Linear Algebra
Final Exam (paper a)

SIST, SYSU, 2010

1.
Let $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 2 & 1 & 3 & 6 \\ -1 & 1 & -4 & -2 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$.
(1) Transform $(A\ b)$ into reduced echelon form. (7 marks)
(2) Find the solution sets of the matrix equations $Ax = \underline{0}$ and $Ax = b$. (3 marks)
(3) Find a unit lower triangular matrix L and a matrix U in echelon form such that $A = LU$. (4 marks)
(4) Let B be any invertible square matrix with LU factorisations $B = L_1U_1$ and $B = L_2U_2$. Show that $L_1 = L_2$ and $U_1 = U_2$. (6 marks)

Solution:
(1) We have

$$(A\ b) \overset{r_2:=r_2-2r_1, r_3:=r_3+r_1}{\longrightarrow} \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 3 & -1 & 4 & 3 \\ 0 & 0 & -2 & -1 & -1 \end{pmatrix} = (U\ b').$$

Then $(U\ b')$ is in EF.

$$(U\ b') \overset{r_3:=\frac{r_3}{-2}}{\longrightarrow} \begin{pmatrix} 1 & -1 & 2 & 1 & 0 \\ 0 & 3 & -1 & 4 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \overset{r_2:=r_2+r_3, r_1:=r_1-2r_3}{\longrightarrow} \begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ 0 & 3 & 0 & \frac{9}{2} & \frac{7}{2} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$\overset{r_2:=\frac{r_2}{3}}{\longrightarrow} \begin{pmatrix} 1 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{7}{6} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \overset{r_1:=r_1+r_2}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & \frac{3}{2} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{3}{2} & \frac{7}{6} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

which is in REF.
(2) The solution sets are $\{x_4 \in \mathbb{R} \mid x_4 \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} & -\frac{1}{2} & 1 \end{pmatrix}^T\}$ and $\{x_4 \in \mathbb{R} \mid \begin{pmatrix} \frac{1}{6} & \frac{7}{6} & \frac{1}{2} & 0 \end{pmatrix}^T + x_4 \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} & -\frac{1}{2} & 1 \end{pmatrix}^T\}$ respectively.
(3) Since we have used $r_2 := r_2 - 2r_1$, $r_3 := r_3 + r_1$ in transforming A into U. By letting $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$, we have $A = LU$.
(4) We have $B = L_1U_1 = L_2U_2$. Since B is invertible, so are L_1, U_2. So $L_1^{-1}L_2 = U_1U_2^{-1} \in \mathbb{L}^1 \cap \mathbb{U} = \{I\}$, which forces that $L_1^{-1}L_2 = U_1U_2^{-1} = I$. So the result follows.

2.

Define $F(x) = \begin{pmatrix} x & 0 & -3 & 7 \\ 0 & 1 & 2 & 1 \\ -3 & 4 & 0 & 3 \\ 1 & -2 & 2 & -1 \end{pmatrix}$.

(1) For what values of x, $\det F(x) = 0$. (7 marks)
(2) Find the cofactor C_{12} of $F(x)$. (4 marks)
(3) Compute $C_{11}-2C_{12}+2C_{13}-C_{14}$, where $C_{11},\ C_{12},\ C_{13},\ C_{14}$ are all cofactors of $F(x)$. (4 marks)
(4) Describe how to compute the determinant of a square matrix in general. (4 marks)

Solution:
(1) $x = -38$.

(2) $C_{12} = (-1)^{1+2}\det\begin{pmatrix} 0 & 2 & 1 \\ -3 & 0 & 3 \\ 1 & 2 & -1 \end{pmatrix} = 6$.

(3) $C_{11} - 2C_{12} + 2C_{13} - C_{14} = \det\begin{pmatrix} 1 & -2 & 2 & -1 \\ 0 & 1 & 2 & 1 \\ -3 & 4 & 0 & 3 \\ 1 & -2 & 2 & -1 \end{pmatrix} = 0$

(4) Book work! See Textbook or Lecture Notes.

3.
Define $\mathbb{R}_n[x] = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \mid a_0, a_1, \ldots, a_n \in \mathbb{R}\}$, that is the set of polynomials over $\mathbb{R}$ with degrees no more than n.
(1) Show that $\mathbb{R}_n[x]$ is a vector space under the usual polynomial addition and scalar multiplication. (5 marks)
(2) Show that $\{1, x, \ldots, x^n\}$ is a basis of $\mathbb{R}_n[x]$. (3 marks)
(3) Define $T_n : \mathbb{R}_n[x] \to \mathbb{R}_n[x]$, $T_n(p(x)) = p(x) - p'(x)$, where $p'(x)$ is the first order derivative of $p(x)$. Show that T_n is linear and find the matrix A_n of T_n with respect to basis $\{1, x, \ldots, x^n\}$. (5 marks)
(4) Find A_1^{-1}, A_2^{-1}. Furthermore, show that A_n^{-1} is a unit upper triangular matrix such that for $1 \leqslant i \leqslant j \leqslant n$, the (i,j)–th entry of A_n^{-1} is $\frac{(j-1)!}{(i-1)!}$. (8 marks)

Solution:
(1) Easy, only routine verification.
(2) It obvious that $\mathbb{R}_n[x] = \mathrm{Span}\{1, \ldots, x^n\}$. Also, if $c_0 + c_1 x + \cdots + c_n x^n = 0$ then $c_0 = \cdots = c_n = 0$ by equality of polynomials. So $\{1, x, \ldots, x^n\}$ is a basis.
(3) It is routine verification for T_n being linear. $A_n = I_n - H_n$, where

$$[H_n]_{ij} = \begin{cases} i & \text{if } j = i+1, \\ 0 & \text{otherwise.} \end{cases}$$

(4) It can be proved by induction on n. The base case is trivial. Suppose the statement hold for $n-1$. Then we have

$$A_n \begin{pmatrix} A_{n-1}^{-1} & b \\ \underline{0}_n^T & 1 \end{pmatrix} = \begin{pmatrix} A_{n-1} & -c \\ \underline{0}_n^T & 1 \end{pmatrix} \begin{pmatrix} A_{n-1}^{-1} & b \\ \underline{0}_n^T & 1 \end{pmatrix} = \begin{pmatrix} I_n & A_{n-1}b - c \\ \underline{0}_n^T & 1 \end{pmatrix},$$

where $b = \left(\frac{n!}{0!} \ \cdots \ \frac{n!}{(n-1)!}\right)^T$, $c = (0 \ \cdots \ 0 \ n)^T$. It suffices to show $A_{n-1}b - c = \underline{0}$, which is equivalent to show $b = A_{n-1}^{-1}c$. But $A_{n-1}^{-1}c = n\left(\frac{(n-1)!}{0!} \ \cdots \ \frac{(n-1)!}{(n-1)!}\right)^T$, so the result follows.

4.
(1) What is an eigenvalue and its corresponding eigenvectors of a square matrix. (4 marks)
Let $A = \begin{pmatrix} -4 & -10 & 0 \\ 1 & 3 & 0 \\ 3 & 6 & 1 \end{pmatrix}$.
(2) Compute the characteristic polynomial of A and find all eigenvalues of A over $\mathbb{R}$. (5 marks)
(3) Find all eigenspaces of A over $\mathbb{R}$. (8 marks)
(4) Can A be diagonalised over $\mathbb{R}$? If your answer is positive, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. (4 marks)

Solution:
(1) Book work!
(2) $c_A(\lambda) = (\lambda - 1)^2(\lambda + 2)$. So $\lambda_1 = 1$, $\lambda_2 = -2$ are all eigenvalues of A.
(3) 1–eigenspace $= \mathrm{Span}\{(-2 \ 1 \ 0)^T, (0 \ 0 \ 1)^T\}$, -2–eigenspace $= \mathrm{Span}\{(-5 \ 1 \ 3)^T\}$.
(4) A can be diagonalised over $\mathbb{R}$. Take $P = \begin{pmatrix} -2 & 0 & -5 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Then $P^{-1}AP = \mathrm{diag}\{1, 1, -2\}$.

5.
(1) Explain what orthogonal matrices are? (3 marks)
(2) Let U be an orthogonal matrix. Show that $|\det U| = 1$. (4 marks)
(3) Let $v_1 = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} c \\ d \\ 0 \end{pmatrix}$. Find the area of the parallelogram with vertices $\underline{0}$, v_1, v_2 and $v_1 + v_2$. (4 marks)
(4) Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be linearly independent. Recall that the length of the line segment with end points $\underline{0}$ and v_1 is

$$\sqrt{v_1 \cdot v_1} = \sqrt{v_1^T v_1} = \sqrt{\det\left(v_1^T v_1\right)},$$

and the volume of the parallelepiped determined by $\underline{0}$, v_1, v_2 and v_3 is

$$|\det(v_1 \ v_2 \ v_3)| = \sqrt{\det\left((v_1 \ v_2 \ v_3)^T (v_1 \ v_2 \ v_3)\right)}.$$

Show that the area of the parallelogram with vertices $\underline{0}$, v_1, v_2 and $v_1 + v_2$ is $\sqrt{\det\left((v_1 \ v_2)^T (v_1 \ v_2)\right)}$. (8 marks)

Solution:
(1) Book work!
(2) We have $1 = \det I = \det(U^{-1}U) = \det(U^T U) = \det U^T \det U = (\det U)^2$. So $|\det U| = 1$.
(3) $|ad - bc|$.

(4) Let $W = \mathrm{Span}\{v_1, v_2\}$ and $\{u_1, u_2\}$ an orthonormal basis of W. Define $T := \mathbb{R}^2 \to \mathbb{R}^3$, $T(x) = Ux$, where $U = (u_1\ u_2) \in \mathbb{M}_{3\times 2}$. Then $T \in \mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^3)$ is orthogonal and $\mathrm{Im}(T) = W$. So there exist $x_1, x_2 \in \mathbb{R}^2$ such that $v_1 = Ux_1$, $v_2 = Ux_2$. Since T is orthogonal, so it preserves distance and angel between vectors, hence the area of the parallelogram determined by vectors. So The area of the parallelogram determined by v_1, v_2 is $|\det(x_1\ x_2)| = \sqrt{\det\big((x_1\ x_2)^T(x_1\ x_2)\big)} = \sqrt{\det\big((v_1\ v_2)^T U^T U(v_1\ v_2)\big)} = \sqrt{\det\big((v_1\ v_2)^T(v_1\ v_2)\big)}$.