1 Symmetrization of Rank-2 Tensors

Defn 1.1 (Symmetric and Antisymmetric Tensor)**.** A rank-2 tensor $S^{\mu\nu}$ is **symmetric** if it's the same under exchange of its indices μ and ν:

$$S^{\mu\nu} = S^{\nu\mu}$$

Examples: stress-energy tensor $\mathcal{T}^{\mu\nu}$; Kronecker-Delta $\delta^{\mu\nu}$; Lorentz metric $\eta^{\mu\nu}$.

A rank-2 tensor $A^{\mu\nu}$ is **antisymmetric** if it's the opposite under exchange of its indices:

$$A^{\mu\nu} = -A^{\nu\mu}$$

Examples: electromagnetic field tensor $F^{\mu\nu}$

Defn 1.2 (Symmetrization and Antisymmetrization)**.** Given an arbitrary rank-2 tensor $T^{\mu\nu}$, we may define its **symmetrization** as

$$\text{Sym } T^{\mu\nu} = T^{[\mu\nu]} = \frac{T^{\mu\nu} + T^{\nu\mu}}{2}$$

and its **antisymmetrization** as

$$\text{AntiSym } T^{\mu\nu} = T^{\{\mu\nu\}} = \frac{T^{\mu\nu} - T^{\nu\mu}}{2}$$

For the following corollaries, let $T^{\mu\nu}$ be an arbitrary tensor, $S^{\mu\nu}$ be an arbitrary symmetric tensor, and $A^{\mu\nu}$ be an arbitrary antisymmetric tensor.

Proposition 1.1 (Symmetric/Antisymmetric Decomposition)**.** Symmetrization and antisymmetrization form a complete, disjoint set of projection operators of the set of rank-2 tensors onto the sets of symmetric and antisymmetric rank-2 tensors respectively. That is, for any given tensor $T^{\mu\nu}$, we may uniquely write it terms of symmetric and antisymmetric components:

$$T^{\mu\nu} = T^{[\mu\nu]} + T^{\{\mu\nu\}}$$

Proof: We'll first show that Sym and AntiSym map tensors to symmetric and antisymmetric tensors respectively:

$$\begin{aligned}
\text{Sym } T^{\nu\mu} &= \frac{T^{\nu\mu} + T^{\mu\nu}}{2} = \text{Sym } T^{\mu\nu} \\
\text{AntiSym } T^{\nu\mu} &= \frac{T^{\nu\mu} - T^{\mu\nu}}{2} = -\text{AntiSym } T^{\mu\nu}
\end{aligned}$$

Now we'll show that Sym and AntiSym are projection operators:

$$\begin{aligned}
\text{Sym Sym } T^{\mu\nu} &= \text{Sym } \frac{T^{\mu\nu} + T^{\nu\mu}}{2} = \frac{1}{2}\left(\text{Sym } T^{\mu\nu} + \text{Sym } T^{\nu\mu}\right) \\
&= \frac{1}{2}\left(\text{Sym } T^{\mu\nu} + \text{Sym } T^{\mu\nu}\right) = \text{Sym } T^{\mu\nu} \\
\text{AntiSym AntiSym } T^{\mu\nu} &= \text{AntiSym } \frac{T^{\mu\nu} - T^{\nu\mu}}{2} = \frac{1}{2}\left(\text{AntiSym } T^{\mu\nu} - \text{AntiSym } T^{\nu\mu}\right) \\
&= \frac{1}{2}\left(\text{AntiSym } T^{\mu\nu} + \text{AntiSym } T^{\mu\nu}\right) = \text{AntiSym } T^{\mu\nu}
\end{aligned}$$

And that they're disjoint projections:

$$\begin{aligned}
\text{Sym AntiSym } T^{\mu\nu} &= \text{Sym } \frac{T^{\mu\nu} - T^{\nu\mu}}{2} = \frac{T^{\mu\nu} + T^{\nu\mu} - T^{\nu\mu} - T^{\mu\nu}}{4} = 0 \\
\text{AntiSym Sym } T^{\mu\nu} &= \text{AntiSym } \frac{T^{\mu\nu} + T^{\nu\mu}}{2} = \frac{T^{\mu\nu} - T^{\nu\mu} + T^{\nu\mu} - T^{\mu\nu}}{4} = 0
\end{aligned}$$

And finally, they're complete:

$$(\text{Sym } + \text{AntiSym })T^{\mu\nu} = \frac{T^{\mu\nu} + T^{\nu\mu}}{2} + \frac{T^{\mu\nu} - T^{\nu\mu}}{2} = T^{\mu\nu}$$

This should match your intuition for these words: there's no difference between a symmetric tensor and a symmetrized tensor, and antisymmetric tensors have no symmetric component (and mutatus mutandi for antisymmetric tensors).

For some more examples, here's rank-2 tensors made out of rank-1 tensors (vectors):

$$\begin{aligned}
\text{Sym } p^\mu q^\nu &= \frac{p^\mu q^\nu + p^\nu q^\mu}{2} \\
\text{AntiSym } p^\mu q^\nu &= \frac{p^\mu q^\nu - p^\nu q^\mu}{2} \\
\text{Sym } p^\mu p^\nu &= p^\mu p^\nu \\
\text{AntiSym } p^\mu p^\nu &= 0
\end{aligned}$$

Please not that multiplying by a scalar doesn't change whether a tensor is symmetric or antisymmetric or neither, so $p^\mu q^\nu + p^\nu q^\mu$ is still and symmetric tensor, it's just twice the symmetrization of $p^\mu q^\nu$.

Proposition 1.2 (Contraction Symmetric and Antisymmetric Tensors)**.** The full contraction of a symmetric tensor and antisymmetric tensor is zero.

Proof:

$$\begin{aligned}
(\text{Sym } T^{\mu\nu})(\text{AntiSym } T_{\mu\nu}) &= \frac{1}{4}\left(T^{\mu\nu} + T^{\nu\mu}\right)\left(T_{\mu\nu} - T_{\nu\mu}\right) \\
&= \frac{1}{4}\left(T^{\mu\nu}T_{\mu\nu} + T^{\nu\mu}T_{\mu\nu} - T^{\mu\nu}T_{\nu\mu} - T^{\nu\mu}T_{\nu\mu}\right) \\
&= \frac{1}{4}\left(T^{\mu\nu}T_{\mu\nu} + T^{\nu\mu}T_{\mu\nu} - T^{\nu\mu}T_{\mu\nu} - T^{\mu\nu}T_{\mu\nu}\right) \\
&= 0
\end{aligned}$$

Proposition 1.3 (Contraction of Arbitrary Tensors)**.** For two rank-2 tensors $T^{\mu\nu}$ and $U^{\mu\nu}$, their contraction is just the sum of the contractions of their (anti)symmetrized components:

$$T^{\mu\nu}U_{\mu\nu} = (\text{Sym } T^{\mu\nu})(\text{Sym } U_{\mu\nu}) + (\text{AntiSym } T^{\mu\nu})(\text{AntiSym } U_{\mu\nu})$$

Proof: Follows immediately from expanding the tensors into their symmetric and antisymmetric components, multiplying out the terms, and cancelling the cross terms by the previous propositions.

Proposition 1.4 (Contractions of Products of Vectors)**.** We often come across contractions of sets of symmetric or antisymmetric vectors. Here's some useful identities.

$$\begin{aligned}
(p^\mu q^\nu + p^\nu q^\mu)(a_\mu b_\nu + a_\nu b_\mu) &= 4(\text{Sym } p^\mu q^\nu)(\text{Sym } a_\mu b_\nu) \\
&= [(p\cdot a)(q\cdot b) + (p\cdot b)(q\cdot a) + (p\cdot b)(q\cdot a) + (p\cdot a)(q\cdot b)] \\
&= 2\left[(p\cdot a)(q\cdot b) + (p\cdot b)(q\cdot a)\right]
\end{aligned}$$

$$
\begin{aligned}
\left(p^\mu q^\nu - p^\nu q^\mu\right)\left(a_\mu b_\nu - a_\nu b_\mu\right) &= 4(\text{AntiSym } p^\mu q^\nu)(\text{AntiSym } a_\mu b_\nu) \\
&= \left[(p\cdot a)(q\cdot b) - (p\cdot b)(q\cdot a) - (p\cdot b)(q\cdot a) + (p\cdot a)(q\cdot b)\right] \\
&= 2\left[(p\cdot a)(q\cdot b) - (p\cdot b)(q\cdot a)\right]
\end{aligned}
$$

$$
\begin{aligned}
(p^\mu q^\nu + p^\nu q^\mu)(p_\mu q_\nu + p_\nu q_\mu) &= 4(\text{Sym } p^\mu q^\nu)(\text{Sym } p_\mu q_\nu) \\
&= 2\left[p^2 q^2 + (p\cdot q)^2\right]
\end{aligned}
$$

Note that because of symmetry, the only terms we can make is $(p\cdot a)(q\cdot b)$ and $(p\cdot b)(q\cdot a)$ up to signs and constant factors.

2 Decomposition of Antisymmetric Rank-2 Tensors

We'll be working in Minkowski space with the metric

$$\eta = \text{diag}(+,-,-,-)$$

Consider the (arbitrary) rank-2 antisymmetric tensor $F^{\mu\nu}$. It may be uniquely expressed by the following 3-vectors $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$
\begin{aligned}
F^{\mu\mu} &= 0 \\
F^{0i} &= E^i \\
F^{ij} &= -\epsilon^{ijk} B^k
\end{aligned}
$$

The $F^{\mu\nu}$ tensor can then be written as:

$$
F^{\mu\nu} = \begin{pmatrix} 0 & E^x & E^y & E^z \\ & 0 & -B^z & B^y \\ & & 0 & -B^x \\ & & & 0 \end{pmatrix}
$$

where the lower half is the negative of the upper half. We may also write the vector components in terms of the tensor:

$$
\begin{aligned}
E^i &= F^{0i} \\
B^i &= -\frac{1}{2}\epsilon^{ijk} F^{jk}
\end{aligned}
$$

The signs and normalizations are chosen to match the familiar electromagnetic field tensor, electric fields, and magnetic fields.

Proof: The time-like field is trivially satisfied by these equations:

$$E^i = F^{0i} = E^i$$

And the space-like field:

$$
\begin{aligned}
B^i &= -\frac{1}{2}\epsilon^{ijk} F^{jk} \\
&= -\frac{1}{2}\epsilon^{ijk}\left(-\epsilon^{jkl} B^l\right) \\
&= \frac{1}{2}(2\delta^{il}) B^l = B^i
\end{aligned}
$$

Proposition 2.1 (Contractions of Antisymmetric Tensors Into Contractions of 3-Vectors)**.** Consider two antisymmetric rank-2 $F^{\mu\nu}$ and $G^{\mu\nu}$ with decompositions into 3-vectors

$$\begin{aligned} F^{0i} &= a^i & G^{0i} &= \alpha^i \\ F^{ij} &= \epsilon^{ijk} b^k & G^{ij} &= \epsilon^{ijk} \beta^k \end{aligned}$$

Then the total contraction of these two tensors in terms of the three vectors is:

$$\begin{aligned} F^{\mu\nu} G_{\mu\nu} &= 2(a^i G_{0i} + b^k \frac{1}{2} \epsilon^{ijk} G_{ij}) \\ &= -2(\boldsymbol{a} \cdot \boldsymbol{\alpha} - \boldsymbol{b} \cdot \boldsymbol{\beta}) \end{aligned}$$

Proof:

$$\begin{aligned} F^{\mu\nu} G_{\mu\nu} &= F^{0i} G_{0i} + F^{i0} G_{i0} + F^{ij} G_{ij} \\ &= 2a^i G_{0i} + \epsilon^{ijk} b^k G_{ij} \\ &= 2(a^i G_{0i} + b^k \frac{1}{2} \epsilon^{ijk} G_{ij}) \\ &= 2(a^i(-\alpha^i) + b^k(\beta^k)) \\ &= -2(\boldsymbol{a} \cdot \boldsymbol{\alpha} - \boldsymbol{b} \cdot \boldsymbol{\beta}) \end{aligned}$$

Example: Consider the generators of the Lorentz algebra

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

which is an antisymmetric rank-2 tensor. This may be rewritten as two 3-vectors of generators

$$\begin{aligned} L^i &= \frac{1}{2} \epsilon^{ijk} J^{jk} \\ K^i &= J^{0i} \end{aligned}$$

and the Lorentz generators in terms of the 3-vectors

$$\begin{aligned} J^{0i} &= K^i \\ J^{ij} &= \epsilon^{ijk} L^k \end{aligned}$$

And so we have the component representation of the $J^{\mu\nu}$ tensor as

$$F^{\mu\nu} = \begin{pmatrix} 0 & K^x & K^y & K^z \\ & 0 & L^z & -L^y \\ & & 0 & L^x \\ & & & 0 \end{pmatrix}$$

Example: Consider the parameterization of the infinitesimal transformation of Lorentz transformations in the 1/2-representation:

$$\Lambda_{1/2} = \exp\left(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}\right) \approx 1 - \frac{i}{2}\omega_{\mu\nu} S^{\mu\nu}$$

Where $S^{\mu\nu}$ is an antisymmetric tensor

$$S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

Which in the Weyl basis we may decompose into

$$B^i = S^{0i} = \frac{i}{4}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$
$$\Sigma^i = \epsilon^{ijk} S^{jk}$$

and $\omega_{\mu\nu}$ is a set of infinitesimal parameters, which we decompose into two 3-vectors as above:

$$\omega^{0i} = \beta^i$$
$$\omega^{jk} = \epsilon^{ijk}\theta^k$$

And so the infinitesimal transformation is:

$$\Lambda_{1/2} \approx 1 + i\left(\boldsymbol{\beta} \cdot \boldsymbol{B} - \boldsymbol{\theta} \cdot \boldsymbol{\Sigma}\right) = \exp\left(i(\boldsymbol{\beta} \cdot \boldsymbol{B} - \boldsymbol{\theta} \cdot \boldsymbol{\Sigma})\right)$$