EE 382V: Social Computing	**Fall 2018**

Lecture 1: August 24

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1.1 Proof that no constricted set $\Rightarrow$ a Perfect Match (PM)

Use induction on n $|A| = |B| = n$
Base Case $n = 1$

$$a_1 \odot \leftrightarrow \odot b_1, \text{ There is } PM$$

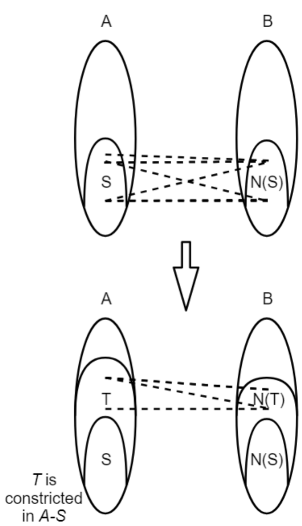
Induction Case
Assume that the theorem is true for all values above $|A| < n$

Case 1
$\exists S$ so that $|S| \equiv |N(S)|$

Case 2
$\forall S : |S| < |N(S)|$

1.1.1 Case 1.

For a selected subset S of vertices from set A, the size of a subset created with the neighbors of the elements of S, i.e. $N(S)$, equals the size of subset S.



It is needed to confirm that for the selected subsets $(S, N(S), E_s)$, the remaining subset $A-S$ does not have any constricted set when matched to subset $B-N(S)$.

This allows to iteratively break the problem into smaller parts, checking that remainder is not constricted after removing any given subset of A and its their neighbors.

If T is constricted in A-S(as shown in the lower section of Figure 1.1) $\Rightarrow S \cup T$ is constricted in A $\Rightarrow$ **contradiction**

Figure 1.1: Splitting sets.

1.1.2 Case 2.

For a selected subset S of vertices from set A, the size of a subset created with the neighbors of the elements of S, i.e. $N(S)$, is less than the size of subset S.

Pick any vertex a in A and match to any vertex b in $N(a)$, remove both a and b from the consideration.

- Verify that there is no constriction after removal.
- Now use induction as the set size has decreased.

In implementation, you want your algorithm to make matching maxim*um* by the time it is maxim*al*.

1.2 Alternating Path

An alternating path is one that alternates between matched and non-matched edges.

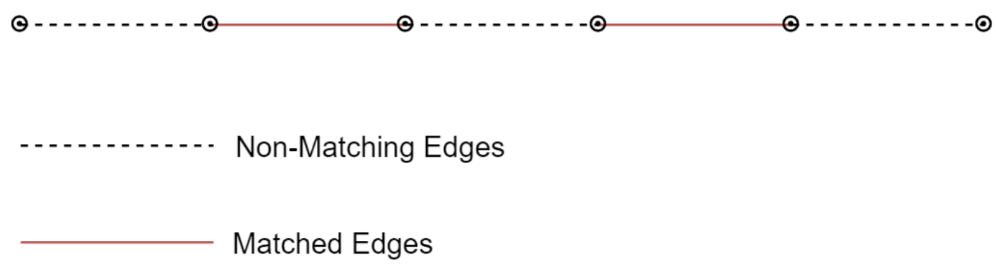


Figure 1.2: Alternating path.

1.2.1 Augmenting Path

An *augmenting path* is an alternating path starts with an exposed vertex in A and ends with and exposed vertex in B. An **exposed vertex** is one that has only non-matching edges attached to it.

If an augmenting path is found, the size of the matching can be augmented (incremented by 1) by exchanging the matched edges for the non matched edges.

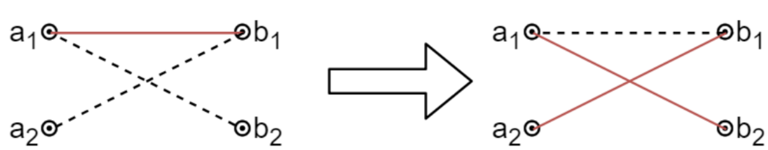


Figure 1.3: Augmenting path edge exchange.

All-edges start as unmatched. Consider using the B.F.S. algorithm to find a match. Always try to find an augmenting path.

B.F.S. Tree(with a twist)

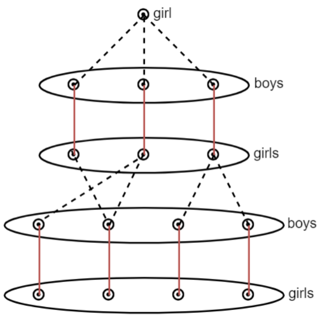


Figure 1.4: Example of a B.F.S.-Tree

This particular example has a constricted set, since there is one more girl than boys.

With B.F.S.-Tree construction, either an augmenting path is found or there is a constricted set. (By B.F.S.-Tree property).

TIP
If there is no constricted set, then you know there is an augmenting path.

Theorem 1.1 (Theorem) *A matching M is maximum if and only if there are no augmenting paths with respect to M.*

Proof:

- The existence of augmenting path indicates matching is not maxim*um*, since matched and non-matching edges can be inverted, increasing matching size by 1.

- To prove that M is not maximum $\Rightarrow \exists$ an augmenting path:

Let M' be a maximum matching,

- Compute $Q = M \triangle M' \leftarrow$ *symmetrical* $\quad (\triangle \equiv xor$ of M and $M')$

  $$M \triangle M' = (M' - M) \bigcup (M - M')$$

- Since $|M'| > |M|$, Q has more edges from M' than from M.

- Each vertex is adjacent to at most one edge in $M \cap Q$ and at most one edges in $M' \cap Q$

- Q is composed of cycles(even number of edges) and paths(odd number of edges) that alternate between edges from M and M'. Cycles will always be constructed by an even number of edges since we are working with bipartite graphs and a cycle with an odd number of edges would only be achievable by matching elements of the same set. Paths found must have more edges from M' since its size is larger than that of M, hence representing the existence of an alternating augmenting path with reference to M, such as the one shown in Figure 1.5.

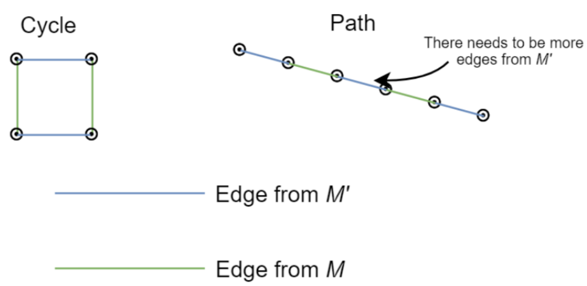


Figure 1.5: Cycle vs path construct

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1.3 Weak Duality

Maximum matching problem with bipartite graphs:

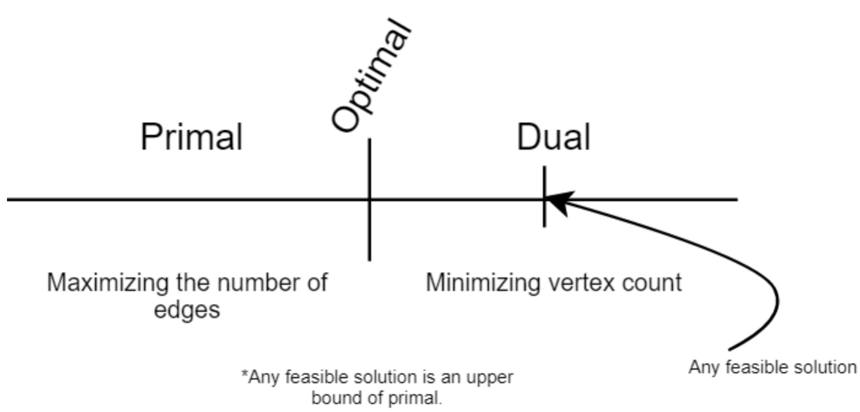

Figure 1.6: Weak duality in matching.

Let G be a bipartite graph, described as $G = (A, B, E)$. $X \subseteq A \cup B$ is a vertex cover of $G = (A, B, E)$ if every edge is adjacent to at least one vertex in X.

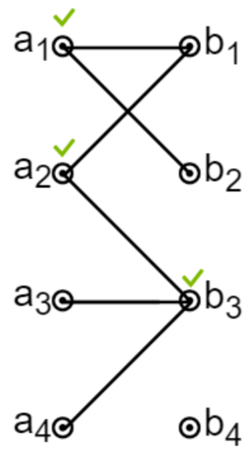

Figure 1.7: Example of a vertex cover. Analyzing what vertices combined are attached to all edges, we can establish $X = (a_1, a_2, b_3)$ for a cover of size 3.

Theorem 1.2 *Given any vertex cover X, and any matching M, $|M| \leq |X|$*

Proof: Each element of X can only have one match. All its other edges are removed once it matches. ■

Theorem 1.3 (König, Egerváry Theorem) *For any bipartite graph, the maximum size of a matching equals the size of the minimum vertex cover.*

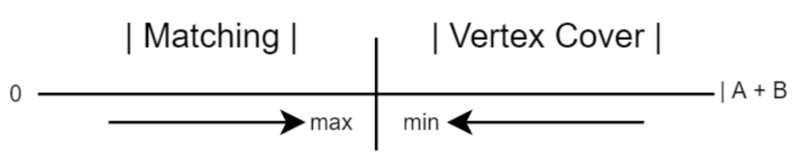

Figure 1.8: Min-Max theorem applied to weak duality in matching.

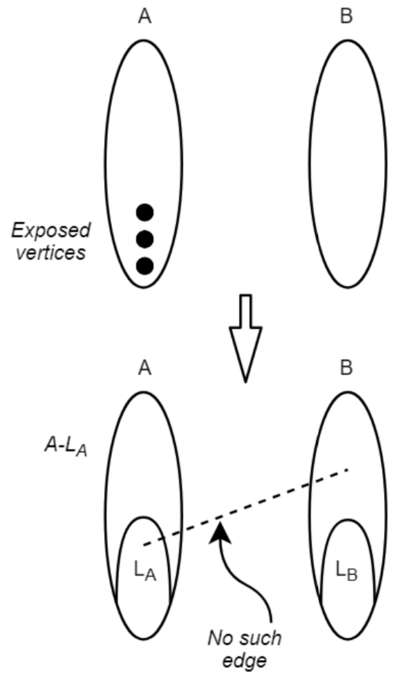

Figure 1.9: Abstraction of sets into L. There may not be any edges between L_A and $B - L$ as all the vertices in L_A are either unmatched or have categorized edges they are connected to in B into L_B, per L's definition.

Let
L: Vertices that can be reached by B.F.S.-tree from an exposed vertex.

M' : maximum matching of a bipartite graph.

$C' = (A - L_A) \cup L_B$

$L = L_A \cup L_B$
$L_A = A \cap L$
$L_B = B \cap L$

Claim

- C' is a vertex cover.
  Every edge in a bipartite graph belongs to either an alternating path, which places its b_x endpoint in $B \cap L$, or it has an a_x endpoint in $A - L_A$. If said edge is matched but not in an alternating path, then its a_x endpoint cannot be in an alternating path, since said path must have included the edge and thus belongs to $A - L_A$. If the edge is unmatched but not in an alternating path, then its a_x endpoint cannot be in an alternating path, for such a path could be extended by adding the edge to it. In consequence, C' forms a vertex cover.

- $|C'| = |M'|$
  Every vertex in C' is an endpoint of a matched edge: every vertex in $A - L_A$ is matched because L includes all unmatched A vertices, and every vertex in $B \cap L$ must also be matched, for if there existed an alternating path to an unmatched vertex then changing the matching by removing the matched edges from this path and adding the unmatched edges in their place would increase the size of the matching. However, no matched edge can have both of its endpoints in C'. Thus, C' is a vertex cover of size equal to M', and must be a minimum vertex cover.