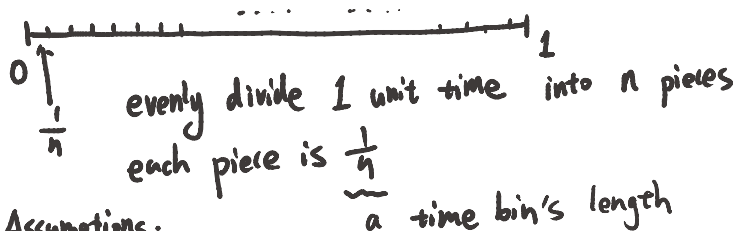


From binomial distribution to Poisson distribution

Binomial distribution $f(k, n, p) = P_r(k; n, p) = P_r(X=k)$
($n=1$) \rightarrow special case: Bernoulli distribution $= \binom{n}{k} p^k (1-p)^{n-k}$



Assumptions:

- ① An event occurs in a time bin independently of events occurring in other time bins.
- ② At most one event can occur in one time bin

$$P_r\{X=k\} = \binom{n}{k} p^k (1-p)^{n-k}$$

$EX = np \triangleq \lambda$ average number of events that occur in a fixed interval (unit time)

$$\Downarrow$$
$$p = \frac{\lambda}{n}$$

$$P_r\{X=k\} = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

We need to let $n \rightarrow \infty$, then λ can be an average rate applicable at any time point.

$$P_r\{X=k\} = \lim_{n \rightarrow \infty} \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$Pr\{X=k\} = \lim_{n \rightarrow \infty} \underbrace{\frac{C_n^k}{n^k}}_{\downarrow} \lambda^k \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{e^{-\lambda}} \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-k}}_1$$

$$\lim_{n \rightarrow \infty} \frac{C_n^k}{n^k} = \frac{n(n-1) \dots (n-k+1)}{k! \cdot \underbrace{n^k}_{=1}} = \frac{1}{k!}$$

$$Pr\{X=k\} = \frac{\lambda^k e^{-\lambda}}{k!} \quad \text{Poisson distribution}$$

We can define λ over any unit time, that is, the unit of λ can be per day, per year, or per second.

Actually, we can rewrite this equation as

$$Pr\{k, \lambda\} = \frac{\lambda^k e^{-\lambda}}{k!}, \text{ then we have}$$

$$Pr\{k, \lambda t\} = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \rightarrow \text{This is the probability of } k \text{ events occurring in time } t, \text{ equivalent to resetting } t \text{ as the unit time.}$$

↓
We can easily derive this equation by repeating the process from binomial

distribution to Poisson distribution

by setting the unit time as t instead of 1.

The time interval between two consecutive events in a Poisson distribution follows the exponential distribution.

Proof: $\Pr\{0 \text{ events in } (0, x]\} = \Pr\{0, \lambda x\} = e^{-\lambda x}$

$$\Pr\{\text{at least 1 event in } (0, x]\} = 1 - e^{-\lambda x}$$

Let Y be the time interval between first and second events.
 $f(y) \triangleq$ the probability density function of Y . \rightarrow Can be obtained by differentiating the cumulative distribution of Y .

$$f(y) = \frac{d \Pr\{Y > y\}}{dy} = \frac{d [1 - \Pr\{Y \leq y\}]}{dy} = \frac{d [1 - \Pr\{\text{at least 1 event in } (0, y]\}]}{dy}$$

$$f(y) = \lambda e^{-\lambda y}$$

\hookrightarrow exponential distribution

Add up multiple Poisson distribution results in a new Poisson distribution with a mean equal to the sum of the means of individual distributions.

Let X and Y be two independent Poisson random variables with means λ and μ , respectively.

$Z = X + Y$ is also a Poisson random variable with mean $\lambda + \mu$.

$$Pr\{Z=k\} = \frac{(\lambda + \mu)^k e^{-(\lambda + \mu)}}{k!}$$

Adding up an infinite number of identical Poisson distributions results in a Gaussian distribution.

This conclusion comes from the central limit theorem.
(CLT)