EE 382V: Social Computing	Fall 2018

Lecture 3 Session 1: September 28

Lecturer: Vijay Garg	*Scribe: Bhargavi Karthik*

3.1 LP-Duality

In Kuhn-Munkres (KM) algorithm, we find matching using the concept of labeling the vertices such that it satisfies the feasibility constraints. Using LP-Duality, we can come up with similar ideas for solving problems in a more intuitive way.

3.1.1 Example for LP Duality

There is an objective function which we are trying to minimize,

$$7x_1 + x_2 + 5x_3$$

subject to constraints,

$$x_1 - x_2 + 3x_3 \geq 10$$

$$5x_1 + 2x_2 - x_3 \geq 6$$

where x_1, x_2, x_3 are the decision variables and assumption is that,
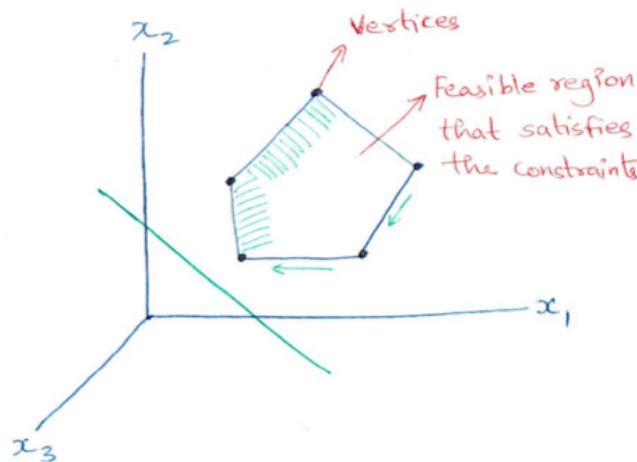
$$x_1, x_2, x_3 \geq 0$$

Figure 3.1: Finding optimal solution in the feasible space

Every constraint is in some sense dividing up the feasible region into two half spaces. The goal here is to find the optimal solution. If we can draw level sets for the given objective function, which means we are trying to optimize along some dimension. In this case we are trying to find the minimum. While trying to optimize along some direction, the point one always hits will be a corner. One of the important properties as per the Simplex algorithm is - "The extremal point is always a vertex".

The approach is to figure out one feasible point and start from there. If it is not a corner, keep moving until we find a corner and keep optimizing. However, if we are at a corner and if there are n other corners to go to, then go in the direction that has the most advantage.

Suppose we have solution, x = (2, 1, 3). Lets substitute in the objective function,

$$7(2) + 1 + 5(3) = 14 + 1 + 15 = 30$$

this solution is also feasible as it satisfies the constraints,

$$2 - 1 + 3(3) = 10$$

$$5(2) + 2(1) - 3 = 9$$

Trying to find a bound on primal:

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \geq 10 \text{ by comparing coefficients point-wise and } x_i \geq 0$$

solution has to be ≥ 10 but we are trying to get a better or greater lower bound

Add up the constraints equations,

$$6x_1 + x_2 + 2x_3 \geq 16$$

The objective function is still greater than the above equation. Hence new lower bound is 16.
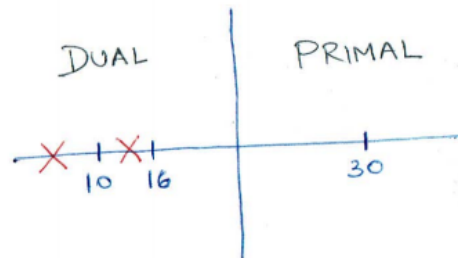

Figure 3.2: Representation of the example as duality

The goal of primal was to minimize and so now the goal of dual will be to maximize. We have to figure out what combination of each constraint should we add such that it gives the tightest bound on the objective function.

We have n decision variables and m constraints in our primal. So dual will have m variables (one variable for each constraint in primal) and n constraints (since n decision variables in primal).

Objective function is $10y_1 + 6y_2$ subject to constraints,

$$y_1 + 5y_2 \leq 7$$

$$-y_1 + 2y_2 \leq 1$$

$$3y_1 - y_2 \leq 5$$

In a generalized representation of the primal, objective function can be written as

$$\sum_{i=1}^{n} c_i x_i$$

constraints can be written as

$$\sum_{j} A_{ij} x_j \geq b_j$$

where A is a matrix of dimension m*n, c and b are vectors of size n and m respectively.

For dual, b is translated to c, c is translated to b and matrix A becomes A^T.

3.1.2 Strong Duality

Theorem 3.1 *The primal program has finite optimum if and only if its dual has finite optimum. Moreover, if x* and y* are optimal solutions for (P) and (D) then,*

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*$$

Note: This translates into all the min-max theorems such as max flow = min cut, max matching = min vertex cover and min number of chains to cover a poset = max sized anti-chain

3.1.3 Weak Duality

Theorem 3.2 *If x and y are feasible solutions for (P) and (D) then,*

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

Proof:

Claim 3.3

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} (\sum_{i=1}^{m} a_{ij} y_i) x_j$$

This claim is true because y is feasible and $x \geq 0$

Claim 3.4

$$\sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} x_j) y_i \geq \sum_{i=1}^{m} b_i y_i$$

This claim is true because x is feasible and $y \geq 0$

Equating claim 3.3 to claim 3.4

Hence proved,

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i$$

∎

3.1.4 Complementary Slackness Conditions

Theorem 3.5 *Given that x,y are (P) and (D) feasible solutions. Then x and y are both optimal if and only if the following complementary slackness conditions are satisfied:*

Claim 3.6

$$\forall j \quad either(x_j = 0) \quad or \quad \sum_{i=1}^{m} a_{ij} y_i = c_j$$

Claim 3.7

$$\forall i \quad either(y_i = 0) \quad or \quad \sum_{j=1}^{n} a_{ij} x_j = b_i$$

3.2 LP Duality and Maximum Matching

For the Kuhn-Munkres Algorithm, we started with x variables and came up with labels through initial feasible labeling. These labels are the dual variables. And we tried to get complementary slackness.

3.2.1 Maximum Matching Problem As Linear Program

Primal:

$$max \sum_{i,j} W_{ij} x_{ij}$$

subject to:

$$\sum_{j} x_{ij} \leq 1$$

and

$$\sum_{i} x_{ij} \leq 1$$

where x_{ij} is an edge between A and B, i is an index from A and j is an index from B. For each edge, there is a weight W_{ij}. We need to pick the edges such that the sum of their weights is maximum.

Replacing integral constraint by linear constraint,

$$x_{ij} \in \{0, 1\} \Rightarrow x_{ij} \geq 0$$

The above transformation is called LP-Relaxation. This can be used only when we can prove that the optimal solution is always going to be integral.

When $|A| = |B| = n$

Number of decision variables $= n^2$

Number of constraints $= 2n$

Dual:

Number of decision variables $= 2n$

Number of constraints $= n^2$

Objective function (minimizing vertex labels):

$$min \sum u_i + \sum v_j$$

subject to: $u_i + v_j \geq w_{ij}$

$u_i \geq 0 \quad and \quad v_j \geq 0$

where u_i is constraints for side A and v_j is constraints for side B