Lecture 3.

In one direction the theorem is obvious. Any composition $(f,g)_w$; $f,g \in R$, is $=0$ in the algebra A. If it reduces to a nontrivial combination of irreducible words then these irreducible words are not linearly independent modulo $id(R)$.

The other direction is: suppose that any composition $(f,g)_w$; $f,g \in R$, reduces to 0. Then all irreducible words are linearly independent modulo $id(R)$. We will prove this assertion later. Now we will consider many examples.

Example 1 (Weyl Algebra). $A = F\langle x,y \mid yx-xy=1\rangle$.

Let $x < y$.

Leading monomial: yx.

It does not admit compositions with itself, so, no composition.

Irr = {words, not containing yx as subwords} = $\{x^i y^j \,;\; i,j \geq 0\}$.

This is a basis of the algebra A.

<u>Example 2.</u> $A = F\langle x,y,z \mid zx - xz = y,\ yz - zy = 2z,\ yx - xy = -2x\rangle$.

Let $x < y < z$.

<u>Reduction System</u>:

$$\begin{cases} zx \to xz + y, \\ zy \to yz - 2z, \\ yx \to xy - 2x \end{cases}$$

There is only one composition: zy and yx, $w = zyx$.

Setting $f = zy - yz + 2z$, $g = yx - xy + 2x$, we have

$$(f,g)_w = (zy - yz + 2z)x - z(yx - xy + 2x)$$

$$= -yzx + 2zx + zxy - 2zx =$$

$$= -y\underline{zx} + \underline{zx}y \to -y(xz+y) + (xz+y)y$$

$$= -yxz - y^2 + xzy + y^2 = -\underline{yx}z + x\underline{zy}$$

$$\to -(xy - 2x)z + x(yz - 2z) =$$

$$= -xyz + 2xz + xyz - 2xz = 0.$$

<u>Irreducible words</u>: not having zx, zy, yx as subwords $= \{x^i y^j z^k \mid i,j,k \geq 0\}$, a basis of A.

<u>Example 3</u> (<u>order matters</u>). $A = F\langle x, y \mid y^2x - xyx = $

If $x < y$ then y^2x is the leading monomial.

It does not admit compositions with itself, so no compositions.

Now let $y < x$. Then xyx is the leading monomial $\overline{xy\underline{x}}\underline{yx}$

$$(xyx - y^2x)yx - xy(xyx - y^2x) =$$

$$= -y^2\underline{xyx} + xy^3x \to -y^4x + xy^3x.$$

Both words y^4x and xy^3x are irreducible

4. Important applications: Lie, commutative and graded algebras.

Lie Algebras.

A *Lie algebra* is a vector space L over F with a binary bilinear product $L \times L \to L$, $a \times b \to [a, b]$ such that

$$\begin{cases} [a,b] = -[b,a] \\ [[a,b],c] + [[b,c],a] + [[c,a],b] = 0 \end{cases}$$

for arbitrary elements $a,b,c \in L$.

<u>Example</u>. Let A be an associative F-algebra. Keep the structure of the F-vector space on A and define a new product

$$[a,b] = ab - ba.$$

Then A becomes a Lie algebra which is denoted as $A^{(-)} = (A, [\ ,\ ])$.

If $A = M_n(F)$ is the algebra of $n \times n$ matrices over F then $M_n(F)^{(-)}$ is a very important Lie algebra denoted as $gl_n(F)$.

The subspace

$$sl_n(F) = \{ a \in M_n(F) \,|\, tr(a) = 0 \}$$

of A is closed under commutators, hence $sl_n(F)$ is a Lie subalgebra of $gl_n(F) = A^{(-)}$.

The subspace of skew-symmetric matrices is also closed under commutators, hence a subalgebra of $sl_n(F)$.

Universal Enveloping Algebras.

Let L be a Lie algebra. A homomorphism $L \to A^{(-)}$, where A is an associative algebra is called a <u>representation</u> of the Lie algebra L.

A <u>morphism</u> from a representation $L \xrightarrow{\varphi} A^{(-)}$ to a representation $L \xrightarrow{\psi} B^{(-)}$

is a homomorphism of associative algebras $A \xrightarrow{x} B$ such that the diagram

$$\begin{tikzcd} & A \arrow[d, "x"] \\ L \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & B \end{tikzcd}$$

is commutative.

<u>Theorem I.4.1.</u> For an arbitrary Lie algebra L there exists a unique (up to isomorphism) universal representation $L \xrightarrow{u} U^{(-)}$, such that for any representation $L \xrightarrow{\varphi} A^{(-)}$ there exists a unique morphism
$$\begin{tikzcd} & U \arrow[d, "x"] \\ L \arrow[ur, "u"] \arrow[r, "\varphi"'] & A \end{tikzcd}.$$

<u>Proof.</u> Let us first show that any two universal representations are

isomorphic.

Let $u: L \to U^{(-)}$ be a universal representati-on. We claim that the associative algebra U is generated by the image $u(L)$. Indeed, let $\langle u(L) \rangle$ be the subalgebra of the algebra U generated by $u(L)$. Then

$$\bar{u}: L \xrightarrow{u} \langle u(L) \rangle$$

is also a representation. By universality of $L \xrightarrow{u} U$ there exists a homomorphism $\chi: U \to \langle u(L) \rangle$ of associative algebras that is identical on $u(L)$. We have $\chi^2 = \chi$. Hence $U = \langle u(L) \rangle \oplus \operatorname{Ker} \chi$ is a direct sum of ideals. If $\operatorname{Ker} \chi \neq (0)$ then we have two morphisms

$$\begin{tikzcd} & U \arrow[d, shift right, "\chi_1"'] \arrow[d, shift left, "\chi_2"] \\ L \arrow[ur, "u"] \arrow[dr, "u"'] & \\ & U \end{tikzcd}$$

$, \chi_1 = id_U$, χ_2 is identical on $\langle u(L) \rangle$, $\chi_2(\operatorname{Ker}\chi) = (0)$, a contradiction.

We showed that the algebra U is generate by $u(L)$.

If $L \xrightarrow{u_1} U_1^{(-)}$, $L \xrightarrow{u_2} U_2^{(-)}$ are universal representations then there exist morphism $\varphi : U_1 \to U_2$, $\psi : U_2 \to U_1$ such that $\psi\varphi$ is identical on $u_1(L)$, hence $\psi\varphi = id_{U_1}$. Similarly $\varphi\psi = id_{U_2}$. Hence φ, ψ are isomorphisms.

Now we will prove existence of a universal representation.

Let e_i, $i \in I$, be a basis of the vector space L. Let

$$[e_i, e_j] = \sum_k \gamma_{ij}^{\ k} e_k \ , \ \gamma_{ij}^{\ k} \in F.$$

Consider the alphabet $X = \{x_i, i \in I\}$ and the subset

$$R = \{ x_i x_j - x_j x_i - \sum_k \gamma_{ij}^{\ k} x_k \mid i, j \in I \}$$

of $F\langle X \rangle$. Consider the associative algebra

$$U = F\langle X \mid R = 0 \rangle = F\langle X \rangle / id(R).$$

The mapping

$$u : e_i \to x_i + id(R) \in U \ , \ i \in I,$$

extends to a representation $u: L \to U^{(-)}$.

Indeed, we only need to check that

$$u([e_i, e_j]) = [u(e_i), u(e_j)]$$

$$LHS = u\left(\sum_k \gamma_{ij}^{\ k} e_k\right) = \sum_k \gamma_{ij}^{\ k} x_k + id(R)$$

$$RHS = [x_i + id(R), x_j + id(R)] = x_i \cdot x_j - x_j \cdot x_i$$

$$+ id(R) = \sum_k \gamma_{ij}^{\ k} x_k + id(R).$$

Let us show that the representation u is universal.

Let A be an associative algebra and let $\varphi: L \to A$ be a representation.

We claim that the mapping

$\chi : x_i + id(R) \to \varphi(e_i) \in A \ , \ i \in I,$

extends to a homomorphism of associative algebras $U \to A$.

We know the presentation of the algebra U: $\quad U = F\langle X \mid R = 0 \rangle$.

~~By the Proposition~~ We only need to check that the mapping χ "respects" the defining relations.

Choose a defining relation

$$x_i x_j - x_j x_i - \sum_k \gamma_{ij}^k x_k = 0$$

We need to check that

$$\varphi(e_i)\varphi(e_j) - \varphi(e_j)\varphi(e_i) - \sum_k \gamma_{ij}^k \varphi(e_k) = 0$$

in A.

this equality follows from the fact that φ is a Lie algebra homomorphism

$$\varphi([e_i, e_j]) = \varphi(e_i)\varphi(e_j) - \varphi(e_j)\varphi(e_i)$$

$$\shortparallel$$

$$\varphi\Big(\sum_k \gamma_{ij}^k e_k\Big) = \sum_k \gamma_{ij}^k \varphi(e_k).$$

this completes the proof of the Proposition.

The algebra $U = U(L)$ is called the universal enveloping algebra of the Lie algebra L.

<u>Theorem</u> (<u>Poincare</u>-<u>Birkhoff</u>-<u>Witt</u>, in short: PBW). Let L be a Lie algebra with a basis $\{e_i, i \in I\}$, the set of indices I is equipped with an order $<$

satisfying minimality condition. ~~then the set of products 1, u~~ Let $u : L \to U(L)$ be the universal representation of L. Then the set of ordered products $1, u(e_{i_1}) \cdots u(e_{i_r})$, $i_1 \leq \cdots \leq i_r$, is a basis of $U(L)$.

__Proof__. Let $[e_i, e_j] = \sum_k \gamma_{ij}^k e_k$. The scalars $\gamma_{ij}^k \in F$ are called __structural constants__.

The algebra $U(L)$ has a presentation

$F\langle x_i, i \in I \mid R = 0 \rangle$, $R = \{x_i x_j - x_j x_i - \sum_k \gamma_{ij}^k x_k\}$.

The set R is closed with respect to compositions.

Indeed, denote $\{x_i, x_j\} = \sum_k \gamma_{ij}^k x_k$.

Consider defining relations

$$f = x_i x_j - x_j x_i - \{x_i, x_j\},\ g = x_j x_k - x_k x_j - \{x_j, x_k\} \quad ; \quad k < j < i.$$

that admit a composition, $w = x_i x_j x_k$

$$(f,g)_w = (x_i x_j - x_j x_i - \{x_i, x_j\}) x_k -$$

$$- x_i (x_j x_k - x_k x_j - \{x_j, x_k\}) = - x_j \underline{x_i x_k}$$

$$- \{x_i, x_j\} x_k + \underline{x_i x_k} x_j + x_i \{x_j, x_k\}$$

$$\to - x_j (x_k x_i + \{x_i, x_k\}) - \{x_i, x_j\} x_k$$

$$+ (x_k x_i + \{x_i, x_k\}) x_j + x_i \{x_j, x_k\}$$

$$= - \underline{x_j x_k} x_i - x_j \{x_i, x_k\} - \{x_i, x_j\} x_k$$

$$+ x_k \underline{x_i x_j} + \{x_i, x_k\} x_j + x_i \{x_j, x_k\}$$

$$\to -\left(x_k x_j + \{x_j, x_k\}\right) x_i - x_j \cdot \{x_i, x_k\}$$

$$-\{x_i, x_j\} x_k + x_k \left(x_j \cdot x_i + \{x_i, x_j\}\right) +$$

$$\{x_i, x_k\} x_j + x_i \cdot \{x_j, x_k\} =$$

$$= -\{x_j, x_k\} x_i - x_j \{x_i, x_k\} - \{x_i, x_j\} x_k$$

$$+ x_k \{x_i, x_j\} + \{x_i, x_k\} x_j + x_i \{x_j, x_k\} =$$

$$[x_i, \{x_j, x_k\}] - [x_j, \underbrace{\{x_i, x_k\}}_{= -\{x_k, x_i\}}] +$$

$$+ [x_k, \{x_i, x_j\}] =$$

$$= [x_i, \{x_j, x_k\}] + [x_j, \{x_k, x_i\}] + [x_k, \{x_i, x_j\}]$$

$$\to \{x_i, \{x_j, x_k\}\} + \{x_j, \{x_k, x_i\}\} + \{x_k, \{x_i, x_j\}\}$$

This expression $= 0$ because it has the same coefficients as

$[e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] = 0$.

We checked that the set R is closed with respect to compositions.

Leading monomials in R: $x_i x_j$, where $i > j$.

<u>Irreducible words</u>: $x_{i_1} x_{i_2} \cdots x_{i_r}$, $i_1 \le i_2 \le \cdots \le i_r$

Now it remains to use Theorem .

<u>Corollary</u>. For an arbitrary Lie algebra L the universal representation

$u: L \to U(L)$ is injective.

Indeed, $u: e_i \to x_i + id(R)$.

The words x_i are irreducible, hence

they are linearly independent modulo $id(R)$. If $a = \sum_i \alpha_i e_i \xrightarrow{u} 0$ then $\sum \alpha_i x_i = 0 \mod id(R)$, hence $\alpha_i = 0$, $a = 0$.

Since the homomorphism u is injective we can identify L with $u(L)$. Then $L \subset U(L)$ and

$$1, e_{i_1} e_{i_2} \cdots e_{i_r}, \quad i_1 \leq i_2 \leq \ldots \leq i_r, \; r \geq 1,$$

is a basis of $U(L)$.