}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})p_iL_jp_lL_m \\
&= \sum_{ij}(p_iL_jp_iL_j - p_iL_jp_jL_i) \\
&= \sum_{ij}p_i\left(p_iL_j + i\hbar\sum_{k}\varepsilon_{jik}p_k\right)L_j - \sum_{i}p_i(\boldsymbol{L}\cdot\boldsymbol{p})L_i \\
&= \sum_{ij}p_i\left(p_iL_j - i\hbar\sum_{k}\varepsilon_{ikj}p_k\right)L_j - \sum_{i}p_i[(\boldsymbol{r}\times\boldsymbol{p})\cdot\boldsymbol{p}] \\
&= \boldsymbol{p}^2\boldsymbol{L}^2 + i\hbar\sum_{ijk}\varepsilon_{ikj}p_ip_kL_j - \sum_{i}p_i[\boldsymbol{r}\cdot(\boldsymbol{p}\times\boldsymbol{p})] \\
&= \boldsymbol{p}^2\boldsymbol{L}^2 + i\hbar\sum_{ijk}\varepsilon_{ikj}p_ip_kL_j
\end{aligned}
$$

$$
\begin{aligned}
&= \boldsymbol{p}^2\boldsymbol{L}^2 + i\hbar(\boldsymbol{p}\times\boldsymbol{p})\cdot\boldsymbol{L} \\
&= \boldsymbol{p}^2\boldsymbol{L}^2. && [16.0.41]
\end{aligned}
$$

[16.0.38][16.0.39][16.0.40][16.0.41] から

$$
\begin{aligned}
(\boldsymbol{p}\times\boldsymbol{L})\cdot(\boldsymbol{L}\times\boldsymbol{p}) &= (\boldsymbol{p}\times\boldsymbol{L})\cdot(-\boldsymbol{p}\times\boldsymbol{L}+2i\hbar\boldsymbol{p}) \\
&= -\boldsymbol{p}^2\boldsymbol{L}^2 + 2i\hbar(\boldsymbol{p}\times\boldsymbol{L})\cdot\boldsymbol{p} \\
&= -\boldsymbol{p}^2\boldsymbol{L}^2 + 2i\hbar\boldsymbol{p}\cdot(\boldsymbol{L}\times\boldsymbol{p}) \\
&= -\boldsymbol{p}^2\boldsymbol{L}^2 - 4\hbar^2\boldsymbol{p}^2, && [16.0.42] \\
(\boldsymbol{L}\times\boldsymbol{p})\cdot(\boldsymbol{p}\times\boldsymbol{L}) &= (-\boldsymbol{p}\times\boldsymbol{L}+2i\hbar\boldsymbol{p})\cdot(\boldsymbol{p}\times\boldsymbol{L}) \\
&= -\boldsymbol{p}^2\boldsymbol{L}^2 + 2i\hbar\boldsymbol{p}\cdot(\boldsymbol{p}\times\boldsymbol{L}) \\
&= -\boldsymbol{p}^2\boldsymbol{L}^2, && [16.0.43] \\
(\boldsymbol{L}\times\boldsymbol{p})\cdot(\boldsymbol{L}\times\boldsymbol{p}) &= (-\boldsymbol{p}\times\boldsymbol{L}+2i\hbar\boldsymbol{p})\cdot(-\boldsymbol{p}\times\boldsymbol{L}+2i\hbar\boldsymbol{p}) \\
&= \boldsymbol{p}^2\boldsymbol{L}^2 - 2i\hbar(\boldsymbol{p}\times\boldsymbol{L})\cdot\boldsymbol{p} - 2i\hbar\boldsymbol{p}\cdot(\boldsymbol{p}\times\boldsymbol{L}) - 4\hbar^2\boldsymbol{p}^2 \\
&= \boldsymbol{p}^2\boldsymbol{L}^2 - 2i\hbar\boldsymbol{p}\cdot(\boldsymbol{L}\times\boldsymbol{p}) - 4\hbar^2\boldsymbol{p}^2 \\
&= \boldsymbol{p}^2\boldsymbol{L}^2. && [16.0.44]
\end{aligned}
$$

また,

$$
\begin{aligned}
\boldsymbol{L}\frac{1}{r} &= \sum_{ijk} r_i p_j \boldsymbol{e}_k \varepsilon_{ijk}\frac{1}{r} + \frac{1}{r}\boldsymbol{L} \\
&= -i\hbar\sum_{ijk}\varepsilon_{ijk} r_i \boldsymbol{e}_k \partial_j \frac{1}{r} + \frac{1}{r}\boldsymbol{L} \\
&= \frac{i\hbar}{r^3}\sum_{ijk}\varepsilon_{ijk} r_i r_j \boldsymbol{e}_k + \frac{1}{r}\boldsymbol{L} \\
&= \frac{i\hbar}{r^2}\sum_{k}(r_{k+1}r_{k-1} - r_{k-1}r_{k+1})\boldsymbol{e}_k + \frac{1}{r}\boldsymbol{L} \\
&= \frac{1}{r}\boldsymbol{L}
\end{aligned}
$$

となるので,

$$
\begin{aligned}
\boldsymbol{L}^2\frac{1}{r} &= \boldsymbol{L}\left(\boldsymbol{L}\frac{1}{r}\right) \\
&= \boldsymbol{L}\left(\frac{1}{r}\boldsymbol{L}\right) \\
&= \boldsymbol{L}\frac{1}{r} + \frac{1}{r}\boldsymbol{L}^2 \\
&= \frac{1}{r}\boldsymbol{L}^2. && [16.0.45]
\end{aligned}
$$

[16.0.37][16.0.38][16.0.45] から

$$
\begin{aligned}
(\boldsymbol{L}\times\boldsymbol{p})\cdot\frac{\boldsymbol{r}}{r} &= \boldsymbol{L}\cdot(\boldsymbol{p}\times\boldsymbol{r})\frac{1}{r} && [16.0.46] \\
&= -\boldsymbol{L}^2\frac{1}{r}
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{r}\boldsymbol{L}^2, && [16.0.47]\\
(\boldsymbol{p}\times\boldsymbol{L})\cdot\frac{\boldsymbol{r}}{r} &= (-\boldsymbol{L}\times\boldsymbol{p}+2i\hbar\boldsymbol{p})\cdot\frac{\boldsymbol{r}}{r}\\
&= \boldsymbol{L}^2\frac{1}{r}+2i\hbar\boldsymbol{p}\cdot\boldsymbol{r}\frac{1}{r}\\
&= \frac{1}{r}\boldsymbol{L}^2+2i\hbar(\boldsymbol{r}\cdot\boldsymbol{p}-3i\hbar)\frac{1}{r}\\
&= \frac{1}{r}\boldsymbol{L}^2+\frac{6\hbar^2}{r}+2\hbar^2\boldsymbol{r}\cdot\boldsymbol{\nabla}\frac{1}{r}\\
&= \frac{1}{r}\boldsymbol{L}^2+\frac{6\hbar^2}{r}+2\hbar^2\frac{\boldsymbol{r}}{r}\cdot\boldsymbol{\nabla}-\frac{2\hbar^2}{r}\\
&= \frac{1}{r}\boldsymbol{L}^2+\frac{4\hbar^2}{r}+\frac{2i\hbar}{r}\boldsymbol{r}\cdot\boldsymbol{p}, && [16.0.48]\\
\frac{\boldsymbol{r}}{r}\cdot(\boldsymbol{p}\times\boldsymbol{L}) &= \frac{1}{r}(\boldsymbol{r}\times\boldsymbol{p})\cdot\boldsymbol{L}\\
&= \frac{1}{r}\boldsymbol{L}^2, && [16.0.49]\\
\frac{\boldsymbol{r}}{r}\cdot(\boldsymbol{L}\times\boldsymbol{p}) &= \frac{\boldsymbol{r}}{r}\cdot(-\boldsymbol{p}\times\boldsymbol{L}+2i\hbar\boldsymbol{p})\\
&= -\frac{1}{r}\boldsymbol{r}\cdot(\boldsymbol{p}\times\boldsymbol{L})+\frac{2i\hbar}{r}\boldsymbol{r}\cdot\boldsymbol{p}\\
&= -\frac{1}{r}(\boldsymbol{r}\times\boldsymbol{p})\cdot\boldsymbol{L}+\frac{2i\hbar}{r}\boldsymbol{r}\cdot\boldsymbol{p}\\
&= -\frac{1}{r}\boldsymbol{L}^2+\frac{2i\hbar}{r}\boldsymbol{r}\cdot\boldsymbol{p}. && [16.0.50]
\end{aligned}
$$

[16.0.42][16.0.41][16.0.43][16.0.44] から

$$
\begin{aligned}
(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})^2 &= (\boldsymbol{L}\times\boldsymbol{p})\cdot(\boldsymbol{L}\times\boldsymbol{p})-(\boldsymbol{L}\times\boldsymbol{p})\cdot(\boldsymbol{p}\times\boldsymbol{L})-(\boldsymbol{p}\times\boldsymbol{L})\cdot(\boldsymbol{L}\times\boldsymbol{p})-(\boldsymbol{p}\times\boldsymbol{L})\cdot(\boldsymbol{p}\times\boldsymbol{L})\\
&= 4\boldsymbol{p}^2\boldsymbol{L}^2+4\hbar^2\boldsymbol{p}^2. && [16.0.51]
\end{aligned}
$$

[16.0.47][16.0.48][16.0.49][16.0.50] から

$$
(\boldsymbol{L}\times\boldsymbol{p})\cdot\frac{\boldsymbol{r}}{r}-(\boldsymbol{p}\times\boldsymbol{L})\cdot\frac{\boldsymbol{r}}{r}+\frac{\boldsymbol{r}}{r}\cdot(\boldsymbol{L}\times\boldsymbol{p})-\frac{\boldsymbol{r}}{r}\cdot(\boldsymbol{p}\times\boldsymbol{L}) = -\frac{4}{r}\boldsymbol{L}^2-\frac{4\hbar^2}{r}. \quad [16.0.52]
$$

[16.0.51][16.0.52] から

$$
\begin{aligned}
&\boldsymbol{R}^2 && [16.0.53]\\
&= \frac{1}{4m}(\boldsymbol{L}\times\boldsymbol{p}-\boldsymbol{p}\times\boldsymbol{L})^2+\frac{k_0Ze^2}{2m}\left[(\boldsymbol{L}\times\boldsymbol{p})\cdot\frac{\boldsymbol{r}}{r}-(\boldsymbol{p}\times\boldsymbol{L})\cdot\frac{\boldsymbol{r}}{r}+\frac{\boldsymbol{r}}{r}\cdot(\boldsymbol{L}\times\boldsymbol{p})-\frac{\boldsymbol{r}}{r}\cdot(\boldsymbol{p}\times\boldsymbol{L})\right]+(k_0Ze^2)^2\\
&= \frac{2}{m}\left(\frac{\boldsymbol{p}^2}{2m}-\frac{k_0Ze^2}{r}\right)(\boldsymbol{L}^2+\hbar^2)+(k_0Ze^2)^2\\
&= \frac{2}{m}H(\boldsymbol{L}^2+\hbar^2)+(k_0Ze^2)^2 && [16.0.54]
\end{aligned}
$$

が得られる．