

## Q&A (4.01-4.10)

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**Exercise 4.1:** In Exercise 2.11, which you should do now if you haven't already done it, you computed the eigenvectors of the Pauli matrices. Find the points on the Bloch sphere which correspond to the normalized eigenvectors of the different Pauli matrices.

**Answer:**

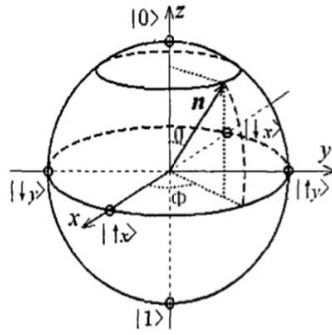


图 1: Bloch vector of pure state quantum system

According to Exercise 2.11, The Pauli matrix  $\sigma_X, \sigma_Y, \sigma_Z$  can be written as

$$\begin{aligned}\sigma_X &= |+\rangle \langle +| - |-\rangle \langle -| \\ \sigma_Y &= \frac{1}{2} \left[ \begin{bmatrix} 1 \\ i \end{bmatrix} \begin{bmatrix} 1 & i \end{bmatrix} - \begin{bmatrix} 1 \\ -i \end{bmatrix} \begin{bmatrix} 1 & -i \end{bmatrix} \right] \\ \sigma_Z &= |0\rangle \langle 0| - |1\rangle \langle 1|.\end{aligned}\tag{1}$$

The eigenvectors of  $\sigma_X$  are  $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$  and  $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ .

$$\begin{aligned}
\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) &= \left| \sigma_X\left(\frac{\pi}{2}, 0\right) \right\rangle \\
&= \cos \frac{\pi}{4} |0\rangle + e^{i0} \sin \frac{\pi}{4} |1\rangle \\
&= \begin{bmatrix} \cos \frac{\pi}{4} \\ e^{i0} \sin \frac{\pi}{4} \end{bmatrix} \leftrightarrow \\
n &= (\sin \Theta \cos \phi, \sin \Theta \sin \phi, \cos \Theta) \\
&= (1, 0, 0) \\
\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) &= \left| \sigma_X\left(\frac{\pi}{2}, \pi\right) \right\rangle \\
&= \cos \frac{\pi}{4} |0\rangle + e^{i\pi} \sin \frac{\pi}{4} |1\rangle \\
&= \begin{bmatrix} \cos \frac{\pi}{4} \\ e^{i\pi} \sin \frac{\pi}{4} \end{bmatrix} \leftrightarrow \\
n &= (\sin \Theta \cos \phi, \sin \Theta \sin \phi, \cos \Theta) \\
&= (-1, 0, 0)
\end{aligned} \tag{2}$$

The eigenvectors of  $\sigma_Y$  are  $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$  and  $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ .

$$\begin{aligned}
\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) &= \left| \sigma_Y\left(\frac{\pi}{2}, \frac{\pi}{2}\right) \right\rangle \\
&= \cos \frac{\pi}{4} |0\rangle + e^{i\frac{\pi}{2}} \sin \frac{\pi}{4} |1\rangle \\
&= \begin{bmatrix} \cos \frac{\pi}{4} \\ e^{i\frac{\pi}{2}} \sin \frac{\pi}{4} \end{bmatrix} \leftrightarrow \\
n &= (\sin \Theta \cos \phi, \sin \Theta \sin \phi, \cos \Theta) \\
&= (0, 1, 0) \\
\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) &= \left| \sigma_Y\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \right\rangle \\
&= \cos \frac{\pi}{4} |0\rangle + e^{i\frac{3\pi}{2}} \sin \frac{\pi}{4} |1\rangle \\
&= \begin{bmatrix} \cos \frac{\pi}{4} \\ e^{i\frac{3\pi}{2}} \sin \frac{\pi}{4} \end{bmatrix} \leftrightarrow \\
n &= (\sin \Theta \cos \phi, \sin \Theta \sin \phi, \cos \Theta) \\
&= (0, -1, 0)
\end{aligned} \tag{3}$$

The eigenvectors of  $\sigma_Z$  are *ket0* and *ket1*.

$$\begin{aligned}
|0\rangle &= \left| \sigma_Z(0, \frac{\pi}{2}) \right\rangle \\
&= \cos 0 |0\rangle + e^{i\frac{\pi}{2}} \sin 0 |1\rangle \\
&= \begin{bmatrix} \cos 0 \\ e^{i\frac{\pi}{2}} \sin 0 \end{bmatrix} \leftrightarrow \\
n &= (\sin \Theta \cos \phi, \sin \Theta \sin \phi, \cos \Theta) \\
&= (0, 0, 1) \\
|1\rangle &= \left| \sigma_Z(\pi, \frac{3\pi}{2}) \right\rangle \\
&= \cos \frac{\pi}{4} |0\rangle + e^{i\frac{3\pi}{2}} \sin \frac{\pi}{4} |1\rangle \\
&= \begin{bmatrix} \cos \frac{\pi}{2} \\ e^{i\frac{3\pi}{2}} \sin \frac{\pi}{2} \end{bmatrix} \leftrightarrow \\
n &= (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \\
&= (0, 0, -1)
\end{aligned} \tag{4}$$

**Exercise 4.2:** Let  $x$  be a real number and  $A$  a matrix such that  $A^2 = I$ . Show that

$$\exp(iAx) = \cos(x)I + i\sin(x)A. \tag{5}$$

Use this result to verify Equations (4.4) through (4.6).

**Answer:**

According to Taylor series expansion formula,

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!} + \dots; \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!} + \dots; \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots (-1)^k \frac{x^{2k}}{(2k)!} + \dots; \\
\exp(iAx) &= I + iAx + \frac{(iAx)^2}{2!} + \frac{(iAx)^3}{3!} + \dots + \frac{iAx^k}{k!} + \dots; \\
&= I + iAx - \frac{Ix^2}{2!} - \frac{(iA)x^3}{3!} + \dots (-1)^{(k-1)/2} \frac{iAx^k}{k!} + \dots; \\
\cos(x)I &= I - \frac{Ix^2}{2!} + \frac{Ix^4}{4!} - \dots (-1)^k \frac{Ix^{2k}}{(2k)!} + \dots; \\
iA\sin(x) &= iAx - \frac{iAx^3}{3!} + \frac{iAx^5}{5!} - \dots (-1)^{(k-1)/2} \frac{x^{2k-1}}{(2k-1)!} + \dots;
\end{aligned} \tag{6}$$

Thus we proved that  $\exp(iAx) = \cos(x)I + iA\sin(x) = \cos(x)I + i\sin(x)A$ .

**Exercise 4.3:** Show that, up to a global phase, the  $\pi/8$  gate satisfies  $T = R_z(\pi/4)$ .

**Answer:**

$$\begin{aligned}
R_z &= \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix} \\
T &= e^{-i\pi/8} \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix} \\
R_z(\frac{\pi}{4}) &= \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix}
\end{aligned} \tag{7}$$

Thus, up to a global phase, the  $\pi/8$  gate satisfies  $T = R_z(\pi/4)$ .

**Exercise 4.4:** Express the Hadamard gate  $H$  as a product of  $R_x$  and  $R_z$  rotations and  $e^i$  for some .

**Answer:**

Hadamard is rotated  $90^\circ$  around the y-axis and  $180^\circ$  around the z-axis. According to Trigonometric function formula,  $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$  and  $\cos^2(\alpha) = 2\cos^2(\alpha) - 1$ .

$$\begin{aligned}
R_x(\theta)R_z(\theta) &= \begin{bmatrix} \cos \theta/2 & -i \sin \theta/2 \\ -i \sin \theta/2 & \cos \theta/2 \end{bmatrix} \begin{bmatrix} \cos \theta/2 - i \sin \theta/2 & 0 \\ 0i \sin \theta/2 + \cos \theta/2 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \cos^2 \theta/2 - i \sin \theta/2 \cos \theta/2 & \cos \theta/2 - i \sin^2 \theta/2 \\ \cos \theta/2 - i \sin^2 \theta/2 & \cos^2 \theta/2 - i \sin \theta/2 \cos \theta/2 \end{bmatrix} \\
&= \begin{bmatrix} 1 + \exp(-i\theta) & 1 + \exp(i\theta) \\ -1 + \exp(-i\theta) & 1 + \exp(i\theta) \end{bmatrix} \\
H &= \frac{X + Z}{\sqrt{2}} \\
&= R_x(\pi)R_y(\pi)
\end{aligned} \tag{8}$$

**Exercise 4.5:** Prove that  $(\hat{n} \cdot \vec{\sigma})^2 = I$ , and use this to verify Equation (4.8).

**Answer:**

$$\begin{aligned}
(\hat{n} \cdot \vec{\sigma})^2 &= [(n_x, n_y, n_z)(\sigma_1, \sigma_2, \sigma_3)]^2 \\
&= [n_x\sigma_1 + n_y\sigma_2 + n_z\sigma_3]^2 \\
&= (n_x^2\sigma_1^2 + n_xn_y\sigma_1\sigma_2 + n_xn_z\sigma_1\sigma_3 + n_y^2\sigma_2^2 + n_yn_x\sigma_2\sigma_1 + n_yn_z\sigma_2\sigma_3) \\
&\quad + (n_z^2\sigma_3^2 + n_zn_x\sigma_3\sigma_1 + n_zn_y\sigma_3\sigma_2) \\
&= n_x^2I + in_xn_y\sigma_3 - in_xn_z\sigma_2 + n_y^2I - in_yn_x\sigma_3 + in_yn_z\sigma_1 + n_z^2I + in_zn_x\sigma_2 - in_zn_y\sigma_1 \\
&= I(n_x^2 + n_y^2 + n_z^2) = I \\
R_{\hat{n}}(\theta) &\equiv \exp(-i\theta\hat{n} \cdot \sigma/2) \\
&= \cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})\theta\hat{n} \cdot \sigma \\
&= \cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})(n_xX + n_yY + n_zZ)
\end{aligned} \tag{9}$$

**Exercise 4.6: (Bloch sphere interpretation of rotations)** One reason why the  $R_{\hat{n}}(\theta)$  operators are referred to as rotation operators is the following fact, which you are to prove. Suppose a single qubit has a state represented by the Bloch vector  $\vec{\lambda}$ . Then the effect of the rotation  $R_{\hat{n}}(\theta)$  on the state is to rotate it by an angle  $\theta$  about the  $\hat{n}$  axis of the Bloch sphere. This fact explains the rather mysterious looking factor of two in the definition of the rotation matrices.

**Answer:**

**Exercise 4.7:** Show that  $XYX = -Y$  and use this to prove that  $XR_y(\theta)X = R_y(-\theta)$ .

**Answer:**

$$\begin{aligned}
 XYX &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -Y \\
 XR_y(\theta)X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
 &= \begin{bmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \\
 R_y(-\theta) &= e^{i\theta Y/2} \\
 &= \cos(\theta/2)I + i\sin(\theta/2)Y \\
 &= \begin{bmatrix} \cos(\theta/2) & 0 \\ 0 & \cos(\theta/2) \end{bmatrix} + \begin{bmatrix} 0 & \sin(\theta/2) \\ -\sin(\theta/2) & 0 \end{bmatrix} \\
 &= \begin{bmatrix} \cos(\frac{\theta}{2}) & \sin(\frac{\theta}{2}) \\ -\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix}
 \end{aligned} \tag{10}$$

Thus, we proved that  $XR_y(\theta)X = R_y(-\theta)$ .

**Exercise 4.8:** An arbitrary single qubit unitary operator can be written in the form

$$U = \exp(i\alpha)R_{\hat{n}}(\theta) \tag{11}$$

for some real numbers  $\alpha$  and  $\theta$ , and a real three-dimensional unit vector  $\hat{n}$ .

1. Prove this fact.
2. Find values for  $\alpha$ ,  $\theta$ , and  $\hat{n}$  giving the Hadamard gate  $H$ .

3. Find values for  $\alpha$ ,  $\theta$ , and  $\hat{n}$  giving the phase gate

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad (12)$$

**Answer:**

1.

$$\begin{aligned} \exp(i\alpha)R_{\hat{n}}(\theta) &= \exp(i\alpha)\exp(-i\theta\hat{n} \cdot \sigma/2) \\ &= (\cos(\alpha) + i\sin(\alpha))\left(\cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)(n_xX + n_yY + n_zZ)\right) \end{aligned} \quad (13)$$

2.  $U = \exp(i\alpha)R_{\hat{n}}(\theta) = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$ , among then,

$$\begin{aligned} U_{11} &= \left(\cos\alpha\cos\left(\frac{\theta}{2}\right) + n_z\sin\alpha\sin\left(\frac{\theta}{2}\right)\right) + i\left(\sin\alpha\cos\left(\frac{\theta}{2}\right) - n_z\cos\alpha\sin\left(\frac{\theta}{2}\right)\right) \\ U_{12} &= \left(n_x\sin\alpha\sin\left(\frac{\theta}{2}\right) - n_y\cos\alpha\sin\left(\frac{\theta}{2}\right)\right) \\ &\quad - i\left(n_x\cos\alpha\sin\left(\frac{\theta}{2}\right) + n_y\sin\alpha\sin\left(\frac{\theta}{2}\right)\right) \\ U_{21} &= \left(n_x\sin\alpha\sin\left(\frac{\theta}{2}\right) + n_y\cos\alpha\sin\left(\frac{\theta}{2}\right)\right) - i\left(n_x\cos\alpha\sin\left(\frac{\theta}{2}\right) - n_y\sin\alpha\sin\left(\frac{\theta}{2}\right)\right) \\ U_{22} &= \left(\cos\alpha\cos\left(\frac{\theta}{2}\right) - n_z\sin\alpha\sin\left(\frac{\theta}{2}\right)\right) + i\left(\sin\alpha\cos\left(\frac{\theta}{2}\right) + n_z\cos\alpha\sin\left(\frac{\theta}{2}\right)\right) \\ H &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ U_{11} &= \frac{1}{\sqrt{2}}, U_{12} = 0 \\ U_{21} &= 0, U_{22} = -\frac{1}{\sqrt{2}} \end{aligned} \quad (14)$$

The solution is  $\alpha = \frac{\pi}{2}, \theta = \pi, \hat{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$ , then  
 $H = \exp\left(\frac{i\pi}{2}\right)R_{\hat{n}}(\pi)$ , for  $\hat{n} = \frac{1}{\sqrt{2}}(1, 0, 1)$ .

3.  $S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$ , then

$$U_{11} = \frac{1}{\sqrt{2}}, U_{12} = 0$$

$$U_{21} = 0, U_{22} = -\frac{i}{\sqrt{2}}.$$

The solution is  $\alpha = \frac{\pi}{4}, \theta = \pi, \hat{n} = \frac{1}{\sqrt{2}}(0, 0, 1)$ , then

$$S = \exp\left(\frac{i\pi}{4}\right)R_{\hat{n}}(\pi), \text{ for } \hat{n} = \frac{1}{\sqrt{2}}(0, 0, 1).$$

**Exercise 4.9:** Explain why any single qubit unitary operator may be written in the form (4.12).

**Answer:**

**Exercise 4.10: ( $X - Y$  decomposition of rotations)** Give a decomposition analogous to Theorem 4.1 but using  $R_x$  instead of  $R_z$ .

**Answer:**

$$\begin{aligned}
 U &= \begin{bmatrix} \exp(i(\alpha - \frac{\beta}{2} - \frac{\delta}{2}))\cos(\frac{\gamma}{2}) & -\exp(i(\alpha - \frac{\beta}{2} + \frac{\delta}{2}))\sin(\frac{\gamma}{2}) \\ \exp(i(\alpha + \frac{\beta}{2} - \frac{\delta}{2}))\sin(\frac{\gamma}{2}) & \exp(i(\alpha + \frac{\beta}{2} + \frac{\delta}{2}))\cos(\frac{\gamma}{2}) \end{bmatrix} \\
 &= \exp(i\alpha) \begin{bmatrix} \cos(\frac{\beta}{2}) - i\sin(\frac{\beta}{2}) & 0 \\ 0 & \cos(\frac{\beta}{2}) + i\sin(\frac{\beta}{2}) \end{bmatrix} \begin{bmatrix} \cos(\frac{\gamma}{2}) & \sin(\frac{\gamma}{2}) \\ \sin(\frac{\gamma}{2}) & \cos(\frac{\gamma}{2}) \end{bmatrix} \begin{bmatrix} \cos(\frac{\delta}{2}) - i\sin(\frac{\delta}{2}) & 0 \\ 0 & \cos(\frac{\delta}{2}) + i\sin(\frac{\delta}{2}) \end{bmatrix} \\
 &\quad (15)
 \end{aligned}$$