Q&A（2.71-2.80）

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2020 年 3 月 5 日

Exercise 2.71: (Criterion to decide if a state is mixed or pure) Let ρ be a density operator. Show that $tr(\rho_2) \leq 1$, with equality if and only if ρ is a pure state.
Answer:

1. Since ρ is positive, it must have a spectral decomposition,$\rho = \sum_i \lambda_i |i\rangle \langle i|$
   the result of $\sum_i |i\rangle \langle i|$ is a matrix, and the Diagonal elements is $\sum_i \lambda_i \langle i|i\rangle$ for $0 \leq \lambda_i \leq 1$.

$$\begin{aligned}
\rho^2 =& \sum_{ij} \lambda_i \lambda_j |i\rangle \langle i|j\rangle \langle j| \\
=& \sum_i \lambda_i^2 |i\rangle \langle i| \\
tr(\rho^2) =& tr(\sum_i \lambda_i^2 |i\rangle \langle i|) \\
=& \sum_i \lambda_i^2 tr(|i\rangle \langle i|) \\
=& \sum_i \lambda_i^2 \langle i|i\rangle \\
=& \sum_i \lambda_i^2
\end{aligned} \tag{1}$$

   Since $\lambda_i \geq \lambda_i^2$ and $\sum_i \lambda_i = 1$, then $\sum_i \lambda_i^2 \leq \sum_i \lambda_i = 1$. When $\lambda_i = 1$, it's a pure state.
   $\rho = \sum_i \lambda_i |i\rangle \langle i| = \sum_i |i\rangle \langle i|$
   $tr(\rho^2) = tr(\sum_{ij} |i\rangle \langle i|j\rangle \langle j|)$

Exercise 2.72: (Bloch sphere for mixed states) The Bloch sphere picture for pure states of a single qubit was introduced in Section 1.2. This description has an important generalization to mixed states as follows.

(a) Show that an arbitrary density matrix for a mixed state qubit may be written as

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}, \tag{2}$$

where $\vec{r}$ is a real three-dimensional vector such that $||\vec{r}|| \leq 1$. This vector is known as the *Bloch vector* for the state ρ.

(b) What is the *Bloch vector* representation for the state $\rho = I/2$?

(c) Show that a state ρ is pure if and only if $||\vec{r}|| = 1$.

(d) Show that for pure states the description of the *Bloch vector* we have given coincides with that in Section 1.2.

Answer:

Since density matrix is Hermitain, matrix representation is $\rho = \begin{bmatrix} a & b \\ b^* & d \end{bmatrix}$, $a, d \in R$ and $b \in C$. Because ρ is density matrix, $\rho = a + d = 1$.
Define $a = \frac{(1+r_3)}{2}$, $d = \frac{(1-r_3)}{2}$ and $b = \frac{(r_1 - ir_2)}{2}$, $r_i \in R$.
In this case,

$$\begin{aligned}
\vec{r} \cdot \vec{\sigma} =& r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3 \\
=& \begin{bmatrix} r_3 & (r_1 - ir_2) \\ (r_1 + ir_2) & -r_3 \end{bmatrix} \\
\rho =& \begin{bmatrix} a & b \\ b^* & d \end{bmatrix} \\
=& \frac{1}{2}\begin{bmatrix} (1+r_3) & (r_1 - ir_2) \\ (r_1 + ir_2) & (1 - r_3) \end{bmatrix} \\
=& \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} r_3 & (r_1 - ir_2) \\ (r_1 + ir_2) & -r_3 \end{bmatrix} \\
=& \frac{1}{2}(I + \vec{r} \cdot \vec{\sigma}).
\end{aligned} \tag{3}$$

Thus for arbitrary density matrix   can be written as $\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$.
Next, we prove the condition that $||\vec{r}|| \leq 1$.
Since ρ is a positive operator, then the eigenvalues of ρ are non-negative.

$$\begin{aligned}
det(\rho - \lambda I) =& det\left(\frac{1}{2}\begin{bmatrix} (1+r_3) - \lambda & (r_1 - ir_2) \\ (r_1 + ir_2) & (1 - r_3) - \lambda \end{bmatrix}\right) \\
=& (\frac{1}{4}(1 + r_3) - \lambda)((1 - r_3) - \lambda) - \frac{1}{4}(r_1 - ir_2)(r_1 + ir_2) \\
=& \frac{1}{4}(\lambda^2 - \lambda + 1 - r_3^2 - (r_1^2 + r_2^2)) \\
=& \frac{1}{4}(\lambda^2 - \lambda + 1 - |\vec{r}|^2) \\
=& 0 \\
\lambda =& \frac{1 \pm \sqrt{1 - 4 * \frac{1}{4}(1 - |\vec{r}|^2)}}{2} \\
=& \frac{1 \pm |\vec{r}|}{2} \\
\geq& 0
\end{aligned} \tag{4}$$

Since $\frac{1-|\vec{r}|}{2} \geq 0 \to |\vec{r}| \leq 1$.

2. $\rho = \frac{I}{2} = \frac{1}{2}\begin{bmatrix} (1+r_3) & (r_1 - ir_2) \\ (r_1 + ir_2) & (1 - r_3) \end{bmatrix}$

$r_3 = 0$, $r_1 = ir_2 = 0$, thus *Bloch vector* is $\vec{r} = (0,0,0)$ and in the center of the ball.

3.

$$\begin{aligned}
\rho^2 =& \frac{I+\vec{r}\cdot\vec{\sigma}}{2}\frac{I+\vec{r}\cdot\vec{\sigma}}{2} \\
=& \frac{1}{4}\left(I + 2\vec{r}\cdot\vec{\sigma} + \vec{r}\cdot\vec{\sigma}\vec{r}\cdot\vec{\sigma}\right) \\
=& \frac{1}{4}\left(I + 2\vec{r}\cdot\vec{\sigma} + (r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3)(r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3)\right) \\
=& \frac{1}{4}\left(I + 2\vec{r}\cdot\vec{\sigma} + (\sum_{ij} r_i r_j(\delta_{ij} I + \sum_{k=1}^{3}\epsilon_{ijk}\sigma_k))\right) \\
=& \frac{1}{4}\left(I + 2\vec{r}\cdot\vec{\sigma} + (r_1r_2\sigma_3 - r_2r_1\sigma_3 - r_1r_3\sigma_2 + r_3r_1\sigma_2 + r_2r_3\sigma_1 - r_3r_2\sigma_1 + r_1r_1 I + r_2r_2 I + r_3r_3 I)\right) \\
=& \frac{1}{4}\left(I + 2\vec{r}\cdot\vec{\sigma} + ||\vec{r}||^2 I\right) \\
tr(\rho^2) =& \frac{1}{4}(2 + 2||\vec{r}||^2)(\because tr(\sigma_i) = 0, i = 1,2,3) \\
=& \frac{1}{2}(1 + ||\vec{r}||^2)
\end{aligned} \tag{5}$$

If ρ is pure, then $tr(rho^2) = 1$.

$$\begin{aligned}
tr(rho^2) =& \frac{1}{2}(1 + ||\vec{r}||^2) = 1 \\
& 1 + ||\vec{r}||^2 = 2 \\
& ||\vec{r}||^2 = 1 \\
& ||\vec{r}|| = 1
\end{aligned} \tag{6}$$

Conversely, if $||\vec{r}|| = 1$, then $tr(\rho^2) = \frac{1}{2}(1 + ||\vec{r}||^2) = 1$. Therefore, ρ is pure.

4.

Exercise 2.73: Let ρ be a density operator. A *minimal ensemble* for ρ is an ensemble $\{p_i, |\psi_i\rangle\}$ containing a number of elements equal to the rank of ρ. Let $|\psi\rangle$ be any state in the support of ρ. (The support of a Hermitian operator A is the vector space spanned by the eigenvectors of A with non-zero eigenvalues.) Show that there is a minimal ensemble for ρ that contains $|\psi\rangle$, and moreover that in any such ensemble $|\psi\rangle$ must appear with probability

$$p_i = \frac{1}{\langle\psi_i|\,\rho^{-1}\,|\psi_i\rangle}, \tag{7}$$

where ρ^{-1} is defined to be the inverse of ρ, when ρ is considered as an operator acting only on the support of ρ. (This definition removes the problem that ρ may not have an inverse.)

Answer:

The density operator ρ can be spectral decomposition, $\rho = \sum_i \lambda_i |i\rangle\langle i|, \lambda_i > 0$, then $\rho^{-1} = \sum_i \frac{1}{\lambda_i}|i\rangle\langle i|$, Obviously, $\{\sqrt{\lambda_i}, |i\rangle\}$ is a minimal ensemble for ρ (note: a number of $\sqrt{\lambda_i}$ equal to the rank of density operator ρ). Suppose $\{\sqrt{p_i}, |\psi_i\rangle\}$ is the *minimal ensemble* of density operator ρ and $\{p_i = \lambda_i, |\psi_i\rangle\}$, Since $|\psi_i\rangle = \sum_j a_j |j\rangle$, then $a_j = \langle j|\psi_j\rangle$, According to postulate 2, There is a unitary operator U and probability p_i, so that $|\psi_i\rangle$ enters state $U|\psi_i\rangle$ with probability $\sqrt{p_i}$, so there is

$$\widetilde{|\psi_i\rangle} = \sqrt{p_i}\,|\psi_i\rangle = \sqrt{p_i}(\sum_j a_j\,|j\rangle) = \sum_j u_{ij}\widetilde{|j\rangle} = \sum_j u_{ij}\sqrt{\lambda_j}\,|j\rangle\,. \tag{8}$$

Then $\sqrt{p_i}a_j = u_{ij}\sqrt{\lambda_j}$, after squaring both sides of the equation, $|u_{ij}|^2 = p_i\frac{|a_j|^2}{\lambda_j}$. Since the sum of the squares of the elements in each row and column of an arbitrary unitary matrix is equal to 1, $\sum_i |u_{ij}|^2 = 1$.

$$p_i\sum_j \frac{|a_{j|^2}}{\lambda_j} = \lambda_j|u_{ij}|^2 = 1 \tag{9}$$

Also because $\rho^{-1} = \sum_i \frac{1}{\lambda_i}|i\rangle\langle i|$, $\sum_j \frac{|a_j|^2}{\lambda_j} = \langle\psi_i|\,\rho^{-1}\,|\psi_i\rangle$, then:

$$p_i = \frac{1}{\sum_j \frac{|a_j|^2}{\lambda_j}} = \frac{1}{\langle\psi_i|\,\rho^{-1}\,|\psi_i\rangle} \tag{10}$$

Exercise 2.74: Suppose a composite of systems A and B is in the state $|a\rangle|b\rangle$, where $|a\rangle$ is a pure state of system A, and $|b\rangle$ is a pure state of system B. Show that the reduced density operator of system A alone is a pure state.

Answer:

$$\begin{aligned}
\rho^{AB} =& |a\rangle\langle a| \otimes |b\rangle\langle b| \\
\rho^{A} =& tr_B(\rho^{AB}) \\
=& |a\rangle\langle a|\, tr(|b\rangle\langle b|) \\
=& |a\rangle\langle a|\,\langle b|b\rangle \\
=& |a\rangle\langle a| \\
tr((\rho^A)^2) =& tr(|a\rangle\langle a|a\rangle\langle a|) \\
=& tr(|a\rangle\langle a|) \\
=& \langle a|a\rangle \\
=& 1
\end{aligned} \tag{11}$$

Thus ρ^A is pure.

Exercise 2.75: For each of the four Bell states, find the reduced density operator for each qubit.

Answer:

Suppose the four Bell states which $|\psi_i\rangle$, i=1,2,3,4 are as follows.

$$
\begin{aligned}
|\psi_1\rangle =& \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
|\psi_2\rangle =& \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
|\psi_3\rangle =& \frac{|10\rangle + |01\rangle}{\sqrt{2}} \\
|\psi_4\rangle =& \frac{|01\rangle - |10\rangle}{\sqrt{2}}
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\rho^1 =& \frac{|00\rangle + |11\rangle}{\sqrt{2}} \frac{\langle 00| + \langle 11|}{\sqrt{2}} \\
=& \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2} \\
\rho^A =& tr_B\left(\frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2}\right) \\
=& \frac{ket0\langle 0|\langle 0|0\rangle + |0\rangle\langle 1|\langle 0|1\rangle + |1\rangle\langle 0|\langle 1|0\rangle + |1\rangle\langle 1|\langle 1|1\rangle}{2} \\
=& \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} \\
=& \frac{I}{2}
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\rho^B =& tr_A\left(\frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2}\right) \\
=& \frac{|0\rangle\langle 0|\langle 0|0\rangle + |0\rangle\langle 1|\langle 0|1\rangle + |1\rangle\langle 0|\langle 1|0\rangle + |1\rangle\langle 1|\langle 1|1\rangle}{2} \\
=& \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} \\
=& \frac{I}{2}
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
\rho^2 =& \frac{|00\rangle - |11\rangle}{\sqrt{2}} \frac{\langle 00| - \langle 11|}{\sqrt{2}} \\
=& \frac{|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|}{2} \\
\rho^A =& tr_B\left(\frac{|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|}{2}\right) \\
=& \frac{|0\rangle\langle 0|\langle 0|0\rangle - |0\rangle\langle 1|\langle 0|1\rangle - |1\rangle\langle 0|\langle 1|0\rangle + |1\rangle\langle 1|\langle 1|1\rangle}{2} \\
=& \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} \\
=& \frac{I}{2} \\
\rho^B =& tr_A\left(\frac{|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|}{2}\right) \\
=& \frac{|0\rangle\langle 0|\langle 0|0\rangle - |0\rangle\langle 1|\langle 0|1\rangle - |1\rangle\langle 0|\langle 1|0\rangle + |1\rangle\langle 1|\langle 1|1\rangle}{2} \\
=& \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} \\
=& \frac{I}{2}
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
\rho^3 =& \frac{|10\rangle + |01\rangle}{\sqrt{2}} \frac{\langle 10| + \langle 01|}{\sqrt{2}} \\
=& \frac{|10\rangle \langle 10| - |10\rangle \langle 01| - |01\rangle \langle 10| + |01\rangle \langle 01|}{2} \\
\rho^A =& tr_B(\frac{|10\rangle \langle 10| - |10\rangle \langle 01| - |01\rangle \langle 10| + |01\rangle \langle 01|}{2}) \\
=& \frac{|1\rangle \langle 1| \langle 0|0\rangle - |1\rangle \langle 0| \langle 0|1\rangle - |0\rangle \langle 1| \langle 1|0\rangle + |0\rangle \langle 0| \langle 1|1\rangle}{2} \\
=& \frac{|0\rangle \langle 0| + |1\rangle \langle 1|}{2} \\
=& \frac{I}{2} \\
\rho^B =& tr_A(\frac{|10\rangle \langle 10| - |10\rangle \langle 01| - |01\rangle \langle 10| + |01\rangle \langle 01|}{2}) \\
=& \frac{|0\rangle \langle 0| \langle 1|1\rangle - |0\rangle \langle 1| \langle 1|0\rangle - |1\rangle \langle 0| \langle 0|1\rangle + |1\rangle \langle 1| \langle 0|0\rangle}{2} \\
=& \frac{|0\rangle \langle 0| + |1\rangle \langle 1|}{2} \\
=& \frac{I}{2}
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
\rho^4 =& \frac{|01\rangle - |10\rangle}{\sqrt{2}} \frac{\langle 01| - \langle 10|}{\sqrt{2}} \\
=& \frac{|01\rangle \langle 01| - |01\rangle \langle 10| - |10\rangle \langle 01| + |10\rangle \langle 10|}{2} \\
\rho^A =& tr_B(\frac{|01\rangle \langle 01| - |01\rangle \langle 10| - |10\rangle \langle 01| + |10\rangle \langle 10|}{2}) \\
=& \frac{|0\rangle \langle 0| \langle 1|1\rangle - |0\rangle \langle 1| \langle 1|0\rangle - |1\rangle \langle 0| \langle 0|1\rangle + |1\rangle \langle 1| \langle 0|0\rangle}{2} \\
=& \frac{|0\rangle \langle 0| + |1\rangle \langle 1|}{2} \\
=& \frac{I}{2} \\
\rho^B =& tr_A(\frac{|01\rangle \langle 01| - |01\rangle \langle 10| - |10\rangle \langle 01| + |10\rangle \langle 10|}{2}) \\
=& \frac{|1\rangle \langle 1| \langle 0|0\rangle - |1\rangle \langle 0| \langle 0|1\rangle - |0\rangle \langle 1| \langle 1|0\rangle + |0\rangle \langle 0| \langle 1|1\rangle}{2} \\
=& \frac{|0\rangle \langle 0| + |1\rangle \langle 1|}{2} \\
=& \frac{I}{2}
\end{aligned} \tag{17}
$$

Exercise 2.76: Extend the proof of the Schmidt decomposition to the case where A and B may have state spaces of different dimensionality.

Answer:

Exercise 2.77: Suppose ABC is a three component quantum system. Show by example that there are quantum states $|\psi\rangle$ of such systems which can not be written in the form

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle \tag{18}$$

where λ_i are real numbers, and $|i_A\rangle, |i_B\rangle, |i_C\rangle$ are orthonormal bases of the respective systems. **Answer:**

Exercise 2.78: Prove that a state $|\psi\rangle$ of a composite system AB is a product state if and only if it has Schmidt number 1. Prove that $|\psi\rangle$ is a product state if and only if ρ_A (and thus ρ_B) are pure states.

Answer:

If $|\psi\rangle$ of a composite system AB is a product state, then the state $|i_A\rangle$ for systerm A and $|i_B\rangle$ for systerm B, so thst $|\psi\rangle = |i_A\rangle |i_B\rangle$. Therefore the Schmidt number is 1.

Conversely, if Schmidt number is 1. $|\psi\rangle$ is written as $|\psi\rangle = |i_A\rangle |i_B\rangle$, thus $|\psi\rangle$ is a product state.

If $|\psi\rangle$ is a product state, $|\psi\rangle = |i_A\rangle |i_B\rangle$.

$$\begin{aligned}
\rho^{AB} &= |i_A\rangle \langle i_A| \otimes |i_B\rangle \langle i_B| \\
\rho^{A} &= tr_B(\rho^{AB}) \\
&= |i_A\rangle \langle i_A| \, tr(|i_B\rangle \langle i_B|) \\
&= |i_A\rangle \langle i_A| \langle i_B|i_B\rangle \\
&= |i_A\rangle \langle i_A| \\
tr((\rho^A)^2) &= tr(|i_A\rangle \langle i_A|i_A\rangle \langle i_A|) \\
&= tr(|i_A\rangle \langle i_A|) \\
&= \langle i_A|i_A\rangle \\
&= 1 \\
tr((\rho^B)^2) &= tr(|i_B\rangle \langle i_B|i_B\rangle \langle i_B|) \\
&= tr(|i_B\rangle \langle i_B|) \\
&= \langle i_B|i_B\rangle \\
&= 1
\end{aligned} \tag{19}$$

Thus ρ_A (and thus ρ_B) are pure states.

Conversely,If ρ_A (and thus ρ_B) are pure states. The state is written as $|\psi\rangle = \sum_i \lambda_i |i_A\rangle |i_B\rangle$.

$$\begin{aligned}
\rho^{A} &= tr_B(|\psi\rangle \langle\psi|) \\
&= \sum_i \lambda_i |i_A\rangle \langle i_A| \, tr_B(\sum_i \lambda_i^* |i_B\rangle \langle i_B|) \\
&= \sum_i (\lambda_i)^2 |i_A\rangle \langle i_A|
\end{aligned} \tag{20}$$

Since ρ_A is pure states, $tr((\rho^A)^2) = 1$.

$$
\begin{aligned}
tr((\rho^A)^2) =& tr(\sum_{i,j} (\lambda_i)^2 (\lambda_j)^2 |i_A\rangle \langle i_A|j_A\rangle \langle j_A|) \\
=& tr(\sum_i \lambda_i^4 |i_A\rangle \langle i_A|) \\
=& \sum_i \lambda_i^4 \langle i_A|i_A\rangle \\
=& \sum_i \lambda_i^4 = 1
\end{aligned} \tag{21}
$$

Because of $\sum_i \lambda_i^4 = 1$ and $\sum_i (\lambda_i)^2 = 1$, Thus we can get $\lambda_i^2 = \lambda_i^4$ where λ_i are non-negative real numbers. Then, Only one i is equal to 1, and the other i is equal to 0. Thus, we proved that $|\psi\rangle = |i_A\rangle |i_B\rangle$.

Exercise 2.79: Consider a composite system consisting of two qubits. Find the Schmidt decompositions of the states
$\frac{|00\rangle+|11\rangle}{\sqrt{2}}$; $\frac{|00\rangle+|01\rangle+|10\rangle+|11\rangle}{\sqrt{2}}$; and $\frac{|00\rangle+|01\rangle+|10\rangle}{2}$.

Answer:

$$
\begin{aligned}
&\frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
=& \frac{|0\rangle \otimes |0\rangle}{\sqrt{2}} + \frac{|1\rangle \otimes |1\rangle}{\sqrt{2}}
\end{aligned} \tag{22}
$$

The composite system is consisted by the state $|i_A\rangle$ for systerm A and the state $|i_B\rangle$ for systerm B. The standard orthogonal basis of the A and B systems consist of $|0\rangle$ and $|1\rangle$.

$$
\begin{aligned}
&\frac{|00\rangle + |01\rangle + |10\rangle + |11\rangle}{\sqrt{2}} \\
=& \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}
\end{aligned} \tag{23}
$$

Suppose $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$, then the above equation equal $|\psi\rangle |\psi\rangle$.

$$
\begin{aligned}
&\frac{|00\rangle + |01\rangle + |10\rangle}{2} \\
=&
\end{aligned} \tag{24}
$$

Exercise 2.80: Suppose $|\psi\rangle$ and $|\phi\rangle$ are two pure states of a composite quantum system with components A and B, with identical Schmidt coefficients. Show that there are unitary transformations U on system A and V on system B such that $|\psi\rangle = (U \otimes V) |\phi\rangle$.

Answer:

Suppose $|\psi\rangle = \sum_i \lambda_i |\psi_i\rangle_A |\psi_i\rangle_B$ and $|\phi\rangle = \sum_i \lambda_i |\phi_i\rangle_A |\phi_i\rangle_B$. Define $U = \sum_i |\psi_i\rangle_A \langle\phi_i|_A$ and

$V = \sum_j |\psi_j\rangle_B \langle\phi_j|_B$.

$$\begin{aligned}(U \otimes V)|\phi\rangle &= \sum_i \lambda_i U |\phi_i\rangle_A V |\phi_i\rangle_B \\ &= \sum_{i,j,k} \lambda_i |\psi_j\rangle_A \langle\phi_j|_A |\phi_i\rangle_A |\psi_k\rangle_B \langle\phi_k|_B |\phi_i\rangle_B \\ &= \sum_i \lambda_i |\psi_i\rangle |\psi_i\rangle \\ &= |\psi\rangle\end{aligned} \tag{25}$$