Q&A（2.31-2.40）

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Exercise 2.31: Show that the tensor product of two positive operators is positive.
Answer:
Suppose $|v\rangle, |w\rangle$ are arbitrary vectors of the V, W space, and A, B are two positive operators of the V, W space, respectively. Suppose $|v\rangle \otimes |w\rangle$ is an arbitrary vectors of the $V \otimes W$ space. Then we can get the following derivation:
$(|v\rangle \otimes |w\rangle, A \otimes B(|v\rangle \otimes |w\rangle)) = (|v\rangle \otimes |w\rangle, A|v\rangle \otimes B|w\rangle) = (|v\rangle, A|v\rangle)(|w\rangle, B|w\rangle) \geq 0.$
Thus we proved that the tensor product of two positive operators is positive.

Exercise 2.32: Show that the tensor product of two projectors is a projector.
Answer:
Suppose A, B are two projectors of the V, W space, respectively, so it can be written as $A = \sum_i |i\rangle\langle i|, B = \sum_j |j\rangle\langle j|$. Thus we can get the following derivation:

$$\begin{aligned}(A \otimes B)^2 &= (A \otimes B)(A \otimes B)\\ &= (AA \otimes BB)\\ &= (A^2 \otimes B^2)\\ &= A \otimes B.\end{aligned}$$

Thus we proved that the tensor product of two projectors is a projector.

Exercise 2.33: The Hadamard operator on one qubit may be written as

$$H = \frac{1}{\sqrt{2}}[(|0\rangle + |1\rangle)\langle 0| + (|0\rangle - |1\rangle)\langle 1|]$$

Show explicitly that the Hadamard transform on n qubits, $H \otimes n$, may be written as

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x,y} (-1)^{x \cdot y} |x\rangle\langle y|.$$

Write out an explicit matrix representation for $H^{\otimes 2}$.

Answer:

Exercise 2.34: Find the square root and logarithm of the matrix $\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$.
Answer:

Since $A^\dagger = A, A$ is a Hermitian and normal. A can be written as $A = \sum_i \lambda_i |i\rangle \langle i|$. We first solve the eigenvalues and eigenvectors of A. We set $\det|A - \lambda E| = 0$, then $|A - \lambda E| = 0$,

$$\begin{bmatrix} 4-\lambda & 3 \\ 3 & 4-\lambda \end{bmatrix} = 0.$$

The solution of λ is 1 or 7.

When $\lambda = 1$, calculate $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$, then we can get $3x_1 + 3x_2 = 0$.

Suppose $x_1 = 1$, then $x_2 = -1$. $|\lambda_1\rangle$ can be written as $|\lambda_1\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

After normalization, the eigenvector is obtained: $|\lambda_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

When $\lambda = 7$, calculate $\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$, then we get $3x_1 - 3x_2 = 0$.

Suppose $x_1 = 1$, then $x_2 = 1$, we can get $|\lambda_7\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

After normalization, the eigenvector is obtained: $|\lambda_7\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So the diagonal representations of A is $\frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \frac{7}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}$.

square root:

$$\sqrt{A} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \frac{\sqrt{7}}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{7}+1 & \sqrt{7}-1 \\ \sqrt{7}-1 & \sqrt{7}+1 \end{bmatrix}.$$

logarithm:

$$\log_2 A = \frac{1}{2} \log_2 7 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Exercise 2.35: (Exponential of the Pauli matrices) Let $\vec{v}$ be any real, three-dimensional unit vector and θ a real number. Prove that

$$\exp(i\theta \vec{v} \cdot \vec{\sigma}) = \cos(\theta) I + i \sin(\theta) \vec{v} \cdot \vec{\sigma},$$

where $\vec{v} \cdot \vec{\sigma} = \sum_{i=1}^{3} \vec{v}_i \vec{\sigma}_i$. This exercise is generalized in Problem 2.1 on page 117.

Answer:

According to the meaning of the question, we can get $\vec{v} \cdot \vec{\sigma} = \sum_{i=1}^{3} \vec{v}_i \vec{\sigma}_i$

$$= v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix}.$$

Since $\vec{v}$ be any real, three-dimensional unit vector, we can get $v_1^2 + v_2^2 + v_3^2 = 1$.
We first solve the eigenvalues and eigenvectors of A.

Let $\begin{bmatrix} v_3 - \lambda & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 - \lambda \end{bmatrix} = 0$, we can get $\lambda^2 = v_1^2 + v_2^2 + v_3^2$.

Thus the solution of λ is 1 or -1.
When $\lambda = 1$, we set the eigenvector is $|\lambda_1\rangle$, and when $\lambda = -1$, we set the eigenvector is $|\lambda_{-1}\rangle$.
Then we can get the following derivation:
$\vec{v} \cdot \vec{\sigma} = |\lambda_1\rangle \langle\lambda_1| - |\lambda_{-1}\rangle \langle\lambda_{-1}|$.
$(\vec{v} \cdot \vec{\sigma})^\dagger = |\lambda_1\rangle \langle\lambda_1| - |\lambda_{-1}\rangle \langle\lambda_{-1}|$.
Since $\vec{v} \cdot \vec{\sigma} = (\vec{v} \cdot \vec{\sigma})^\dagger$, $\vec{v} \cdot \vec{\sigma}$ is Hermitian.
$i\theta\vec{v} \cdot \vec{\sigma} = i\theta |\lambda_1\rangle \langle\lambda_1| - i\theta |\lambda_{-1}\rangle \langle\lambda_{-1}|$,
$\exp(i\theta\vec{v} \cdot \vec{\sigma}) = e^{i\theta} |\lambda_1\rangle \langle\lambda_1| + e^{-i\theta} |\lambda_{-1}\rangle \langle\lambda_{-1}|$
$= (\cos(\theta) + i\sin(\theta)) |\lambda_1\rangle \langle\lambda_1| + (\cos(\theta) - i\sin(\theta)) |\lambda_{-1}\rangle \langle\lambda_{-1}|$
$= \cos(\theta)(|\lambda_1\rangle \langle\lambda_1| + |\lambda_{-1}\rangle \langle\lambda_{-1}|) + i\sin(\theta)(|\lambda_1\rangle \langle\lambda_1| - |\lambda_{-1}\rangle \langle\lambda_{-1}|)$.
According to the completeness relation, we can get $|\lambda_1\rangle \langle\lambda_1| + |\lambda_{-1}\rangle \langle\lambda_{-1}| = I$.
Thus, we proved that $\exp(i\theta\vec{v} \cdot \vec{\sigma}) = \cos(\theta)I + i\sin(\theta)\vec{v} \cdot \vec{\sigma}$.

Exercise 2.36: Show that the Pauli matrices except for I have trace zero.
Answer:
$\mathrm{tr}\,(\sigma_0) = \mathrm{tr}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) = 2$.
$\mathrm{tr}\,(\sigma_1) = \mathrm{tr}(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = 0$.
$\mathrm{tr}\,(\sigma_2) = \mathrm{tr}(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}) = 0$.
$\mathrm{tr}\,(\sigma_3) = \mathrm{tr}(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}) = 0$.
Above all, the Pauli matrices except for I have trace zero.

Exercise 2.37: (Cyclic property of the trace) If A and B are two linear operators show that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA).$$

Answer:
$\mathrm{tr}(AB) = \sum_i \langle i| AB |i\rangle = \sum_i \langle i| AIB |i\rangle = \sum_{ij} \langle i| A |j\rangle \langle j| B |i\rangle = \sum_{ij} \langle j| B |i\rangle \langle i| A |j\rangle$
$= \sum_j \langle j| BA |j\rangle = \mathrm{tr}(BA)$.
Thus we proved that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Exercise 2.38: (Linearity of the trace) If A and B are two linear operators, show that

$$\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$$

and if z is an arbitrary complex number show that

$$\mathrm{tr}(zA) = z\,\mathrm{tr}(A).$$

Answer:
$\mathrm{tr}(A) = \sum_i (A)_{ii}, \mathrm{tr}(B) = \sum_i (B)_{ii}$.
$\mathrm{tr}(A+B) = \sum_i (A+B)_{ii} = \sum_i (A_{ii} + B_{ii}) = \sum_i (A_{ii}) + \sum_i (B_{ii}) = \mathrm{tr}(A) + \mathrm{tr}(B)$.
Thus we proved that $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.
$\mathrm{tr}(zA) = \sum_i (zA)_{ii} = z \sum_i (A)_{ii} = z\,\mathrm{tr}(A)$.
Thus we proved that $\mathrm{tr}(zA) = z\,\mathrm{tr}(A)$.

Exercise 2.39: (The Hilbert−Schmidt inner product on operators) The set L_v of linear operators on a Hilbert space V is obviously a vector space −the sum of two linear operators is a linear operator, zA is a linear operator if A is a linear operator and z is a complex number, and there is a zero element 0. An important additional result is that the vector space L_v can be given a natural inner product structure, turning it into a Hilbert space.
(1) Show that the function $(\ \cdot\ ,\ \ \cdot\ )$ on $L_v \times\ L_v$ defined by

$$(A, B) = \mathrm{tr}(A^\dagger B)$$

is an inner product function. This inner product is known as the Hilbert−Schmidt or trace inner product.
(2) If V has d dimensions show that L_v has dimension d^2.
(3) Find an orthonormal basis of Hermitian matrices for the Hilbert space L_v.
Answer:
(1) Prove as follows:
According to the definition of inner product:
1. $(\ \cdot\ ,\ \cdot\ )$ is linear in the second argument.
$(A, \sum_i \lambda_i B_i) = \mathrm{tr}(A^\dagger \sum_i \lambda_i B_i) = \mathrm{tr}(\sum_i \lambda_i A^\dagger B_i) = \sum_i \lambda_i\,\mathrm{tr}(A^\dagger B_i)$.
Thus $(\ \cdot\ ,\ \cdot\ )$ is linear in the second argument.
2. $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$.
$(A, B)^* = (\mathrm{tr}(A^\dagger B))^* = (\sum_{ij} \langle i| A |j\rangle \langle j| B |i\rangle)^* = \sum_{ij} (\langle j| B |i\rangle)^* (\langle i| A^\dagger |j\rangle)^* = \sum_{ij} (\langle i| B^\dagger |j\rangle)(\langle j| A |i\rangle)$
$= \sum_i \langle i| B^\dagger A |i\rangle = \mathrm{tr}(B^\dagger A) = (B, A)$.
Thus we proved that $(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$.
3. $(|v\rangle, |v\rangle) \geq 0$ with equality if and only if $|v\rangle = 0$.
$(A, A) = \mathrm{tr}(A^\dagger A) = \mathrm{tr}(\sum_i \lambda_i^* \lambda_i |i\rangle \langle i|) = \mathrm{tr}(\sum_i ||\lambda_i||^2 |i\rangle \langle i|) = \sum_i ||\lambda_i||^2\,\mathrm{tr}(|i\rangle \langle i|)$
$= \sum_i ||\lambda_i||^2 \langle i|i\rangle = \sum_i ||\lambda_i||^2 \geq 0$. When $A = 0$, we can get $(A, A) = 0$.
Thus we proved that $(|v\rangle, |v\rangle) \geq 0$ with equality if and only if $|v\rangle = 0$.
Above all, the function $(\ \cdot\ ,\ \ \cdot\ )$ on $L_v \times\ L_v$ is an inner product function.

(2) Since $A = \sum_{ij} \langle i| A |j\rangle |i\rangle \langle j| = \sum_{ij} A_{ji} |i\rangle \langle j|$, then $|i\rangle \langle j|$ as a set of bases. $|i\rangle$ has d, then $|i\rangle \langle j|$ has d^2.
(3)According to the second question, we can suppose that $|i\rangle \langle j|$ is an orthonormal basis. Verification as follows:
$(|i\rangle \langle j| , |i\rangle \langle j|) = \mathrm{tr}(|j\rangle \langle i|i\rangle \langle j|) = \mathrm{tr}(|j\rangle \langle j|) = \langle j|j\rangle = 1.$
$(|i\rangle \langle j| , |k\rangle \langle l|) = \mathrm{tr}(|j\rangle \langle i|k\rangle \langle l|) = \mathrm{tr}(|j\rangle\, 0\, \langle j|) = 0, (k, l \neq i, j).$
Above all, thus $|i\rangle \langle j|$ is the orthonormal basis.

Exercise 2.40: (Commutation relations for the Pauli matrices) Verify the commutation relations

$$[X, Y] = 2iZ; [Y, Z] = 2iX; [Z, X] = 2iY.$$

Answer:
$[X, Y] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$= \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$
$= \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = 2iZ.$
$[Y, Z] = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = 2iX.$
$[Z, Y] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
$= \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = 2iY.$
Thus we verified the commutation relations for the Pauli matrices.