EE 382V: Introduction to Social Computing **Fall 2018**

Lecture 4 Session 1: September 29
Stable Marriage

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4.1 Review

Recall the algorithm to find a stable marriage from last class.

4.1.1 Inputs

1. mPref: data structure containing preferences for each man
2. wPref: data structure containing preferences for each woman
3. rank: where rank[i][j] indicates the rank of man j for woman i; higher preference correlates to lower rank

4.1.2 Stable Marriage Algorithm

1. $freelist$ = list of men who are free (initially has all men)
2. while ($freelist$ is not empty):
    3. choose some man m from $freelist$
    4. m proposes to his most preferred woman w that he has not already proposed to
    5. if w is free:
        6. w accepts proposal (assign m to w)
    7. else: // w is already engaged to m'
        8. if $rank[w][m] > rank[w][m']$: // w prefers m'
            9. add m back to $freelist$ // reject m; w still engaged to m'
        10. else:
            11. add m' back to the $freelist$ // reject m'
            12. w gets engaged to m
13. print assignment

4.2 Stable Marriage Algorithm Correctness

Theorem 4.1 *The Algorithm terminates with a matching.*

Proof:

Claim 4.2 *The last woman in a man's preference list must be free*

Invariant: Once a woman is engaged, she stays engaged.

In order for a man to make it to his last preference, the previous $n-1$ women must already be engaged since the only reason they can reject his proposal is if they are engaged to someone that they prefer more. Since the number of men and women is equal, the last woman in the list can never be engaged.

Since the last woman in a man's preference list will always be free, every man is guaranteed that he will match with one woman, so a matching will exist.

■

Theorem 4.3 *The algorithm (if it terminates) ends with a stable matching.*

Proof: Assume that $(m, w')(m', w)$ is a blocking pair where m is married to w' and w is married to m'.

In order for $(m, w')(m', w)$ to be a blocking pair, this means that m must prefer w to w' and that w must prefer m to m'.

Since m prefers w to w', he would have proposed to w first and gotten rejected.

Invariant: A woman's assignment can only improve with execution of the algorithm.

Therefore, w can only have rejected m if she had a proposal from someone she preferred more. If that person was m', we know that she must prefer m' to m. If that person was not m', we know that she prefers that person to m and that she prefers m' to that person, so by extension, she prefers m' to m. Generally, w will always prefer her current fiance to all men she rejected, so she must prefer m' to m.

The requirement for (m, w) to be a blocking pair was that m must prefer w to w' and that w must prefer m to m'. These two requirements cannot co-exist, so by contradiction, we know that $(m, w')(m', w)$ cannot be a blocking pair. ■

4.3 Posets: Partially Ordered Sets

4.3.1 Definition

A poset is defined as $(X, \leq)$ where X is a set and $\leq$ is a binary relation on that set.

A reflexive poset has the following properties:

1. reflexive - for all vertices v in V: $v \leq v$
2. anti-symmetric - for all pairs of vertices u, v in V: if $v \leq u$ and $u \leq v$ then $u == v$

3. transitive - for all groups of vertices u, v, w in V: $u \leq v \leq w$ implies that $u \leq w$

Consider all following posets to be reflexive posets.

4.3.2 Example 1

X: $\mathbb{R}^n$

$\leq$: $U \leq V \equiv$ for all i: $u[i] \leq v[i]$

Claim 4.4 $(X, \leq)$ *is a poset.*

4.3.2.1 Numerical Example 1

$M1 = (2, 5, 3)$; $M2 = (6, 8, 5)$

$2 \leq 6$, $5 \leq 8$, $3 \leq 5$ says that $M1 \leq M2$.

4.3.2.2 Numerical Example 2

$M1 = (1, 3, 1)$; $M2 = (2, 1, 2)$

$1 \leq 2$, $3 > 1$. What does this mean?

Definition 4.5 $M1$ *is incomparable to* $M2$ *if* $(M1 \nleq M2)$ *and* $(M2 \nleq M1)$.

4.3.3 Example 2

$z = \{a, b, c\}$

X: { {}, {a}, {b}, {c}, {a,b}, {a,c}, {b,c}, {a,b,c} }

$\leq$: $\subseteq$

$\{a, b\} \leq \{a, b, c\}$

$\{a, b\}$ is incomparable to $\{b, c\}$

4.3.4 Dealing with Incomparable Items

If $M1$ and $M2$ are incomparable, try to find $M3$ which is better than both $M1$ and $M2$.

$M1 = (1, 1, 2, 3)$

$M2 = (2, 2, 1, 2)$

Note: the numbers here indicate preference where 1 indicates first choice, 2 indicates second choice, etc

find $M3 = (1, 1, 1, 2)$ such that $M3 \leq M1$ and $M3 \leq M2$.

4.3.5 Hasse Diagrams: Pictoral Representations of Posets

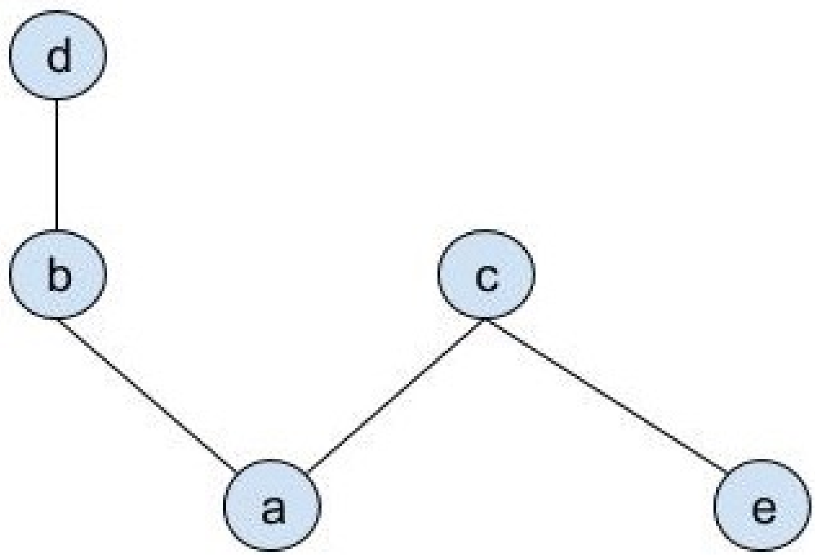

Figure 4.1: Hasse Diagram example

This diagram shows that:

$a \leq b$

$a \leq c$

$b \leq d$

$e \leq c$

c and d are incomparable

a and e are incomparable

4.3.6 Operations on Posets

Given a poset $(X, \leq)$:

Definition 4.6 *Let $x, y \in X$. $z \in X$ is meet of x and y if:*

1. $z \leq x$ *and* $z \leq y$

   AND

2. *for all* z': $z' \leq x$ *and* $z' \leq y \implies z' \leq z$

Where meet is also known as the greatest lower bound. The first condition is the lower bound, and the second condition is that the lower bound must be greater than all other lower bounds.

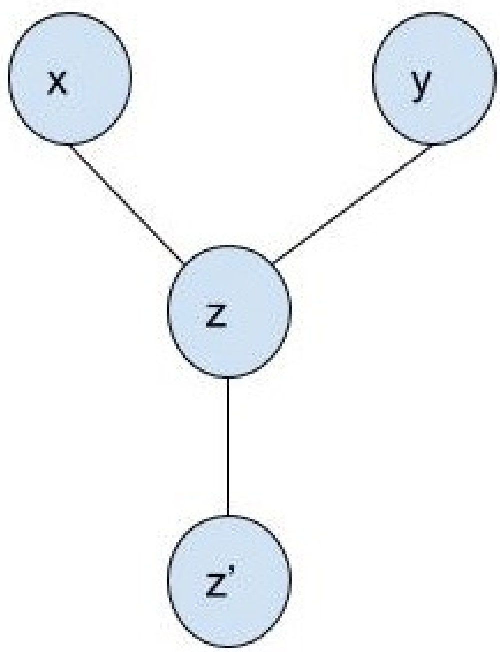

Figure 4.2: Meet Example; z is the meet of x and y

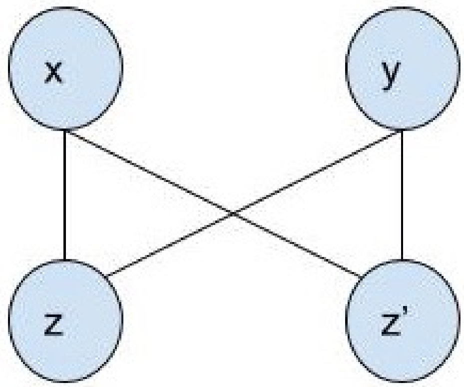

Figure 4.3: z is not a meet; z and z' are incomparable

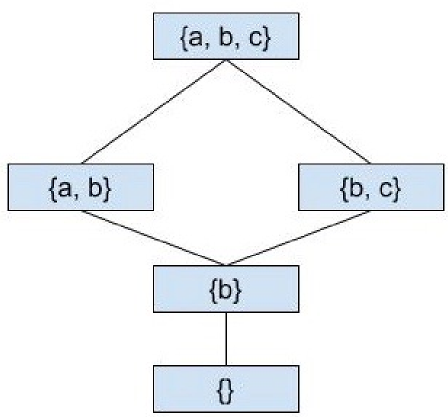

Figure 4.4: meet corresponds to intersection of sets

4.3.7 Lattice Posets

Definition 4.7 *A poset $(X, \leq)$ is lattice (nice) if for all $a, b \in X$, $meet(a, b)$ exists and $join(a, b)$ exists.*

Definition 4.8 *$join(a, b)$ is the greatest lower bound (reverse of meet).*

$(a \vee b) = join(a, b)$

$(a \wedge b) = meet(a, b)$

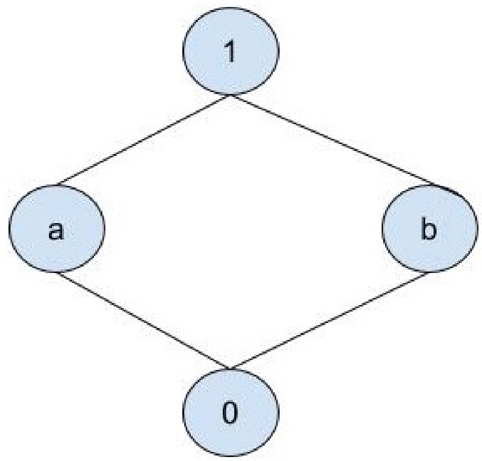

Figure 4.5: meet(a,1) = a

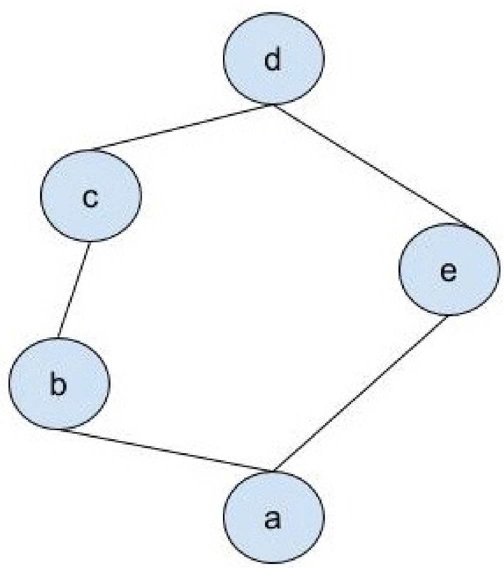

Figure 4.6: meet(c,e) = a; join(c,e) = d; meet(b,e) = a; join(b,e) = d

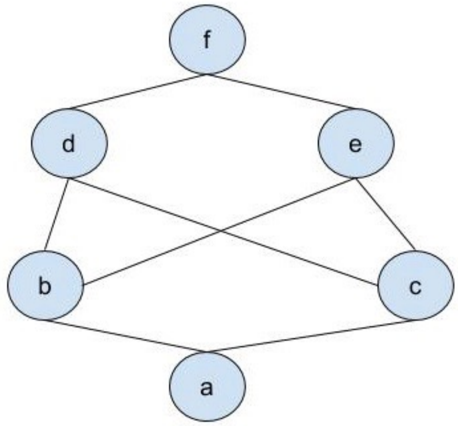

Figure 4.7: meet(b,c) = a; There is no meet of d and e since b and c are incomparable. a cannot be the meet since b and c are bigger than a.

The definition of a lattice can be extended to apply to all finite subsets of X.

Definition 4.9 *A lattice $L = (X, \leq$ is distributive if for all $x, y, z \in X$:*

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
   OR
2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

Referencing Figure 11.6:

$e \wedge (b \vee c) = e \wedge c = a$

$(e \wedge b) \vee (e \wedge c) = a \vee a = a$

This works out. However, we need to show this for all x, y, z.

$c \wedge (b \vee e) = c \wedge d = c$

$(c \wedge b) \vee (c \wedge e) = b \vee a = b.$

This does not work. This lattice is not distributive.

There is a theorem which states that lattices of certain shapes cannot be distributive.