This homework does not provide Python code. Instead, we encourage you to write your own code to help you answer some of these problems, and/or to test and debug the code components we do ask for. Some of the problems below are simple enough that hand calculation should be possible; your hand solutions can serve as test cases for your code. You may also find that including utilities written in previous labs (like an `sd` or signed distance function) will be helpful, as you build up additional functions and utilities for calculation of margins, different loss functions, gradients, and other functions needed for margin maximization and gradient descent here.

For your convenience, we have copied the hands-on section into a colab notebook, which may be found here.

1) Margin

When we train a classifier, it is desirable for the classifier to have a large margin with regard to the points in our data set, in the hope that this will make the classifier more robust to any new points we might see.

We have previously defined the margin of a single example (a single data point) with respect to a separator, but that does not directly indicate whether a separator will perform well on a large data set. Thus, we would like to find a score function S for a separator (θ, θ_0), such that maximizing S leads to a better separator.

Marge Inovera suggests that because big margins are good, we should maximize the sum of the margins. So, she defines:

$$S_{sum}(\theta, \theta_0) = \sum_i \gamma(x^{(i)}, y^{(i)}, \theta, \theta_0).$$

Minnie Malle suggests that it would be better to just worry about the points closest to the margin, and defines:

$$S_{min}(\theta, \theta_0) = \min_i \gamma(x^{(i)}, y^{(i)}, \theta, \theta_0).$$

Maxim Argent suggests:

$$S_{max}(\theta, \theta_0) = \max_i \gamma(x^{(i)}, y^{(i)}, \theta, \theta_0).$$

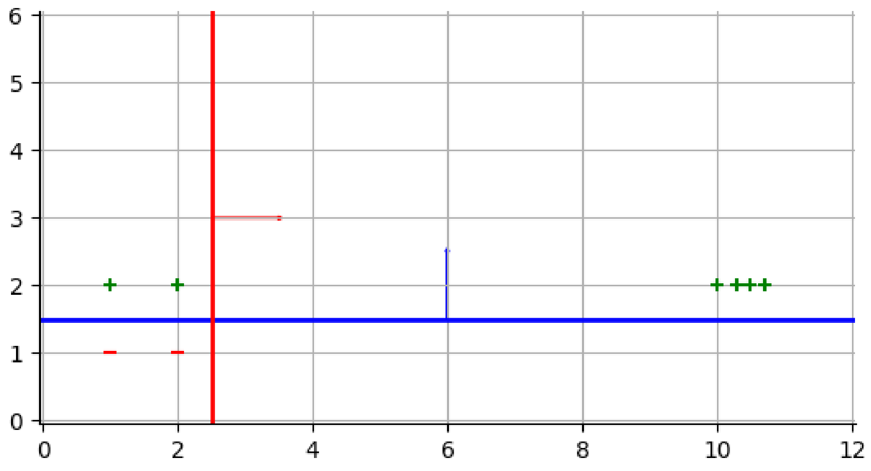
Recall that the margin of a given point is defined as

$$\gamma(x, y, \theta, \theta_0) = \frac{y(\theta \cdot x + \theta_0)}{||\theta||}.$$

Consider the following data, and two potential separators (red and blue).

```
data = np.array([[1, 2, 1, 2, 10, 10.3, 10.5, 10.7],
                 [1, 1, 2, 2,  2,  2,  2, 2]])
labels = np.array([[-1, -1, 1, 1, 1, 1, 1, 1]])
blue_th = np.array([[0, 1]]).T
blue_th0 = -1.5
red_th = np.array([[1, 0]]).T
red_th0 = -2.5
```

The situation is illustrated in the figure below.



1A) What are the values of each score $(S_{sum}, S_{min}, S_{max})$ on the red separator?

Enter a Python list of three numbers. [31.5, -1.5, 8.2]

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1B) What are the values of each score $(S_{sum}, S_{min}, S_{max})$ on the blue separator?

Enter a Python list of three numbers. [4, 0.5, 0.5]

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1C) Which of these separators maximizes S_{sum}?

Choose one: red

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1D) Which separator maximizes S_{min}?

Choose one: blue

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1E) Which separator maximizes S_{max}?

Choose one: red

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1F) Which score function should we prefer if our goal is to find a separator that generalizes better to new data?

Choose one: S_min

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2) What a loss

Based on the previous part, we've decided to try to find a linear separator θ, θ_0 that **maximizes** the **minimum margin** (the distance between the separator and the points that come closest to it.) We define the margin of a data set (X, Y), with respect to a separator as

$$\gamma(X, Y, \theta, \theta_0) = \min_i \gamma(x^{(i)}, y^{(i)}, \theta, \theta_0).$$

As discussed in the notes, an approach to this problem is to specify a value γ_{ref} for the margin of the data set, and then seek to find a linear separator that maximizes γ_{ref}.

2A) We can think about a (not necessarily maximal) margin γ_{ref} for the data set as a value such that:

- ○ for at least one point $x^{(i)}, y^{(i)}$, we have $\gamma(x^{(i)}, y^{(i)}, \theta, \theta_0) \geq \gamma_{ref}$
- [✓ for every point $x^{(i)}, y^{(i)}$, we have $\gamma(x^{(i)}, y^{(i)}, \theta, \theta_0) \geq \gamma_{ref}$]
- ○ for at least one point $x^{(i)}, y^{(i)}$, we have $\gamma(x^{(i)}, y^{(i)}, \theta, \theta_0) \leq \gamma_{ref}$
- ○ for every point $x^{(i)}, y^{(i)}$, we have $\gamma(x^{(i)}, y^{(i)}, \theta, \theta_0) \leq \gamma_{ref}$

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2B) Suppose for our data set we find that the maximum γ_{ref} across all linear separators is 0.

Is our data linearly separable? No

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2C) For this subproblem, assume that $\gamma_{ref} > 0$ (i.e., the data is linearly separable). Note that in this case, the Perceptron algorithm is guaranteed to find a separator that correctly classifies all of the data points.

What is the **largest minimum margin** guaranteed by running the Perceptron algorithm on a data set that has a maximum margin equal to $\gamma_{ref} > 0$?

- ○ 0
- ○ ∞
- ○ γ_{ref}
- [✓ some ϵ where $\epsilon > 0$]

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Now we want to improve on the (infinitesimally small) guaranteed margin of the Perceptron algorithm. We saw in the lecture that a powerful way of designing learning algorithms is to describe them as optimization problems, then use relatively general-purpose optimization strategies to solve them.

A typical form of the optimization problem is to minimize an objective that has the form

$$J(\theta, \theta_0) = \frac{1}{n}\sum_{i=1}^{n} L(x^{(i)}, y^{(i)}, \theta, \theta_0) + \lambda R(\theta, \theta_0)$$

where L is a per-point *loss function* that characterizes how much error was made by the hypothesis (θ, θ_0) on the point, and R is a *regularizer* that describes some prior knowledge or general preference over hypotheses.

We first consider the objective of finding a maximum-margin separator using the format above, using the so-called "zero-infinity" loss, $L_{0,\infty}$:

$$L_{0,\infty}(\gamma(x, y, \theta, \theta_0), \gamma_{ref}) = \begin{cases} \infty & \text{if } \gamma(x, y, \theta, \theta_0) < \gamma_{ref} \\ 0 & \text{otherwise} \end{cases}$$

and

$$J_{0,\infty}(\theta, \theta_0) = \frac{1}{n}\sum_{i=1}^{n} L_{0,\infty}(x^{(i)}, y^{(i)}, \theta, \theta_0) + \lambda R(\theta, \theta_0).$$

2D) For a linearly separable data set, positive λ, and positive R given nonzero θ, what is true about the **minimal** value of $J_{0,\infty}$?

Which of the following is true: It is always finite and positive

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2E) For a **non** linearly separable data set, and positive λ and γ_{ref}, what is true about the **minimal** value of $J_{0,\infty}$:

Which of the following is true: It is infinite

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3) Simply inseparable

We would prefer a loss function that helps steer optimization toward a solution, in the case when the data is linearly separable. Furthermore, in real data sets it is relatively rare that the data is linearly separable, so our algorithm should be able to handle this case also and still work toward an optimal, though imperfect, linear separator. Instead of using $(0, \infty)$ loss, we should design a loss function that will let us "relax" the constraint that all of the points have margin bigger than γ_{ref}, while still encouraging large margins.

The *hinge loss* is one such more relaxed loss function; we will define it in a way that makes a connection to the problem we are facing:

$$L_h\left(\frac{\gamma(x, y, \theta, \theta_0)}{\gamma_{ref}}\right) = \begin{cases} 1 - \frac{\gamma(x,y,\theta,\theta_0)}{\gamma_{ref}} & \text{if } \gamma(x, y, \theta, \theta_0) < \gamma_{ref} \\ 0 & \text{otherwise} \end{cases}$$

When the margin of the point is greater than or equal to γ_{ref}, we are happy and the loss is 0; while when the margin is less than γ_{ref}, we have a positive loss that increases the further away the margin is from γ_{ref}.

3A) Given this definition, if γ_{ref} is positive what can we say about $L_h(\gamma(x, y, \theta, \theta_0)/\gamma_{ref})$, no matter what finite values θ and θ_0 take on?

Which of the following is true about L_h: It is always >= 0

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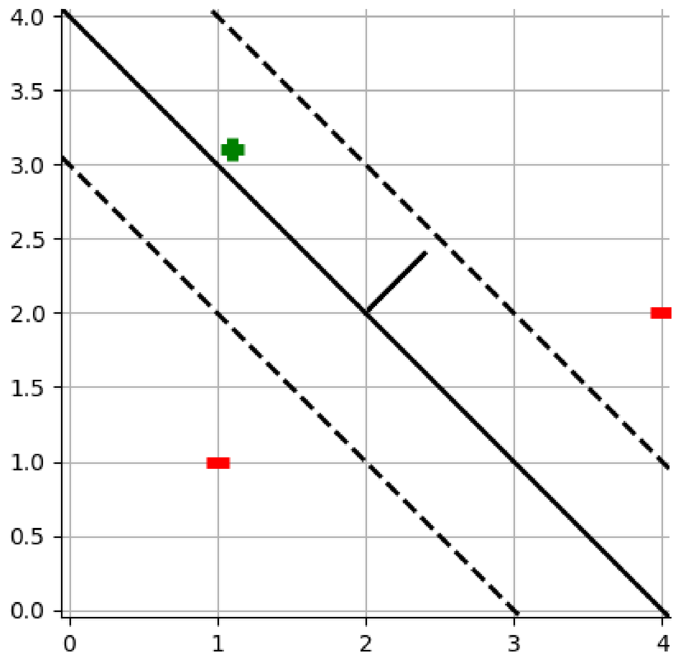
Here is a separator and three points. The dotted lines represent the margins determined by γ_{ref}.

```
data = np.array([[1.1, 1, 4],[3.1, 1, 2]])
labels = np.array([[1, -1, -1]])
th = np.array([[1, 1]]).T
th0 = -4
```



3B) What is $L_h(\gamma(x, y, \theta, \theta_0)/\gamma_{ref})$ for each point, where $\gamma_{ref} = \sqrt{2}/2$? Enter the values in the same order as the respective points are listed in `data`. You can do this computationally or by hand.

Enter the three hinge loss values in order as a Python list of three numbers:

[0.8, 0, 3]

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4) It hinges on the loss

Putting hinge loss and regularization together, we can look at regularized average hinge loss:

$$\frac{1}{n}\sum_{i=1}^{n} L_h\left(\frac{\gamma(x^{(i)}, y^{(i)}, \theta, \theta_0)}{\gamma_{ref}}\right) + \lambda\frac{1}{\gamma_{ref}^2} \quad .$$

We only need to minimize this over two parameters θ, θ_0, since the third parameter γ_{ref} can be expressed as $\frac{1}{\|\theta\|}$, as they both represent the distance from the decision boundary to the margin boundary. Plugging in $\gamma_{ref} = \frac{1}{\|\theta\|}$ and also expanding γ, we arrive at the SVM (support vector machine) objective:

$$J(\theta, \theta_0) = \frac{1}{n}\sum_{i=1}^{n} L_h(y^{(i)}(\theta^T x^{(i)} + \theta_0)) + \lambda\|\theta\|^2 \quad .$$

4A) If the data is linearly separable and we use the SVM objective, if we now let $\lambda = 0$ and find the minimizing values of θ, θ_0, what will happen?

Which of the following is true: The minimal objective value will be 0

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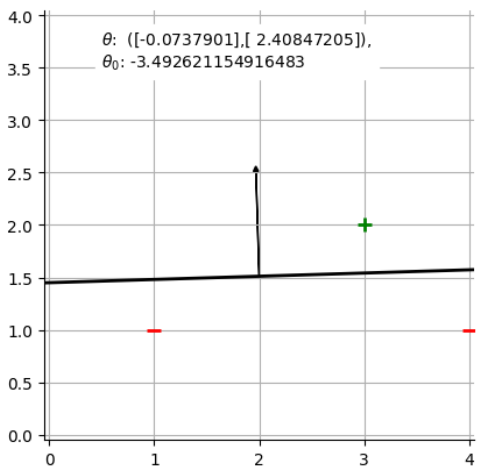
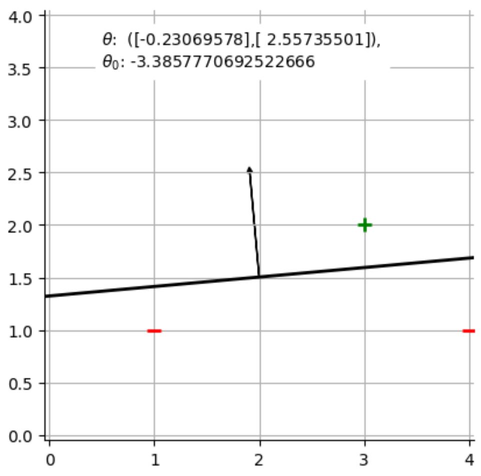
4B) Consider the following plots of separators. They are for λ values of 0 and 0.001. Match each λ to a plot.



A



B

Enter a Python list with the values of λ for the two graphs above (i.e., `[lambdaA, lambdaB]`).

[0.001, 0]

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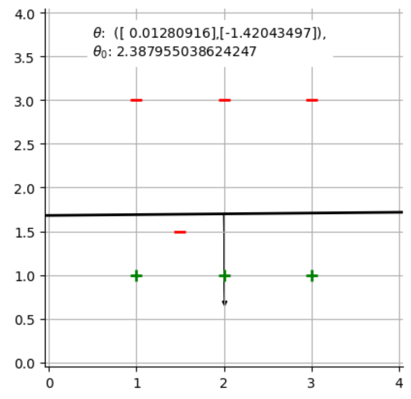
Solution: `[0.001, 0.0]`

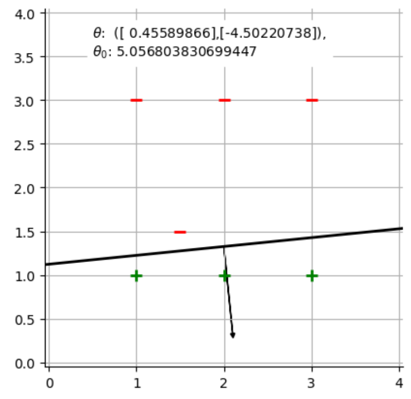
Explanation:

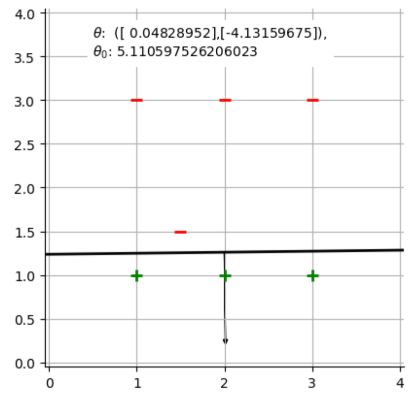
The separators in both plot A and plot B achieve zero hinge loss (for some θ). (Note: this is not immediately obvious; one needs to plug in the numbers to confirm at least for non-zero $lambda$. Remember that even if a separator has no errors, there may still be non-zero hinge loss if some of the correctly classified points are "too close" to the separator (within the margin).

With zero hinge loss in these plots, the operative term in the cost function becomes $\lambda||\theta||^2$. Considering the norm of the θ vector in the two cases, for plot A the norm of θ is 2.4096, while for plot B the norm of θ is 2.5677. So plot A has non-zero λ operating to shrink θ, while plot B has zero λ allowing the norm of θ in plot B to be bigger.

4C) Consider the following three plots of separators. They are for λ values of 0, 0.001, and 0.03. Match to the plot.







A B C

Enter a Python list with the values of λ for the three graphs above (i.e., `[lambdaA, lambdaB, lambdaC]`).

[0.03, 0, 0.001]

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Solution: `[0.03, 0.0, 0.001]`

Explanation:

When we increase λ, we penalize larger values of θ. In some cases, the may mean we incur non-zero or larger hinge loss (where points are closers to the separator, or even where some points are misclassified).

Plot A has minimized θ to the degree that a point is misclassified; this corresponds to the largest value of λ, 0.3. (Note that for Plot A, $||\theta|| = 1.4205$, average hinge loss is 0.1861, and the margin is negative, -2.0735).

Plot B, in contrast, has the largest $||\theta||$, corresponding to the smallest λ, or $\lambda = 0$. (Note that for Plot B, $||\theta|| = 4.525$, average hinge loss is 0, and the margin is positive, 0.2233.)

Finally, for Plot C we see that again zero hinge loss can be achieved, but here a non-zero $\lambda = 0.001$ has acted to reduce $||\theta||$ or equivalent increase the margin. (Note that for Plot C, $||\theta|| = 4.1319$, average hinge loss is 0, and the margin is positive, 0.2455, slightly larger than in Plot B.)

Another observation we may make is that when $\lambda = 0$, only the hinge loss term remains, so we only care to linearly separate the data. Since the data are separable, the optimization process will find a separator that perfectly separates the data. This gives us either B or C could correspond to $\lambda = 0$ from visual inspection. To choose between them for which corresonds to $\lambda = 0$ or λ very small, we have to look more closely at the corresponding margin or norm of θ.

5) Linear Support Vector Machines

The training objective for the Support Vector Machine (with slack) can be seen as optimizing a balance between the average hinge loss over the examples and a regularization term that tries to keep θ small (or equivalently, increase the margin). This balance is set by the regularization parameter λ. Here we only consider the case without the offset parameter θ_0 (setting it to zero) and rewrite the training objective as an average so that it is given by

$$\left[\frac{1}{n}\sum_{i=1}^{n} L_h(y^{(i)}\,\theta\cdot x^{(i)}\,)\right] + \frac{\lambda}{2}||\theta||^2 = \frac{1}{n}\sum_{i=1}^{n}\left[L_h(y^{(i)}\,\theta\cdot x^{(i)}\,) + \frac{\lambda}{2}||\theta||^2\right]$$

where $L_h(y(\theta\cdot x)) = \max\{0, 1-y(\theta\cdot x)\}$ is the hinge loss. (Note that we will also sometimes write the hinge loss as $L_h(v) = \max(0, 1-v)$.) Now we can minimize the above overall objective function with the Pegasos algorithm that iteratively selects a training point at random and applies a gradient descent update rule based on the corresponding term inside the brackets on the right hand side.

In this problem we will optimize the training objective using a single training example, so that we can gain a better understanding of how the regularization parameter, λ, affects the result. To this end, we refer to the single training example as the feature vector and label pair, (x, y). We will then try to find a θ that minimizes

$$J_\lambda^1(\theta) \equiv L_h(y(\theta\cdot x)) + \frac{\lambda}{2}||\theta||^2\ .$$

In the next subparts, we will try to show that the θ minimizing J_λ^1, denoted $\hat\theta$, is necessarily of the form

$$\hat\theta = \eta\, y x$$

for some real $\eta > 0$.

In the expressions below, you can use `lambda` to stand for λ, `x` to stand for x, `transpose(x)` for transpose of an array, `norm(x)` for the length (norm) of a vector, `x@y` to indicate a matrix product of two arrays, and `x*y` for elementwise (or scalar) multiply.

5A) Consider first the case where the loss is positive: $L_h(y(\theta\cdot x)) > 0$. We can minimize J_λ^1 with respect to θ by computing a formula for its gradient with respect to θ, and then solving for the θ for which the gradient is equal to 0. Let us denote that value as $\hat\theta$.

Enter an expression for $\hat\theta$: y*x/lambda

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5B) Now find the smallest (in the norm sense) $\hat\theta$ for which $L_h(y(\theta\cdot x)) = 0$.

Note: Be careful -- you cannot simply divide by a vector!

Enter your answer as a Python expression: $\hat\theta =$ x/norm(x)**2/y

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Note that if the hinge loss is zero, the point is correctly classified.

5C) Let $\hat{\theta} = \hat{\theta}(\lambda)$ be the minimizer of $J^1_\lambda(\theta)$. Is it possible to pick a value for λ so that the training example x, y wil be misclassified by $\hat{\theta}(\lambda)$?

To answer this question, recall that a point is misclassified when $y(\theta \cdot x) \leq 0$. Use your result from part 5A where you found the $\hat{\theta}$ that minimizes $J^1_\lambda(\theta)$ to write an expression for $y(\hat{\theta} \cdot x)$ in terms of x, y and λ.

For θ that minimizes $J^1_\lambda(\theta)$, enter an expression for $y(\hat{\theta} \cdot x)$:

norm(x)**2/lambda

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5D)

Under what conditions is $y(\hat{\theta} \cdot x) \leq 0$? Select all that are true.

- ☑ $x = 0$
- ☐ $y < 0$
- ☐ $y > 0$
- ☐ $\lambda = 0$
- ☑ $\lambda = \infty$

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5E) You will notice that if $y(\hat{\theta} \cdot x) \leq 0$, then $\hat{\theta}$ will misclassify x. The above result shows that our optimal classifier $\hat{\theta}$ won't misclassify (except in edge cases); however, we might still be concerned about correctly classified points that are "too close" to the separator, and thereby increase our regularized loss function.

Suppose we have a linear classifier described by θ. We say a correctly classified datapoint $\hat{x}, \hat{y}$ is on the margin boundary of the classifier if

$$\hat{y}(\theta \cdot \hat{x}) = 1.$$

When a classifier is determined by minimizing a regularized loss function with a single training example, like J^1_λ above, too much regularization can result in a classifier that puts a correctly classified training point *inside* the margin, and thus incur hinge loss. That is, if we have a single training example (x, y) and regularize with a λ that is too large, we may discover that $y(\hat{\theta} \cdot x) < 1$. Fortunately, for this single training example case, we can ensure that λ is not too large.

Write an expression for the maximum value of λ, in terms of x and y, that ensures that the (x, y) example is NOT inside the margin: norm(x)**2

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So, where are we? We now have a good objective function, the SVM objective, that will strive to correctly classify data points, but also seek to maximize the margin (minimize the norm of θ), given a judicious choice of $lambda$. This objective function can be used in either batch optimization (calculating average losses across the whole data set), or on a data point by data point basis. This gives us powerful flexibility in optimizing (minimizing) this objective function using gradient descent, which we will consider next.

6) Implementing gradient descent

In this section we will implement generic versions of gradient descent and apply these to the SVM objective.

Reminder: For your convenience, we have copied the hands-on section into a colab notebook, which may be found here.

6.1) Gradient descent

Note: If you need a refresher on gradient descent, you may want to reference this week's notes.

We want to find the x that minimizes the value of the *objective function* $f(x)$, for an arbitrary scalar function f. The function f will be implemented as a Python function of one argument, that will be a numpy column vector. For efficiency, we will work with Python functions that return not just the value of f at $f(x)$ but also return the gradient vector at x, that is, $\nabla_x f(x)$.

We will now implement a generic gradient descent function, `gd`, that has the following input arguments:

- `f`: a function whose input is an `x`, a column vector, and returns a scalar.
- `df`: a function whose input is an `x`, a column vector, and returns a column vector representing the gradient of `f` at `x`.
- `x0`: an initial value of x, `x0`, which is a column vector.
- `step_size_fn`: a function that is given the iteration index (an integer) and returns a step size.
- `max_iter`: the number of iterations to perform

Our function `gd` returns a tuple:

- `x`: the value at the final step
- `fs`: the list of values of `f` found during all the iterations (including `f(x0)`)
- `xs`: the list of values of `x` found during all the iterations (including `x0`)

Hint: This is a short function!

Hint 2: If you do `temp_x = x` where `x` is a vector (numpy array), then `temp_x` is just another name for the same vector as `x` and changing an entry in one will change an entry in the other. You should either use `x.copy()` or remember to change entries back after modification.

Some test or example functions that you may find useful are included below. You may also find `rv` and `cv` (from previous weeks) useful, though not necessary.

```
def f1(x):
    return float((2 * x + 3)**2)

def df1(x):
    return 2 * 2 * (2 * x + 3)

def f2(v):
    x = float(v[0]); y = float(v[1])
    return (x - 2.) * (x - 3.) * (x + 3.) * (x + 1.) + (x + y -1)**2
```

```
def df2(v):
    x = float(v[0]); y = float(v[1])
    return cv([(-3. + x) * (-2. + x) * (1. + x) + \
               (-3. + x) * (-2. + x) * (3. + x) + \
               (-3. + x) * (1. + x) * (3. + x) + \
               (-2. + x) * (1. + x) * (3. + x) + \
               2 * (-1. + x + y),
               2 * (-1. + x + y)])
```

To evaluate results, we also use a simple `package_ans` function, which checks the final `x`, as well as the first and last values in `fs`, `xs`.

```
def package_ans(gd_vals):
    x, fs, xs = gd_vals
    return [x.tolist(), [fs[0], fs[-1]], [xs[0].tolist(), xs[-1].tolist()]]
```

The test cases are provided below, but you should feel free (and are encouraged!) to write more of your own.

```
# Test case 1
ans=package_ans(gd(f1, df1, cv([0.]), lambda i: 0.1, 1000))

# Test case 2
ans=package_ans(gd(f2, df2, cv([0., 0.]), lambda i: 0.01, 1000))
```

```
def gd(f, df, x0, step_size_fn, max_iter):

    #itinitialize some paramaters
    num_iter = 0
    list_x = []
    list_f = []
    #copy the input so we dont modify it on each iter
    x = x0.copy()
    #exceute updates intil we reach out max num of updates
    while num_iter < max_iter:
      #calculate the current gradient
      grad = df(x)
      list_x.append(x)
      list_f.append(f(x))
      #next selection is the current - the gradient times a step size
```

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Here is the solution we wrote:

```
def gd(f, df, x0, step_size_fn, max_iter):
    prev_x = x0
    fs = []; xs = []
    for i in range(max_iter):
        prev_f, prev_grad = f(prev_x), df(prev_x)
        fs.append(prev_f); xs.append(prev_x)
        if i == max_iter-1:
            return prev_x, fs, xs
        step = step_size_fn(i)
        prev_x = prev_x - step * prev_grad
```

6.2) Numerical Gradient

Getting the analytic gradient correct for complicated functions is tricky. A very handy method of verifying the analytic gradient or even substituting for it is to estimate the gradient at a point by means of *finite differences*.

Assume that we are given a function $f(x)$ that takes a column vector as its argument and returns a scalar value. In gradient descent, we will want to estimate the gradient of f at a particular x_0.

The i^{th} component of $\nabla_x f(x_0)$ can be estimated as

$$\frac{f(x_0 + \delta^i) - f(x_0 - \delta^i)}{2\delta}$$

where δ^i is a column vector whose i^{th} coordinate is δ, a small constant such as 0.001, and whose other components are zero. Note that adding or subtracting δ^i is the same as incrementing or decrementing the i^{th} component of x_0 by δ, leaving the other components of x_0 unchanged. Using these results, we can estimate the i^{th} component of the gradient.

For example, if $x_0 = (1, 1, \ldots, 1)^T$ and $\delta = 0.01$, we may approximate the first component of $\nabla_x f(x_0)$ as

$$\frac{f((1,1,1,\dots)^T + (0.01,0,0,\dots)^T) - f((1,1,1,\dots)^T - (0.01,0,0,\dots)^T)}{2 \cdot 0.01}.$$

(We add the transpose so that these are column vectors.) **This process should be done for each dimension independently, and together the results of each computation are compiled to give the estimated gradient, which is d dimensional.**

Implement this as a function `num_grad` that takes as arguments the objective function `f` and a value of `delta`, and returns a new **function** that takes an `x` (a column vector of parameters) and returns a gradient column vector.

Note: As in the previous part, make sure you do not modify your input vector.

The test cases are shown below; these use the functions defined in the previous exercise.

```
x = cv([0.])
ans=(num_grad(f1)(x).tolist(), x.tolist())

x = cv([0.1])
ans=(num_grad(f1)(x).tolist(), x.tolist())

x = cv([0., 0.])
ans=(num_grad(f2)(x).tolist(), x.tolist())

x = cv([0.1, -0.1])
ans=(num_grad(f2)(x).tolist(), x.tolist())
```

```
def num_grad(f, delta=0.001):
    def df(x):
        x_temp = x.copy()
        d, n = x.shape
        grad = np.zeros((d, 1))

        for row in range(d):
            increment = np.zeros((d,1))
            increment[row, 0] = delta
            grad_row = (f(x + increment) - f(x - increment))/(2*delta)
            grad[row, 0] = grad_row
        return grad
    return df

```

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A faster (one function evaluation per entry), though sometimes less accurate, estimate is to use:

$$\frac{f(x_0 + \delta^i) - f(x_0)}{\delta}$$

for the i^{th} component of $\nabla_x f(x_0)$.

6.3) Using the Numerical Gradient

Recall that our generic gradient descent function takes both a function `f` that returns the value of our function at a given point, and `df`, a function that returns a gradient at a given point. Write a function `minimize` that takes only a function `f` and uses this function and numerical gradient descent to return the local minimum. We have provided you with our implementations of `num_grad` and `gd`, so you should not redefine them in the code box below. You may use the default of `delta=0.001` for `num_grad`.

Hint: Your definition of `minimize` should call `num_grad` exactly once, to return a function that is called many times. You should return the same outputs as `gd`.

The test cases are:

```
ans = package_ans(minimize(f1, cv([0.]), lambda i: 0.1, 1000))

ans = package_ans(minimize(f2, cv([0., 0.]), lambda i: 0.01, 1000))
```

```
def minimize(f, x0, step_size_fn, max_iter):
    #itinitialize some paramaters
    num_iter = 0
    list_x = []
    list_f = []
    #copy the input so we dont modify it on each iter
    x = x0.copy()
    num_grad_func = num_grad(f)
    #exceute updates intil we reach out max num of updates
    while num_iter < max_iter:
      #calculate the current gradient
      grad = num_grad_func(x)
      list_x.append(x)
      list_f.append(f(x))
      #next selection is the current - the gradient times a step size
```

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7) Applying gradient descent to SVM objective

Now that we've implemented gradient descent in the general case, let's go back to hinge loss and the SVM objective. Our goal in this section will be to derive and implement appropriate gradient calculations that we can use with `gd` for optimization of the SVM objective. In the derivations below, we'll consider linear classifiers *with* offset, i.e., θ, θ_0.

Recall that hinge loss is defined as:

$$L_h(v) = \begin{cases} 1 - v & \text{if } v < 1 \\ 0 & \text{otherwise} \end{cases} \; .$$

This is usually implemented as:

$$hinge(v) = \max(0, 1 - v) \; .$$

The hinge loss function, in the context of our problem, takes in the distance from a point to the separator as x. This loss function helps us penalize a model for leaving points within a distance to our separator:

$$L_h(y\,(\theta \cdot x + \theta_0)) = \begin{cases} 1 - y\,(\theta \cdot x + \theta_0) & \text{if } y\,(\theta \cdot x + \theta_0) < 1 \\ 0 & \text{otherwise} \end{cases}$$

The SVM objective function incorporates the mean of the hinge loss over all points and introduces a regularization term to this equation to make sure that the magnitude of θ stays small (and keeps the margin large). Note that we have used λ instead of $\frac{\lambda}{2}$ for simplicity (and without loss of generality).

$$J(\theta, \theta_0) = \left[\frac{1}{n}\sum_{i=1}^{n} L_h(y^{(i)}\,(\theta \cdot x^{(i)} + \theta_0\,)\,)\right] + \lambda||\theta||^2$$

We're interested in applying our gradient descent procedure to this function in order to find the 'best' separator for our data, where 'best' is measured by the lowest possible SVM objective.

7.1) Calculating the SVM objective

Implement the single-argument function `hinge`, which computes L_h, and use that to implement hinge loss for a data point and separator. Using the latter function, implement the SVM objective. Note that these functions should work for matrix/vector arguments, so that we can compute the objective for a whole dataset with one call.

Note that `x` is $d \times n$, `y` is $1 \times n$, `th` is $d \times 1$, `th0` is 1×1, `lam` is a scalar.

Hint: Look at `np.where` for implementing `hinge`.

In the test cases for this problem, we'll use the following `super_simple_separable` test dataset and test separator for some of the tests. A couple of the test cases are also shown below.

```
def super_simple_separable():
    X = np.array([[2, 3, 9, 12],
                  [5, 2, 6, 5]])
    y = np.array([[1, -1, 1, -1]])
    return X, y

sep_e_separator = np.array([[-0.40338351], [1.1849563]]), np.array([[-2.26910091]])

# Test case 1
x_1, y_1 = super_simple_separable()
th1, th1_0 = sep_e_separator
ans = svm_obj(x_1, y_1, th1, th1_0, .1)

# Test case 2
ans = svm_obj(x_1, y_1, th1, th1_0, 0.0)
```

Note: In this section, you will code many individual functions, each of which depends on previous ones. We **strongly recommend** that you test each of the components on your own to debug.

```
def hinge(v):
    #hinge loss calculate for each data point so v is 1xn
    return np.where(v>0, v, 0)

# x is dxn, y is 1xn, th is dx1, th0 is 1x1
def hinge_loss(x, y, th, th0):
    return 1 - y * (np.dot(th.T, x) + th0)

# x is dxn, y is 1xn, th is dx1, th0 is 1x1, lam is a scalar
def svm_obj(x, y, th, th0, lam):
    d, n = x.shape
    h_loss = hinge_loss(x, y, th, th0)
    h_loss_list = hinge(loss_list)
    reg_loss = lam * (np.linalg.norm(th)**2)

    return np.sum(loss_list)/n + reg_loss


```

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Here is the solution we wrote:

```
def hinge(v):
    return np.where(v >= 1, 0, 1 - v)

def hinge_loss(x, y, th, th0):
    return hinge(y * (np.dot(th.T, x) + th0))

def svm_obj(X, y, th, th0, lam):
    return np.mean(hinge_loss(X, y, th, th0)) + lam * np.linalg.norm(th) ** 2
```

7.2) Calculating the SVM gradient

Define a function `svm_obj_grad` that returns the gradient of the SVM objective function with respect to θ and θ_0 in a single column vector. The last component of the gradient vector should be the partial derivative with respect to θ_0. Look at `np.vstack` as a simple way of stacking two matrices/vectors vertically. We have broken it down into pieces that mimic steps in the chain rule; this leads to code that is a bit inefficient but easier to write and debug. We can worry about efficiency later.

Some test cases that may be of use are shown below:

```
X1 = np.array([[1, 2, 3, 9, 10]])
y1 = np.array([[1, 1, 1, -1, -1]])
th1, th10 = np.array([[-0.31202807]]), np.array([[1.834     ]])
X2 = np.array([[2, 3, 9, 12],
               [5, 2, 6, 5]])
y2 = np.array([[1, -1, 1, -1]])
th2, th20=np.array([[ -3.,  15.]]).T, np.array([[ 2.]])
```

```
d_hinge(np.array([[ 71.]])).tolist()
d_hinge(np.array([[ -23.]])).tolist()
d_hinge(np.array([[ 71, -23.]])).tolist()

d_hinge_loss_th(X2[:,0:1], y2[:,0:1], th2, th20).tolist()
d_hinge_loss_th(X2, y2, th2, th20).tolist()
d_hinge_loss_th0(X2[:,0:1], y2[:,0:1], th2, th20).tolist()
d_hinge_loss_th0(X2, y2, th2, th20).tolist()

d_svm_obj_th(X2[:,0:1], y2[:,0:1], th2, th20, 0.01).tolist()
d_svm_obj_th(X2, y2, th2, th20, 0.01).tolist()
d_svm_obj_th0(X2[:,0:1], y2[:,0:1], th2, th20, 0.01).tolist()
d_svm_obj_th0(X2, y2, th2, th20, 0.01).tolist()

svm_obj_grad(X2, y2, th2, th20, 0.01).tolist()
svm_obj_grad(X2[:,0:1], y2[:,0:1], th2, th20, 0.01).tolist()
```

Note: In this section, you will code many individual functions, each of which depends on previous ones. We **strongly recommend** that you test each of the components on your own to debug.

```
# Returns the gradient of hinge(v) with respect to v.
def d_hinge(v):
    return np.where(v<1, -1, 0)

# Returns the gradient of hinge_loss(x, y, th, th0) with respect to th
def d_hinge_loss_th(x, y, th, th0):
    return y * x * d_hinge(y * (np.dot(th.T, x) + th0))

# Returns the gradient of hinge_loss(x, y, th, th0) with respect to th0
def d_hinge_loss_th0(x, y, th, th0):
    return y * d_hinge(y * (np.dot(th.T, x) + th0))

# Returns the gradient of svm_obj(x, y, th, th0) with respect to th
def d_svm_obj_th(x, y, th, th0, lam):
    return np.mean(d_hinge_loss_th(x, y, th, th0), axis = 1, keepdims = True) + 2*lam*
# Returns the gradient of svm_obj(x, y, th, th0) with respect to th0
def d_svm_obj_th0(x, y, th, th0, lam):
    return np.mean(d_hinge_loss_th0(x, y, th, th0), axis = 1, keepdims = True)

# Returns the full gradient as a single vector (which includes both th, th0)
def svm_obj_grad(x, y, th, th0, lam):
    return np.vstack((d_svm_obj_th(x, y, th, th0, lam), d_svm_obj_th0(x, y, th, th0, l
```

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7.3) Batch SVM minimize

Putting it all together, use the functions you built earlier to write a gradient descent minimizer for the SVM objective. You do not need to paste in your previous definitions; you can just call the ones defined by the staff. You will need to call `gd`, which is already defined for you as well; your function `batch_svm_min` should return the values that `gd` does.

- Initialize all the separator parameters to zero,
- use the step size function provided below, and
- specify 10 iterations.

Test cases are shown below, where an additional separable test data set has been specified.

```
def separable_medium():
    X = np.array([[2, -1, 1, 1],
                  [-2, 2, 2, -1]])
    y = np.array([[1, -1, 1, -1]])
    return X, y
sep_m_separator = np.array([[ 2.69231855], [ 0.67624906]]), np.array([[-3.02402521]])

x_1, y_1 = super_simple_separable()
ans = package_ans(batch_svm_min(x_1, y_1, 0.0001))

x_1, y_1 = separable_medium()
ans = package_ans(batch_svm_min(x_1, y_1, 0.0001))
```

```
def batch_svm_min(data, labels, lam):
    def svm_min_step_size_fn(i):
       return 2/(i+1)**0.5
    d, n = data.shape
    th = np.zeros((d+1, 1))

    def f(th):
        return svm_obj(data, labels, th[:-1, :], th[-1:, :], lam)

    def df(th):
        return svm_obj_grad(data, labels, th[:-1, :], th[-1:, :], lam)


    return gd(f, df, th, svm_min_step_size_fn, 10)
```

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7.4) Numerical SVM objective (Optional)

Recall from the previous question that we were able to closely approximate gradients with numerical estimates. We may apply the same technique to optimize the SVM objective.

Using your definition of `minimize` and `num_grad` from the previous problem, implement a function that optimizes the SVM objective through numeric approximations.

How well does this function perform, compared to the analytical result? Consider both accuracy and runtime.