Preface

Mathematical modeling is the link between mathematics and the rest of the world. You ask a question. You think a bit, and then you refine the question, phrasing it in precise mathematical terms. Once the question becomes a mathematics question, you use mathematics to find an answer. Then finally (and this is the part that too many people forget), you have to reverse the process, translating the mathematical solution back into a comprehensible, no–nonsense answer to the original question. Some people are fluent in English, and some people are fluent in calculus. We have plenty of each. We need more people who are fluent in both languages and are willing and able to translate. These are the people who will be influential in solving the problems of the future.

This text, which is intended to serve as a general introduction to the area of mathematical modeling, is aimed at advanced undergraduate or beginning graduate students in mathematics and closely related fields. Formal prerequisites consist of the usual freshman–sophomore sequence in mathematics, including one–variable calculus, multivariable calculus, linear algebra, and differential equations. Prior exposure to computing and probability and statistics is useful, but is not required.

Unlike some textbooks that focus on one kind of mathematical model, this book covers the broad spectrum of modeling problems, from optimization to dynamical systems to stochastic processes. Unlike some other textbooks that assume knowledge of only a semester of calculus, this book challenges students to use *all* of the mathematics they know (because that is what it takes to solve real problems).

The overwhelming majority of mathematical models fall into one of three categories: optimization models; dynamic models; and probability models. The type of model used in a real application might be dictated by the problem at hand, but more often, it is a matter of choice. In many instances, more than one type of model will be used. For example, a large Monte Carlo simulation model may be used in conjunction with a smaller, more tractable deterministic dynamic model based on expected values.

This book is organized into three parts, corresponding to the three main categories of mathematical models. We begin with optimization models. A five-step method for mathematical modeling is introduced in Section 1 of Chapter 1, in the context of one–variable optimization problems. The remainder of the first chapter is an introduction to sensitivity analysis and robustness. These fundamentals of mathematical modeling are used in a consistent way throughout the rest of the book. Exercises at the end of each chapter require students to master them as well. Chapter 2, on multivariable optimization, introduces decision variables, feasible and optimal solutions, and constraints. A review of the method of Lagrange multipliers is provided for the benefit of those students who were not exposed to this important technique in multivariable calculus. In the section on sensitivity analysis for problems with constraints, we learn that Lagrange multipliers represent shadow prices (some authors call them dual variables). This sets the stage for our discussion of linear programming later in Chapter 3. At the end of Chapter 3 is a section on discrete optimization that was added in the second edition. Here we give a practical introduction to integer programming using the branch–and–bound method. We also explore the connection between linear and integer programming problems, which allows an earlier introduction to the important issue of discrete versus continuous models. Chapter 3 covers some important computational techniques, including Newton's method in one and several variables, and linear and integer programming.

In the next part of the book, on dynamic models, students are introduced to the concepts of state and equilibrium. Later discussions of state space, state variables, and equilibrium for stochastic processes are intimately connected to what is done here. Nonlinear dynamical systems in both discrete and continuous time are covered. There is very little emphasis on exact analytical solutions in this part of the book, since most of these models admit no analytic solution. At the end of Chapter 6 is a section on chaos and fractals that was added in the second edition. We use both analytic and simulation methods to explore the behavior of discrete and continuous dynamic models, to understand how they can become chaotic under certain conditions. This section provides a practical and accessible introduction to the subject. Students gain experience with sensitive dependence to initial conditions, period doubling, and strange attractors that are fractal sets. Most important, these mathematical curiosities emerge from the study of real–world problems.

Finally, in the last part of the book, we introduce probability models. No prior knowledge of probability is assumed. Instead we build upon the material in the first two parts of the book, to introduce probability in a natural and intuitive way as it relates to real–world problems. Chapter 7 introduces the basic notions of random variables, probability distributions, the strong law of large numbers, and the central limit theorem. At the end of Chapter 7, Introduction to Probability Models, is a section on diffusion, which was added in the third edition. Here we give a gentle introduction to partial differential equations by focusing on the diffusion equation. We provide a simple derivation of the point source solution to this partial differential equation, using Fourier transforms, to arrive at the normal density. Then we connect the diffusion model to the central limit theorem introduced in the previous Section 7.3, Introduction to Statistics. This new section on diffusion grew out of a class taught at the University of Nevada for beginning graduate students in the earth sciences. The applications are to contaminant migration in the atmosphere and ground water. Chapter 8 covers the basic models of stochastic processes, including Markov chains, Markov processes, and linear regression. At the end of Chapter 8, Stochastic Models, a new section on time series was added in the third edition. This section also serves as an introduction to multivariate regression models with more than one predictor. As a natural follow-up to the discussion in Section 8.3, Linear Regression, the new section on time series introduces the important idea of correlation. It also shows how to recognize correlated variables and include the dependence structure in a time series model. The discussion is focused on autoregressive models, since these are the most generally useful time series models. They are also the most convenient, in that they can be handled using widely available linear regression software. For the benefit of students with access to a statistical package, this section illustrates the proper application and interpretation of advanced methods including autocorrelation plots and sequential sums of squares. However, the entire section can also be covered using only a basic implementation of regression that allows multiple predictors and outputs the two basic measures: R^2 and the residual standard deviation s. This can all be done with a good spreadsheet or hand calculator. Chapter 9 treats simulation methods for stochastic models. The Monte Carlo method is introduced, and Markov property is applied to create efficient simulation algorithms. Analytic simulation methods are also explored, and compared to the Monte Carlo method. In the fourth edition, two new sections were added to the end of Chapter 9. The first new section covers particle tracking methods, for solving partial differential equations via Monte carlo simulation of the underlying stochastic process. The final section of the book introduces fractional calculus in the context of anomalous diffusion. The fractional diffusion equation is solved by particle tracking, and applied to a problem in ground water pollution. This section ties together the concepts of fractals, fractional derivatives, and probability distributions with heavy tails.

Each chapter in this book is followed by a set of challenging exercises. These exercises require significant effort, as well as a certain amount of creativity, on the part of the student. I did not invent the problems in this book. They are real problems. They were not designed to illustrate the use of any particular mathematical technique. Quite the opposite. We will occasionally go over some new mathematical techniques in this book *because the problem demands it.* I was determined that there would be no place in this book where a student could look up and ask, “What is all of this for?” Although typically oversimplified or grossly unrealistic, story problems embody the fundamental challenge in applying mathematics to solve real problems. For most students, story problems present plenty of challenge. This book teaches students how to solve story problems. There is a general method that can be applied successfully by any reasonably capable student to solve any story problem. It appears in Chapter 1, Section 1. This same general method is applied to problems of all kinds throughout the text.

Following the exercises in each chapter is a list of suggestions for further reading. This list includes references to a number of UMAP modules in applied mathematics that are relevant to the material in the chapter. UMAP modules can provide interesting supplements to the material in the text, or extra credit projects. All of the UMAP modules are available at a nominal cost from the Consortium for Mathematics and Its Applications (`www.comap.com`).

One of the major themes of this book is the use of appropriate technology for solving mathematical problems. Computer algebra systems, graphics, and numerical methods all have their place in mathematics. Many students have not had an adequate introduction to these tools. In this course we introduce modern technology in context. Students are motivated to learn because the new technology provides a more convenient way to solve real–world problems. Computer algebra systems and 2–D graphics are useful throughout the course. Some 3–D graphics are used in Chapters 2 and 3 in the sections on multivariable optimization. Students who have already been introduced to 3–D graphics should be encouraged to use what they know. Numerical methods covered in the text include, among others, Newton's method, linear programming, the Euler method, linear regression, and Monte Carlo simulation.

The text contains numerous computer–generated graphs, along with instruction on the appropriate use of graphing utilities in mathematics. Computer algebra systems are used extensively in those chapters where significant algebraic calculation is required. The text includes computer output from the computer algebra systems Maple and Mathematica in Chapters 2, 4, 5, and 8. The chapters on computational techniques (Chapters 3, 6, and 9) discuss the appropriate use of numerical algorithms to solve problems that admit no analytic solution. Sections 3.3 and 3.4 on linear-integer programming include computer output from the popular linear programming package LINDO. Sections 8.3 and 8.4 on linear regression and time series include output from the commonly used statistical package Minitab.

Students need to be provided with access to appropriate technology in order to take full advantage of this textbook. We have tried to make it easy for instructors to use this textbook at their own institution, whatever their situation. Some will have the means to provide students with access to sophisticated computing facilities, while others will have to make do with less. The bare necessities include: (1) a software utility to draw 2–D graphs; and (2) a machine on which students can execute a few simple numerical algorithms. All of this can be done, for example, with a computer spreadsheet program or a programmable graphics calculator. The ideal situation would be to provide all students access to a good computer algebra system, a linear programming package, and a statistical computing package. The following is a partial list of appropriate software packages that can be used in conjunction with this textbook.

Computer Algebra Systems:

- Derive, Chartwell-Yorke Ltd., `www.chartwellyorke.com/derive`
- Maple, Waterloo Maple, Inc., `www.maplesoft.com`
- Mathcad, Parametric Technology Corp., `www.ptc.com/products/mathcad`
- Mathematica, Wolfram Research, Inc., `www.wolfram.com/mathematica`
- MATLAB, The MathWorks, Inc., `www.mathworks.com/products/matlab`
- Maxima, free download, `maxima.sourceforge.net`

Statistical Packages:

- Minitab, Minitab, Inc., `www.minitab.com`
- SAS, SAS Institute, Inc., `www.sas.com`
- SPSS, IBM Corp., `www.ibm.com/software/analytics/spss`
- S-PLUS, TIBCO Corp., `spotfire.tibco.com`
- R, R Foundation for Statistical Computing, free download, `www.r-project.org`

Linear Programming Packages:

- LINDO, LINDO Systems, Inc., `www.lindo.com`
- MPL, Maximal Software, Inc., `www.maximal-usa.com`
- AMPL, AMPL Optimization, LLC, `www.ampl.com`
- GAMS, GAMS Development Corp., `www.gams.com`

The numerical algorithms in the text are presented in the form of pseudocode. Some instructors will prefer to have students implement the algorithms on their own. On the other hand, if students are not going to be required to program, we want to make it easy for instructors to provide them with appropriate software. All of the algorithms in the text have been implemented on a variety of computer platforms that can be made available to users of this textbook at no additional cost. If you are interested in obtaining a copy, please contact the author, or go to `www.stt.msu.edu/users/mcubed/modeling.html` where you can download these implementations. Also, if you are willing to share your own implementation with other instructors and students, please send us a copy. With your permission, we will make copies available to others at no charge.

A complete and detailed solutions manual for instructors is available from the author or the publisher, for instructors who adopt the text for classroom use. Computer implementations of the algorithms used in the text can be downloaded for a variety of platforms, along with the computer files used to produce all of the graphics and computer outputs included in the text. These downloads are all available at `www.stt.msu.edu/users/mcubed/modeling.html`

The response to the first three editions of the text has been gratifying. The best part of this job is interacting with students and instructors who use this book. Please feel free to contact me with any comments or suggestions.

Mark M. Meerschaert
Department of Statistics and Probability
Michigan State University
C430 Wells Hall
East Lansing, MI 48824-1027 USA

Phone: (517) 353-8881
Fax: (517) 432-1405
Email: mcubed@stt.msu.edu
Web: www.stt.msu.edu/users/mcubed