Q&A（2.61-2.70）

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Exercise 2.61: Calculate the probability of obtaining the result +1 for a measurement $\vec{v}\cdot\vec{\sigma}$, given that the state prior to measurement is $|0\rangle$. What is the state of the system after the measurement if +1 is obtained?

Answer:

$$\begin{aligned}
p(+1) &= \langle 0|\lambda_1\rangle \langle\lambda_1|0\rangle \\
&= \frac{1}{2}\langle 0|\left(I+\vec{v}\cdot\vec{\sigma}\right)|0\rangle \\
&= \frac{1}{2}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1+v_3 & v_1-iv_2\\ v_1+iv_2 & 1-v_3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \\
&= \frac{1}{2}(1+v_3).
\end{aligned} \tag{1}$$

The state of the quantum system immediately after the measurement is

$$\begin{aligned}
\frac{|\lambda_1\rangle\langle\lambda_1|0\rangle}{\sqrt{p(+1)}} &= \frac{|\lambda_1\rangle\langle\lambda_1|0\rangle}{\sqrt{\frac{1}{2}(1+v_3)}} \\
&= \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}} * \frac{1}{2}\begin{bmatrix}1+v_3 & v_1-iv_2\\ v_1+iv_2 & 1-v_3\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \\
&= \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}}\frac{1}{2}\begin{bmatrix}1+v_3\\ v_1+iv_2\end{bmatrix} \\
&= \sqrt{\frac{1+v_3}{2}}\begin{bmatrix}1\\ \frac{v_1+iv_2}{1+v_3}\end{bmatrix}.
\end{aligned} \tag{2}$$

Exercise 2.62: Show that any measurement where the measurement operators and the POVM elements coincide is a projective measurement.

Answer:

Exercise 2.53: What are the eigenvalues and eigenvectors of H?

Answer:

$$det|A-\lambda I| = det\left|\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right| = \begin{bmatrix}\frac{1}{\sqrt{2}}-\lambda & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}-\lambda\end{bmatrix} = 0$$

Eigenvalues are $\pm = \pm 1$ and associated eigenvectors are $|\lambda_\pm\rangle = \frac{1}{\sqrt{4\pm 2\sqrt{2}}}\begin{bmatrix}1\\ -1\pm\sqrt{2}\end{bmatrix}$.

Exercise 2.54: Suppose A and B are commuting Hermitian operators. Prove that $exp(A)exp(B) = exp(A+B)$. (Hint: Use the results of Section 2.1.9.)

Answer:

Since $[A,B]=0$, A and B are simultaneously diagonalize, $A=\sum_i a_i |i\rangle\langle i|, B=\sum_j b_i |j\rangle\langle j|$.

$$\begin{aligned}exp(A)exp(B) &= \sum_i exp(a_i)|i\rangle\langle i| B = \sum_j exp(b_i)|j\rangle\langle j|\\ &=\sum_{i,j} exp(a_i+b_j)|i\rangle\langle i|j\rangle\langle j|\\ &=\sum_{i,j} exp(a_i+b_j)|i\rangle\langle j|\delta_{i,j}\\ &=\sum_i exp(a_i+b_i)|i\rangle\langle i|\\ &=exp(A+B).\end{aligned} \tag{3}$$

Exercise 2.55: Prove that $U(t_1,t_2)$ defined in Equation (2.91) is unitary.

Answer:

$$\begin{aligned}U(t_1,t_2)^\dagger U(t_1,t_2) =&exp\left(\frac{iH(t_2-t_1)}{\hbar}\right)\left(\frac{-iH(t_2-t_1)}{\hbar}\right)\\ =&exp\left(\frac{i\sum_{E_1}E_1|E_1\rangle\langle E_1|(t_2-t_1)}{\hbar}\right)exp\left(\frac{-i\sum_{E_2}E_2|E_2\rangle\langle E_1|(t_2-t_1)}{\hbar}\right)\\ =&\sum_{E_1,E_2} exp\left(\frac{iE_1(t_2-t_1)}{\hbar}\right)|E_2\rangle\langle E_2| exp\left(\frac{-iE_1(t_2-t_1)}{\hbar}\right)|E_1\rangle\langle E_1|\\ =&\sum_{E_1,E_2} exp\left(\frac{i(E_1-E_2)(t_2-t_1)}{\hbar}\right)|E_1\rangle\langle E_1|E_2\rangle\langle E_2|\\ =&\sum_{E_1,E_2} exp\left(\frac{i(E_1-E_2)(t_2-t_1)}{\hbar}\right)|E_1\rangle\langle E_2|\delta_{E_1,E_2}\\ =&\sum_{E_1}|E_1\rangle\langle E_1|\\ =&I.\end{aligned} \tag{4}$$

Thus $U(t_1,t_2)$ is unitary.

Exercise 2.56: Use the spectral decomposition to show that $K - i\log(U)$ is Hermitian for any unitary U, and thus $U = exp(iK)$ for some Hermitian K.

Answer:

Since U is unitary, then U can perform spectral decomposition, $U = \sum_i \lambda_i |i\rangle \langle i|$

$$\begin{aligned}
K^\dagger =& (-i \log(U))^\dagger \\
=& (-i \log\left(\sum_i \lambda_i |i\rangle \langle i|\right))^\dagger \\
=& (i \sum_i \log(\lambda_i) |i\rangle \langle i|).
\end{aligned} \tag{5}$$

Exercise 2.57: (Cascaded measurements are single measurements) Suppose L_l and M_m are two sets of measurement operators. Show that a measurement defined by the measurement operators L_l followed by a measurement defined by the measurement operators M_m is physically equivalent to a single measurement defined by measurement operators N_{lm} with the representation $N_{lm} = M_m L_l$.

Answer:

If the state of the quantum system is $|\psi\rangle$ immediately before the measurement. The state of the system after the first measurement is $|\psi_L\rangle = \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}$ and the second measurement is $|\psi_M\rangle = \frac{M_m|\psi_L\rangle}{\sqrt{\langle\psi_L|M_m^\dagger M_m|\psi_L\rangle}}$.

$$\begin{aligned}
\langle\psi_L| =& \frac{\langle\psi| L_l^\dagger}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}} \\
|\psi_M\rangle =& \frac{M_m |\psi_L\rangle}{\sqrt{\langle\psi_L| M_m^\dagger M_m |\psi_L\rangle}} \\
=& \frac{M_m \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}}{\frac{\langle\psi|L_l^\dagger}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} M_m^\dagger M_m \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}} \\
=& \frac{M_m L_l |\psi\rangle}{\langle\psi| L^\dagger M_m^\dagger M_m L_l |\psi\rangle}.
\end{aligned} \tag{6}$$

The state of the system after the measurement operators N_{lm} ($N_{lm} = M_m L_l$.) is

$$\begin{aligned}
|\psi_N\rangle =& \frac{N_{lm} |\psi\rangle}{\sqrt{\langle\psi| N_{lm}^\dagger N_{lm} |\psi\rangle}} \\
=& \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger M_m^\dagger M_m L_l |\psi\rangle}} = \quad |\psi_M\rangle .
\end{aligned} \tag{7}$$

Thus we proved that Cascaded measurements are single measurements.

Exercise 2.58: Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M , with corresponding eigenvalue m. What is the average observed value of M, and the standard deviation?

Answer:

$$
\begin{aligned}
\langle M\rangle &= \langle\psi| M |\psi\rangle \\
&= \langle\psi| m |\psi\rangle \\
&= m \langle\psi|\psi\rangle \\
= m[\Delta M]^2 &= \left\langle M^2\right\rangle - \langle M\rangle^2 \\
&= \langle\psi| m^2 |\psi\rangle - m^2 \\
&= m^2 - m^2 \\
&= 0.
\end{aligned} \tag{8}
$$

Exercise 2.49: Suppose we have qubit in the state $|0\rangle$, and we measure the observable X. What is the average value of X? What is the standard deviation of X?
Answer:

$$
\begin{aligned}
\langle X\rangle &= \langle 0| X |0\rangle \\
&= 0 \\
\left\langle X^2\right\rangle &= \langle 0| X^2 |0\rangle \\
&= 1 \\
[\Delta X] &= \sqrt{\langle X^2\rangle - \langle X\rangle^2} = 1.
\end{aligned} \tag{9}
$$

Exercise 2.50: Show that $v \cdot \sigma$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by $P_\pm = (I \pm \vec{v} \cdot \vec{\sigma})/2$.
Answer:

$$
\begin{aligned}
\vec{v} \cdot \vec{\sigma} &= \sum_{i=1}^{3} v_i \sigma_i \\
&= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
&= v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \\
det(\vec{v} \cdot \vec{\sigma} - \lambda I) &= (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\
&= \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\
&= \lambda^2 - 1.
\end{aligned} \tag{10}
$$

Eigenvalues are $= \pm 1$. 未完