1 Coulomb Scattering

2 Bhabha Scattering

$$e^-(p) + e^+(k) \to e^-(p') + e^+(k')$$

The t-channel matrix element:

$$\begin{aligned}
-i\mathcal{M}_t &= [\overline{u}(p')(-ie\gamma^\mu)u(p)]\,\frac{-i\eta_{\mu\nu}}{(p-p')^2+i\epsilon}\,[\overline{v}(k)(-ie\gamma^\nu)v(k')] \\
&= \frac{ie^2}{t}\overline{u}(p')\gamma^\mu u(p)\overline{v}(k)\gamma_\mu v(k')
\end{aligned}$$

and the s-channel matrix element has a relative minus sign with the t-channel diagram:

$$\begin{aligned}
i\mathcal{M}_s &= [\overline{v}(k)(-ie\gamma^\mu)u(p)]\,\frac{-i\eta_{\mu\nu}}{s^2+i\epsilon}\,[\overline{u}(p')(-ie\gamma^\nu)v(k')] \\
&= \frac{ie^2}{s}\overline{v}(k)\gamma^\mu u(p)\overline{u}(p')\gamma_\mu v(k')
\end{aligned}$$

The sum-squared matrix elements are:

$$\begin{aligned}
|\mathcal{M}_t+\mathcal{M}_s|^2 &= |\mathcal{M}_t|^2 + 2\mathrm{Re}[\mathcal{M}_t^\dagger\mathcal{M}_s] + |\mathcal{M}_s|^2 \\
\sum_{spins}|\mathcal{M}_t|^2 &= \frac{e^4}{t^2}\mathrm{Tr}\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right]\mathrm{Tr}\left[\not{k}\gamma_\mu\not{k}'\gamma_\nu\right] \\
&= \frac{e^4}{t^2}4\left(p'^\mu p^\nu + p'^\nu p^\mu - (p\cdot p')\eta^{\mu\nu}\right)4\left(k_\mu k'_\nu + k_\nu k'_\mu - (k\cdot k')\eta_{\mu\nu}\right) \\
&= \frac{32e^4}{t^2}\left((p'\cdot k')(p\cdot k) + (p'\cdot k)(p\cdot k')\right) \\
&= 8e^4\frac{s^2+u^2}{t^2}
\end{aligned}$$

$$\begin{aligned}
\sum_{spins}|\mathcal{M}_s|^2 &= \frac{e^4}{s^2}\mathrm{Tr}\left[(\not{k}-m)\gamma^\mu(\not{p}+m)\gamma^\nu\right]\mathrm{Tr}\left[(\not{p}'+m)\gamma_\mu(\not{k}'-m)\gamma_\nu\right] \\
&= \frac{e^4}{s^2}4\left[k^\mu p^\nu + k^\nu p^\mu - (k\cdot p)\eta^{\mu\nu}\right]4\left[p'_\mu k'_\nu + p'_\nu k'_\mu - (p'\cdot k')\eta_{\mu\nu}\right] \\
&= \frac{32e^4}{s^2}\left[(k\cdot p')(p\cdot k') + (k\cdot k')(p\cdot p')\right] \\
&= 8e^4\frac{u^2+t^2}{s^2}
\end{aligned}$$

$$
\begin{aligned}
\sum_{spins} \mathcal{M}_t^\dagger \mathcal{M}_s &= -\frac{e^4}{ts} \mathrm{Tr}\left[\overline{v}(k')\gamma_\mu v(k)\overline{v}(k)\gamma^\nu u(p)\overline{u}(p)\gamma^\mu u(p')\overline{u}(p')\gamma_\nu v(k')\right] \\
&= -\frac{e^4}{ts} \mathrm{Tr}\left[\not{k}'\gamma_\mu \not{k}\gamma^\nu \not{p}\gamma^\mu \not{p}'\gamma_\nu\right] \\
&= -\frac{e^4}{ts}(-2) \mathrm{Tr}\left[\not{k}'\not{p}\gamma^\nu \not{k}\not{p}'\gamma_\nu\right] \\
&= -\frac{e^4}{ts}(-8)(k\cdot p') \mathrm{Tr}\left[\not{k}'\not{p}\right] \\
&= \frac{32e^4}{st}(k\cdot p')(k'\cdot p) \\
&= 8e^4\frac{u^2}{st}
\end{aligned}
$$

$$
\frac{1}{4}\sum_{spins}|\mathcal{M}_s+\mathcal{M}_t|^2 = 2e^4\left[\frac{u^2+t^2}{s^2}+2\frac{u^2}{st}+\frac{s^2+u^2}{t^2}\right]
$$

The differential cross-section is then

$$
\begin{aligned}
\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \frac{1}{(8\pi)^2}\frac{\overline{|\mathcal{M}|^2}}{s} \\
&= \frac{e^4}{32\pi^2 s}\left(u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2+\frac{s^2}{t^2}+\frac{t^2}{s^2}\right) \\
&= \frac{\alpha^2}{2s}\left(u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2+\frac{s^2}{t^2}+\frac{t^2}{s^2}\right)
\end{aligned}
$$

3 Muon Decay

$$
\mu^-(p) \to e^-(p') + \nu_\mu(k) + \overline{\nu}_e(k')
$$

Approximating the propagator as $\sim -i/M_W^2$

$$
\begin{aligned}
-i\mathcal{M} &= \left[\frac{g}{\sqrt{2}}\overline{u}_{\nu_\mu}(k)\gamma^\mu\frac{1}{2}(1-\gamma^5)u_\mu(p)\right]\frac{-i}{M_W^2}\left[\frac{g}{\sqrt{2}}\overline{v}_{\nu_e}(k')\gamma_\mu\frac{1}{2}(1-\gamma^5)u_e(p')\right] \\
&= \frac{-ig^2}{8M_W^2}\overline{u}_{\nu_\mu}(k)\gamma^\mu(1-\gamma^5)u_\mu(p)\overline{v}_{\nu_e}(k')\gamma_\mu(1-\gamma^5)u_e(p')
\end{aligned}
$$

$$
\sum_{spins}|\mathcal{M}|^2 = \frac{G^2}{2}\mathrm{Tr}\left[\not{k}\gamma^\mu(1-\gamma^5)(\not{p}+m_\mu)\gamma^\nu(1-\gamma^5)\right]\mathrm{Tr}\left[\not{k}'\gamma_\mu(1-\gamma^5)\not{p}'\gamma_\nu(1-\gamma^5)\right]
$$

The first trace is

$$
\begin{aligned}
L^{\mu\nu}_{muon} &= \mathrm{Tr}\left[\not{k}\gamma^\mu(1-\gamma^5)\not{p}\gamma^\nu(1-\gamma^5)\right] \\
&= \mathrm{Tr}\left[\not{k}\gamma^\mu\not{p}\gamma^\nu\right]-\mathrm{Tr}\left[\not{k}\gamma^\mu\gamma^5\not{p}\gamma^\nu\right]-\mathrm{Tr}\left[\not{k}\gamma^\mu\not{p}\gamma^\nu\gamma^5\right]+\mathrm{Tr}\left[\not{k}\gamma^\mu\gamma^5\not{p}\gamma^\nu\gamma^5\right] \\
&= 2\mathrm{Tr}\left[\not{k}\gamma^\mu\not{p}\gamma^\nu\right]-2\mathrm{Tr}\left[\not{k}\gamma^\mu\gamma^5\not{p}\gamma^\nu\right] \\
&= 8(k^\mu p^\nu+k^\nu p^\mu-(k\cdot p)\eta^{\mu\nu})-8i\epsilon^{\mu\nu\alpha\beta}k_\alpha p_\beta
\end{aligned}
$$

The second is similarly

$$
L^{\mu\nu}_{elec} = 8(k'^\mu p'^\nu + k'^\nu p'^\mu - (k'\cdot p')\eta^{\mu\nu} - i\epsilon^{\mu\nu\alpha\beta}k'_\alpha p'_\beta)
$$

The amplitude squared is then:

$$\begin{aligned}
\sum_{spins} |\mathcal{M}|^2 &= 32G^2 \left(2(k\cdot k')(p\cdot p') + 2(k\cdot p')(k'\cdot p) - \epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\rho\sigma}k_\alpha p_\beta k'^\rho p'^\sigma\right) \\
&= 64G^2 \left((k\cdot k')(p\cdot p') + (k\cdot p')(k'\cdot p) - (\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho)k_\alpha p_\beta k'^\rho p'^\sigma\right) \\
&= 128G^2(k\cdot p')(p\cdot k') \\
\frac{1}{2}\sum_{spins} |\mathcal{M}|^2 &= 64G^2(k\cdot p')(p\cdot k')
\end{aligned}$$

The decay rate is then

$$\begin{aligned}
\mathrm{d}\Gamma &= \frac{1}{2m_\mu}\overline{|\mathcal{M}|^2}(2\pi)^4\delta^4(p-p'-k-k')\frac{d^3p'}{(2\pi)^3 2E'}\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega}\frac{\mathrm{d}^3k'}{(2\pi)^3 2\omega'} \\
&= \frac{1}{2m_\mu}\overline{|\mathcal{M}|^2}(2\pi)^4\delta^4(p-p'-k-k')\frac{d^3p'}{(2\pi)^3 2E'}\frac{\mathrm{d}^3k'}{(2\pi)^3 2\omega'}\,\mathrm{d}^4k\,\theta(\omega)\delta(k^2) \\
&= \frac{1}{2m_\mu}\overline{|\mathcal{M}|^2}\frac{1}{(2\pi)^5}\frac{\mathrm{d}^3p'}{2E'}\frac{\mathrm{d}^3k'}{2\omega'}\theta(m_\mu - E' - \omega')\delta((p-p'-k')^2)
\end{aligned}$$

We may use the 4-momentum conservation to rewrite the amplitude as

$$\begin{aligned}
k\cdot p' &= \frac{(k-p')^2 - k^2 - (p')^2}{2} = \frac{1}{2}(k'-p)^2 = \frac{1}{2}((\omega'-m_\mu)^2 - (\omega')^2) = \frac{1}{2}m_\mu(m_\mu - 2\omega') \\
p\cdot k' &= \omega' m_\mu \\
\overline{|\mathcal{M}|^2} &= 32G^2\omega' m_\mu^2(m_\mu - 2\omega')
\end{aligned}$$

$$\begin{aligned}
\Gamma &= \frac{m_\mu G^2}{2\pi^5}\int \omega'(m_\mu - 2\omega')\frac{\mathrm{d}^3p'}{2E'}\frac{\mathrm{d}^3k'}{2\omega'}\theta(m_\mu - E' - \omega')\delta\left(m_\mu^2 - 2m_\mu(E'+\omega') - 2p'\cdot k'\right) \\
&= \frac{m_\mu G^2}{2\pi^5}\int \omega'(m_\mu - 2\omega')\frac{4\pi(E')^2\,\mathrm{d}E'}{2E'}\frac{2\pi(\omega')^2\,\mathrm{d}\omega'\,\mathrm{d}\cos\theta}{2\omega'} \\
&\qquad \times\theta(m_\mu - E' - \omega')\delta\left(m_\mu^2 - 2m_\mu(E'+\omega') + 2E'\omega'(1-\cos\theta)\right)
\end{aligned}$$

The integration of the δ-function of $\cos\theta$ is over the interval $[-1,1]$, so we gain the conditions:

$$\begin{aligned}
0 &= m_\mu^2 - 2m_\mu(E'+\omega') + 2E'\omega'(1-\cos\theta) \\
|\cos\theta| &= \left|1 + \frac{m_\mu(m_\mu - 2E' - 2\omega')}{2E'\omega'}\right| \le 1
\end{aligned}$$

$$\begin{aligned}
1 + \frac{m_\mu(m_\mu - 2E' - 2\omega')}{2E'\omega'} &\le 1 \\
m_\mu - 2E' - 2\omega' &\le 0 \\
\frac{1}{2}m_\mu - E' &\le \omega' \\
1 + \frac{m_\mu(m_\mu - 2E' - 2\omega')}{2E'\omega'} &\ge -1 \\
m_\mu^2 - 2m_\mu E' - 2m_\mu\omega' &\ge -4E'\omega' \\
m_\mu(m_\mu - 2E') &\ge 2\omega'(m_\mu - 2E') \\
\omega' \le \frac{1}{2}m_\mu \qquad &\text{and} \qquad 0 \le E' \le \frac{1}{2}m_\mu
\end{aligned}$$

Furthermore there's a $1/2E'\omega'$ factor from integrating the δ-function over $\cos\theta$:

$$\begin{aligned}
\Gamma &= \frac{m_\mu G^2}{2\pi^5}\int_0^{m_\mu/2}\mathrm{d}E'\int_{\frac{1}{2}m_\mu-E'}^{m_\mu/2}\mathrm{d}\omega'\frac{4\pi(E')^2}{2E'}\frac{2\pi(\omega')^2}{2\omega'}\omega'(m_\mu-2\omega')\frac{1}{2E'\omega'}\\
&= \frac{m_\mu G^2}{2\pi^3}\int_0^{m_\mu/2}\mathrm{d}E'\left[m_\mu\frac{\omega'^2}{2}-\frac{2\omega'^3}{3}\right]_{\frac{1}{2}m_\mu-E'}^{m_\mu/2}\\
&= \frac{m_\mu G^2}{2\pi^3}\int_0^{m_\mu/2}\mathrm{d}E'\left[\frac{m_\mu^3}{8}-\frac{m_\mu^3}{12}-\frac{m_\mu}{2}\left(\frac{1}{2}m_\mu-E'\right)^2+\frac{2}{3}\left(\frac{1}{2}m_\mu-E'\right)^3\right]\\
&= \frac{m_\mu G^2}{2\pi^3}\int_0^{m_\mu/2}\mathrm{d}E'\left[\frac{1}{2}m_\mu E'^2-\frac{2}{3}E'^3\right]\\
&= \frac{m_\mu G^2}{2\pi^3}\left[\frac{1}{6}m_\mu E'^3-\frac{1}{6}E'^4\right]_0^{m_\mu/2}\\
&= \frac{m_\mu^5 G^2}{196\pi^3}\approx 3.0019\times 10^{-19}\text{ GeV}=\frac{\hbar}{2.187\ \mu\text{s}}
\end{aligned}$$

The actual mean lifetime is 2.2 μs, so first order is pretty close!