Q&A（2.11-2.20）

Luotingyu　Jianghui

11th September 2019

Exercise 2.11: (Eigenedecomposition of the Pauli matrices) Find the eigenvectors, eigenvalues, and diagonal representations of the Pauli matrices X, Y, Z.

Answer:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

For any matrices A, through $\det|A - \lambda I| = 0$, we can caculate the eigenvalues.
Firstly, we discuss X. We set $\det|X - \lambda I| = 0$, then

$$\left| \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0, \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0.$$

The solution of λ is 1 or -1.
When $\lambda_1 = 1$, calculate $(X - \lambda_1 I)|\lambda_1\rangle = 0$,

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we get the solution $x_1 = x_2$.
We assume that $x_1 = 1$, then $x_2 = 1$.
After normalization, the following eigenvector is obtained:

$$|\lambda_1\rangle = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \tfrac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |+\rangle.$$

When $\lambda_2 = -1$, calculate $(X - \lambda_2 I)|\lambda_2\rangle = 0$,

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Thus we get the solution $x_1 = -x_2$.
We assume that $x_1 = 1$, then $x_2 = -1$.
After normalization, the following eigenvector is obtained:

$$|\lambda_2\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} - \tfrac{1}{\sqrt{2}}\begin{bmatrix}0\\1\end{bmatrix} = |-\rangle.$$

Thus the diagonal representation of X is$|+\rangle\langle+| - |-\rangle\langle-|$.
Secondly, we discuss Y. We set $\det|Y - \lambda I| = 0$, then

$$\left|\begin{bmatrix}1 & -i\\ i & 0\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right| = 0, \begin{vmatrix}-\lambda & -i\\ i & -\lambda\end{vmatrix} = 0.$$

The solution of λ is 1 or -1.

When $\lambda_1 = 1$, calculate $(Y - \lambda_1 I)|\lambda_1\rangle = 0$,

$$\begin{bmatrix}-1 & -i\\ i & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

So we get the solution $x_1 = -ix_2$.
We assume that $x_1 = 1$, then $x_2 = i$.
After normalization, the following eigenvector is obtained:

$$|\lambda_1\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}.$$

When $\lambda_2 = -1$, calculate $(Y - \lambda_2 I)|\lambda_2\rangle = 0$,

$$\begin{bmatrix}1 & -i\\ i & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}.$$

So we get the solution $x_1 = ix_2$. We assume that $x_1 = 1$, thus $x_2 = -i$.
After normalization, the following eigenvector is obtained:

$$|\lambda_2\rangle = \tfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}.$$

So we can get $Y = \begin{bmatrix}\frac{\sqrt{2}}{2}\\ \frac{\sqrt{2}}{2}i\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}i\end{bmatrix} - \begin{bmatrix}\frac{\sqrt{2}}{2}\\ -\frac{\sqrt{2}}{2}i\end{bmatrix}\begin{bmatrix}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}i\end{bmatrix}$

$$= \tfrac{1}{2}\left[\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1 & i\end{bmatrix} - \begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1 & -i\end{bmatrix}\right].$$

Thirdly, we discuss Z. We set $\det|Z - \lambda I| = 0$, then

$$\left|\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right| = 0, \begin{vmatrix}1-\lambda & 0\\ 0 & -1-\lambda\end{vmatrix} = 0.$$

The solution of λ is 1 or -1.

When $\lambda_1 = 1$, calculate $(Z - \lambda_1 I)|\lambda_1\rangle = 0$,

$$\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution is $x_2 = 0$.
We assume that $x_1 = 1$, thus $|\lambda_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
When $\lambda_2 = -1$, calculate $(Z - \lambda_2 I)|\lambda_2\rangle = 0$,

$$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The solution is $x_1 = 0$.
We assume that $x_2 = 1$,

thus $|\lambda_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

So the diagonal representation of Z is $|0\rangle\langle 0| - |1\rangle\langle 1|$.

Exercise 2.12: Prove that the matrix $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is not diagonalizable.
Answer:
The necessary and sufficient condition for diagonalization is that there are n linearly independent eigenvectors for n-order square matrices. According to the knowledge of linear algebra elementary transformation,

$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \xrightarrow{second\ line\ minus\ first\ line} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. They have same eigenvalue $\lambda = 1$.
Then we compute the eigenvectors, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Thus we get the solution is $x_1 = 0$, and we set $x_2 = 1$. So the eigenvector is $k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $k \neq 0$.

Only have one eigenvalue and one anti-linearly dependent eigenvector of the matrix does not satisfy the necessary and sufficient conditions, so the matrix is not diagonalizable.

Exercise 2.13: If $|w\rangle$ and $|v\rangle$ are any two vectors, show that $(|w\rangle\langle v|)^\dagger = |v\rangle\langle w|$.
Answer:
Suppose A , B is any linear operator on a Hilbert space V. For all vectors $|w\rangle, |v\rangle \in V$, since $(AB)^\dagger = B^\dagger A^\dagger$ and $|v\rangle^\dagger \equiv \langle v|$, we can get $(|v\rangle\langle w|)^\dagger = \langle w|^\dagger |v\rangle^\dagger = |w\rangle\langle v|$.

Exercise 2.14:(Anti-linearity of the adjoint) Show that the adjoint operation is anti-linear, $\left(\sum_i a_i A_i\right)^\dagger = \sum_i a_i^* A_i^\dagger$.

Answer:

Suppose A_i is any linear operator on a Hilbert space V and we set all vectors $|w\rangle, |v\rangle \in V$. According to inner product is linear in the second argument and conjugate-linear in the first argument, we can make the following derivation:

$\left(|v\rangle, \sum_i a_i^* A_i^\dagger \, |w_i\rangle\right) = \sum_i a_i^* \left(|v\rangle, A_i^\dagger \, |w_i\rangle\right)$
$= \sum_i a_i^* \left(A_i|v\rangle, |w_i\rangle\right) = \left(\sum_i a_i A_i|v\rangle, |w_i\rangle\right) = \left(|v\rangle, (\sum_i a_i A_i)^\dagger \, |w_i\rangle\right)$.
Thus the adjoint operation is anti-linear, $(\sum_i a_i A_i)^\dagger = \sum_i a_i^* A_i^\dagger$.

Exercise 2.15: Show that $\left(A^\dagger\right)^\dagger = A$.

Answer:

Suppose A is any linear operator on a Hilbert space V and we set $|w_i\rangle, |v\rangle \in V$.
Since $\left(|v\rangle, A^\dagger|w\rangle\right) = \left(A^\dagger|w\rangle, |v\rangle\right)^* = \left(|w\rangle, A|v\rangle\right)^* = \left(A|v\rangle, |w\rangle\right)$ and $\left(|v\rangle, A^\dagger|w\rangle\right) = \left[\left(A^\dagger\right)^\dagger |v\rangle, |w\rangle\right]$.
Thus we proved that $\left(A^\dagger\right)^\dagger = A$.

Exercise 2.16: Show that any projector P satisfies the equation $P^2 = P$.

Answer:

Suppose V is a Hermite space, W is the k-dimensional subspace of d-dimensional vector space V. Using the Gram-Schimdt process, we can construct $|1\rangle, |2\rangle, \ldots |d\rangle$ as a set of standard orthogonal basis of V, so that $|1\rangle, |2\rangle, \ldots |k\rangle$is a standard orthogonal basis of $W, P \equiv \sum_{i=1}^k |i\rangle\langle i|$.

The proof process is as follows:

$$P^2 = (\sum_{i=1}^k |i\rangle\langle i|)(\sum_{j=1}^k |j\rangle\langle j|) = \sum_{i=1}^k \sum_{j=1}^k |i\rangle\langle i|j\rangle\langle j|$$
$$= \sum_{i=1}^k \sum_{j=1}^k |i\rangle\delta_{ij}\langle j| = \sum_{i=1}^k |i\rangle\langle i| = P.$$

Exercise 2.17: Show that a normal matrix is Hermitian if and only if it has real eigenvalues.

Answer:

Suppose A is a normal matrix, so it can be given a spectral decomposition, $A = \sum_i \lambda_i |i\rangle\langle i|$ and $A^\dagger = \sum_i \lambda_i^* |i\rangle\langle i|, (\lambda_i \geq 0)$.
Since A is a Hermitian operators, we have $A = A^\dagger$, then $\sum_i \lambda_i |i\rangle\langle i| = \sum_i \lambda_i^* |i\rangle\langle i|$. Thus we can get $\lambda_i = \lambda_i^*$, then $\lambda_i \in R$. Since $\lambda_i \in R$, we can get $\lambda_i = \lambda_i^*$ and $\sum_i \lambda_i |i\rangle\langle i| = \sum_i \lambda_i^* |i\rangle\langle i|$. Thus we can get $A = A^\dagger$. A is Hermitian.
Above all, we proved that a normal matrix is Hermitian if and only if it has real eigenvalues.

Exercise 2.18: Show that all eigenvalues of a unitary matrix have modulus 1, that is, can be written in the form $e^{i\theta}$ for some real θ.

Answer:

Suppose U is a unitary matrix, thus U is normal and can be given a spectral decomposition, $U = \sum_i \lambda_i |i\rangle\langle i|, (\lambda_i \geq 0)$. The derivation process is as follows:
U satisfies $UU^\dagger = I(I \equiv \sum_i |i\rangle\langle i|)$.
$UU^\dagger = \left(\sum_i \lambda_i |i\rangle\langle i|\right)\left(\sum_i \lambda_i |i\rangle\langle i|\right)^\dagger = \sum_i \lambda_i \lambda_i^* |i\rangle\langle i| = I$.
$\sum_i \lambda_i \lambda_i^* |i\rangle\langle i| = \sum_i |i\rangle\langle i| \Rightarrow \forall i, \lambda_i \lambda_i^* = 1$.
Since $\lambda_i \lambda_i^* = 1$, we can get $\|\lambda_i\| = 1$.
Let $\lambda_i = e^{i\theta} = \cos\theta + i\sin\theta$, then $\lambda_i^* = e^{-i\theta} = \cos\theta - i\sin\theta$.
Since $e^{i\theta} * (e^{i\theta})^* = (\cos\theta + i\sin\theta) * (\cos\theta - i\sin\theta) = 1$, λ_i can be written in the form $e^{i\theta}$ for some real θ.

Exercise 2.19: (Pauli matrices: Hermitian and unitary) Show that the Pauli matrices are Hermitian and unitary.
Answer:
For Y is an example.
Hermitian:

$$Y^\dagger = (Y^*)^{\mathrm{T}} = \left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^*\right)^{\mathrm{T}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = Y.$$

Unitary:

$$Y^\dagger Y = (Y^*)^T Y = \left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^*\right)^T \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^T \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

Thus Y is Hermitian and unitary.
Other cases will also reach corresponding conclusions according to the above calculations.

Exercise 2.20: (Basis changes) Suppose A' and A'' are matrix representations of an operator A on a vector space V with respect to two different orthonormal bases,$|v_i\rangle$ and $|w_j\rangle$.Then the elements of A' and A'' are $A'_{ij} = \langle v_i|A|v_j\rangle$ and $A''_{ij} = \langle w_i|A|w_j\rangle$. Characterize the relationship between A' and A''.
Answer:
Suppose $U \equiv \sum_i |w_i\rangle\langle v_i|, U$ is unitary operator, then we can make the following derivation:

$$
\begin{aligned}
A'_{ij} &= \langle v_i | A | v_j \rangle \\
&= \left\langle v_i \left| U U^\dagger A U U^\dagger \right| v_j \right\rangle \\
&= \sum_{p,q,r,s} \langle v_i | w_p \rangle \langle v_p | v_q \rangle \langle w_q | A | w_r \rangle \langle v_r | v_s \rangle \langle w_s | v_j \rangle \\
&= \sum_{p,q,r,s} \langle v_i | w_p \rangle \delta_{pq} A''_{qr} \delta_{rs} \langle w_s | v_j \rangle \\
&= \sum_{p,r} \langle v_i | w_p \rangle \langle w_r | v_j \rangle A''_{pr}.
\end{aligned}
$$

Suppose P=$\sum_{ij} p_{ij}$ is a row elementary matrix and its elements are $p_{ij} = \langle w_j | v_i \rangle$, we can get $A'_{ij} = \sum_{p,r} p^*_{ip} A''_{pr} p_{jr}$. Thus there is a row elementary matrix P between A and B such that $A' = P^\dagger A'' P$ holds.