中山大学计算机学院高等代数 2022 年期中开卷题

1. Let A and B be $n \times n$ matrices. Prove that if A is symmetric, then $B^T AB$ is symmetric.

2. Let A and B be $n \times n$ matrices. Prove that if A, B and $A + B$ are all invertible, then $A^{-1} + B^{-1}$ is invertible, via the following two steps:
   a) Prove that $A^{-1} + B^{-1} = A^{-1}(A + B)B^{-1}$.
   b) Prove that $A^{-1}(A + B)B^{-1}$ is invertible.

3. Let A be an $n \times n$ matrix and n is an odd number, which satisfies $A^T A = I$ and $|A| \neq -1$. Prove that $I - A$ is not invertible.

4. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be symmetric matrices. Prove that $\boldsymbol{AB}$ is a symmetric matrix if and only if $\boldsymbol{AB} = \boldsymbol{BA}$.

5. Let $\boldsymbol{\alpha_1}$, $\boldsymbol{\alpha_2}$, $\boldsymbol{\alpha_3}$, $\boldsymbol{\alpha_4}$ be n-dimensional vectors, which satisfy that $\boldsymbol{\alpha_4}$ is not a linear combination of $\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \boldsymbol{\alpha_3}$, but $\boldsymbol{\alpha_1}$ is a linear combination of $\boldsymbol{\alpha_2}, \boldsymbol{\alpha_3}, \boldsymbol{\alpha_4}$. Prove that $\boldsymbol{\alpha_1}$ is a linear combination of $\boldsymbol{\alpha_2}, \boldsymbol{\alpha_3}$.

6. Let A be an $n \times n$ matrix and A is invertible. A^* is the adjugate of A, aka $adj\, A$.
   a) Please show that $|A^*| = |A|^{n-1}$.
   b) Prove that A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

7. Let A be an $n \times n$ matrix satisfying $A^2 + 2A - 4I = O$. Prove that $A - I$ is invertible, and find the inverse matrix of $A - \boldsymbol{I}$.

8. Let A be an $m \times n$ matrix. Prove: Rank($A^T A$)=Rank(A).