Please discuss in your buzz group

We have seen that approximating a N-dimensional measurement vector $\mathbf{y}$ with a regression model $\mathbf{\Phi}\boldsymbol{\beta}$, where $\mathbf{\Phi}$ is a $N \times M$ matrix of M basis functions stored in its columns, and β is a M-dimensional parameter vector, can be obtained by minimizing

$$\|\mathbf{y} - \mathbf{\Phi}\boldsymbol{\beta}\|^2 + \lambda\boldsymbol{\beta}^T\mathbf{\Theta}\boldsymbol{\beta},$$

where $\mathbf{\Theta}$ is a regularization matrix and λ is a regularization strength.

Here we will consider the special case where there is no regularization (e.g., λ is set to zero), so that the regression problem reduces to minimizing

$$\|\mathbf{y} - \mathbf{\Phi}\boldsymbol{\beta}\|^2.$$

As we have seen, the solution to this problem is given by

$$\hat{\boldsymbol{\beta}} = \left(\mathbf{\Phi}^T\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^T\mathbf{y},$$

so that $\mathbf{y}$ is approximated by

$$\hat{\mathbf{y}} = \mathbf{\Phi}\hat{\boldsymbol{\beta}} = \mathbf{\Phi}\left(\mathbf{\Phi}^T\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^T\mathbf{y} = \mathbf{H}\mathbf{y},$$

where $\mathbf{H} = \mathbf{\Phi}\left(\mathbf{\Phi}^T\mathbf{\Phi}\right)^{-1}\mathbf{\Phi}^T$ is a $N \times N$ matrix that transforms $\mathbf{y}$ into $\hat{\mathbf{y}}$.

Task 1

Consider the case

$$\mathbf{y} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

What is $\hat{\boldsymbol{\beta}}$?

Task 2

In the same case

$$\mathbf{y} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

what is $\mathbf{H}$, what is the number of degrees of freedom $\mathrm{trace}(\mathbf{H})$, and what is $\hat{\mathbf{y}}$? Can you explain (e.g., draw) what is happening?

Task 3

Now consider the slightly harder problem

$$\mathbf{y} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{\Phi} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}.$$

Again: what is $\mathbf{H}$, what is the number of degrees of freedom $\mathrm{trace}(\mathbf{H})$, and what is $\hat{\mathbf{y}}$? Can you explain (e.g., draw) what is happening?