Q&A (2.21-2.30)

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Exercise 2.21: Repeat the proof of the spectral decomposition in Box 2.2 for the case when M is Hermitian, simplifying the proof wherever possible.

answer 2.21:

Exercise 2.22: Prove that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

answer 2.22:
Firstly, we set A is Hermitian, then we set λ_i and λ_j are different eigenvalues of A, and the corresponding eigenvectors are v_i and v_j.
Secondly, we have that
$(|v_i\rangle, A|v_j\rangle) = (|v_i\rangle, \lambda_j|v_j\rangle) = \lambda_j (|v_i\rangle, |v_j\rangle) = \lambda_j\langle v_i|v_j\rangle$,
$(|v_i\rangle, A|v_j\rangle) = \left(A^\dagger|v_i\rangle, |v_j\rangle\right) = (A|v_i\rangle, |v_j\rangle) = (\lambda_i|v_i\rangle, |v_j\rangle) = \lambda_i\langle v_i|v_j\rangle$,
from above we can know that $\lambda_i\langle v_i|v_j\rangle = \lambda_j\langle v_i|v_j\rangle$.
Since $\lambda_i \neq \lambda_i$, we have that $\langle v_i|v_j\rangle = 0$. Finally, we can conclude that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

Exercise 2.23: Show that the eigenvalues of a projector P are all either 0 or 1.

answer 2.23:
Firstly, we set λ_i is one of the eigenvalues of a projector P, and the corresponding eigenvector is v_i.
Then we have
$P|v_i\rangle = \lambda_i|v_i\rangle$, $P^2|v_i\rangle = P\left(P|v_i\rangle\right) = P\left(\lambda_i|v_i\rangle\right) = \lambda_i P|v_i\rangle = \lambda_i^2|v_i\rangle$.
Since $P^2 = P$, then we can have
$\lambda_i|v_i\rangle = \lambda_i^2|v_i\rangle \Rightarrow \left(\lambda_i^2 - \lambda_i\right)|v_i\rangle = 0$.
We get the result of λ_i is 1 or 0, thus we can conclude that the eigenvalues of a projector P are all either 0 or 1.

Exercise 2.24: (Hermiticity of positive operators) Show that a positive operator is necessarily Hermitian. (Hint: Show that an arbitrary operator A can be written $A = B + iC$ where B and C are Hermitian.)

answer 2.24:
Firstly, we set A is a positive operator, and we have
$A = \dfrac{A + A^\dagger}{2} + \dfrac{A - A^\dagger}{2} = \dfrac{A + A^\dagger}{2} + i\left(\dfrac{-iA + iA^\dagger}{2}\right)$.

Then we let $B = \frac{A+A^\dagger}{2}$ and $C = \frac{-iA+iA^\dagger}{2}$, and have that
$B^\dagger = \frac{1}{2}\left(A + A^\dagger\right)^\dagger = \frac{1}{2}\left(A + A^\dagger\right) = B,$
$C^\dagger = \frac{1}{2}\left(-iA + iA^\dagger\right)^\dagger = \frac{1}{2}\left[-\left(iA\right)^\dagger + \left(iA^\dagger\right)^\dagger\right] = \frac{1}{2}\left(-iA + iA^\dagger\right) = C.$
Thus B and C are Hermitian. And for any vertor $|\varphi\rangle$, we have $\langle\varphi|B|\varphi\rangle \in \mathbf{R}$ and $\langle\varphi|C|\varphi\rangle \in \mathbf{R}$.
Since A is a positive operator, and for any vector $|v\rangle$ we have that
$\langle v|A|v\rangle = \langle v|\left(B + iC\right)|v\rangle = \langle v|B|v\rangle + i\langle v|C|v\rangle \geq 0.$
Thus $C = 0$, and we have $A^\dagger = B^\dagger = B = A$.
So we can conclude that a positive operator is necessarily Hermitian.

Exercise 2.25: Show that for any operator A, $A^\dagger A$ is positive.

answer 2.25:
For any vertor $|v\rangle$, we have
$\left(|v\rangle, A^\dagger A|v\rangle\right) = \left(A|v\rangle, A|v\rangle\right) \geq 0.$
Thus we can conclude that for any operator A, $A^\dagger A$ is positive.

Exercise 2.26: Let $|\psi\rangle = \left(|0\rangle + |1\rangle\right)/\sqrt{2}$. Write out $|\psi\rangle^{\otimes 2}$ and $|\psi\rangle^{\otimes 3}$ explicitly, both in terms of tensor products like $|0\rangle|1\rangle$, and using the Kronecker product.

answer 2.26:
We have that

$$
\begin{aligned}
|\psi\rangle^{\otimes 2} &= |\psi\rangle \otimes |\psi\rangle \\
&= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{2}\left[|0\rangle \otimes \left(|0\rangle + |1\rangle\right) + |1\rangle \otimes \left(|0\rangle + |1\rangle\right)\right] \\
&= \frac{1}{2}\left(|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right) \\
&= \frac{1}{2}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right) \\
&= \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix};
\end{aligned}
$$

$$
\begin{aligned}
|\psi\rangle^{\otimes 3} &= |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \\
&= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \\
&= \frac{1}{2\sqrt{2}} \left(|0\rangle|0\rangle|0\rangle + |0\rangle|0\rangle|1\rangle + \cdots + |1\rangle|1\rangle|0\rangle + |1\rangle|1\rangle|1\rangle\right) \\
&= \frac{1}{2\sqrt{2}} \left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right) \\
&= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}.
\end{aligned}
$$

Exercise 2.27: Calculate the matrix representation of the tensor products of the Pauli operators (a) X and Z; (b) I and X; (c) X and I. Is the tensor product commutative?

answer 2.27:
Firstly, we Calculate the matrix representation of the tensor products of the Pauli operators (a) X and Z; (b) I and X; (c) X and I:

$$
\begin{aligned}
X \otimes Z &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 1 \times \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ 1 \times \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & 0 \times \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix};
\end{aligned}
$$

$$
\begin{aligned}
I \otimes X &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0 \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 1 \times \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \\
X \otimes I &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0 \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Secondly, since

$$
\begin{aligned}
I \otimes X - X \otimes I &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \\
&\neq \mathbf{0}.
\end{aligned}
$$

We can conclude that the tensor product is not commutative.

Exercise 2.28: Show that the transpose, complex conjugation, and adjoint operations distribute over the tensor product,
$(A \otimes B)^* = A^* \otimes B^*; (A \otimes B)^T = A^T \otimes B^T; (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$.

answer 2.28:
Suppose the matrix representation of A is m by n, then we have

$$
\begin{aligned}
A^* \otimes B^* &= \begin{bmatrix} A_{11}^* B^* & A_{12}^* B^* & \dots & A_{1n}^* B^* \\ A_{21}^* B^* & A_{22}^* B^* & \dots & A_{2n}^* B^* \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}^* B^* & A_{m2}^* B^* & \dots & A_{mn}^* B^* \end{bmatrix} \\
&= \begin{bmatrix} (A_{11} B)^* & (A_{12} B)^* & \cdots & (A_{1n} B)^* \\ (A_{21} B)^* & (A_{22} B)^* & \cdots & (A_{2n} B)^* \\ \vdots & \vdots & \vdots & \vdots \\ (A_{m1} B)^* & (A_{m2} B)^* & \cdots & (A_{mn} B)^* \end{bmatrix} \\
&= (A \otimes B)^* ; \\
A^T \otimes B^T &= \begin{bmatrix} A_{11} B^T & A_{21} B^T & \dots & A_{m1} B^T \\ A_{12} B^T & A_{22} B^T & \dots & A_{m2} B^T \\ \vdots & \vdots & \vdots & \vdots \\ A_{1n} B^T & A_{2n} B^T & \dots & A_{mn} B^T \end{bmatrix} \\
&= \begin{bmatrix} (A_{11} B)^T & (A_{21} B)^T & \cdots & (A_{m1} B)^T \\ (A_{12} B)^T & (A_{22} B)^T & \cdots & (A_{m2} B)^T \\ \vdots & \vdots & \vdots & \vdots \\ (A_{1n} B)^T & (A_{2n} B)^T & \cdots & (A_{mn} B)^T \end{bmatrix} \\
&= (A \otimes B)^T ; \\
A^\dagger \otimes B^\dagger &= (A^*)^T \otimes (B^*)^T = (A^* \otimes B^*)^T = \left[(A \otimes B)^*\right]^T = (A \otimes B)^\dagger .
\end{aligned}
$$

Exercise 2.29: Show that the tensor product of two unitary operators is unitary.

answer 2.29:
We set A is an unitary operator in m dimensions, and B is an unitary operator in n dimensions. And we let $U = A \otimes B$, then we have

$$
\begin{aligned}
U^\dagger U &= (A \otimes B)^\dagger (A \otimes B) \\
&= \left(A^\dagger \otimes B^\dagger\right) (A \otimes B) \\
&= A^\dagger A \otimes B^\dagger B \\
&= I_{m \times m} \otimes I_{n \times n} \\
&= I_{(mn) \times (mn)} .
\end{aligned}
$$

Thus we can conclude that the tensor product of two unitary operators is unitary.

Exercise 2.30: Show that the tensor product of two Hermitian operators is Hermitian.

answer 2.30:
We Suppose that A and B are Hermitian operators, and we can konw that $A^\dagger = A, B^\dagger = B$.
Then we have $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger = A \otimes B$, so we can conclude that the tensor product of two Hermitian operators is Hermitian.