Beginning with Linear Homogeneous Differential Equations with Constant Coefficients

Second-order differential equations

$$y'' + py' + qy = 0 \quad \cdots\cdots\cdots (1)$$

characteristic equation $r^2 + pr + q = 0$, roots r_1, r_2

(1) $r_1 \neq r_2$ $\quad y = c_1 e^{r_1 x_1} + c_2 e^{r_2 x_2}$

(2) $r_1 = r_2$ $\quad y = (c_1 + c_2 x) e^{r_1 x}$

only when $r_1 = r_2$, $c_2 x e^{r_1 x}$ is a solution to (1).

substitute $c_2 x e^{r_1 x}$ into (1) and it shows only when $r_1 = -\frac{p}{2}$, the equation holds. (In this situation, p and q are not independent)

(3) $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$ $\quad y = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$$e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i \sin\beta)$$

$e^{\alpha x}(c_1 \cos\alpha + c_2 \cos\beta)$ is a linear combination of $e^{r_1 x}$ and $e^{r_2 x}$.

The system of differential equations $\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = A\begin{bmatrix}x\\ y\end{bmatrix}$

solve
$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = A\begin{bmatrix}x\\ y\end{bmatrix} \cdots\cdots (2) \qquad \left(\begin{aligned}\dot{x} &= \frac{dx}{dt}\\ \dot{y} &= \frac{dy}{dt}\end{aligned}\right)$$

Note: $e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ ☆

solution $\begin{bmatrix}x\\ y\end{bmatrix} = e^{At}\begin{bmatrix}x(0)\\ y(0)\end{bmatrix}$

(1) A has two different eigenvalues λ_1 and λ_2 (including the situation that $\lambda_1=\lambda_2$ and the eigenvalue space has two dimensions, i.e., the geometric multiplicity is two, equal to the algebraic multiplicity) directly substituting ① or ② into (2) shows they are correct

$$\begin{bmatrix}x\\ y\end{bmatrix} = \underbrace{e^{At}\begin{bmatrix}x(0)\\ y(0)\end{bmatrix}}_{①} = \underbrace{c_1 e^{\lambda_1 t}\begin{bmatrix}x_1\\ y_1\end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix}x_2\\ y_2\end{bmatrix}}_{②}$$

$\begin{bmatrix}x_1\\ y_1\end{bmatrix}$, $\begin{bmatrix}x_2\\ y_2\end{bmatrix}$ are eigenvectors of λ_1 and λ_2, respectively.

Proof: let $P = \begin{bmatrix}x_1 & x_2\\ y_1 & y_2\end{bmatrix}$ $\quad A = P\cdot D\cdot P^{-1}$, $D = \begin{bmatrix}\lambda_1 & \\ & \lambda_2\end{bmatrix}$

$$e^{At} = e^{PDP^{-1}t} = P e^{Dt} P^{-1} = P\begin{bmatrix}e^{\lambda_1 t} & \\ & e^{\lambda_2 t}\end{bmatrix} P^{-1}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{-1} \underbrace{\left(c_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right)}_{\text{let } t=0 \text{ in } ② \;=\; \begin{bmatrix} x(0) \\ y(0) \end{bmatrix}}$$

$$= \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \underbrace{\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{-1} c_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{= \begin{bmatrix} 1 \\ 0 \end{bmatrix}} +$$

$$\begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{\lambda_1 t} \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} \\ 0 \end{bmatrix} = e^{\lambda_1 t} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & \\ & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{-1} c_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$= c_1 e^{\lambda_1 t} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + c_2 e^{\lambda_2 t} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

(2) $\lambda_1 = \lambda_2$ and the geometric multiplicity is one

only one independent eigenvector

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{At} \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \ ?$$

Proof: $A = \lambda v \cdot v^T$ $\quad \left(v = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \right)$ ← eigenvector

This pare is not completed.

Because I don't know the right solution now.