Q&A（2.51-2.60）

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Exercise 2.51: Verify that the Hadamard gate H is unitary.

Answer:

$H^\dagger H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I.$

Thus the Hadamard gate H is unitary.

Exercise 2.52: Verify that $H^2 = I$.

Answer:

$H^2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I.$

Exercise 2.53: What are the eigenvalues and eigenvectors of H?

Answer:

$det|A - \lambda I| = det\left|\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right| = \begin{bmatrix}\frac{1}{\sqrt{2}} - \lambda & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} - \lambda\end{bmatrix} = 0$

Eigenvalues are $\pm = \pm 1$ and associated eigenvectors are $|\lambda_\pm\rangle = \frac{1}{\sqrt{4\pm 2\sqrt{2}}}\begin{bmatrix}1\\ -1\pm\sqrt{2}\end{bmatrix}$.

Exercise 2.54: Suppose A and B are commuting Hermitian operators. Prove that $exp(A)exp(B) = exp(A+B)$. (Hint: Use the results of Section 2.1.9.)

Answer:

Since $[A, B] = 0$, A and B are simultaneously diagonalize, $A = \sum_i a_i |i\rangle\langle i|, B = \sum_j b_i |j\rangle\langle j|$.

$$\begin{aligned}
exp(A)exp(B) &= \sum_i exp(a_i)|i\rangle\langle i| B = \sum_j exp(b_i)|j\rangle\langle j| \\
&= \sum_{i,j} exp(a_i + b_j)|i\rangle\langle i|j\rangle\langle j| \\
&= \sum_{i,j} exp(a_i + b_j)|i\rangle\langle j|\delta_{i,j} \\
&= \sum_i exp(a_i + b_i)|i\rangle\langle i| \\
&= exp(A+B).
\end{aligned} \tag{1}$$

Exercise 2.55: Prove that $U(t_1, t_2)$ defined in Equation (2.91) is unitary.

Answer:

$$
\begin{aligned}
U(t_1,t_2)^{\dagger}U(t_1,t_2) =& exp\left(\frac{iH(t_2-t_1)}{\hbar}\right)\left(\frac{-iH(t_2-t_1)}{\hbar}\right) \\
=& exp\left(\frac{i\sum_{E_1} E_1 \left|E_1\right\rangle \left\langle E_1\right| (t_2-t_1)}{\hbar}\right) exp\left(\frac{-i\sum_{E_2} E_2 \left|E_2\right\rangle \left\langle E_1\right| (t_2-t_1)}{\hbar}\right) \\
=& \sum_{E_1,E_2} exp\left(\frac{iE_1(t_2-t_1)}{\hbar}\right) \left|E_2\right\rangle \left\langle E_2\right| exp\left(\frac{-iE_1(t_2-t_1)}{\hbar}\right) \left|E_1\right\rangle \left\langle E_1\right| \\
=& \sum_{E_1,E_2} exp\left(\frac{i(E_1-E_2)(t_2-t_1)}{\hbar}\right) \left|E_1\right\rangle \left\langle E_1|E_2\right\rangle \left\langle E_2\right| \\
=& \sum_{E_1,E_2} exp\left(\frac{i(E_1-E_2)(t_2-t_1)}{\hbar}\right) \left|E_1\right\rangle \left\langle E_2\right| \delta_{E_1,E_2} \\
=& \sum_{E_1} \left|E_1\right\rangle \left\langle E_1\right| \\
=& I.
\end{aligned} \tag{2}
$$

Thus $U(t_1,t_2)$ is unitary.

Exercise 2.56: Use the spectral decomposition to show that $K \; -i\log(U)$ is Hermitian for any unitary U, and thus $U = exp(iK)$ for some Hermitian K.

Answer:

Since U is unitary, then U can perform spectral decomposition, $U = \sum_i \lambda_i \left|i\right\rangle \left\langle i\right|$

$$
\begin{aligned}
K^{\dagger} =& (-i\log(U))^{\dagger} \\
=& (-i\log\left(\sum_i \lambda_i \left|i\right\rangle \left\langle i\right|\right))^{\dagger} \\
=& (i\sum_i \log(\lambda_i) \left|i\right\rangle \left\langle i\right|).
\end{aligned} \tag{3}
$$

Exercise 2.57: (Cascaded measurements are single measurements) Suppose L_l and M_m are two sets of measurement operators. Show that a measurement defined by the measurement operators L_l followed by a measurement defined by the measurement operators M_m is physically equivalent to a single measurement defined by measurement operators N_{lm} with the representation $N_{lm} = M_m L_l$.

Answer:

If the state of the quantum system is $\left|\psi\right\rangle$ immediately before the measurement. The state of the system after the first measurement is $\left|\psi_L\right\rangle = \frac{L_l\left|\psi\right\rangle}{\sqrt{\left\langle\psi\right|L_l^{\dagger}L_l\left|\psi\right\rangle}}$ and the second measurement is

$|\psi_M\rangle = \frac{M_m|\psi_L\rangle}{\sqrt{\langle\psi_L|M_m^\dagger M_m|\psi_L\rangle}}$.

$$\begin{aligned}
\langle\psi_L| =& \frac{\langle\psi| L_l^\dagger}{\sqrt{\langle\psi| L_l^\dagger L_l |\psi\rangle}} \\
|\psi_M\rangle =& \frac{M_m |\psi_L\rangle}{\sqrt{\langle\psi_L| M_m^\dagger M_m |\psi_L\rangle}} \\
=& \frac{M_m \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}}{\frac{\langle\psi|L_l^\dagger}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}} M_m^\dagger M_m \frac{L_l|\psi\rangle}{\sqrt{\langle\psi|L_l^\dagger L_l|\psi\rangle}}} \\
=& \frac{M_m L_l |\psi\rangle}{\langle\psi| L^\dagger M_m^\dagger M_m L_l |\psi\rangle}.
\end{aligned} \tag{4}$$

The state of the system after the measurement operators N_{lm} ($N_{lm} = M_m L_l$.) is

$$\begin{aligned}
|\psi_N\rangle =& \frac{N_{lm} |\psi\rangle}{\sqrt{\langle\psi| N_{lm}^\dagger N_{lm} |\psi\rangle}} \\
=& \frac{M_m L_l |\psi\rangle}{\sqrt{\langle\psi| L_l^\dagger M_m^\dagger M_m L_l |\psi\rangle}} = \quad |\psi_M\rangle .
\end{aligned} \tag{5}$$

Thus we proved that Cascaded measurements are single measurements.

Exercise 2.58: Suppose we prepare a quantum system in an eigenstate $|\psi\rangle$ of some observable M , with corresponding eigenvalue m. What is the average observed value of M, and the standard deviation?

Answer:

$$\begin{aligned}
\langle M\rangle =& \langle\psi| M |\psi\rangle \\
=& \langle\psi| m |\psi\rangle \\
=& m \langle\psi|\psi\rangle \\
= m[\Delta M]^2 =& \left\langle M^2\right\rangle - \langle M\rangle^2 \\
=& \langle\psi| m^2 |\psi\rangle - m^2 \\
=& m^2 - m^2 \\
=& 0.
\end{aligned} \tag{6}$$

Exercise 2.49: Suppose we have qubit in the state $|0\rangle$, and we measure the observable X. What is the average value of X? What is the standard deviation of X?

Answer:

$$
\begin{aligned}
\langle X \rangle =& \langle 0| X |0\rangle \\
=& 0 \\
\left\langle X^2 \right\rangle =& \langle 0| X^2 |0\rangle \\
=& 1 \\
[\Delta X] =& \sqrt{\langle X^2 \rangle - \langle X \rangle^2} = 1.
\end{aligned} \tag{7}
$$

Exercise 2.50: Show that $v \cdot \sigma$ has eigenvalues ± 1, and that the projectors onto the corresponding eigenspaces are given by $P_{\pm} = (I \pm \vec{v} \cdot \vec{\sigma})/2$.

Answer:

$$
\begin{aligned}
\vec{v} \cdot \vec{\sigma} =& \sum_{i=1}^{3} v_i \sigma_i \\
=& v_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
=& v_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \\
=& v_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
=& \begin{bmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{bmatrix} \\
det(\vec{v} \cdot \vec{\sigma} - \lambda I) =& (v_3 - \lambda)(-v_3 - \lambda) - (v_1 - iv_2)(v_1 + iv_2) \\
=& \lambda^2 - (v_1^2 + v_2^2 + v_3^2) \\
=& \lambda^2 - 1.
\end{aligned} \tag{8}
$$

Eigenvalues are $= \pm 1$. if $\lambda = 1$

$$
\begin{aligned}
\vec{v} \cdot \vec{\sigma} - \lambda I =& \vec{v} \cdot \vec{\sigma} - I \\
=& \begin{bmatrix} v_3 - 1 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 - 1 \end{bmatrix}
\end{aligned} \tag{9}
$$

Normalized eigenvector is $|\lambda_1\rangle = \sqrt{\frac{1+v_3}{2}} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix}$.

$$\begin{aligned}
|\lambda_1\rangle \langle\lambda_1| =& \frac{1+V_3}{2} \begin{bmatrix} 1 \\ \frac{1-v_3}{v_1-iv_2} \end{bmatrix} \begin{bmatrix} 1 & \frac{1-v_3}{v_1+iv_2} \end{bmatrix} \\
=& \frac{1+v_3}{2} \begin{bmatrix} 1 & \frac{v_1-iv_2}{1+v_3} \\ \frac{v_1+iv_2}{1+v_3} & \frac{1-v_3}{1+v_3} \end{bmatrix} \\
=& \frac{1}{2} \begin{bmatrix} 1+v_3 & v_1-iv_2 \\ v_1+iv_2 & 1-v_3 \end{bmatrix} \\
=& \frac{1}{2} \left(I + \begin{bmatrix} v_3 & v_1-iv_2 \\ v_1+iv_2 & -v_3 \end{bmatrix} \right) \\
=& \frac{1}{2}(I + \vec{v}\cdot\vec{\sigma}).
\end{aligned} \tag{10}$$

If $\lambda = -1$.

Normalized eigenvalue is$|\lambda_{-1}\rangle = \sqrt{\frac{1-v_3}{2}} \begin{bmatrix} 1 \\ -\frac{1+v_3}{v_1-iv_2} \end{bmatrix}$.

Similarly, we can get the $|\lambda_{-1}\rangle \langle\lambda_{-1}| = \frac{1}{2}(I - \vec{v}\cdot\vec{\sigma})$.