

## Q&A (2.31-2.40)

LuoTingyu JiangHui

2020 年 2 月 15 日

**Exercise 2.41: (Anti-commutation relations for the Pauli matrices)** Verify the anti-commutation relations

$$\{\sigma_i, \sigma_j\} = 0$$

where  $i \neq j$  are both chosen from the set 1,2,3. Also verify that ( $i = 0, 1, 2, 3$ )

$$\sigma_i^2 = I.$$

**Answer:**

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

When  $i = 1, j = 2$ , we can get the follow equation,

$$\begin{aligned} \{\sigma_1, \sigma_2\} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\ &= 0. \end{aligned} \tag{1}$$

Similarly available, we verify that  $\{\sigma_i, \sigma_j\} = 0$ .

When  $i = 1, j = 2$ , we can get the follow equation,

$$\begin{aligned} \sigma_0^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I. \end{aligned} \tag{2}$$

Similarly available, we verify that  $\sigma_i^2 = I$ .

**Exercise 2.42:** Verify that

$$AB = \frac{[A, B] + \{A, B\}}{2}.$$

**Answer:**

$$\frac{[A, B] + \{A, B\}}{2} = \frac{AB - BA + AB + BA}{2} = AB. \quad (3)$$

**Exercise 2.43:** Show that for  $j, k = 1, 2, 3$ ,

$$\sigma_j \sigma_k = \delta_{jk} I + i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l.$$

**Answer:**

$\sigma_j \sigma_j = \delta_{jj} I + i \sum_{l=1}^3 \epsilon_{jjl} \sigma_l$ , according to the exercise 2.40, we can know the equations  $\epsilon_{jjl} = 0 (l = 1, 2, 3, j = 1, 2, 3)$  and  $\sigma_j \sigma_j = \sigma_j^2 = I$ .  
Thus when  $j = k$ ,  $\sigma_j \sigma_k = \delta_{jk} I + i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l$ .

When  $j = 1, k = 2$ , we can get the follow equations,

$$\sigma_1 \sigma_2 = 0 + i \epsilon_{121} \sigma_1 + i \epsilon_{122} \sigma_2 + i \epsilon_{123} \sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

$$\sigma_1 \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Thus  $\sigma_1 \sigma_2 = \sigma_{12} I + i \sum_{l=1}^3 \epsilon_{12l} \sigma_l$ .

When  $j = 2, k = 1$ , we can get the follow equations,

$$\sigma_2 \sigma_1 = 0 + i \epsilon_{211} \sigma_1 + i \epsilon_{212} \sigma_2 + i \epsilon_{213} \sigma_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix},$$

$$\sigma_2 \sigma_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}.$$

Thus  $\sigma_2 \sigma_1 = \sigma_{21} I + i \sum_{l=1}^3 \epsilon_{21l} \sigma_l$ .

Similarly, we can get the equations:

$$\sigma_1 \sigma_3 = \sigma_{13} I + i \sum_{l=1}^3 \epsilon_{13l} \sigma_l$$

$$\sigma_3 \sigma_1 = \sigma_{31} I + i \sum_{l=1}^3 \epsilon_{31l} \sigma_l$$

$$\sigma_2 \sigma_3 = \sigma_{23} I + i \sum_{l=2}^3 \epsilon_{23l} \sigma_l$$

$$\sigma_3 \sigma_2 = \sigma_{32} I + i \sum_{l=2}^3 \epsilon_{32l} \sigma_l.$$

In summary, we proved the  $\sigma_j \sigma_k = \delta_{jk} I + i \sum_{l=1}^3 \epsilon_{jkl} \sigma_l$ . for  $j, k, l = 1, 2, 3$ .

**Exercise 2.44:** Suppose  $[A, B] = 0$ ,  $\{A, B\} = 0$ , and  $A$  is invertible. Show that  $B$  must be 0s.

**Answer:**

We can get the follow equations:

$$[A, B] = 0 \rightarrow AB - BA = 0 \quad (1)$$

$$\{A, B\} = 0 \rightarrow AB + BA = 0 \quad (2).$$

Add up the above equations, the solution is  $AB = 0$ .

Since  $A$  is invertible,  $A$  can't be zero matrix, then  $B$  must be 0s.

**Exercise 2.45:** Show that  $[A, B]^\dagger = [B^\dagger, A^\dagger]$ .

**Answer:**

$$[A, B]^\dagger = (AB - BA)^\dagger = (AB)^\dagger - (BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger].$$

**Exercise 2.46:** Show that  $[A, B] = -[B, A]$ .

**Answer:**

$$[A, B] = AB - BA = -(BA - AB) = -[B, A].$$

**Exercise 2.47:** Suppose  $A$  and  $B$  are Hermitian. Show that  $i[A, B]$  is Hermitian.

**Answer:**

If we want to prove that  $i[A, B]$  is Hermitian, we can prove that  $(i[A, B])^\dagger = i[A, B]$ . Known that  $A$  and  $B$  are Hermitian, according to exercise 2.45 and exercise 2.46 we can do the following derivation:

$$(i[A, B])^\dagger = -i[B^\dagger, A^\dagger] = -i[B, A] = i[A, B].$$

**Exercise 2.48:** What is the polar decomposition of a positive matrix  $P$ ? Of a unitary matrix  $U$ ? Of a Hermitian matrix,  $H$ ?

**Answer:**

Since  $P$  is a positive matrix and it is diagonalizable. Then  $P = \sum_i \lambda_i |i\rangle \langle i|$ ,  $\lambda_i \geq 0$ .

$$J = \sqrt{P^\dagger P} = \sqrt{P^2} = \sum_i \lambda_i^2 |i\rangle \langle i| = P.$$

Therefore polar decomposition of  $P$  is  $P = UP$  for all  $P$ . Thus  $U = I$ , then  $P = P$ .

Since  $U$  is a unitary matrix, then  $U$  can be decomposed by  $U = WJ$  where  $W$  is unitary and  $J$  is positive,  $J = \sqrt{U^\dagger U}$ .  $J = \sqrt{U^\dagger U} = \sqrt{I} = I$ .

Since unitary operators are invertible,  $W = UJ^{-1} = UI^{-1} = UI = U$ . Thus polar decomposition of  $U$  is  $U = U$ .

$$\text{Suppose } H = UJ, J = \sqrt{H^\dagger H} = \sqrt{H^2}.$$

For spectral decomposition,  $H = \sum_i \lambda_i |i\rangle \langle i|$ ,  $\lambda_i \in \mathbb{R}$ .

$$\sqrt{H^\dagger H} = \sum_i \sqrt{\lambda_i^2} |i\rangle \langle i| = \sum_i |\lambda_i| |i\rangle \langle i| \neq H.$$

Thus  $H = U\sqrt{H^2}$ .

**Exercise 2.49:** Express the polar decomposition of a normal matrix in the outer product representation.

**Answer:**

Suppose  $A$  is a normal matrix, then  $A$  is diagonalizable,  $A = \sum_i \lambda_i |i\rangle \langle i|$ .

$$\begin{aligned}
J &= \sqrt{A^\dagger A} = \sum_i |\lambda_i| |i\rangle \langle i| \\
U &= \sum_i |e_i\rangle \langle i| \\
A &= UJ \\
&= \sum_i |e_i\rangle \langle i| * \sum_i |\lambda_i| |i\rangle \langle i| \\
&= \sum_i |\lambda_i| |e_i\rangle \langle i|
\end{aligned} \tag{4}$$

**Exercise 2.50:** Find the left and right polar decompositions of the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

**Answer:**

Suppose  $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ ,  $A = UJ$ .

$$\begin{aligned}
J &= \sqrt{A^\dagger A} = \sqrt{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} \\
U &= \sum_i |e_i\rangle \langle i| \\
A &= UJ = \sum_i |e_i\rangle \langle i| * \sum_i |\lambda_i| |i\rangle \langle i| \\
&= \sum_i |\lambda_i| |e_i\rangle \langle i|
\end{aligned} \tag{5}$$