

Q&A (2.51-2.60)

LuoTingyu JiangHui

2020 年 2 月 19 日

Exercise 2.51: Verify that the Hadamard gate H is unitary.

Answer:

$$H^\dagger H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Thus the Hadamard gate H is unitary.

Exercise 2.52: Verify that $H^2 = I$.

Answer:

$$H^2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Exercise 2.53: What are the eigenvalues and eigenvectors of H ?

Answer:

$$\det|A - \lambda I| = \det \left| \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda \end{vmatrix} = 0$$

Eigenvalues are ± 1 and associated eigenvectors are $|\lambda_{\pm}\rangle = \frac{1}{\sqrt{4 \pm 2\sqrt{2}}} \begin{bmatrix} 1 \\ -1 \pm \sqrt{2} \end{bmatrix}$.

Exercise 2.54: Suppose A and B are commuting Hermitian operators. Prove that $\exp(A)\exp(B) = \exp(A+B)$. (Hint: Use the results of Section 2.1.9.)

Answer:

Since $[A, B] = 0$, A and B are simultaneously diagonalize, $A = \sum_i a_i |i\rangle \langle i|$, $B = \sum_j b_j |j\rangle \langle j|$.

$$\begin{aligned} \exp(A)\exp(B) &= \sum_i \exp(a_i) |i\rangle \langle i| B = \sum_j \exp(b_j) |j\rangle \langle j| \\ &= \sum_{i,j} \exp(a_i + b_j) |i\rangle \langle i|j\rangle \langle j| \\ &= \sum_{i,j} \exp(a_i + b_j) |i\rangle \langle j| \delta_{i,j} \\ &= \sum_i \exp(a_i + b_i) |i\rangle \langle i| \\ &= \exp(A+B). \end{aligned} \tag{1}$$

Exercise 2.55: Prove that $U(t_1, t_2)$ defined in Equation (2.91) is unitary.

Answer:

$$\begin{aligned}
U(t_1, t_2)^\dagger U(t_1, t_2) &= \exp\left(\frac{iH(t_2 - t_1)}{\hbar}\right) \exp\left(\frac{-iH(t_2 - t_1)}{\hbar}\right) \\
&= \exp\left(\frac{i \sum_{E_1} E_1 |E_1\rangle \langle E_1| (t_2 - t_1)}{\hbar}\right) \exp\left(\frac{-i \sum_{E_2} E_2 |E_2\rangle \langle E_2| (t_2 - t_1)}{\hbar}\right) \\
&= \sum_{E_1, E_2} \exp\left(\frac{i E_1 (t_2 - t_1)}{\hbar}\right) |E_2\rangle \langle E_2| \exp\left(\frac{-i E_2 (t_2 - t_1)}{\hbar}\right) |E_1\rangle \langle E_1| \\
&= \sum_{E_1, E_2} \exp\left(\frac{i(E_1 - E_2)(t_2 - t_1)}{\hbar}\right) |E_1\rangle \langle E_1| E_2 \rangle \langle E_2| \\
&= \sum_{E_1, E_2} \exp\left(\frac{i(E_1 - E_2)(t_2 - t_1)}{\hbar}\right) |E_1\rangle \langle E_2| \delta_{E_1, E_2} \\
&= \sum_{E_1} |E_1\rangle \langle E_1| \\
&= I.
\end{aligned} \tag{2}$$

Thus $U(t_1, t_2)$ is unitary.

Exercise 2.56: Use the spectral decomposition to show that $K - i \log(U)$ is Hermitian for any unitary U , and thus $U = \exp(iK)$ for some Hermitian K .

Answer:

Since U is unitary, then U can perform spectral decomposition, $U = \sum_i \lambda_i |i\rangle \langle i|$

$$\begin{aligned}
K^\dagger &= (-i \log(U))^\dagger \\
&= (-i \log\left(\sum_i \lambda_i |i\rangle \langle i|\right))^\dagger \\
&= (i \sum_i \log(\lambda_i) |i\rangle \langle i|).
\end{aligned} \tag{3}$$

Exercise 2.57: (Cascaded measurements are single measurements) Suppose L_l and M_m are two sets of measurement operators. Show that a measurement defined by the measurement operators L_l followed by a measurement defined by the measurement operators M_m is physically equivalent to a single measurement defined by measurement operators N_{lm} with the representation $N_{lm} = M_m L_l$.

Answer:

If the state of the quantum system is $|\psi\rangle$ immediately before the measurement. The state of the system after the first measurement is $|\psi_L\rangle = \frac{L_l |\psi\rangle}{\sqrt{\langle \psi | L_l^\dagger L_l | \psi \rangle}}$ and the second measurement is

$$|\psi_M\rangle = \frac{M_m |\psi_L\rangle}{\sqrt{\langle\psi_L| M_m^\dagger M_m |\psi_L\rangle}}.$$

$$\begin{aligned} \langle\psi_L| &= \frac{\langle\psi| L_l^\dagger}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}} \\ |\psi_M\rangle &= \frac{M_m |\psi_L\rangle}{\sqrt{\langle\psi_L| M_m^\dagger M_m |\psi_L\rangle}} \\ &= \frac{M_m \frac{L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}}}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}} \\ &= \frac{\frac{\langle\psi| L_l^\dagger}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}} M_m^\dagger M_m \frac{L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}}}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}} \\ &= \frac{M_m L_l |\psi\rangle}{\langle\psi| L_l^\dagger M_m^\dagger M_m L_l |\psi\rangle}. \end{aligned} \tag{4}$$

The state of the system after the measurement operators N_{lm} ($N_{lm} = M_m L_l$) is

$$\begin{aligned} |\psi_N\rangle &= \frac{N_{lm} |\psi\rangle}{\sqrt{\langle\psi| N_{lm}^\dagger N_{lm} |\psi\rangle}} \\ &= \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger M_m^\dagger M_m L_l |\psi\rangle}} = |\psi_M\rangle. \end{aligned} \tag{5}$$

Thus we proved that Cascaded measurements are single measurements.

Exercise 2.58: Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M , with corresponding eigenvalue m . What is the average observed value of M , and the standard deviation?

Answer:

$$\begin{aligned} \langle M \rangle &= \langle\psi| M |\psi\rangle \\ &= \langle\psi| m |\psi\rangle \\ &= m \langle\psi| \psi\rangle \\ &= m [\Delta M]^2 = \langle M^2 \rangle - \langle M \rangle^2 \\ &= \langle\psi| m^2 |\psi\rangle - m^2 \\ &= m^2 - m^2 \\ &= 0. \end{aligned} \tag{6}$$

Exercise 2.49: Suppose we have qubit in the state $|0\rangle$, and we measure the observable X . What is the average value of X ? What is the standard deviation of X ?

Answer:

$$\begin{aligned}
\langle X \rangle &= \langle 0 | X | 0 \rangle \\
&= 0 \\
\langle X^2 \rangle &= \langle 0 | X^2 | 0 \rangle \\
&= 1 \\
[\Delta X] &= \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = 1.
\end{aligned} \tag{7}$$

Exercise 2.50: Show that $\vec{v} \cdot \vec{\sigma}$ has eigenvalues ± 1 , and that the projectors onto the corresponding eigenspaces are given by $P_{\pm} = (I \pm \vec{v} \cdot \vec{\sigma})/2$.

Answer:

$$\begin{aligned}
\vec{v} \cdot \vec{\sigma} &= \sum_{i=1}^3 v_i \sigma_i \\
&= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \\
\det(\vec{v} \cdot \vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\
&= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\
&= \lambda^2 - 1.
\end{aligned} \tag{8}$$

Eigenvalues are ± 1 . if $\lambda = 1$

$$\begin{aligned}
\vec{v} \cdot \vec{\sigma} - \lambda I &= \vec{v} \cdot \vec{\sigma} - I \\
&= \begin{bmatrix} v_3 - 1 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 - 1 \end{bmatrix}
\end{aligned} \tag{9}$$

Normalized eigenvector is $|\lambda_1\rangle = \sqrt{\frac{1+v_3}{2}} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix}$.

$$\begin{aligned}
 |\lambda_1\rangle \langle \lambda_1| &= \frac{1+V_3}{2} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1-v_3}{v_1+iv_2} \end{bmatrix} \\
 &= \frac{1+v_3}{2} \begin{bmatrix} 1 & \frac{v_1-iv_2}{1+v_3} \\ \frac{v_1+iv_2}{1+v_3} & \frac{1-v_3}{1+v_3} \end{bmatrix} \\
 &= \frac{1}{2} \begin{bmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{bmatrix} \\
 &= \frac{1}{2} \left(I + \begin{bmatrix} v_3 & v_1-iv_2 \\ v_1+iv_2 & -v_3 \end{bmatrix} \right) \\
 &= \frac{1}{2} (I + \vec{v} \cdot \vec{\sigma}).
 \end{aligned} \tag{10}$$

If $\lambda = -1$.

Normalized eigenvalue is $|\lambda_{-1}\rangle = \sqrt{\frac{1-v_3}{2}} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix}$.

Similarly, we can get the $|\lambda_{-1}\rangle \langle \lambda_{-1}| = \frac{1}{2} (I - \vec{v} \cdot \vec{\sigma})$.