1. Prove that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

Answer:

Firstly, we set A is Hermitian, then we set i and j are different eigenvalues of A, and the corresponding eigenvectors are |vi and |vj . Secondly, we have that

$$\begin{aligned}
(|v_i\rangle, A|v_j\rangle) =& (|v_i\rangle, \lambda|v_j\rangle) \\
=& \lambda_j(|v_i\rangle, |v_j\rangle) \\
=& \lambda_j \langle v_i|v_j\rangle \\
(|v_i\rangle, A|v_j\rangle) =& (A^\dagger|v_i\rangle, |v_j\rangle) \\
=& (A|v_i\rangle, |v_j\rangle) \\
=& (\lambda_i|v_i\rangle, |v_j\rangle) \\
=& \lambda_i \langle v_i|v_j\rangle .
\end{aligned} \tag{1}$$

From above we can know that $\lambda_j \langle v_i|v_j\rangle = \lambda_i \langle v_i|v_j\rangle$. Since $\lambda_i \neq \lambda_j$, we have that $\langle v_i|v_j\rangle = 0$. Finally, we can conclude that two eigenvectors of a Hermitian operator with different eigenvalues are necessarily orthogonal.

2. Suppose a composite of systems A and B is in the state $|a\rangle|b\rangle$, where $|a\rangle$ is a pure state of system A, and $|b\rangle$ is a pure state of system B. Show that the reduced density operator of system A alone is a pure state.

Answer:

$$\begin{aligned}
\rho^{AB} =& |a\rangle\langle a| \otimes |b\rangle\langle b| \\
\rho^{A} =& tr_B(\rho^{AB}) \\
=& |a\rangle\langle a| \, tr(|b\rangle\langle b|) \\
=& |a\rangle\langle a| \langle b|b\rangle \\
=& |a\rangle\langle a| \\
tr((\rho^A)^2) =& tr(|a\rangle\langle a|a\rangle\langle a|) \\
=& tr(|a\rangle\langle a|) \\
=& \langle a|a\rangle \\
=& 1
\end{aligned} \tag{2}$$

Thus ρ^A is pure.