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Nonlinear Registration

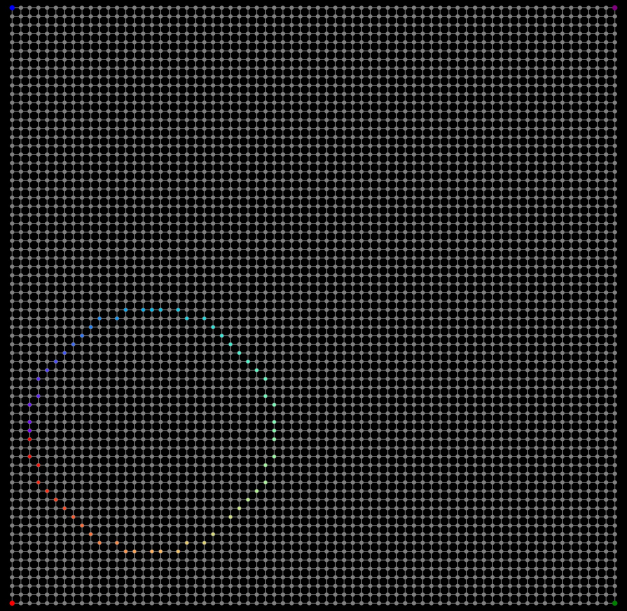
Aalto-yliopisto
Aalto-universitetet
Aalto University

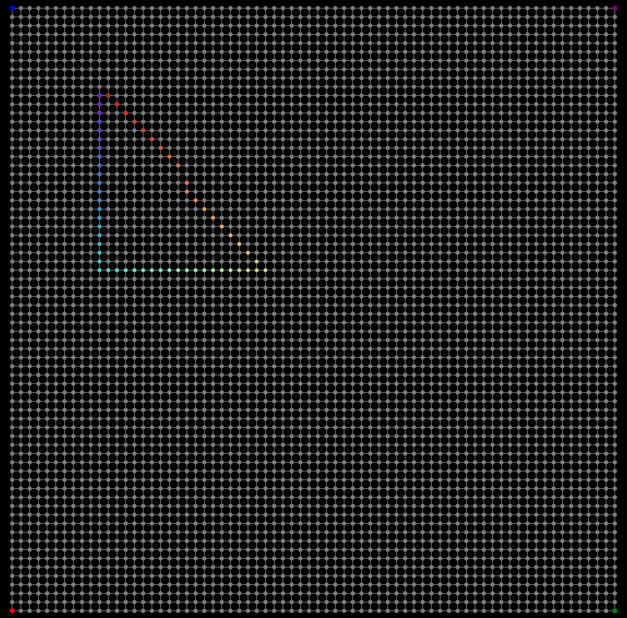
Medical Image Analysis

Original slides by Koen Van Leemput

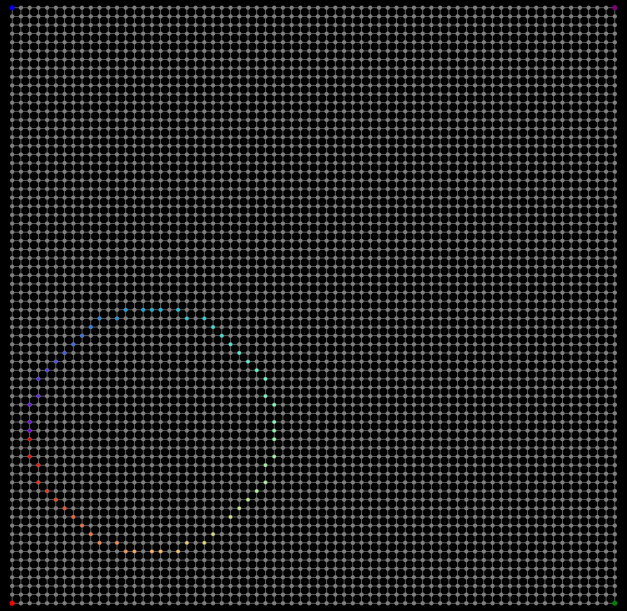
Small modifications by Oula Puonti

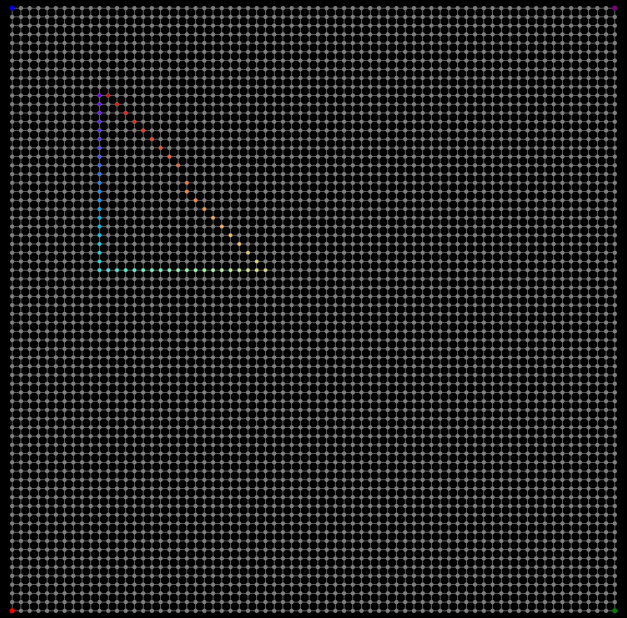
Registration: bits and pieces





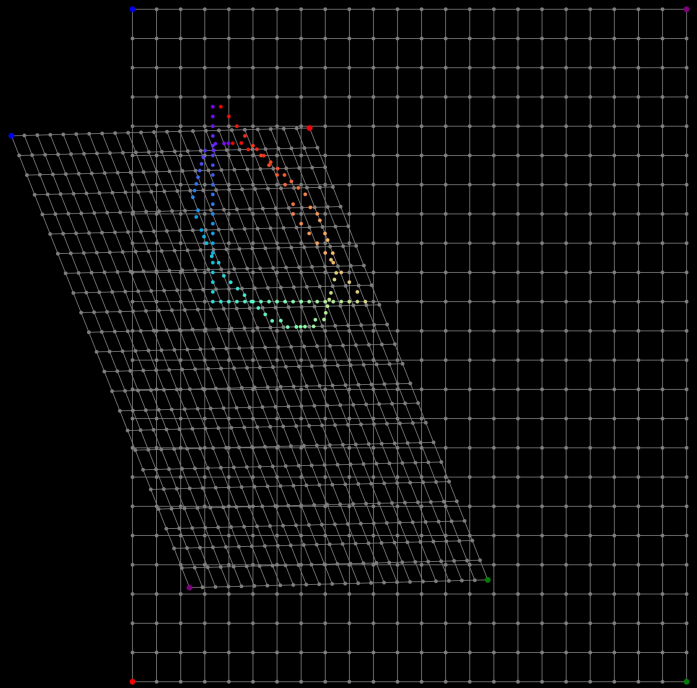
Registration: bits and pieces

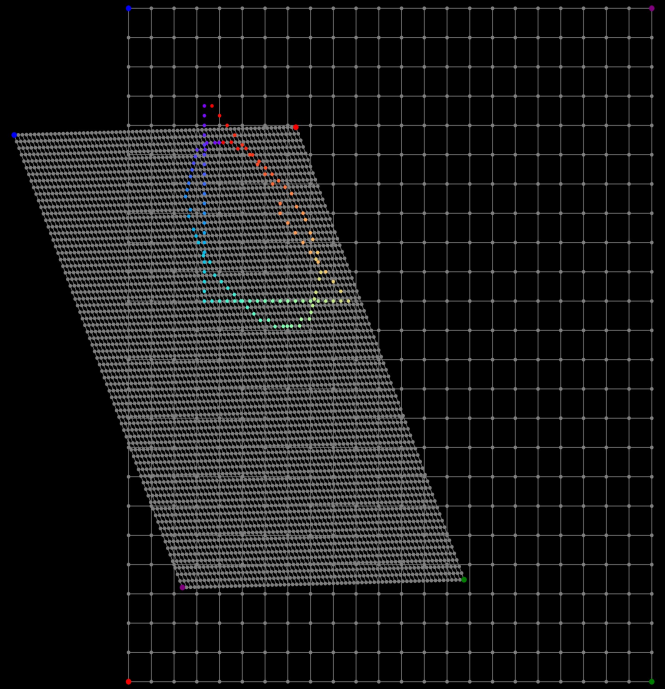




Similarity metric:
Coordinate difference

Registration: bits and pieces

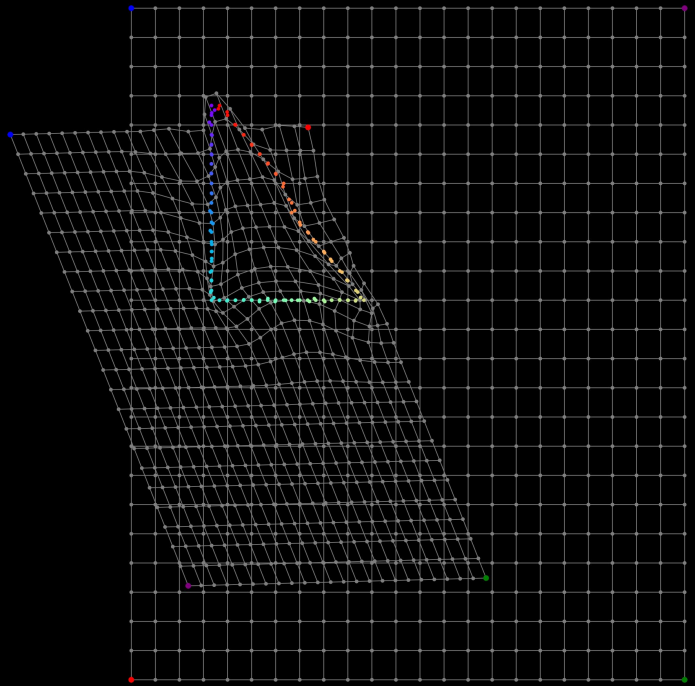


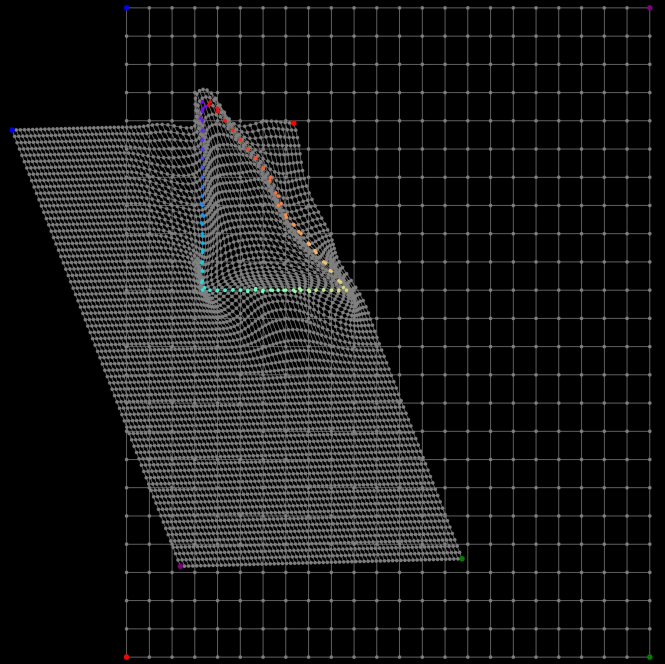


Similarity metric:
Coordinate difference

Transformation:
Affine

Registration: bits and pieces

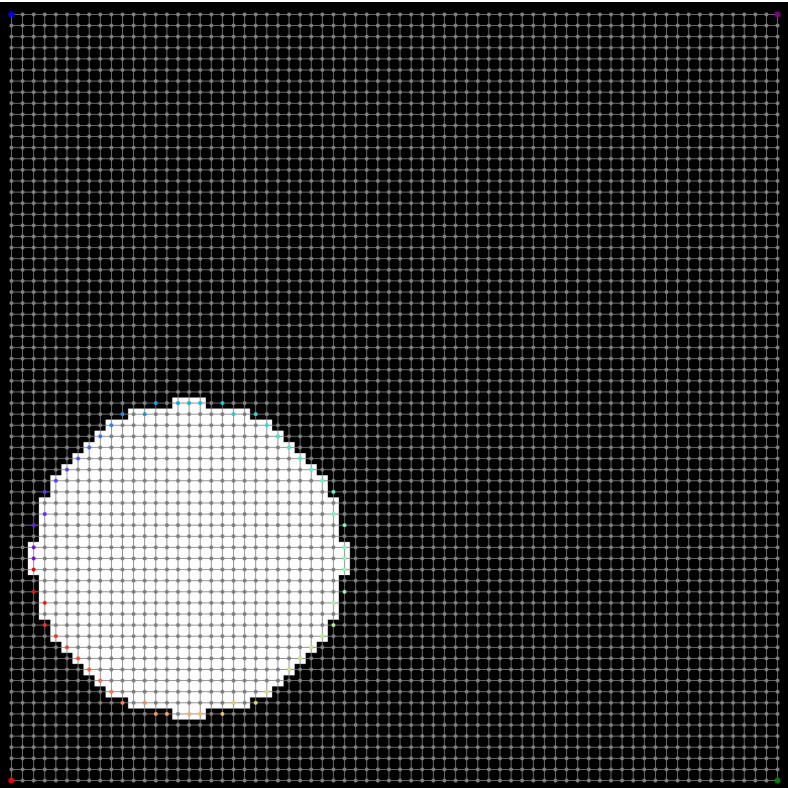




Similarity metric:
Coordinate difference

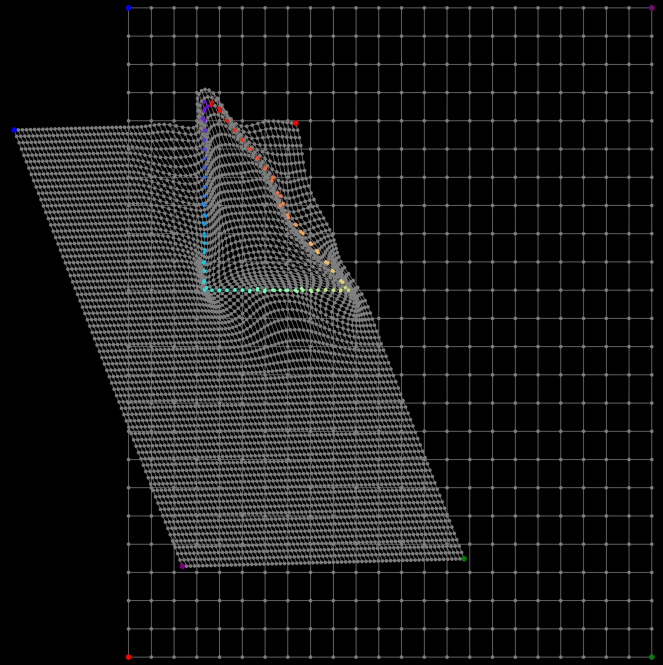
Transformation:
Affine+Nonlinear

Registration: bits and pieces



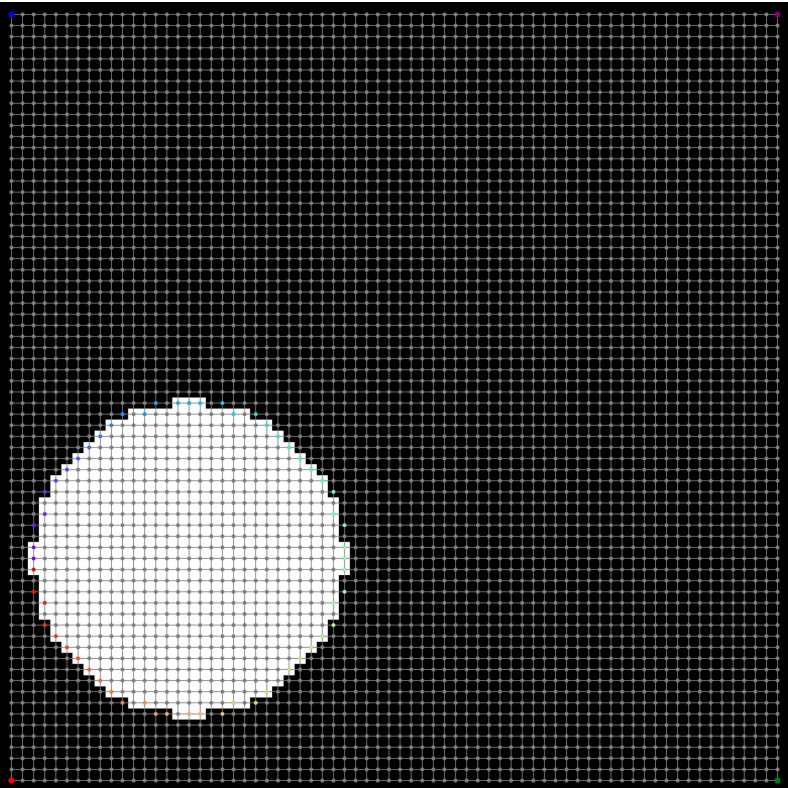
Similarity metric:
Coordinate difference

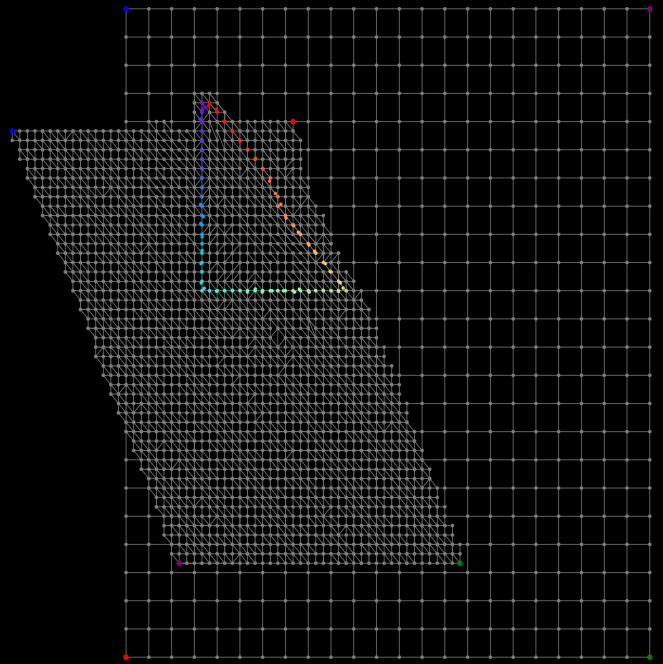
Transformation:
Affine+Nonlinear



Interpolation?

Registration: bits and pieces



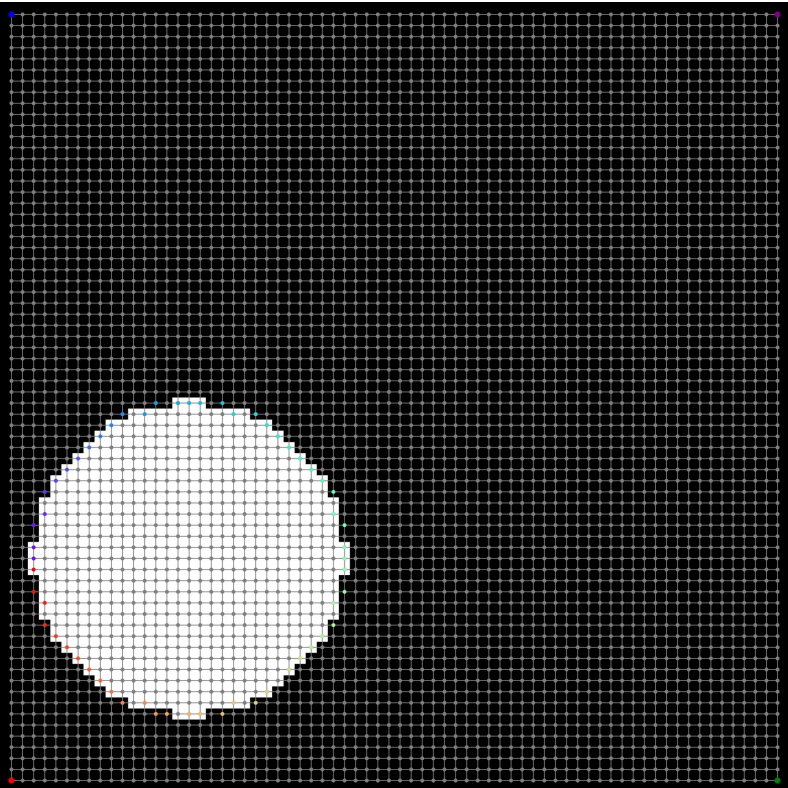


Similarity metric:
Coordinate difference

Transformation:
Affine+Nonlinear

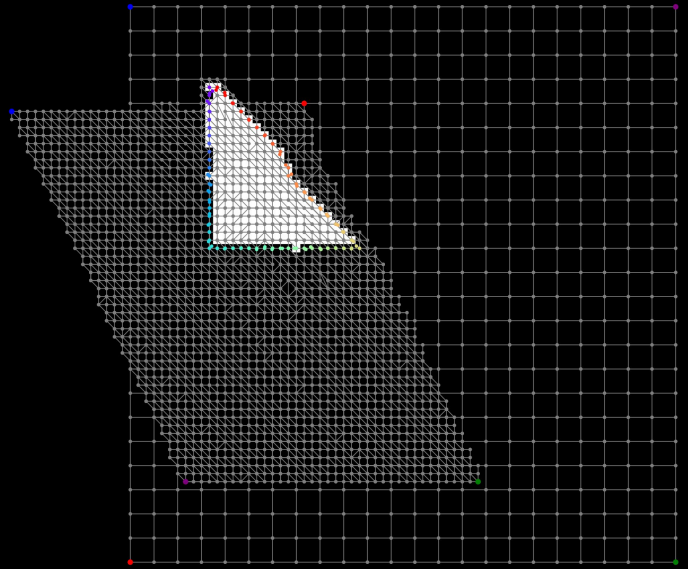
Interpolation?

Registration: bits and pieces



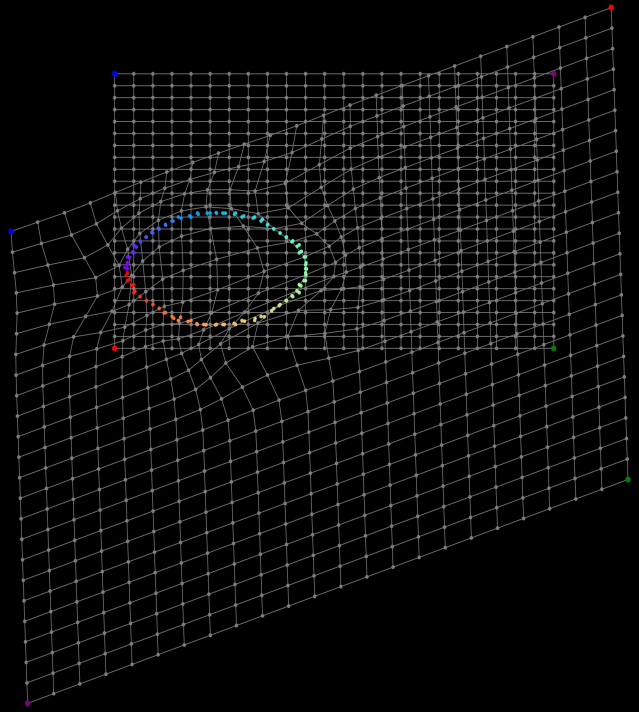
Similarity metric:
Coordinate difference

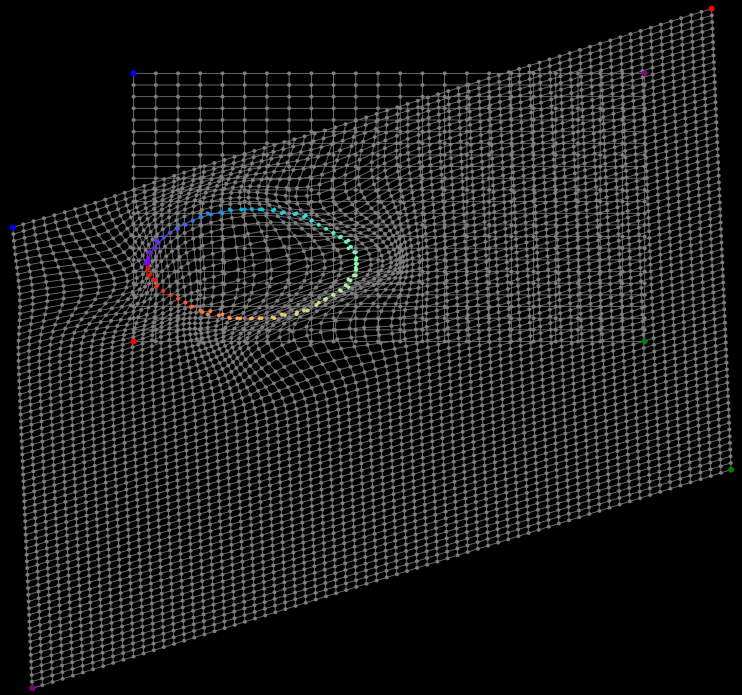
Transformation:
Affine+Nonlinear



Interpolation?

Registration: bits and pieces



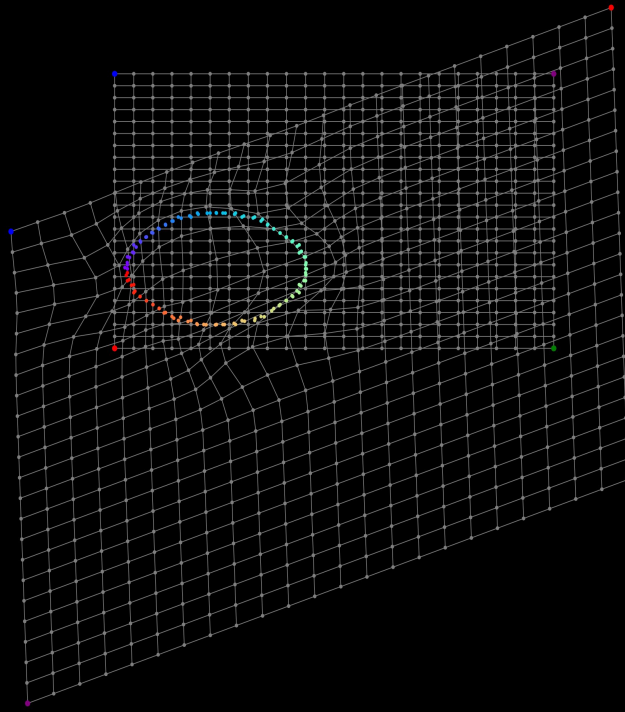


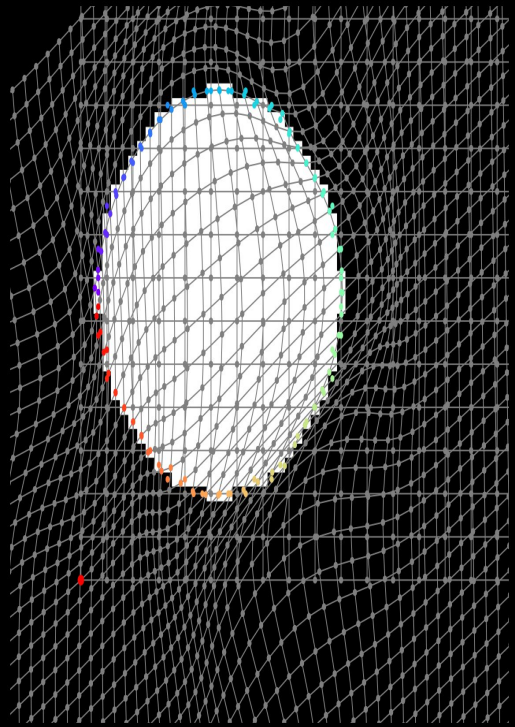
Similarity metric:
Coordinate difference

Transformation:
Affine+Nonlinear

Interpolation?

Registration: bits and pieces



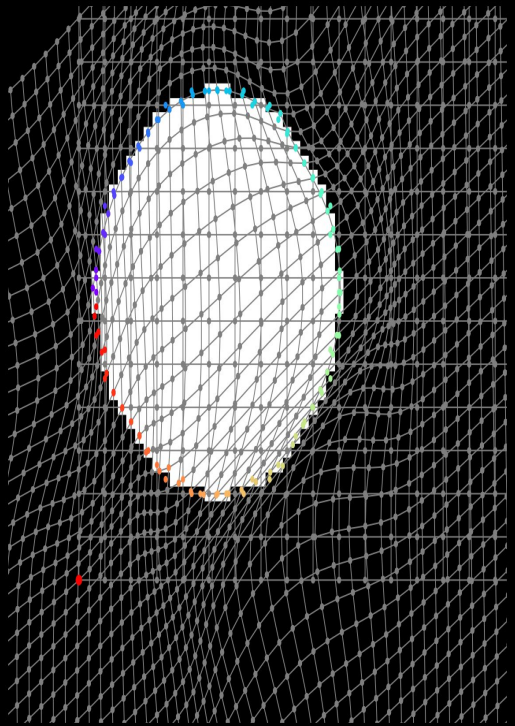


Similarity metric:
Coordinate difference

Transformation:
Affine+Nonlinear

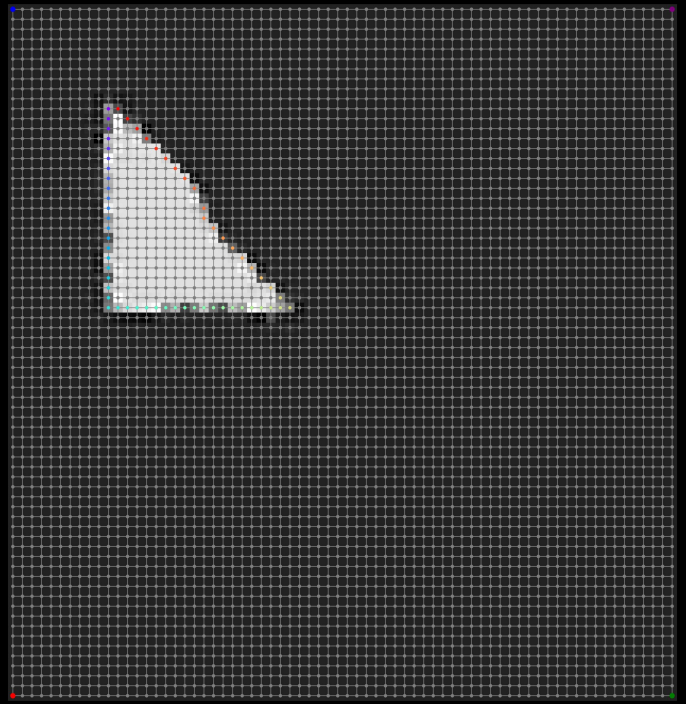
Interpolation:
Pull from “moving image”

Registration: bits and pieces



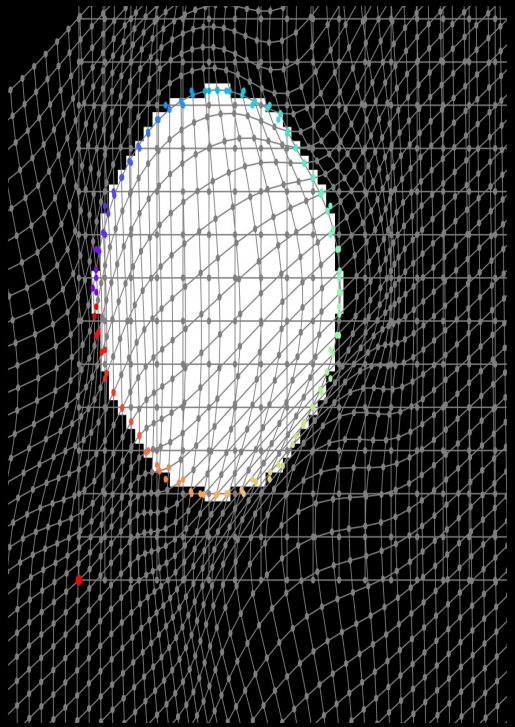
Similarity metric:
Coordinate difference

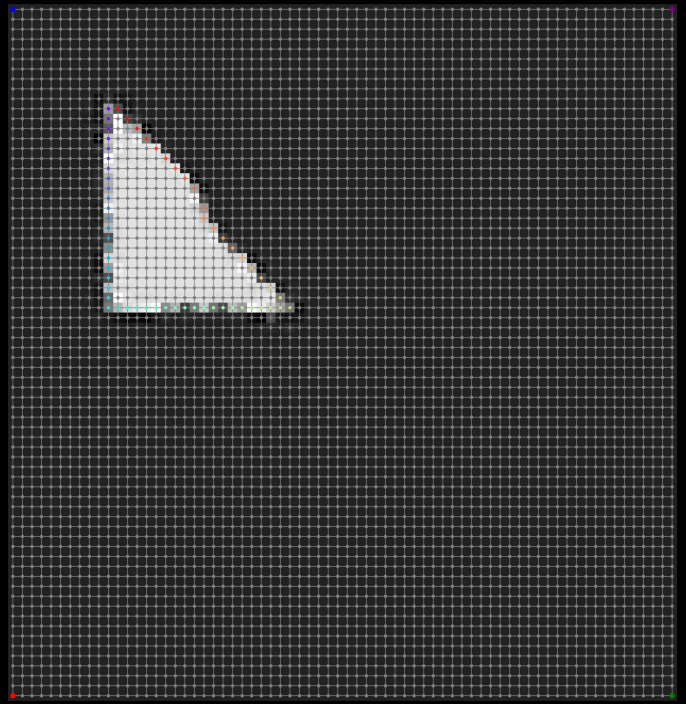
Transformation:
Affine+Nonlinear



Interpolation:
Pull from “moving image”

Registration: bits and pieces





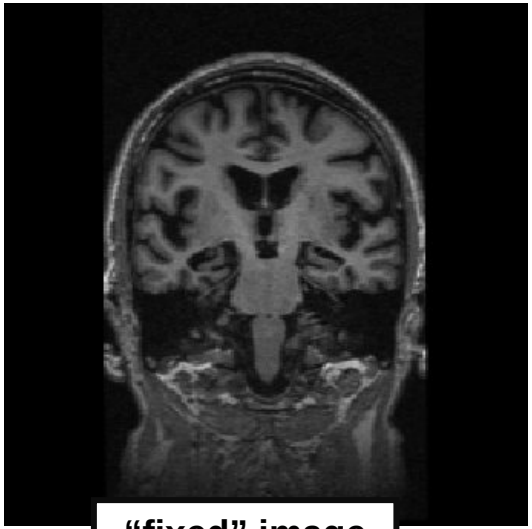
Similarity metric:
Coordinate difference

Transformation:
Affine+Nonlinear

Interpolation:
Pull from “moving image”

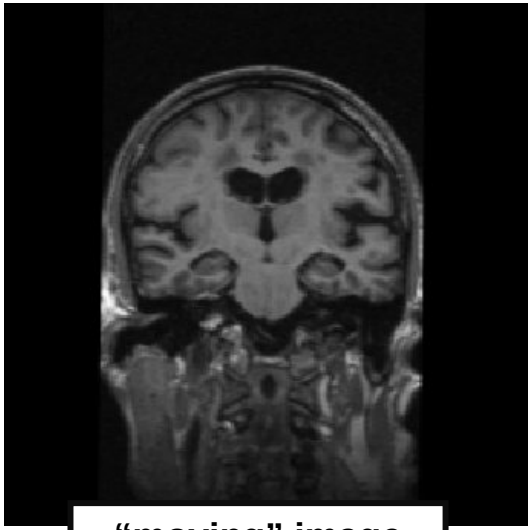
Optimization:
Least-square

Spatial transformations



“fixed” image

$$\mathbf{x} = (x_1, \ldots, x_D)^{\mathrm{T}}$$

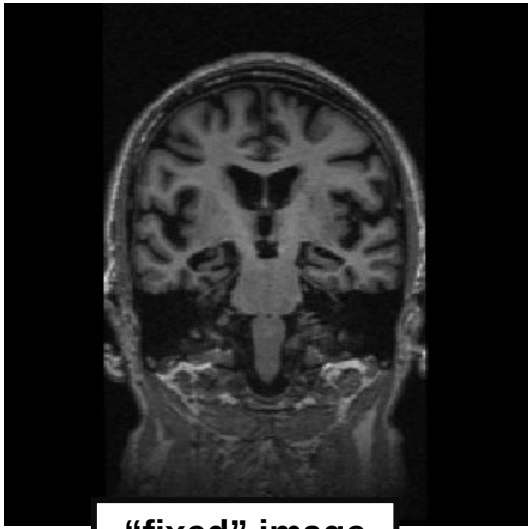


“moving” image

$$\mathbf{y} = (y_1, \ldots, y_D)^{\mathrm{T}}$$

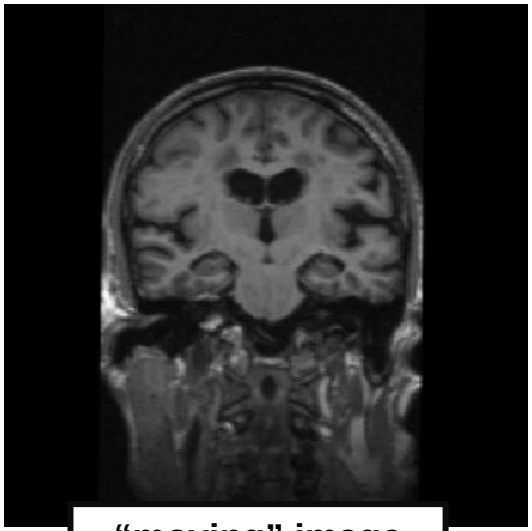
Spatial transformations

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \begin{pmatrix} y_1(\mathbf{x}, \mathbf{w}) \\ \vdots \\ y_D(\mathbf{x}, \mathbf{w}) \end{pmatrix}$$



“fixed” image

$$\mathbf{x} = (x_1, \ldots, x_D)^{\mathrm{T}}$$

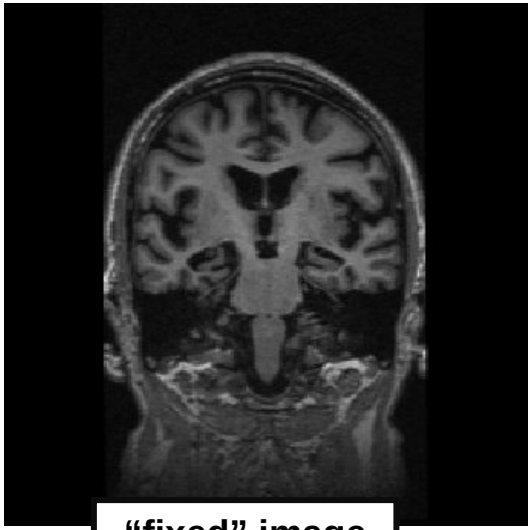


“moving” image

$$\mathbf{y} = (y_1, \ldots, y_D)^{\mathrm{T}}$$

Spatial transformations

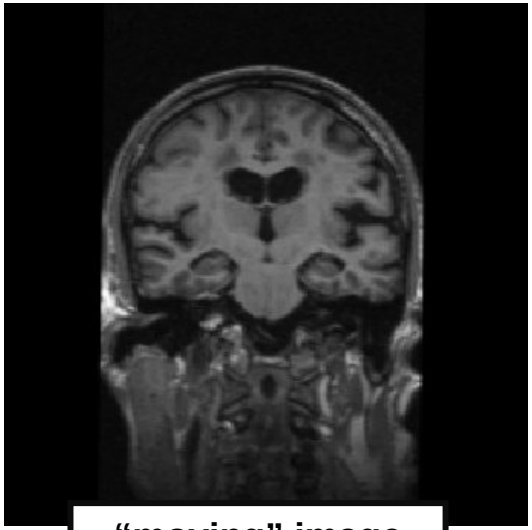
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \begin{pmatrix} y_1(\mathbf{x}, \mathbf{w}) \\ \vdots \\ y_D(\mathbf{x}, \mathbf{w}) \end{pmatrix}$$



“fixed” image

$\mathbf{x} = (x_1, \ldots, x_D)^{\mathrm{T}}$

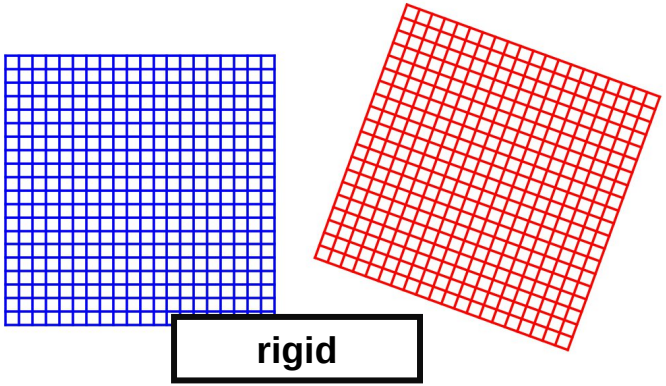
$y_d(\mathbf{x}, \mathbf{w})$
controls how points $\mathbf{x}$ in the fixed image
move along the d-th direction in the moving image
as the parameters $\mathbf{w}$ are varied

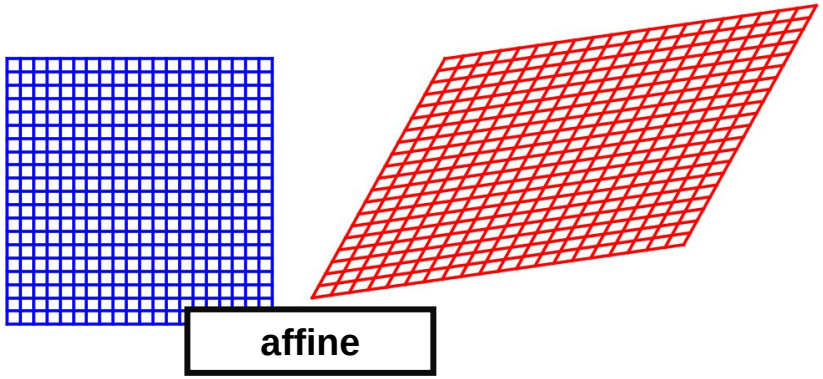


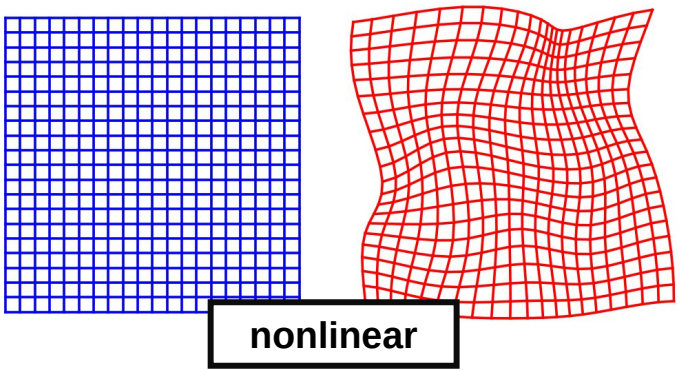
“moving” image

$\mathbf{y} = (y_1, \ldots, y_D)^{\mathrm{T}}$

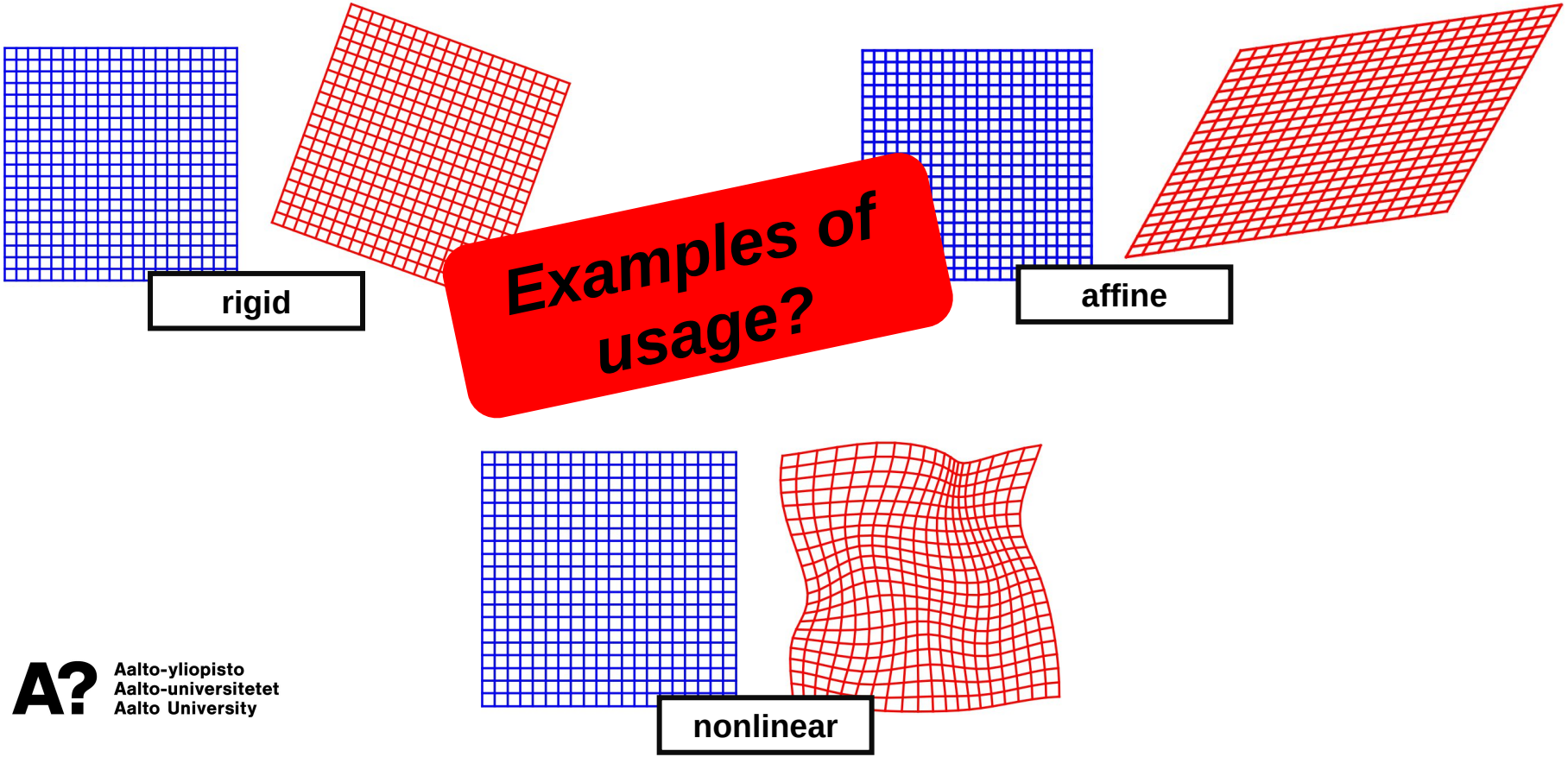
Spatial transformations







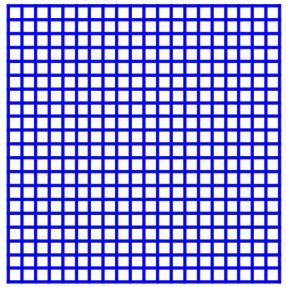
Spatial transformations

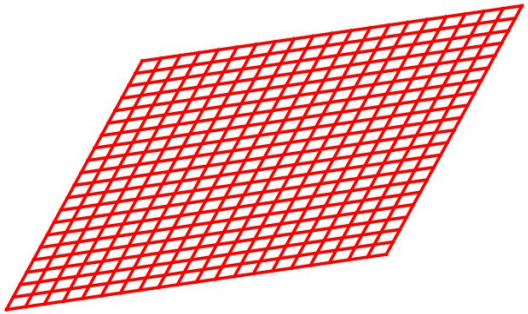


Affine transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t},$$

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \text{and} \quad \mathbf{t} = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$





Affine transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t},$$

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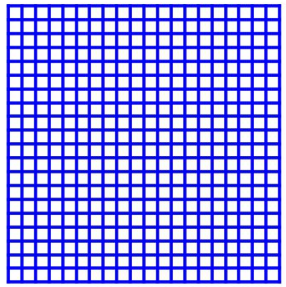
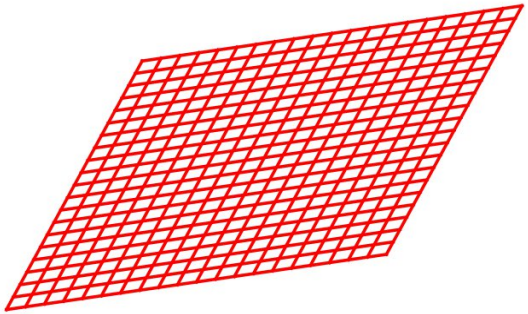




$y_d(\mathbf{x}, \mathbf{w})$
controls how points $\mathbf{x}$ in the fixed image
move along the d-th direction in the moving image
as the parameters $\mathbf{w}$ are varied

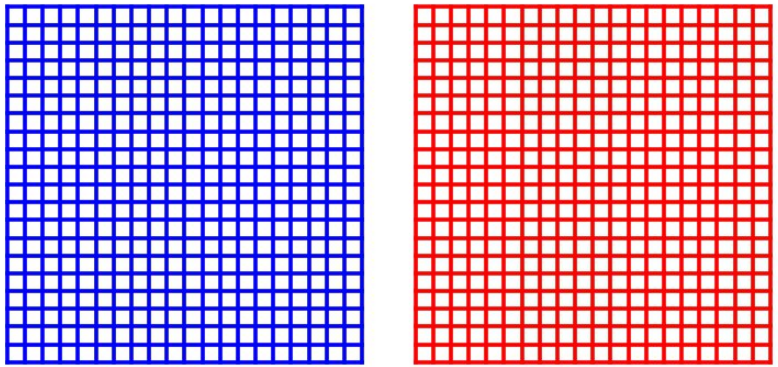
$$y_d(\mathbf{x}, \mathbf{w}_d) = t_d + a_{d,1}x_1 + \ldots + a_{d,D}x_D,$$

$$\mathbf{w}_d = (t_d, a_{d,1}, \ldots, a_{d,D})^{\mathrm{T}}$$

$$\mathbf{w} = (\mathbf{w}_1^{\mathrm{T}}, \ldots, \mathbf{w}_D^{\mathrm{T}})^{\mathrm{T}}$$

Affine transformation

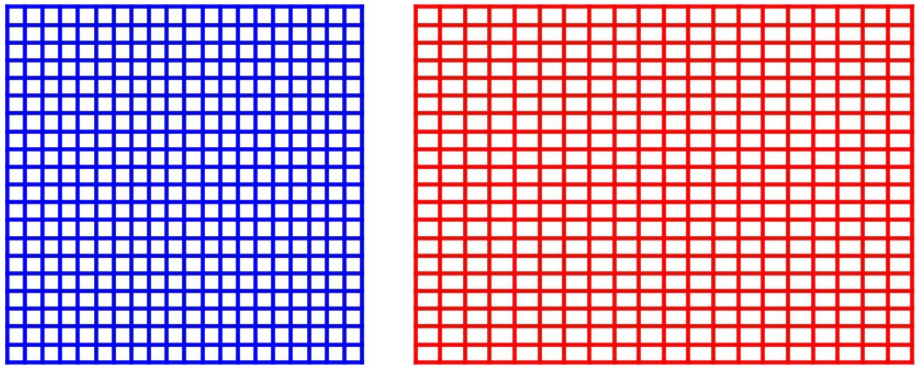
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t},$$



$$\mathbf{A} = \begin{pmatrix} 1.0 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

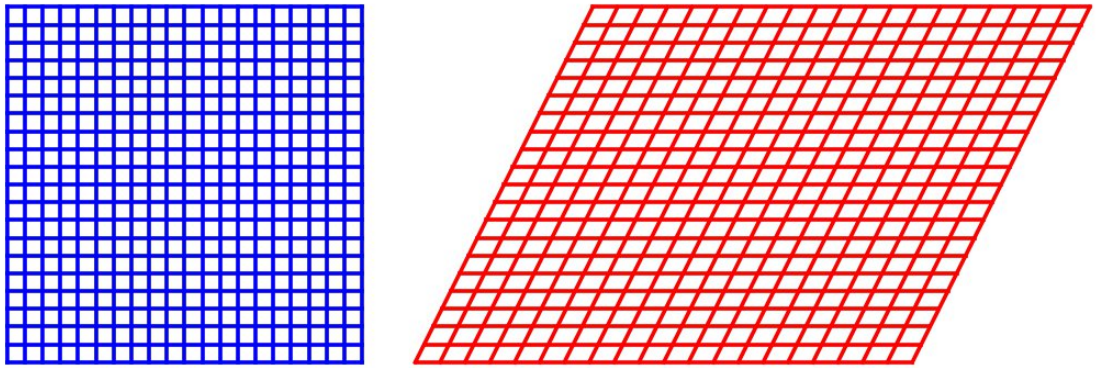
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t},$$



$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.0 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

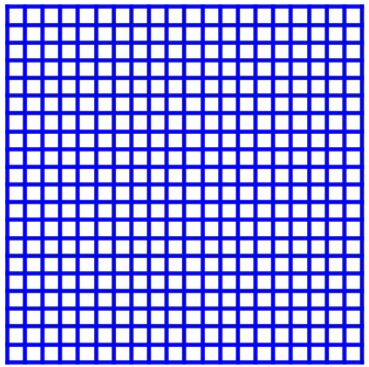
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t},$$

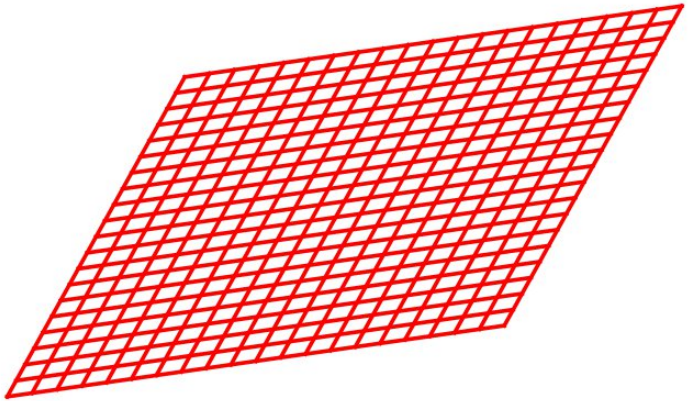


$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.5 \\ 0.0 & 1.0 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 0 \end{pmatrix}$$

Affine transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t},$$

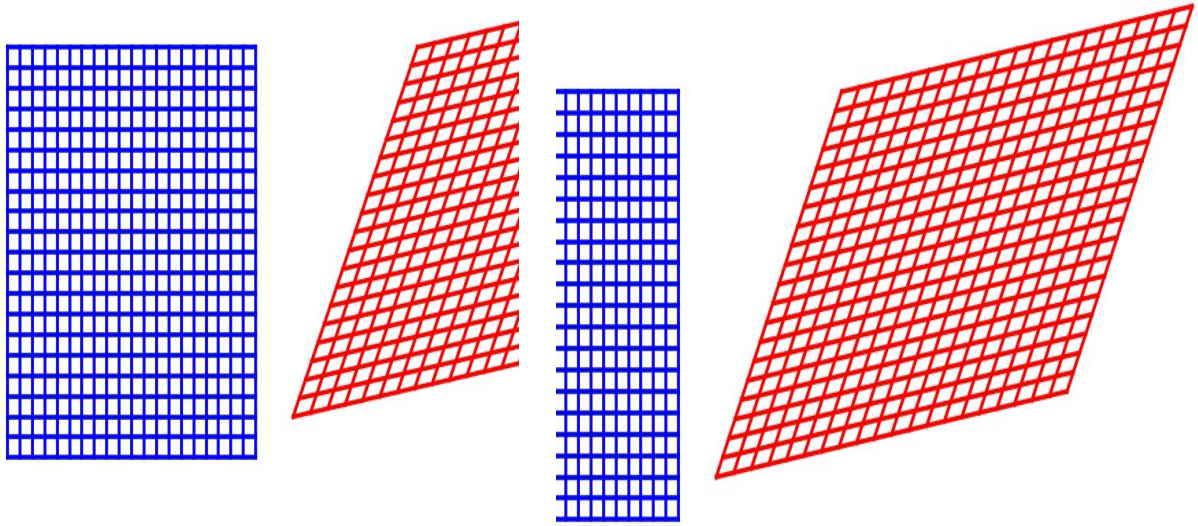




$$\mathbf{A} = \begin{pmatrix} 1.4 & 0.5 \\ 0.2 & 0.9 \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} 23 \\ 6 \end{pmatrix}$$

Affine (*linear*) transformation...

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{A}\mathbf{x} + \mathbf{t},$$



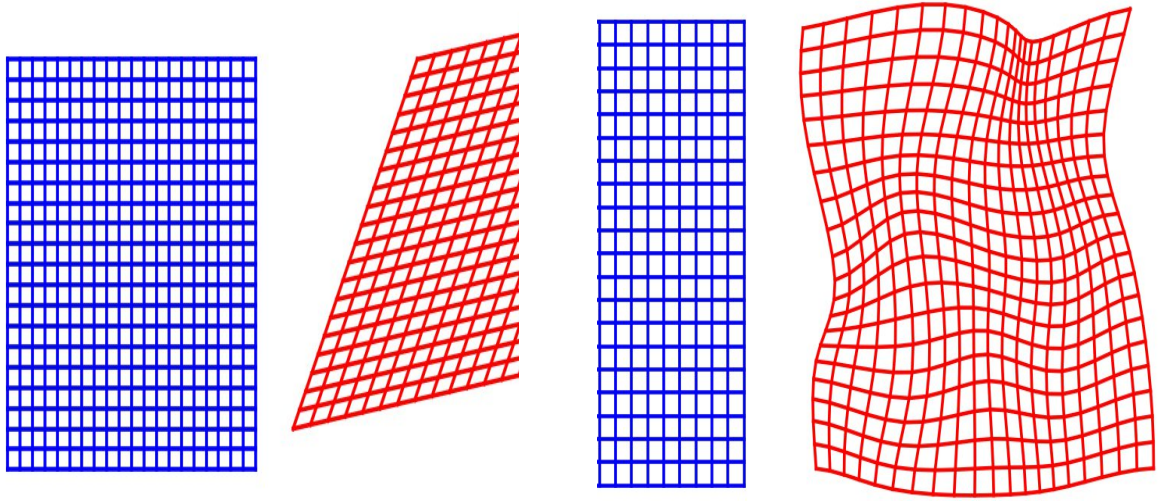
$$y_d(\mathbf{x}, \mathbf{w}_d) = t_d + a_{d,1}x_1 + \ldots + a_{d,D}x_D,$$

$$\mathbf{w}_d = (t_d, a_{d,1}, \ldots, a_{d,D})^{\mathrm{T}}$$

$$\mathbf{w} = (\mathbf{w}_1^{\mathrm{T}}, \ldots, \mathbf{w}_D^{\mathrm{T}})^{\mathrm{T}}$$

...vs. *nonlinear* transformation

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{x} + \boldsymbol{\delta}(\mathbf{x}, \mathbf{w})$$



$$y_d(\mathbf{x}, \mathbf{w}_d) = x_d + \delta_d(\mathbf{x}, \mathbf{w}_d), \quad \text{with} \quad \delta_d(\mathbf{x}, \mathbf{w}_d) = \sum_{m=0}^{M-1} w_{d,m}\phi_m(\mathbf{x}).$$

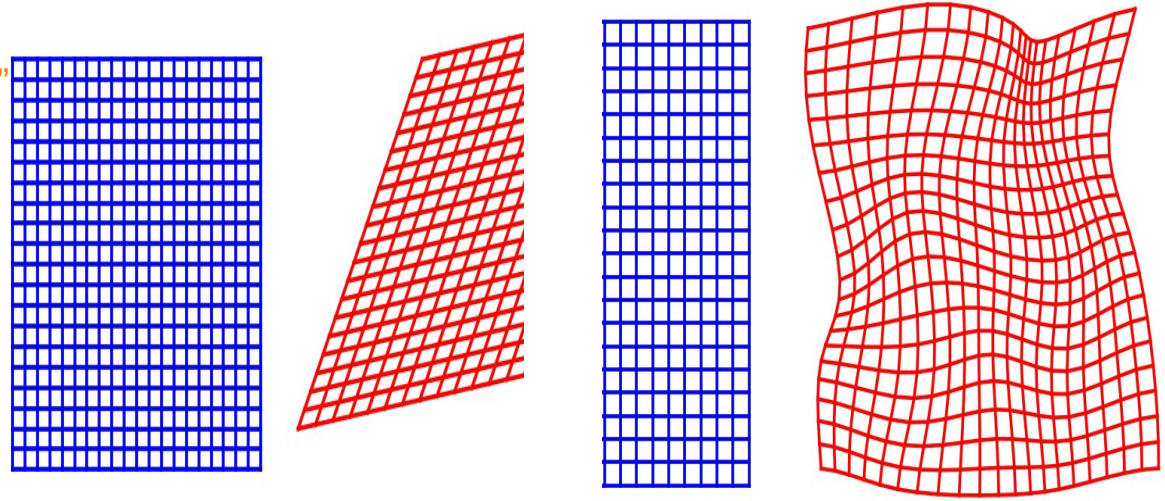
$$\mathbf{w}_d = (w_{d,0}, \ldots w_{d,M-1})^{\mathrm{T}}$$

$$\mathbf{w} = (\mathbf{w}_1^{\mathrm{T}}, \ldots, \mathbf{w}_D^{\mathrm{T}})^{\mathrm{T}}$$

...vs. *nonlinear* transformation

"residual deformation"

$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{x} + \boldsymbol{\delta}(\mathbf{x}, \mathbf{w})$$



$$y_d(\mathbf{x}, \mathbf{w}_d) = x_d + \delta_d(\mathbf{x}, \mathbf{w}_d), \quad \text{with} \quad \delta_d(\mathbf{x}, \mathbf{w}_d) = \sum_{m=0}^{M-1} w_{d,m}\phi_m(\mathbf{x}).$$

$$\mathbf{w}_d = (w_{d,0}, \ldots w_{d,M-1})^{\mathrm{T}}$$

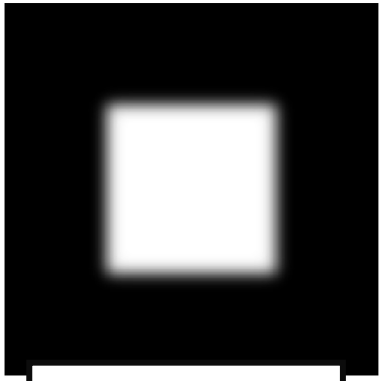
$$\mathbf{w} = (\mathbf{w}_1^{\mathrm{T}}, \ldots, \mathbf{w}_D^{\mathrm{T}})^{\mathrm{T}}$$

nonlinear basis functions

fixed image

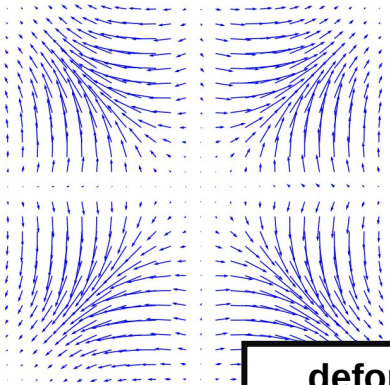
$\mathcal{F}(\mathbf{x})$

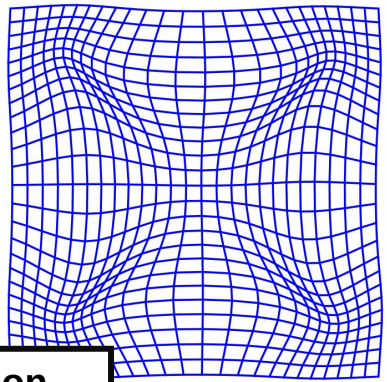
$$\mathbf{y}(\mathbf{x}, \mathbf{w}) = \mathbf{x} + \boldsymbol{\delta}(\mathbf{x}, \mathbf{w})$$



moving image

$\mathcal{M}(\mathbf{y})$





deformation

$= \mathrm{x} + \delta(\mathrm{x}, \mathrm{w})$

interpolated moving image

$\mathcal{M}(\mathbf{y}(\mathbf{x}, \mathbf{w}))$

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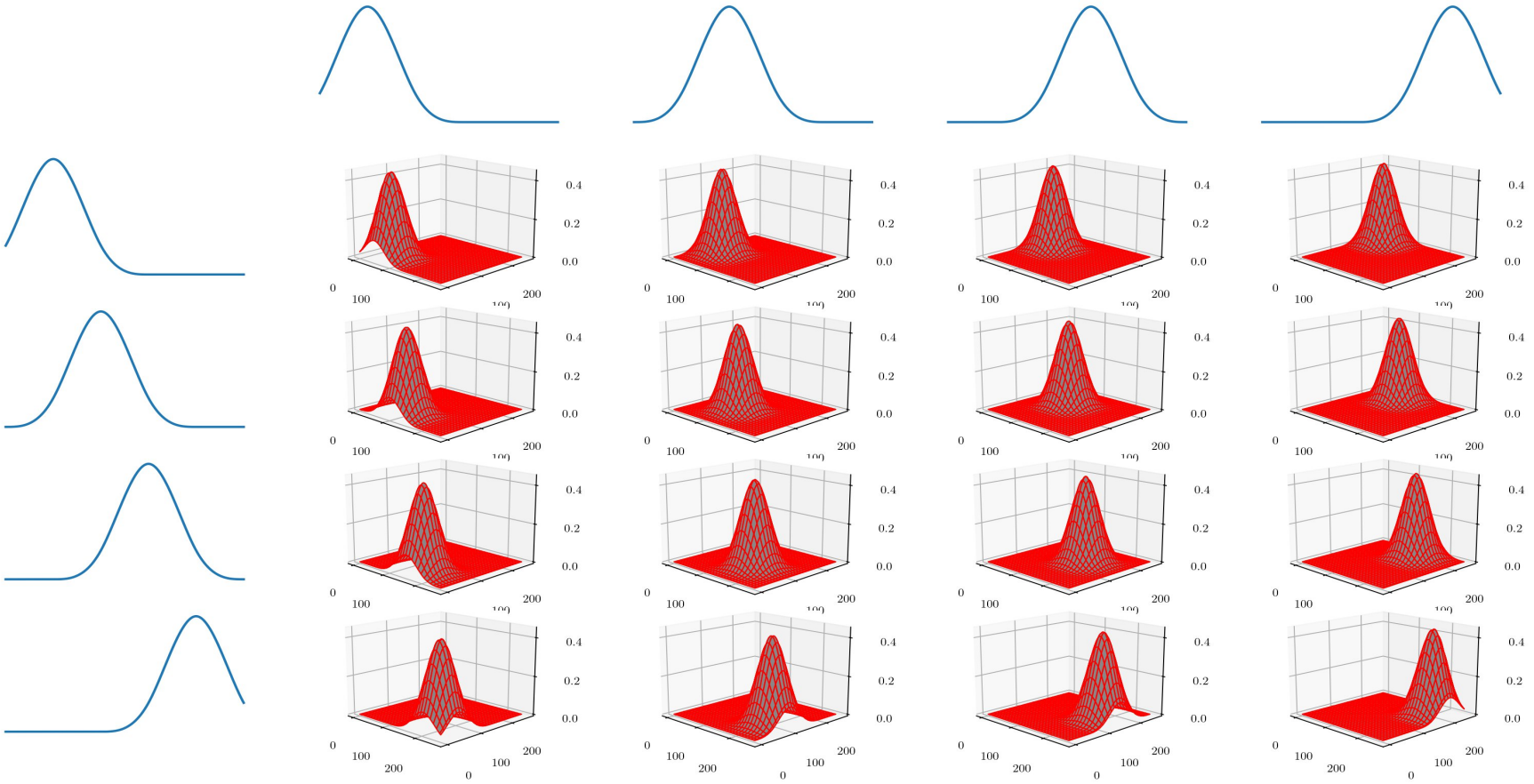
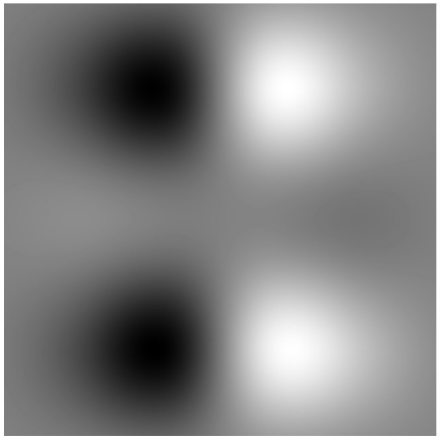
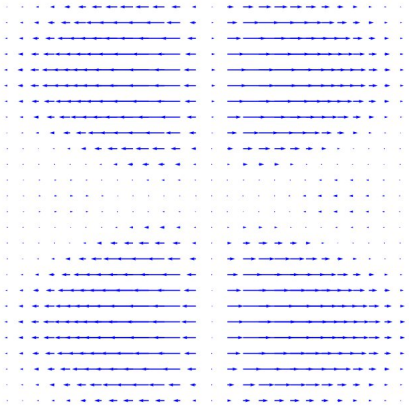


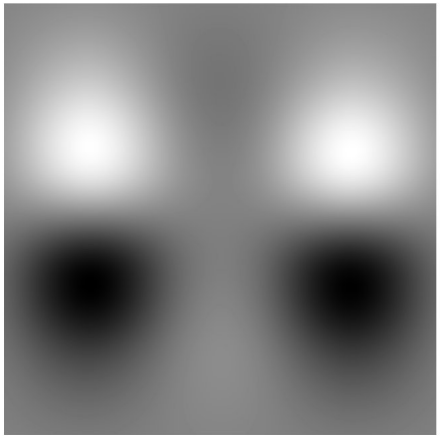
16 nonlinear basis functions



direction 1

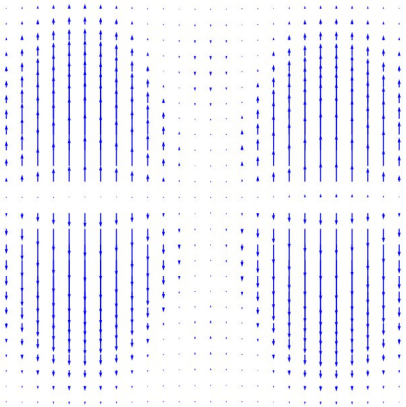
$\delta_1(\mathbf{x}, \mathbf{w}_1)$

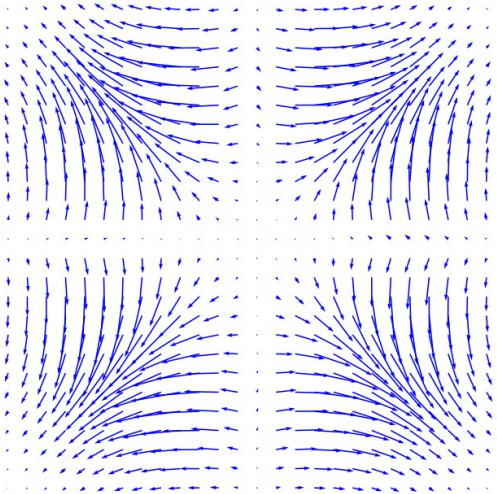




direction 2

$\delta_2(\mathbf{x}, \mathbf{w}_2)$



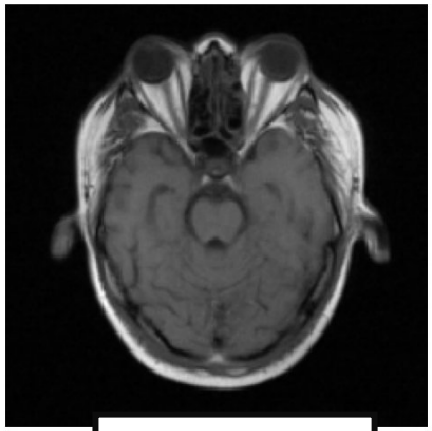


deformation

$= \mathrm{x} + \delta(\mathrm{x}, \mathrm{w})$

Focus on intra-modal registration

Images have similar intensity characteristics



$\mathcal{F}(\mathbf{x})$

$\mathcal{M}(\mathbf{y}(\mathbf{x}, \mathbf{w}))$

$[\mathcal{F}(\mathbf{x}) - \mathcal{M}(\mathbf{y}(\mathbf{x}, \mathbf{w}))]^2$

$$E(\mathbf{w}) = \sum_{n=1}^{N} [\mathcal{F}(\mathbf{x}_n) - \mathcal{M}(\mathbf{y}(\mathbf{x}_n, \mathbf{w}))]^2 .$$

sum over all voxels

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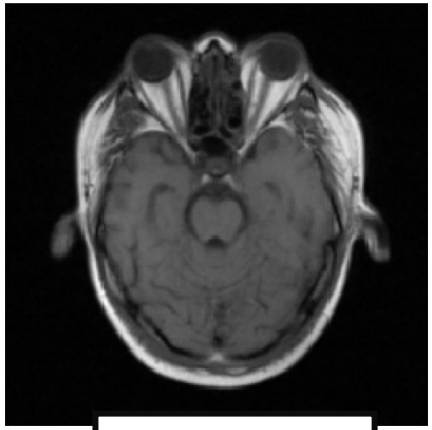
Focus on intra-modal registration

Images have similar intensity characteristics



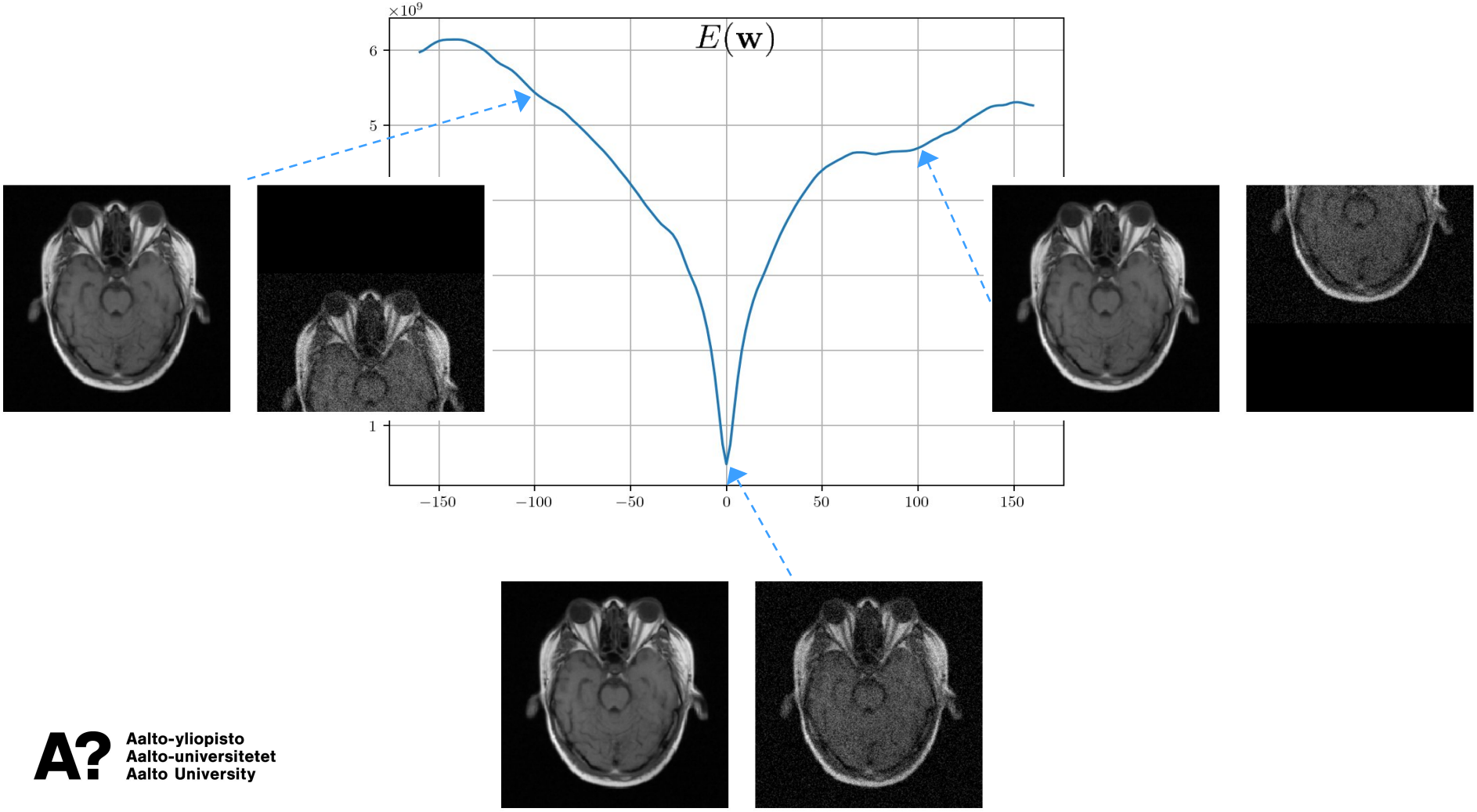
$\mathcal{F}(\mathbf{x})$

$\mathcal{M}(\mathbf{y}(\mathbf{x}, \mathbf{w}))$

$[\mathcal{F}(\mathbf{x}) - \mathcal{M}(\mathbf{y}(\mathbf{x}, \mathbf{w}))]^2$

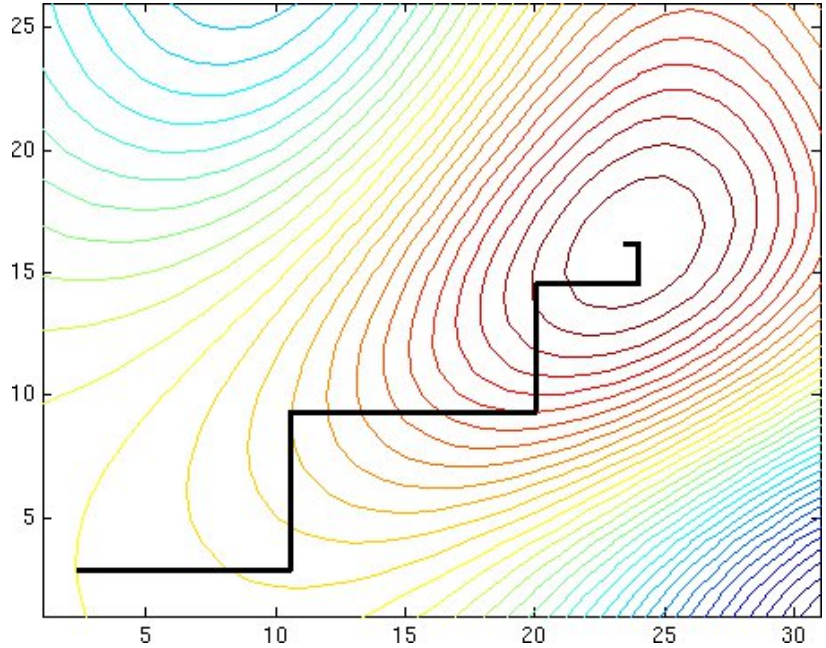
$$E(\mathbf{w}) = \sum_{n=1}^{N} [\mathcal{F}(\mathbf{x}_n) - \mathcal{M}(\mathbf{y}(\mathbf{x}_n, \mathbf{w}))]^2 .$$

sum over all voxels


$\times 10^9$
$E(\mathbf{w})$
6
5
1
−150
−100
−50
0
50
100
150

Numerical optimization

Find transformation parameters $\mathbf{w}$ **that minimize** $E(\mathbf{w})$

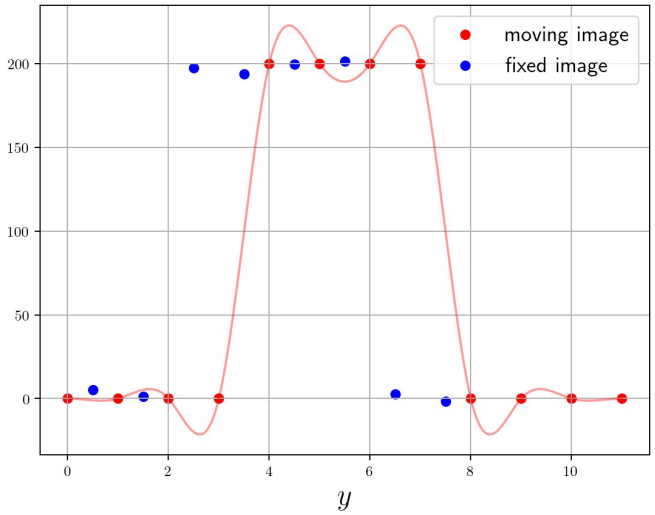


Gauss-Newton optimization

Toy example: 1D and translation only

transformation model: $y(x,t) = x + t$ — parameter to be optimized

- ✔ energy: $E(t) = \sum_{n=1}^{N} E_n(t)$ with $E_n(t) = [\mathcal{F}(x) - \mathcal{M}(y(x_n,t))]^2$



before optimization

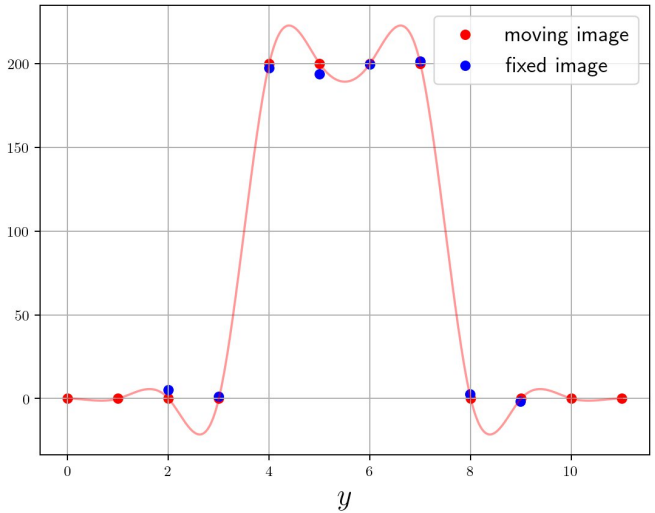
Gauss-Newton optimization

Toy example: 1D and translation only

transformation model: $y(x,t) = x + t$

parameter to be optimized

- ✔ energy: $E(t) = \sum_{n=1}^{N} E_n(t)$ with $E_n(t) = [\mathcal{F}(x) - \mathcal{M}(y(x_n,t))]^2$

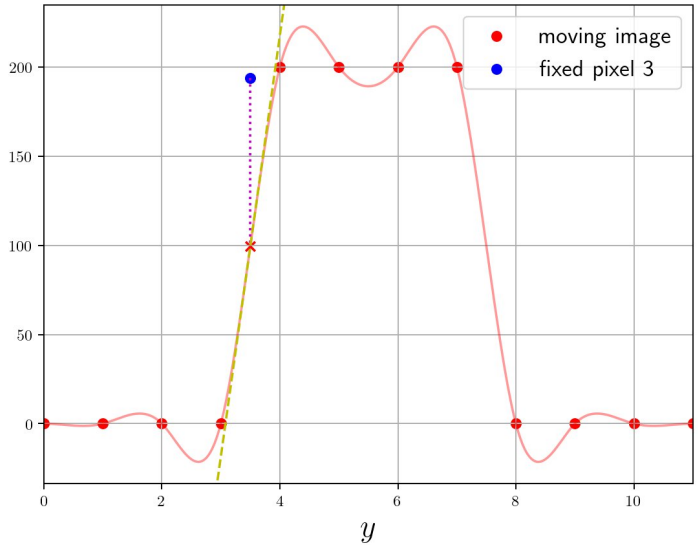


after optimization

Gauss-Newton optimization

$$g_n = \left.\frac{\mathrm{d}\mathcal{M}(y)}{\mathrm{d}y}\right|_{y=x_n+t}$$

Idea: for a *small* deviation ϵ around current estimate of t: $\mathcal{M}(y(x_n, t+\epsilon)) \simeq \mathcal{M}(y(x_n, t)) + g_n \cdot \epsilon$



Energy:

$$E(t+\epsilon) = \sum_{n=1}^{N} E_n(t+\epsilon)$$

$$E_n(t+\epsilon) = [\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t+\epsilon))]^2$$

$$\simeq [\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t)) - g_n \cdot \epsilon]^2$$

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Task: what is ϵ minimizing $E(t+\epsilon) = \sum_{n=1}^{N}$?

Gauss-Newton optimization

Solution: standard linear regression!

$$\epsilon = (\boldsymbol{\psi}^T \boldsymbol{\psi})^{-1} \boldsymbol{\psi}^T \boldsymbol{\tau}$$

Now update t:

$$t \leftarrow t + \epsilon$$

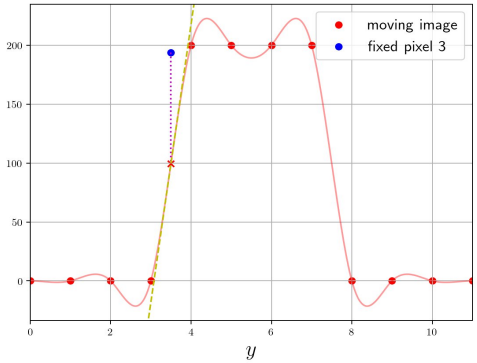
where $\boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_N \end{pmatrix}$ with $\tau_n = \mathcal{F}(x_n) - \mathcal{M}(y(x_n, t))$

and $\psi = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix} = \begin{pmatrix} g_1 & & \\ & \ddots & \end{pmatrix}$

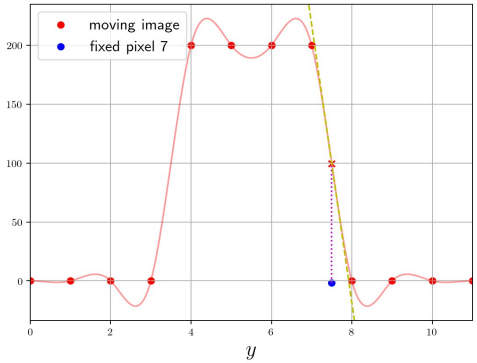
one basis function

initialization

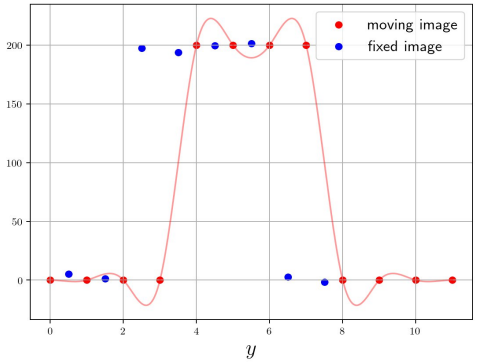
pixel 3



pixel 7



all pixels



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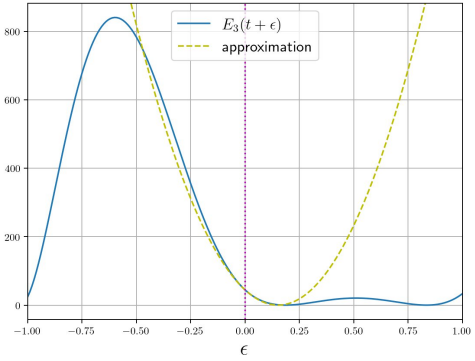
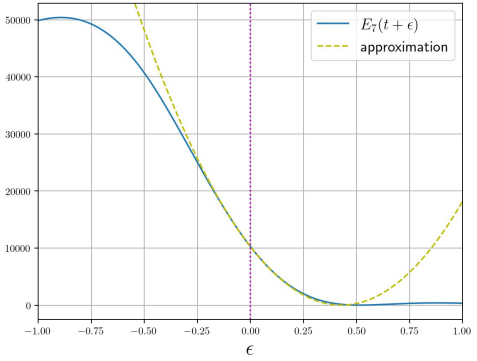
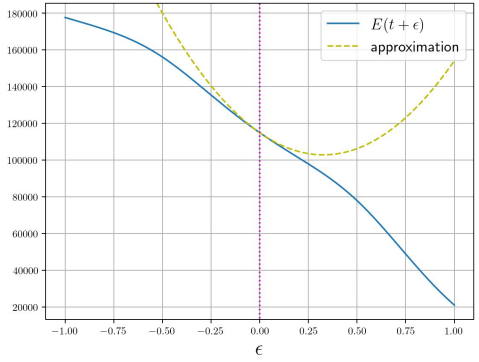
... + ...

+ ... =







Energy:

$$E(t+\epsilon) = \sum_{n=1}^{N} E_n(t+\epsilon)$$

with

$$E_n(t+\epsilon) = \left[\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t+\epsilon))\right]^2$$

$$\simeq \left[\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t)) - g_n \cdot \epsilon\right]^2$$

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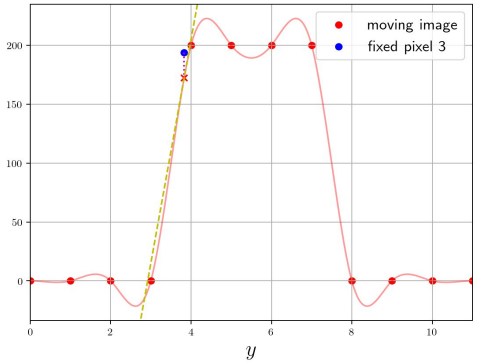
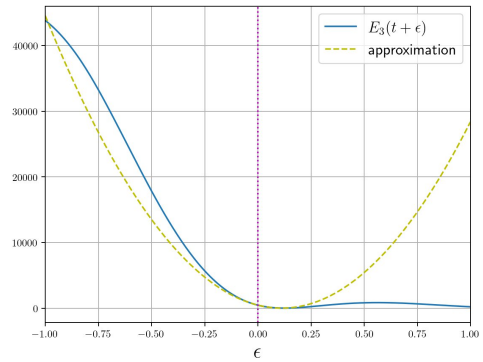
after 1 iteration

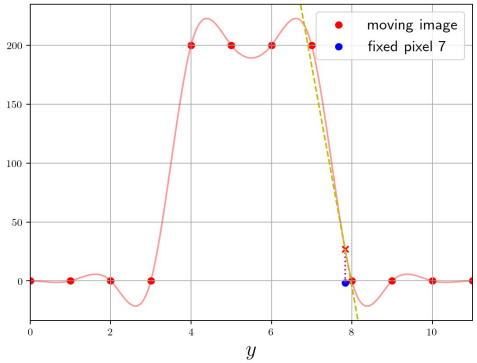
pixel 3

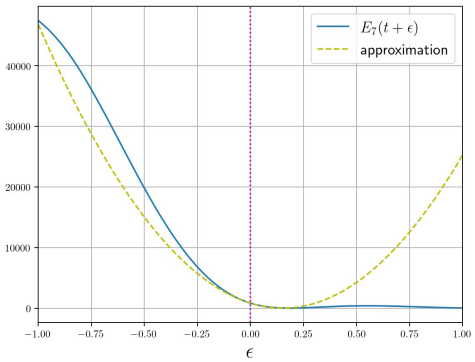




... +

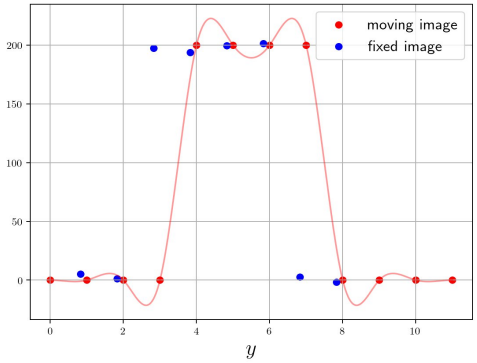
pixel 7

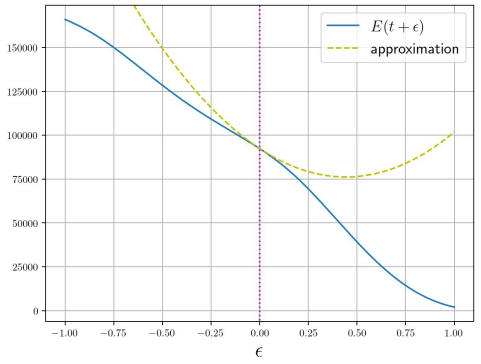




... + ... + ... =

all pixels





Energy:

$$E(t+\epsilon) = \sum_{n=1}^{N} E_n(t+\epsilon)$$

with

$$E_n(t+\epsilon) = \left[\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t+\epsilon))\right]^2$$

$$\simeq \left[\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t)) - g_n \cdot \epsilon\right]^2$$

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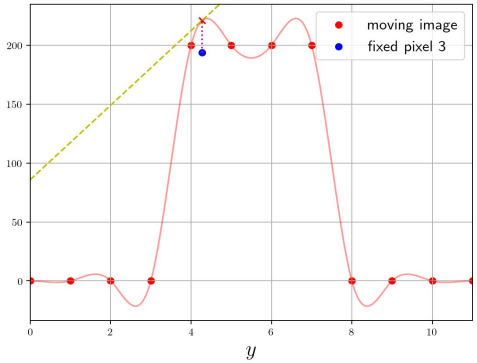
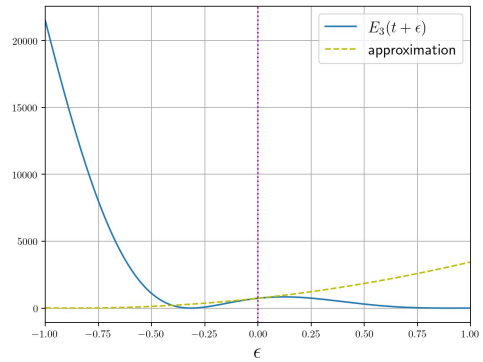
after 2 iterations

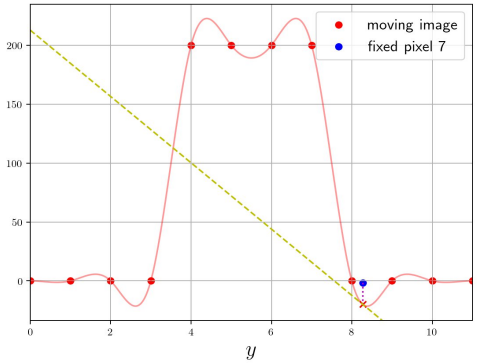
pixel 3

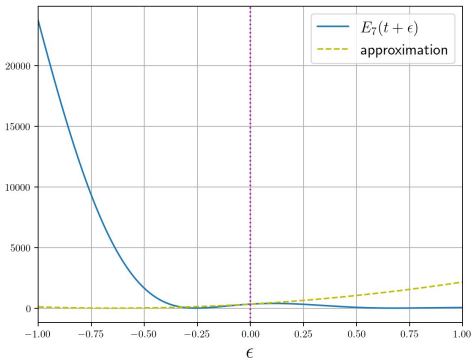




... +

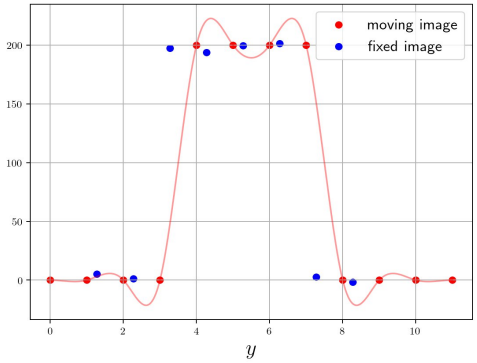
pixel 7

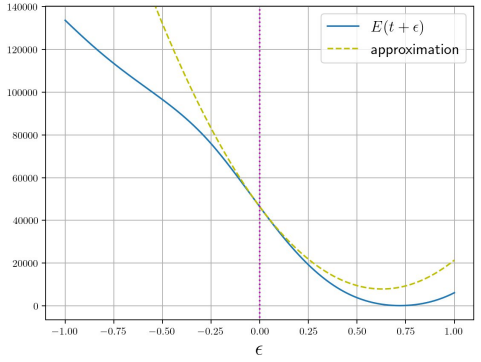




... + ... + ... =

all pixels





Energy: $E(t+\epsilon) = \sum_{n=1}^{N} E_n(t+\epsilon)$ with

$$E_n(t+\epsilon) = \left[\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t+\epsilon))\right]^2$$
$$\simeq \left[\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t)) - g_n \cdot \epsilon\right]^2$$

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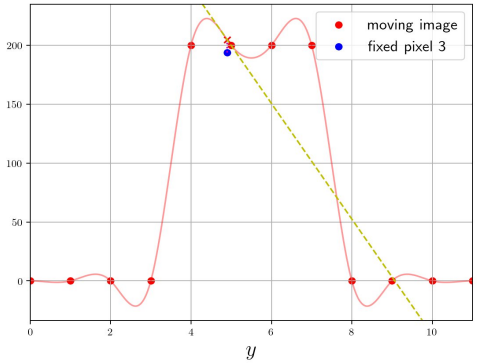
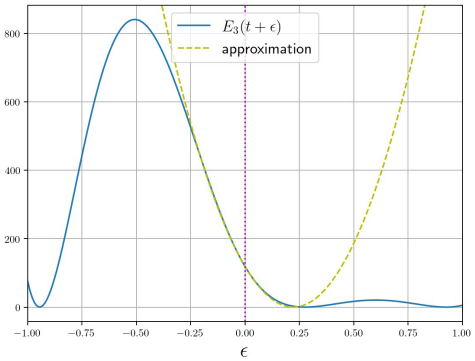
after 3 iterations

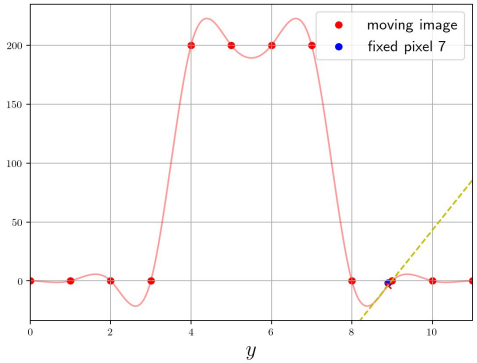
pixel 3

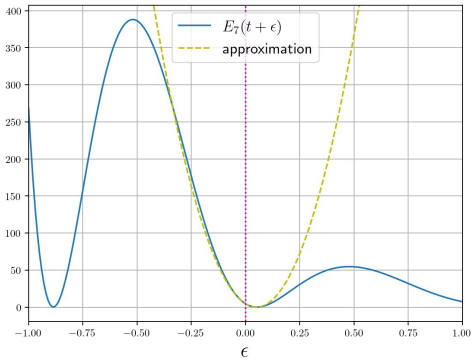




... +

pixel 7





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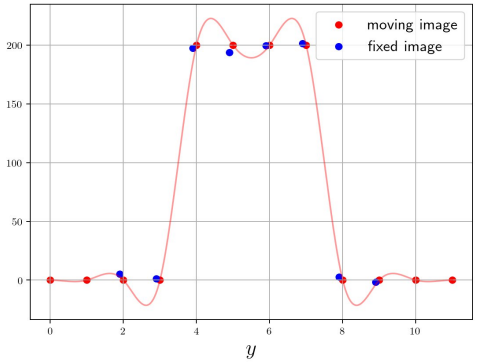
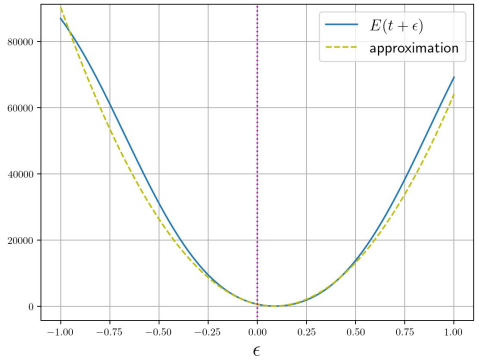
all pixels





Energy:

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$$\simeq \left[\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t)) - g_n \cdot \epsilon\right]^2$$

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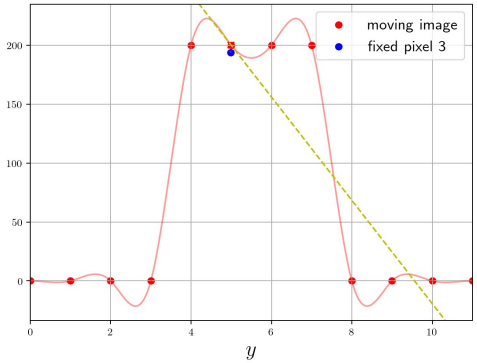
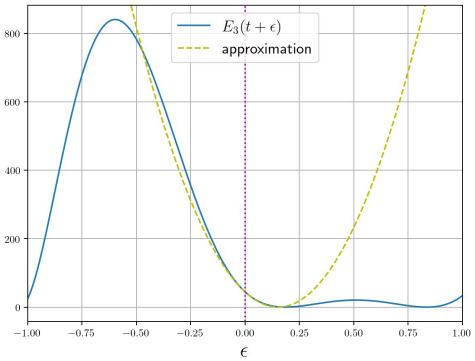
after 4 iterations

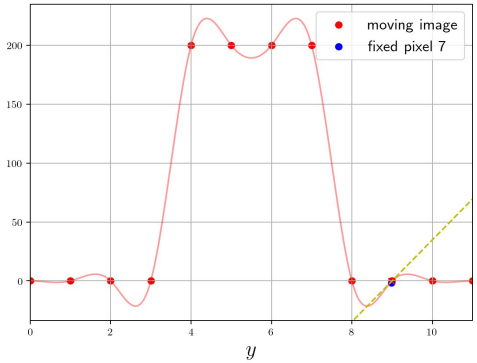
pixel 3

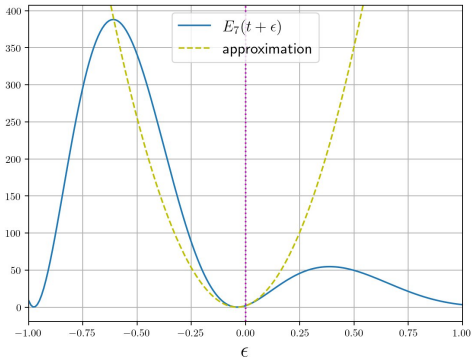




... +

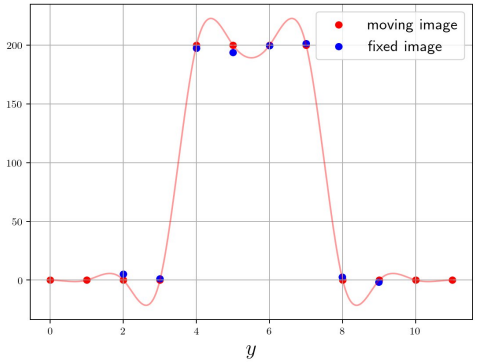
pixel 7

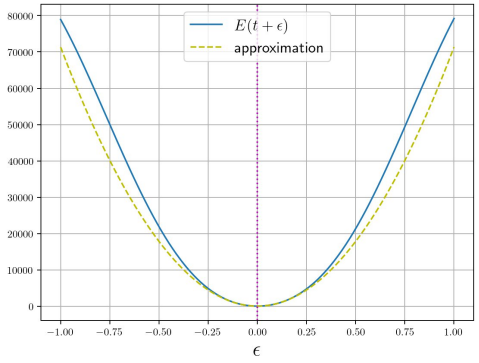




... + ... + ... =

all pixels





Energy: $E(t+\epsilon) = \sum_{n=1}^{N} E_n(t+\epsilon)$ with

$$E_n(t+\epsilon) = \left[\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t+\epsilon))\right]^2$$
$$\simeq \left[\mathcal{F}(x_n) - \mathcal{M}(y(x_n, t)) - g_n \cdot \epsilon\right]^2$$

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Gauss-Newton optimization

Solution: standard linear regression!

$$\epsilon = (\boldsymbol{\psi}^T \boldsymbol{\psi})^{-1} \boldsymbol{\psi}^T \boldsymbol{\tau}$$

Now update t:

$$t \leftarrow t + \epsilon$$

where $\boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_N \end{pmatrix}$ with $\tau_n = \mathcal{F}(x_n) - \mathcal{M}(y(x_n, t))$

and $\psi = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix} = \begin{pmatrix} g_1 & & \\ & \ddots & \end{pmatrix}$

one basis function

Gauss-Newton optimization

Solution: standard linear regression!

$$\epsilon = (\boldsymbol{\psi}^T \boldsymbol{\psi})^{-1} \boldsymbol{\psi}^T \boldsymbol{\tau}$$

Now update t:

$$t \leftarrow t + \epsilon$$

where $\boldsymbol{\tau} = \begin{pmatrix} \tau_1 \\ \vdots \\ \tau_N \end{pmatrix}$ with $\tau_n = \mathcal{F}(x_n) - \mathcal{M}(y(x_n, t))$

and $\boldsymbol{\psi} = \begin{pmatrix} g_1 \\ \vdots \\ g_N \end{pmatrix} = \underbrace{\begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_N \end{pmatrix}}_{\mathbf{G}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

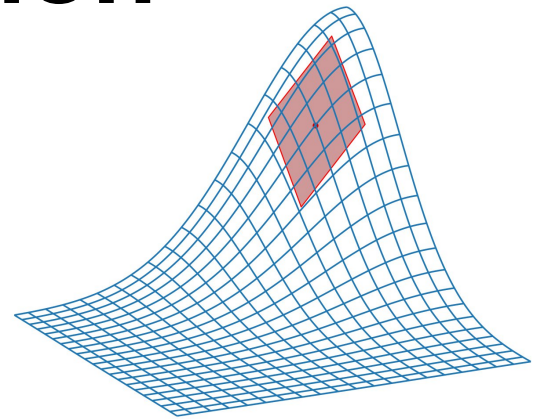
one basis function

Gauss-Newton optimization

In real situations, exactly the same idea but:

$D > 1$ spatial dimensions

$M > 1$ basis functions



$$\mathcal{M}(\mathbf{y}(\mathbf{x}_n, \mathbf{w} + \boldsymbol{\epsilon})) \simeq \mathcal{M}(\mathbf{y}(\mathbf{x}_n, \mathbf{w})) + \sum_{d=1}^{D} \sum_{m=0}^{M} \left(g_{d,n} \phi_m(\mathbf{x}_n)\right) \epsilon_{d,m}.$$

Solution: where

$$\boldsymbol{\epsilon} = \left(\boldsymbol{\Psi}^{\mathrm{T}} \boldsymbol{\Psi}\right)^{-1} \boldsymbol{\Psi}^{\mathrm{T}} \boldsymbol{\tau}$$

and

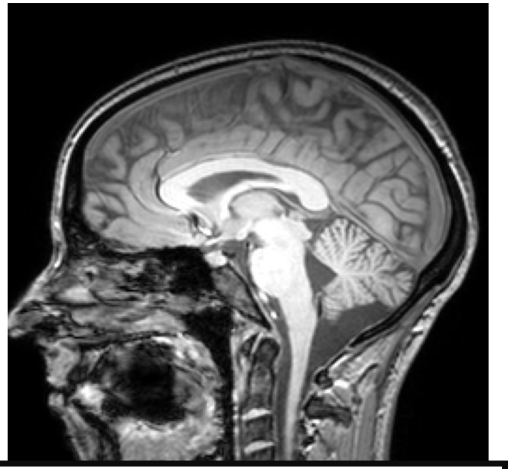
$$\boldsymbol{\Psi} = \left(\begin{array}{c|c|c|c} \mathbf{G}_1 \boldsymbol{\Phi} & \cdots & \mathbf{G}_D \boldsymbol{\Phi} \end{array} \right)$$

$$\boldsymbol{\Phi} = \begin{pmatrix} \phi_0(\mathbf{x}_1) & \phi_1(\mathbf{x}_1) & \cdots & \phi_{M-1}(\mathbf{x}_1) \\ \phi_0(\mathbf{x}_2) & \phi_1(\mathbf{x}_2) & \cdots & \phi_{M-1}(\mathbf{x}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_0(\mathbf{x}_N) & \phi_1(\mathbf{x}_N) & \cdots & \phi_{M-1}(\mathbf{x}_N) \end{pmatrix}$$

initialization

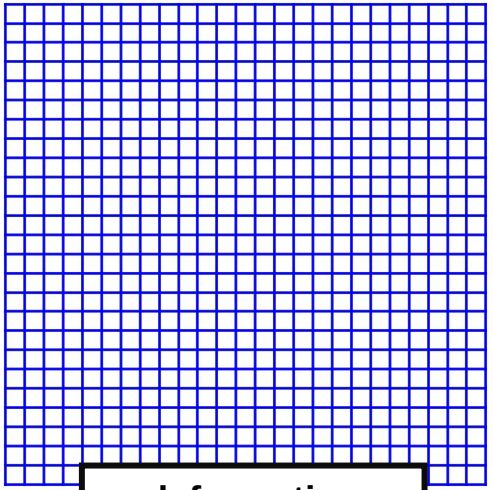
fixed image

$\mathcal{F}(\mathbf{x})$



interpolated moving image

$\mathcal{M}(\mathbf{y}(\mathbf{x}, \mathbf{w}))$



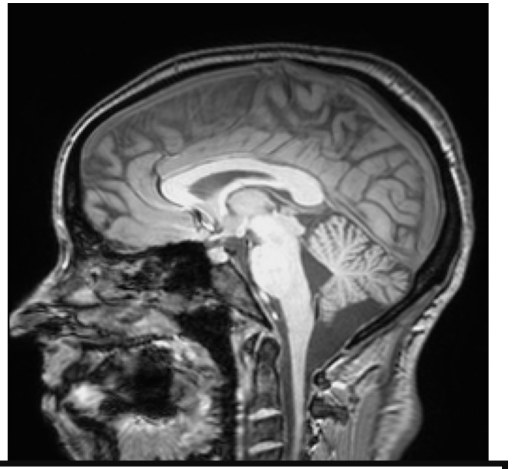
deformation

$= \mathrm{x} + \delta(\mathrm{x}, \mathrm{w})$

after 10 iterations

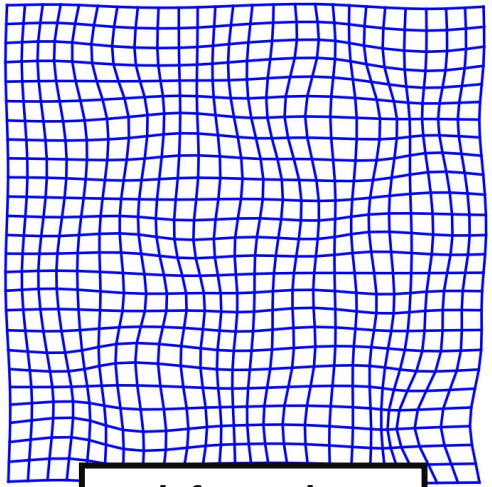
fixed image

$\mathcal{F}(\mathbf{x})$



interpolated moving image

$\mathcal{M}(\mathbf{y}(\mathbf{x}, \mathbf{w}))$



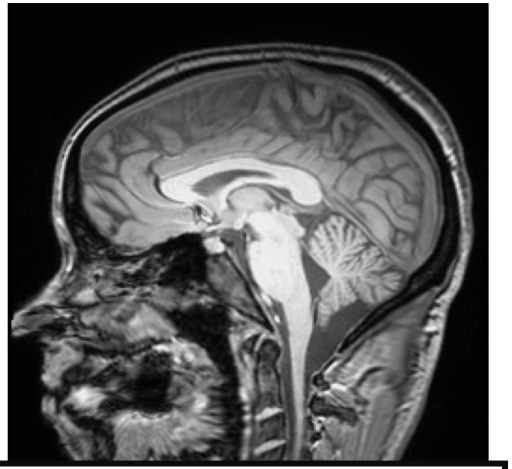
deformation

$= \mathrm{x} + \delta(\mathrm{x}, \mathrm{w})$

after 30 iterations

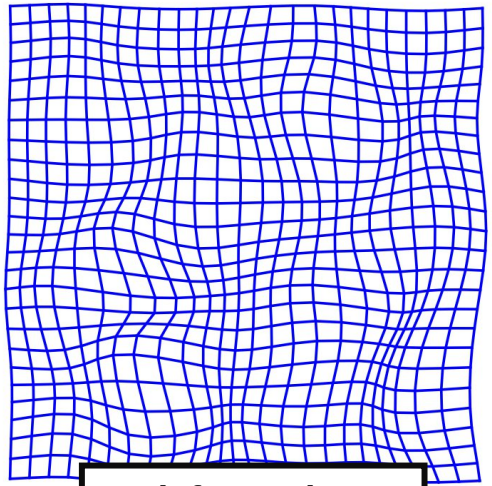
fixed image

$\mathcal{F}(\mathbf{x})$



interpolated moving image

$\mathcal{M}(\mathbf{y}(\mathbf{x}, \mathbf{w}))$



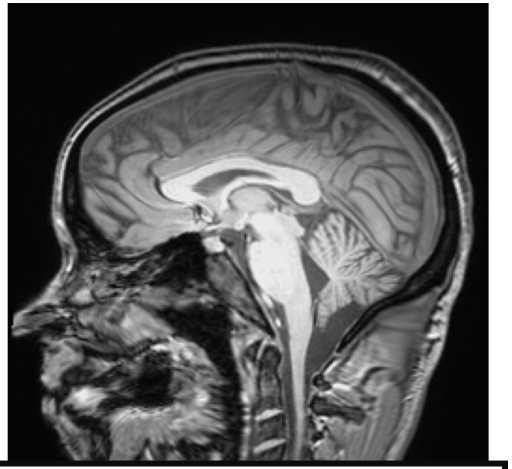
deformation

$= \mathrm{x} + \delta(\mathrm{x}, \mathrm{w})$

after convergence

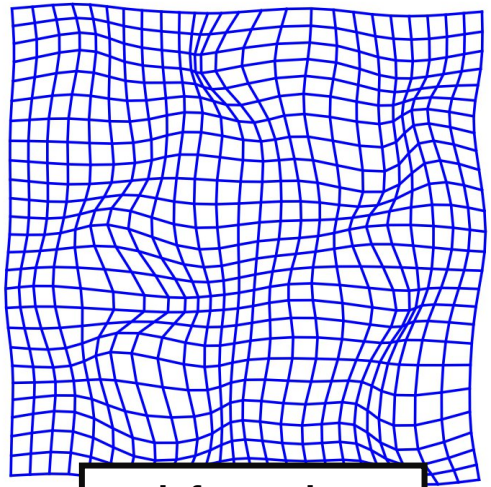
fixed image

$\mathcal{F}(\mathbf{x})$



interpolated moving image

$\mathcal{M}(\mathbf{y}(\mathbf{x}, \mathbf{w}))$



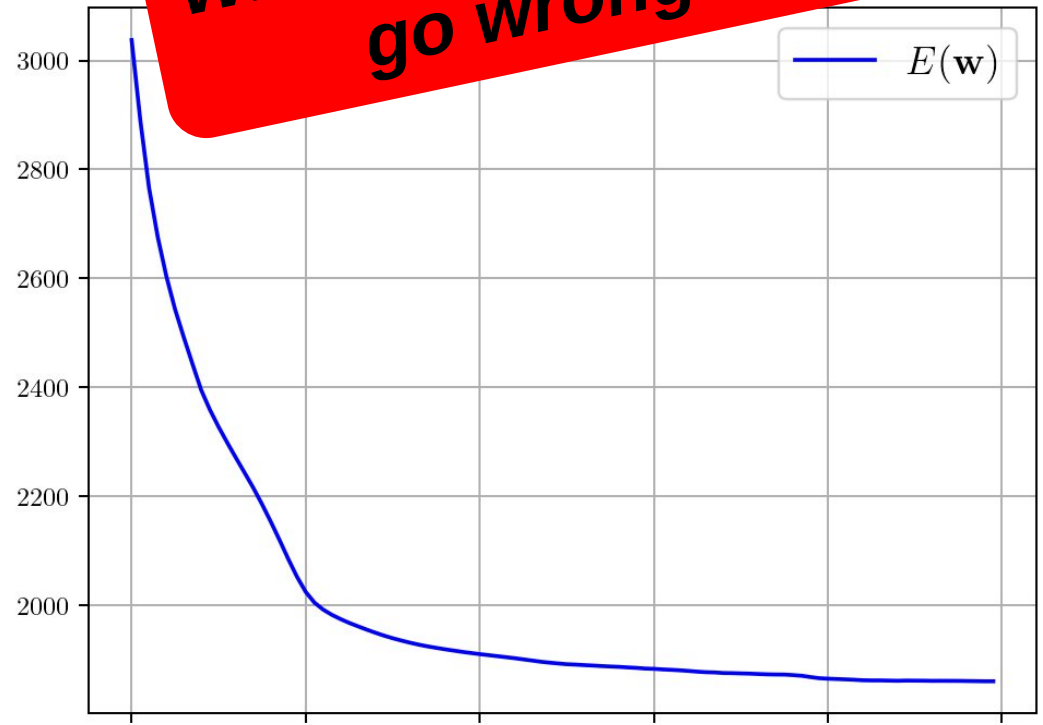
deformation

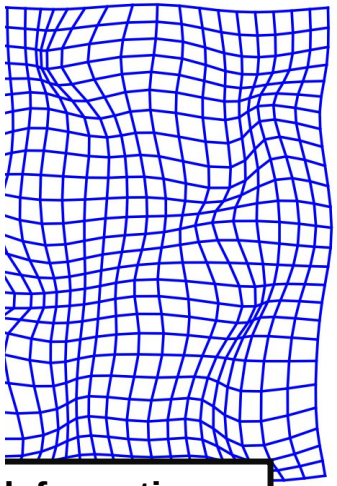
$= \mathrm{x} + \delta(\mathrm{x}, \mathrm{w})$

What could possibly go wrong?

fixed i

$\mathcal{F}(\mathbf{x}$





leformation

$= \mathrm{x} + \delta(\mathrm{x}, \mathrm{w})$

Safety rails...

We have assumed that ϵ is small

What if it isn't? Energy $E(\mathbf{w})$ could go *up* instead of *down*!

Solution: Levenberg–Marquardt

$$\boldsymbol{\epsilon} = \left(\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\Psi} + \lambda\mathbf{I}\right)^{-1}\boldsymbol{\Psi}^{\mathrm{T}}\boldsymbol{\tau},$$

tunable