CAS MA293
Spring 21

Challenge Problem 2- Rows With Rods

Introduction:
Length of the Cuisinair Rods varies by their colors.

Length (cm)	Color	Length (cm)	Color
1	White	6	Dark Green
2	Red	7	Black
3	Light Green	8	Brown
4	Purple	9	Blue
5	Yellow	10	Orange

For each game, a row with Cuisanaire Rods is defined as one or more Cuisanaire Rods being placed end to end, with the defining length of the row. For example, for the row of length 3, it can be formed by 3 white, 1 white+1red, 1red+1white, and 1 green. Note, the order of rods does matter in the game.

<u>Conditions include:</u>
a. Rows use only red and white rods
b. Rows use only red, white, and light green
c. Rows use only red and yellow rods
d. Rows use exactly 3 rods of any positive integers
e. Total number of rows of length n
f. Rows use any rod except white rods

Solutions & Justification:
a. Rows use only red and white rods

Length (cm)	Number of Rows	Math Terminology
1	1	1C1
2	2	2C2+1C0
3	3	3C3+2C1
4	5	4C4+3C2+2C0
5	8	5C5+4C3+3C2
6	13	6C6+5C4+4C2+3C0

Recursive equation: R(n)=R(n-1)+R(n-2), n>=1
Base case: R(1)=1, R(2)=2

In this game, only red(2) and white(1) rods are used for each length of the row. Since white 1 is the smallest rod, the game starts with row of length 1 that has 1 possible combination. Set the first tile to be 1 for white first, and the remaining tile will be n-1 tiles. However, if set the first tile to be 2, and the remaining tile will be n-2. Thus, these two cases contain possible combinations as R(n-1) and R(n-2).

Because the problem only asks for red and white rods, the first tile can only be red or white, thus the number of combinations of rows will be R(n)=R(n-1)+R(n-2).

Mathematical Induction:
Show R(n) is true for all integers that are greater than or equal to 1

Base case: R(1) is the statement that n=1 is true because the row only contains 1 white rod and there's no other way to form the row.

Inductive case: Assume n=k is true, that is, assume R(k)=R(k-1)+R(k-2) is true. We will prove that R(k+1) is also true. To do so, we must prove that R(k+1)=R(k+1-1)+R(k+1-2)= R(k)+R(k-1)

Consider a row with k+1 cm length. The first rod can be white or red. If the first rod is white, the remaining row will be (k+1)-1 cm and the combinations will be R(k). And if the first rod is red, the remaining row will be (k+1)-2 cm and the combinations will be R(k-1). So the total number of rows of k+1 length will be R(k+1)= R(k)+R(k-1). Thus, by the mathematical induction, R(n) is true for all non-zero positive integers n, and the theorem is true for all n=k+1. QED.

b. Rows use only red, white, and light green rods

Length (cm)	Number of Rows	Math Terminology
1	1	1C1
2	2	2C2+1C0
3	4	3C3+2C1+1C0
4	7	4C4+3C2+2C1+2C0
5	13	5C5+4C3+3C2+3C1+2C0
6	24	6C6+5C4+4C2+4C2+3C1+3C0+ 2C0

Recursive equation: R(n)=R(n-1)+R(n-2)+R(n-3), n>=1
Base case: R(1)=1, R(2)=2, R(3)=4

In this game, green(3), red(2) and white(1) rods are used for each length of the row. Since white 1 is the smallest rod, the game starts with row of length 1 that has 1 possible combination. Set the first tile to be 1 for white first, and the remaining tile will be n-1 tiles. However, if set the first tile to be 2, and the remaining tile will be n-2. Last, if the set begins with a green tile (3), there are n-3 tiles remaining. Thus, these three cases contain possible combinations as R(n-1), R(n-2) and R(n-3).

Because the problem only asks for red, white, and green rods, the first tile can only be red, white or green, thus the number of combinations of rows will be R(n)=R(n-1)+R(n-2)+R(n-3).

Mathematical Induction:
Show R(n) is true for all integers that are greater than or equal to 1

Base case: R(1) is the statement that n=1 is true because the row only contains 1 white rod and there's no other way to form the row.

Inductive case: Assume n=k is true, that is, assume R(k)=R(k-1)+R(k-2)+R(k-3) is true. We will prove that R(k+1) is also true. To do so, we must prove that
R(k+1)=R(k+1-1)+R(k+1-2)+R(k+1-3) = R(k)+R(k-1)+R(k-2)

Consider a row with k+1 cm length. The first rod can be white, red, or yellow. If the first rod is white, the remaining row will be (k+1)-1 cm and the combinations will be R(k). And if the first rod is red, the remaining row will be (k+1)-2 cm and the combinations will be R(k-1). Similarly, if the first rod is yellow, the remaining row will be (k+1)-3 cm and the combination will be R(k-2). So the total number of rows of k+1 length will be R(k+1)= R(k)+R(k-1)+R(k-2). Thus, by the mathematical induction, R(n) is true for all non-zero positive integers n, and the theorem is true for all n=k+1. QED.

c. Rows use only red and yellow rods

Length (cm)	Number of Rows
1	0
2	1
3	0
4	1
5	1
6	1
7	2

Length (cm)	Number of Rows
8	1
9	3
10	2
11	4
12	4

Recursive equation: $R(n)=R(n-2)+R(n-5)$, $n=2$ & $n>=5$
Base case: $R(1)=0$, $R(2)=1$, $R(3)=0$, $R(4)=1$, $R(5)=1$

In this game, red and yellow rods are used for each length of the row. Because red has a length of 2 and yellow is a length of 5, rows of length 1 and 3 are not able to formed since they are the common multiples of 2 & 5. And length 2 is also a base case because it can be formed with 1 red tile. As for length 5 of a row, it can be formed by 1 yellow tile.

Starts from length 5 of a row, if the first tile is red, the remaining tiles will be n-2. And if the first tile is yellow, the remaining tiles will be n-5. Thus, these two cases contain possible combinations as $R(n-2)$ and $R(n-5)$.

Because the problem only asks for red and yellow tiles, the first tile can only be red or yellow, thus the number of combinations of rows will be $R(n)=R(n-2)+R(n-5)$.

d. Rows use exactly 3 rods of any positive integer length

Length (cm)	Number of Rows
1	0
2	0
3	1
4	3
5	6
6	10
7	15
8	21

Explicit equation: R(n)= C(n-1,2)

In this game, 3 rods are used for any positive length. Because the smallest length of rod is 1 cm for white, the row began from length of 3 in which rows with length 1 and 2 are 0 under such condition. Assume the length of the row is n, there will be n-1 spaces between rods. You have to choose two places from n-1 space to divide the row into 3 pieces. Since order does not matter in this case, we apply C(n-1,2) for the explicit equation of this problem.

e. Total number of rows of length n

Length (cm)	Number of Rows
1	1
2	2
3	4
4	8
5	16
6	32

Recursive equation: R(n)=1+ the sum of Ri from i=1 to i=n-1
Base case: R(0)=0

In this game, all of the rods will be used for length n in which all the tile combinations can be applied in such scenario. However, the tile combinations for n contains the same tile combination with n-1 but with the increasing number, new tile will be included. For example, for length 1, it is only white. And for length 2, it can be white and yellow. Thus, the difference between the tile combinations is the existence of the new tile, which results in (the sum of Ri from i=1 to I=n-1)+1.

f. Rows use any rod except white rods

Length (cm)	Number of Rows
0	0
1	0
2	1
3	1

Length (cm)	Number of Rows
4	2
5	3
6	5
7	8

Recursive equation: R(n)=1 + the sum of Ri from i=1 to i=n-2
Base case: R(0)=0, R(1)=0

In this game, all the rods will be used except for the white rods. So with the increasing number of length, red rods are the smallest length in which the new existence of the red tile occurs once for each two increasing numbers. For example, for length 2, it is red. For length 3, it is green. And for length 4, it is 2 reds and a purple. Thus, the total number of tile combinations will be the sum of Ri from i=1 and i=n-2. And because for every new length, a new tile can be used like 3 for green and 4 for a purple, there is a plus one in the equation.

Process:

At the beginning, I have no idea about what I should do about this problem. So my teammate and I start list the total number of rods for length n from n to 6. It is tedious but after we listed all of them, we then look at the instruction of certain conditions and put the possible combinations of tiles into each condition. In this way, the table is gradually formed. However, one thing confused us the most is for some of the conditions, the base cases are not connected with each other. For example, part c has length 1 is 0, length 2 is 1 and length 3 is 0 again. Originally, I just began the table with length 4 but after discussing with the professor, I made a more complete table that could help me to see the problem in a more comprehensible way, in which later I added a condition for my recursive equation that made it more understandable and complete (n=2 & n>=5).

As for finding the recursive equations for each table, it is quite obvious by having a complete table, observing the difference between each length, and adding their common combinations together. However, I think the exercise helps me noticing the importance of mathematical conditions for constructing the recursive equations through making tables and writing the base case for each problem.

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I worked with my team member Yunhong Yang, Xinyu Zhu, and Xintong Jiang. The four of us listed the combinations of rods together during class time and formulated a table to record our

discoveries. We discussed the patterns and tried to find the recursive equation for each one of them. And then, we worked individually for our paper.

I acknowledge that I collaborated on this problem only with classmates from our section of CAS MA 293, I only used allowable resources, and I wrote this paper by myself.

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