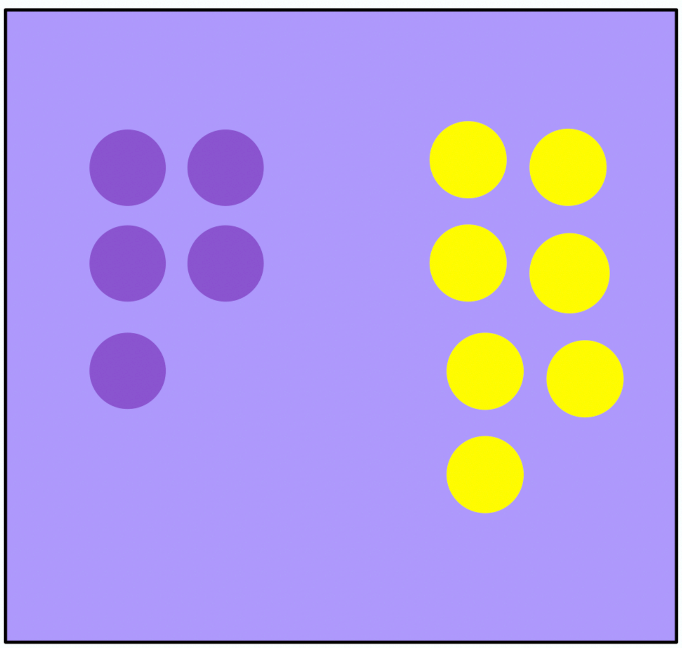
CAS MA293
Spring 21

Challenge Problems 1 – Two-Color Tile Game

Introduction:

In this paper, I will discuss two-color tile game. The strategies to win the game will be stated and justified. In addition, the process of winning the game with conclusions will be shown.

There are two colors of tiles—p purple tiles and y yellow tiles (p and y are non-negative integers) in this game. Two players turn turns to make one move on each turn. On a turn, players are legal to take any number of tiles with the same color or take the same number of purple and yellow tiles. And the last player to take tiles or a tile wins.



For example, if there are 5 purple and 7 yellow tiles, players can take any number that is smaller or equal to 5 of the purple one or any number is smaller or equal to 7 of the yellow tile group. And they could also take same number of both different color tiles, such as 2 yellow tiles and 2 purple tiles. However, they cannot take different numbers of two different color tiles, like 1 purple tile and 2 yellow tiles or take numbers that are greater than the total number of each tile group, including 8 yellow tiles (since there are only 7 yellow tiles).

Through classifying different situations of the two-color tile game where p+y smaller or equal to 50, losing positions where if it's your turn you loose are found. Patterns are observed in the numbers of the losing positions. And also, in some scenarios, going first and second has the ability to decide who takes control or wins the game. So in this paper, I will discuss all the losing positions that the total number of tiles is less than or equal to 50 and patterns in the losing positions.

Solution:

The losing positions are the positions that will definitely lose if it is your turn. And I assume all the players will make their best moves in the game. I list the all the losing positions within 50 tiles by putting the smaller number in the front. In this way, it is easier to understand the pattern of the game later.

However, in other situations, you can only win if you go to losing positions in one step so you opponents will not have chance to put you on the losing positions again. Therefore, they will definitely lose the game.

Through eliminating tiles beyond the losing positions in which no matter how the other player moves tiles when he/she faces the losing positions, he/she loses. For example, when there are 20 purple tiles and 15 yellow tiles, through eliminating tiles above the “choosing losing positions,” you go first and move 11 purple tiles. Then, it is the 9:15 losing position. No matter how your opponents move, he/she loses.

Difference of two tiles	Losing positions (p+y ≤ 50)
1	1:2
2	3:5
3	4:7
4	6:10
5	8:13
6	9:15
7	11:18
8	12:20
9	14:23
10	16:26
11	17:28
12	19:31

Justification:

I start finding losing positions of the game from the most basic position which is 1:2 position. For 1:2, the second player wins. And for 1:3 tiles, I can take 1 tile to make it 1:2 losing position so I wins. And because all the number of all losing positions are consistent, the first player can move to one of the losing positions first in one step and wins.

Because the losing positions contain number consistently from 1, the player can observe the number of tiles he/she faces and choose the losing position that contains the same initial number and move the game to losing positions in one step so his/her opponents lose definitely. For example, there are 28 purple tiles and 19 yellow tiles, the number 28 corresponds to the losing position of 17:28 so the first player could take 2 yellow tiles and move the game to the 17:28 losing position. Therefore, the second player loses definitely.

What is more, I also observe the effectiveness of the difference of two tiles. The player could choose the losing positions through the difference of the two color tiles on the chessboard. For example, there are 35 purple tiles and 30 yellow tiles. The difference is 5, which corresponds to the losing position of 8:13. So the first player could take 22 purple and 22 yellow tiles. And the second player is moved to the losing position of 8:13 in which he/she loses definitely.

And by listing the losing positions that have been found so far, there is a pattern involved in the number of tiles for the losing positions in regards to their difference. For difference of 1, the losing position is 1:2. For difference of 2, the losing position is 3:5. For difference of 3, losing position is 4:7 and for difference of 4, losing position is 6:10. By putting the smaller number at the front, we can see 1,2,3,4,5 have already involved in the losing positions. With an existence of 7, it is clear that 6 will be occurred soon in the losing position as the smaller number of that pattern. By adding the difference subsequently, we find 6+4=10 and test it as a new losing position. As for the following losing positions, it could be calculated by adding the difference to the smallest real number that have not included in the previous losing positions. For example, since 6, 7 and 10 are included, the next smallest real number is 8. And we can add 8 to its corresponding difference in the table and gets a losing position of 8:13. In conclusion, the most important thing is to find the smallest number that did not appear in the front losing positions. This number will be the smallest number of the new losing position and the bigger number of that losing position will be the sum of the corresponding difference and the smaller number.

PS: the reason the losing position for difference 4 is (6,10) instead of (5,9) is that 5 has already occurred in the preceding losing position for (3,5). Therefore, if we start with (5,9), we can take out 6 from 9, which creates a losing position for the opponents in which it is not a losing position. Since the smallest number that did not appear in the front losing positions is 6, by adding its corresponding difference of 4, the losing position is (6,10)

In order to find all the losing positions that the total number of tiles is smaller or equal to 50, the remaining pairs of losing positions could be found with the above pattern based on the rules of consistency of the numbers.

Process:

I played this game with my group members. At the beginning, we have no ideas about how to find the strategies of two-color tiles game even though we know how to find the losing position for the one-color tile game. So we focused on the first player, whether the first player decides the game. Although this is convincing that the first player can take control of the game, we do not know how to move tiles to a determined losing position. So we began with the example that is 5 purple and 7 yellow tiles. By following the rules of the game, we reached 1:2 and found that no matter the player does what, he/she loses. So we recorded the number and increase the yellow tiles to 8, which leads to an increasing difference of 1. For the first round of 5:8, we still focused on the ending losing position of 1:2, instead of discovering a new pair of losing position that is 3:5. However, after playing this game for a few more times with changing moves, we finally noticed the new pair of losing position that is 3:5. With the increasing difference between two colors of tiles, we eventually found the first 4 sets of losing positions and noticed a consistency pattern involved in the numbers of losing positions. Based on the difference and one number based on the pattern, we formed several groups of losing positions and verified them step by step as listed above for all the p+y smaller or equal to 50.

Revision:

I mainly changed my solution and justification parts of this paper. Originally, they were not quite logically described. I added too many examples to show how I proceeded the problem but they did not show the key points of the losing positions, which are closely related to the initial number position and the difference between two color tiles. So I changed my words in this revision and only include useful examples to make it more general for discussing a pattern, which could make it easier to be understood and followed.

For the second revision, I include more details of how I constructed the table of losing positions with the game. And also, I add more process of how I pick the next losing positions and also examples to show my process. Besides that, I used counterexamples to show my process of getting the next new losing positions.

Acknowledgements:

I worked with my team member Yunhong Yang and Xinyu Zhu. The three of us played together during class time and formulated a table to record our discoveries. We discussed rules and tried different moves for each game to verify the losing position. And then, we worked individually for the final paper.

I acknowledge that I collaborated on this problem only with classmates from our section of CAS MA 293, I only used allowable resources, and I wrote this paper by myself.

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