Contents

1 Notation

- $I = [0, 1]$ the unit interval.
- $C^\infty(M)$ the set of smooth functions $f : M \to \mathbb{R}$ on a manifold M.
- $\mathfrak{X}(M)$ the set of smooth vector fields on M.

2 Vector Calculus

2.1 Divergence and Stokes Theorem for Scalar Functions

Let $f \in C^\infty(\mathbb{R})$. We wish to integrate f over some volume $V \subset \mathbb{R}^3$. Introduce constant vector field $\vec{c} \in \mathfrak{X}(\mathbb{R}^3)$, and consider the new vector field $\vec{F} = f\vec{c}$. Applying the divergence theorem:

$$\begin{aligned}\int_V \nabla \cdot \vec{F} \, \mathrm{d}^3 r &= \int_{\partial V} \vec{F} \cdot \mathrm{d}\vec{S} \\ \vec{c} \cdot \int_V \nabla f \, \mathrm{d}^3 r &= \vec{c} \cdot \int_{\partial V} f \, \mathrm{d}\vec{S} \\ \int_V \nabla f \, \mathrm{d}^3 r &= \int_{\partial V} f \, \mathrm{d}\vec{S}\end{aligned}$$

For a vector field $\vec{A} \in \mathfrak{X}(\mathbb{R}^3)$, we may repeat the process to derive a corresponding Stokes' theorem:

$$\int_V \nabla \times \vec{A} \, \mathrm{d}^3 r = \int_{\partial V} \mathrm{d}\vec{S} \times \vec{A}$$

3 Integration Techniques

3.1 Contour Integration

Let $z : I \to \mathbb{C}$ be a curve defining the contour $C \subset \mathbb{C}$ in the complex plane. Consider the integral of a complex function $w(z) = u(x, y) + iv(x, y)$ where $z(t) = x(t) + iy(t)$. We may write it as a linear combination of real integrals:

$$\begin{aligned}\int_C w(z) \, \mathrm{d}z &= \int_C (u \, \mathrm{d}x - v \, \mathrm{d}y) + i \int_C (v \, \mathrm{d}x + u \, \mathrm{d}y) \\ &= \int_0^1 \left(u\frac{dx}{dt} - v\frac{dy}{dt} \right) \mathrm{d}t + i \int_0^1 \left(v\frac{dx}{dt} + u\frac{dy}{dt} \right) \mathrm{d}t\end{aligned}$$

Theorem 3.1 (Cauchy-Goursat)**.** If $f(z)$ is analytic wihtin and on a contour C, then

$$\oint_C f(z) \, \mathrm{d}z = 0$$

Corollary 3.1 (Path Independence)**.** If $f(z)$ is analytic within a region R, and C_1, C_2 lie within R with the same endpoints, then

$$\int_{C_1} f \, \mathrm{d}z = \int_{C_2} f \, \mathrm{d}z$$

(c.f. path independence of conservative vector fields)

Example: Integral of $1/z$ along the unit circle.
Let $f(z) = 1/z$, and C be the unit circle in the complex plane. Rewriting $z = e^{i\theta}$ along the unit circle:

$$\int_C \frac{\mathrm{d}z}{z} = \int_0^{2\pi} \frac{\mathrm{d}\left(e^{i\theta}\right)}{e^{i\theta}} \, \mathrm{d}\theta = i \int_0^{2\pi} \mathrm{d}\theta = 2\pi i$$

Theorem 3.2 (Cauchy Integral Formula)**.** If $f(z)$ is analytic within and on a closed contour C, then for any z_0 interior to C,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, \mathrm{d}z$$

Proof outline: deform C to an arbitrarily small circle about z_0. Then $f(z) = f(z_0) + (f(z) - f(z_0))$ with a factor of $2\pi i$ coming from integrating $1/z$. The error term $f(z) - f(z_0)$ vanishes along an arbitrarily small circle.

This may be used to calculate the nth derivative of an analytic function:

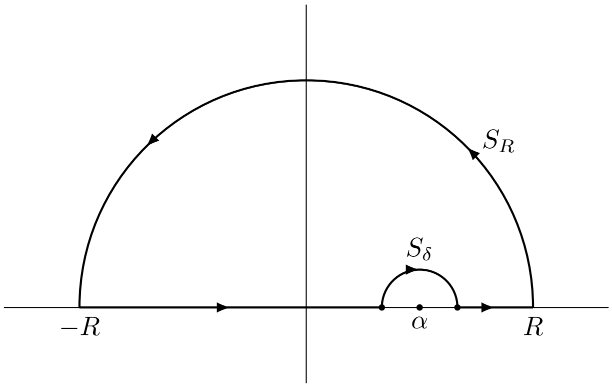
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z$$

3.2 Principal Value

Consider a function $f(z)$, analytic in the upper half of the complex plane, which satisfies $|f(z)| \to 0$ as $|z| \to \infty$. Consider the contour integral

$$\oint_C \frac{f(z)}{z-\alpha} dz = 0$$

as shown in the following:



Expressing the contour integral in terms of pieces:

$$\int_{-R}^{\alpha-\delta} \frac{f(z)}{z-\alpha}\,\mathrm{d}z + \int_{\alpha+\delta}^{R} \frac{f(z)}{z-\alpha}\,\mathrm{d}z = -\int_{S_\delta} \frac{f(z)}{z-\alpha}\,\mathrm{d}z - \int_{S_R} \frac{f(z)}{z-\alpha}\,\mathrm{d}z$$

As $R \to \infty$, $|f(Re^{i\theta})| \to 0$ along the contour, which vanishes. With the same method of $f(z) = f(\alpha) + (f(z) - f(\alpha))$, the S_δ integral becomes $-i\pi f(\alpha)$ (c.f. half the Cauchy integral formula). Define the **principal value** as:

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{f(z)}{z-\alpha}\,\mathrm{d}z = \lim_{\delta\to 0}\left[\int_{-\infty}^{\alpha-\delta} \frac{f(z)}{z-\alpha}\,\mathrm{d}z + \int_{\alpha+\delta}^{\infty} \frac{f(z)}{z-\alpha}\,\mathrm{d}z\right] = i\pi f(\alpha)$$

This is an equation for complex valued functions. Writing $f(x) = f_R(x) + i f_I(x)$ where $f_{R,I}$ are real valued functions:

$$f_R(\alpha) = \frac{1}{\pi}\text{p.v.} \int_{-\infty}^{\infty} \frac{f_I(x)}{x-\alpha}\,\mathrm{d}x$$
$$f_I(\alpha) = -\frac{1}{\pi}\text{p.v.} \int_{-\infty}^{\infty} \frac{f_R(x)}{x-\alpha}\,\mathrm{d}x$$

N.B. when taking the principal value of $1/x$, we cannot use $f(x) = 1$ since it doesn't vanish as $|x| \to \infty$, so we need to use the $\lim_{\delta\to 0}$ prescription of the principal value to try and compute it.

3.3 Residues

3.4 Delta Function

Properties of the δ-function:

1. (Translation)

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\,\mathrm{d}x = f(a)$$

2. (Scaling)

$$\delta(ax) = \frac{\delta(x)}{|a|}$$

3. (Composition) Let $\{x_i\}$ be the set of isolated zeroes of g with non-vanishing derivative $g'(x_i)$. Then:

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$$

Note the absolute value comes from the integration bounds swapping when $g'(x_i) < 0$.

4. (Derivative)

$$\int_{-\infty}^{\infty} f(x)\delta'(x-a) = -f'(a)$$

Useful representations of the δ-function:

1. (Fourier Transformation)

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,\mathrm{d}k$$
$$\delta^3(\vec{x}) = \frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{x}}\,\mathrm{d}^3k$$

2. (Laplacian of $1/r$)

$$\delta^3(\vec{r}) = -4\pi\nabla^2\frac{1}{r}$$

3.5 Regularization

Useful for turning divergent integrals of the form e^{ikr} into limits of convergent ones. Basic idea: perform the conversion

$$\int e^{\pm ikr}\,\mathrm{d}k \mapsto \lim_{\epsilon\to 0}\int e^{\pm ik(r\pm i\epsilon)}\,\mathrm{d}k = \lim_{\epsilon\to 0}\int e^{\pm ikr}e^{-k\epsilon}\,\mathrm{d}k$$

This adds the exponential supression term $e^{k\epsilon}$ which goes to 0 very quickly as $k\to\infty$. After integrating, we then can attempt to evaluate the limit.

Example: Regularizing the Fourier transformation in 3 dimensions

$$\begin{aligned}
\delta^3(\vec{r}) &= \frac{1}{(2\pi)^3}\int e^{i\vec{k}\cdot\vec{r}}\,\mathrm{d}^3k \\
&= \frac{1}{(2\pi)^3}\int_0^{\infty} k^2\,\mathrm{d}k\int_{-1}^{1}\mathrm{d}\cos\theta\int_0^{2\pi}\mathrm{d}\phi\,e^{ikr\cos\theta} \\
&= \frac{1}{(2\pi)^2}\int_0^{\infty}\frac{k}{ir}\left(e^{ikr}-e^{-ikr}\right)\mathrm{d}k \\
&= \frac{1}{(2\pi)^2}\lim_{\epsilon\to 0}\int_0^{\infty}\frac{k}{ir}\left(e^{ik(r+i\epsilon)}-e^{-ik(r-i\epsilon)}\right)\mathrm{d}k
\end{aligned}$$

Integrating by parts on the first term:

$$\int_0^{\infty} ke^{ik(r+i\epsilon)}\,\mathrm{d}k = \frac{k}{i(r+i\epsilon)}e^{ik(r+i\epsilon)}\bigg|_0^{\infty} - \frac{1}{i(r+i\epsilon)}\int_0^{\infty} e^{ik(r+i\epsilon)}\,\mathrm{d}k = -\frac{1}{(r+i\epsilon)^2}$$

And for the second term:

$$\int_0^{\infty} ke^{-ik(r-i\epsilon)}\,\mathrm{d}k = \frac{1}{(r-i\epsilon)^2}$$

Gives the result

$$\begin{aligned}
\delta^3(\vec{r}) &= \frac{1}{(2\pi)^2}\lim_{\epsilon\to 0}\left[\frac{1}{(r-i\epsilon)^2}-\frac{1}{(r+i\epsilon)^2}\right] \\
&= \lim_{\epsilon\to 0}\frac{\epsilon}{\pi^2}\frac{1}{(r^2+\epsilon^2)^2}
\end{aligned}$$

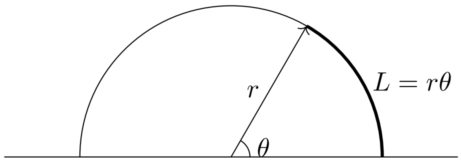
We see indeed that the integral vanishes for $r\neq 0$ since the denominator is positive while the numerator vanishes. When $r=0$, the the limit becomes

$$\lim_{\epsilon\to 0}\frac{1}{\pi^2}\frac{1}{\epsilon^3}$$

which diverges, as we expect of the δ-function.

3.6 Solid Angle

Radian agnles are the ratio of an arc length with the angle that subtends it:



By definition of radians:

$$\theta = \frac{L}{r}$$

For a more general curve with length element $\mathrm{d}\vec{l}$, the element must be projected onto the arc length $\mathrm{d}L = \hat{r} \cdot \mathrm{d}\vec{l}$:

$$\mathrm{d}\theta = \frac{\hat{r} \cdot \mathrm{d}\vec{l}}{r}$$

The solid angle is defined analagously, except with respect to the surface of a sphere:

$$\mathrm{d}\Omega = \frac{\hat{r} \cdot \mathrm{d}\vec{A}}{r^2} = \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

4 Fourier Transformations

Asymmetric definition:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, \mathrm{d}k \qquad \mathcal{F}[f](k) = \tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} \, \mathrm{d}x$$

Symmetric definition:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} \, \mathrm{d}k \qquad \mathcal{F}[f](k) = \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, \mathrm{d}x$$

4.1 Convolution Theorem

A **convolution** of two functions f and g is:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x') g(x - x') \, \mathrm{d}x'$$

Fourier transformations turn convolutions in x-space to products in k-space:

$$\begin{aligned}
\mathcal{F}[f * g](k) &= \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}x' \, f(x') g(x - x') e^{-ikx} \\
&= \int_{-\infty}^{\infty} \mathrm{d}x' \, f(x') e^{-ikx'} \int_{-\infty}^{\infty} \mathrm{d}x \, g(x - x') e^{-ik(x - x')} \\
&= \mathcal{F}[f](k) \cdot \mathcal{F}[g](k)
\end{aligned}$$