Q&A（2.31-2.40）

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Exercise 2.41: (Anti-commutation relations for the Pauli matrices) Verify the anti-commutation relations

$$\{\sigma_i, \sigma_j\} = 0$$

where $i \neq j$ are both chosen from the set 1,2,3. Also verify that $(i = 0, 1, 2, 3)$

$$\sigma_i^2 = I.$$

Answer:
$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, $\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.
When $i = 1, j = 2$,we can get the follow equation,

$$\begin{aligned} \{\sigma_1, \sigma_2\} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\ &= 0. \end{aligned} \tag{1}$$

Similarly avaliable, we verify that $\{\sigma_i, \sigma_j\} = 0$.
When $i = 1, j = 2$, we can get the follow equation,

$$\begin{aligned} \sigma_0^2 &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I. \end{aligned} \tag{2}$$

Similarly avaliable, we verify that $\sigma_i^2 = I$.

Exercise 2.42: Verify that

$$AB = \frac{[A,B]+\{A,B\}}{2}.$$

Answer:

$$\begin{aligned}\frac{[A,B]+\{A,B\}}{2} =& \frac{AB-BA+AB+BA}{2} \\ =& AB.\end{aligned} \tag{3}$$

Exercise 2.43: Show that for $j,k=1,2,3,$

$$\sigma_j\sigma_k = \delta_{jk}I + i\textstyle\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l.$$

Answer:

$\sigma_j\sigma_j = \delta_{jj}I + i\sum_{l=1}^{3}\epsilon_{jjl}\sigma_l$, according to the exercise 2.40, we can know the equations $\epsilon_{jjl}=0(l=1,2,3,j=1,2,3)$ and $\sigma_j\sigma_j=\sigma_j^2=I$.
Thus when $j=k$, $\sigma_j\sigma_k = \delta_{jk}I + i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l$.

When $j=1,k=2$,we can get the follow equations,
$\sigma_1\sigma_2 = 0 + i\epsilon_{121}\sigma_1 + i\epsilon_{122}\sigma_2 + i\epsilon_{123}\sigma_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$,
$\sigma_1\sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.
Thus $\sigma_1\sigma_2 = \delta_{12}I + i\sum_{l=1}^{3}\epsilon_{12l}\sigma_l$.
When $j=2,k=1$,we can get the follow equations,
$\sigma_2\sigma_1 = 0 + i\epsilon_{211}\sigma_1 + i\epsilon_{212}\sigma_2 + i\epsilon_{213}\sigma_3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$,
$\sigma_2\sigma_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$.
Thus $\sigma_2\sigma_1 = \delta_{21}I + i\sum_{l=1}^{3}\epsilon_{21l}\sigma_l$.
Similarly, we can get the equations:
$\sigma_1\sigma_3 = \delta_{13}I + i\sum_{l=1}^{3}\epsilon_{13l}\sigma_l$
$\sigma_3\sigma_1 = \delta_{31}I + i\sum_{l=1}^{3}\epsilon_{31l}\sigma_l$
$\sigma_2\sigma_3 = \delta_{23}I + i\sum_{l=2}^{3}\epsilon_{23l}\sigma_l$
$\sigma_3\sigma_2 = \delta_{3}I + i\sum_{l=2}^{3}\epsilon_{32l}\sigma_l$.
In summary, we proved the $\sigma_j\sigma_k = \delta_{jk}I + i\sum_{l=1}^{3}\epsilon_{jkl}\sigma_l$. for $j,k,l=1,2,3,$.

Exercise 2.44: Suppose $[A,B]=0, \{A,B\}=0$, and A is invertible. Show that B must be 0s.

Answer:
We can get the follow equations:
$[A,B]=0 \to AB-BA=0$ (1)
$\{A,B\}=0 \to AB+BA=0$ (2).
Add up the above equations, the solution is $AB=0$.

Since A is invertible, A can't be zero matrix, then B must be 0s.

Exercise 2.45: Show that $[A, B]^\dagger = [B^\dagger, A^\dagger]$.
Answer:
$[A, B]^\dagger = (AB - BA)^\dagger = (AB)^\dagger - (BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger]$.

Exercise 2.46: Show that $[A, B] = -[B, A]$.
Answer:
$[A, B] = AB - BA = -(BA - AB) = -[B, A]$.

Exercise 2.47: Suppose A and B are Hermitian. Show that $i[A, B]$ is Hermitian.
Answer:
If we want to prove that $i[A, B]$ is Hermitian, we can prove that $(i[A, B])^\dagger = i[A, B]$. Konwn that A and B are Hermitian, according to exercise 2.45 and exercise 2.46 we can do the following derivation:
$(i[A, B])^\dagger = -i[B^\dagger, A^\dagger] = -i[B, A] = i[A, B]$.

Exercise 2.48: What is the polar decomposition of a positive matrix P ? Of a unitary matrix U? Of a Hermitian matrix, H?
Answer:
Since P is a positive matrix and it is diagonalizable. Then $P = \sum_i \lambda_i |i\rangle \langle i|, \lambda_i \geq 0$.
$J = \sqrt{P^\dagger P} = \sqrt{P^2} = \sum_i \lambda_i^2 |i\rangle \langle i| = P$.
Therefore polar decomposition of P is $P = UP$ for all P. Thus $U = I$, then $P = P$.
Since U is a unitary matrix,then U can be decomposed by $U = WJ$ where W is unitary and J is positive, $J = \sqrt{U^\dagger U}$. $J = \sqrt{U^\dagger U} = \sqrt{I} = I$.
Since unitary operators are invertible, $W = UJ^{-1} = UI^{-1} = UI = U$. Thus polar decomposition of U is $U = U$.
Suppose $H = UJ.J = \sqrt{H^\dagger H} = \sqrt{H^2}$.
For spectral decomposition, $H = \sum_i \lambda_i |i\rangle \langle i|, \lambda_i \in R$.
$\sqrt{H^\dagger H} = \sum_i \sqrt{\lambda_i^2} |i\rangle \langle i| = \sum_i |\lambda_i| |i\rangle \langle i| \neq H$.
Thus $H = U\sqrt{H^2}$.

Exercise 2.49: Express the polar decomposition of a normal matrix in the outer product representation.
Answer:
Suppose A is a normal matrix, then A is diagonalizable, $A = \sum_i \lambda_i |i\rangle \langle i|$.

$$
\begin{aligned}
J =& \sqrt{A^\dagger A} = \sum_i |\lambda_i| \left| i \right\rangle \left\langle i \right| \\
U =& \sum_i \left| e_i \right\rangle \left\langle i \right| \\
A =& UJ \\
=& \sum_i \left| e_i \right\rangle \left\langle i \right| * \sum_i |\lambda_i| \left| i \right\rangle \left\langle i \right| \\
=& \sum_i |\lambda_i| \left| e_i \right\rangle \left\langle i \right|
\end{aligned} \tag{4}
$$

Exercise 2.50: Find the left and right polar decompositions of the matrix

$$
\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.
$$

Answer:
Suppose $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A = UJ$.

$$
\begin{aligned}
J =& \sqrt{A^\dagger A} = \sqrt{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}} \\
U =& \sum_i \left| e_i \right\rangle \left\langle i \right| \\
A =& UJ = \sum_i \left| e_i \right\rangle \left\langle i \right| * \sum_i |\lambda_i| \left| i \right\rangle \left\langle i \right| \\
=& \sum_i |\lambda_i| \left| e_i \right\rangle \left\langle i \right|
\end{aligned} \tag{5}
$$