EE 382V: Social Computing **Fall 2018**

Scribe Assignment 1: September 29

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1.1 The LP formulation of Max-Flow Min-Cut problem

1.1.1 Problem Domain

We are given a connected graph which is directed from an origin node s to a destination node t, where each edge in the graph has some non-negative capacity which represents the maximum flow across the edge.

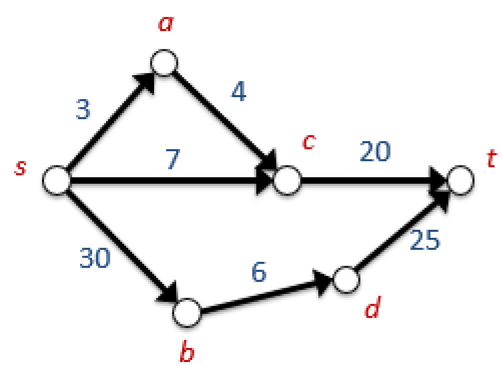

Figure 1.1: sample flow

1.1.2 Primal Problem

We want to calculate the *maximum flow* which can be sent from s to t. Another way to state this is that we want to maximize the flow through each edge from s subject to these constraints:

- Conservation constraints: The size of a the incoming flow for a vertex must be the same as the outgoing flow
- Capacity constraints: No edge may carry more than its maximum capacity. Define $c(u, v)$ as the maximum capacity for an edge $(u, v) \in E$
- The flow through each edge must be non-negative

This can be stated as the following linear program:

Maximize:

$$\sum_{v:(s,v)\in E} f(s,v)$$

Subject to:

$$\begin{gathered}\sum_{u:(u,v)\in E} f(u,v) = \sum_{w:(v,w)\in E} f(v,w) \\ f(u,v) \leq c(u,v) \qquad \forall (u,v) \in E \\ f(u,v) \geq 0 \qquad \forall (u,v) \in E\end{gathered}$$

1.1.3 Dual Problem

The dual problem is to *cut* the graph into two components with one containing s and the other containing t by removing the edges with the minimum total capacity. Let S be the component containing s and T be the component containing t. Every node in the graph (except for s and t) has a dual variable $y(u)$ which is 1 if $u \in S$ and 0 otherwise, and every edge in the graph has a dual variable $y(u,v)$ which is 1 if $u \in S$ and $v \in T$ and 0 otherwise. The dual problem can be stated as:

Minimize:

$$\sum_{(u,v)\in E} c(u,v) * y(u,v)$$

Subject to:

$$\begin{gathered}y(v) + y(s,v) \geq 1 \qquad \forall v : (s,v) \in E \\ y(v) - y(u) + y(u,v) \geq 0 \qquad \forall (u,v) \in E (s \neq u, t \neq v) \\ -y(u) + y(u,t) \geq 0 \qquad \forall u : (u,t) \in E\end{gathered}$$

1.1.4 Primal Example

For figure 1.1, the primal problem can be stated as:

Maximize:

$$f(s,a) + f(s,c) + f(s,b)$$

Subject to:

$$\begin{gathered}
f(s,a) - f(a,c) = 0 \\
f(s,b) - f(b,d) = 0 \\
f(a,c) + f(s,c) - f(c,t) = 0 \\
f(b,d) - f(d,t) = 0 \\
f(s,a) \leq 3 \\
f(s,c) \leq 7 \\
f(s,b) \leq 30 \\
f(a,c) \leq 4 \\
f(b,d) \leq 6 \\
f(c,t) \leq 20 \\
f(d,t) \leq 25 \\
f(u,v) \geq 0 \qquad \forall (u,v) \in E
\end{gathered}$$

1.1.5 Dual Example

Minimize:

$$c(s,a)*y(s,a)+c(s,b)*y(s,b)+c(s,c)*y(s,c)+c(a,c)*y(a,c)+c(b,d)*y(b,d)+c(c,t)*y(c,t)+c(d,t)*y(d,t)$$

Subject to:

$$\begin{gathered}
y(a) + y(s,a) \geq 1 \\
y(b) + y(s,b) \geq 1 \\
y(c) + y(s,c) \geq 1 \\
y(c) - y(a) + y(a,c) \geq 0 \\
y(d) - y(b) + y(b,d) \geq 0 \\
-y(c) + y(c,t) \geq 0 \\
-y(d) + y(d,t) \geq 0
\end{gathered}$$

1.1.6 Solution

The following mapping satisfies the primal with a value of 16:

$$f(s,a) = 3; f(s,c) = 7; f(s,b) = 6; f(a,c) = 3; f(c,t) = 10; f(b,d) = 6; f(d,t) = 6$$

And the following 'cut' satisfies the dual with a value of 16:

$$y(s,a) = 1; y(s,c) = 1; y(s,b) = 0; y(a,c) = 0; y(c,t) = 0; y(b,d) = 1; y(d,t) = 0$$

Since the primal and dual solutions have the same value, we know that these solutions are optimal.

A value of 1 for $y(s,a), y(s,c), y(b,c)$ in the dual problem indicates that the edge belongs to the 'cut' set. Finding these constraining edges is the intuitive and natural approach to solving the maximization problem. This is interesting because even without formal knowledge of LP duality in this case, you would have a tendency to construct the dual in order to solve the primal problem.