1 Free Particle Wavefunctions

1.1 Non-Relativistic Particles

Schrodinger equation

$$0 = i\partial_t\psi - H\psi = \left(i\partial_t + \frac{\nabla^2}{2m}\right)\psi$$

Solution, where $E = \boldsymbol{p}^2/2m$:

$$\psi_p = Ne^{-iEt+i\boldsymbol{p}\cdot\boldsymbol{x}}$$

Density and current

	ρ	$\boldsymbol{j}$
Equation	$\lvert\psi\rvert^2$	$\frac{1}{2mi}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)$
Free	$\lvert N\rvert^2$	$\frac{\boldsymbol{p}}{m}\lvert N\rvert^2$

1.2 Scalar Field Equation

Klein-Gordon Equation

$$(-\partial_t^2 + \nabla^2)\phi = m^2\phi$$

Solution, where $p^0 = E = \sqrt{\boldsymbol{p}^2 + m^2}$:

$$\phi = Ne^{-iEt+i\boldsymbol{p}\cdot\boldsymbol{x}} = Ne^{-ip\cdot x}$$

Density and current

	ρ	$\boldsymbol{j}$	j^μ
Equation	$i\left(\phi^*\partial_t\phi - \phi\partial_t\phi^*\right)$	$-i\left(\phi^*\nabla\phi - \phi\nabla\phi^*\right)$	$i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)$
Free	$2E\lvert N\rvert^2$	$2\boldsymbol{p}\lvert N\rvert^2$	$2p^\mu\lvert N\rvert^2$

Antiparticles (Feynman-Stuckelberg Interpretation)

$$j^\mu(\phi, -p) = 2p^\mu|N|^2 = j^\mu(\phi^*, p)$$
$$\phi^* = Ne^{-i(-p)\cdot x} = Ne^{ip\cdot x}$$

Identify antiparticles ϕ^* as having opposite charge in the current. Antiparticles have positive energy and can be used instead of equivalent negative energy solutions.

1.3 Electrodynamics

Minimal coupling prescription

$$i\partial^\mu \to i\partial^\mu + eA^\mu$$

1.4 Scalar QED

Perturbative potential

$$V = -ie(\partial_\mu A^\mu + A^\mu \partial_\mu) - e^2 A^2$$

Electromagnetic current between states $i \to f$

$$j_\mu^{fi} = -ie(\phi_f^* \partial_\mu \phi_i - \phi_i \partial_\mu \phi_f^*) = -eN_i N_f (p_i + p_f)_\mu e^{i(p_f - p_i)\cdot x}$$

Photon equation of motion and solution (from Maxwell's equations)

$$\begin{aligned}(-\partial_t^2 + \nabla^2)A^\mu &= j^\mu \\ A^\mu &= -\frac{1}{q^2} j^\mu \\ &= \frac{-ig^{\mu\nu}}{q^2} j_\nu\end{aligned}$$

Photon propagator

$$\frac{-ig^{\mu\nu}}{q^2}$$

2 Perturbation Theory

State transistion amplitude $i \to f$:

$$\begin{aligned}V_{fi}(t) &= \int \mathrm{d}^3 x\, \phi_f^*(\boldsymbol{x}, t) V(\boldsymbol{x}, t) \phi_i(\boldsymbol{x}, t) \\ T_{fi} &= -i \int \mathrm{d}^4 x \phi_f^*(x) V(x) \phi_i(x) \\ &= -i \int_{-\infty}^{\infty} V_{fi}(t)\, \mathrm{d}t\end{aligned}$$

Fermi's Golden Rule (assuming V is time-independent)

$$\begin{aligned}W_{fi} &= \int \rho(E_f)\, \mathrm{d}E_f \lim_{T\to\infty} \frac{|T_{fi}|^2}{T} \\ &= 2\pi |V_{fi}|^2 \int \rho(E_f)\, \mathrm{d}E_f\, \delta(E_f - E_i) \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{d}t \\ &= 2\pi |V_{fi}|^2 \rho(E_i)\end{aligned}$$

State transition amplitude for $2 \to 2$ scattering (scalar QED):

$$\begin{aligned}T_{fi} &= -i \int j_\mu^{(1)} A^\mu\, \mathrm{d}^4 x \\ &= -i \int j_\mu^{(1)} \frac{-ig^{\mu\nu}}{q^2} j_\nu^{(2)}\, \mathrm{d}^4 x \\ &= N_A N_B N_C N_D (2\pi)^4 \delta(p_A + p_B - p_C - p_D)(-i\mathcal{M}) \\ -i\mathcal{M} &= ie(p_A + p_C)^\mu \left(-i\frac{g_{\mu\nu}}{q^2}\right) ie(p_B + p_D)^\nu\end{aligned}$$

3 Cross-Section

The transition matrix element is

$$T_{n\to m} = -i \langle m| \int_{-\infty}^{\infty} H_I(t)\,\mathrm{d}t\,|n\rangle = -i\langle m| H_{int}|n\rangle\, 2\pi\delta(\omega)$$

The probability is then

$$P_{n\to m} = |T_{n\to m}|^2 = |\langle m| H_{int}|n\rangle|^2(2\pi)^2\delta(\omega)^2$$

To compute the squared delta function, rewrite it as a Fourier transform over some finite time T and use the first delta function to set the second:

$$\begin{aligned}(2\pi)^2\delta(\omega)^2 &= 2\pi\delta(\omega)\lim_{T\to\infty}\int_{-T/2}^{T/2} e^{-i\omega t}\,\mathrm{d}t\\ &= 2\pi\delta(\omega)\lim_{T\to\infty}\int_{-T/2}^{T/2}\mathrm{d}t\\ &= 2\pi\delta(\omega)\lim_{T\to\infty} T\end{aligned}$$

This is obviously divergent, so instead we take the probability rate

$$W_{n\to m} = \lim_{T\to\infty}\frac{P_{n\to m}}{T} = |\langle m| H_{int}|n\rangle|^2 2\pi\delta(\omega)$$