Lecture 1.

Chapter I. Generators & Relations.

1. Free semigroups. Let X be a nonempty set. We will call X an alphabet and each of its elements a letter in the alphabet.

A word in X is either a finite sequence of letters written juxtaposed next to one another or the empty word

For example, if $X = \{x_1, x_2, \dots\}$ then $x_3 x_4 x_3 x_1$ is a word in X.

We denote the set of all words in the alphabet X as X^*.

Then X^* is a multiplicative semigroup with multiplication defined by concatenation of words.

Example. Let $v = x_3 x_4 x_1$, $w = x_3$.

Then $v \cdot w = x_3 x_4 x_1 x_3$, $w \cdot v = x_3^2 x_4 x_1$.

Proposition I.1.1 Let S be a semigroup. An arbitrary mapping $X \xrightarrow{\varphi} S$ uniquely extends to a homomorphism $X^* \to S$.

Proof. Given $\varphi : X \to S$ define $\overline{\varphi}$ to sending a word $v = x_{i_1} x_{i_2} \cdots x_{i_k}$ to

$$\overline{\varphi}(v) = \underbrace{\varphi(x_{i_1}) \cdot \varphi(x_{i_2}) \cdots \varphi(x_{i_k})}_{\text{here multiplication in } S}$$

<u>Def.</u> Let S be a semigroup. An equivalence relation $\sim \subseteq S \times S$ is called a <u>congruence</u> if $a \sim b$, $c \sim d \Rightarrow ac \sim bd$.

<u>In</u> this case we can define the semigroup on $S/\sim = \{$equivalence classes$\}$. Moreover $S \to S/\sim$, $a \to a/\sim$ is a homomorphism of semigroups.

Given a homomorphism $\varphi : S_1 \to S_2$ of semigroups

$a \sim b$ if and only if $\varphi(a) = \varphi(b)$

is a congruence.

Congruence = analog of a normal subgroup in GROUPS and an ideal in RINGS.

Suppose that a semigroup S is generated by a subset $\{s_i, i \in I\} \subset S$.

Consider the alphabet

$$X = \{x_i , i \in I\}$$

and the mapping

$$\varphi : x_i \to s_i , \quad i \in I.$$

The mapping φ extends to an epimorphism

$$\overline{\varphi} : X^* \to S$$

Let $a \sim b$ iff $\overline{\varphi}(a) = \overline{\varphi}(b)$; $a, b \in X^*$.

Then $$S \cong X^*/\sim$$

So,

every semigroup is a homomorphic image of a free semigroup of an appropriate rank.

Remember: $\sim \subset X^* \times X^*$

We say that $R \subset \sim$ generates $\sim$ if $\sim$ is the smallest congruence that contains $\sim$, so

$\sim = \bigcap$ (all congruences on X^* that contain R)

Then R uniquely determines $\sim$ and, hence, uniquely determines the semigroup S up to isomorphism.

Let $R = \{a_j \times b_j\}_{j \in J} \subset X^* \times X^*$; a_j, b_j are words. We write:

$S = \langle X \mid a_j = b_j,\ j \in J \rangle$.

why is it important to have a nice presentation by generators and relations?

Let S be a semigroup and let $s_1, \dots, s_n$ be a set of generators of S.

Let T be another semigroup.

Not every mapping $s_i \to T$, $1 \le i \le n$, extends to a homomorphism $S \to T$.
How can we find out if it extends or not?
Suppose that we know a presentation of the semigroup S in these generators.

$$S = \langle x_1, \dots, x_n \mid a_1(x) = b_1(x), \dots, a_m(x) = b_m(x) \rangle$$

It means the following:

let $\sim$ be the congruence ~~that~~ on X^* that corresponds to the homomorphism $x_i \to s_i$, $1 \le i \le n$. The congruence $\sim$ is generated by $a_1 \times b_1, \dots, a_m \times b_m$.

<u>Proposition I.1.2.</u> Let $\varphi : s_i \to t_i \in T$, $1 \le i \le n$ be a mapping. This mapping extends to a homomorphism $S \to T$ if and only if $a_i(t_1, t_2, \dots, t_n) = b_i(t_1, \dots, t_n)$, $1 \le i \le m$.

<u>Proof</u>. In one direction the assertion is clear: if φ extends to a homomorphism then $a_i(t_1, t_2, \dots, t_n) = b_i(t_1, \dots, t_n)$.

Now suppose that these equalities

hold. Consider the homomorphisms

$$\begin{tikzcd} & Y^* \arrow[dl, "y_i \to s_i"', "\sim"] \arrow[dr, "y_i \to t_i", "\sim'"'] & \\ S & & T \end{tikzcd}$$

Let $\sim'$ be the congruence that corresponds to the homomorphism $y_i \to t_i$, $1 \le i \le n$.

We have $a_i(t_1, \dots, t_n) = b_i(t_1, \dots, t_n)$, $1 \le i \le n$, hence

hence $a_i(Y) \sim' b_i(Y)$, $1 \le i \le n$, hence $\sim'$ contains $a_i \times b_i$, $1 \le i \le n$. Hence

$$\sim \,\subseteq\, \sim'$$

This implies that the mapping

$$u(s_1, \dots, s_n) \to u(t_1, \dots, t_n), \quad u \in Y^*$$

is well defined.

Indeed, if $u(s_1,\dots,s_n) = v(s_1,\dots,s_n)$ then $u \sim v$. This implies ~~$u(t_1,\dots,t_n) = v(t_1,\dots$~~ that $u \sim' v$, i.e. $u(t_1,\dots,t_n) = v(t_1,\dots t_n)$.
This mapping is a homomorphism.
This completes the proof of the Proposition.

Equalities $a_j = b_j$, $j \in J$, are called <u>defining relations</u>.

Let v, w be words in the alphabet X. We say that <u>w is obtained from v by substitution</u> if some word a_j is a subword of v, $v = v' a_j v''$ and $w = v' b_j v''$ or some word b_j is a subword of v, $v = v' b_j v''$ and $w = v' a_j v''$.

In this case we write $v \to w$.

This relation is symmetric, if $v \to w$, then $w \to v$.

<u>Proposition</u> I.1.3. Let $S = \langle X \mid a_j = b_j, j \in J \rangle$.

words v, w are equal in S if and only if there is a finite sequence

$$v = v_0 \to v_1 \to v_2 \to \cdots \to v_k = w.$$

<u>Proof.</u> Let $\sim$ be the congruence in X^* generated by all elements $a_j \times b_j$, $j \in J$. We need to prove that $v \sim w$ if and only if there is a finite sequence

$$v = v_0 \to v_1 \to v_2 \to \cdots \to v_k = w.$$

Define another congruence $\approx$ as follows: $v \approx w$ if and only if there exists a finite sequence

$$v = v_0 \to v_1 \to v_2 \to \cdots \to v_k = w.$$

It is easy to see that $\approx$ is a congruence, All elements $a_j \times b_j$, $j \in J$, belong to $\tilde{\sim}$. Since $\sim$ is a minimal congruence containing all $a_j \times b_j$, $j \in J$, we conclude that

$$\sim \subseteq \tilde{\sim}$$

On the other hand if

$$v = v_0 \to v_1 \to \dots \to v_k = w$$

then $v \sim v_1, v_1 \sim v_2, \dots, v_{k-1} \sim w$, hence

$$v \sim w.$$

In other words, $\tilde{\sim} \subseteq \sim$. We proved that

$$\sim = \tilde{\sim}.$$

A semigroup S is called <u>finitely presented</u> if it has a finite presentation

$$S \cong \langle x_1, \dots, x_n \mid a_1 = b_1, \dots, a_m = b_m \rangle,$$

$a_i, b_i \in X^*$.

It means that S is generated by a finite subset ~~$a_1, \ldots$~~ $s_1, \ldots, s_n$ and the conguence on X^*, $X = \{x_1, \ldots, x_n\}$, defined by the homomorphism $x_i \to s_i$, $1 \le i \le n$, is finitely generated.

<u>Proposition</u> I.1.4 Let S be a semigroup. Let $\{s_1, \ldots, s_n\}$ and $\{s'_1, \ldots, s'_k\}$ be finite generating subsets of S. If S is finitely presented in $\{s_1, \ldots, s_n\}$ then it is finitely presented also in $\{s'_1, \ldots, s'_k\}$.

<u>Proof</u>. Let $\sim$ be the congruence on $\{x_1, \ldots, x_n\}^*$ that corresponds to the homomorphism $x_i \to s_i$, $1 \le i \le n$.

Let $\sim'$ be the congruence on $\{y_1,\dots,y_k\}^*$ that corresponds to the homomorphism $y_j \to s_j'$, $1\le j\le k$.

Let $\sim$ be generated by the relations

$$a_\mu(x_1,\dots,x_n)\sim b_\mu(x_1,\dots,x_n),$$

μ runs over a finite set of numbers.

Let $s_i = c_i(s_1',\dots,s_k')$, $1\le i\le n$, $c_i\in\{y_1,\dots,y_k\}^*$

~~We claim that the congruence~~ $\cancel{\sim'}$ ~~is generated by~~

$$\cancel{a_\mu(c_1(y_1,\dots,y_k),\dots,c_n(y_1,\dots,y_k))\sim'}$$
$$\cancel{b_\mu(c_1(y_1,\dots,y_k),\dots,c_n(y_1,\dots,y_k)).}$$

~~First, we notice that~~

Let $s_j' = d_j(s_1, \dots, s_n)$, $1 \le j \le k$, $d_j \in \{x_1, \dots, x_n\}^*$.

Then

$$a_\mu(c_1(y_1, \dots, y_k), \dots, c_n(y_1, \dots, y_k)) \sim' \qquad (\mathrm{I})$$
$$b_\mu(c_1(y_1, \dots, y_k), \dots, c_n(y_1, \dots, y_k)),$$

$$y_j \sim' d_j(c_1(y_1, \dots, y_k), \dots, c_n(y_1, \dots, y_k)) \qquad (\mathrm{II})$$

We claim that the congruence $\sim'$ is generated by (I) and (II).

In other words: if $\sim''$ is a congruence on Y^* and

$$a_\mu(c_1(Y), \dots, c_n(Y)) \sim'' b_\mu(c_1(Y), \dots, c_n(Y)),$$
$$y_j \sim'' d_j(c_1(Y), \dots, c_n(Y))$$

then $\sim' \subseteq \sim''$. Let $T = Y^*/\sim''$.

Consider the mapping $s_i \to C_i(Y)/\sim''$.

By the Proposition this mapping extends to a homomorphism $\varphi : S \to T$.

Since $s_j' = d_j(s_1, \dots, s_n)$ ~~and~~ we have

$\varphi(s_j') = d_j(C_1(Y), \dots, C_n(Y))/\sim'' = y_j/\sim''$.

Now, if $u(y_1, \dots, y_k) \sim' v(y_1, \dots, y_k)$ i.e.

$u(s_1', \dots, s_k') = v(s_1', \dots, s_k')$ then

$u(y_1, \dots, y_k) \sim'' v(y_1, \dots, y_k)$. We proved that

$\sim' \subseteq \sim''$.