Q&A（2.11-2.20）

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Exercise 2.11: (Eigenedecomposition of the Pauli matrices) Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y, Z.

Answer:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Due to $c(\lambda) \equiv$ det $|A - \lambda I|$, When $c(\lambda) = 0$,we can get det$|A - \lambda I| = 0$.

Firstly, we discuss X, thus det$|X - \lambda I| = 0$, then

$$\left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0, \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0.$$

The solutions of λ are 1 and -1.
When $\lambda_1 = 1$, $(X - \lambda_1 I)|\lambda_1\rangle = 0$,

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we get the solution $x_1 = x_2$.
We assume that $x_1 = 1$, thus $x_2 = 1$.
After normalization, the following eigenvector is obtained:

$$|\lambda_1\rangle = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \tfrac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |+\rangle.$$

When $\lambda_2 = -1$, $(X - \lambda_2 I)|\lambda_2\rangle = 0$,

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we get the solution $x_1 = -x_2$.
We assume that $x_1 = 1$, thus $x_2 = -1$.

After normalization, the following eigenvector is obtained:

$$|\lambda_2\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} - \tfrac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} = |-\rangle.$$

So the diagonal representations of X is $X = |+\rangle\langle+| - |-\rangle\langle-|$.
Secondly, we discuss Y, thus $\det|Y - \lambda I| = 0$, then

$$\left|\begin{bmatrix}1 & -i\\ i & 0\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right| = 0, \begin{vmatrix}-\lambda & -i\\ i & -\lambda\end{vmatrix} = 0.$$

The solutions of λ are 1 and -1.

When $\lambda_1 = 1$, $(Y - \lambda_1 I)|\lambda_1\rangle = 0$,

$$\begin{bmatrix}-1 & -i\\ i & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

So we get the solution $x_1 = -ix_2$.
We assume that $x_1 = 1$, thus $x_2 = i$.
After normalization, the following eigenvector is obtained:

$$|\lambda_1\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}.$$

When $\lambda_2 = -1$, $(Y - \lambda_2 I)|\lambda_2\rangle = 0$,

$$\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

So we get the solution $x_1 = ix_2$. We assume that $x_1 = 1$, thus $x_2 = -i$.
After normalization, the following eigenvector is obtained:

$$|\lambda_2\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}.$$

So $Y = \begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}i\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}i\end{bmatrix} - \begin{bmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}i\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}i\end{bmatrix}$

$= \frac{1}{2}\left[\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1 & i\end{bmatrix} - \begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1 & -i\end{bmatrix}\right].$

Thirdly, we discuss Z, thus $\det|Z - \lambda I| = 0$, then

$$\left|\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right| = 0, \begin{vmatrix}1-\lambda & 0\\ 0 & -1-\lambda\end{vmatrix} = 0.$$

The λ solutions are 1 and -1.

When $\lambda_1 = 1$, $(Z - \lambda_1 I)|\lambda_1\rangle = 0$,

$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution $x_2 = 0$.

We assume that $x_1 = 1$, thus $\lambda_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

When $\lambda_2 = -1$, $(Z - \lambda_2 I)|\lambda_2\rangle = 0$,

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution $x_1 = 0$.

We assume that $x_2 = 1$,

thus $\lambda_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

So the diagonal representations of Z is $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$.

Exercise 2.12: Prove that the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is not diagonalizable.

Answer:

The necessary and sufficient condition for diagonalization is that there are n linearly independent eigenvectors for n-order square matrices. According to the knowledge of linear algebra elementary transformation,

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{first\ line\ minus\ second\ line} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. They have same eigenvalue $\lambda = 1$ and the eigenvector that $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So the eigenvector is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Only have one eigenvalue and eigenvector of the matrix does not satisfy the necessary and sufficient conditions, so the matrix is not diagonalizable.

Exercise 2.13: If $|w\rangle$ and $|v\rangle$ are any two vectors, show that $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.

Answer:

Suppose A is any linear operator on a Hilbert space V. It turns out that there exists a unique linear operator $A^\dagger$ on V such that for all vectors $|w\rangle, |v\rangle \in V$.

Since $(AB)^\dagger = B^\dagger A^\dagger$ and $|v\rangle^\dagger \equiv \langle v|$, $(|v\rangle\langle w|)^\dagger = \langle w|^\dagger |v\rangle^\dagger = |w\rangle\langle v|$.

Exercise 2.14:(Anti-linearity of the adjoint) Show that the adjoint operation is antilinear, $\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* A_i^\dagger$.

Answer:

Suppose A is any linear operator on a Hilbert space V. It turns out that there exists a unique linear operator $A^\dagger$ on V such that for all vectors $|w\rangle, |v\rangle \in V$. According to inner product is linear in the second argument and conjugate-linear in the first argument, we can make the following derivation:

$\left(|v\rangle, \sum_i a_i^* A_i^\dagger \, |w_i\rangle\right) = \sum_i a_i^* A_i^\dagger \left(|v\rangle, |w_i\rangle\right)$
$= \sum_i a_i^* \left(A_i|v\rangle, |w_i\rangle\right) = \left(\sum_i a_i A_i|v\rangle, |w_i\rangle\right) = \left(\sum_i a_i A_i\right)^\dagger \left(|v\rangle, |w_i\rangle\right) = \left(|v\rangle, (\sum_i a_i A_i)^\dagger \, |w_i\rangle\right)$.
Thus $(\sum_i a_i A_i)^\dagger = \sum_i a_i^* A_i^\dagger$.

Exercise 2.15: Show that $\left(A^\dagger\right)^\dagger = A$.

Answer:

Suppose A is any linear operator on a Hilbert space V. It turns out that there exists a unique linear operator $A^\dagger$ on V such that for all vectors $|w\rangle, |v\rangle \in V$.
Since $\left(|v\rangle, A^\dagger|w\rangle\right) = \left(A^\dagger|w\rangle, |v\rangle\right)^* = \left(|w\rangle, A|v\rangle\right)^* = \left(A|v\rangle, |w\rangle\right)$ and $\left(|v\rangle, A^\dagger|w\rangle\right) = \left[\left(A^\dagger\right)^\dagger |v\rangle, |w\rangle\right]$.
Thus $\left(A^\dagger\right)^\dagger = A$.

Exercise 2.16: Show that any projector P satisfies the equation $P^2 = P$

Answer:

Suppose V is a Hermite space, W be the k-dimensional subspace of d-dimensional vector space V. Using the gram Schimdt process, we can construct $|1\rangle\, |2\rangle \ldots |d\rangle$ is a set of standard orthogonal basis of V, so that $|1\rangle\, |2\rangle \ldots |k\rangle$is a standard orthogonal basis of $W, P \equiv \sum_{i=1}^k |i\rangle\langle i|$

$$
\begin{aligned}
P^2 &= (\sum_{i=1}^k |i\rangle\langle i|)(\sum_{j=1}^k |j\rangle\langle j|) = \sum_{i=1}^k \sum_{j=1}^k |i\rangle\langle i|j\rangle\langle j| \\
&= \sum_{i=1}^k \sum_{j=1}^k |i\rangle \delta_{ij} \langle j| = \sum_{i=1}^k |i\rangle\langle i| = P.
\end{aligned}
$$

Exercise 2.17: Show that a normal matrix is Hermitian if and only if it has real eigenvalues.

Answer:

Suppose $P \equiv \sum_i \lambda_i |i\rangle\langle i|$, thus $P^\dagger = \sum_i \lambda_i^* |i\rangle\langle i|$. Since P is a Hermitian operators, we have $P = P^\dagger$, then $\sum_i \lambda_i |i\rangle\langle i| = \sum_i \lambda_i^* |i\rangle\langle i|$. Thus $\lambda_i = \lambda_i^*$, $\lambda_i \in R$.

Exercise 2.18: Show that all eigenvalues of a unitary matrix have modulus 1, that is, can be written in the form $e^{i\theta}$ for some real θ.

Answer:

Suppose U is a unitary matrix, $U \equiv \sum_i \lambda_i |i\rangle\langle i|$, thus $U^\dagger = \sum_i \lambda_i^* |i\rangle\langle i|$ and U satisfies $U^\dagger U = I, I \equiv \sum_i |i\rangle\langle i|$.
$UU^\dagger = \left(\sum_i \lambda_i |i\rangle\langle i|\right) \left(\sum_i \lambda_i |i\rangle\langle i|\right)^\dagger = \sum_i \lambda_i \lambda_i^* |i\rangle\langle i| = I$

$\sum_i \lambda_i\lambda_i^*|i\rangle\langle i| = \sum_i |i\rangle\langle i| \Rightarrow \forall i, \lambda_i\lambda_i^* = 1$
Because $\lambda_i\lambda_i^* = 1$, thus $\|\lambda_i\| = 1$.
Let $\lambda_i = e^{i\theta} = \cos\theta + i\sin\theta$, then $\lambda_i^* = e^{-i\theta} = \cos\theta - i\sin\theta$.
$e^{i\theta} * e^{-i\theta} = (\cos\theta + i\sin\theta) * (\cos\theta - i\sin\theta) = 1$

Exercise 2.19: (Pauli matrices: Hermitian and unitary) Show that the Pauli matrices are Hermitian and unitary. **Answer:**

For Y be an example,
Hermitian:

$$Y^\dagger = (Y^*)^{\mathrm{T}} = \left(\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}^*\right)^{\mathrm{T}}$$
$$= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y.$$

Unitary:

$$Y^\dagger Y = (Y^*)^T Y = \left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^*\right)^T \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$
$$= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

So Y is Hermite operator and unitary matrix.
Other cases will also reach corresponding conclusions according to the above calculations.

Exercise 2.20: (Basis changes) Suppose A' and A'' are matrix representations of an operator A on a vector space V with respect to two different orthonormal bases,$|v_i\rangle$ and $|w_j\rangle$.Then the elements of A' and A'' are $A'_{ij} = \langle v_i|A|v_j\rangle$ and $A''_{ij} = \langle w_i|A|w_j\rangle$. Characterize the relationship between A' and A''.

Answer:

$$U \equiv \sum_i |w_i\rangle \langle v_i|$$

$$\begin{aligned}
A'_{ij} &= \langle v_i|A|v_j\rangle \\
&= \left\langle v_i \left| UU^\dagger AUU^\dagger \right| v_j \right\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \langle v_p|v_q\rangle \langle w_q|A|w_r\rangle \langle v_r|v_s\rangle \langle w_s|v_j\rangle \\
&= \sum_{p,q,r,s} \langle v_i|w_p\rangle \delta_{pq} A''_{qr} \delta_{rs} \langle w_s|v_j\rangle \\
&= \sum_{p,r} \langle v_i|w_p\rangle \langle w_r|v_j\rangle A''_{pr}
\end{aligned}$$