CAS MA293
Spring 21

Challenge Problems 3 - Boxes and Salads

Introduction:
In this paper, the number of possible ways to color the box under certain conditions will be discussed. This problem consists six parts in which two different types of coloring will be found for three different shapes of boxes, including code, square prism, and rect prism with 3 different face shapes. And boxes changing orientation with the same coloring does not count as a separate coloring. The process of finding the possible ways of coloring each box will be stated and justified.

Types of coloring:

1. There are six possible colors, and each face is a different color.
2. There are two colors —black and white—and each face is painted one of the colors. Repeats are allowed, and all faces could be the same color.

Solution & Justification:

Cube with six possible colors:

We can determine bottom face as any color so there are 5C1 ways to select color for the top face, which leaves 4! ways for the other four faces of the cube. However, because the rotational symmetry which is stated in the game that when boxes change their orientation with the same coloring does not count as a separate coloring, 4! Needs to divide 4. And the total possible ways to color the boxes with different color of each side is 5C1*4!/4=30.

Cube with two colors (black and white):

When there are 1 black and 5 white or 1 white and 5 black, the total ways of coloring are 2 since there is 1 way for each.

When there are 2 black and 4 white or 2 white and 4 black, the total ways of coloring are 4 since there are 2 ways for each.

When there are 3 black and 3 white, the total ways of coloring are 2 since there are 1 way for each.

When there are 6 black or 6 white sides of the cube, the total ways of coloring are 2.

Thus, the total number of coloring the cube with two colors (black and white) is 2+2+4+2=10.

Square Prism (Not Cube) with six possible colors:

Since the square prism has 2 squared faces and 4 rectangular faces, we can determine the squared faces by any color we want, for example, purple. When the squared face is on the top, there are 4 ways of rotational conformation in which the total ways of rotational conformation for two squared faces are 4*2=8. And for the coloring with six different colors, the first face will have 6 choices, second will have 5 choices and so on. The total ways of coloring the square prism are 6!. With the consideration of 8 rotational conformations, the total ways of coloring the square prism with six possible colors are 6!/8=90.

Square Prism (Not Cube) with two colors (black and white):
Assume: S=Square faces, R=Rectangular faces, B=Black, W=White

When there are 6 black or 6 white, there are 2 ways of coloring since one way for each.

When there are 1 black and 5 white or 1 white and 5 black, there are 4 ways of coloring because for the first condition, the possible ways of coloring will be 1WS+1BS+4WR & 2WS+3WR+1BR. And for the second condition, the possible ways of coloring will be 1WS+1BS+4BR & 2BS+3B3+1BS.

When there are 2 black and 4 white or 2 white and 4 black, there are 8 ways of coloring because for the first condition, the possible ways of coloring will be 2BS+4WR, 1BS+1WS+1BR+3WR, 2WS+2BR+2WR (either 2BR across to each other or 2BR next to each other). And for the second condition, the possible ways of coloring will be 2WS+4BR, 1WS+1BS+1WR+3BR, 2BS+2WR+2BR (either 2WR across to each other or 2WR next to each other).

When there are 3 black and 3 white, the total ways of coloring are 4 because for the first condition, the possible ways of coloring will be 2BS+1BR+3WR, 2WS+1WR+3BR, 1BS+1WS+2BR+2WR (BR across or next to each other).

Thus, the total number of coloring the square prism with two colors (black and white) is 2+4+8+4=18.

Rect Prism with 3 Different Face Shapes with six possible colors:

Since the rect prism has two identical width faces and there are 2 identical rotations for each rectangular side, there are 2*2=4 configurations. And for the coloring with six different colors, the first face will have 6 choices, second will have 5 choices and so on. The total ways of coloring the rect prism are 6!. With the consideration of 4 possible configurations, the total ways of coloring the square prism with six possible colors are 6!/4=180.

Rect Prism with 3 Different Face Shapes with two colors (black and white):

When there are 6 white or 6 black, there are 2 ways of coloring since one way for each.

When there are 1 white and 5 black or 1 black and 5 white, there are 6 ways of coloring because for the first condition, white can be either on length, width, or the height of the rect prism, and the rest of the faces are black color. This results in 3 ways for the first condition. And it is the same with the second condition, which is also 3 ways of coloring.

When there are 2 white and 4 black or 2 black and 4 white, there are 12 ways of coloring because for the first condition, 2 white can be either on length, width, or the height of the rect prism; also, it can be 1 white on length & 1 white on width, 1 white on width & 1 white on height, and 1 white on height & 1 white on length. So there are 6 ways for the first condition. Since the second condition is the same with the first one, there are 12 ways in total for coloring the rect prism.

When there are 3 white and 3 black, there are 7 ways of coloring in which they involve 2 white on width & 1 white on height, 2 white on width & 1 white on length, 2 white on length & 1 white on height, 2 white on length & 1 white on width, 2 white on height & 1 white on width, 2 white on height & 1 white on length, and 1 white on width & 1 white on length & 1 white on height.

Process:
At the beginning, I viewed this problem from an incorrect perspective. I thought since there are six faces for each shape, there are 6! ways to color the shapes with six possible colors. However, I didn't take the fact that the rotational configuration will affect the result of the number of coloring each shape of box.

With the help of drawing the layout of each shape of box, it is quite clear to identify different sides, for example, rectangular sides and square sides. And it also helps me to classify the sides of shapes into groups and gives it a name, like bottom or top. It sounds simple but these labels indeed help me to better illustrate the problem and solve it in a proper manner.

Acknowledgments:
I worked with my team member Yunhong Yang, Xinyu Zhu, and Xintong Jiang. The four of us drew the layout of the shape together and found the mistakes in our first try. And then, we labeled each side of the shape to better understand this problem together. After counting the number of ways for each shape of box under different conditions, we work individually to finish our own paper.

I acknowledge that I collaborated on this problem only with classmates from our section of CAS MA 293, I only used allowable resources, and I wrote this paper by myself.

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