EE 382V: Social Computing **Fall 2018**

Lecture 29: November 10

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29.1 Answers to Problems on Auctions

The following two exercises focus on second-price, sealed-bid auctions where a seller is selling some object that bidders each have their own independent, private valuations for.

29.1.1 Exercise 9-3

(a) Four possible outcomes of two bidders having valuations 0 or 1:

- $(v_1, v_2) = (0, 0)$: Bidder 1 or 2 wins and pays 0.
- $(v_1, v_2) = (0, 1)$: Bidder 2 wins and pays 0.
- $(v_1, v_2) = (1, 0)$: Bidder 1 wins and pays 0.
- $(v_1, v_2) = (1, 1)$: Bidder 1 or 2 wins and pays 1.

If we equally weight the probability of each of the four possible outcomes above, then the probability of each outcome is $\frac{1}{4}$. With this knowledge, we can then show the seller's expected revenue (ER_S) is also $\frac{1}{4}$.

$$ER_S = \frac{1}{4}(0) + \frac{1}{4}(0) + \frac{1}{4}(0) + \frac{1}{4}(1) = \frac{1}{4}$$

(b) Eight possible outcomes of three bidders having valuations 0 or 1:

- $(v_1, v_2, v_3) = (0, 0, 0)$: Bidder 1, 2, or 3 wins and pays 0.
- $(v_1, v_2, v_3) = (0, 0, 1)$: Bidder 3 wins and pays 0.
- $(v_1, v_2, v_3) = (0, 1, 0)$: Bidder 2 wins and pays 0.
- $(v_1, v_2, v_3) = (0, 1, 1)$: Bidder 2 or 3 wins and pays 1.
- $(v_1, v_2, v_3) = (1, 0, 0)$: Bidder 1 wins and pays 0.
- $(v_1, v_2, v_3) = (1, 0, 1)$: Bidder 1 or 3 wins and pays 1.
- $(v_1, v_2, v_3) = (1, 1, 0)$: Bidder 1 or 2 wins and pays 1.
- $(v_1, v_2, v_3) = (1, 1, 1)$: Bidder 1, 2, or 3 wins and pays 1.

If we equally weight the probability of each of the eight possible outcomes above, then the probability of each outcome is $\frac{1}{8}$. With this knowledge, we can then show the seller's expected revenue (ER_S) has increased to $\frac{1}{2}$ with the additional bidder.

$$ER_S = \frac{1}{8}(0) + \frac{1}{8}(0) + \frac{1}{8}(0) + \frac{1}{8}(1) + \frac{1}{8}(0) + \frac{1}{8}(1) + \frac{1}{8}(1) + \frac{1}{8}(1) = \frac{4}{8} = \frac{1}{2}$$

(c) Comparing the solutions for (a) and (b), we see that going from two bidders to three bidders increased the seller's expected revenue from $\frac{1}{4}$ to $\frac{1}{2}$. If we were to add in an additional fourth bidder, we would again expect to see the seller's expected revenue increase. The same again with a fifth bidder, and a sixth, and so on.

To prove this, we note that the seller will earn 0 anytime less than two bidders bid 1, and 1 otherwise. Thus, in this particular case the seller's expected revenue happens to be equal to the probability that at least two bidders bid 1, or 1 minus the probability that 1 or 0 bidders bid 1.

$$ER_S = P(2 \; or \; more) = 1 - P(1 \; or \; 0)$$

$P(1 \; or \; 0)$ is equal to the number of outcomes where 1 or 0 bidders bid 1 divided by the total number of possible outcomes. Let n be the number of bidders. Then, there are n different outcomes where a single bidder bids 1, and 1 outcome where every bidder bids 0. Additionally, since each bidder has only 2 possible choices, we see there are 2^n total possible outcomes.

$$P(1 \; or \; 0) = \frac{number \; of \; outcomes \; where \; 1 \; or \; 0 \; bidders \; bid \; 1}{total \; number \; of \; possible \; outcomes} = \frac{n+1}{2^n}$$

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After substituting, we get a function of n, the total number of bidders, to easily calculate the seller's expected revenue for any positive integer number of bidders.

$$ER_S(n) = 1 - \frac{n+1}{2^n}$$

Analyzing the function, we see as n grows, so too does ER_S. We also see as n grows to infinity, ER_S converges to 1, never actually reaching 1.

$$ER_S(1) = 0, \;\; ER_S(2) = \frac{1}{4}, \;\; ER_S(3) = \frac{1}{2}, \;\; ER_S(4) = \frac{11}{16}, \;\; ER_S(5) = \frac{13}{16}, \;\; \ldots$$

29.1.2 Exercise 9-4

(a) We know a always bids its valuation, but b sometimes makes a mistake with its bid. When b's actual valuation is 0, half the time it bids 1 mistakenly. Below we outline the possible outcomes of valuations and bids.

- $(v_a, v_b) = (0, 0)$:
  - **Outcome 1.** $(b_a, b_b) = (0, 0)$: Bidder a (50%) or b (50%) wins and pays 0.
  - **Outcome 2.** $(b_a, b_b) = (0, 1)$: **Mistake bid.** Bidder b wins and pays 0.
- $(v_a, v_b) = (0, 1)$:
  - **Outcome 3.** $(b_a, b_b) = (0, 1)$: Bidder b wins and pays 0.
- $(v_a, v_b) = (1, 0)$: Bidder 1 wins and pays 0.
  - **Outcome 4.** $(b_a, b_b) = (1, 0)$: Bidder a wins and pays 0.
  - **Outcome 5.** $(b_a, b_b) = (1, 1)$: **Mistake bid.** Bidder a (50%) or b (50%) wins and pays 1.
- $(v_a, v_b) = (1, 1)$:
  - **Outcome 6.** $(b_a, b_b) = (1, 1)$: Bidder a or b wins and pays 1.

Given a bids 0 (Outcomes 1, 2, and 3), we see there is a $\frac{1}{8}$ probability it wins and pays 0, and a 0 probability it wins and pays 1. To calculate these values note that given a bids 0, outcomes 1 and 2 each have a $\frac{1}{4}$ probability of occurrence, while outcome 3 has a $\frac{1}{2}$ probability of occurrence.

$$P(a\ wins\ and\ pays\ 0 \mid a\ bids\ 0) = \frac{1}{4}(\frac{1}{2}) + \frac{1}{4}(0) + \frac{1}{2}(0) = \frac{1}{8}$$

$$P(a\ wins\ and\ pays\ 1 \mid a\ bids\ 0) = \frac{1}{4}(0) + \frac{1}{4}(0) + \frac{1}{2}(0) = 0$$

Given a bids 1 (Outcomes 4, 5, and 6), we see there is a $\frac{1}{4}$ probability it wins and pays 0, and a $\frac{3}{8}$ probability it wins and pays 1. To calculate these values note that given a bids 1, outcomes 4 and 5 each have a $\frac{1}{4}$ probability of occurrence, while outcome 6 has a $\frac{1}{2}$ probability of occurrence.

$$P(a\ wins\ and\ pays\ 0 \mid a\ bids\ 1) = \frac{1}{4}(1) + \frac{1}{4}(0) + \frac{1}{2}(0) = \frac{1}{4}$$

$$P(a\ wins\ and\ pays\ 1 \mid a\ bids\ 1) = \frac{1}{4}(0) + \frac{1}{4}(\frac{1}{2}) + \frac{1}{2}(\frac{1}{2}) = \frac{3}{8}$$

To determine if remaining true is still the dominant strategy for a given b's mistake bids, for each of the possible true valuations for a, we will use the probabilities calculated above to analyze whether staying true or lying is better.

Case that a's true valuation is 0:

When a's true valuation is 0, by staying true and bidding 0 we see using the probabilities above it only has a $\frac{1}{8}$ probability of winning and paying its valuation or less, but has 0 probability of winning and having to pay more than its valuation.

If instead it lies and bids 1, we see it has an increased $\frac{1}{4}$ probability of winning and paying its valuation or less, but it also now has a $\frac{3}{8}$ probability of winning and having to pay more than its valuation, which makes this strategy inadvisable.

Thus, it is better to stay true given a's valuation is 0.

Case that a's true valuation is 1:

When a's true valuation is 1, by staying true and bidding 1 we see using the probabilities above it has a $\frac{1}{4} + \frac{3}{8} = \frac{5}{8}$ probability of winning and paying its valuation or less.

If instead it lies and bids 0, we see it only has a $\frac{1}{8}$ probability of winning and paying its valuation or less.

Thus, it is better to stay true given a's valuation is 1.

Conclusion:

In each possible valuation for a, staying true remains the dominant strategy for a even considering it is aware of the mistake bids made by b.

(b) If we equally weight the probability of each of the four possible valuations above at $\frac{1}{4}$, and equally the possible bids in each valuation case, we can then show the seller's expected revenue (ER_S) is $\frac{3}{8}$.

$$ER_S = \frac{1}{4}(\frac{1}{2}(0) + \frac{1}{2}(0)) + \frac{1}{4}(0) + \frac{1}{4}(\frac{1}{2}(0) + \frac{1}{2}(1)) + \frac{1}{4}(1) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$