Digraph Clustering by Marginal Propagation

CPS 521-321 Duke University
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Haoran Zhang[1]

[1] Duke University, USA

Outline

Introduction

Community within graphs features both **stronger similarities and differences** within and between other communities.

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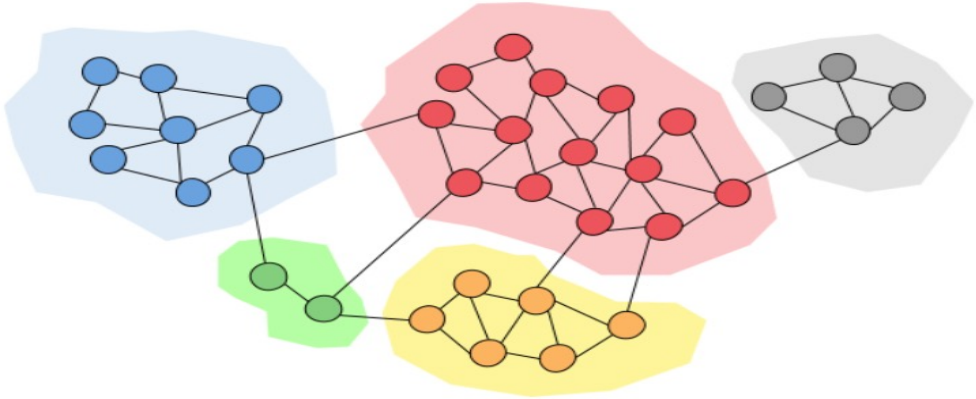

Introduction & Method

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1. Remove the self-loops, isolated nodes

Algorithm 1: Remove the self-loops, isolated nodes in the graph (MATLAB)

Data: Graph Adjacency Matrix
Result: Self-loops, isolated nodes free adjacency matrix
(1) Load the data and obtain adjacency matrix as wells as graph object

```
A = Problem.A;
G = digraph(A);
```

(2) Remove the self-loops for all the nodes

```
diag = diag(diag(A));
A = A - diag;
```

(3) Remove the isolated nodes (indegree + outdegree = 0)

```
outdeg = indegree(dig_A);
indeg = outdegree(dig_A);
mixdeg = indeg + outdeg;
zero_pos = find(~mixdeg);
G = rmnode(G, zero_pos);
A = adjacency(G);
```

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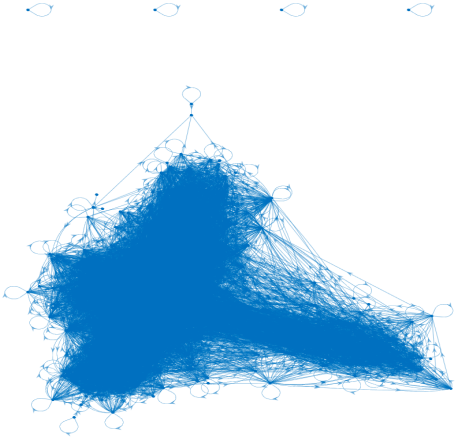

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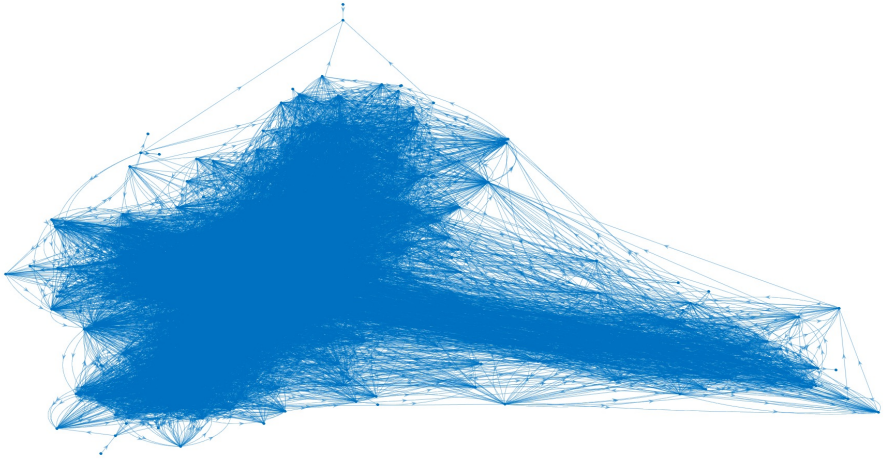

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Algorithm 2: Generate connectivity adjacency matrix after N steps/hops (MATLAB)

Data: Adjacency matrix without isolated nodes from **Algorithm 1**
Result: Pairwise connectivity matrix from one to N step/hops
(1) Calculate the connectivity matrix after N steps/hops

```
A_org = A;
A_sum = A_org;
while N>0 do
    A = A*A_org;
    A_sum = A_sum + A;
    N = N - 1;
end
```

(2) Get the graph corresponding to the matrix

```
G = digraph(A_sum);
```

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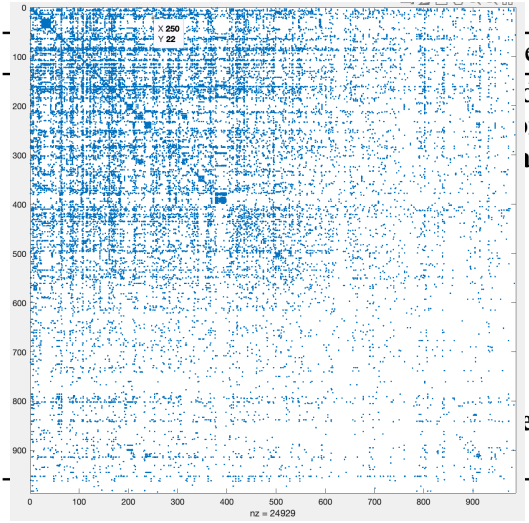

ency matrix after N steps/hops (MATLAB)

d nodes from **Algorithm 1**

m one to N step/hops

fter N steps/hops

e matrix

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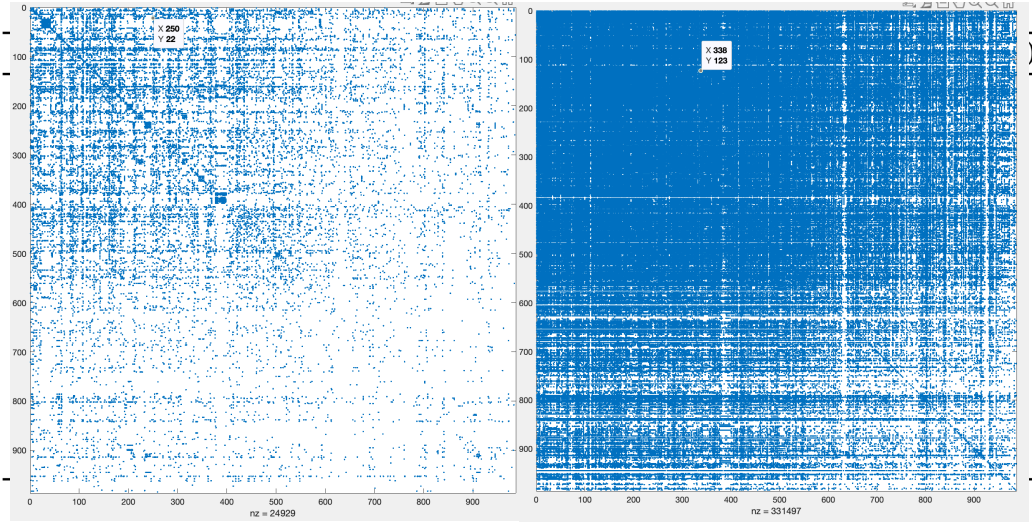

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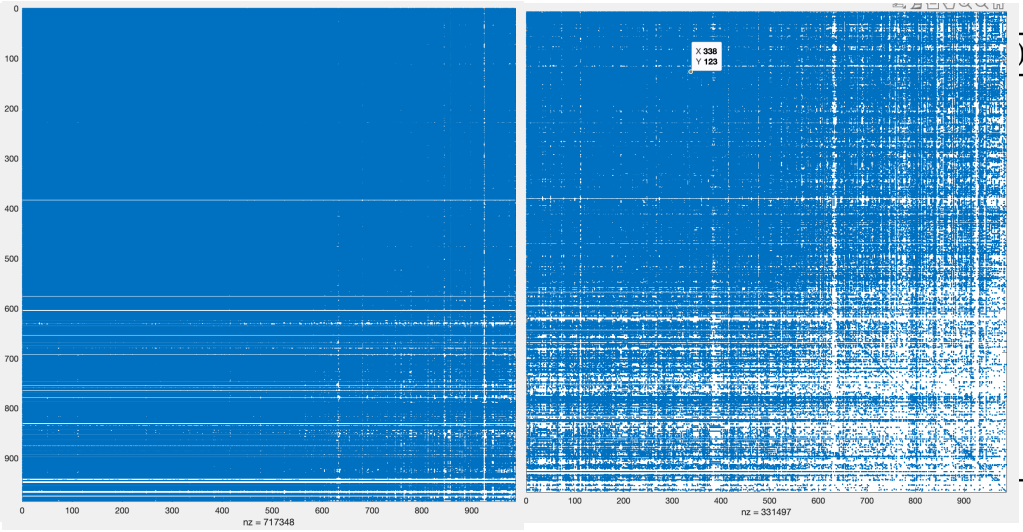

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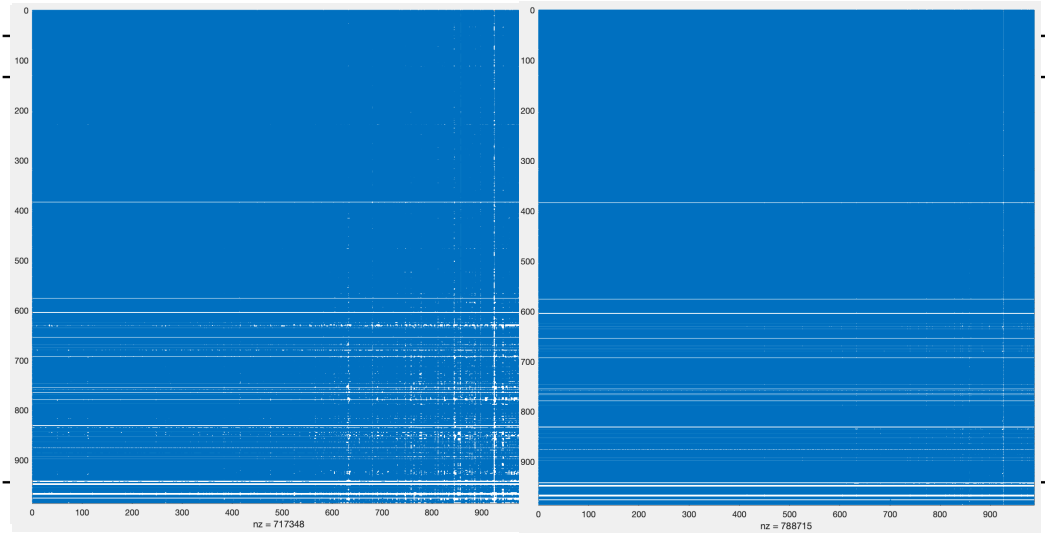

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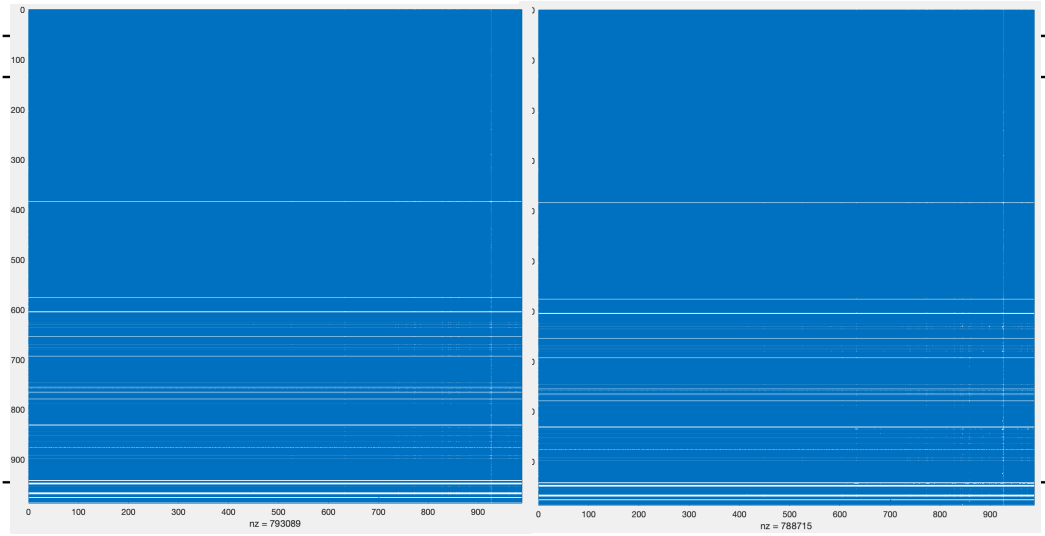

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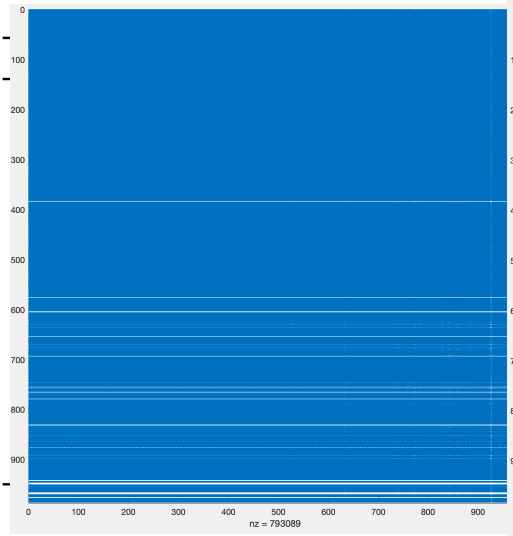

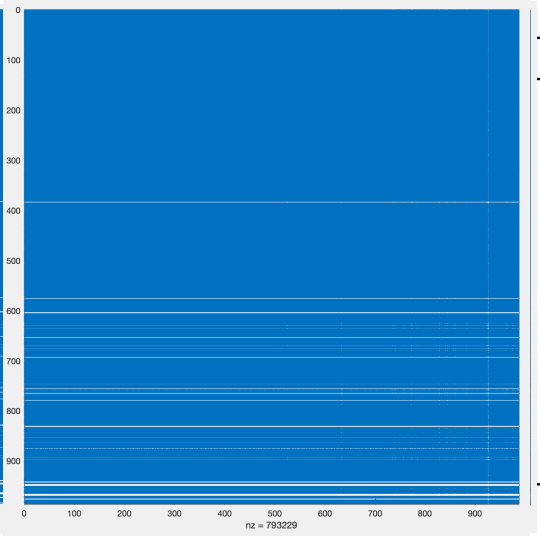

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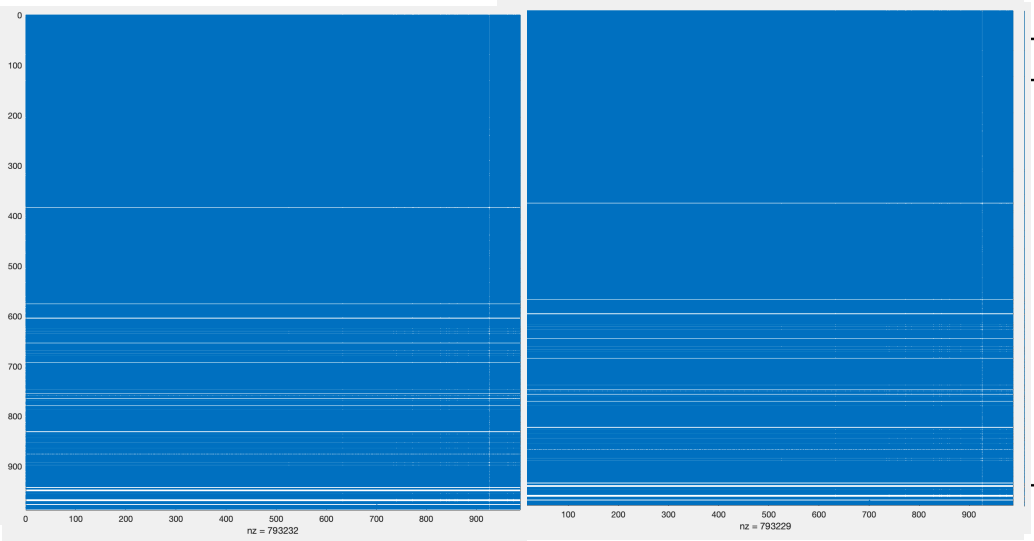

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3. Obtain zero values in both degree and whole matrix (after select best N)

Algorithm 3: Find the positions of zeros in both types of degree and whole matrix

Data: Pairwise connectivity matrix from one to N steps/hops from **Algorithm 2**
Result: Positions of zero values in degrees and whole matrix

```
A_Connectivity = (A_sum ~= 0);
A_Connectivity_graph = digraph(A_Connectivity);
InDegree = outdegree(A_Connectivity_graph);
OutDegree = indegree(A_Connectivity_graph);
zeros = find(~A_Connectivity);
```

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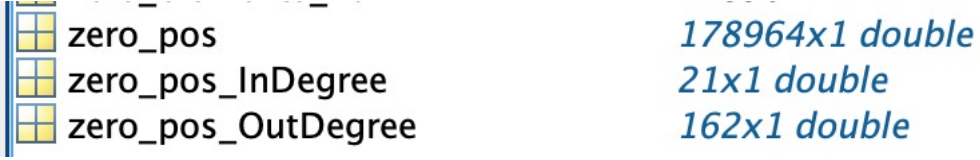

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```
>> zero_pos_InDegree.'

ans =

  Columns 1 through 11

   525   633   735   740   757   772   774   785   828   840   845

  Columns 12 through 21

   857   861   883   923   925   926   961   964   974   977
```

zero_pos	178964x1 double
zero_pos_InDegree	21x1 double
zero_pos_OutDegree	162x1 double

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4. Filter unique mutually disconnected nodes

Algorithm 4: Find unconnected nodes indexes after N steps/hops (MATLAB)

```
Data: Indexes of zeroes from Algorithm 3
Result: Mutual disconnected nodes indexes after N steps/hops
(1) Create all zeros matrix with same shape of A
Disc_matrix = zeros(length(A_1), length(A_1));
(2) Calculate the x and y coordinate of zero node
for i = 1:length(zero_pos) do
    x = ceil(zero_pos(i)/length(A_1));
    y = zero_pos(i) - (x-1)*length(A_1);
    if ismember(y, zero_pos_InDegree) | ismember(x, zero_pos_OutDegree) then
        continue;
    else
        Disc_matrix(x,y) = 1;
    end
end
(3) Keep only symmetric nodes in the matrix (i.e. when the disconnectivity is mutual)
total_zeros = sum(Disc_matrix, 'all');
Disc_matrix_sym = floor((Disc_matrix + Disc_matrix.')/2);
L = tril(Disc_matrix_sym);
L_total_num = sum(L, 'all');
Disc_node_id = find(L);
```

Algorithm 5: Record the positions of disconnected nodes and obtain unique node id list (MATLAB)

Data: Unconnected node indexes from **Algorithm 4**
Result: Indexes of unique nodes that are mutually unconnected

```
unique_disconnected_node = [0];
index = 1;
for i = 1:length(Dis_node_id) do
    x = ceil(Dis_node_id/length(A));
    y = Dis_node_id(i) - (x-1)*length(A);
    unique_disconnected_node(index) = x;
    unique_disconnected_node(index+1) = y;
    index = index + 2;
end
unique_nodes = unique(unique_disconnected_node);
```

The total number of communities assumed to be *c*:

```
c = length(unique_nodes);
```

Algorithm 5: Record the positions of disconnected nodes and obtain unique node id list (MATLAB)

```
unique_disconnected_node =

  Columns 1 through 11

   203   415   433   464   559   568   617   632   683   724   728

  Columns 12 through 22

   730   818   820   834   880   885   902   905   924   940   944

  Column 23

   985
```

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4. Filter unique mutually disconnected nodes
5. Label the marginal nodes as different communities and propagate

Algorithm 6: Assign nodes into communities and update their probabilities (MATLAB)

Data: Unique disconnected nodes from **Algorithm 5** and adjacency matrix A
Result: Probabilities of each node belong to Community from 1 to c
(1) Store the probabilities of each nodes belonging to community in a $N \times C$ matrix:

```
c_label = 1;
c_prob = zeros(length(A), length(unique_nodes));
for y = 1:length(A) do
    if ismember(y, unique_nodes) then
        c_prob(y, c_label) = 1.0
    else
        c_prob(y, :) = c_prob(y, :) + 1.0/length(unique_nodes);
    end
end
```

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(2) Add community probabilities to neighbours proportional to its out-degree

```
EPOCHS = E;
for epochs = 1:EPOCHS do
    for node = length(A):-1:1 do
        sucs_nodes = successor(G, node);
        infl_prob = c_(node, :)
        for i = 1:length(sucs_nodes) do
            sucs_node = sucs_nodes(i);
            if ismember(sucs_node, unique_nodes) then
                continue;
            else
                c_prob(sucs_node, :) = c_prob(sucs_node, :) +
                 1.0*infl_prob/length(sucs_nodes)
            end
        end
    end
    for node = length(A):-1:1 do
        c_prob(node, :) = c_prob(node, :) / sum(c_prob(node, :));
    end
end
```

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198	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
199	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0

```
y = 203 Assign community label 1
y = 415 Assign community label 2
y = 433 Assign community label 3
y = 464 Assign community label 4
y = 559 Assign community label 5
y = 568 Assign community label 6
y = 617 Assign community label 7
y = 632 Assign community label 8
y = 683 Assign community label 9
y = 724 Assign community label 10
```

```
y = 728 Assign community label 11
y = 730 Assign community label 12
y = 818 Assign community label 13
y = 820 Assign community label 14
y = 834 Assign community label 15
y = 880 Assign community label 16
y = 885 Assign community label 17
y = 902 Assign community label 18
y = 905 Assign community label 19
y = 924 Assign community label 20
y = 940 Assign community label 21
y = 944 Assign community label 22
y = 985 Assign community label 23
```

212	0.0435	0.0435	0.0435	0.0435	0.04
213	0.0435	0.0435	0.0435	0.0435	0.04

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198	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
199	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
200	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
201	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
202	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
203	1	0	0	0	0	0	0	0	0	
204	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
205	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
206	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
207	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
208	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
209	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
210	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
211	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
212	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0
213	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0.0435	0

(3) We normalize all the probabilities to make their sums equal to one

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counts = hist(c_label_pred, $[1 : C]$);

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```
>> Community_Prob_A(1,:)

ans =

  Columns 1 through 8

    0.0416    0.0630    0.0432    0.0355    0.0500    0.0354    0.0356    0.0630

  Columns 9 through 16

    0.0384    0.0372    0.0649    0.0500    0.0477    0.0421    0.0334    0.0376

  Columns 17 through 23

    0.0570    0.0349    0.0331    0.0441    0.0391    0.0305    0.0426
```

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    0.0384    0.0372    0.0649    0.0500    0.0477    0.0421    0.0334    0.0376

  Columns 17 through 23

    0.0570    0.0349    0.0331    0.0441    0.0391    0.0305    0.0426
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3. Obtain zero values in both degree and whole matrix (after select best N)
4. Filter unique mutually disconnected nodes
5. Label the marginal nodes as different communities and propagate
6. Calculate modularity scores on predicted communities

$$Modularity: \quad Q = \frac{1}{2m} \sum_{ij} (A_{ij} - P_{ij}) \, \delta \left(C_i, C_j \right)$$

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The δ-function returns one if vertices i and j are in the same community ($C_i = C_j$), zero otherwise.

Result

The test dataset for this project is:

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Table 1: Statistics for the test dataset Email EU Core

Dataset Statistics			
Nodes	1005	Edges in largest SCC	24729 (0.967)
Edges	25571	Average clustering coefficient	0.3994
Number of communities	42	Number of triangles	105461
Nodes in largest WCC	986 (0.981)	Fraction of closed triangles	0.1085
Edges in largest WCC	25552 (0.999)	Diameter (longest shortest path)	7
Nodes in largest SCC	803 (0.799)	90-percentile effective diameter	2.9

Result

Predicted Community Distribution:

Result

Predicted Community Distribution:

Table 2: Community Label distribution between the predicted and ground truth for Email EU Core

Dataset	label	#	label	#	Dataset	label	#	label	#	label	#	label	#
Email EU Core	**1**	263	**13**	8	Ground Truth	**1**	111	**13**	27	**25**	12	**37**	3
	2	135	**14**	8		**2**	107	**14**	26	**26**	10	**38**	3
	3	122	**15**	7		**3**	91	**15**	25	**27**	10	**39**	2
	4	82	**16**	5		**4**	56	**16**	24	**28**	9	**40**	1
	5	80	**17**	4		**5**	54	**17**	22	**29**	9	**41**	1
	6	42	**18**	4		**6**	49	**18**	19	**30**	8	**42**	0
	7	40	**19**	4		**7**	39	**19**	18	**31**	8		
	8	39	**20**	3		**8**	34	**20**	15	**32**	6		
	9	38	**21**	2		**9**	31	**21**	13	**33**	6		
	10	35	**22**	2		**10**	29	**22**	13	**34**	5		
	11	35	**23**	1		**11**	29	**23**	13	**35**	4		
	12	27				**12**	28	**24**	12	**36**	4		

Result

Predicted Community Distribution:

Modularity Score!

Table 2: Community Label distribution between the predicted and ground truth for Email EU Core

Dataset	label	#	label	#	Dataset	label	#	label	#	label	#	label	#
Email EU Core	**1**	263	**13**	8	Ground Truth	**1**	111	**13**	27	**25**	12	**37**	3
	2	135	**14**	8		**2**	107	**14**	26	**26**	10	**38**	3
	3	122	**15**	7		**3**	91	**15**	25	**27**	10	**39**	2
	4	82	**16**	5		**4**	56	**16**	24	**28**	9	**40**	1
	5	80	**17**	4		**5**	54	**17**	22	**29**	9	**41**	1
	6	42	**18**	4		**6**	49	**18**	19	**30**	8	**42**	0
	7	40	**19**	4		**7**	39	**19**	18	**31**	8		
	8	39	**20**	3		**8**	34	**20**	15	**32**	6		
	9	38	**21**	2		**9**	31	**21**	13	**33**	6		
	10	35	**22**	2		**10**	29	**22**	13	**34**	5		
	11	35	**23**	1		**11**	29	**23**	13	**35**	4		
	12	27				**12**	28	**24**	12	**36**	4		

Abundant Features

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Marginal Propagation outputs a N × C matrix with:

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Membership Similarity

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Marginal Propagation outputs a N × C matrix with:

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Membership Similarity & Community Similarity

Similarity Formula

$$Membership\ Similarity\ M_S = V_1 \cdot V_2$$

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Overlapping Community Detection

Limitations

The number of communities outputted is restricted by the diameter being selected in Algorithm 2 (D/2)

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If the diameter of the graph is either too large or too short, this method's performance will be severely depreciated

Communities can be dominated after long iterations

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Thanks for your attention