Q&A（2.61-2.70）

LuoTingyu JiangHui

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Exercise 2.61: Calculate the probability of obtaining the result +1 for a measurement $\vec{v}\cdot\vec{\sigma}$, given that the state prior to measurement is $|0\rangle$. What is the state of the system after the measurement if +1 is obtained?

Answer:

$$\begin{aligned} p(+1) =& \langle 0|\lambda_1\rangle \langle \lambda_1|0\rangle \\ =& \frac{1}{2} \langle 0| (I+\vec{v}\cdot\vec{\sigma}) |0\rangle \\ =& \frac{1}{2} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ =& \frac{1}{2}(1+v_3). \end{aligned} \tag{1}$$

The state of the quantum system immediately after the measurement is

$$\begin{aligned} \frac{|\lambda_1\rangle \langle\lambda_1|0\rangle}{\sqrt{p(+1)}} =& \frac{|\lambda_1\rangle \langle\lambda_1|0\rangle}{\sqrt{\frac{1}{2}(1+v_3)}} \\ =& \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}} * \frac{1}{2} \begin{bmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ =& \frac{1}{\sqrt{\frac{1}{2}(1+v_3)}} \frac{1}{2} \begin{bmatrix} 1+v_3 \\ v_1+iv_2 \end{bmatrix} \\ =& \sqrt{\frac{1+v_3}{2}} \begin{bmatrix} 1 \\ \frac{v_1+iv_2}{1+v_3} \end{bmatrix}. \end{aligned} \tag{2}$$

Exercise 2.62: Show that any measurement where the measurement operators and the POVM elements coincide is a projective measurement.

Answer:

Suppose M_m is a measurement operator, From the assumption, $E_m = M_m^\dagger M_m = M_m$.

$$
\begin{aligned}
p(m) &= |\psi\rangle E_m \langle\psi| \\
&= |\psi\rangle M_m \langle\psi| \\
&\geq 0 \\
&= (|\psi\rangle, M_m |\psi\rangle)
\end{aligned} \tag{3}
$$

Thus, M_m is an positive operator for all $|\psi\rangle$, then M_m is Hermitian, $M_m^\dagger = M_m$.

$$
E_m = M_m^\dagger M_m = M_m^2 = M_m \tag{4}
$$

Thus, any measurement where the measurement operators and the POVM elements coincide is a projective measurement.

Exercise 2.63: Suppose a measurement is described by measurement operators M_m. Show that there exist unitary operators U_m such that $M_m = U_m sqrt E_m$, where E_m is the POVM associated to the measurement.

Answer:

Since E_m is a positive operator.

$$
\begin{aligned}
M_m^\dagger M_m &= (U_m sqrt E_m)^\dagger U_m sqrt E_m \\
&= \sqrt{E_m}^\dagger U_m^\dagger U_m sqrt E_m \\
&= \sqrt{E_m}^\dagger \sqrt{E_m} \\
&= E_m
\end{aligned} \tag{5}
$$

Since E_m is POVM, for arbitrary unitary U, $M_m^\dagger M_m$ is POVM.

Exercise 2.64: Suppose Bob is given a quantum state chosen from a set $|\psi_1\rangle, ..., |\psi_m\rangle$ of linearly independent states. Construct a POVM $\{E_1, E_2, ..., E_{m+1}\}$ such that if outcome E_i occurs, $1 \leq i \leq m$, then Bob knows with certainty that he was given the state $|\psi_i\rangle$. (The POVM must be such that $\langle\psi_i| E_i |\psi_i\rangle > 0$ for each i.) **Answer:**

Exercise 2.65: Express the states $\frac{(|0\rangle+|1\rangle)}{2}$ and $\frac{(|0\rangle-|1\rangle)}{2}$ in a basis in which they are not the same up to a relative phase shift.

Answer:

$$
|+\rangle = \frac{(|0\rangle + |1\rangle)}{2} \quad |-\rangle = \frac{(|0\rangle - |1\rangle)}{2} \tag{6}
$$

Exercise 2.66: Show that the average value of the observable $X_1 Z_2$ for a two qubit system measured in the state $\frac{(|00\rangle+|11\rangle)}{\sqrt{2}}$ is zero.

Answer:

The X matrix takes $|0\rangle$ to $|1\rangle$, and $|1\rangle$ to $|0\rangle$, and the Z matrix leaves $|0\rangle$ invariant, and takes $|1\rangle$ to $-|1\rangle$.

$$\begin{aligned} E(m) =& \frac{(\langle 00| + \langle 11|)}{\sqrt{2}} X_1 Z_2 \frac{(|00\rangle + |11\rangle)}{\sqrt{2}} \\ =& \frac{(\langle 00| + \langle 11|)}{\sqrt{2}} \frac{(|10\rangle - |01\rangle)}{\sqrt{2}} \\ =& \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \\ =& 0 \end{aligned} \tag{7}$$

Exercise 2.67: Suppose V is a Hilbert space with a subspace W . Suppose $U : W \to V$ is a linear operator which preserves inner products, that is, for any $|w_1\rangle$ and $|w_2\rangle$ in W,

$$\langle w_1| U^\dagger U |w_2\rangle = \langle w_1|w_2\rangle \tag{8}$$

Prove that there exists a unitary operator $U' : V \to V$ which *extends* U . That is, $U' |w\rangle = U |w\rangle$ for all $|w\rangle$ in W, but U' is defined on the entire space V. Usually we omit the prime symbol and just write U to denote the extension.
Answer:

Exercise 2.68: Prove that $|\psi\rangle \neq |a\rangle |b\rangle$ for all single qubit states $|a\rangle$ and $|b\rangle$.
Answer:
Suppose $|a\rangle = a_0 |0\rangle + a_1 |1\rangle$ and $|b\rangle = b_0 |0\rangle + b_1 |1\rangle$.

$$|a\rangle |b\rangle = a_0 b_0 |00\rangle + a_0 b_1 |01\rangle + a_1 b_0 |10\rangle + a_1 b_1 |11\rangle \tag{9}$$

If $|\psi\rangle = |a\rangle |b\rangle$, then $a_0 b_0 = \frac{1}{\sqrt{2}}$, $a_0 b_1 = 0$, $a_1 b_0 = 0$, and $a_1 b_1 = \frac{1}{\sqrt{2}}$.
If $a_0 b_1 = 0$, then $a_0 = 0$ or $b_1 = 0$.
When $a_0 = 0$,this does not satisfy $a_0 b_0 = \frac{1}{\sqrt{2}}$. When $b_1 = 0$,this does not satisfy $a_1 b_1 = \frac{1}{\sqrt{2}}$.
If $a_1 b_0 = 0$, then $a_1 = 0$ or $b_0 = 0$.
When $a_1 = 0$,this does not satisfy $a_1 b_1 = \frac{1}{\sqrt{2}}$. When $b_0 = 0$,this does not satisfy $a_0 b_0 = \frac{1}{\sqrt{2}}$.
Thus $|\psi\rangle \neq |a\rangle |b\rangle$.

Exercise 2.69: Verify that the Bell basis forms an orthonormal basis for the two qubit state space.
Answer:
Suppose the Bell basis as follows,

$$\begin{aligned}
|\psi_1\rangle =& \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
|\psi_2\rangle =& \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \\
|\psi_3\rangle =& \frac{|01\rangle + |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \\
|\psi_4\rangle =& \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}
\end{aligned} \tag{10}$$

We need to prove $\{|\psi_1\rangle\}$ is linearly independent basis.

$$\begin{aligned}
&a_1 |\psi_1\rangle + a_2 |\psi_2\rangle + a_3 |\psi_3\rangle + a_4 |\psi_4\rangle = 0 \\
&\begin{bmatrix} a_1 + a_2 \\ a_3 + a_4 \\ a_3 - a_4 \\ a_1 - a_2 \end{bmatrix} = 0 \\
&a_1 = a_2 = a_3 = a_4 = 0
\end{aligned} \tag{11}$$

Moreover, $\langle\psi_i|\psi_j\rangle = \delta_{ij} for i, j = 1\ 2\ 3\ 4$ and the norm of $\{|\psi_1\rangle\}$ is

$$\begin{aligned}
|\psi_1\rangle =& \sqrt{\langle\psi_i|\psi_i\rangle} \\
=& 1
\end{aligned} \tag{12}$$

Thus, $\{|\psi_1\rangle\}$ forms an orthonormal basis.

Exercise 2.70: Suppose E is any positive operator acting on Alice's qubit. Show that $\langle\psi| E \otimes I |\psi\rangle$ takes the same value when $|\psi\rangle$ is any of the four Bell states. Suppose some malevolent third party ('Eve') intercepts Alice's qubit on the way to Bob in the superdense coding protocol. Can Eve infer anything about which of the four possible bit strings 00, 01, 10, 11 Alice is trying to send? If so, how, or if not, why not?

Answer:

$$
\begin{aligned}
\langle\psi_1| E \otimes I |\psi_1\rangle =& \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} E \otimes I \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\
=& \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix} E \otimes \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}
\end{aligned} \tag{13}
$$